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MELVYN L. GOLDSTEIN

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GSFC
GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

A NONLINEAR THEORY OF COSMIC RAY PITCH ANGLE DIFFUSION IN HOMOGENEOUS MAGNETOSTATIC TURBULENCE

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Melvyn L. Goldstein
Laboratory for Extraterrestrial Physics
NASA-Goddard Space Flight Center
Greenbelt, MD

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ABSTRACT

A plasma strong turbulence, weak coupling, theory is applied to the problem of cosmic ray pitch angle scattering in magnetostatic turbulence. The theory used is a rigorous generalization of Weinstock's "resonance-broadening" theory and contains no ad hoc approximations. A detailed calculation is presented for a model of "slab" turbulence with an exponential correlation function. The results agree well with numerical simulations. The rigidity dependence of the pitch angle scattering coefficient differs from that found by previous researchers. The differences result from an inadequate treatment of particle trajectories near 90^{o} pitch angle in earlier work.

Subject headings: cosmic rays-hydromagnetics-plasmas

I. INTRODUCTION

Central to the problem of understanding the origin of cosmic rays is the difficulty of computing the motion of charged particles through regions of strongly turbulent electromagnetic fields that characterize interstellar and interplanetary space. The basic problem is to compute a spatial diffusion coefficient starting from knowledge of the statistical properties of the electromagnetic turbulence. To do this one needs a valid kinetic theory that describes particle motion in strongly turbulent fields. In general, such a theory is not available. In this paper, we are interested in the special circumstance in which the cosmic ray distribution neither modifies the fields with which it interacts, nor undergoes Coulomb collisions with other particles. The first partially successful efforts to develop such a kinetic theory (Jokipii 1966; Roelof 1966; Hall and Sturrock 1967; Hasselmann and Wiberenz 1968) made use of the quasilinear approximation (v., e.g. Rowlands, Shapiro, and Shevchenko 1966). In this approximation, it is assumed that the deviation of a particle's trajectory from its helical motion in the mean magnetic field is small because the strength of the fluctuation field is assumed to be weak. It is then possible to compute a pitch angle scattering coefficient D_{μ}, that describes particle diffusion in turbulent magnetic fields (μ is the cosine of the particles' pitch angle with respect to the mean magnetic field). From D_{μ} one can compute a spatial diffusion coefficient (Earl 1974). In this paper we will discuss only the derivation of the pitch angle scattering coefficient, D_{μ}.

Recently, several aspects of the quasilinear approximation have been questioned (Klimas and Sandri 1973; Jones, Birmingham and Kaiser 1973c; Kaiser, Jones and Birmingham 1973; Volk 1973). The difficulties with quasilinear theory are associated with the behavior of particles with pitch angles, θ, nar $\pi/2$, where either D_{μ} ($\mu = 0$) = 0 (when the fluctuation fields cannot mirror particles to first order), or $D_{\mu}(\mu=0)$ $\propto \delta(\mu)$ [when first order mirroring forces do exist (Fisk, Goldstein, Klimas, and Sandri 1974; Goldstein, Klimas and Sandri 1975; also cf. Lee and Völk 1975)].

In the past two years, several attempts have been made to improve quasilinear theory (Jones, Kaiser and Birmingham 1973a,b; Völk 1973; Owens 1974). Owens' work replaces two of the nonlinear terms that are dropped in quasilinear theory with a constant. This results in a small amount of scattering at $\theta = \pi/2$, which is insufficient to explain the significant scattering through $\theta = \pi/2$ observed in numerical simulations of the problem (Kaiser 1974). Völk 1973 modified quasilinear theory by incorporating a better treatment of the particle orbits. He recognized the importance of magnetic mirroring (particle trapping) effects in his description of the scattering. His approach is similar to that of Dupree (1966, 1967) for electrostatic turbulence. His results are also in rather poor agreement with the numerical simulations (Kaiser 1974). This apparently results from a breakdown in several of the simplifying approximations made in his analysis.

The most successful theoretical effort to date to improve quasilinear theory is that of Jones, Kaiser and Birmingham (1973a,b). Their analysis is motivated by the nonlinear (electrostatic) turbulence theory of Weinstock (1969). Jones, et al. (1973a,b) introduce the concept of partial averaging which enables them to include mirroring effects in a physically plausible way. The basic approximation made by Jones, et al.(1973a,b) is to introduce the propagator U_p, which describes the motion of particles through the partially averaged field. They argue, but do not prove, that U_p provides a useful approximation to U_A, the exact propagator in Weinstock's theory. (A mathematical definition of U_A is given in §II). The results of these computations appear to agree nicely with the three numerical simulations presented by Kaiser (1974). However, both the simulations and the theoretical analysis are presently being revised (Kaiser, private communication).

The analysis presented in this paper is a derivation that starts with the Vlasov equation and proceeds in a rigorous manner. The particle trajectories that are computed are more general than those in previous work. We show that these higher order corrections are necessary to obtain a good approximation to D_μ at μ near zero. Our analysis follows the ideas of Weinstock (1969), Piran (1972) and Ben-Israel, Piran, Eviatar and Weinstock (1975). These latter two papers are generalizations of Weinstock's weak coupling electrostatic turbulence theory to include electromagnetic effects. As shown below, this generalization is not a trivial exercise, and it is not

surprizing that the results of Völk (1973) are quantitatively inaccurate. It is not necessary to introduce partial averaging or any other heuristic ideas in the analysis that follows. The results shown in §V are in good agreement with the simulations that are presently available for comparison.

In the next section we outline the theoretical framework of the turbulence theory under the fundamental assumption that the coupling is weak (Kadomstev 1965). Much of the general theory has been developed by Piran (1972) and Ben-Israel, et al. (1975) and we will not repeat their detailed formulation here. The general theory leads to equations that cannot be solved without limiting one's attention to specific models of wave turbulence. One such model of interest for the problem of cosmic ray pitch angle scattering, is homogeneous "slab" magnetostatic turbulence with an exponential correlation function.

Because cosmic ray velocities are typically much greater than the phase velocities of the magnetohydrodynamic waves that scatter them, the magnetostatic approximation should be very well satisfied. This is equivalent to dropping the Maxwell "curl" equations in the analysis. In confining our attention to "slab" turbulence, we consider only magnetic fluctuations that vary along the direction of the mean magnetic field. This greatly simplifies the analysis. Waves observed in the interplanetary medium near 1 AU are thought to be predominantly Alfvèn waves propagating along the mean field direction. A magnetostatic slab model approximates this situation quite well. The specific examples computed below are limited to an exponential correlation function for

the turbulence. This is done to compare this theory with numerical simulations of this problem that have thus far been confined to an exponential correlation function (Kaiser 1974). Generalization to other correlation functions and to isotropic turbulence is straightforward and will be considered in future papers.

Section III is devoted to a detailed calculation of $D_\mu(\mu = o)$ valid to second order in the strength of the fluctuation field. All nonlinear terms are included except those that are proportional to the fluctuation in the parallel mementum. They are expected to be unimportant for the reasons discussed below.

In section IV, we estimate D_μ at all values of μ. In doing so, we neglect most terms that are proportional to the deviations of the mean of the momentum from the value expected from quasilinear theory. The validity of this approximation is estimated by comparing the results of section IV at $\mu = o$ with those of section III. For the range of parameters considered, the approximation is good.

In the last section, we compare our results to the simulations and to previous theoretical work. We find that the results of Völk (1973) and Owens (1974) fit neither the numerical simulations (Kaiser 1974) nor our theory. The published numerical simulations do not cover a sufficiently wide range of parameters to distinguish between this theory and that of Jones, et al. (1973a,b). However, the two theories predict different dependencies on the parameters of the theory. We show that the results of Jones et al. (1973a,b) can be recovered from our analysis if we drop terms important for

particle mirroring. We conclude that our analysis contains the first accurate evaluation of particle orbits near 90° pitch angle.

II THEORY

The derivation presented here follows, in broad outline, the work of Piran (1972) and Ben-Israel, Piran, Eviatar and Weinstock (1975). We begin with the Vlasov equation, written in dimensionless form (Klimas and Sandri 1971, 1973),

$$\frac{\partial F}{\partial \tau} + KF + \mathfrak{L}F + \eta \mathfrak{L}' F = 0 \tag{1}$$

for the cosmic ray distribution function $F(\underline{x},\hat{\underline{p}},\tau)$. In equation (1) $\tau = t\omega_c$ where t is dimensional time and ω_c is the gyrofrequency of the particle. In addition, the differential operators K, $\mathfrak{L}$ and $\mathfrak{L}'$ are defined as follows:

$$K = \hat{p}\cdot\frac{\partial}{\partial \underline{x}} \tag{2}$$

$$\mathfrak{L} = -\hat{p}\cdot\underline{\Omega}\cdot\frac{\partial}{\partial \hat{p}} \tag{3}$$

$$\mathfrak{L}' = -\hat{p}\cdot\underline{\Omega}'\cdot\frac{\partial}{\partial \hat{p}} \tag{4}$$

where $\hat{\underline{p}} = \underline{p}/p, \underline{x} = \underline{x}^*/r_g$ ($\underline{x}^*$ is dimensional length and r_g is the Larmor radius) and

$$\frac{\partial}{\partial \hat{p}} \equiv p(\underline{I} - \hat{p}\hat{p})\cdot\frac{\partial}{\partial \underline{p}} \equiv p\hat{n}\cdot\frac{\partial}{\partial \underline{p}} \tag{5}$$

The skew-symmetric tensors $\underline{\Omega}$ and $\underline{\Omega}'$ are defined as $\Omega_{ij} = \epsilon_{ijk}\,\beta_k$ and $\Omega'_{ij} = \epsilon_{ijk}\,\beta'_k$ where $\hat{\underline{\beta}} = \underline{B}_o/B_o$, $\underline{\beta}' = \underline{B}'/\langle \underline{B}'\cdot\underline{B}'\rangle^{1/2}$ and the magnetic field is separated into a mean part, $\underline{B}_o$, plus a fluctuation, $\underline{B}'$. The small parameter is $\eta = \langle \underline{B}'\cdot\underline{B}'\rangle^{1/2}/B_o$. The symbol $\langle\ \rangle$ denotes ensemble average.

We proceed to derive a diffusion-like equation for the average part of the distribution function $f \equiv \langle F\rangle$. We assume that $Kf = 0$. If one takes the ensemble average of equation (1) and subtracts the resulting equation from equation (1), the following relation for $f' \equiv F - \langle F\rangle$ results:

$$\left[\frac{\partial}{\partial\tau} + (1 - A)\mathcal{H}\right] f' = -\eta\mathcal{L}'f \qquad (6)$$

where $\mathcal{H} \equiv K + \mathcal{L} + \eta\mathcal{L}'$ is the total Hamiltonian differential operator and the operator A takes the ensemble average of everything to its right (i.e. $AF \equiv \langle F\rangle$). Owens (1974) takes the nonlinear terms (1-A) $\eta\mathcal{L}'$ on the left hand side of equation (6) and sets them equal to a constant, $\hat{\alpha}$. This severe approximation results in a gross undirestimation of the scattering through $\mu = o$.

Equation (6) can be solved formally in terms of the propagator $U_A(\tau, \tau_o)$, which is defined by the following differential equation (Weinstock 1970; Ben-Israel, et al. 1975):

$$\left[\frac{\partial}{\partial\tau} + (1 - A)\mathcal{H}\right] U_A\,(\tau - \tau_o) = 0 \qquad (7)$$

$$U_A\,(0) = 1$$

In writing equation (7), we used the fact that for magnetostatic turbulence $U_A(\tau,\tau_o) = U_A(\tau-\tau_o)$.

Equation (6) has the formal solution

$$f'(\tau) = -\eta \int_{\tau_o}^{\tau} d\lambda\, U_A(\tau-\lambda)\, \mathcal{L}' f(\lambda) \qquad (8)$$

where we have taken $f'(o)=0$.

If we substitute this solution back into the ensemble average of equation (1), we have the exact master equation for f:

$$\frac{\partial f}{\partial \tau} + \mathcal{L} f = \eta^2 \langle \mathcal{L}' \int_{\tau_o}^{\tau} d\lambda\, U_A(\tau-\lambda)\, \mathcal{L}' \rangle f(\lambda) \qquad (9)$$

Because it is not possible to find an exact representation for $U_A(\tau-\tau_o)$, one proceeds by finding approximations that retain as much information as possible about the effects of the stochastic fields on the particle trajectories. With this goal in mind, we do not make the quasilinear approximation in which it is assumed that the stochastic fields do not affect a particle's motion during an interaction.

