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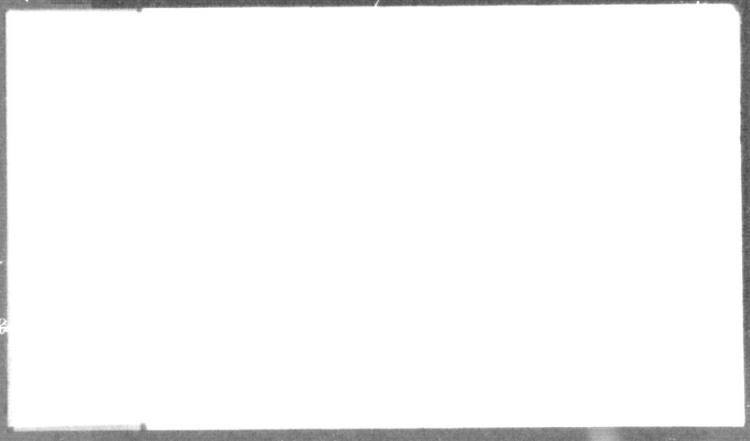

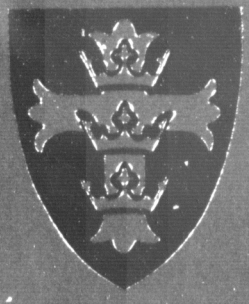

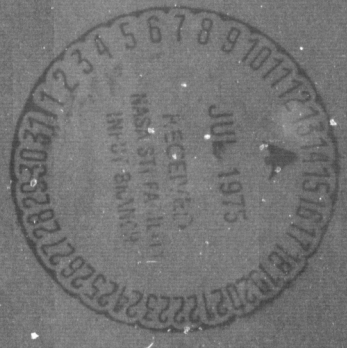

DEPARTMENT OF AEROSPACE ENGINEERING

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STEADY AND OSCILLATORY, SUBSONIC AND SUPERSONIC, AERODYNAMIC PRESSURE AND GENERALIZED FORCES FOR COMPLEX AIRCRAFT CONFIGURATIONS AND APPLICATION TO FLUTTER

by

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ABSTRACT

A general method for steady and oscillatory, subsonic and supersonic, potential linearized aerodynamic flow around complex configurations is introduced. By applying the Green function method, a linear integral equation relating the unknown, small perturbation potential on the surface of the body, to the known downwash is obtained. The surfaces of the aircraft, wake and diaphragm (if necessary) are divided into small quadrilateral elements which are approximated with hyperboloidal surfaces. The potential and its normal derivative are assumed to be constant within each element. This yields a set of linear algebraic equations. The coefficients are evaluated analytically. By using Gaussian elimination method, equations are solved for the potentials at the centroids of elements. The pressure coefficient is evaluated by the finite different method. The lift and moment coefficients are evaluated by numerical integration. Numerical results are presented. Applications to flutter are also included.

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NOMENCLATURE

$\bar{a}_i$	Equations (3.13) and (3.14)
A	The area of the wing planform, or ω_α^2/ω^2 in Equation (7.35)
Arg	Phase angle
AR	Aspect ratio
a	Free-stream speed of sound
b	Span
b_{hk}	Equations (3.4) and (5.4)
$\hat{b}_{hk}$	Equations (4.6) and (6.4)
C_k	Equation (7.44)
c_p	Pressure coefficient, $2(p - p_\infty)/\rho_\infty U_\infty^2$
c_l	Lift distribution coefficient, $C_{p_l} - C_{p_u}$
C_l	Sectional lift coefficient
C_L	Total lift coefficient
$\tilde{C}_{L\alpha}$, $\tilde{C}_{M\alpha}$	Total lift coefficient and total moment coefficient about axis $X = X_{EA}$ for the wing in oscillation in pitch about axis $X = X_{EA}$ per unit angle amplitude
$\tilde{C}_{Lh}$, $\tilde{C}_{Mh}$	Total lift coefficient and total moment coefficient about axis $X = X_{EA}$ for the wing in oscillation in plunge
C	Chord
$c_{l\alpha}$	$\left(\partial c_l/\partial\alpha\right)_{\alpha=0}$
C_{hk}	Equations (3.5) and (5.5)
$\hat{C}_{hk}$	Equations (4.7) and (6.5)
E	Equation (2.29)
$F_j(\xi,\eta)$	Equation (5.13)

G	Green function, Equations (2.18), (2.20), (2.22) and (2.24)
H	Heaviside function, Equation (2.34)
$I_D(\xi,\eta)$	Doublet integral, Equations (3.10), (5.8), (B.36), (B.38) and (B.40)
$I_s(\xi,\eta)$	Source integral, Equations (3.11) and (5.9)
$I_{s1}(\xi,\eta)$	Equations (5.10), (B.26), (B.27) and (B.28)
$I_{s2}(\xi,\eta)$	Equations (5.11), (B.30), (B.31) and (B.32)
$I_{s3}(\xi,\eta)$	Equations (5.12), (B.35), (B.37) and (B.39)
$I_{w1}(\eta)$	Equation (3.17)
$I_{w2}(\eta)$	Equation (3.18)
K	Reduced frequency, $\omega \ell / U_\infty$
ℓ	Reference length
LW	Truncated wake length
M	Mach number, U_∞ / a_∞
$\bar{n}$	Equation (3.15)
$\bar{N}$	Outward unit normal to surface Σ_A
$\bar{N}_U$	Upper unit normal to surface Σ_W
$\bar{N}^c$	Conormal, defined in Equation (2.37)
NX, NY	Number of body elements in X, Y direction
ND	Number of diaphragm elements
NW	Number of wake elements in Σ_K'
$\bar{P}$	Point on the surface Σ , Equation (2.40)
$\bar{P}_*$	Control point, (X_*, Y_*, Z_*)
$\bar{q}$	Equation (3.12)

$\bar{P}_C, \bar{P}_1, \bar{P}_2, \bar{P}_3$	Equation (2.41) (see Figure 1)
$\bar{P}_m, \bar{P}_d, \bar{P}_{md}$	Equations (3.19) through (3.21) (see Figure 2b)
$\bar{P}_+, \bar{P}_-$	Figure 2b
$\bar{R}$	Equation (2.27)
R	Equation (2.28)
$\|\| R \|\|$	Equation (2.35)
γ_α	Equation (7.8)
t	Time
T	$a_\infty \beta t / \ell$
TR	Taper ratio
x, y, z	Space coordinate
X, Y, Z	Nondimensional Prandtl-Glauert coordinates, Equation (2.6)
(x_M, y_M, z_M)	Moment center
X_{EA}	Location of elastic axis (see Figure 3)
X_{TE}	X-component of the trailing-edge vector (see Figure 2a)
X_α	Equation (7.8)
$\Delta(X_w)$	Incremental length of wake element
$\tilde{z}$	Vibration mode, Equation (C.18)
U_∞	Free-stream velocity
$W_{hk}, \hat{W}_{hk}$	Wake coefficient
$W_{hk}^{(TE)}$	Equation (3.16)
$\hat{W}_{hk}^{(TE)}$	Equation (4.12)

α	Angle of attack
β	$\lvert 1 - M^2 \rvert^{\frac{1}{2}}$
Σ	Surface surrounding body and wake
Σ_k	Surface element
Σ_k'	Wake strip emanating from the trailing element
Σ_A	Surface of aircraft
Σ_w	Surface of wake
τ	Thickness ratio
Φ	Velocity potential, $U_\infty X + \varphi$
φ	Perturbation velocity potential
ϕ	$\varphi / U_\infty l$
ϕ_D	ϕ on the diaphragm
ψ	Normal derivative of ϕ
ψ_D	ψ on the diaphragm
$\hat{\phi}$	Equations (2.31) and (2.39)
$\tilde{\phi}$	Equations (2.31) and (2.39)
ϕ_k, $\hat{\phi}_k$	Values of ϕ and $\hat{\phi}$ at the centroid of surface element Σ_k
ψ_k	Values of ψ_k at the centroid of surface element Σ_k
Λ	Sweep angle
ω	Frequency of oscillation
ω_h	Equation (7.12)
ω_α	Equation (7.13)
Ω	Compressible reduced frequency, $\omega l / a_\infty \beta = k M / \beta$

$(\ )$	$(\ )_u - (\ )_l$
∇	Gradient in X, Y, Z space
$\odot$	Super product, Equation (2.36)
$\|\| \bar{a} \|\|$	Supernorm of vector $\bar{a}$, Equation (2.35)

SECTION 1

INTRODUCTION

1.1 GENESIS OF THE PROBLEM

The structural design of aircraft requires the evaluation of the aerodynamic loads. Aircraft structures are flexible and this flexibility is responsible for various types of aeroelastic phenomena. For instance, the static aerodynamic loads depend upon the geometry of the surface of the aircraft, while the geometry depends upon the aerodynamic loads. Therefore, it is necessary to evaluate these loads by an iterative process. In addition, the structural design is subject to aeroelastic constraints such as divergence and flutter. These constraints require accurate evaluations of steady and oscillatory loads. Again, the optimal design requires an iterative process, since the resizing of the structure changes the stability boundary. These iterative processes are best performed by computer analysis. This implies that considerable advantages are obtained if the mathematical modeling of the problem is general, flexible and efficient. The method used in this work is based upon the theoretical formulation developed by Morino (References 1 and 2). A preliminary analysis is described in Reference 3.

It may be worth noting that the usual methods for the evaluation of aerodynamic loads are the computational lifting-surface (zero-thickness wing) theories. These methods are efficient and flexible but not sufficiently general for the purpose of automated design of complex aircraft configurations. In addition, several computational methods around complex aircraft configurations have already been developed. These methods are general, but usually quite cumbersome to use and always require human intervention to define the suitable types of elements to be used. In other words, it may be safely stated that none of the other methods currently available satisfies all the requirements of generality, flexibility and efficiency necessary for automated design. Furthermore, for oscillatory flow around complex configurations, only techniques based upon the doublet-lattice method exist in subsonic range (see References 4 and 5), while no other method is available in the supersonic one.

The method used here is applicable to steady and oscillatory subsonic and supersonic flows (References 6 to 10) and provides an efficient, general, and flexible aerodynamic tool to be used for the design of aircrafts. These advantages of the present method are particularly relevant in the use of automated design.

1.2 OUTLINE OF THE METHOD

The present method is mainly based on the Green function method. By applying the Green function method, the equation of the velocity potential yields a linear equation relating the velocity potential φ, at any point, $\bar{P}_*$, in the flow field with the values of φ and its normal derivative on the surface Σ, surrounding the body and the wake. The integral equation is obtained by imposing that the value of the potential at $\bar{P}_*$ approaches the value of φ on the surface, if $\bar{P}_*$ approaches a point on the surface. In order to solve the integral equation, the surface of the aircraft and its wake is divided into small quadrilateral elements. Each element is replaced by a hyperboloidal surface defined by the four corner points of the element. The unknown φ is assumed to be constant within each element. Therefore, the integral equation reduces to a system of algebraic equations with unknown φ_K at the centroids of the elements. The coefficients are evaluated analytically. Once the distribution of the velocity potential is obtained, the pressure distribution, the lift coefficient as well as the generalized forces can be easily evaluated.

In Reference 3, the surface element is replaced by its tangent plane at the element centroids. The improvement obtained by using hyperboloidal element are that the formulation becomes general, flexible, simpler to use and more accurate since the continuity of the surface is maintained (see References 6 to 10).

The work presented here includes further developments with respect to the ones presented in Reference 8. In particular, it includes refinements and applications of the code SOSSA ACTS (Steady and Oscillatory Subsonic and Supersonic Aerodynamics for

Aerospace Complex Transportation Systems). By using this program, several numerical results have been obtained. These include comparison with ones obtained by lifting surface theory, as well as results for finite thickness wings (in both subsonic and supersonic, steady and oscillatory flow) and results with wing-body configuration (in both subsonic and supersonic steady flow). Also the evaluation of generalized forces in oscillatory subsonic and supersonic flow is introduced. The numerical results presented here include the evaluation of lift and moment coefficient. Also included are a study of the role of diaphragms in supersonic flow, a study of convergence in both subsonic and supersonic, steady and oscillatory flow and a study of the effect of the truncation of the wake in oscillatory subsonic flow. Finally, a preliminary application to flutter problem is presented.

1.3 OUTLINE OF WORK

In Section 2 the theoretical formulation is introduced. In Section 3, numerical formulation as well as numerical results for steady subsonic flow are considered. In Section 4, numerical formulation and results for oscillatory subsonic flow are considered. In Section 5, numerical formulation and results for steady supersonic flow are considered. In Section 6, numerical formulations and results for oscillatory supersonic flow are considered. Preliminary flutter analysis for a rigid model with two degrees of freedom is considered in Section 7. Finally, conclusions are given in Section 8. Details of the proof of the formulation used and a list of the computer program are given in the Appendices.

SECTION 2

THEORETICAL FORMULATION

2.1 INTRODUCTION

In this section, the theoretical formulation is briefly presented. Basic flow equations are considered in Subsection 2.2, while the boundary condition is considered in Subsection 2.3. Green theorems for subsonic and supersonic, steady and oscillatory flow, derived in References 1 and 2, are summarized in Subsection 2.4. The numerical formulation is considered in Subsection 2.5. The evaluation of pressure coefficient and generalized forces are considered in Subsections 2.6 and 2.7, respectively.

2.2 BASIC FLOW EQUATION

The aerodynamic forces are derived from the equations which relates the flow properties to the geometry of the lifting bodies. For many practical applications, the flow may be assumed to be inviscid, isentropic, and initially irrotational. In this case, the flow is described by the velocity potential Φ. In addition, the perturbation of the flow is assumed to be small, so that the linearized equation may be used.

The equation of the unsteady aerodynamic potential, is

$$a^2 \nabla^2 \Phi = \frac{D_c^2 \Phi}{D t^2} \tag{2.1}$$

where a is the speed of sound, ∇^2 is the Laplacian operator, and

$$\frac{D_c}{Dt} = \frac{\partial}{\partial t} + \nabla \Phi \cdot \nabla \tag{2.2}$$

is the total time derivative. The subscript "c" reminds that $\nabla\Phi$ should be treated as a constant in order to obtain the second total time derivative.

Consider a frame of reference such that the undisturbed flow has velocity, U_∞, in the direction of the positive x-axis. By introducing the perturbation potential φ, such that

$$\Phi = U_\infty x + \varphi \tag{2.3}$$

(where φ represents perturbation velocity which is assumed to be small with respect to U_∞) and neglecting higher order terms, Equation (2.1) reduces to the linearized equation for the potential flow

$$\nabla^2\varphi - \frac{1}{a_\infty^2}\frac{d^2\varphi}{dt^2} = 0 \tag{2.4}$$

where a_∞ is the free-stream speed of sound, while

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} \tag{2.5}$$

is the linearized total time derivative. For simplicity, the following nondimensional Prandtl-Glauert coordinates are introduced

$$X = x/\beta\ell \quad , Y = y/\ell \quad , Z = z/\ell \quad , \phi = \varphi/U_\infty\ell \tag{2.6}$$

with

$$\beta = \sqrt{1 - M^2} \tag{2.7}$$

for subsonic flow and

$$\beta = \sqrt{M^2 - 1} = B \tag{2.8}$$

for supersonic flow.

2.3 BOUNDARY CONDITION

The lifting body considered here has arbitrary shape and its motion consists of

small vibrations of arbitrary nature. Thus, the surface of the body is represented in general form by

$$S(x, y, z, t) = 0 \tag{2.9}$$

Then, the boundary condition on the body is given by

$$\frac{D_e S}{Dt} = \frac{\partial S}{\partial t} + \nabla \Phi \cdot \nabla S = 0 \tag{2.10}$$

By using Equation (2.3), Equation (2.10) yields the boundary condition for unsteady flow,

$$\nabla \varphi \cdot \nabla S = -\frac{1}{U_\infty}\left(\frac{\partial S}{\partial t} + U_\infty \frac{\partial S}{\partial X}\right) \tag{2.11}$$

Furthermore, the boundary condition at infinity is given by $\varphi \equiv 0$, because the flow is assumed to be uniform at infinity. In the following, the analysis is limited to steady state flow. In this case, Equation (2.11) yields the steady-state boundary condition

$$\bar{\nabla} \varphi \cdot \bar{n}_s = -n_x \tag{2.12a}$$

Consider the subsonic case first. By using the subsonic nondimensional Prandtl-Glauert coordinates given by Equations (2.6) and (2.7), Equation (2.12a) yields

$$\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial S}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial S}{\partial z}\frac{\partial \varphi}{\partial z}\right) + \frac{\partial S}{\partial X}$$

$$= \left(\frac{1}{\beta^2}\frac{\partial S}{\partial X}\frac{\partial \phi}{\partial X} + \frac{\partial S}{\partial Y}\frac{\partial \phi}{\partial Y} + \frac{\partial S}{\partial Z}\frac{\partial \phi}{\partial Z}\right) + \frac{1}{\beta}\frac{\partial S}{\partial X}$$

$$= \left(\frac{\partial S}{\partial X}\frac{\partial \phi}{\partial X} + \frac{\partial S}{\partial Y}\frac{\partial \phi}{\partial Y} + \frac{\partial S}{\partial Z}\frac{\partial \phi}{\partial Z}\right) + \left(\frac{1}{\beta^2} - 1\right)\frac{\partial S}{\partial X}\frac{\partial \phi}{\partial X} + \frac{1}{\beta}\frac{\partial S}{\partial X}$$

$$= \frac{\partial \phi}{\partial N} + \frac{M^2}{1-M^2} N_x \frac{\partial \phi}{\partial X} + \frac{N_x}{\beta} = 0 \tag{2.12b}$$

where N_x is the component of $\bar{N}$ along x-axis (direction of the undisturbed flow). Therefore,

$$\frac{\partial \phi}{\partial N} = -N_x\left(\frac{1}{\beta} + \frac{M^2}{1-M^2}\frac{\partial \phi}{\partial x}\right) \tag{2.13a}$$

The second term on the right hand side is of higher order (see Subsection C.1)*. Therefore, Equation (2.13a) may be approximated as

$$\frac{\partial \phi}{\partial N} = -\frac{N_x}{\beta} \tag{2.13b}$$

Next, consider the steady supersonic flow. In this case, the boundary condition is still given by Equation (2.12a). By using the supersonic nondimensional Prandtl-Glauert coordinates, given by Equations (2.6) and (2.8), and following the same procedure to obtain Equation (2.12b), one obtains

$$\frac{\partial \phi}{\partial N^c} = -N_x\left(\frac{1}{\beta} + \frac{M^2}{M^2-1}\frac{\partial \phi}{\partial x}\right) \tag{2.14a}$$

where $\bar{N}^c = \left(-N_x, N_y, N_z\right)$ is the conormal derivative.

Similarly, by neglecting the second term on the right-hand side, Equation (12.14a) may be approximated as

$$\frac{\partial \phi}{\partial N^c} = -\frac{N_x}{\beta} \tag{2.14b}$$

For clarity and conciseness, the boundary condition as well as the evaluation of the pressure coefficient for oscillatory flow are not derived here, but are presented in Appendix C.

2.4 GREEN FUNCTION METHOD

Observe that Equation (2.4) is a second order linear partial differential

*This is true everywhere on the surface of the aircraft except at the stagnation point.

equation with boundary conditions given by Equation (2.11) on the surface of the body and $\varphi = 0$ at infinity.