Note that equation (9) can be rewritten as

$$\frac{\partial f}{\partial \tau} + \mathcal{L} f = \eta^2 \langle \mathcal{L}' \int_{\tau_o}^{\tau} d\lambda\, U_A(\tau-\lambda)\,(1-A)\, \mathcal{L}' \rangle f(\lambda) \qquad (10)$$

An approximation to $U_A(\tau)\,(1-A)$ can be found in terms of the propagator $U(\tau-\tau_o)$ defined by

$$\left[\frac{\partial}{\partial\tau} + \mathcal{H}\right] U(\tau-\tau_o) = 0$$
$$U(0) = 1 \qquad (11)$$

This can be written (Ben-Israel, et al. 1975):

$$\left[\frac{\partial}{\partial\tau} + (1-A)\mathcal{H}\right] (1-A)\, U(\tau) = -\eta(1-A)\, \mathcal{L}'AU(\tau) \qquad (12)$$

with the formal solution

$$(1-A)\, U(\tau) = U_A(\tau)\, (1-A) - \eta \int_o^\tau d\lambda\, U_A(\tau-\lambda)(1-A)\, \mathcal{L}'AU(\lambda) \qquad (13)$$

Here, and in the remainder of this paper, we take $\tau_o = 0$. To proceed, we drop the second term on the right-hand side of equation (13). This approximation is valid in the limit of weak coupling (Kadomtsev 1965; Klimas and Sandri 1973), i.e. in the limit $\eta^2 \ll 1$. The representation of $U(\tau)$ is also unknown, so it is necessary to approximate $U(\tau)$ in terms of the propagator $\bar{U} \equiv \langle U \rangle$, which satisfies the ensemble average of equation (11):

$$\frac{\partial}{\partial\tau} \bar{U}(\tau) = - \langle \mathcal{H} U(\tau) \rangle \qquad (14)$$

or

$$\left[\frac{\partial}{\partial\tau} + \mathcal{H}\right] \bar{U}(\tau) = -\eta \langle \mathcal{L}'U(\tau) \rangle + \eta\, \mathcal{L}'\bar{U}(\tau) \qquad (15)$$

The formal solution is

$$\bar{U}(\tau) = U(\tau) + \eta \int_o^\tau d\lambda\, U(\tau-\lambda) \left[\mathcal{L}'\bar{U}(\lambda) - \langle \mathcal{L}'U(\lambda) \rangle \right] \qquad (16)$$

Therefore, to lowest order in η, $\bar{U}(\tau) \simeq U(\tau)$ and equation (10) can be rewritten as

$$\frac{\partial f}{\partial \tau} + \mathcal{L} f \cong \eta^2 < \mathcal{L}' \int_0^\tau d\lambda \; \bar{U}(\lambda) \; \mathcal{L}' > f(\tau - \lambda) \tag{17}$$

To reduce equation (17) to a diffusion equation, we make the adiabatic approximation. The nature of this approximation is to restrict our interest in equation (17) to times $\tau \gg 1$ when $f(\tau-\lambda) \simeq f(\tau)$.

One now substitutes equations (3)-(5) into equation (17) and averages over gyrophase (cf. Goldstein, Klimas and Sandri 1975). After some manipulation, equation (17) becomes

$$\frac{\partial f(\tau)}{\partial \tau} = \frac{\partial}{\partial \mu} D_{\mu} \frac{\partial}{\partial \mu} f(\tau) \tag{18}$$

where

$$D_{\mu}(\tau) \equiv \frac{-\eta^2}{2\pi} \int_0^\tau d\lambda \int_0^{2\pi} d\phi \; < \hat{p} \cdot \Omega \cdot \underline{B}'(\underline{x}) \; \bar{U}(\lambda) \; \underline{B}'(\underline{x}) \cdot \Omega \cdot \hat{p} > \tag{19}$$

$$D_{\mu} = \lim_{\tau \to \infty} D_{\mu}(\tau) \tag{20}$$

In writing equation (18), we assumed that the turbulence is axially symmetric about $\hat{\beta}$. Also $f(\tau)$ in equation (18) now denotes the gyrophase average of $f(\tau)$ in equation (17).

The assumption in writing equation (18) is that equation (19) converges 'rapidly" to D_{μ}. However, in quasilinear theory, for particle pitch angles near $\pi/2$, equation (19) takes an arbitrarily long time to saturate. (For a more comprehensive discussion of the

adiabatic approximation, we refer the reader to Weinstock 1970; Jokipii 1971; Klimas and Sandri 1973; Jones and Birmingham 1974; and Fisk, et al. 1974). A consequence of the present nonlinear analysis is that the integral in equation (19) saturates "quickly", even for $\mu = 0$. Thus we expect the adiabatic approximation to be a good one. This will be examined in more detail in a future paper. The remainder of this paper will be devoted to the derivation of $D_{\mu}(\tau)$.

We proceed by writing $\underline{B}'(\underline{x})$ as a Fourier integral transform over wave number, $\underline{k}$. It follows that

$$\langle \underline{B}'(\underline{k})\ \underline{B}'(\underline{k}')\overline{U}\ \exp[i\underline{k}'\cdot\underline{x}\quad]\cdot\underline{\Omega}\cdot\hat{p}\quad\rangle = \langle \underline{B}'(\underline{k})\ \underline{B}'(\underline{k}')\rangle \langle \exp\left[i\underline{k}'\cdot\underline{x}(\lambda)\right]\cdot\underline{\Omega}\cdot\hat{p}\ (\lambda)\rangle \tag{21}$$

This permits writing the theory solely in terms of the two-point correlation tensor of the magnetic turbulence.

We define $\underline{R}(\underline{k})$ through the relation

$$\langle \underline{B}'(\underline{k})\ \underline{B}'(\underline{k}')\rangle = (2\pi)^{3/2}\ \underline{R}(\underline{k})\ \delta\ (\underline{k}+\underline{k}') \tag{22}$$

where $\underline{R}(\underline{k})$ is the Fourier integral transform of the two-point correlation tensor,

$$\underline{R}(\underline{r}) = \langle \underline{B}'(\underline{x})\ \underline{B}'(\underline{x}+\underline{r})\rangle \tag{23}$$

(In writing equation (22), we have restricted ourselves to homogeneous turbulence.) $D_{\mu}(\tau)$ can now be written as

$$D_{\mu}(\tau) = \frac{-\eta^2}{(2\pi)^2} \int_0^{\tau} d\lambda \int_0^{2\pi} d\phi \int d^3k \exp(i\underline{k}\cdot\underline{x}) \hat{p}\cdot \underline{\Omega}\cdot\underline{R}(\underline{k})\cdot\underline{\Omega}\cdot\langle \hat{p}(\lambda) \exp[-i\underline{k}\cdot\underline{x}(\lambda)] \rangle \tag{24}$$

The basic mathematical problem is that of computing the ensemble average in equation (24). If $\underline{p}(\tau)$ is written as the sum of a fluctuation part, $\underline{p}'(\tau)$, plus a mean part $\langle \hat{p}(\tau) \rangle$, then the correlation becomes

$$\langle \hat{p}(\tau) \exp[-i\underline{k}\cdot\underline{x}(\tau)] \rangle = \langle \hat{p}(\tau) \rangle \langle \exp[-i\underline{k}\cdot\underline{x}(\tau)] \rangle + \langle \hat{p}'(\tau) \exp[-i\underline{k}\cdot\underline{x}'(\tau)] \rangle \tag{25}$$

We define

$$\Delta\underline{p}(\lambda) \equiv \langle \hat{p}(\lambda) \rangle - \hat{p}_o(\lambda) \tag{26}$$

where $\hat{p}_o(\lambda)$ is the streamed velocity of the particle in the absence of fluctuation fields. The propagator of this trajectory, $U_o(\tau)$, satisfies

$$\left[\frac{\partial}{\partial\tau} + \mathcal{L}\right] U_o(\tau-\tau_o) = 0 \tag{27}$$

$$U_o(0) = 1$$

The solution is the well-known helical trajectory used in quasilinear theory, which we can write as (<u>v.</u>, <u>e.g.</u>, Goldstein, Klimas and Sandri 1975).

$$U_o(\tau)\ \underline{x} \equiv \underline{x}_o(\tau) = \underline{x} - \underline{r}_o(\tau) \tag{28}$$

$$\underline{r}_o(\tau) = \underline{\mathcal{D}}(\tau)\cdot\hat{p} \equiv \hat{p}\cdot\underline{\mathcal{D}}^{+}(\tau) \tag{29}$$

$$U_o(\tau)\ \hat{p} \equiv \hat{p}_o(\tau) = \underline{\mathcal{C}}(\tau)\cdot\hat{p} = \hat{p}\cdot\underline{\mathcal{C}}^{+}(\tau) \tag{30}$$

$$\underline{\mathcal{C}}(\tau) = \underline{P} + \underline{N}\cos\tau - \underline{\Omega}\sin\tau \tag{31}$$

$$\underline{\mathcal{D}}(\tau) = \underline{P}\tau + \underline{N}\sin\tau + \underline{\Omega}(\cos\tau - 1) \tag{32}$$

$$P_{ij} \equiv \hat{\beta}_i \hat{\beta}_j \tag{33}$$

$$N_{ij} = \delta_{ij} - P_{ij} \tag{34}$$

Equation (27) can be rewritten (Ben-Israel, et al. 1975) as

$$\left[\frac{\partial}{\partial\tau} + \mathcal{H}\right] U_o(\tau) = \eta\, \mathcal{L}' U_o(\tau) \tag{35}$$

with the solution

$$U_o(\tau) = U(\tau) + \eta \int_o^{\tau} d\lambda\ U(\tau - \lambda)\, \mathcal{L}' U_o(\lambda) \tag{36}$$

The quantity $\Delta\underline{p}(\lambda)$ in equation (26) is unique to the magnetic turbulence theory. In the electrostatic problem (Dupree 1966; Weinstock 1969, 1970), the momentum trajectories do not appear

explicitly. The velocity dependent nature of the magnetic force greatly complicates the computation of the nonlinear wave-particle interaction in strong turbulence (Ben-Israel, et al. 1975). However, some simplifications are possible for the test particle problem in magnetostatic turbulence. For example, we will argue below that the second term on the right-hand side of equation (25) is unimportant in the particular examples that one can compute. We now outline the computation of $\prec \exp -i [\, \underline{k}\cdot\underline{x}(\lambda) \,] \succ$ and $\Delta\underline{p}(\lambda)$.

Following Ben-Israel, et al. (1975) and Weinstock (1970), we expand $\prec \exp -i [\underline{k}\cdot\underline{x}(\lambda)\,] \succ$ in a cumulant expansion (Kubo 1962). To second order we have

$$\prec \exp -i\,[\underline{k}\cdot\underline{x}(\lambda)] \succ \cong \exp\,[-i\underline{k}\cdot\prec \underline{x}(\lambda)\succ - \frac{1}{2}\,\underline{k}\cdot\underline{D}_1(\lambda)\cdot\underline{k}] \qquad (37)$$

where $\underline{D}_1(\lambda,\lambda) \equiv \underline{D}_1(\lambda) \equiv \prec \underline{x}'(\lambda)\ \underline{x}'(\lambda) \succ$ and $\underline{x}'(\lambda) \equiv (1\text{-}A)\ \underline{x}(\lambda)$. [Ben-Israel, et al. (1975) are missing the factor of 1/2 on the rhs of equation (37)].

To our knowledge, the general convergence properties of this expansion are unknown. However, if the fluctuation, $\underline{x}'(\lambda)$, obey Gaussian statistics, then equation (37) is exact.

We define

$$\Delta\underline{x}(\tau) \equiv \prec \underline{x}(\tau) \succ - \underline{x}_o(\tau) \qquad (38)$$

where, again, subscript "o" denotes streaming by the propagator $U_o(\tau)$. Then, from equations (36), (3) (4) and (16), we find to $\mathcal{O}(\eta^2)$

$$\Delta\underline{x}(\tau) \cong -\eta^2 \int_o^\tau d\lambda \int_\lambda^{\tau'} d\nu \prec \bar{U}(\tau-\nu)\hat{p}\cdot\underline{\Omega}'\cdot \frac{\partial}{\partial\hat{p}}\, U(\nu-\lambda)\ \hat{p}\cdot\underline{\Omega}'\cdot\underline{\mathcal{D}}^{+}(\lambda) \succ \qquad (39)$$

Similarly,

$$\Delta\underline{p}(\tau) \cong \eta^2 \int_0^\tau d\lambda \int_\lambda^\tau d\nu \prec \bar{U}(\tau-\nu)\hat{p}\cdot\underline{\Omega}' \cdot \frac{\partial}{\partial\hat{p}} U(\nu-\lambda)\hat{p}\cdot\underline{\Omega}'\cdot\underline{\mathcal{C}}^+(\lambda) \succ \qquad (40)$$

To compute $\underline{D}_1(\tau)$, note that one can write

$$\bar{U}(\tau) = U_o(\tau) - \eta \int_o^\tau d\lambda\ U_o(\tau-\lambda) \prec \mathcal{L}'U(\lambda) \succ \simeq U_o(\tau) + \mathcal{O}(\eta^2) \qquad (41)$$

Then, from the definition of $\underline{D}_1(\tau)$ and equation (41), one has

$$\underline{D}_1(\tau) = \eta^2 \prec \int_o^\tau d\nu \left[\!\left[U(\tau-\nu)\ \hat{p}\cdot\underline{\Omega}'\cdot\underline{\mathcal{O}}^+(\nu) \right]\!\right] \int_o^\tau d\lambda\ U(\tau-\lambda)\ \hat{p}\cdot\underline{\Omega}'\cdot\underline{\mathcal{O}}^+(\lambda) \succ \qquad (42)$$

where the symbol $\left[\!\left[\ \right]\!\right]$ means that operators within the bracket act only on variables contained within the bracket.