The definition of the Green function of Equation (2.4) is given as

$$\nabla^2 G - \frac{1}{a_\infty^2}\frac{d^2 G}{dt^2} = \delta(x-x_1, y-y_1, z-z_1, t-t_1) \tag{2.15}$$

with

$$G = 0 \text{ at infinity} \tag{2.16}$$

where δ is the Dirac delta function defined by

$$\int_{-\infty}^{\infty}\int\int\int F(x_1, y_1, z_1, t_1)\,\delta(x-x_1, y-y_1, z-z_1, t-t_1)\,dx_1 dy_1 dz_1 dt_1 = F(x, y, z, t) \tag{2.17}$$

The solution of G are derived for instance in Reference 1. For steady subsonic flow, G is given by

$$G = \frac{1}{4\pi R} \tag{2.18}$$

with

$$= \left\{(x-x_1)^2 + \beta^2\left[(y-y_1)^2 + (z-z_1)^2\right]\right\}^{1/2} \tag{2.19}$$

For oscillatory subsonic flow, G is given by

$$G = -\frac{1}{4\pi R}\,\delta(t_1 - t + T) \tag{2.20}$$

where $\delta(t_1 - t + T)$ is the usual Dirac delta function and

$$T = \frac{1}{a_\infty \beta^2}\left[R - M(x-x_1)\right] \tag{2.21}$$

with R given by Equation (2.19) and $\beta = \sqrt{1-M^2}$

For steady supersonic flow, G is given by

$$G = -\frac{1}{2\pi R} \tag{2.22}$$

with

$$R = \left\{ (X-X_1)^2 - \beta^2 \left[(y-y_1)^2 + (z-z_1)^2 \right] \right\}^{1/2} \tag{2.23}$$

For oscillatory supersonic flow, G is given by

$$G = -\frac{1}{4\pi R} \left[\delta(t_1 - t + T^-) + \delta(t_1 - t + T^+) \right] \tag{2.24}$$

where

$$T^{\pm} = \frac{1}{a_\infty \beta^2} \left[M(X - X_1) \pm R \right] \tag{2.25}$$

with R given by Equation (2.23) and $\beta = \sqrt{M^2 - 1}$

2.4.1 Green Theorem for Steady Subsonic Flow

By applying the Green function method and using Equation (2.18) the linearized equation of the steady subsonic aerodynamic potential flow can be rewritten as

$$4\pi E(\bar{P}_*)\phi(\bar{P}_*) = -\iint_{\Sigma_A} \frac{\partial \phi}{\partial N} \frac{1}{R} d\Sigma_A + \iint_{\Sigma_A} \phi \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_A$$

$$+ \iint_{\Sigma_W} \Delta\phi \frac{\partial}{\partial N_U}\left(\frac{1}{R}\right) d\Sigma_W \tag{2.26}$$

with nondimensional Prandtl-Glauert coordinates. Where Σ_A and Σ_W are the surfaces of the aircraft and the wake in X, Y, Z coordinate system, while $\bar{N}$ is the outward unit normal to Σ_A, $\bar{N}_U$ is the upper unit normal to Σ_W and $\bar{R}$ is the vector pointing from any point, $\bar{P}_* = (X_*, Y_*, Z_*)$, in the flow field to the dummy point on surface Σ_A or Σ_W,

$$\bar{R} = (X - X_*)\,\bar{i} + (Y - Y_*)\,\bar{j} + (Z - Z_*)\,\bar{k} \tag{2.27}$$

Moreover

$$R = |\bar{R}| = \left[(X - X_*)^2 + (Y - Y_*)^2 + (Z - Z_*)^2\right]^{1/2} \tag{2.28}$$

while E is the domain function, which is defined as (see Reference 1)

$$\begin{aligned} E(\bar{P}_*) &= 1 && \text{outside } \Sigma_A \\ &= 1/2 && \text{on } \Sigma_A \\ &= 0 && \text{inside } \Sigma_A \end{aligned} \tag{2.29}$$

2.4.2 Green Theorem for Oscillatory Subsonic Flow

By applying Green function method and using Equation (2.20), the linearized equation of the oscillatory subsonic aerodynamic potential flow yields the Green theorem for oscillatory subsonic flow as

$$4\pi E(P_*)\hat{\Phi}(P_*) = -\iint_{\Sigma_A} \frac{\partial \hat{\Phi}}{\partial N} \frac{e^{-i\Omega R}}{R}\, d\Sigma_A + \iint_{\Sigma_A} \hat{\Phi} \frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma_A$$

$$+ \iint_{\Sigma_W} \Delta\hat{\Phi} \frac{\partial}{\partial N_U}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma_W \tag{2.30}$$

where X, Y, Z, and Σ_A, Σ_W, N, N_U, R, E are defined in Subsection 2.4.1 while $\hat{\Phi}$ is such that

$$\begin{aligned} \Phi(X, Y, Z, T) &= \tilde{\Phi}(X, Y, Z)\, e^{i\Omega T} \\ &= \hat{\Phi}(X, Y, Z)\, e^{i\Omega(T + MX)} \end{aligned} \tag{2.31}$$

with

$$T = a_\infty \beta t / \ell \qquad \Omega = \omega \ell / a_\infty \beta = KM/\beta \tag{2.32}$$

where ω is natural frequency and K is the dimensionless reduced frequency, $K = \omega \ell / U_\infty$.

2.4.3 Green Theorem for Steady Supersonic Flow

By applying the Green function method and using Equation (2.22), the linearized equation of the steady supersonic aerodynamic potential flow can be rewritten as

$$2\pi E(\bar{P}_*)\phi(\bar{P}_*) = -\iint_{\Sigma_A} \frac{\partial \phi}{\partial N^c} \frac{H}{\|\bar{R}\|} d\Sigma_A + \iint_{\Sigma_A} \phi \frac{\partial}{\partial N^c}\left(\frac{H}{\|\bar{R}\|}\right) d\Sigma_A \tag{2.33}$$

where

$$H = 1 \quad \text{for} \quad X_* - X > \left[(Y_*-Y)^2+(Z_*-Z)^2\right]^{1/2}$$

$$= 0 \quad \text{for} \quad X_* - X \le \left[(Y_*-Y)^2+(Z_*-Z)^2\right]^{1/2} \tag{2.34}$$

$\|\bar{R}\|$ is the "supernorm" of vector $\bar{R}$ and is defined by

$$\|\bar{R}\| = |\bar{R} \odot \bar{R}|^{1/2} \tag{2.35}$$

with the "super-product," $\odot$, defined by

$$\bar{a} \odot \bar{b} = a_X b_X - a_Y b_Y - a_Z b_Z \tag{2.36}$$

and the conormal derivative $\partial/\partial N^c$ is given by

$$\frac{\partial}{\partial N^c} \equiv -N_X \frac{\partial}{\partial X} + N_Y \frac{\partial}{\partial Y} + N_Z \frac{\partial}{\partial Z} = -N \odot \nabla \tag{2.37}$$

while $\bar{R}$ and E are defined in Subsection 2.4.1

2.4.4 Green Theorem for Oscillatory Supersonic Flow

By applying the Green function method and using Equation (2.24) the linearized equation for the oscillatory supersonic aerodynamic potential flow yields the Green Theorem for oscillatory supersonic flow,

$$2\pi E(\bar{P}_*)\phi(\bar{P}_*) = -\iint_\Sigma \frac{\partial \hat{\phi}}{\partial N^c}\frac{H}{\|R\|}\cos\Omega\|R\|\,d\Sigma$$

$$+\iint_\Sigma \hat{\phi}\,\frac{\partial}{\partial N^c}\left(\frac{H}{R}\cos\Omega\|R\|\right)d\Sigma \qquad (2.38)$$

where $\hat{\phi}$ is such that

$$\phi(X,Y,Z,T) = \tilde{\phi}(X,Y,Z)e^{i\omega T} = \hat{\phi}(X,Y,Z)\,e^{i\Omega(T-MX)} \qquad (2.39)$$

while the other parameters are defined in Subsections 2.4.1 and 2.4.3.

2.5 NUMERICAL FORMULATION

The integral equations given in Equations (2.26), (2.30), (2.33) and (2.38) can be solved by replacing the surface of the aircraft and wake by a finite number of quadrilateral hyperboloidal elements. Thus, the integral over Σ becomes a summation of a finite number of integrals over hyperboloidal surfaces which represent the surface of each element on Σ.

The general expression for the hyperboloidal surface is

$$\bar{P}(\xi,\eta) = \bar{P}_c + \xi\bar{P}_1 + \eta\bar{P}_2 + \xi\eta\bar{P}_3$$

$$-1 \le \xi \le 1 \quad , \quad -1 \le \eta \le 1 \qquad (2.40)$$

where $\bar{P}_c$, $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ are defined in terms of the four corner position vectors, $\bar{P}(1,1)$, $\bar{P}(1,-1)$, $\bar{P}(-1,1)$ and $\bar{P}(-1,-1)$.

$$\bar{P}_c = \frac{1}{4}\left[\bar{P}(1,1) + \bar{P}(1,-1) + \bar{P}(-1,1) + \bar{P}(-1,-1)\right]$$

$$\bar{P}_1 = \frac{1}{4}\left[\bar{P}(1,1) + \bar{P}(1,-1) - \bar{P}(-1,1) - \bar{P}(-1,-1)\right]$$

$$\bar{P}_2 = \frac{1}{4}\left[\bar{P}(1,1) - \bar{P}(1,-1) + \bar{P}(-1,1) - \bar{P}(-1,-1)\right]$$

$$\bar{P}_3 = \frac{1}{4}\left[\bar{P}(1,1) - \bar{P}(1,-1) - \bar{P}(-1,1) + \bar{P}(-1,-1)\right] \qquad (2.41)$$

Assume that the values of the velocity potential and its normal derivative within each element are constant. Then the integrand of each surface integral become purely a function of the vector, $\bar{P}_*$ and the geometry of the surface element on Σ. By using Equation (2.40), the analytical expression for each element-integral can be obtained. Finally, a system of linear equation relate computed coefficients to the unknown, ϕ_k, at the centroid of element, Σ_k. The system is solved for ϕ_k by using the Gaussian elimination method.

2.6 PRESSURE AND PRESSURE COEFFICIENT

The pressure on the aircraft is evaluated from the linearized Bernoulli Theorem

$$P - P_\infty = -\rho_\infty \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} \right) \tag{2.42a}$$

Therefore, the pressure coefficient is

$$C_p = (P - P_\infty) / \tfrac{1}{2} \rho_\infty U_\infty^2 = -\frac{2}{U_\infty^2} \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} \right)$$

$$= -\frac{2}{U_\infty^2} \frac{d\varphi}{dt} \tag{2.42b}$$

For steady flow, by using nondimensional Prandtl-Glauert coordinates given in Equation (2.6), Equation (2.42b) is reduced to

$$C_p = -2 \frac{\partial \phi}{\partial x} \tag{2.43}$$

For oscillatory flow, details on the calculation of C_p are given in Appendix C. For all the results presented here, the pressure coefficient is evaluated by finite different method.

2.7 GENERALIZED FORCES

The evaluation of the generalized forces, used for the flutter analysis, is presented in this section. The generalized forces are defined as

$$Q_h = -\iint p\,\bar{n}\cdot\bar{U}_h\,d\Sigma \tag{2.44}$$

where $p\bar{n}$ is the force acting on the surface of the body, and $\bar{U}_h$ is a prescribed shape-function (mode). Two rigid body modes are considered in this section: they correspond to the lift force and the pitch moment. For the lift force,

$$\bar{U}_h = \bar{k} \tag{2.45}$$

therefore,

$$\begin{aligned}\bar{L} &= -\iint p\,n_z\,d\Sigma \\ &= \iint (p_\ell - p_u)\,dx\,dy\end{aligned} \tag{2.46}$$

where n_z is the z component of the unit surface normal. The subscript "ℓ" and "n" refer to the lower and upper elements. For a thin wing, $\tilde{C}_L$ is evaluated as follows (see Equation C.31))

$$\begin{aligned}\tilde{C}_L &= -\frac{1}{A}\iint_{\Sigma_A}\Delta\tilde{C}_p\,dx\,dy \\ &= +2\,\frac{\ell}{A}\int_{-\frac{b}{2}}^{\frac{b}{2}}dy\int_{X_{LE}}^{X_{TE}}\frac{\partial}{\partial X}(\Delta\tilde{\phi})\,dX + \frac{2K_i}{A}\iint_{\Sigma A}(\Delta\tilde{\phi})\,dx\,dy \\ &= \frac{2\ell}{A}\int_{-\frac{b}{2}}^{\frac{b}{2}}(\Delta\tilde{\phi})_{X_{TE}}\,dy + \frac{2K_i}{A}\iint_{\Sigma A}(\Delta\tilde{\phi})\,dX\,dy \\ &= \frac{2\ell}{A}\sum_{j=1}^{NY}\left[(\Delta\tilde{\phi})_{X_{TE}}\right]_j(\Delta y)_j + \frac{2K_i}{A}\sum_{h=1}^{NX}\sum_{j=1}^{NY}(\Delta\phi)_{hj}A_{hj}\end{aligned} \tag{2.47}$$

where

$$\Delta C_{\quad} = C_{\quad}{}_{U} - C_{\quad}{}_{\ell} \tag{2.48}$$

and A is the total projected area of the wing on x-y plane, ℓ is the reference length, $X_{LE}(y)$ and $X_{TE}(y)$ are the x-component of vectors pointing to the leading edge and the trailing edge.

For the pitch moment, the mode is given by

$$\bar{U}_h = -(z - z_M)\,\bar{i} + (x - x_M)\,\bar{k} \tag{2.49}$$

and corresponds to a rigid body rotation (around the axis $X = X_M$, $Z = Z_M$) which is positive if the leading edge moves downward. Therefore,

$$\bar{M} = \iint -p\bar{n} \cdot \left[-(z - z_M)\,\bar{i} + (x - x_M)\,\bar{k}\right] d\Sigma$$

$$= -\iint p\left[-(z - z_M)n_x + (x - x_M)n_z\right] d\Sigma$$

$$\cong +\iint (P_\ell - P_u)(x - x_M)\,dx\,dy \tag{2.50}$$

Note that the term with $p(z - z_M)n_x$ is negligible because both $z - z_M$ and n_x are small. For a thin wing the pitch moment coefficient is evaluated as follows (see Equation C.31))

$$\tilde{C}_M = +\frac{1}{A\ell}\iint_{\Sigma_A} \Delta\tilde{C}_p\,(x - x_M)\,dx\,dy$$

$$= -\frac{2}{A\ell}\int_{-\frac{b}{2}}^{\frac{b}{2}} dy \int_{X_{LE}}^{X_{TE}} \left[\ell\frac{\partial}{\partial x}(\Delta\phi) + ik(\Delta\phi)\right](x - x_M)\,dx\,dy$$

$$= -\frac{2}{A\ell}\int_{-\frac{b}{2}}^{\frac{b}{2}} dy \left\{\ell\,(\Delta\tilde{\phi})_{X_{TE}}(X_{TE} - x_M) + \int_{X_{LE}}^{X_{TE}} \left[\ell - ik(x - x_M)\right](\Delta\tilde{\phi})\,dx\right\}$$

$$= +2 \frac{1}{A} \sum_{j=1}^{NY} (\Delta\tilde{\phi})_j (x_{TE} - x_M)_j (\Delta y)_j$$

$$+2 \frac{1}{A} \sum_{h=1}^{NX} \sum_{j=1}^{NY} (\Delta\phi)_{h,j} \left[1 - Ki \frac{(x_c - x_M)}{\ell}\right] A_{hj} \qquad (2.51)$$

SECTION 3

STEADY SUBSONIC FLOW

3.1 INTRODUCTION

In Subsection 3.2, the numerical formulation for steady subsonic flow, derived in Reference 6, is outlined. For completeness, the proof of this formulation is given in Appendix A. Numerical results are presented in Subsection 3.4.

3.2 NUMERICAL FORMULATION

Consider the Green theorem for steady subsonic flow. According to Equation (2.26)

$$4\pi E(\bar{P}_*)\phi(\bar{P}_*) = -\iint_{\Sigma_A} \frac{\partial \phi}{\partial N}\frac{1}{R}\,d\Sigma_A + \iint_{\Sigma_A} \phi \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_A + \iint_{\Sigma_W} \Delta\phi \frac{\partial}{\partial N_U}\left(\frac{1}{R}\right) d\Sigma_W \tag{3.1}$$

with boundary condition, given by Equation (2.13b) as

$$\frac{\partial \phi}{\partial N} = -N_X/\beta \tag{3.2}$$

By imposing that the value of the potential at $\bar{P}*$ approaches that at a point $\bar{P}$ on the surface Σ_A, if $\bar{P}*$ approaches $\bar{P}$, the value of $E(\bar{P}*)$ on the surface is found to be 1/2 (Reference 1), and an integral equation relating the potential on the surface Σ_A to its normal derivative is obtained.

Divide the surface Σ_A into N small quadrilateral elements, Σ_k, assume that the values of ϕ and $\frac{\partial \phi}{\partial N}$ are constant within each element, and impose that

Equation (3.1) is satisfied at the centroid, $\overline{P_h}$, of each element Σ_h. This yields N simultaneous linear equations, which can be expressed in the following matrix form:

$$[\delta_{hK} - C_{hK} - W_{hK}] \left\{ \phi_K \right\} = [b_{hK}] \left\{ \left(\frac{\partial \phi}{\partial N}\right)_K \right\} \tag{3.3}$$

where δ_{hK} is the Kronecker delta

$$C_{hK} = \left[\frac{1}{2\pi} \iint_{\Sigma K} \frac{\partial}{\partial N} \left(\frac{1}{R} \right) d\Sigma_K \right]_{\bar{P}_* = \bar{P}_h} \tag{3.4}$$

$$b_{hK} = \left[-\frac{1}{2\pi} \iint_{\Sigma K} \frac{1}{R} \, d\Sigma_K \right]_{\bar{P}_* = \bar{P}_h} \tag{3.4}$$

The coefficients W_{hK} represent the influence of the wake. These are obtained as follows[1]: the wake is assumed to be composed of straight vortex lines emanating from the trailing edge; the wake surface is divided into strips. The coefficient $W_{hK} = 0$, if the element Σ_K is not in contact with the trailing edge, while for the elements Σ_K in contact with the trailing edge

$$W_{hK} = W_{hK}^{(TE)} \tag{3.6}$$

with

$$W_{hK}^{(TE)} = \left[\pm \frac{1}{2\pi} \iint_{\Sigma K'} \frac{\partial}{\partial N_u} \left(\frac{1}{R} \right) d\Sigma_K \right]_{\bar{P}_* = \bar{P}_h} \tag{3.7}$$

where Σ'_K is the strip of the wake, bounded by two streamlines emanating from the element Σ_K. The upper (lower) sign holds for point $\bar{P}_h$ on the upper (lower) surface of the aircraft.

Since no pressure difference can exist on the wake, $\Delta\phi$ of the wake is constant along a streamline emanating from the trailing edge, and therefore, equal to $\Delta\phi$ at the trailing edge, or approximately, $\Delta\phi$ at the centroids of the elements adjacent to the trailing edge. Therefore, in deriving Equation (3.3), the values of $\Delta\phi$ on the wake are approximated with the values of $\Delta\phi$ at the centroids of the elements adjacent to the trailing edge.

3.3 ANALYTICAL EXPRESSIONS OF C_{hK}, b_{hK} and W_{hK}

The integrals in Equations (3.4), (3.5) and (3.7) are evaluated analytically by approximating the surface of each element with a hyperboloidal surface described in Subsection 2.5. The derivations are given in Reference 6 and are outlined here in Appendix A. The expressions for C_{hK} and b_{hK} are given by

$$C_{hK} = I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1) \tag{3.8}$$

$$b_{hK} = I_S(1,1) - I_S(1,-1) - I_S(-1,1) + I_S(-1,-1) \tag{3.9}$$

with

$$I_D(\xi,\eta) = \frac{1}{2\pi} \tan_p^{-1} \left(\frac{-\bar{q}\times\bar{a}_1 \cdot \bar{q}\times\bar{a}_2}{|\bar{q}|\ \bar{q}\cdot\bar{a}_1\times\bar{a}_2} \right) \tag{3.10}$$

(The subscript p reminds that the principal value of $\tan^{-1}$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ should be considered) and

$$I_s(\xi,\eta) = -\frac{1}{2\pi}\left\{ -\bar{q}\times\bar{a}_1\cdot\bar{n}\,\frac{1}{|\bar{a}_1|}\ln\left||\bar{q}||\bar{a}_1| + \bar{q}\cdot\bar{a}_1\right| \right.$$

$$+ \bar{q}\times\bar{a}_2\cdot\bar{n}\,\frac{1}{|\bar{a}_2|}\ln\left||\bar{q}||\bar{a}_2| + \bar{q}\cdot\bar{a}_2\right| \qquad (3.11)$$

$$\left. + \bar{q}\cdot\bar{n}\,\tan_p^{-1}\left(\frac{-\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2}{|\bar{q}|\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2}\right)\right\}$$

where

$$\bar{q}(\xi,\eta) = \bar{P}_0 + \xi\bar{P}_1 + \eta\bar{P}_2 + \xi\eta\bar{P}_3 - \bar{P}_h \qquad (3.12)$$

$$\bar{a}_1 = \bar{P}_1 + \eta\bar{P}_3 \qquad (3.13)$$

$$\bar{a}_2 = \bar{P}_2 + \xi\bar{P}_3 \qquad (3.14)$$

$$\bar{n} = \frac{\bar{a}_1\times\bar{a}_2}{|\bar{a}_1\times\bar{a}_2|} \qquad (3.15)$$

Furthermore, the expression for $W_{hK}^{(TE)}$ may be obtained by taking the limit of I_D in Equation (3.10) by letting $|\bar{a}_1|$ go to infinity. The derivation is given in Appendix A.4. The expression of $W_{hK}^{(TE)}$ is then given as

$$W_{hK}^{(TE)} = I_{w_1}(1) - I_{w_2}(1) - I_{w_1}(-1) + I_{w_2}(-1) \qquad (3.16)$$

with

$$I_{w1}(\eta) = \mp \frac{1}{2\pi} \tan^{-1}_{\rho} \frac{\bar{P}_{md} \cdot \bar{P}_m - (\bar{P}_{md} \cdot \bar{i}_x)(\bar{P}_d \cdot \bar{i}_x)}{\bar{P}_{md} \cdot \bar{i}_x \times \bar{P}_d} \tag{3.17}$$

$$I_{w2}(\eta) = \mp \frac{1}{2\pi} \tan^{-1}_{\rho} \frac{-(\bar{P}_{md} \cdot \bar{P}_{md})(\bar{i}_x \cdot \bar{P}_d) + (\bar{P}_{md} \cdot \bar{P}_d)(\bar{P}_{md} \cdot \bar{i}_x)}{|\bar{P}_{md}|\, \bar{P}_{md} \cdot \bar{i}_x \times \bar{P}_d} \tag{3.18}$$

$$(\eta = 1, -1)$$

where

$$\bar{P}_m = (\bar{P}_+ + \bar{P}_-)/2 - \bar{P}_h \tag{3.19}$$

$$\bar{P}_d = (\bar{P}_+ - \bar{P}_-)/2 \tag{3.20}$$

$$\bar{P}_{md} = \bar{P}_m + \eta \bar{P}_d \tag{3.21}$$

In the above equations, $\bar{P}+$ and $\bar{P}-$ are the two corner points of the element, Σ_h, which are on the trailing edge. (See Figure 2b). Equation (3.3) can be solved numerically to yield the values of ϕ_K. Once the ϕ_K are obtained, the pressure can be evaluated from the linearized Bernoulli Theorem, given by Equation (2.43).

3.4 NUMERICAL RESULTS

Numerical results obtained for steady subsonic flow are presented here. In Subsection 3.4.1, the results are compared with those obtained by the lifting surface theory. Comparison with experimental results are considered in Subsection 3.4.2. Analysis of convergence is considered in Subsection 3.4.3. The evaluation of the lift coefficient is given in Subsection 3.4.4. Finally, the pressure distribution coefficient of a wing-body configuration is presented in Subsection 3.4.5.

3.4.1 Comparison With Lifting Surface Theories

Comparison with various lifting surface theories are presented in Figures 4 - 7. Figure 4 shows the lift distribution per unit angle of attack, $C_{l\alpha}$, for a rectangular wing with AR = 1 and M = .2. The results, obtained with NX = NY = 7 are compared with the ones of Cunningham[11] and Kulakowski and Haskell.[12] Figure 5 shows the distribution of C_l for a tapered swept wing with aspect ratio AR = 3, taper ratio TR = .5, $\alpha = 5°$, sweep angle along quarter chord $\Lambda\ 1/4 = 45°$ and Mach number M = .8. The results obtained with NX = NY = 7 are compared with ones of Cunningham[11] and Kolbe and Boltz.[13] Figure 6 shows the distribution of the section lift coefficient per unit angle of attack, $C_{l\alpha}$, for a rectangular wing with aspect ratio, AR = 4 and Mach number M = .507. The result, obtained with NX = NY = 7 and 10, are compared with the ones by Yates.[14]

All these results are in good agreement with the existing one. Thickness ratio used in evaluating the above results is chosen to be 0.1%. The reasons to choose this value is that, as shown in Reference 3, the solution with $\tau = 0.1\%$ is a good approximation for the zero thickness solution and that values of thickness ratio smaller than 0.01% may cause elimination of significant figures.

3.4.2 Comparison With Experimental Results

Figues 7, 8a and 8b present a comparison with the experimental results of Lessing, Troutman and Menees.[15] The results are relative to a rectangular wing with aspect ratio AR = 3. The airfoil consists of a biconvex circular arc section with 5 percent thickness and with sharp leading and trailing edges. Figure 7 shows the pressure

distribution on the upper and the lower surfaces, C_{pu} and $C_{p\ell}$, versus x/c at 2y/b = 0., 0.5, 0.7 and 0.9 for the above mentioned wing with $\alpha = 0^\circ$ and M = 0.24. Figure 8a shows the lift distribution, C_ℓ, versus x/c at 2y/b = 0, 0.5, 0.7 and 0.9 for the above mentioned wing with $\alpha = 5^\circ$ and M = 0.24, while Figure 8b shows the pressure distribution on the upper and the lower surface, C_{pu} and $C_{p\ell}$, versus x/c at 2y/b = 0, 0.5, 0.7 and 0.9 for the above mentioned wing with $\alpha = 5^\circ$ and M = 0.24. It should be mentioned that the results for Figure 8a are obtained by taking the advantage of antisymmetry. If the velocity potential, ϕ, is separated into two parts, symmetric part, ϕ_s, and antisymmetric part, ϕ_{as}, then the difference of the velocity potential on the upper and lower surfaces of the wing is twice the value of ϕ_{as} while the velocity potential on the upper surface of the wing is $\phi_u = \phi_s + \phi_{as}$ and the velocity potential on the lower surface is $\phi_\ell = \phi_s - \phi_{as}$. The value of ϕ_{as} can be obtained by taking advantage of antisymmetry while the value of ϕ_s can be obtained by taking advantage of symmetry. This is mainly because the doublet, source and wake integrals are nearly independent of angle of attack, α. Therefore C_{hK}, b_{hK} and W_{hK} can be evaluated at $\alpha = 0$, while the only α-dependent term in Equation (3.3) is the linearized boundary term, $\frac{\partial \phi}{\partial n}$, which can be separated into symmetric and antisymmetric as follows: Set

$$\frac{\partial \phi}{\partial N} = -n_x = -\left[(n_x)_{\alpha=0}\cos\alpha \pm (n_z)_{\alpha=0}\sin\alpha\right] \simeq -(n_x)_{\alpha=0} \mp (n_z)_{\alpha=0}\alpha \tag{3.22}$$

and as mentioned before,

$$\phi = \phi_s \pm \phi_{as} \tag{3.23}$$

where $(n_x)_{\alpha=0}$ and $(n_z)_{\alpha=0}$ are the x-component and z-component of the surface unit normal. The upper (or lower) sign holds for the upper (or lower) surface. A similar example of the symmetric and antisymmetric problem is given in Section B.5. It may be worth noting that the results for Figure 8b are obtained by directly solving ϕ, or with C_{hK}, b_{hK} and W_{hK} evaluated at $\alpha = 5^{\circ}$. However, the lift coefficient obtained by subtracting C_{pl} from C_{pu} is consistent with results obtained for Figure 8a.

3.4.3 Analysis of Convergence

Figure 9 presents a study of convergence of the values of ϕ_K in terms of the number of elements. The wing is the same as that of Figure 7 with $\alpha = 0^{\circ}$ and M = 0.24. In the results shown in Figure 9, there is a sharp change of the values near the leading edge and the wing tip, therefore, in order to increase the rate of convergence, smaller elements along the leading edge and the wing tip are used. For this purpose, the coordinates of the nodes (x, y) are defined by

$$X_i = \left(\frac{1}{NX}\right)^2 \qquad i = 0, 1, 2 \text{ --- } NX$$

$$Y_j = \left[1 - \left(\frac{1}{NY}\right)^2\right]^2 \qquad j = 0, 1, 2 \text{ --- } NY \tag{3.24}$$

Result shown is the distribution of ϕ along y = 0. It may be noted that 100 elements on the whole wing (i.e., NX = NY = 5) are sufficient for convergence.

3.4.4 Total Lift Coefficient

The total lift coefficient for steady flow is evaluated from Equation (2.47) without the imaginary terms, i.e.,

$$C_L = 2 \frac{\ell}{A} \sum_{j=1}^{NY} \left[(\phi_u - \phi_\ell)_{X_{TE}} \right]_j (\Delta y)_j \qquad (3.25)$$

where ϕ_ℓ and ϕ_u are the extrapolated values of the velocity potential along the lower and upper trailing edges. Figure 10 shows the values of C_L of a triangular wing with different aspect ratios in incompressible flow. The result obtained at $\tau = 0.1\%$ are compared with the analytical ones.[16]

3.4.5 Wing-Body Configurations

In Figure 11, results for the pressure distribution coefficient of a wing-body configuration in steady subsonic flow are compared, with the results presented by Labrujere, Loeve and Slooff.[17] The results were obtained for M = 0 and a rectangular midpositioned wing with chord c = 1, span b = 6, thickness ratio $\tau = 0.9$, and angle of attack $\alpha_\omega = 6^\circ$. The body is at zero angle of attack and is composed of a forebody with length $L_A = 2$ and radius

$$r = 0.5 - 0.125(x - x_{LE}) \qquad (3.26)$$

a midsection of length $L_M = 1$ and radius r = 0.5 and an aftbody of length $L_A = 9$, and radius r = 0.5. It is worth note that the extra long aftbody is used to simulate the wake emanating from the circular fuselage. The number of elements is 200 on the whole configuration (NX = 5, NY = 5 on the wing, NX = 2, NY = 3 on the forebody, NX = 5, NY = 3 on the midsection, and NX = NY = 3 on the aftbody). Figure 11b presents the distribution of $\phi_u - \phi_\ell$ along three circumferential stations (see Figure 11b) for the same wing-body configuration.

SECTION 4

OSCILLATORY SUBSONIC FLOW

4.1 INTRODUCTION

In subsection 4.2, the numerical formulations for oscillatory subsonic flow are considered. The boundary condition and the pressure coefficient are considered in Appendix C. In subsection 4.3, numerical results obtained are presented.

4.2 NUMERICAL FORMULATION

Consider the Green Theorem for oscillatory subsonic flow, given by Eq. (2.30)

$$4\pi E(\bar{P}_*)\hat{\phi}(\bar{P}_*) = -\iint_{\Sigma_A} \frac{\partial \hat{\phi}}{\partial N} \frac{e^{-i\Omega R}}{R} d\Sigma_A + \iint_{\Sigma_A} \hat{\phi} \frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma_A$$

$$+ \iint_{\Sigma_W} \Delta\hat{\phi} \frac{\partial}{\partial N_u}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma_W \qquad (4.1)$$

with boundary condition given by Equation (C.24)

$$\frac{\partial \hat{\phi}}{\partial N} = N_z \left(ik\tilde{\bar{z}} + \frac{1}{\beta}\frac{\partial \tilde{\bar{z}}}{\partial X}\right) e^{-i\Omega MX} \qquad (4.2)$$

where $\tilde{\bar{z}}(X,Y)$ is the vibration mode defined by Equation (C.18)

By imposing that the value of the velocity potential at $\bar{P}_*$ approaches that at a point, $\bar{P}$, on the surface Σ_A, if $\bar{P}_*$ approaches $\bar{P}$, the value of $E(\bar{P}_*)$ is found to be 1/2 (Reference 1), and an integral equation relating the potential on the surface Σ_A to its normal derivative is obtained. Noting that no pressure difference can exist on the wake, and using Equation (C.32)

$$\tilde{C}_p = -\frac{2}{\beta} e^{-ik\beta X} \frac{\partial}{\partial X}\left(\hat{\phi}\, e^{ikX/\beta}\right) \qquad (4.3)$$

one can conclude that $(\Delta\hat{\phi})e^{ikx/\beta}$ is constant along a streamline emanating from the trailing edge and equal to the value at the trailing edge

$$(\Delta\hat{\phi})\,e^{ikx/\beta} = (\Delta\hat{\phi}_{TE})\,e^{ikx_{TE}/\beta} \simeq (\Delta\hat{\phi}_K)\,e^{ikx_K/\beta} \tag{4.4}$$

In Equation 4.4, the value of $\hat{\phi}_{TE}\,e^{ikx_{TE}/\beta}$ is approximated with this value $\Delta\phi\,e^{ikx_K/\beta}$ at the centroid of the element, Σ_K, adjacent to the trailing edge. In Equation 4.4, X_K represents the X component of the position vector of the centroid of elements, Σ_K, adjacent to the trailing edge where the wake starts to develop. Using Equation (4.4) and applying the same procedures used for the steady flow, one obtains

$$[\delta_{hK} - \hat{C}_{hK} - \hat{W}_{hK}]\left\{\hat{\phi}_K\right\} = [\hat{b}_{hK}]\left\{\left(\frac{\partial\hat{\phi}}{\partial N}\right)_K\right\} \tag{4.5}$$

where

$$\hat{C}_{hK} = \left[\frac{1}{2\pi}\iint_{\Sigma K}\frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right)d\Sigma_K\right]_{\bar{P}_* = \bar{P}_h} \tag{4.6}$$

$$\hat{b}_{hK} = \left[\frac{-1}{2\pi}\iint_{\Sigma K}\frac{e^{-i\Omega R}}{R}\,d\Sigma_K\right]_{\bar{P}_* = \bar{P}_h} \tag{4.7}$$

and for elements not adjacent to the trailing edge, $\hat{W}_{hK}=0$, otherwise,

$$\hat{W}_{hK} = \left[\pm\frac{1}{2\pi}\iint_{\Sigma_K'} e^{-ik(x-x_K)/\beta}\,\frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right)d\Sigma_K\right]_{\bar{P}_* = \bar{P}_h} \tag{4.8}$$

where X_K is defined previously. Σ_K' is the strip of the wake, bounded by two streamlines emanating from the element Σ_K. The upper (or lower) sign holds for point, $\bar{P}_h$, on the upper (or lower) surface of the aircraft. Using the following equation

$$\frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right) = -\frac{1}{R^2}(1 + i\Omega R)\frac{\partial R}{\partial N} = (1 + i\Omega R)\,e^{-i\Omega R}\frac{\partial}{\partial N}\left(\frac{1}{R}\right) \tag{4.9}$$

the approximate evaluations of $\hat{C}_{hK}$ and $\hat{b}_{hk}$ are obtained by replacing the exponential term with its value at the centroid of the element. This yields

$$\hat{C}_{hK} = e^{-i\Omega R_c}(1 + i\Omega R_c) C_{hK} \tag{4.10}$$

$$\hat{b}_{hK} = e^{-i\Omega R_c} b_{hK} \tag{4.11}$$

where C_{hK} and b_{hK} are given by Equations (3.8) and (3.9) whereas R_c is the distance between the centroids of elements Σ_h and Σ_K. For the wake contribution one obtains $\hat{W}_{hK=0}$, if the element Σ_K is not in contact with the trailing edge, while, for the element Σ_K in contact with the trailing edge, dividing the strip Σ'_K into small elements, one obtains

$$\hat{W}_{hK} = \hat{W}^{(TE)}_{hK} = \sum_{j=1}^{NW} e^{-ik(X_j - X_K + MR_c)/\beta}(1 + i\Omega R_c) C_{hj} \tag{4.12}$$

where, j refers to the jth wake element emanating from the element Σ_K, C_{hj} is given by Equation (3.8) R_c is the distance between the centroids of element, Σ_h, and the ith wake element; NW is the number of elements of the truncated wake emanating from Σ_K. (A more refined analysis is obtained by taking the limit of the present one when NW tends to infinity).

4.3 NUMERICAL RESULT

Results for oscillatory subsonic flow are presented here. In Subsection 4.3.1 the results are compared with the experimental ones. In Subsection 4.3.2 the analysis of convergence is considered. In oscillatory flow, instead of using semi-infinite wake, trancated wake is used. Therefore a study of the wake length LW as well as the

analysis of the convergence due to the number of wake elements are considered in Subsection 4.3.3.

4.3.1. Comparison with Experimental Results

Figure 12 presents a comparison with the experimental results of Lessing, Troutman and Menees[15]. The results are relating to a rectangular wing with aspect ratio AR = 3. The airfoil consists of a biconvex circular-arc section, 5% thickness with sharp leading and trailing edges. The wing considered has the same geometry as the wing considered for Figures 7 and 8 and is oscillating in bending mode described by

$$\bar{z} = .18043 \left|\frac{2y}{b}\right| + 1.70255 \left|\frac{2y}{b}\right|^2 - 1.13688 \left|\frac{2y}{b}\right|^3 + .25389 \left|\frac{2y}{b}\right|^4 \qquad (4.13)$$

with

$$K = \frac{\omega c}{2 U_\infty} = 0.47 \quad \text{and} \quad M = 0.24$$

The values showed are the real and imaginary part of the distribution of $\tilde{C}_L$ along $2y/b = 0$ and $2y/b = 0.5$. They are obtained with NX=NY=7, NW=20, and LW=2 C. The node coordinates are given by Equation (3.22).

4.3.2 Analysis of Convergence

A convergence study of the problem considered in above subsection are presented in Figures 13a and 13b. The values showed are the real and imaginary parts of the distribution of $\tilde{C}_L$ along $2y/b$ =0.5 and 2y/b=.1328

respectively. They are obtained with different numbers of wing elements, NX=NY= 5, 6, 7, while the number of wake element emenating from the trailing edge and the wake length, LW, are fixed such that NW=30, LW =3.5C. Using 64 elements on the whole wing (i.e. NX=NY=4) is sufficient for convergence.

4.3.3 Analysis of Wake Effect

The wake effect of the oscillatory subsonic flow is evaluated from Equation (4.12). The values of $\hat{W}_{hk}$ depend on the number of wake elements and the wake length. Note that, C_{hi} in Equation (4.12) is a doublet integral which represents the solid angle. The value of the solid angle decreases as the distance between the control point and wake element increases. Therefore, the wake effect is dominated by the wake elements closest to the trailing edge. For this reason, it is necessary to determine the effective wake length. Also after the effective wake length is determined, it is necessary to study the convergence due to the number of wake elements. Figure 14 shows the distribution of $\tilde{C}_L$, along $2Y/b$ =0.13265, of the same problem considered for Figure 12a. The results are obtained with different number of NW=5, 10, 20, 30, 40, while using 196 elements on the whole wing, i.e., NX=NY=7, and using four times chord length for the wake length, or LW =4.0C. The length of the wake elements varies from 0.8C to 0.1C from the figure shown. At least NW=20 should be used for convergence. Figure 15 shows the distribution of $\tilde{C}_L$, along $2Y/b$ =0.13265, of the same problem considered for Figure 12a. The results are obtained with different lengths

of wake LW = 0.5C, C, 2C, 3C and 4C. The length of the wake element is the same for each run, or $(\Delta X)_W$ = LW/NW = 0.1C. The numbers of NW for each run are 5, 10, 20 and 40 respectively. From the figure shown, it can be concluded that at least, LW=2C, should be used as effective truncated wake length.