To proceed, and at the same time keep the mathematics tractable, we will make several of the simplifying assumptions alluded to in the introduction. The interested reader is referred to Ben-Israel, et al. (1975) for a more general derivation. However, Ben-Israel et al. make an unnecessary approximation, viz., they write

$$\frac{\partial}{\partial\hat{p}}\hat{p}(\tau) = \frac{\partial}{\partial\hat{p}}\left[\prec \hat{p}(\tau) \succ + \underline{p}'(\tau)\right] \simeq \frac{\partial}{\partial\hat{p}}\hat{p}_o(\tau) \qquad (43)$$

The term $\frac{\partial}{\partial\hat{p}}\Delta\underline{p}(\tau)$, neglected in writing equation (43), is of the same order as $\Delta\underline{p}(\tau)$, which is kept in their analysis. We will find below that an approximate solution can be found, including terms proportional to $\partial\Delta\underline{p}(\tau)/\partial\hat{p}'$ for the special case of $\theta = \pi/2$ ($\mu = 0$) in a "slab" model.

We will follow Ben-Israel et al. (1975) in dropping the term proportional to $\frac{\partial}{\partial \hat{p}} \underline{p}'(\tau)$. We confine the remainder of the discussion to the "slab" correlation function

$$\underline{R}(\underline{k}) = \underline{N}\, \delta(\underline{k}_\perp) R(k_\parallel) \tag{44}$$

where $R(k_\parallel)$ is the power spectrum of the turbulence and

$$\int d\underline{k}_\perp\ \delta(\underline{k}_\perp) = 2\pi \tag{45}$$

In the following section, we develop the theory for the single point $\mu = 0$. This is the region in which quasilinear theory gives the poorest approximation. It is also the case for which the most complete nonlinear analysis can be done.

III. SCATTERING THROUGH $\mu=0$ IN A SLAB MODEL

An immediate consequence of restricting the discussion to $\mu=0$ in a slab model is that $\Delta x_{\perp}(\tau)$ does not appear, $\Delta x_{\parallel}(\tau) = 0$ and the second term in equation (25) can be shown to be proportional to μ and, hence, equal to zero. Using equations (24)-(26), (37) and (44)-(45), $D\mu(\tau)$ can be written as

$$D_{\mu}(\tau) = \frac{\eta^2}{(2\pi)^{3/2}} \int_0^{\tau} d\lambda \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dk_{\parallel} R(k_{\parallel}) [\cos\lambda + \hat{p}_{\perp} \cdot \Delta \underline{p}_{\perp}(\lambda)] \exp[-k_{\parallel}^2 D_1(\lambda)/2] \tag{46}$$

where

$$D_1(\lambda) \equiv D_{1_{\parallel,\parallel}}(\lambda).$$

The two quantities that must be computed are $\Delta \underline{p}_{\perp}(\tau)$ and $D_1(\tau)$. We outline first the derivation of $D_1(\tau)$ to illustrate the techniques used in evaluating all of the nonlinear quantities presented in this discussion. We start with equation (42). In general, $\underline{\Omega}'$ can be expressed as

$$\underline{\Omega}' = (\hat{\beta} \cdot \underline{\beta}')\underline{\Omega} - \hat{\beta}(\underline{\beta}' \cdot \underline{\Omega}) - (\underline{\Omega} \cdot \underline{\beta}')\hat{\beta}. \tag{47}$$

In the case of slab turbulence considered here, the first term in equation (47) is zero. $D_1(\tau)$ becomes

$$D_1(\tau) = \eta^2 \int_0^{\tau} d\nu \int_0^{\tau} d\lambda \; \nu\lambda \, \langle [\![U(\tau-\nu) \; \hat{p} \cdot \underline{\Omega} \cdot \underline{\beta}']\!] \; U(\tau-\nu) \; \hat{p} \cdot \underline{\Omega} \cdot \underline{\beta}' \rangle \tag{48}$$

One now writes the fluctuation fields as Fourier integral transforms, and uses equations (21) and (22) to arrive at

$$D_1(\tau) = \frac{-\eta^2}{(2\pi)^{3/2}} \int_0^{\tau} d\nu \int_0^{\tau} d\lambda \int d^3k \, \langle \exp[i \, \underline{k} \cdot (\underline{x}(\tau-\nu) - \underline{x}(\tau-\lambda))] \; \nu\lambda \; \hat{p}(\tau-\nu) \cdot \underline{\Omega} \cdot \underline{R}(\underline{k}) \cdot \underline{\Omega} \cdot \hat{p} \; (\tau-\lambda) \rangle \tag{49}$$

Substitution of $\underline{R}(k)$ from equation (44) yields

$$D_1(\tau) = \frac{\eta^2}{\sqrt{2\pi}} \int_0^\tau d\nu \int_0^\tau d\lambda \int_{-\infty}^{\infty} dk_\parallel \; R(k_\parallel) \; \nu\lambda \; \langle \exp[ik_\parallel x'_\parallel(\nu-\lambda)] \hat{p}_\perp(\tau-\nu) \cdot \hat{p}_\perp(\tau-\lambda) \rangle \tag{50}$$

where we have used the approximation (Ben-Israel, et al. 1975)

$$x'_\parallel(\tau-\nu) - x'_\parallel(\tau-\lambda) \simeq x'_\parallel(\nu-\lambda) \tag{51}$$

To evaluate the correlation that appears here, one must expand the integrand of equation (50) as follows

$$\begin{aligned} \langle \exp[ik_\parallel x'_\parallel(\nu-\lambda)] \hat{p}_\perp(\tau-\nu) \cdot \hat{p}_\perp(\tau-\lambda) \rangle &= \langle \hat{p}_\perp(\tau-\nu) \rangle \cdot \langle \hat{p}_\perp(\tau-\lambda) \rangle \times \\ &\langle \exp ik_\parallel x'_\parallel(\nu-\lambda) \rangle + \langle \hat{p}'_\perp(\tau-\nu) \cdot \hat{p}'_\perp(\tau-\lambda) \rangle \langle \exp[ik_\parallel x'_\parallel(\nu-\lambda)] + \\ &\langle \hat{p}'_\perp(\tau-\nu) \exp[ik_\parallel x'(\nu-\lambda)] \rangle \cdot \langle \hat{p}_\perp(\tau-\lambda) \rangle + \\ &\langle \hat{p}'_\perp(\tau-\lambda) \exp[ik_\parallel x'_\parallel(\nu-\lambda)] \rangle \cdot \langle \hat{p}_\perp(\tau-\nu) \rangle . \end{aligned} \tag{52}$$

In Appendix A, we show that for $\mu=0$, only the first term in equation (52) is expected to be important. Consequently, equation (50) becomes

$$\begin{aligned} D_1(\tau) = \frac{\eta^2}{\sqrt{2\pi}} \int_0^\tau d\nu \int_0^\tau d\lambda \int_{-\infty}^{\infty} dk_\parallel \; R(k_\parallel) \nu\lambda \; \exp[-k_\parallel^2 D_1(\nu-\lambda)] [\hat{p}_{\perp o}(\tau-\nu) + \Delta\underline{\hat{p}}_\perp(\tau-\nu)] \\ \cdot [\hat{p}_{o\perp}(\tau-\lambda) + \Delta\underline{p}_\perp(\tau-\lambda)] \end{aligned} \tag{53}$$

Finally, we must derive an expression for $\Delta p_\perp(\tau)$. From the definition of $\Delta\underline{p}_\perp(\tau)$ (equation 26) and the identies (35) and (16), we have to $\mathcal{O}(\eta^2)$

$$\Delta\underline{p}_\perp(\tau) = \eta^2 \int_0^\tau d\lambda \int_\lambda^\tau d\nu \; \langle U(\tau-\nu) \hat{p} \cdot \Omega' \cdot \frac{\partial}{\partial \hat{p}} \; \overline{U}(\nu-\lambda) \hat{p} \cdot \underline{\Omega}' \cdot \frac{\partial}{\partial \hat{p}} \; \hat{p}_{\perp o}(\lambda) \rangle \tag{54}$$

In Appendix B, we show that equation (54) may be written approximately as

$$\begin{aligned} \Delta\underline{p}_\perp(\tau) = \frac{-\eta^2}{\sqrt{2\pi}} \int_0^\tau d\lambda \int_\lambda^\tau d\nu \int_{-\infty}^{\infty} dk_\parallel R(k_\parallel) \exp[-\frac{k_\parallel^2}{2} D_1(\nu-\lambda)] [1 + \frac{\partial}{\partial\mu} \Delta p_\parallel(\tau-\lambda) \Big|_{\mu=o}] \\ [\hat{p}_{\perp o}(\tau-\nu) + \Delta\underline{p}_\perp(\tau-\nu)] \cdot \underline{\mathcal{C}}_\perp^+(\lambda) \end{aligned} \tag{55}$$

The approximations involved in deriving equation (55) are discussed in Appendix B. In Appendix C, a derivation of $\Delta p_{\parallel}(\tau)$ is given, From that discussion we have

$$\frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau)\Big|_{\mu=0} = \frac{-\pi^2}{\sqrt{2\pi}}\int_0^{\tau} d\lambda \int_{-\infty}^{\infty} dk_{\parallel} R(k_{\parallel})\cos\lambda(\tau-\lambda)\exp[-k_{\parallel}^2 D_1(\lambda)/2]$$

$$\times\{2+k_{\parallel}^2\lambda[\lambda - (\frac{\partial}{\partial\mu}\Delta x_{\parallel}(\lambda))]\}\Big|_{\mu=0} \tag{56}$$

where $\partial\Delta x_{\parallel}(\tau)/\partial\mu$ is given by equation (D28).

The set of equations (53), (55) and (56) provides a complete system of coupled, nonlinear integral equations, which in principle can be solved, and substituted into equation (46) for $D_{\mu}(\tau)$. Unfortunately, this system cannot be solved analytically. To solve the system numerically, we must specialize to particular forms of the power spectrum, $R(k_{\parallel})$. The numerical simulations developed by Kaiser (1974) are for $R(k_{\parallel})$ of the form

$$R(k_{\parallel}) = \frac{2\epsilon}{\sqrt{2\pi}}(1 + \epsilon^2 k_{\parallel}^2)^{-1} \tag{57}$$

where ϵ is the ratio of the correlation length of the magnetic field turbulence, λ_c, to r_g. If we use equation (57), the integrals over $k_{\parallel}$ can be performed analytically. For these two reasons (comparison with numerical simulations and mathematical simplicity), we have confined our numerical analysis to the particular case of equation (57). There are two integrations over $k_{\parallel}$ to perform. They are

$$g(\tau) \equiv \int_{-\infty}^{\infty} dk_{\parallel} R(k_{\parallel})\exp -k_{\parallel}^2 D_1(\tau)/2$$

$$= \sqrt{2\pi}\ \mathrm{Erfc}\ [\frac{D_1(\tau)}{2\epsilon^2}]^{1/2}\ \exp[D_1(\tau)/2\epsilon^2] \tag{58}$$

and

$$h(\tau) \equiv \int_0^\infty dk_\parallel k_\parallel^2 R(k_\parallel) \exp[-k_\parallel^2 D_1(\tau)/2] \tag{59}$$

$$= (2\pi\epsilon^4)^{-1/2} \{[2\pi\epsilon^2/D_1(\tau)]^{1/2} - \pi \mathrm{Erfc}\,[D_1(\tau)/2\epsilon^2]^{1/2} \exp[D_1(\tau)/2\epsilon^2]\}$$

The system of equations (46), (53), (55) and (56) can now be simplified to

$$D_\mu(\tau) = \frac{\eta^2}{(2\pi)^{3/2}} \int_0^{2\pi} d\phi \int_0^\tau d\lambda g(\lambda) \,[\cos\lambda + \cos\phi\, \Delta p_x(\tau) + \sin\phi\, \Delta p_y(\tau)] \tag{60}$$

$$D_1(\tau) = \frac{\eta^2}{\sqrt{2\pi}} \int_0^\tau d\nu \int_0^\tau d\lambda\, g(\nu-\lambda)\nu\lambda [\hat{p}_{\perp o}(\tau-\nu) + \Delta \underline{p}_\perp(\tau-\nu)] \cdot [\hat{p}_{\perp o}(\tau-\lambda) + \Delta \underline{p}_\perp(\tau-\lambda)] \tag{61}$$

$$\Delta \underline{p}_\perp(\tau) = \frac{-\eta^2}{\sqrt{2\pi}} \int_0^\tau d\lambda \int_\lambda^\tau d\nu\, g(\nu-\lambda) \{1 + [\frac{\partial}{\partial\mu} \Delta p_\parallel(\tau-\lambda)]_{\mu=o}\} \times \tag{62}$$

$$[\hat{p}_{\perp o}(\tau-\nu) + \Delta \underline{p}_\perp(\tau-\nu)] \cdot \underline{\mathcal{C}}_\perp^+(\lambda)$$

$$\frac{\partial}{\partial\mu} \Delta p_\parallel(\tau)\Big|_{\mu=o} = \frac{-2\eta^2}{\sqrt{2\pi}} \int_0^\tau d\lambda(\tau-\lambda)\cos\lambda \{g(\lambda) + \lambda h(\lambda)[\lambda - \frac{\partial}{\partial\mu} \Delta x_\parallel(\lambda)]\}_{\mu=o} \tag{63}$$

$$\frac{\partial}{\partial\mu} \Delta x_\parallel(\tau)\Big|_{\mu=o} = \frac{\eta^2}{\sqrt{2\pi}} \int_0^\tau d\lambda \cos\lambda(\tau-\lambda)^2 \{g(\lambda) + \lambda h(\lambda)\,[\lambda - \frac{\partial}{\partial\mu} \Delta x_\parallel(\lambda)]\}_{\mu=o} \tag{64}$$