4.3.4 Results of Generalized Forces

Figures 16a and 16b shows the absolute values and the phase angles of $\tilde{C}_L$ and $\tilde{C}_M$ for a delta wing in incompressible oscillatory flow, as functions of the reduced frequency, $K = \omega c / 2 U_\infty$. The wing with AR=4 and τ=.5% is oscillating in pitch about the axis $X=X_M$. The vibration mode is described by $\frac{(X-X_m)}{\ell}$ with $X_M = 0.5\,(X_{LE} + X_{TE})$. The results are compared with the experimental, as well as, the numerical results by Laschka [18].

Figures 17a and 17b shows the absolute values and the phase angles of $\tilde{C}_L$ and $\tilde{C}_M$ as function of the Mach number which varies from 0.0 to 2.5. (Section 5 presents the formulation for the supersonic range) for a rectangular wing with AR=2 and τ=0.001. The vibration mode is the same as that of the above problem, and the reduced frequency is set at K=1. The results are compared with the ones obtained by Laschka [18]. Figures 18a and 18b show the thickness effect on the problem of Figure 17. The results are obtained with two thickness ratios τ=0.001 and τ=0.05. The Mach number varies from 0.0 to 2.5.

Figures 19a and 19b show the absolute values and the phase angles of $\tilde{C}_L$ and $\tilde{C}_M$ as functions of the reduced frequencies for a rectangular wing

with AR=2 and τ=0.001. The wing in incompressible flow is oscillating in plunge. The vibration mode is described by $e^{i\omega t}$. The results are compared with the ones by Lawrance and Gerber[19] as well as Laschka[18].

Figures 20a and 20b shows the absolute values and the phase angles of $\tilde{C}_L$ and $\tilde{C}_M$ of the same problem as that given for Figures 19a and 19b, while the reduced frequency is set at K=1.0 and the Mach number varys from 0.0 to 2.5. Results are compared with the ones of Laschka[18].

SECTION 5

STEADY SUPERSONIC FLOW

5.1 INTRODUCTION

In Subsection 5.2 the numerical formulations for steady supersonic flow, derived in Reference.7, is outlined. For completeness, the derivation of these formulations and more detail formulations are summarized in Appendix B. In Subsection 5.3 wings with supersonic leading edges and diaphragms for wings with subsonic leading edges are discussed. In Subsection 5.4 numerical results are presented.

5.2 NUMERICAL FORMULATIONS

Consider the Green theorem for steady supersonic flow. According to Equation (2.33)

$$2\pi E(\bar{P}_*)\phi(\bar{P}_*) = -\iint_{\Sigma_A} \frac{\partial \phi}{\partial N^c} \frac{H}{\|\bar{R}\|} d\Sigma_A + \iint_{\Sigma_A} \phi \frac{\partial}{\partial N^c}\left(\frac{H}{\|\bar{R}\|}\right) d\Sigma_A \tag{5.1}$$

with the boundary condition given by Equation (2.14a)

$$\frac{\partial \phi}{\partial N^c} = -N_x\left(\frac{1}{\beta} + \frac{M^2}{M^2-1}\frac{\partial \phi}{\partial X}\right) \tag{5.2}$$

Applying the same procedure for subsonic flow, one obtains the following equations, expressed in matrix form,

$$[\delta_{hk} - C_{hk} - W_{hk}]\{\phi_k\} = [b_{hk}]\left\{\left(\frac{\partial \phi}{\partial N^c}\right)_k\right\} \tag{5.3}$$

where

$$C_{hK} = \left[\frac{1}{\pi}\iint_{\Sigma_K} \frac{\partial}{\partial N^c}\left(\frac{H}{\|\bar{q}\|}\right) d\Sigma\right]_{\bar{P}_* - \bar{P}_h} \tag{5.4}$$

and

$$b_{hK} = \left[-\frac{1}{\pi}\iint_{\Sigma_K} \frac{H}{\|\bar{q}\|} d\Sigma\right]_{\bar{P}_* = \bar{P}_h} \tag{5.5}$$

For hyperboloidal elements completely inside the Mach forecone, C_{hK} and b_{hK} are still given by

$$C_{hK} = I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1) \tag{5.6}$$

$$b_{bK} = I_S(1,1) - I_S(1,-1) - I_S(-1,1) + I_S(-1,-1) \tag{5.7}$$

For elements completely inside the Mach forecone I_D and I_S are given by

$$I_D(\xi,\eta) = \frac{1}{\pi}\tan_p^{-1}\left(\frac{-\bar{q}\times\bar{a}_1 \odot \bar{q}\times\bar{a}_2}{\|\bar{q}\|\ \bar{q}\cdot\bar{a}_1\times\bar{a}_2}\right) \tag{5.8}$$

and

$$I_S(\xi,\eta) = \frac{1}{\pi}\frac{1}{\bar{n}\odot\bar{n}}\left[I_{S_1}(\xi,\eta) + I_{S_2}(\xi,\eta) + I_{S_3}(\xi,\eta)\right] \tag{5.9}$$

with

$$I_{s_1}(\xi,\eta) = -(\bar{q}\times\bar{a}_1\circ\bar{n})\,F_1(\xi,\eta) \tag{5.10}$$

$$I_{s_2}(\xi,\eta) = (\bar{q}\times\bar{a}_2\circ\bar{n})\,F_2(\xi,\eta) \tag{5.11}$$

$$I_{s_3}(\xi,\eta) = \bar{q}\cdot\bar{n}\,\tan_p^{-1}\left(\frac{-\bar{q}\times\bar{a}_1\circ\bar{q}\times\bar{a}_2}{\|\bar{q}\|\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2}\right) \tag{5.12}$$

where $\bar{q}$, $\bar{a}_1$, $\bar{a}_2$, and $\bar{n}$ are given by Equations (3.12) through (3.15) while

$$F_j(\xi,\eta) = \frac{1}{\|\bar{a}_j\|}\,\ell_n\left|\frac{\|\bar{q}\|\,\|\bar{a}_j\|+\bar{q}\circ\bar{a}_j}{\|\bar{q}\times\bar{a}_j\|}\right| \qquad \bar{a}_j\circ\bar{a}_j>0$$

$$= \frac{\|\bar{q}\|}{\bar{q}\circ\bar{a}_j} \qquad \bar{a}_j\circ\bar{a}_j=0 \tag{5.13}$$

$$= -\frac{1}{\|\bar{a}_j\|}\,\sin^{-1}\left(\frac{\bar{q}\circ\bar{a}_j}{\|\bar{q}\times\bar{a}_j\|}\right) \qquad \bar{a}_j\circ\bar{a}_j<0$$

For elements partially inside the Mach forecone I_D and I_{S1}, I_{S2}, I_{S3} are still given by Equations (5.8), (5.10), (5.11) and (5.12) if the corner point is inside the Mach forecone. However, if the corner point is outside the Mach forecone, integrals in Equations (5.4) and (5.5) involve singularity. Details of these kinds of integrals are discussed in Subsection B.4. For elements completely outside the Mach forecone,

$$H \equiv 0 \tag{5.14a}$$

Therefore

$$C_{hK} = b_{hK} = 0 \tag{5.14b}$$

Note that the wake effect has not been considered for the results presented here, because the geometry of the lifting body under consideration includes only wing, nose, or middle-section fuselage. Therefore, for each element, the wake is always outside the Mach forecone, i.e., $W_{hK} = 0$.

In addition, by neglecting the higher order terms, the boundary condition is given by (see Equation 2.14b)

$$\frac{\partial \phi}{\partial N} = - N_x / \beta \tag{5.15}$$

5.3 DIAPHRAGMS

There are three categories of wing geometry in supersonic flow: 1) supersonic leading edges (the elements Σ_h on the upper or the lower sides are influenced only by the elements on the same sides, therefore, the integration in Equations (5.4) and (5.5) are around the upper or lower surface only for the upper or lower elements); 2) mixed supersonic-subsonic leading edge (for this kind of wing, diaphragms have to be used to separate upper and lower sides of the aircraft; then, the problem can be handled as the case of supersonic leading edge); 3) subsonic leading edge (for this kind of wing, results obtained with or without diaphragms are the same [see Subsection 5.4.5]; however, for small thickness ratio, diaphragms are still needed to avoid the elimination of significant figures).

If diaphragms are used, both values of the velocity potential and its normal derivative, ϕ_D and ψ_D, of the diaphragm element are unknown. However, two

equations are obtained for each element of the diaphragm: one relates ϕ_p and ψ_p of the diaphragm element, Σ_p, to the upper geometry of the aircraft and the upper side diaphragm, the other one relates the same quantities to the lower geometry. Therefore, the total number of equations is equal to the total number of unknowns. Details of the numerical procedures are given in Subsection B.5. As mentioned before, for wing with subsonic leading edge, procedure with or without diaphragm can be used. If the procedure without diaphragm is employed, then the integral equations are solved the same way as for the subsonic case (solutions are obtained by solving Equation (5.3)). However, if the procedure with diaphragm is employed, solutions are obtained by solving Equations (B.41) or (B.46).

5.4 NUMERICAL RESULTS

Results for steady supersonic flow are presented here. In Subsection 5.4.1, results are compared with the experimental ones. In Subsection 5.4.2, analysis of convergence is studied. Problems of airfoils in conical flow are studied in Subsection 5.4.3. The role of diaphragms has already been discussed in Subsection 5.3. In order to show that diaphragms are not necessary for geometries with subsonic leading edge, two results obtained with and without diaphragms for a circular cone in conical flow are compared in Subsection 5.4.4. Finally, in order to study body interference and also to show the generality of the present method, result of wing-body configuration is considered in Subsection 5.4.5.

5.4.1 Comparison With Experimental Results

Figure 21 shows the distribution of the pressure coefficient C_p on the lower and upper surfaces for a rectangular wing with aspect ratio AR = 3. The airfoil consists of a biconvex circular-arc section, 5% thick, with sharp leading and trailing edges. The results are obtained for $\alpha = 0^\circ$, M = 1.3 and NX = NY = 7. Figure 22a shows the

distribution of the lift coefficient for the same wing considered in Figure 21. The results are obtained for $\alpha = 5^{\circ}$, M = 1.3 and NX = NY = 7. These results are compared with the ones obtained by Lessing, et al.[15] In Figure 22b, the pressure distributions on the upper and the lower sides of the wing are shown. It should be mentioned that results in Figure 22a are obtained by taking the advantage of anti-symmetry as in the case described in Subsection 3.4.2.

5.4.2 Analysis of Convergence

Convergence study of the problem considered in Figure 21 is presented in Figure 23 which shows the distribution of the velocity potential along 2y/b = 0 for different numbers of elements. The curves are obtained with NX = NY = 5, 6 and 7. It is shown that 144 elements on the whole wing, i.e., NX = NY = 6, are sufficient for convergence.

5.4.3 Delta Wings

Figure 24 shows the distribution of life coefficient per unit angle of attack for a delta wing with supersonic leading edge and

$$M = \beta / \tan \Lambda = 1.2$$

where Λ is the sweep angle of the leading edge. The results obtained with NX = 8, NY = 12 and M = 1.2 are compared with the exact solution which is given by (Reference 20)

$$\Delta C_p = \frac{4\alpha}{\pi\beta} \frac{m}{\sqrt{m^2-1}} Re\left[\cos^{-1}\frac{1-m\gamma}{m-\gamma} + \cos^{-1}\frac{1+m\gamma}{m+\gamma}\right] \qquad (5.16)$$

where

$$\gamma = \frac{\beta y}{x}$$

Figure 25 shows the distribution of the lift coefficient per unit angle of attack for a delta wing with subsonic leading edges and m = 0.8333. The results obtained with NX = NY = 7 and M = $\sqrt{2}$ are compared with the exact solution, which is given by (Reference 20)

$$\Delta C_p = \frac{4m^2\alpha}{\beta E(K)} \frac{1}{\sqrt{m^2-\gamma^2}} \tag{5.17}$$

where $K = \sqrt{1-m^2}$ and E is the elliptic integral of the second kind. It may be worth noting that there are two types of elements — grids which can be used for delta wings (see Figure 2c). For supersonic delta wings, the flow is conical, therefore, the element-grid shown in Figure 2c (b) appears to be the more natural one. In addition, the use of this grid implies the use of a completely general quadrilateral element. Let $\bar{\xi}$ and $\bar{\eta}$ be the generalized coordinates such that $\bar{\eta}$ has same direction as y while the direction of $\bar{\xi}$ passes the origin and pointed downward. The derivative of ϕ with respect to X is obtained as follows: consider the directional derivatives $\frac{d\phi}{d\xi}$ and $\frac{d\phi}{d\eta}$

$$\frac{d\phi}{d\xi} = \left(\frac{\partial\phi}{\partial X}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j}\right)\cdot\frac{\bar{\xi}}{|\bar{\xi}|} = \frac{\partial\phi}{\partial X}\frac{X}{\sqrt{X^2+Y^2}} + \frac{\partial\phi}{\partial Y}\frac{Y}{\sqrt{X^2+Y^2}} \tag{5.18}$$

$$\frac{d\phi}{d\eta} = \left(\frac{\partial\phi}{\partial X}\bar{i} + \frac{\partial\phi}{\partial y}j\right)\cdot\frac{\bar{\eta}}{|\eta|} = \frac{\partial\phi}{\partial Y} \tag{5.19}$$

where

$$\bar{\xi} = X\,\bar{i} + Y\,\bar{j} \tag{5.20a}$$

$$\bar{\eta} = Y\,\bar{j} \tag{5.20b}$$

Therefore

$$\frac{\partial \phi}{\partial X} = \left(\frac{d\phi}{d\xi}\right) \frac{\sqrt{X^2+y^2}}{X} - \left(\frac{d\phi}{d\eta}\right) \frac{y}{X} \qquad (5.21)$$

The values of $\left(\frac{d\phi}{d\xi}\right)$ and $\left(\frac{d\phi}{d\eta}\right)$ can be evaluated by the finite difference method. By using Equation (5.21), the values of C_p are obtained. The results obtained above by applying this scheme are very accurate. The results obtained for Figure 24 are especially accurate (up to the fifth significant figure).

5.4.4 Circular Cone (Analysis of Diaphragm Existence)

For most cases of supersonic flow considered here, diaphragms seem to be necessary to avoid determinant singularity. However, in this Subsection, an example is given to show that diaphragms are unnecessary, for example, for bodies with subsonic leading edge. This example is related to a circular cone with base radius r = 1, length l = 5, and M = 2.0. One is obtained with diaphragm by solving Equation (B.46), while the other one is obtained without diaphragm by solving Equation (5.3). The results are in good agreement. The pressure coefficient of the body is evaluated from the slope of ϕ-X curve. This value is evaluated as 0.112 which is consistent with the exact solution given in Reference 21 (Figure 17, p. 187, $C_p \cong 0.12$ for M = 2, $\theta_c = \tan^{-1}\left(\frac{1}{5}\right) \simeq 11^\circ$).

5.4.5 Wing-Body Configuration

The present method is sufficiently general to be applie. for any complex configuration. Presented here is an example of this application. A wing-body combination in supersonic flow with M = 1.48 is considered in Figures 27a, 27b, and 27c. The geometry of the configuration is shown in Figure 27c.

The combination is composed of a wing with chord $c = 3$, span $S = 9$, thickness $\tau = 5\%$, a forebody of length $L_A = 6.0$, and radius varying from 0 to 0.75 linearly, and

a midsection of length $L_M = 3.0$ and radius $r = 0.75$. Wake and aftbody are not considered. The angle of attack of the wing is $\alpha_w = 1.92$, while the angle of attack of the body is zero, $\alpha_\beta = 0$. The results are obtained with 580 elements on the whole configuration (NX = NY = 10 on the wing, NX = 5, NY = 3 on the nose, NX = 10, NY = 3 on the middle section). In Figure 27a the distribution of $\beta C_\ell/\alpha_w$ on the wing section is presented, and the results are compared with the ones by Nielsen[22] and Woodward, Tinoco and Larsen,[23] while in Figure 27b the distribution of $\beta C_\ell/\alpha_w$ on the fuselage is shown.

SECTION 6

OSCILLATORY SUPERSONIC FLOW

6.1 INTRODUCTION

In Subsection 6.2, the numerical formulation for oscillatory supersonic flow is considered. The boundary conditions and the pressure coefficient are considered in Appendix C. The role of the diaphragm in oscillatory supersonic flow is the same as in steady supersonic flow. It has been already discussed in Subsection 5.3, therefore it is unnecessary to repeat it here. In Subsection 6.3, numerical results obtained are presented.

6.2 NUMERICAL FORMULATIONS

Consider the Green Theorem for oscillatory supersonic flow. According to Equation (2.38)

$$2\pi E(\bar{P}_*)\phi(\bar{P}_*) = \iint_{\Sigma} \frac{\partial \hat{\phi}}{\partial N^c} \frac{H}{\|\bar{R}\|} \cos(\Omega \|\bar{R}\|)\, d\Sigma + \iint \hat{\phi} \frac{\partial}{\partial N^c}\left(\frac{H}{\|\bar{R}\|} \cos \Omega \|\bar{R}\|\right) d\Sigma \tag{6.1}$$

with the boundary condition, given by Equation (C.28)

$$\frac{\partial \hat{\phi}}{\partial N^c} = N_z \left(iK\tilde{Z} + \frac{1}{\beta} \frac{\partial \tilde{Z}}{\partial X}\right) e^{i\Omega MX} \tag{6.2}$$

where $\tilde{Z}(X,Y)$ is the vibration mode defined by Equation (C.18). Applying the same procedure used for subsonic flow, one obtains the following equations, expressed in matrix form

$$[\delta_{hk} - \hat{C}_{hk} - \hat{W}_{hk}]\{\hat{\phi}_k\} = [\hat{b}_{hk}]\left\{\left(\frac{\partial \hat{\phi}}{\partial N^c}\right)\right\} \tag{6.3}$$

where

$$\hat{C}_{hk} = \left[\frac{1}{\pi}\iint_{\Sigma_k} \frac{\partial}{\partial N^c}\left(\frac{H}{\|\bar{R}\|}\cos(\Omega\|\bar{R}\|)\right) d\Sigma_k\right]_{\bar{P}_* = \bar{P}_h} \tag{6.4}$$

and

$$\hat{b}_{hk} = \left[-\frac{1}{\pi}\iint_{\Sigma_k} \frac{H}{\|\bar{R}\|}\cos(\Omega\|\bar{R}\|)\, d\Sigma_k\right]_{\bar{P}_* = \bar{P}_h} \tag{6.5}$$

The wake effect for supersonic flow is not considered here since only supersonic-trailing-edge wings are presented here. Therefore $\hat{W}_{hk} = 0$.

Noting that

$$\frac{\partial}{\partial N^c}\left[\frac{1}{\|\bar{R}\|}\cos(\Omega\|\bar{R}\|)\right]$$

$$= -\frac{1}{\|\bar{R}\|^2}\left[\cos(\Omega\|\bar{R}\|) + \Omega\|\bar{R}\|\sin(\Omega\|\bar{R}\|)\right]\frac{\partial\|\bar{R}\|}{\partial N^c} \tag{6.6}$$

$$= \left[\cos(\Omega\|\bar{R}\|) + \Omega\|\bar{R}\|\sin(\Omega\|\bar{R}\|)\right]\frac{\partial}{\partial N^c}\left(\frac{1}{\|\bar{R}\|}\right)$$

the approximating evaluations of $\hat{C}_{hk}$ and $\hat{b}_{hk}$ are obtained by replacing the sine and cosine terms with their values at the centroid of the element. This yields

$$\hat{C}_{hk} = \left[\cos(\Omega\|\bar{R}_c\|) + \Omega\|\bar{R}_c\|\sin(\Omega\|\bar{R}_c\|)\right]C_{hk} \tag{6.7}$$

and

$$\hat{b}_{hK} = \cos(\omega \|\bar{R}_{cl}\|)\, b_{hK} \tag{6.8}$$

where C_{hK} and b_{hK} are given by Equations (5.6) and (5.7) with I_D, I_S given by Equations (5.8) through (5.12) for elements completely inside the Mach forecone. For elements partially inside the Mach forecone, I_D and I_S are still given by Equations (5.8) and (5.9) if the corner point is inside the Mach forecone, otherwise, they are given by Equations (B.26) through (B.28) for I_{S1}, Equations (B.30) through (B.32) for I_{S2}, and Equations (B.35) through (B.40) for I_{S3} and I_D. For element completely outside the Mach forecone $\hat{C}_{hK} = \hat{b}_{hK} = 0$.

6.3 NUMERICAL RESULTS

Results for oscillatory supersonic flow are presented here.

In Subsection 6.3.1 a wing oscillating in bending mode is considered. In Subsection 6.3.2, the convergence problem is considered. In Subsection 6.3.3, the evaluation of generalized forces is considered.

6.3.1 Comparison With Experimental Results

Figure 28 shows the distribution of absolute value and phase angle of $\tilde{C}_\ell$ for a rectangular wing oscillating in bending mode

$$\bar{z} = .18043\left|\frac{2y}{b}\right| + 1.70255\left|\frac{2y}{b}\right|^2 - 1.13688\left|\frac{2y}{b}\right|^3 + .25387\left|\frac{2y}{b}\right|^4 \tag{6.9}$$

with $k = \omega c/2 U_\infty = 0.1$

The airfoil, with aspect ratio AR = 3, consists of a biconvex circular-arc section, 5% thick with sharp leading and trailing edges. Results obtained for M = 1.3

and NX = NY = 7 are compared with the ones by Lessing, et al.[15] It should be mentioned that the sign of phase angle of $\tilde{C}_\ell$ obtained by the present method is opposite to that of Ref. 15. The author believes that this is due to the different choice of the sign convention.

6.3.2 Analysis of Convergence

A convergence study of the above problem is presented here. In Figure 29, the distributions of the imaginary and real parts of the velocity potential along 2y/b = 0.5 for the problem given for Figure 28 are shown. The curves are obtained with NX = NY = 5, 6, 7 and 8. It is shown that 100 elements (i.e., NX = NY = 5) are sufficient for convergence.

6.3.3 Results of Generalized Forces

The typical results for the evaluation of generalized forces are presented in Figures 17a, 17b, 18a, 18b, 20a and 20b. In Figures 17a and 17b, a rectangular wing with oscillation in pitch is considered. In Figures 18a and 18b the thickness effect of the problem given for Figures 17a and 17b is considered. In Figure 20a and 20b a rectangular wing with oscillation in plunge is considered. These results are obtained for Mach number varying from 0.0 to 2.0 and $K = \frac{2\omega c}{U_\infty} = 1.0$ and compared with the ones obtained by Laschka.[18] Details of the problems have already been given in Subsection 4.3.4, therefore, they are not repeated here.

SECTION 7

FLUTTER ANALYSIS

7.1 INTRODUCTION

The previous sections present an efficient aerodynamic tool to analyze flutter phenomenon. In this section, a preliminary flutter analysis is presented. In Subsection 7.2, the mathematical model is presented. In Subsection 7.3, the Lagrange's equation forces are discussed. In Subsection 7.5, the evaluation of aerodynamic forces is outlined. In Subsection 7.6, the flutter analysis is described. In Subsection 7.7, numerical results are presented.

7.2 MATHEMATICAL MODEL

As usual, the mathematical model is established by describing the aerodynamic loads due to simple harmonic motion and then studying the stability of infinitesimal vibration of this kind of motion. The model is based on the fact that the motion at the flutter boundary is simple harmonic and that the flutter boundary is by definition the one relative to an infinitesimal oscillation around the next configuration.

The most typical aircraft flutter results from a coupling between the bending and torsion modes of a wing. A typical section (airfoil) analysis is considered for simplicity. This kind of flutter is considered in this section. The airfoil under consideration is permitted freedom to execute small vertical and angular displacements, h and α, and is constrained by two springs representing bending and torsion respectively. Thus, the displacement in Z direction is

$$W = Z - Z(X, y) = h(t) + (X - X_{EA})\alpha(t) \tag{7.1}$$

where X_{EA} denotes the position of elastic axis and h is the vertical displacement of the elastic axis (Figure 3).

Mathematically, it is easier to describe displacement with exponential time dependent $e^{i\omega t}$ (ω real) and assume that all dependent variables are proportional to $e^{i\omega t}$. Therefore, Equation (7.1) can be rewritten as

$$W = \tilde{h}\, e^{i\omega t} + (X - X_{EA})\tilde{\alpha}\, e^{i\omega t} = \tilde{w}\, e^{i\omega t} \tag{7.2}$$

with

$$\tilde{w} = \tilde{h} + \tilde{\alpha}(X - X_{EA}) \tag{7.3}$$

$$h = \tilde{h}\, e^{i\omega t} \tag{7.4}$$

$$\alpha = \tilde{\alpha}\, e^{i\omega t} \tag{7.5}$$

The actual quantities of W, h and α are found by taking the real part of the complex counterparts.

7.3 LAGRANGE'S EQUATION OF MOTION

Equation (7.1) describes the displacement of airfoil having two degree of freedom. The variables h and α are two generalized coordinates. The motion of the unrestrained airfoil is defined by the Lagrange's Equation which states that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} + \frac{\partial U}{\partial q_K} = Q_K \tag{7.6}$$

where $\bar{q}_k$ is the k-th generalized coordinates, T is the kinetic energy of the wing segment, U is the internal strain energy of the wing segment, Q_k is the k-th generalized force corresponding to the k-th generalized coordinate. The kinetic energy of the wing segment is

$$\begin{aligned} T &= \frac{1}{2}\int_{chord} \dot{w}^2\, dm \\ &= \frac{1}{2}\int_{chord} \left[\dot{h} + (x - x_{EA})\dot{\alpha}\right]^2 dm \\ &= \frac{1}{2} m\dot{h}^2 + \frac{1}{2} m\ell^2 r_\alpha^2 \dot{\alpha}^2 + m\ell x_\alpha \dot{\alpha}\dot{h} \end{aligned} \tag{7.7}$$

where ℓ is a reference length, while r_α and X_α are the dimensionless radius of gyration and static balance about the elastic axis,

$$r_\alpha^2 = \frac{1}{m\ell^2}\int_{chord}(x - x_{EA})^2\, dm \;; \quad x_\alpha = \frac{1}{m\ell}\int_{chord}(x - x_{EA})\, dm \tag{7.8}$$

If the stiffness of bending and torsion spring are represented by K_h and K_α respectively, the internal strain energy is given by

$$U_E = \frac{1}{2} K_h h^2 + \frac{1}{2} K_\alpha \alpha^2 \tag{7.10}$$

or

$$U_E = \frac{1}{2} m \omega_h^2 h^2 + \frac{1}{2} m \ell^2 r_\alpha^2 \omega_\alpha^2 \alpha^2 \qquad (7.11)$$

where ω_α and ω_h are the natural frequencies without mass coupling, or

$$\omega_h = \sqrt{\frac{K_h}{m}} \qquad (7.12)$$

$$\omega_\alpha = \sqrt{\frac{K_\alpha}{m \ell^2 r_\alpha^2}} \qquad (7.13)$$

Combining Equations (7.7) and (7.11) with the Lagrange Equation, Equation (7.6), with $q_1 = h$, $q_2 = \alpha$, $Q_1 = Q_h$ and $Q_2 = Q_\alpha$, the following two equations of motion are obtained

$$m \ddot{h} + m \ell x_\alpha \ddot{\alpha} + m \omega_\alpha^2 h = Q_h \qquad (7.14)$$

$$m \ell x \ddot{h} + m \ell^2 r_\alpha^2 \ddot{\alpha} + m \omega_\alpha^2 \ell^2 r_\alpha^2 \alpha = Q_\alpha \qquad (7.15)$$

Combining Equations (7.4) and (7.5) with Equations (7.14) and (7.15) yields

$$-m \omega^2 \tilde{h} - m \omega^2 \ell x_\alpha \tilde{\alpha} + m \omega_h \tilde{h} = \tilde{Q}_h \qquad (7.16)$$

$$-m \omega^2 \ell x_\alpha \tilde{h} - m \omega^2 \ell^2 r_\alpha^2 \tilde{\alpha} + m \omega_\alpha^2 \ell^2 r_\alpha^2 \tilde{\alpha} = \tilde{Q}_\alpha \qquad (7.17)$$

As mentioned above, the symbol tilde above each quantity represents the complex time-independent part.

7.4. GENERALIZED FORCES

The generalized forces are found from the virtual work which is given by

$$\begin{aligned} \tilde{\delta W} &= \iint_A \tilde{C}_p \, \delta w \, dx \, dy \\ &= \delta h \iint_A \tilde{C}_p \, dx \, dy + \delta \alpha \iint_A \tilde{C}_p (X - X_{EA}) \, dx \, dy \\ &\equiv \tilde{Q}_h \, \delta h + \tilde{Q}_\alpha \, \delta \alpha \end{aligned} \qquad (7.18)$$

where

$$\tilde{Q}_h = \tilde{L} = \frac{\rho U_\infty^2}{2} \iint \Delta \tilde{C}_p \, dx \, dy \tag{7.19}$$

and

$$\tilde{Q}_\alpha = \tilde{M} = \frac{c^2 \rho U_\infty^2}{2} \iint \Delta \tilde{C}_p (X - X_{EA}) \, dx \, dy \tag{7.20}$$

Combining Equations (7.19) and (7.20) with Equations (7.14) and (7.15) yields

$$\left[-m\omega^2 + m\omega_h^2 \right] \tilde{h} - m l X_\alpha \omega^2 \tilde{\alpha} = \tilde{L} \tag{7.21}$$

$$-m\omega^2 l X_\alpha \tilde{h} + \left[-m\omega^2 l^2 r_\alpha^2 + m\omega_\alpha^2 l r_\alpha^2 \right] \tilde{\alpha} = \tilde{M} \tag{7.22}$$

7.5 AERODYNAMIC FORCES

Following is the procedure to evaluate $\tilde{L}$ and $\tilde{M}$. Because of the assumption of small vibration, the doublet integral and source integral (see Equations (4.4), (4.5), (6.4), and (6.5)) are not influenced by the displacement, W, while the following boundary condition is a function of W

$$\frac{\partial \phi}{\partial N} = N_z \left(ikW + \frac{1}{\beta} \frac{\partial W}{\partial X} \right) \tag{7.23}$$

Therefore, it is legitimate to write

$$\begin{aligned} C_p &= \mathcal{L}\left(\frac{\partial \phi}{\partial N} \right) = \mathcal{L}(W) \\ &= \mathcal{L}\left[\tilde{h} e^{i\omega t} + (X - X_{EA}) \tilde{\alpha} e^{i\omega t} \right] \\ &= \tilde{h} \mathcal{L}\left[e^{i\omega t} \right] + \tilde{\alpha} \mathcal{L}\left[(X - X_{EA}) e^{i\omega t} \right] \end{aligned} \tag{7.24}$$

where $\mathcal{L}$ denotes the linear operator.

Therefore, by applying Equation (7.24) to Equation (7.19) and (7.20) the lift force and pitch moment are found.

$$\tilde{L} = \frac{\rho U_\infty^2 c}{2} \iint \tilde{C}_p \, dx \, dy$$

$$= \frac{\rho U_\infty^2 c}{2} \left\{ \frac{\tilde{h}}{l} \iint \mathcal{L}[l e^{i\omega T}] \, dx \, dy + \tilde{\alpha} \iint \mathcal{L}[(x - x_{EA}) e^{i\omega t}] \, dx \, dy \right\}$$

$$= \frac{\rho U_\infty^2 c}{2} \left[\frac{\tilde{h}}{l} \tilde{C}_{Lh} + \tilde{\alpha} \tilde{C}_{L\alpha} \right] \qquad (7.25)$$

$$\tilde{M} = \frac{\rho U_\infty^2 c^2}{2} \iint \tilde{C}_p (x - x_{EA}) \, dx \, dy$$

$$= \frac{\rho U_\infty^2 c^2}{2} \left\{ \frac{\tilde{h}}{l} \iint \mathcal{L}[l e^{i\omega t}] (x - x_{EA}) \, dx \, dy \right.$$

$$\left. + \tilde{\alpha} \iint \mathcal{L}[(x - x_{EA}) e^{i\omega t}] (x - x_{EA}) \, dx \, dy \right\}$$

$$= \frac{\rho U_\infty^2 c^2}{2} \left(\frac{\tilde{h}}{l} \tilde{C}_{Mh} + \tilde{\alpha} \tilde{C}_{M\alpha} \right) \qquad (7.26)$$

In Equations (7.25) and (7.26) $\tilde{C}_{Lh}$ and $\tilde{C}_{Mh}$ are lift force and pitch moment coefficients about elastic axis of the airfoil in the motion with oscillation in plunge. The displacement for plunge motion with $\frac{h}{l} = 1$ is described by

$$w = l e^{i\omega t} \qquad (7.27)$$

The boundary condition is then given by

$$\frac{\partial \phi}{\partial N} = N_z (i K \quad e^{i\omega t}) \qquad (7.28)$$

or

$$\frac{\partial \tilde{\phi}}{\partial N} = N_z \, i \, K \qquad (7.29)$$

Similarly, $\tilde{C}_{L\alpha}$ and $\tilde{C}_{M\alpha}$ are lift force and pitch moment about elastic axis of the airfoil in the motion with oscillation in pitch about the elastic axis with unit angle ampitude. The displacement is described by

$$W = (\chi - \chi_{EA})\, e^{i\omega t} \tag{7.30}$$

the boundary condition is then given by

$$\frac{\partial \phi}{\partial N} = N_z \left[1 + iK \frac{(\chi - \chi_{EA})}{\ell}\right] e^{i\omega t} \tag{7.31}$$

i.e.

$$\frac{\partial \tilde{\phi}}{\partial N} = N_z \left[1 + iK \frac{(\chi - \chi_{EA})}{\ell}\right] \tag{7.32}$$

7.6 FLUTTER ANALYSIS

Combining Equations (7.25) and (7.26) with Equations (7.21) and (7.22) yields

$$\left\{m\omega^2\ell\left[-1 + \frac{\omega_h^2}{\omega^2}\right] - \frac{\rho U_\infty^2 c}{2} C_{Lh}\right\} \frac{\tilde{h}}{\ell} - \left\{m\omega^2\ell\chi_\alpha + \frac{\rho U_\infty^2 c}{2} C_{L\alpha}\right\} \tilde{\alpha} = 0 \tag{7.33}$$

and

$$\left\{-m\ell\omega^2(\ell\chi_\alpha) - \frac{\rho U_\infty^2 c}{2} C_{Mh}\right\} \frac{\tilde{h}}{\ell} - \left\{m\omega^2\ell^2\gamma_\alpha^2\left[1 - \frac{\omega_\alpha^2}{\omega^2} + \frac{\rho U_\infty^2 c}{2}\right]\right\} \tilde{\alpha} = 0 \tag{7.34}$$

By introducing

$$A = \frac{\omega_\alpha^2}{\omega^2} \tag{7.35}$$

and

$$K = \omega\ell / U_\infty \tag{7.36}$$

$$\mu = m / \pi \rho \ell^2 \tag{7.37}$$

$$\ell = \frac{c}{2} \tag{7.38}$$

Equations (7.33) and (7.34) may be rewritten in the following matrix form

$$\begin{bmatrix} 1-\left(\frac{\omega_h}{\omega_\alpha}\right)^2 A + \frac{1}{\pi K^2 \mu} C_{Lh} & X_\alpha + \frac{1}{\pi K^2 \mu} C_{L\alpha} \\ X_\alpha + \frac{2}{\pi K^2 \mu} C_{Mh} & \gamma_\alpha^2 (1-A) + \frac{2}{\pi K^2 \mu} C_{M\alpha} \end{bmatrix} \begin{Bmatrix} \frac{\tilde{h}}{\ell} \\ \tilde{\alpha} \end{Bmatrix} = 0 \tag{7.39}$$

In order to yield a non-trivial solution, the determinant of the above left-hand side matrix must be zero. The value of A can, therefore, be evaluated. By plotting the values of the imaginary part of A versus different reduced frequencies, the flutter is found at a specific reduced frequency where the imaginary part of A is zero.

7.7 NUMERICAL RESULTS

In order to compare the numerical results with the analytical ones, a two-dimensional thin airfoil in incompressible, oscillatory flow was studied. For this airfoil, the analytical expressions of $\tilde{C}_{Lh}$, $\tilde{C}_{L\alpha}$, $\tilde{C}_{Mh}$ and $\tilde{C}_{M\alpha}$ are given in several standard text books (see e.g. Reference 24) as

$$\tilde{C}_{Lh} = 2\pi K^2 \left(2 \frac{C_K}{K} i - 1\right) \tag{7.40}$$

$$\tilde{C}_{L\alpha} = \pi K^2 \left\{ X_{EA} + 2 \frac{C_K}{K^2} + \left[(1-2X_{EA}) C_K + 1 \right] \frac{i}{K} \right\} \tag{7.41}$$

$$\tilde{C}_{Mh} = \pi K^2 \left[(2X_{EA} + 1) \frac{C_K}{K} i - X_{EA} \right] \tag{7.42}$$

$$\tilde{C}_{M\alpha} = \frac{1}{2}\pi K^2 \left\{ (0.125 + X_{EA}^2) + (2X_{EA}+1)\frac{C_K}{K^2} + (0.5 - X_2)\left[(2X_{EA}+1)K - 1\right]\frac{i}{K} \right\} \quad (7.43)$$

where

$$C_K = \left(-J_1(K) + i\,Y_1(K)\right) \Big/ \left[-\left(J_1(K)+Y_0(K)\right) + i\left(Y_1(K) - J_0(K)\right)\right] \quad (7.44)$$

where J_1 and J_0 and J-Bessel functions of the first and zero orders and Y_1 and Y_0 are Y-Bessel functions of the first and zero orders and reduced frequency, K, is defined as

$$K = \omega c / 2 U_\infty \quad (7.45)$$

It should be noted that the purpose for this two dimensional (airfoil) flutter analysis is to assess the validity of the three dimensional aerodynamic theory presented in the proceeding sections. For this reason the two dimensional airfoil is approximated by a thin rectangular wing with high aspect ratios. For the current problem rectangular wing with AR=16 can be considered as a two dimensional wing as being verified in Figure 30. In Figures 30a, 30b, 30c and 30d, the curves of $\tilde{C}_{Lh}$, $\tilde{C}_{L\alpha}$, $\tilde{C}_{Mh}$, and $\tilde{C}_{M\alpha}$ as functions of the aspect ratio for a thin rectangular wing with K=0.25 are presented. It is shown that most curves converge at AR=16. Values of $\tilde{C}_{Lh}$, $\tilde{C}_{L\alpha}$, $\tilde{C}_{Mh}$, and $\tilde{C}_{M\alpha}$ versus K varying from 0.1 to 1.5 are obtained with $X_{EA}=-.2C$, NX=8, NY=10, NW=20, LW =3.0C, τ =0.1% and AR=16 are compared with the exact ones in Figures 34a, 34b, 34c and 34d. The numerical results are in good agreement with the exact ones for reduced frequencies higher than 0.15, while the results for lower reduced frequencies are not completely satisfactory.

As mentioned before, by plotting the value of imaginary A versus reduced frequency K, the flutter speed can be located at a specific K where the imaginary A is zero. An example of this work is presented in Figure 31 which is related to a

rectangular wing with AR = 16, τ = .1%, μ = 5, X_{EA} = -.2c, ω_h/ω_α = 0.5, X_α = 0.2 and r_α = 0.5. The flutter boundary occurs at K = 0.53 and $A = \omega_\alpha^2/\omega^2$ = 1.775. Combining Equations (4.35) (4.36) and (4.37) gives the nondimensional flutter speed, $2U_F/c\,\omega_\alpha$ = 1.42 which is in good agreement with the exact solution (Reference 24)* The flutter speed as a function of ω_h/w_α for the same wing with μ= 5, X_α = 0.2, r_α = 0.5 and X_{EA} = -.2c is presented in Figure 32a and compared with the solution given by two dimensional airfoil theory (Figure 9-5(A), Reference 24). Additional results for the same wing with μ= 10, X_α = 0.2, r_α = 0.5 and X_{EA} = -.2c are presented in Figure 32b.