Several additional simplifications can be made. For the axisymmetric turbulence considered here, the integrand of $D_\mu(\tau)$ should be independent of ϕ. From the form of equations (60)-(64), this assertion may not be obvious. However, it can be shown numerically that this is indeed the case. Therefore, to simplify the analysis presented here, we use the fact that the integrand of equation (60) is independent of ϕ and evaluate it at $\phi = 0$. The following time symmetries are then not difficult to prove

$$D_1(\tau) = D_1(-\tau) \tag{65}$$

$$\Delta p_x(\tau) = \Delta p_x(-\tau) \tag{66}$$

$$\Delta p_y(\tau) = -\Delta p_y(-\tau) \tag{67}$$

$$\frac{\partial}{\partial\mu}\,\Delta p_{\parallel}(\tau) = \frac{\partial}{\partial\mu}\,\Delta p_{\parallel}(-\tau) \tag{68}$$

$$\frac{\partial}{\partial\mu}\,\Delta x_{\parallel}(\tau) = -\frac{\partial}{\partial\mu}\,\Delta x_{\parallel}(-\tau) \tag{69}$$

Equations (60)-(62) simplify to

$$D_\mu(\tau) = \frac{\eta^2}{\sqrt{2\pi}} \int_o^\tau d\lambda\; g(\lambda)\,[\cos\lambda + \Delta p_{\perp x}(\lambda)] \tag{70}$$

$$D_1(\tau) = \frac{2\eta^2}{\sqrt{2\pi}} \int_o^\tau d\lambda\; \cos\lambda\; g(\lambda)\,[\tau^3/3 - \lambda/2(\tau^2-\lambda^2/3)] +$$

$$\frac{\eta^2}{\sqrt{2\pi}} \int_o^\tau d\nu \int_o^\tau d\lambda\; g(\nu-\lambda)\;\nu\lambda\,\{\Delta p_{\perp x}(\tau-\nu)[2\cos(\tau-\lambda) + \Delta p_{\perp x}(\tau-\lambda)] +$$

$$\Delta p_{\perp y}(\tau-\nu)[2\sin(\tau-\lambda) + \Delta p_{\perp y}(\tau-\lambda)]\} \tag{71}$$

$$\begin{pmatrix}\Delta p_{\perp x}(\tau)\\ \Delta p_{\perp y}(\tau)\end{pmatrix} = \frac{-\eta^2}{\sqrt{2\pi}} \int_o^\tau d\lambda \int_\lambda^\tau d\nu\; g(\nu-\lambda)\,\left[1 + \frac{\partial}{\partial\mu}\,\Delta p_{\parallel}(\tau-\lambda)\right]_{\mu=o}$$

$$\left\{\begin{pmatrix}\cos(\tau-\nu+\lambda)\\ -\sin(\tau-\nu+\lambda)\end{pmatrix} + \cos\lambda\begin{pmatrix}\Delta p_{\perp x}(\tau-\nu)\\ \Delta p_{\perp y}(\tau-\nu)\end{pmatrix}\right.$$

$$\left. + \sin\lambda\begin{pmatrix}\Delta p_{\perp y}(\tau-\nu)\\ -\Delta p_{\perp x}(\tau-\nu)\end{pmatrix}\right\}. \tag{72}$$

To produce a numerical solution, one would like to transform these equations into a system of differential equations. However, the best that can be done with this system is to derive a set of nonlinear integro-differential equations which can, however, be numerically integrated using a standard Adams-Bashforth predictor-corrector technique (Krogh 1966). In Appendix D, we outline the transformation of the system of equations (70)-(72) and (63)-(64) to a set of integro-differential equations. We defer until §V a discussion of the numerical results of that integration. We turn now to an appropriate derivation of the system of equations that describes scattering at arbitrary pitch angle.

IV SCATTERING THROUGH ARBITRARY PITCH ANGLE - AN APPROXIMATE CALCULATION IN A SLAB MODEL

In this section, we derive a set of approximate equations to describe pitch angle scattering at arbitrary pitch angles in a slab model. Many terms proportional to $\Delta\underline{p}(\tau)$ will be dropped in this analysis for reasons that are basically heuristic. An estimate of the validity of the various approximations made below can be made *a posteriori* by comparing the results of §III with the results of this section. The system of equations derived here is similar to that derived by Ben-Israel, *et al.* (1975) in their discussion of ion-cyclotron turbulence.

The equation for $D_\mu(\tau)$ is now

$$D_\mu(\tau) = \frac{\pi^2}{(2\pi)^{3/2}} \int_0^\tau d\lambda \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dk_\parallel \, R(k_\parallel) \cos[k_\parallel(\mu\lambda - \Delta x_\parallel(\lambda)] \exp[-k_\parallel^2 D_\perp(\lambda)/2]$$

$$[(1-\mu^2)\cos\lambda + \Delta\underline{p}_\perp(\lambda) \cdot \hat{p}_\perp] \tag{73}$$

The first quantity we evaluate is $\Delta x_\parallel(\tau)$, defined by equations (38) and (39). When one uses the approximation (43), $\Delta x_\parallel(\tau)$ becomes

$$\Delta x_\parallel(\tau) = \frac{-\pi^2}{(2\pi)^3} \int_0^\tau d\lambda \int_\lambda^\tau d\nu \int d^3k \int d^3k' \, \langle \exp i[\underline{k}\cdot\underline{x}(\tau-\nu) + \underline{k}'\cdot\underline{x}(\tau-\lambda)] \tag{74}$$

$$\hat{p}_o(\tau-\nu)\cdot\underline{\Omega}'(k)\cdot[\underline{\mathcal{C}}^+(\nu-\lambda)\cdot\underline{\Omega}'(k') + i\underline{\mathcal{D}}^+(\nu-\lambda)\cdot\underline{k}\hat{p}_o(\tau-\nu)\underline{\mathcal{C}}^+(\nu-\lambda)\cdot\underline{\Omega}'(k')]\cdot\hat{\beta}\lambda\rangle$$

In writing equation (74), we have replaced $\hat{p}(\tau-\nu)$ by $\hat{p}_o(\tau-\nu)$ everywhere. The justification for doing this is two-fold. First, one expects that nonlinear terms such as $\Delta\underline{p}(\tau)$ and $\underline{p}'(\tau)$ will be less important for μ not near zero. The reasoning here is that quasilinear theory should be a very good approximation except near $\mu = 0$. As we

shall see in §V, the results of this section are generally in good agreement at $\mu = 0$ with those of §III. Second, the term $\Delta x_{\parallel}(\tau)$ is equivalent to a shift in the wave-particle resonance (Weinstock 1972; Ben-Israel, et al. 1975), which is generally not the most important term in resonant broadening theories. The most important term characteristically is the broadening of the wave-particle resonance, which saturates the interaction and allows particles to propagate through $\mu = 0$. The resonance broadening is given by the term $\exp-[k_{\parallel}^2 D_1(\tau)]$. The contributions to $D_1(\tau)$ from $\Delta\underline{p}(\tau)$ and $\underline{p}'(\tau)$ should be even less important for $\mu \neq 0$, where quasilinear theory becomes a good approximation.

One now substitutes the identity (47) into equation (74) and uses equations (22) and (44) for slab turbulence. After some rather lengthy matrix algebra, $\Delta x_{\parallel}(\tau)$ becomes

$$\Delta x_{\parallel}(\tau) = \frac{\pi^2}{\sqrt{2\pi}} \int_0^{\tau} d\lambda \int_{\lambda}^{\tau} d\nu \int_{-\infty}^{\infty} dk_{\parallel} \exp\{i\, k_{\parallel}[\mu(\nu-\lambda) - \Delta x_{\parallel}(\nu-\lambda)] - k_{\parallel}^2 D_1(\nu-\lambda)/2\} \times$$

$$R(k_{\parallel})\lambda\cos(\nu-\lambda)[2\mu - i\, k_{\parallel}(\nu-\lambda)(1-\mu^2)]\} \qquad (75)$$

Because the integrand in equation (75) is basically a function of $(\nu-\lambda)$, $\Delta x_{\parallel}(\tau)$ can be further simplified to

$$\Delta x_{\parallel}(\tau) = \frac{\pi^2}{\sqrt{2\pi}} \int_0^{\tau} d\lambda \int_0^{\infty} dk_{\parallel} R(k_{\parallel}) \exp\left[-k_{\parallel}^2 \frac{D_1(\lambda)}{2}\right] \cos\lambda(\tau-\lambda)^2$$

$$\{2\mu\cos[k_{\parallel}(\mu\lambda - \Delta x_{\parallel}(\lambda))] + (1-\mu^2)k_{\parallel}\lambda\sin[k_{\parallel}(\mu\lambda - \Delta x_{\parallel}(\lambda))]\} \qquad (76)$$

Piran (1972) and Ben-Israel, et al. (1975) incorrectly write the argument of the trigonometric functions as $[\mu\lambda+\Delta x_{\parallel}(\lambda)]$.

When $R(k_{\|})$ is given by equation (57), the integrals over $k_{\|}$ can be performed analytically with the result

$$\Delta x_{\|}(\tau) = \frac{\pi^2}{\sqrt{2\pi}} \int_0^{\tau} d\lambda (\tau-\lambda)^2 [2\mu G(\lambda) + (1-\mu^2) H(\lambda)] \tag{77}$$

where

$$G(\tau) \equiv \int_0^{\infty} dk\ R(k) \cos\tau\ \cos[k(\mu\tau - \Delta x_{\|}(\tau))] \exp[-k^2 D_1(\tau)/2] \tag{78}$$

$$= \frac{\pi}{2\sqrt{2\pi}} \cos\tau\ e^{z}\{e^{-y}\ \mathrm{Erfc}[\sqrt{z} - y/2\sqrt{z}] + e^{y}\ \mathrm{Erfc}[\sqrt{z} + y/2\sqrt{z}]\} \tag{79}$$

and

$$H(\tau) \equiv \int_0^{\infty} dk R(k) k\tau \cos\tau\ \sin[k(\mu\tau - \Delta x_{\|}(\tau))] \exp[-k^2 D_1(\tau)/2] \tag{80}$$

$$= \frac{\pi}{2\epsilon\sqrt{2\pi}} \tau \cos\tau\ \exp(z^2)\{e^{-y}\ \mathrm{Erfc}[z - y/2z] - e^{y}\ \mathrm{Erfc}[z + y/2z]\} \tag{81}$$

with

$$z \equiv D_1(\tau)/2\epsilon^2 \tag{82}$$

$$y \equiv [\mu\tau - \Delta x_{\|}(\tau)]/\epsilon \tag{83}$$

To evaluate $D_1(\tau)$, one substitutes equation (44) for $\underline{R}(\underline{k})$ into equation (49). After some algebra, one finds

$$D_1(\tau) = \frac{\pi^2}{\sqrt{2\pi}} \int_0^{\tau} d\nu \int_0^{\tau} d\lambda \int_{-\infty}^{\infty} dk_{\|}\, \nu\lambda (1-\mu^2) R(k_{\|}) \cos(\nu-\lambda) \exp[i\, k_{\|}[\mu(\nu-\lambda) -$$

$$\Delta x_{\|}(\nu-\lambda)] - k_{\|}^2 D_1(\nu-\lambda)/2] \tag{84}$$

where we have again set $\hat{p}(\tau) = \hat{p}_o(\tau)$.

The double integral over time can be reduced to a single integral so that $D_1(\tau)$ becomes

$$D_1(\tau) = \frac{\pi^2}{\sqrt{2\pi}} (1-\mu^2)\{\frac{2}{3}\tau^3 \int_{-\tau}^{\tau} d\lambda G(\lambda) + \int_{-\tau}^{o} d\lambda G(\lambda)[\tau^2\lambda - \lambda^3/3]$$

$$- \int_{o}^{\tau} d\lambda G(\lambda)[\tau^2\lambda - \lambda^3/3]\} \qquad (85)$$

By differentiating $D_1(\tau)$ with respect to τ, and setting $\tau' = -\tau$, one can easily show that $D_1(\tau) = D_1(\tau' = -\tau)$. Consequently,

$$D_1(\tau) = \frac{2\pi^2}{\sqrt{2\pi}} (1-\mu^2) \int_{o}^{\tau} d\lambda G(\lambda) [\frac{2}{3}\tau^3 - \tau^2\lambda + \lambda^3/3] \qquad (86)$$

The reader may have noticed that although we have dropped $\Delta\underline{p}(\tau)$ in our derivations of $\Delta x_{\parallel}(\tau)$ and $D_1(\tau)$, we have kept a term proportional to $\Delta p_{\perp}(\tau)\cdot p_{\perp}$ in the expression for $D_{\mu}(\tau)$ (equation 73). We expect that deviations of a particles momentum from the predictions of quasilinear theory would be relatively unimportant in computing the resonance shift ($\Delta x_{\parallel}(\tau)$) or the resonance broadening $\exp-[k_{\parallel}^2 D_1(\tau)]$. However, in computing $D_{\mu}(\tau)$ a contribution from $\Delta\underline{p}(\tau)$ should be important for the following reasons: in quasilinear theory for $\mu = 0$, the wave - particle resonance is with waves of zero wavelength, which have zero power. Therefore, $D_{\mu} = 0$ at $\mu = 0$ for the slab model in quasi-linear theory. The functions $D_1(\tau)$ and $\Delta x_{\parallel}(\tau)$ serve to broaden this resonance and shift it away from $k_{\parallel} = \infty$. This leads to a finite value for D_{μ} at $\mu = 0$. In addition to the resonant effects ($k_{\parallel} \sim \infty$) particles are affected by long wavelength fluctuations ($k_{\parallel} \sim 0$) that tend to mirror them (Völk 1973; Fisk, et al. 1974; Goldstein, Klimas and Sandri 1975; Lee and Völk 1975). For $k_{\parallel} \sim 0$, neither $\Delta x_{\parallel}(\tau)$ nor $\exp[-k_{\parallel}^2 D_1(\tau)]$ provides significant contributions to D_{μ}. The only large nonlinear

contribution to D_μ for $k_\parallel \sim 0$ is the term proportional to $\underline{\Delta p}_\perp(\tau)$ in equation (73). This term was dropped by Ben-Israel, et al. (1975) and similar terms were neglected by Völk (1973) and by Jones, et al. (1973a,b). We will return to this point in §V. We can approximate this term in much the same way in which $\Delta x_\parallel(\tau)$ was computed above. If one drops all factors of $\underline{p}'(\tau)$ and $\underline{\Delta p}(\tau)$ in the integrand of equation (55), one can easily show that

$$\begin{aligned} \underline{\Delta p}_\perp(\tau)\cdot\hat{\underline{p}}_\perp &\equiv (1-\mu^2)\cos\tau\,\delta p_\perp(\tau) \\ &= -\frac{2\eta^2}{\sqrt{2\pi}}(1-\mu^2)\cos\tau\int_0^\tau d\lambda(\tau-\lambda)\cos\lambda(1+\tan\lambda\,\tan\tau)[G(\lambda)-\mu H(\lambda)] \end{aligned} \tag{87}$$

so that

$$D_\mu(\tau) = \frac{2\eta^2(1-\mu^2)}{\sqrt{2\pi}}\int_0^\tau d\lambda G(\lambda)[1+\delta p_\perp(\lambda)] \tag{88}$$

The equations in the system (77)-(83), (86)-(88) now form a coupled nonlinear system of integral equations for $D_\mu(\tau)$. Völk (1973) derives an expression similar to equation (88), but he neglects the term equivalent to $\delta p_\perp(\lambda)$ and sets $D_\mu(\mu)$ equal to a constant when evaluating $\langle \exp -i\, k_\parallel x_\parallel(\tau)\rangle$ (equation 37). He then finds $\Delta x_\parallel(\tau)=0$ and $D_1(\tau) = \frac{1}{3} D_\mu \tau^3$. We find that $D_1(\tau) \propto \tau^3$ only when $\tau \gg 1$, and consequently the magnitude of $D_1(\tau)$ is generally much smaller than would be predicted by Völk (1973). In general, his results do not agree with ours, nor do they fit the simulations (Kaiser 1974).

In Appendix E this system of integro-differential equations is rewritten as a system of coupled differential equations that can be integrated numerically using standard techniques. The results of that integration are discussed in the next section.

V. DISCUSSION

The equations that determine D_μ were integrated numerically and the results are shown in figures 1-5. For comparison, we have included in figures 1-3 the theoretical prediction of Jones, Kaiser and Birmingham (1973a,b), Völk (1973) and Owens (1974). In all of these figures, we have plotted ϵD_μ to conform with the normalization used in the simulations. [If one were to define $\tilde{D}_\mu$ to be the dimensioned value of D_μ, then $\epsilon D_\mu \equiv (\lambda_c/\omega_c r_g)\tilde{D}_\mu$.] The theoretical predictions of quasilinear theory (Jokipii 1971) are also shown in figures 1-3. The simulation results are taken from Kaiser (1974). Since that reference was written, the simulation program has been subject to some revision (Kaiser, private communication) and the simulation results plotted in figures 1-3 may be slightly inaccurate. Consequently, in our discussion, deviations of ~20% of our theoretical curves for ϵD_μ *vs.* μ from the simulation points should not be considered significant.

In figures 1-3, the solid squares are the results of integrating the complete set of equations derived in §III and Appendix D. The solid curve results from integrating the system of equations derived in §IV and Appendix E.

The factor of $\sqrt{2}$ in front of η in the figures arises from the difference between linearly polarization turbulence (Jones, *et al.* 1973a,b) and the planar polarization used here. The parameter ϵ_2 of Jones, *et al.* (1973a,b) is equal to $\sqrt{2}\eta$. However this equivalence between our axisymmetric model and the linearly polarized model used

by Kaiser (1974) in the numerical simulations cannot be exact. Differences arise when the phase average is performed on equation (17). In general, for nonaxisymmetric turbulence, there would be an additional term in equation (18) of the form $\frac{\partial}{\partial\mu} D_{\mu\phi} \frac{\partial}{\partial\phi} f(\tau)$. For axisymmetric turbulence $D_{\mu\phi}$ is not a function of ϕ and equation (18) follows. [One cannot avoid this complication by assuming that f is gyrotropic] Jones (private communication) maintains that the consequences of this lack of axisymmetry in the numerical simulations is expected to be small. We proceed below to compare our results with the simulations on the assumption that this is indeed the case.

The results shown in Figures 1-3 indicate that quasilinear theory is quite good for $\mu \gtrsim 1/2$, as expected. Significant deviations from quasilinear theory are evident for $\mu \sim 0$. Here our results are similar to those of Jones et al. (1973a,b) and Kaiser (1974).

The predictions of Völk (1973) are about a factor of two too high for $\sqrt{2}\eta = .3$ and $\epsilon = 1$. The results of Owens (1974) are consistently low and will not be discussed any further here. All of these theories have very different functional dependencies of the parameters η and ϵ at $\mu = 0$. Figures 4 and 5 show plots of $\epsilon D_\mu(\mu = 0)$ vs. ϵ and η. One can see that

$$\epsilon D_\mu \propto \eta^3 \epsilon \tag{89}$$

whereas, from the summary of Kaiser (1974) one has

$$\epsilon D_\mu^{\text{JKB}} \propto \eta^3 \tag{90}$$

$$\epsilon D_\mu^{\text{VÖLK}} \propto \eta^4/\epsilon \tag{91}$$

$$\epsilon D_\mu^{\text{OWENS}} \propto \epsilon\eta^4/(1+\eta^4) \tag{92}$$

The parameter dependence illustrated in equations (91) and (92) results from the severe approximations made by Völk (1973) and Owens (1974). Within the context of the turbulence theory presented here, there appears to be no consistent way to recover the results of Völk (1973) and Owens (1974). However, a detailed comparison between this work and that of Jones, et al. (1973a,b) is possible. One can recover the parameter dependence shown in equation (90) from our theory when one uses in our analysis the approximations to the particle orbits used by Jones, et al. (1973a,b) and Kaiser (1973). The basic difference between the present work and that of Jones, et al. (1973a,b) is to use a propagator U_p in place of $\bar{U}$ in equation (19) for $D_\mu(\tau)$. U_p is the propagator in the partially averaged field,

$$\underline{\beta}_p(\underline{x},\underline{x}') = \bar{\beta} + \underline{\beta}'(\underline{x}) \cdot \underline{R}(\underline{x}-\underline{x}') \tag{93}$$

A further approximation is then made, namely, that in evaluating trajectories in $\underline{\beta}_p$, a guiding center approximation can be made (Kaiser 1973).

One consequence of this is that $U_p(\tau)\,\phi \cong U_o(\tau)\,\phi=\phi+\tau$ where ϕ is the phase angle.

The propagator $\bar{U}(\tau)$, on the other hand, introduces the quantity $\Delta\underline{p}(\tau)$ through the relation

$$\bar{U}(\tau)\hat{p} = \hat{p}_o(\tau) + \Delta\underline{p}\ (\tau) \tag{94}$$

The approximation made by Jones et al. (1973a,b) that $U_p(\tau)\phi \backsim \phi+\tau$ is a fairly severe one. It is not difficult to show that such an approximation is equivalent to a $\Delta\underline{p}_\perp$ of the form (at $\mu = 0$)

$$\Delta p_x(\tau) = \alpha\cos\ (\phi + \tau) \tag{95}$$

$$\Delta p_y(\tau) = \alpha\sin\ (\phi + \tau) \tag{96}$$

where we have chosen $\hat{\beta}$ to be in the z-direction ($\hat{\beta} = \hat{e}_z$), and where α is an arbitrary function of τ and ϕ. However, Jones, et al. (1973a,b) make the additional approximation that

$$(1-\mu^{*2})^{1/2} \simeq (1-\mu_o^{\ 2})^{1/2} \equiv p_{o\perp}(\tau) \tag{97}$$

where here

$$\begin{aligned} U_p(\tau)\mu &\equiv \mu^* \\ U_o(\tau)\mu &\equiv \mu_o \end{aligned} \tag{98}$$

But because the magnitude of the momentum is a constant, one must have

$$(1-\mu^{*2}) \cong [\hat{p}_{o_\perp}(\tau) + \Delta\underline{p}_{\ \perp}(\tau)]^2 \tag{99}$$

(we have dropped quantities of the form $\langle\underline{p}'(\tau)\cdot\underline{p}'(\tau)\rangle$). Equations (97)-(99) together require that $\alpha = 0$ (or equivalently, $\delta p_{\ \perp}(\tau) = 0$). If one sets the term $\delta\underline{p}_\perp(\tau) = 0$ in equation (88), one finds the results plotted as open circles in figures (4) and (5). Now, in place of

equation (89), one has

$$\epsilon D_{\mu}(\delta p_{\perp} = 0,\ \mu=0) \propto \eta^3 \tag{100}$$

We conclude then that the parameter dependence found by Jones, et al. (1973a,b) and Kaiser (1973, 1974) results from an inadequate approximation to the particles' momentum, $\hat{p}(\tau)$. It remains to be shown whether more refined calculations of orbits in the partially averaged field would bring their theory into closer agreement with the work presented here. From the numerical simulations presently available, one cannot determine the parameter dependence of D_{μ} with any precision. The published simulations are consistent with either an η^3 or $\epsilon\eta^3$ behavior.

Several important questions remain. First, the convergence properties of this theory have not been investigated. In the case of electrostatic turbulence, a convergence proof has been presented by Thompson and Benford (1973), but challenged by Orszag (1975). The electromagnetic case is considerably more complex. Perhaps comparison with numerical simulations is the most fruitful way to test this type of nonlinear theory.

It is not difficult to generalize the results presented here for different correlation functions, including isotropic turbulence, and this work is in progress. The exponential correlation function is a convenient one to use for many reasons, but one should bear in mind that it has some well-known undesirable features.

Also, the validity of the adiabatic approximation has not been investigated in any detail in the context of this resonance - broadening approach. It is known to fail at $\mu = 0$ in quasilinear theory, but is

expected to be a better approximation here because the time integral in equations (70) and (88) for $D_{\mu}(\tau)$ saturates quickly. These questions are presently being investigated.

VI CONCLUSIONS

We have presented a theory for evaluating the pitch angle scattering coefficient for particles moving in magnetostatic turbulence. The theory is a perturbation expansion in the strength of the magnetic field fluctuations, which is equivalent to the weak coupling approximation of Kadomstev (1965). The pitch angle scattering coefficient was defined and evaluated assuming that the adiabatic approximation is valid. This theory is expected to be good for weak field fluctuations ($\eta^2 \ll 1$) and arbitrary particle rigidities. [However, for slab turbulence, the high energy regime, $\epsilon \ll 1$, is not physically interesting.] We believe the derivation to be fairly rigorous with a minimum of *ad hoc* assumptions.

The theory was evaluated in detail for the special case of a slab turbulence model with an exponential correlation function. For $\mu \sim 1$, we find that quasilinear theory provides a good approximation. At $\mu=0$, where quasilinear theory was expected to fail, this strong turbulence theory predicts significant scattering through 90° —much of it due to mirroring forces. At $\mu=0$, D_μ was computed in detail. All of the important nonlinear terms were retained. In that calculation (section III, above), in addition to the weak coupling and adiabatic approximations, it was assumed that terms proportional to $p_\parallel'(\tau)$ could be dropped. A test of the validity of that assumption will have to await more refined numerical simulations. In section IV, D_μ was computed for arbitrary μ. There terms in Δp (τ) were omitted from the analysis; except the one term resulting from mirroring forces. The results of section IV evaluated at $\mu=0$ compare favorably to the more exact calculation of section III (*cf.* figs. 1-3).

We found that $\epsilon D_{\mu}(\mu=0) \propto \eta^3 \epsilon$. A direct comparison of these results with those of Kaiser (1973, 1974) and Jones et al. (1973a,b) showed that their approximate evaluation of their theory neglected the term in Δp (τ) that comes from mirroring. Our theory reduces to theirs (within factors of ~ 2) if one drops all terms in $\Delta p_{\perp}(\tau)$ in the analysis (cf. figs. 4 and 5). It remains to be shown whether a more accurate calculation of particle orbits in the partially averaged field of Jones, et al. (1973a,b) and Kaiser (1973, 1974) will bring their results into agreement with ours. The simulations presently available (Kaiser 1974) cannot distinguish between a pitch angle scattering coefficient proportional to η^3 or $\epsilon\eta^3$.