*See Figure 9-5(A)(j) of Reference 24 (p. 540), which yields $2\,U_F/c\omega_\alpha \cong 1.47$.

SECTION 8
CONCLUSIONS

8.1 INTRODUCTION

The general comments of the present method are made in Subsection 8.2. The outline of the computer program is introduced in Subsection 8.3, while the CPU time and the memory space required for the computer program are listed in Subsection 8.4

8.2 GENERAL COMMENTS

The results obtained in the previous sections are in very good agreement with the existing ones. Especially, compared with the exact solutions, several results are remarkably accurate. For example, values of C_{ℓ_α}, evaluated for a supersonic delta wing in Subsection 5.4.3, is accurate up to fifth significant figure. In addition to the accuracy, the method is very efficient, extremely general and very easy to use. The results obtained include a variety of problems for either steady or oscillatory, subsonic or supersonic flow. Two results for wing-body configurations in subsonic and supersonic flow are included, while the method can be further applied to wing-body-tail combinations, or more complex configurations. A preliminary result of flutter analysis is presented in Section 7. The pressure is evaluated by finite difference method. However, a hyperboloidal distribution of the velocity potential, ϕ, within each element in terms of ϕ's at corners can be obtained by evaluating ϕ at each corner by taking average of ϕ at the centroids of the surrounding elements. Therefore, an increase of generality and efficiency can be attained. In addition, the present method for the oscillatory flow can be generalized for the study of the unsteady flow without additional increase in computational complexity (Reference 25). Also it should be mentioned that the computer code presented here is believed to be the only one available for supersonic oscillatory flows around non-zero-thickness configuration.

8.3 COMPUTER PROGRAM

The formulations in the previous sections are imbedded into a general computer program called SOSSA ACTS (Steady and Oscillatory Subsonic and Supersonic Aerodynamics for Complex Transportation System). The listing of this program is given in Appendix E. In general, the program is separated into three parts. The first part of the program is to define the geometry of the aircraft, wake and diaphragm (if necessary) and divide their surface surfaces into a number of elements. Each element is then replaced by a hyperboloidal surface defined by four corner points. This part of the program includes Subroutines GEOMET and VEC123. The second part of the program is to evaluate the coefficients C_{hK}, b_{hK} and W_{hK} and to solve a system of linear equations by using the Gaussian elimination method for unknowns ϕ, or $\hat{\phi}$. This part of the program includes Subroutines CEOFF and CCGELG. The third part of the program is to evaluate the pressure coefficient and the generalized forces from the computed velocity potential ϕ or $\hat{\phi}$. This part of the program includes Subroutine COEFPR.

8.4 COMPUTER TIME AND MEMORY SPACE

The following table gives the computer time used by the program SOSSA ACTS in terms of the total number of elements on the wing, $N_{ELEM} = 4 \times N_x \times N_y$.

Computer Time (Secs) for SOSSA ACTS

N_{ELEM}	Subsonic		Supersonic	
	Steady	Unsteady	Steady	Unsteady
4 x 4 x 4	29	161	8	21
4 x 5 x 5	70	324	15	38
4 x 6 x 6	143	543	29	65
4 x 7 x 7	268	928	54	119

All the results are obtained by taking advantage of the symmetry (or antisymmetry) in y and z directions. The wake for subsonic flow has $N_W = 1$ for steady and $N_W = 30$ for unsteady flow. The diaphragm for supersonic flow has $N_D = 9$. The computer times are expressed in seconds and were obtained on an IBM 370/145 computer of the Boston University Computer center.

Finally, the memory space required for steady flow is approximately given by

$$N_{words} = 9,500 + N_{EQN} + N^2_{EQN} \tag{8.1}$$

where N_{words} is the number of words, while N_{EQN} is the number of equations. Similarly for unsteady flow, the memory space requirement is approximately given by

$$N_{words} = 12,500 + 22\, N_{EQN} + 2N^2_{EQN} \tag{8.2}$$

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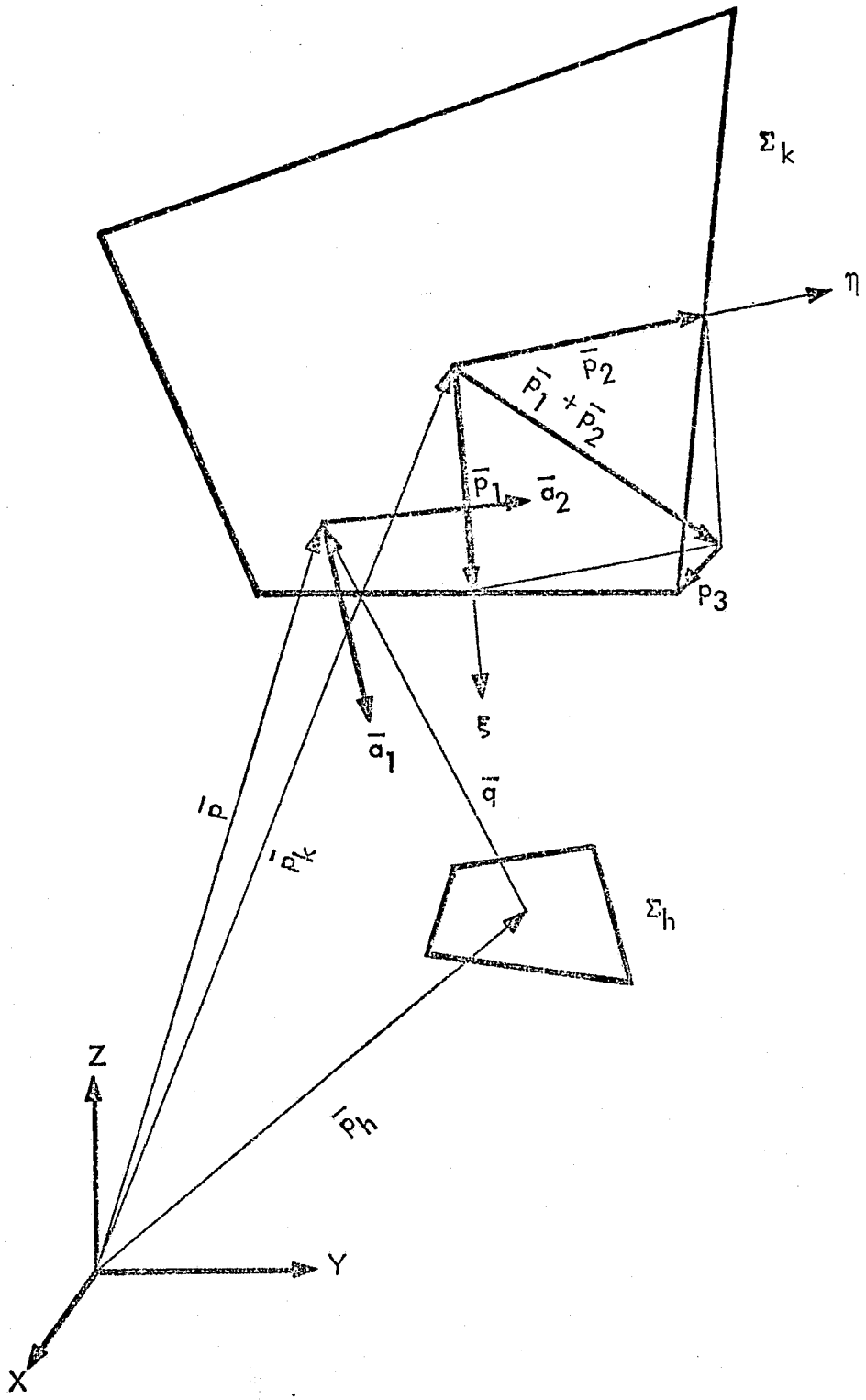

Figure 1. Hyperboloidal Element

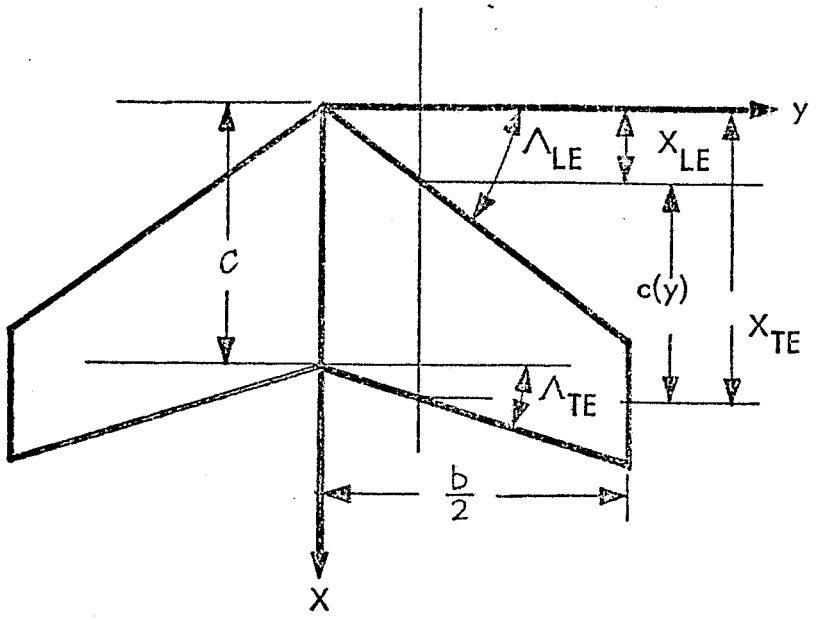

Figure 2a. Wing Planform

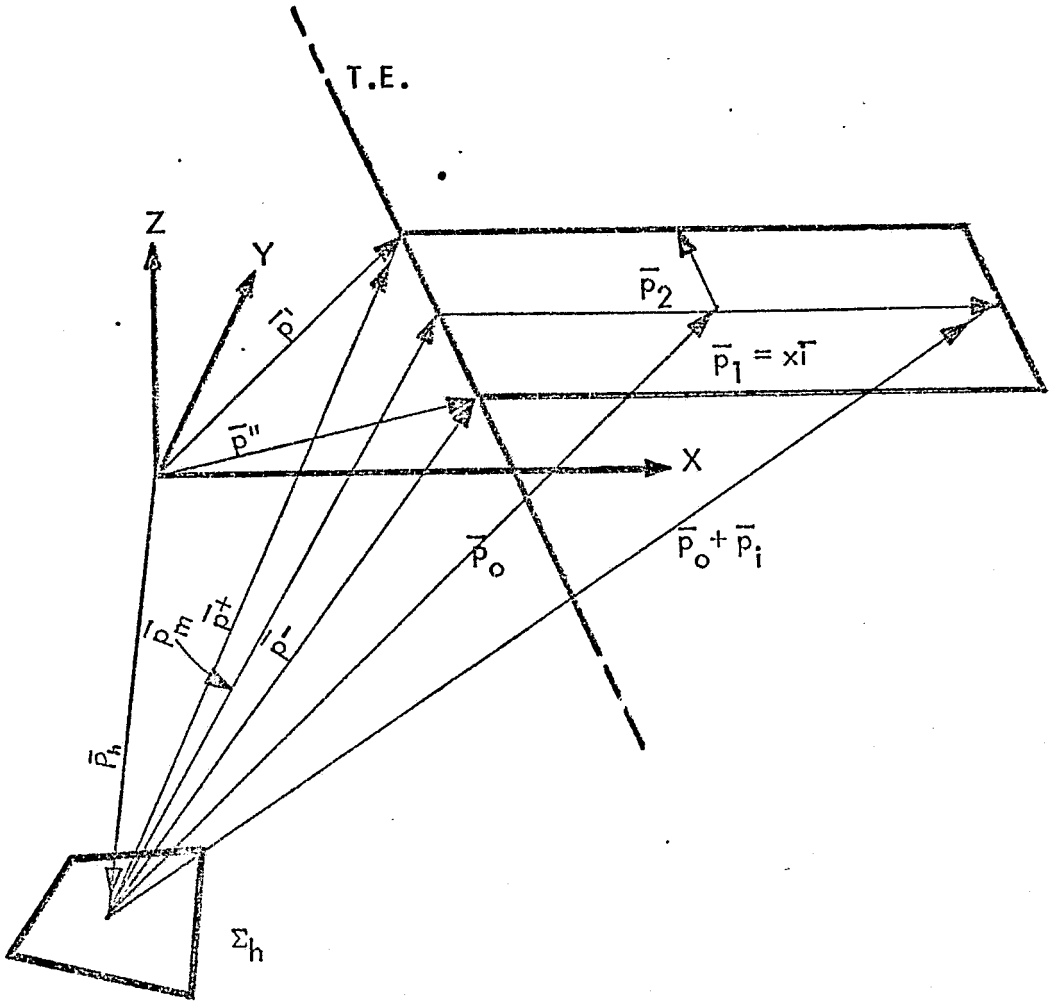

Figure 2b. Wake Element

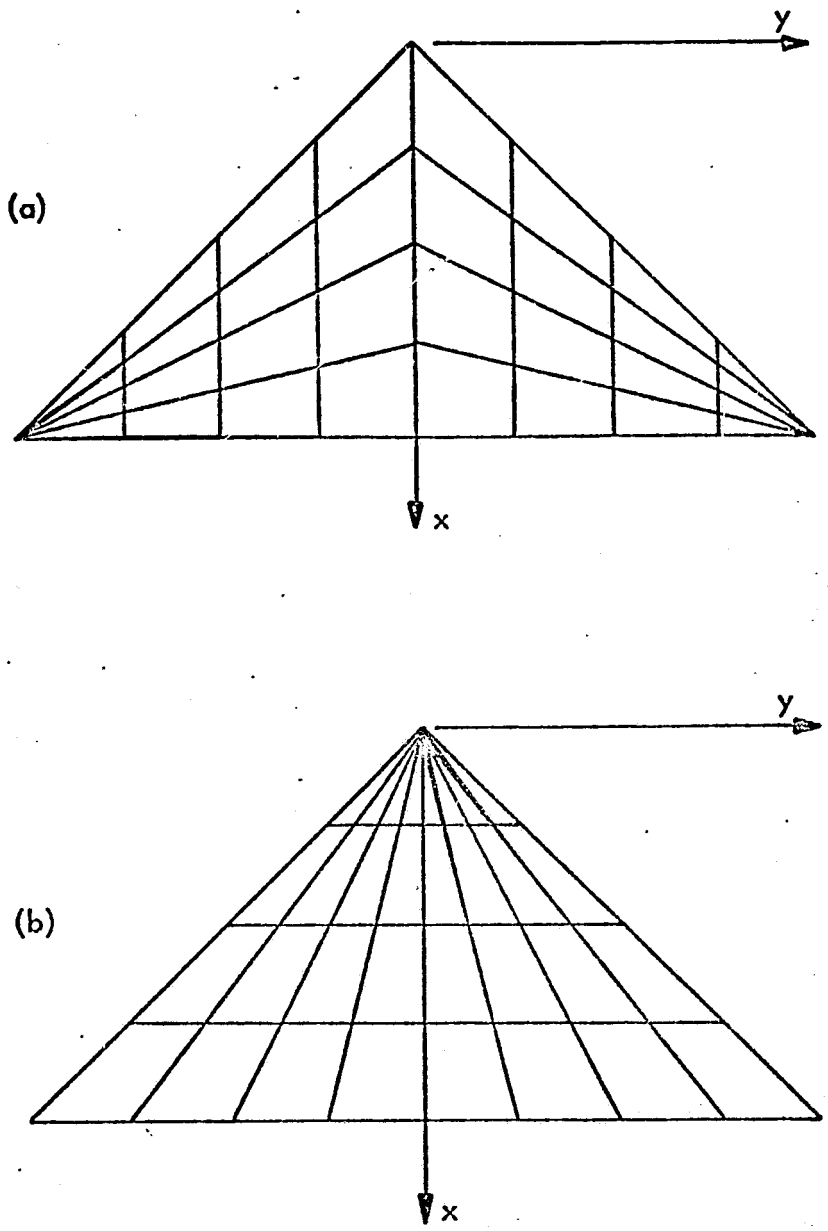

Figure 2c. Two Types of Element-Grids for Delta Wings.