The success of this theory in matching the numerical simulations leads to the conclusion that the weak coupling approximation is a good one in this problem. The adiabatic approximation also appears valid, though more extensive checks of this are possible and are in progress. Our general conclusion is that D_{μ} can be accurately computed within weak coupling, strong turbulence theory, if a sufficiently good evaluation of the particles' momentum (and position) is included. We believe that this is the first calculation of D_{μ} to do this. Generalization to other correlation functions and other turbulence geometrics (e.g. isotropic) is in progress.

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APPENDIX A

In this appendix we show that, for slab turbulence at $\mu=0$, the following terms are approximately zero

$$\langle \underline{p}_\perp'(\nu)\cdot\underline{p}_\perp'(\lambda)\rangle \simeq 0 \tag{A1}$$

$$\langle \underline{p}_\perp'(\nu)\ \exp ik_\parallel x_\parallel(\lambda)\rangle \simeq 0 \tag{A2}$$

In analogy to equation (42) we immediately have

$$\langle \underline{p}_\perp'(\nu)\cdot\underline{p}_\perp'(\lambda)\rangle = -\eta^2 \int_o^\nu d\tau \int_o^\lambda ds \left[\left[U(\nu-\tau)\hat{p}\cdot\underline{\Omega}'\cdot\underline{\mathcal{C}}_\perp^\dagger(\tau)\right]\right]\cdot\underline{\mathcal{C}}_\perp(-s)\cdot\underline{\Omega}'[\underline{x}(\lambda-s)]\cdot p(\lambda-s)\rangle \tag{A3}$$

Using equation (47) for $\underline{\Omega}'(\underline{x})$, one has

$$\langle \underline{p}_\perp'(\nu)\cdot\underline{p}_\perp'(\lambda)\rangle = -\eta^2 \int_o^\nu d\tau \int_o^\lambda ds \langle \hat{p}(\nu-\tau)\cdot\hat{\beta}\underline{\beta}'[\underline{x}(\nu-\tau)]\cdot\underline{\Omega}\cdot\underline{\mathcal{C}}_\perp^\dagger(\tau+s)\cdot\Omega\cdot \underline{\beta}'[\underline{x}(\lambda-s)]\ \hat{\beta}\cdot\hat{p}(\lambda-s)\rangle \tag{A4}$$

If one writes the fluctuation fields as Fourier transforms, and performs the indicated matrix algebra, one finds

$$\langle \underline{p}_\perp'(\nu)\cdot\underline{p}_\perp'(\lambda)\rangle = \frac{2\eta^2}{\sqrt{2\pi}} \int_o^\nu d\tau \int_o^\lambda ds \int_{-\infty}^{\infty} dk_\parallel R(k_\parallel) \cos(\tau+s)\ \exp\left[-k_\parallel^2 D_1(\nu-\tau,\lambda-s)/2\right] \langle p_\parallel'(\nu-\tau)p_\parallel'(\lambda-s)\rangle \tag{A5}$$

where we have expanded the correlation and used the fact that at $\mu=0$, $\langle p_\parallel(\tau)\rangle=0$. In principle, it is possible to retain equation (A5), but to do so would significantly complicate the analysis. However, this term is not expected to produce a significant contribution to $D_1(\tau)$ because the correlations in equation (A5) are in general evaluated at different times. In the context of the weak coupling approximation, such

nonlocal behavior is expected to be small. Also, the correlation $\langle p_{\parallel}'(\nu)p_{\parallel}'(\lambda)\rangle$ is formally proportional to η^2, which implies that $\langle \underline{p}_{\perp}'(\nu)\cdot\underline{p}_{\perp}'(\lambda)\rangle$ is $\mathcal{O}(\eta^4)$. For these reasons we arrive at equation (A1).

We return to equation (A2), which can be written as

$$\langle \underline{p}_{\perp}'(\nu)\exp ik_{\parallel}x_{\parallel}'(\lambda)\rangle = \langle \underline{p}_{\perp}'(\nu)\exp ik_{\parallel}x_{\parallel}(\lambda)\rangle \tag{A6}$$

Note that

$$\underline{p}'(\nu)\equiv[U(\nu)-\bar{U}(\nu)]\hat{\underline{p}}_{\perp} \tag{A7}$$

From the identity (16), we then have

$$\langle \underline{p}_{\perp}'(\nu)\exp ik_{\parallel}x_{\parallel}(\lambda)\rangle = -\eta\langle {}_{o}\int^{\nu} ds\, \left[\left[U(\nu-s)[\mathfrak{L}'\bar{U}(s)-\langle\mathfrak{L}'U(s)\rangle]\hat{\underline{p}}_{\perp}\right]\right] \exp ik_{\parallel}x_{\parallel}(\lambda)\rangle \tag{A8}$$

If one approximates U as $\bar{U}$ in equation (A8), then the second term is clearly of higher order in η and one has

$$\langle \underline{p}_{\perp}'(\nu)\exp ik_{\parallel}x_{\parallel}(\lambda)\rangle \cong -\eta\,\langle {}_{o}\int^{\nu} d\tau\left[\left[\bar{U}(\nu-\tau)\mathfrak{L}'U_{o}(\tau)\hat{\underline{p}}_{\perp}\right]\right] \exp ik_{\parallel}x_{\parallel}(\lambda)\rangle \tag{A9}$$

where we have used equation (36). If we rewrite $\exp i[k_{\parallel}x_{\parallel}(\lambda)]$ as $U(\lambda)\exp i[k_{\parallel}x_{\parallel}]$, and use equation (16), equation (A9) becomes

$$\langle \underline{p}_{\perp}'(\nu)\exp ik_{\parallel}x_{\parallel}(\lambda)\rangle \cong \eta^2\langle {}_{o}\int^{\nu} d\tau\left[\left[\bar{U}(\nu-\tau)\mathfrak{L}'U_{o}(\tau)\hat{\underline{p}}_{\perp}\right]\right]_{o}\int^{\lambda} ds\, \bar{U}(\lambda-s)\mathfrak{L}'\bar{U}(s) \exp ik_{\parallel}x_{\parallel}\rangle \tag{A10}$$

Following Ben-Israel, et al. (1975) we find, to lowest order in η,

$$\langle \underline{p}_{\perp}'(\nu)\exp ik_{\parallel}x_{\parallel}(\lambda)\rangle \cong -i\eta^2\,{}_{o}\int^{\nu} d\tau\,\langle\left[\left[\,\bar{U}(\nu-\tau)\hat{\underline{p}}\cdot\underline{\Omega}'\cdot\underline{\mathcal{E}}_{\perp}^{\dagger}(\tau)\right]\right]_{o}\int^{\lambda} ds\,\bar{U}(\lambda-s) \hat{\underline{p}}\cdot\underline{\Omega}'\cdot\underline{\mathcal{E}}^{\dagger}(s)\cdot\underline{k}\rangle \tag{A11}$$

$$\simeq \frac{-i\eta^2}{\sqrt{2\pi}} \int_0^{\nu} d\tau \int_0^{\lambda} ds\, R(k_\parallel) \langle \exp ik_\parallel [x'_\parallel(\nu-\tau) - x'_\parallel(\lambda-s)] s k_\parallel p'_\parallel(\nu-\tau) \underline{p}_\perp(\lambda-s) \cdot \underline{C}^\dagger_\perp(s) \rangle \tag{A12}$$

In the analysis presented here, we will drop any terms in $\underline{p}'_\parallel(\tau)$. This is equivalent to ignoring nonlocal correlations of the form $\langle \underline{p}'(\tau)\underline{x}'(\lambda)\rangle$. we expect that in the context of the weak coupling approximation, nonlocal correlations between momentum and position fluctuations should be less important than autocorrelations of fluctuation in position evaluated at equal times [e.g. $D_1(\tau,\tau) \equiv \langle \underline{x}'(\tau)\underline{x}'(\tau)\rangle$]. For a more comprehensive discussion of these various correlations, and their role in determining $D_\mu(\tau)$, the reader is referred to Ben-Israel, et al. (1975).

APPENDIX B

In this appendix we derive equation (55) for $\Delta\underline{p}_\perp(\tau)$. The result of taking the Fourier transform of the fluctuation fields in equation (54) is that

$$\Delta\underline{p}_\perp(\tau)=\frac{\eta^2}{(2\pi)^3}\int_0^\tau d\lambda\int_\lambda^\tau d\nu\int d^3k\int d^3k'\langle\exp[i\underline{k}\cdot\underline{x}(\tau-\nu)]\hat{p}(\tau-\nu)\cdot\underline{\Omega}'(\underline{k}')\cdot\underline{\mathcal{C}}(\tau-\nu)\cdot\left[\frac{\partial}{\partial\hat{p}}+\hat{n}\cdot\underline{\mathcal{D}}^\dagger(\tau)\cdot\nabla\right]\hat{p}(\tau-\lambda)\cdot\underline{\Omega}'(\underline{k}')\cdot\underline{\mathcal{C}}_\perp^\dagger(\lambda)\exp[i\underline{k}'\cdot\underline{x}(\tau-\lambda)]\rangle \tag{B1}$$

where we have used the approximation

$$U(\tau)\frac{\partial}{\partial\hat{p}}\cong\frac{\partial}{\partial\hat{p}_0(\tau)}\quad U(\tau)\equiv\underline{C}(\tau)\cdot\left[\frac{\partial}{\partial\hat{p}}+\hat{n}\cdot\underline{\mathcal{D}}^\dagger(\tau)\cdot\nabla\right]U(\tau) \tag{B2}$$

If one substitutes the identity (47) for $\underline{\Omega}'(\underline{k})$ and uses equations (22) and (44), it follows, after some straightforward algebra, that

$$\Delta\underline{p}_\perp(\tau)=\frac{\eta^2}{\sqrt{2\pi}}\int_0^\tau d\lambda\int_\lambda^\tau d\nu\int_{-\infty}^{\infty}dk_\parallel R(k_\parallel)\langle\exp[ik_\parallel x_\parallel'(\nu-\lambda)]\left\{p_\parallel'(\tau-\nu)\dot{\underline{\mathcal{C}}}_\perp(\nu-\lambda-\tau)\cdot\left[\frac{\partial}{\partial\hat{p}_\perp}p_\parallel(\tau-\lambda)+ik_\parallel p_\parallel'(\tau-\lambda)\frac{\partial}{\partial\hat{p}_\perp}r_\parallel(\tau-\lambda)\right]-\hat{p}_\perp(\tau-\nu)\cdot\underline{\mathcal{C}}_\perp^\dagger(\lambda)\cdot\left[\frac{\partial}{\partial\mu}p_\parallel(\tau-\lambda)+ik_\parallel p_\parallel'(\tau-\lambda)\left(\frac{\partial}{\partial\mu}r_\parallel(\tau-\lambda)-(\tau-\nu)\right)\right]\right\}\rangle \tag{B3}$$

Again, we drop terms containing $p_\parallel'(\tau)$. Equation (55) follows immediately when one uses the approximation (A2).

APPENDIX C

From equation (40) for $\Delta\underline{p}(\tau)$ we have

$$\Delta p_{\parallel}(\tau) = \eta^2 \int_0^{\tau} d\lambda \int_{\lambda}^{\tau} d\nu \, \langle \bar{U}(\tau-\nu)\hat{p}\cdot\underline{\Omega}\cdot\frac{\partial}{\partial\hat{p}} U(\nu-\lambda)\hat{p}\cdot\underline{\Omega}'\cdot\hat{\beta}\rangle \tag{C1}$$

Following the derivation of $\Delta x_{\parallel}(\tau)$ given in §IV, one has

$$\Delta p_{\parallel}(\tau) = \frac{-\eta^2}{\sqrt{2\pi}} \int_0^{\tau} d\lambda \int_{\lambda}^{\tau} d\nu \int_{-\infty}^{\infty} dk_{\parallel} R(k_{\parallel}) \exp\{ik_{\parallel}[\mu(\nu-\lambda)] - \Delta x_{\parallel}(\nu-\lambda)] - k_{\parallel}^2 D_1(\nu-\lambda)/2\}$$

$$\cos(\nu-\lambda)[2\mu - ik_{\parallel}(1-\mu^2)(\nu-\lambda)] \tag{C2}$$

which becomes

$$\Delta p_{\parallel}(\tau) = \frac{-2\eta^2}{\sqrt{2\pi}} \int_0^{\tau} d\lambda \int_0^{\infty} dk_{\parallel} R(k_{\parallel}) \exp\left[-k_{\parallel}^2 \frac{D_1(\lambda)}{2}\right] (\tau-\lambda) \cos\lambda$$

$$\{2\mu \cos[k_{\parallel}(\mu\lambda - \Delta x_{\parallel}(\lambda))] + (1-\mu^2) k_{\parallel}\lambda \sin[k_{\parallel}(\mu\lambda - \Delta x_{\parallel}(\lambda))]\} \tag{C3}$$

Equation (56) follows immediately.