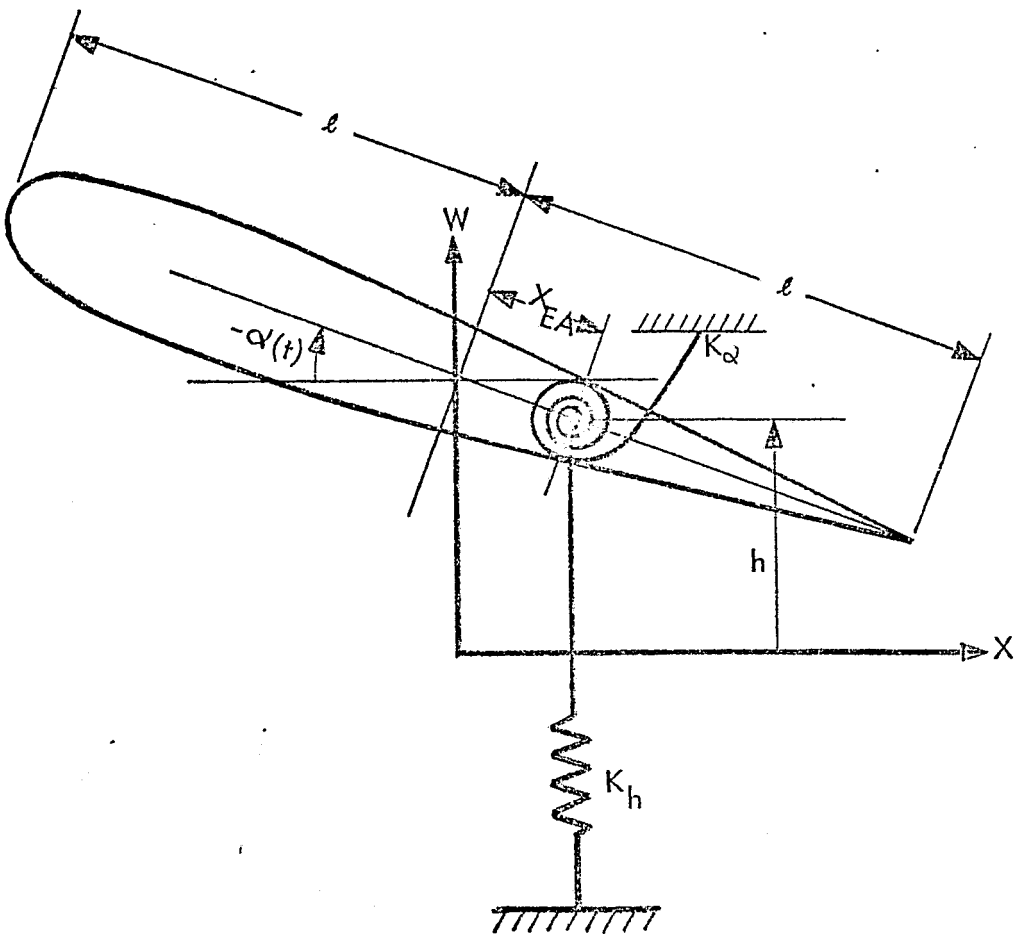

Figure 3. [illegible]l Model for Flutter Analysis

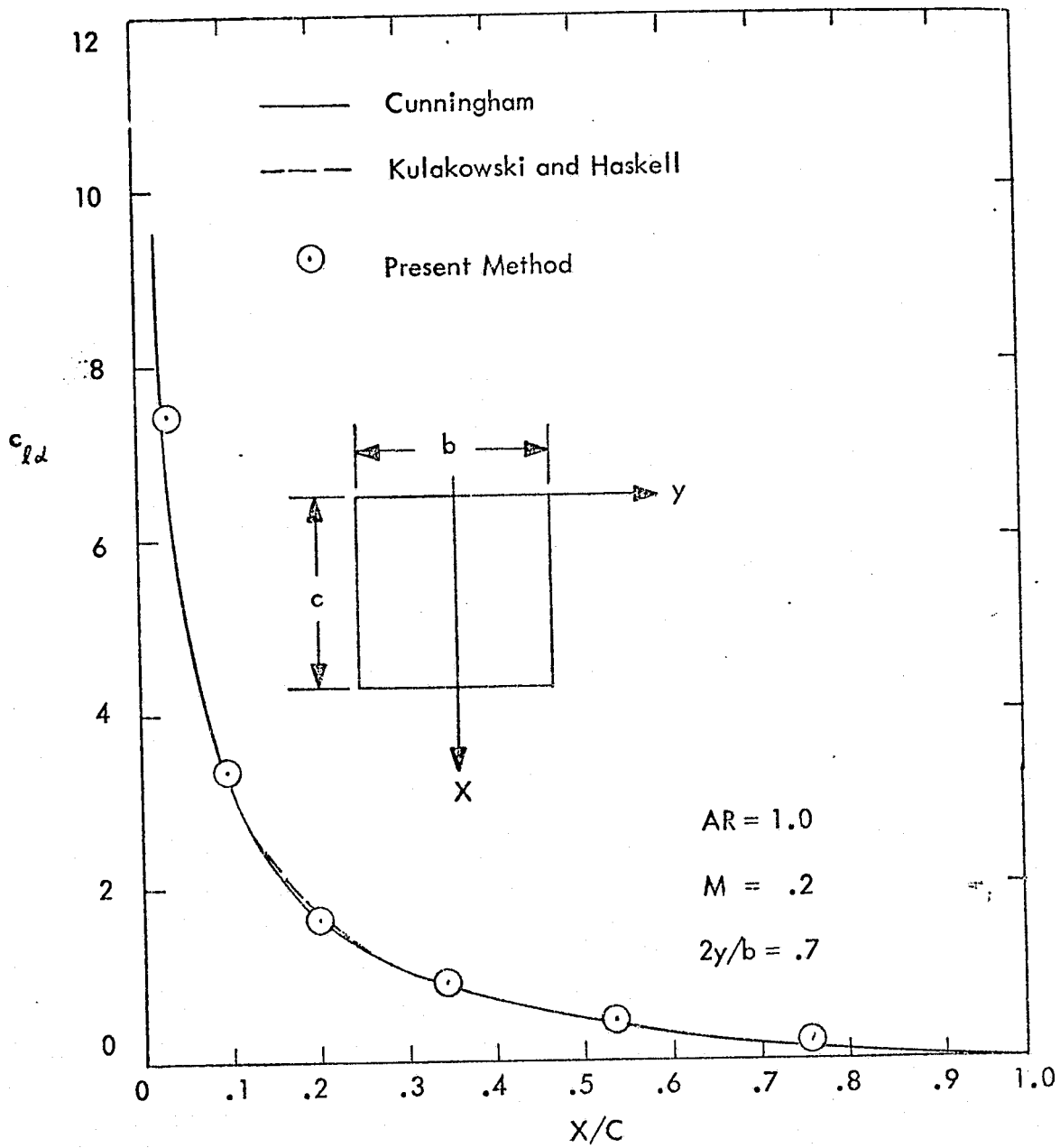

Figure 4. The distribution of $c_{\ell\alpha}$ along 2y/b = .7 for a rectangular wing with AR = 1.0, M = .2, and NX = NY = 7 for comparison with results of Ref. 11.

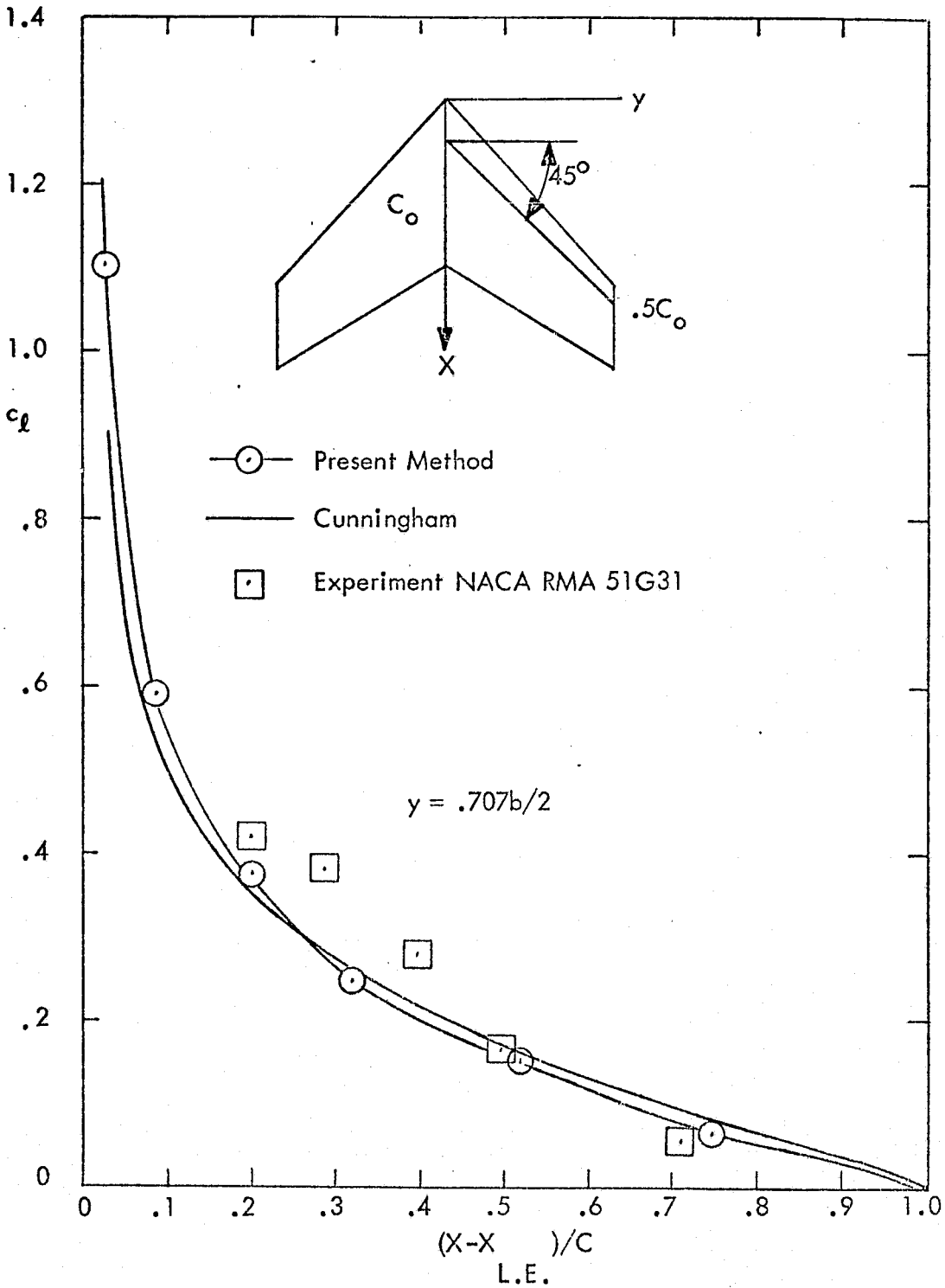

Figure 5. The distribution of c_ℓ along 2y/b = .707 for a tapered swept wing with AR = 3, TR = .5, $\Lambda_{1/4}$ = 45°, M = .8, α = 5° and NX = NY = 7 for comparison with results of Ref. 13

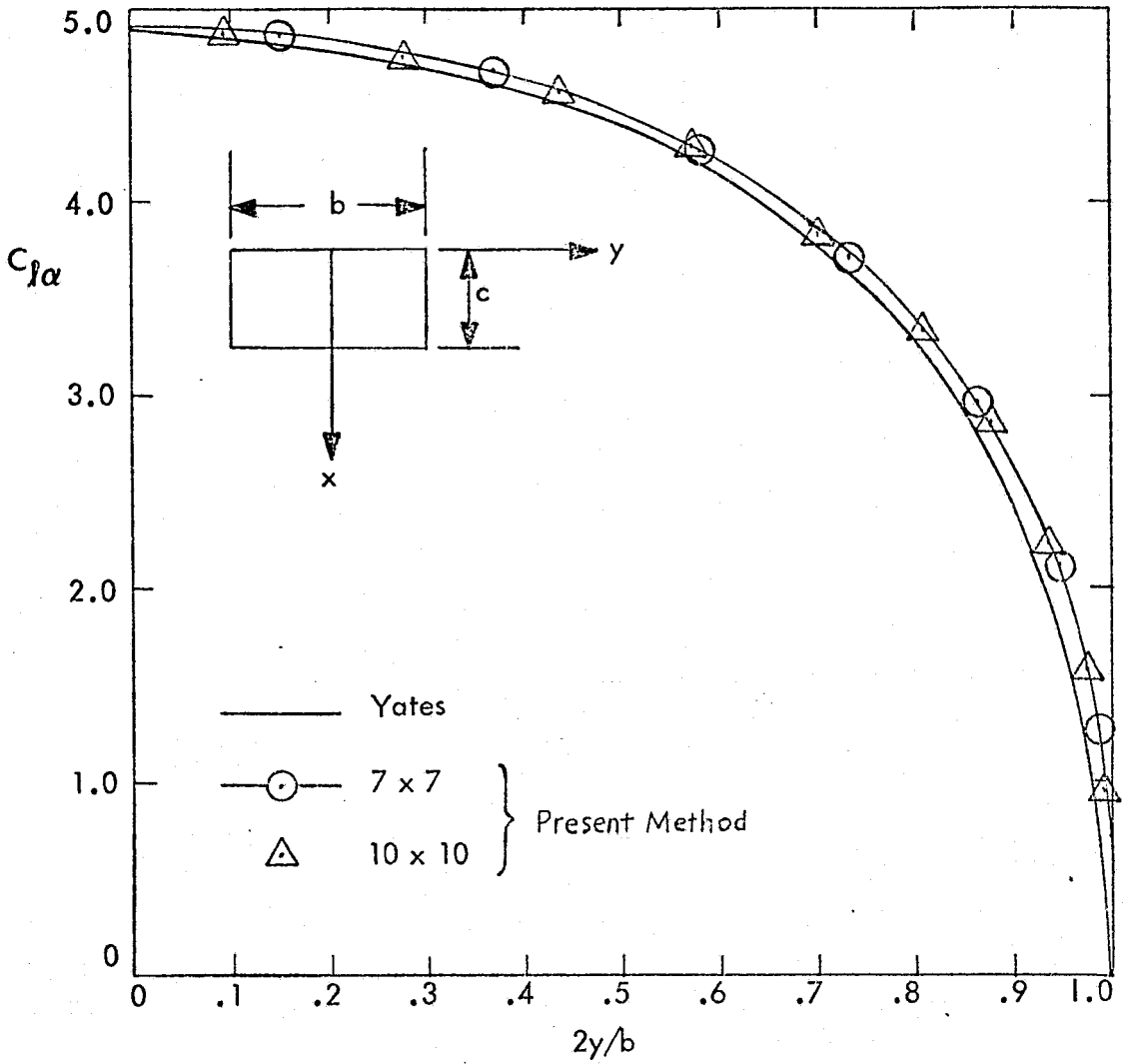

Figure 6. The distribution of the section lift coefficient per unit angle of attack for a rectangular wing with AR = 4, M = .507 and NX=NY=7 and 10 for comparison with results of Ref. 14

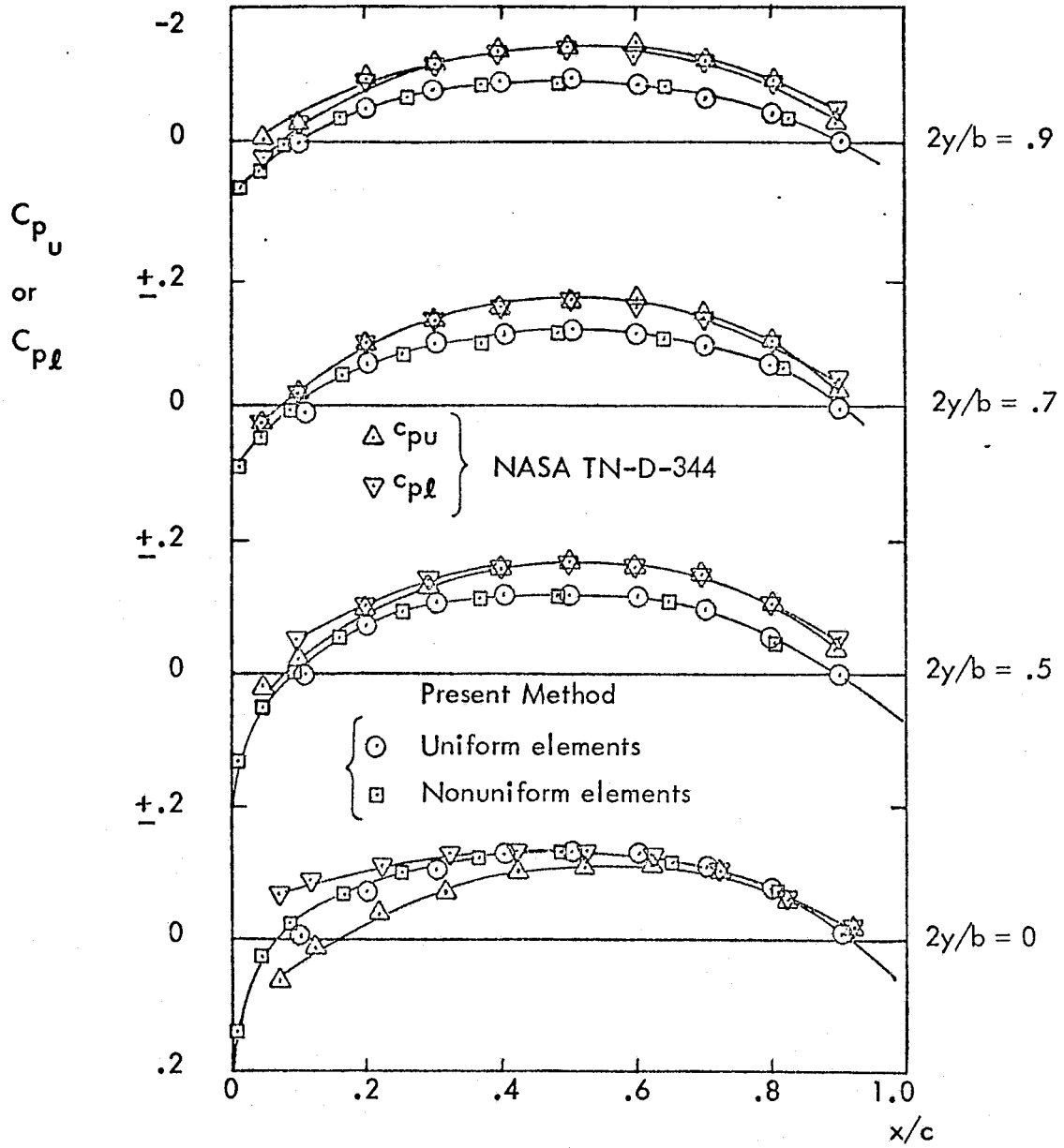

Figure 7. The distribution of c_p on the upper and lower surfaces of a symmetric rectangular wing with AR = 3, τ = 5% α = 0°, M = .24 and NX=NY=10 for comparison with results of Ref. 15

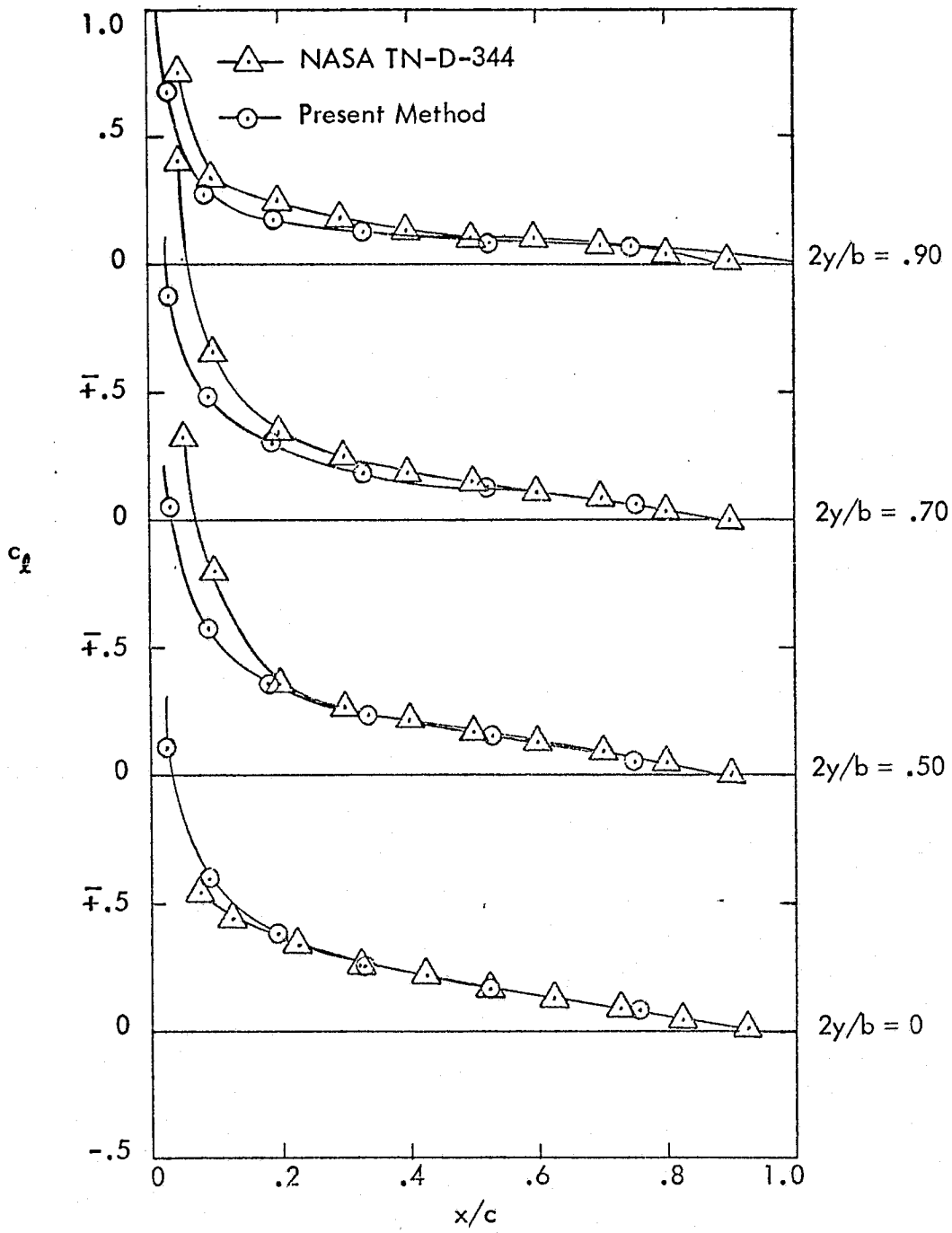

Figure 8a. The distribution of the lift coefficient, c_ℓ, on a symmetric rectangular wing with AR = 3, τ = 0.05 α = 5°, M = .24 and NX=NY=7 for comparison with results of Ref. 15.