APPENDIX D

In the discussion below we outline the transformation of equations (70)-(72) and (63)-(64) to a system of integro-differential equations that can be solved numerically. Equation (70) can be easily converted into a differential equation

$$\dot{D}_\mu(\tau)\equiv\frac{d}{d\tau}D_\mu(\tau)=\frac{\eta^2}{\sqrt{2\pi}}g(\tau)[\cos(\tau)+\Delta p_x(\tau)] \tag{D1}$$

Differentiating equation (71) yields

$$\dot{D}_1(\tau)=\frac{2\eta^2}{\sqrt{2\pi}}\int_0^\tau d\lambda\cos\lambda g(\lambda)(\tau^2-\lambda\tau)+\frac{2\eta^2}{\sqrt{2\pi}}\int_0^\tau d\lambda\int_0^\tau d\nu g(\nu-\lambda)(2\tau-\nu-\lambda)$$

$$[\Delta\underline{p}_\perp(\nu)\cdot\hat{p}_{\perp_0}(\lambda)+\tfrac{1}{2}\Delta\underline{p}_\perp(\nu)\cdot\Delta\underline{p}_\perp(\lambda)] \tag{D2}$$

Now define

$$\dot{f}_1(\tau) = g(\tau)\cos\tau \tag{D3}$$

$$\dot{f}_2(\tau) = tg(\tau)\cos\tau \tag{D4}$$

$$f_3(\tau) = \int_0^\tau d\lambda\int_0^\tau d\nu g(\nu-\lambda)\Delta\underline{p}_\perp(\nu)\cdot[\hat{p}_{\perp_0}(\lambda)+\tfrac{1}{2}\Delta\underline{p}_\perp(\lambda)] \tag{D5}$$

$$f_4(\tau) = \int_0^\tau d\lambda\int_0^\tau d\nu g(\nu-\lambda)(\nu+\lambda)\Delta\underline{p}_\perp(\nu)\cdot[\hat{p}_{\perp_0}(\lambda)+\tfrac{1}{2}\Delta\underline{p}_\perp(\lambda)] \tag{D6}$$

with $f_i(0) = 0$, $i = 1\text{-}4$. Therefore

$$\dot{D}_1(\tau) = \frac{2\eta^2}{\sqrt{2\pi}}[\tau^2 f_1(\tau)-\tau f_2(\tau)+2\tau f_3(\tau)-f_4(\tau)] \tag{D7}$$

Now consider

$$\dot{f}_3(\tau) = \int_0^\tau d\lambda g(\lambda)\Delta\underline{p}_\perp(\lambda)\cdot\hat{p}_{\perp_0}(\tau-\lambda)+\int_0^\tau d\nu g(\nu-\tau)\Delta\underline{p}_\perp(\nu)\cdot\hat{p}_{\perp_0}(\tau)$$

$$+\int_0^\tau d\lambda g(\tau-\lambda)\Delta\underline{p}_\perp(\lambda)\cdot\underline{p}_\perp(\tau) \tag{D8}$$

and define

$$\dot{f}_5(\tau) = g(\tau)\sin\tau \tag{D9}$$

$$\dot{f}_6(\tau) = \tau g(\tau)\sin\tau, \quad [f_5(0) = f_6(0) = 0] \tag{D10}$$

$$q_1(\tau) \equiv \int_0^\tau d\nu g(\tau-\nu)\Delta p_{\perp x}(\nu) \tag{D11}$$

$$q_2(\tau) \equiv \int_0^\tau d\nu g(\tau-\nu)\Delta p_{\perp y}(\nu) \tag{D12}$$

$$q_3(\tau) \equiv \int_0^\tau d\nu \nu g(\tau-\nu)\Delta p_{\perp x}(\nu) \tag{D13}$$

$$q_4(\tau) \equiv \int_0^\tau d\nu \nu g(\tau-\nu)\Delta p_{\perp y}(\nu) \tag{D14}$$

so that

$$\begin{aligned}\dot{f}_3(\tau) = {} & f_1(\tau)[\Delta p_{\perp x}(\tau)\cos\tau + \Delta p_{\perp y}(\tau)\sin\tau] + \\ & f_5(\tau)[\Delta p_{\perp x}(\tau)\sin\tau - \Delta p_{\perp y}(\tau)\cos\tau] + \\ & q_1(\tau)[\cos\tau + \Delta p_{\perp x}(\tau)] + q_2(\tau)[\sin\tau + \Delta p_{\perp y}(\tau)]\end{aligned} \tag{D15}$$

Similarly,

$$\begin{aligned}\dot{f}_4(\tau) = {} & [2\tau f_1(\tau) - f_2(\tau)][\Delta p_{\perp x}(\tau)\cos\tau + \Delta p_{\perp y}(\tau)\sin\tau] + \\ & [2\tau f_5(\tau) - f_6(\tau)][\Delta p_{\perp x}(\tau)\sin\tau - \Delta p_{\perp y}(\tau)\cos\tau] + \\ & [\tau q_1(\tau) + q_3(\tau)][\cos\tau + \Delta p_{\perp x}(\tau)] + [\tau q_2(\tau) + q_4(\tau)][\sin\tau + \Delta p_{\perp y}(\tau)]\end{aligned} \tag{D16}$$

In a completely analogous manner one has

$$\Delta\dot{p}_{\perp x}(\tau) = \frac{-\eta^2}{\sqrt{2\pi}} \{\cos\tau[f_1(\tau)(1 + \frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau))_{\mu=0} + \tau f_5(\tau) - f_6(\tau) + f_7(\tau) +$$

$$+ f_{10}(\tau) + f_{12}(\tau)]$$

$$-\sin\tau[-f_5(\tau)(1 + \frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau))_{\mu=0} + \tau f_1(\tau) - f_2(\tau) + f_8(\tau) +$$

$$+ f_9(\tau) + f_{11}(\tau)] + q_1(\tau)[1 + \frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau)]_{\mu=0}\} \qquad (D17)$$

$$\Delta\dot{p}_{\perp y}(\tau) = \frac{-\eta^2}{\sqrt{2\pi}} \{\cos\tau[f_5(\tau)(1 + \frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau))_{\mu=0} - \tau f_1(\tau) + f_2(\tau) - f_8(\tau)$$

$$- f_9(10) - f_{11}(\tau)]$$

$$-\sin\tau[f_1(\tau)(1 + \frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau))_{\mu=0} + \tau f_5(\tau) - f_6(\tau) + f_7(\tau)$$

$$+ f_{10}(\tau) + f_{12}(\tau)] + q_2(\tau)[1 + \frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau)]\} \qquad (D18)$$

where

$$\dot{f}_7(\tau) = f_5(\tau)\frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau)\Big|_{\mu=0} \qquad (D19)$$

$$\dot{f}_8(\tau) = f_1(\tau)\frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau)\Big|_{\mu=0} \qquad (D20)$$

$$\dot{f}_9(\tau) = q_1(\tau)\cos\tau - q_2(\tau)\sin\tau \qquad (D21)$$

$$\dot{f}_{10}(\tau) = q_2(\tau)\cos\tau + q_1(\tau)\sin\tau \qquad (D22)$$

$$\dot{f}_{11}(\tau) = \dot{f}_9(\tau)[\frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau)]_{\mu=0} \qquad (D23)$$

$$f_{12}(\tau) = \dot{f}_{10}(\tau)[\frac{\partial}{\partial\mu}\Delta p_{\parallel}(\tau)]_{\mu=0} \qquad (D24)$$

Finally

$$\frac{\partial}{\partial \mu} \Delta p_{\parallel}(\tau)\Big|_{\mu=0} = \frac{-2\eta^2}{\sqrt{2\pi}} [\tau f_{13}(\tau) - f_{14}(\tau)] \tag{D25}$$

where

$$\dot{f}_{13}(\tau) = \cos\tau \{g(\tau) + \tau h(\tau)[\tau - \frac{\partial}{\partial \mu} \Delta x_{\parallel}(\tau)]_{\mu=0}\} \tag{D26}$$

$$\dot{f}_{14}(\tau) = \tau \dot{f}_{13}(\tau) \tag{D27}$$

and

$$\frac{\partial}{\partial \tau} [\frac{\partial \Delta x_{\parallel}(\tau)}{\partial \mu}]_{\mu=0} = - \frac{\partial}{\partial \mu} \Delta p_{\parallel}(\tau)\Big|_{\mu=0} \tag{D28}$$

The complete system of equations (D1), (D3), (D4), (D7), (D9) and (D28) can be numerically integrated using a standard Adams-Bashforth predictor - corrector method.

APPENDIX E

The system of equations derived in § IV can be converted to the following set of differential equations. The derivation is straightforward.

$$\dot{D}_{\mu}(\tau) = \frac{2\eta^2}{\sqrt{2\pi}} (1-\mu^2)G(\tau)[1+\delta p_{\perp}(\tau)], \; D_{\mu}(0)=0 \tag{E1}$$

$$\dot{D}_1(\tau) = \frac{4\eta^2(1-\mu^2)}{\sqrt{2\pi}} \tau[\tau e_1(\tau)-e_2(\tau)], \; D_1(0)=0 \tag{E2}$$

$$\dot{e}_1(\tau) = G(\tau) \tag{E3}$$

$$\dot{e}_2(\tau) = \tau G(\tau) \tag{E4}$$

$$\Delta\dot{x}_{\parallel}(\tau)=\frac{2\eta^2}{\sqrt{2\pi}} \{2\mu[\tau e_1(\tau)-e_2(\tau)] + (1-\mu^2)[\tau e_3(\tau)-e_4(\tau)]\} \tag{E5}$$

$$\dot{e}_3(\tau) = H(\tau) \tag{E6}$$

$$\dot{e}_4(\tau) = \tau H(\tau) \tag{E7}$$

$G(\tau)$ and $H(\tau)$ are given by equations (78) and (80). $\delta p_{\perp}(\tau)$ follows from equation (86).

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FIGURE CAPTIONS

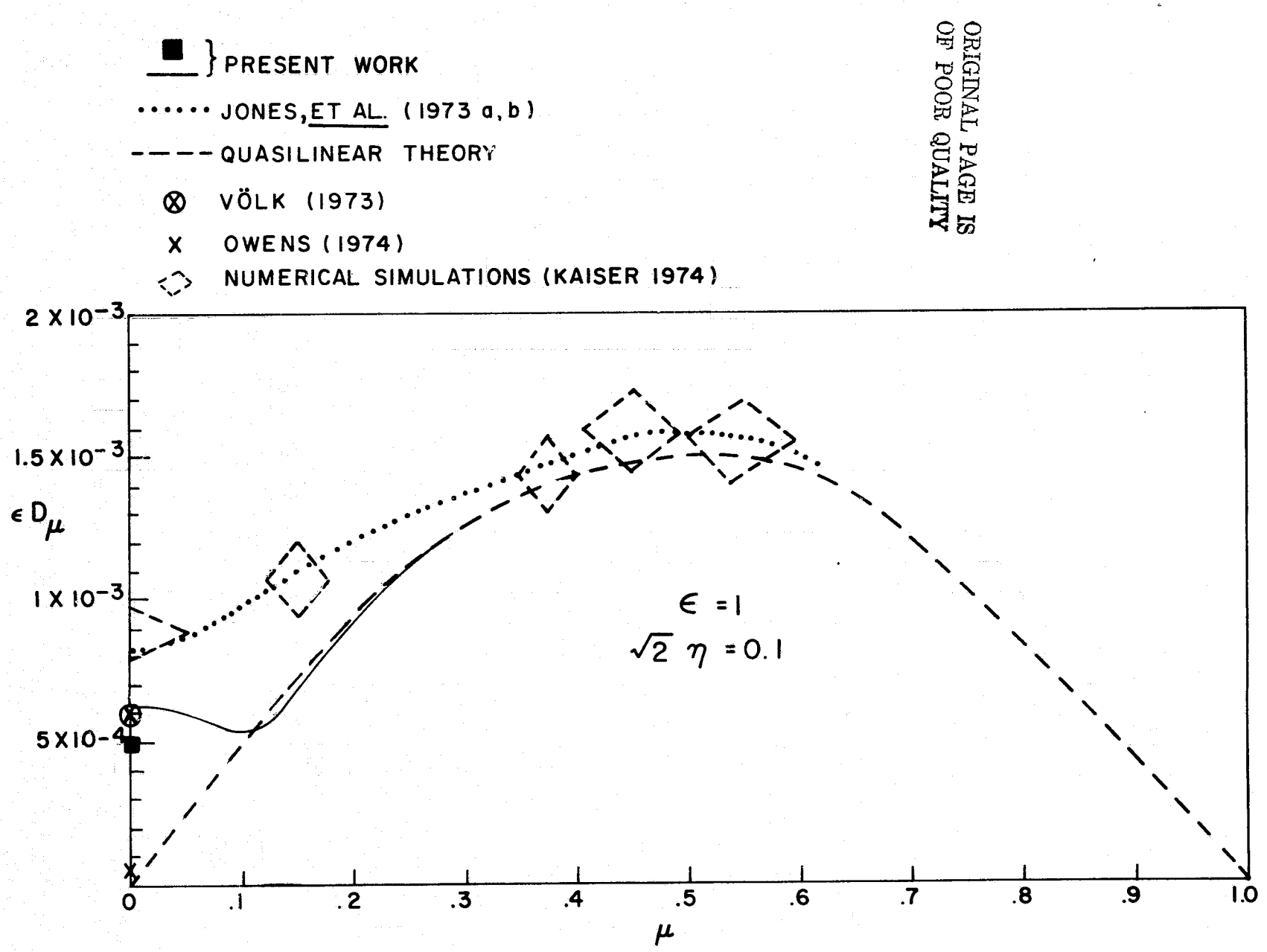
Figure 1. The dimensionless pitch angle diffusion coefficient ϵD_{μ} is plotted as a function of μ. The solid square represents the solution of the set of equations derived in §III, while the solid line follows from the system of equations discussed in §IV. The parameter values are $\epsilon=1$, $\sqrt{2}\eta=0.1$.

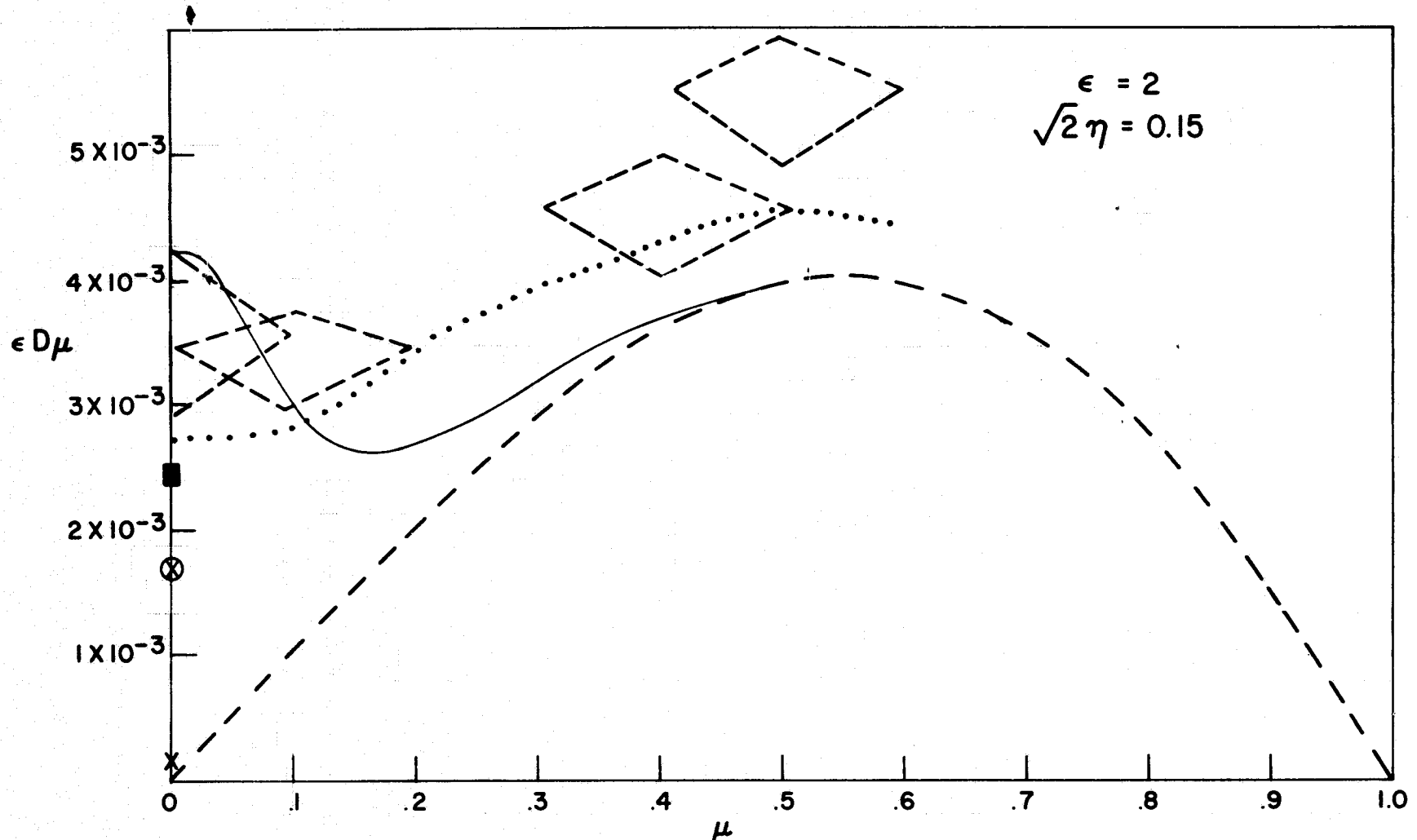
Figure 2. ϵD_{μ} vs. μ for $\epsilon=2$, $\sqrt{2}\eta=0.15$. The notation follows that of figure 1.

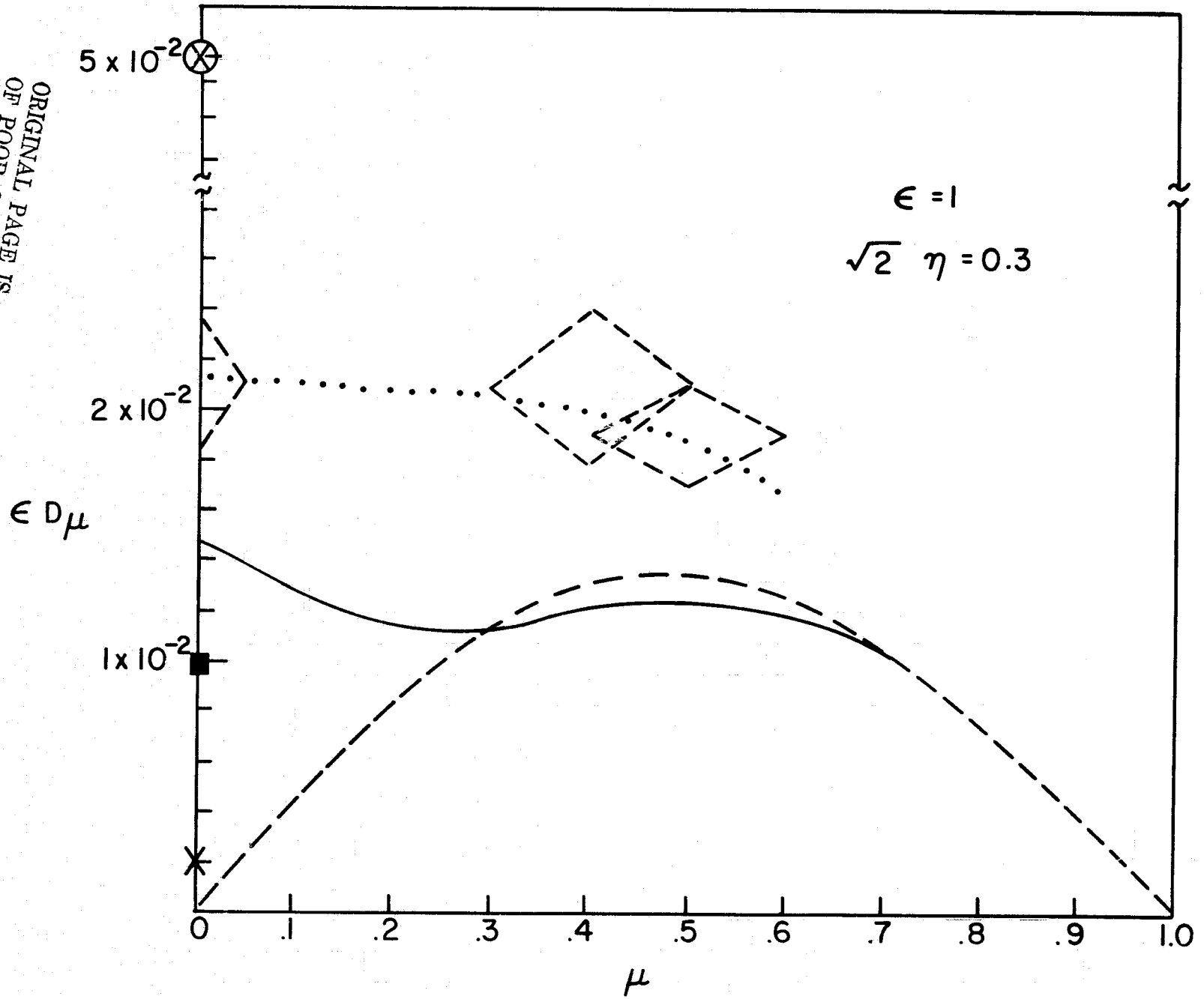
Figure 3. ϵD_{μ} vs. for $\epsilon=1$, $\sqrt{2}\eta=0.3$. The notation follows that of figure 1.

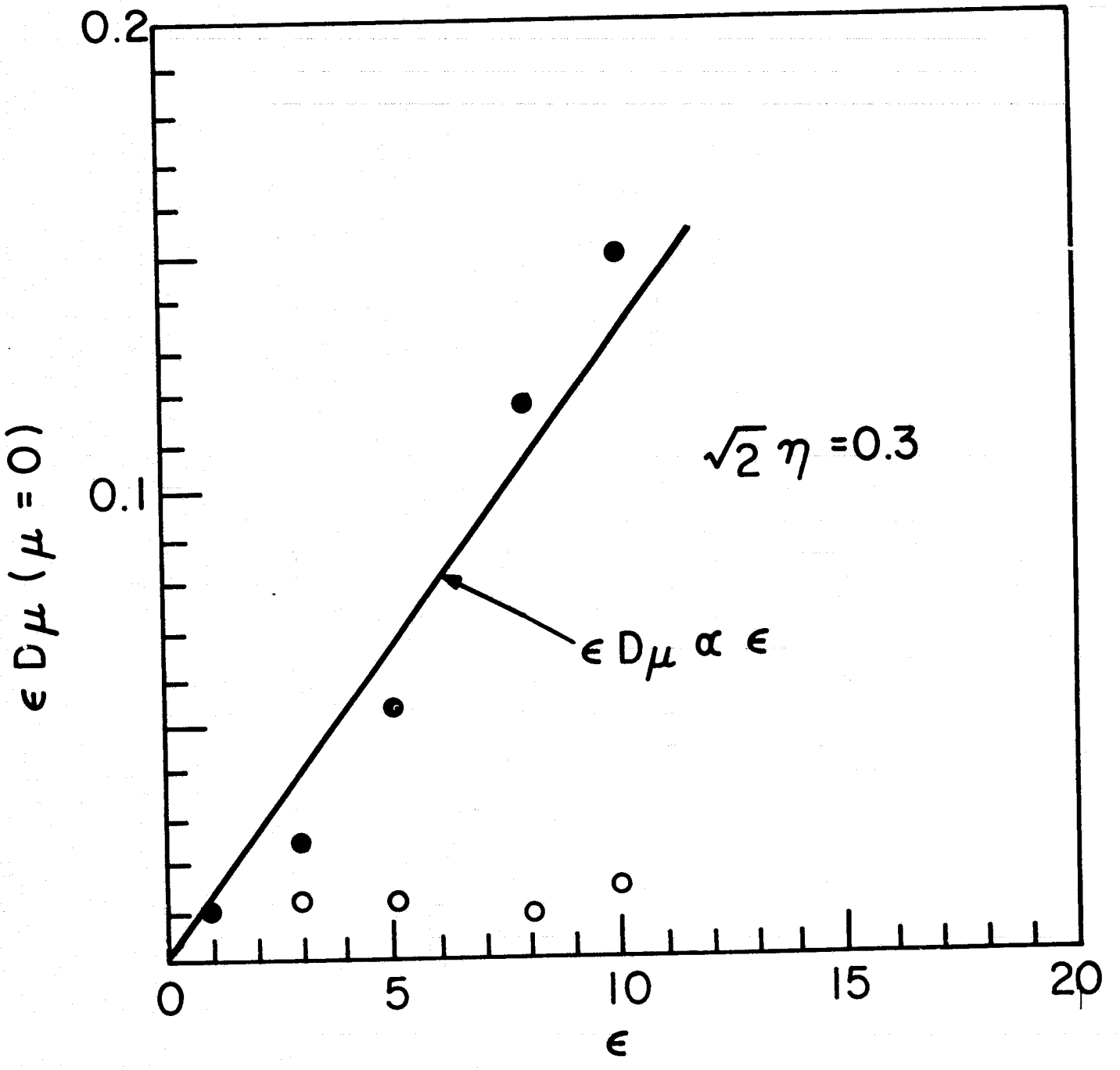
Figure 4. ϵD_{μ} vs. ϵ at $\mu=0$ for $\sqrt{2}\eta=0.3$. When the momentum deviation terms are retained ($\Delta\underline{p}(\tau)\neq 0$), $\epsilon D_{\mu}(\mu=0)$ is approximately proportional to ϵ (solid circles). However, when these momentum terms are dropped ($\delta p_{\perp}=0$ in equation 88) one finds that $\epsilon D_{\mu}(\mu=0)$ is approximately independent of ϵ (open circles), thus recovering the results of Jones, et al. (1973a,b).

Figure 5. $\epsilon D_{\mu}(\mu=0)$ vs. η for $\epsilon=5$. $\epsilon D_{\mu}(\epsilon=0)$ is approximately proportional to η^3 whether the momentum deviation terms are kept (solid circles) of dropped (open circles).




$\epsilon = 2$
$\sqrt{2}\eta = 0.15$
5×10^{-3}
4×10^{-3}
3×10^{-3}
2×10^{-3}
1×10^{-3}
$\epsilon D\mu$
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
μ




0.2
0.1
0
5
10
15
20
$\epsilon D\mu\,(\mu=0)$
ϵ
$\sqrt{2}\,\eta = 0.3$
$\epsilon D\mu \propto \epsilon$