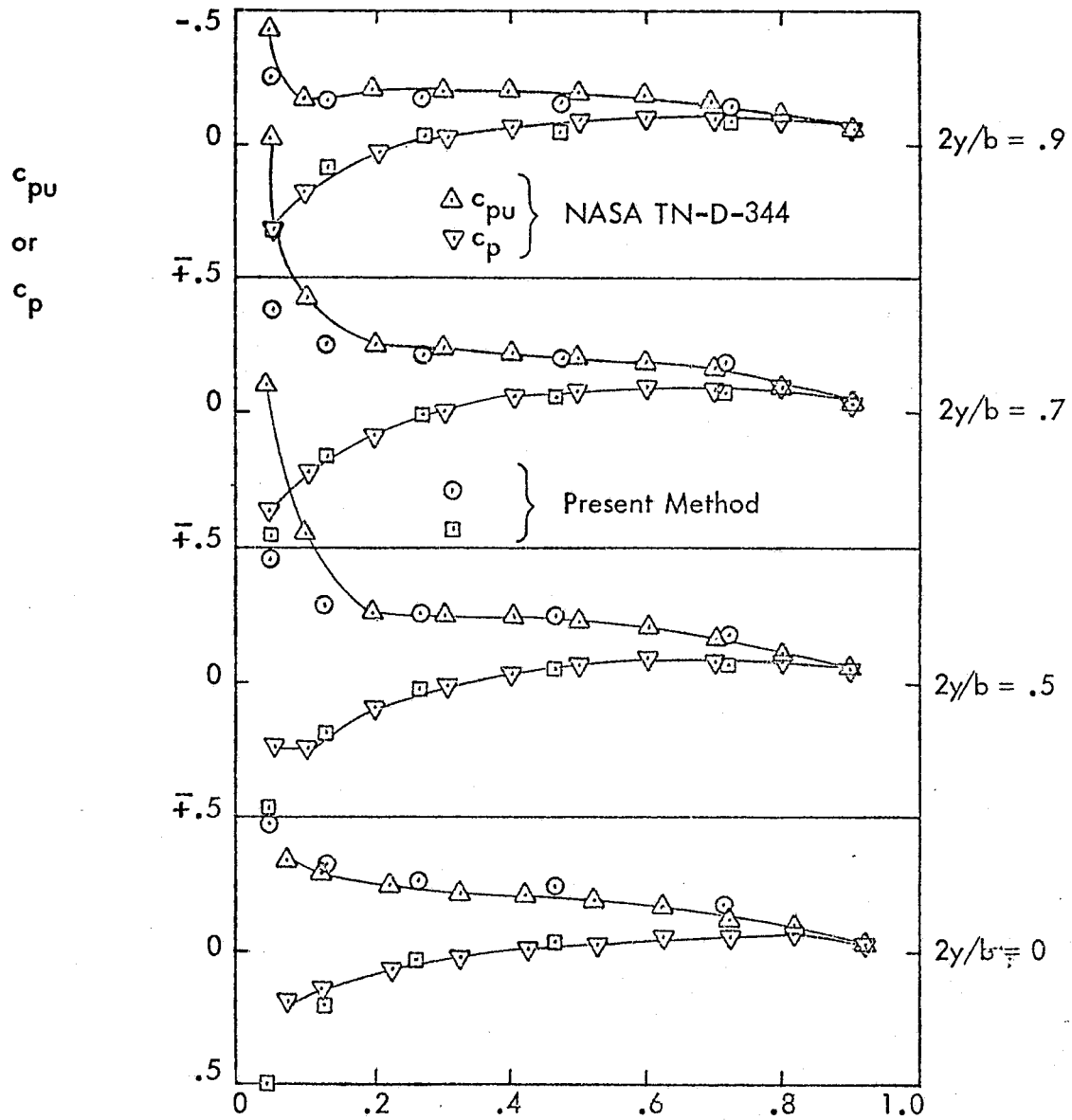

Figure 8b. The distribution of c_p on the upper and lower surfaces of a symmetric rectangular wing with AR = 3, $\tau = 0.05$, $\alpha = 5^o$, M = .24 and NX=NY=6 for comparison with results of Ref. 15.

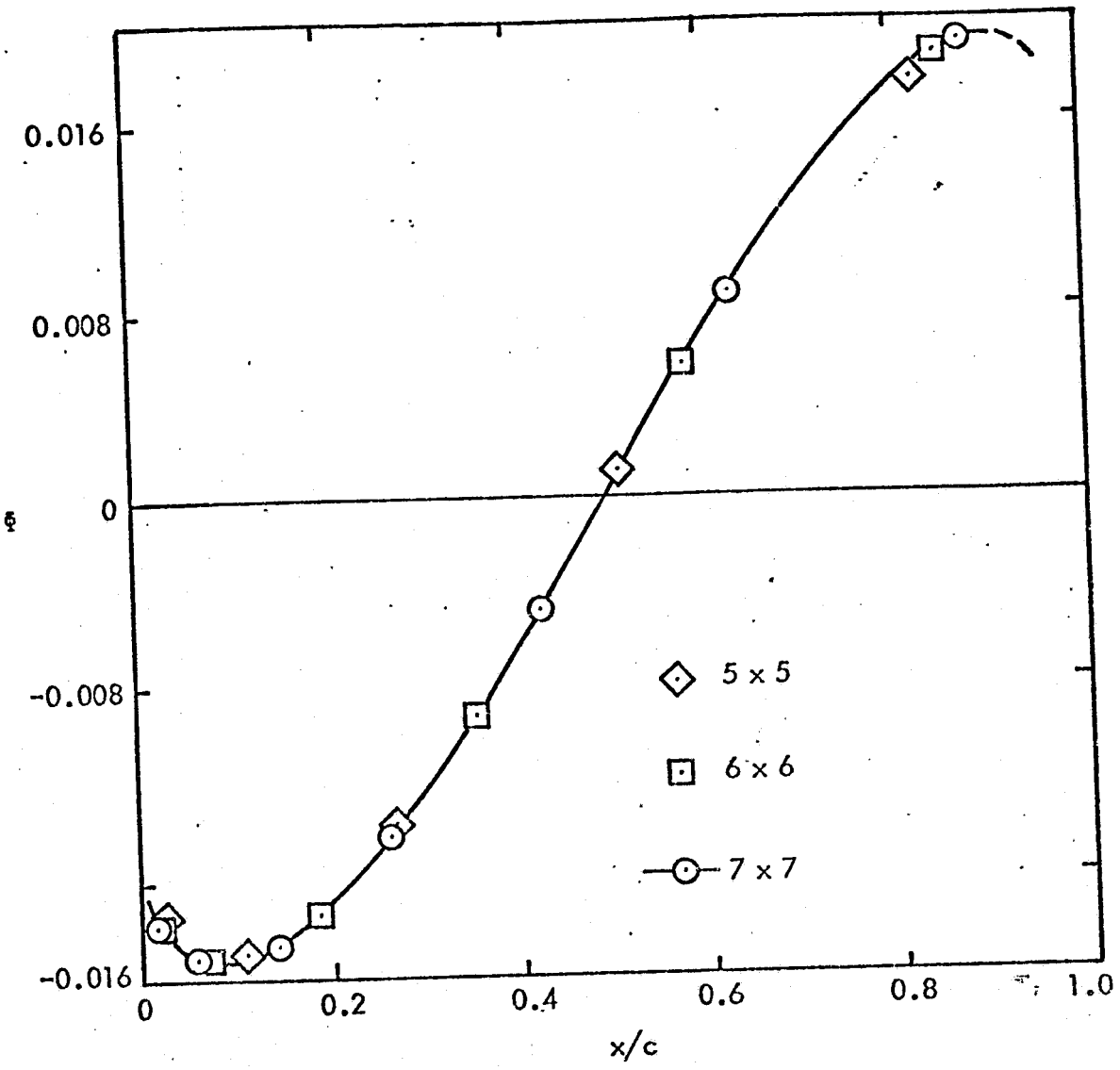

Figure 9. Analysis of Convergence: Potential Distribution, Φ, Versus x/c, at y = 0, for Rectangular Wing With Biconvex Section, in Steady Subsonic Flow, for AR = 3, τ = 0.05, M = 0.24, $\alpha = 0^{o}$. NX = NY = 5, 6, 7.

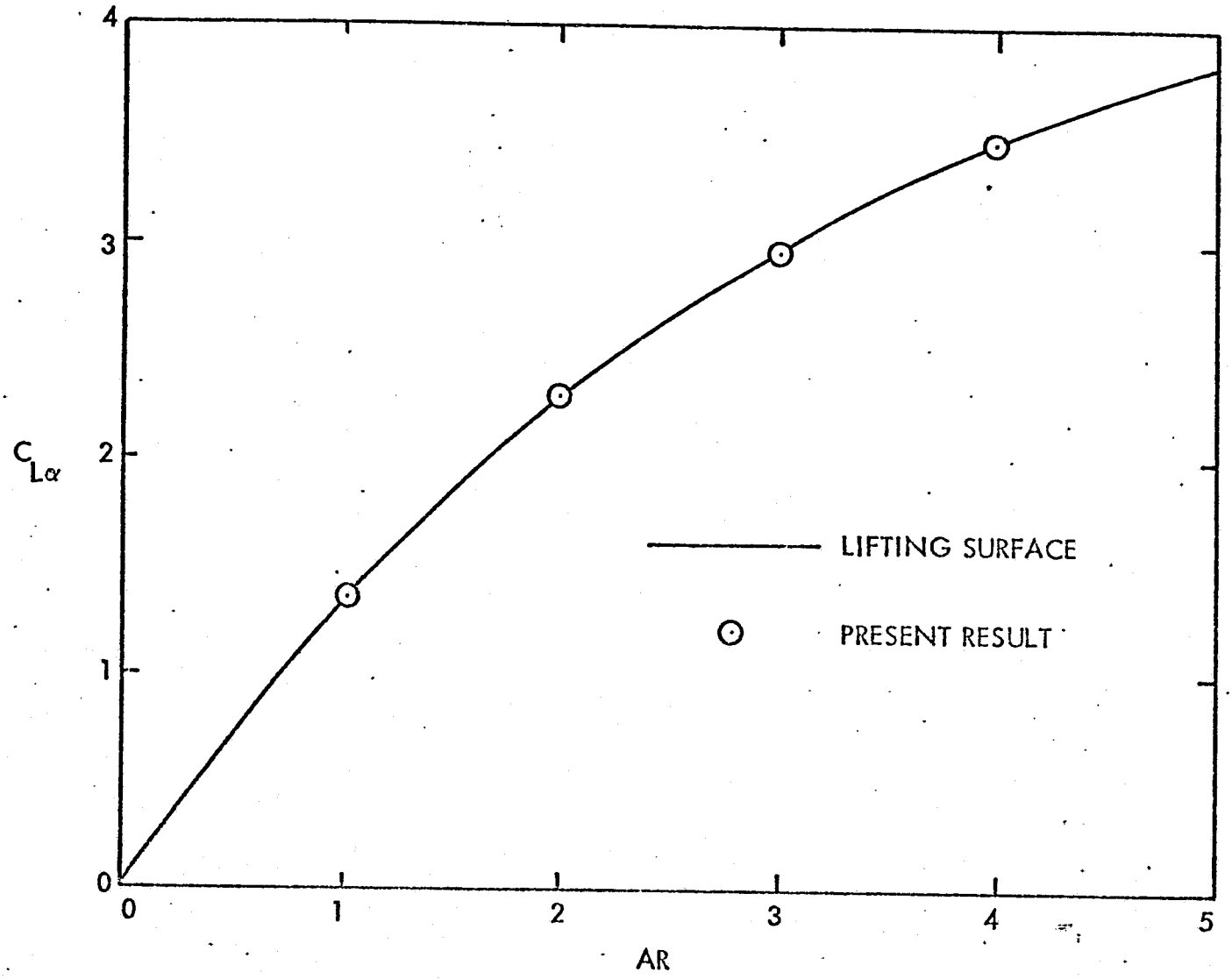

Figure 10. Lift Distribution Coefficient, $C_{L\alpha}$, Versus Aspect Ratio AR, for Delta Wing in Steady Subsonic Flow, With $\tau = 0.001$, $M = 0$, $N_x = N_y = 7$. Comparison With Lifting Surface Theory of Reference 16.

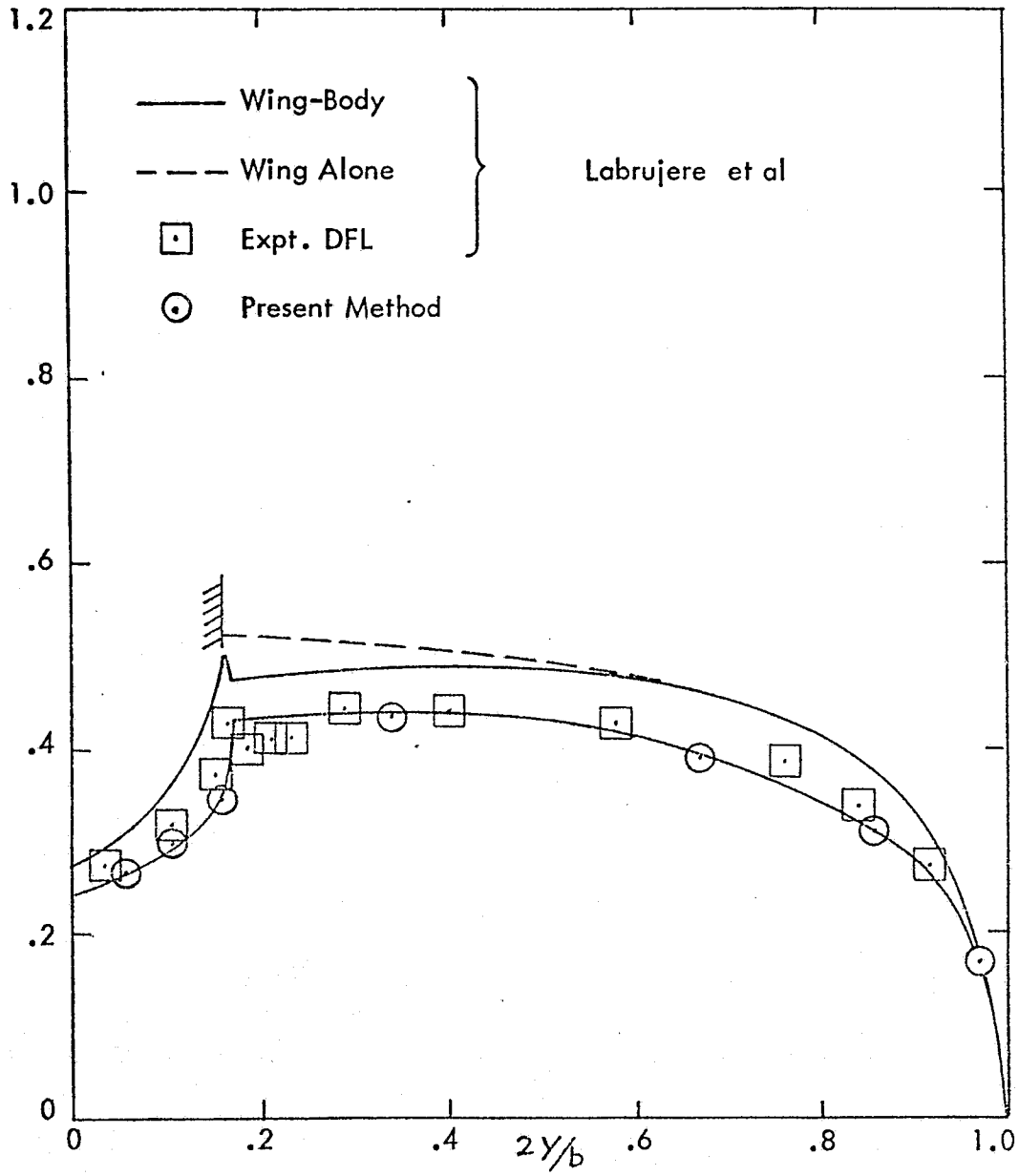

Figure 11a. The distribution of section lift coefficient, C_ℓ, for a wing-body configuration with $\alpha_w = 6^\circ$, $\alpha_B = 0^\circ$, $\tau = 9\%$, M=0, b=6c, r = 0.5c and 200 elements on the whole wing for the comparison with results of Ref. 17

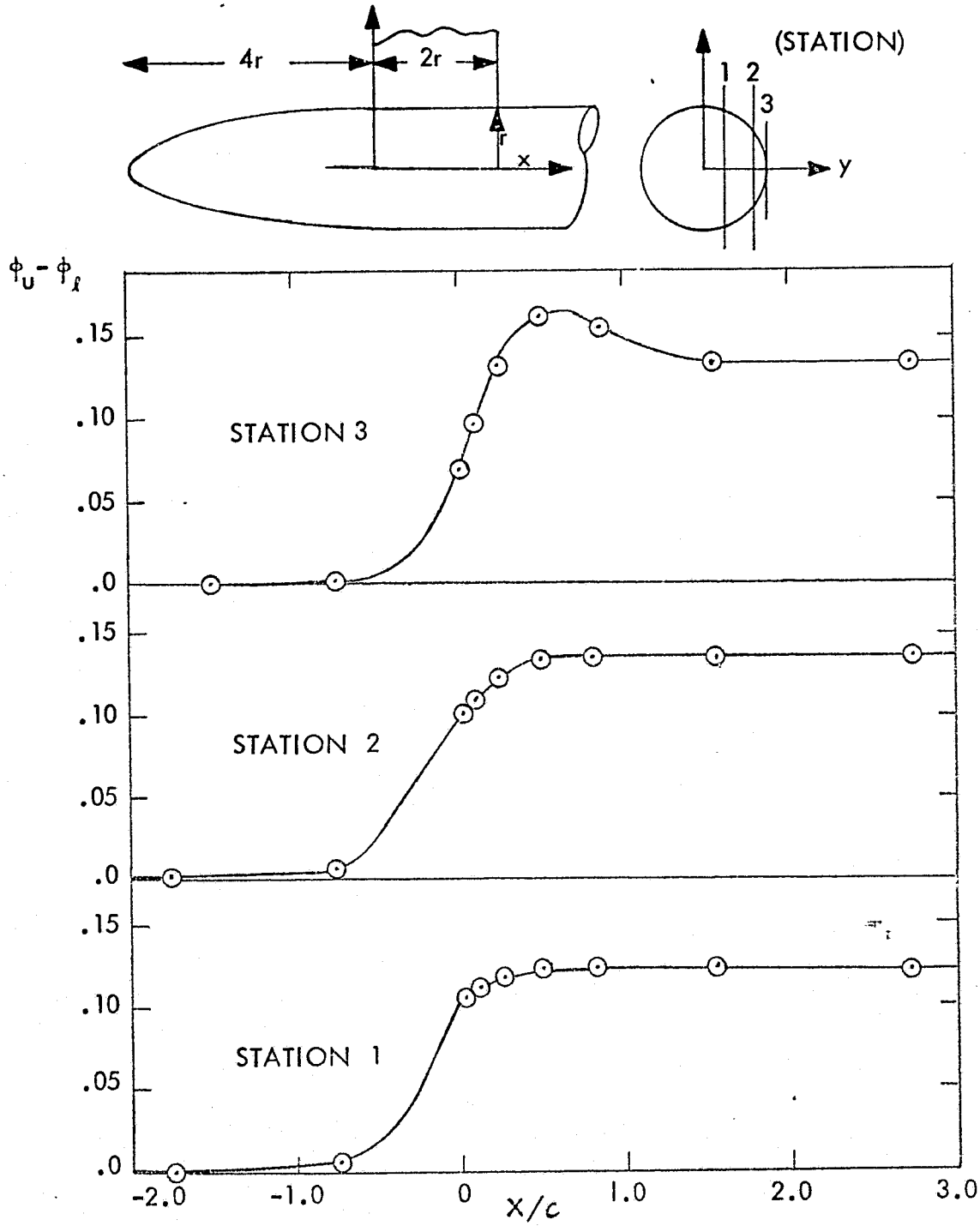

Figure 11b. The distribution of $\phi_U - \phi_L$ along three circumferential stations for a wing-body configuration with $\alpha_W = 6^\circ$ $\alpha_B = 0^\circ$, $\tau = 9\%$, M=0, b=6c, r=0.5c

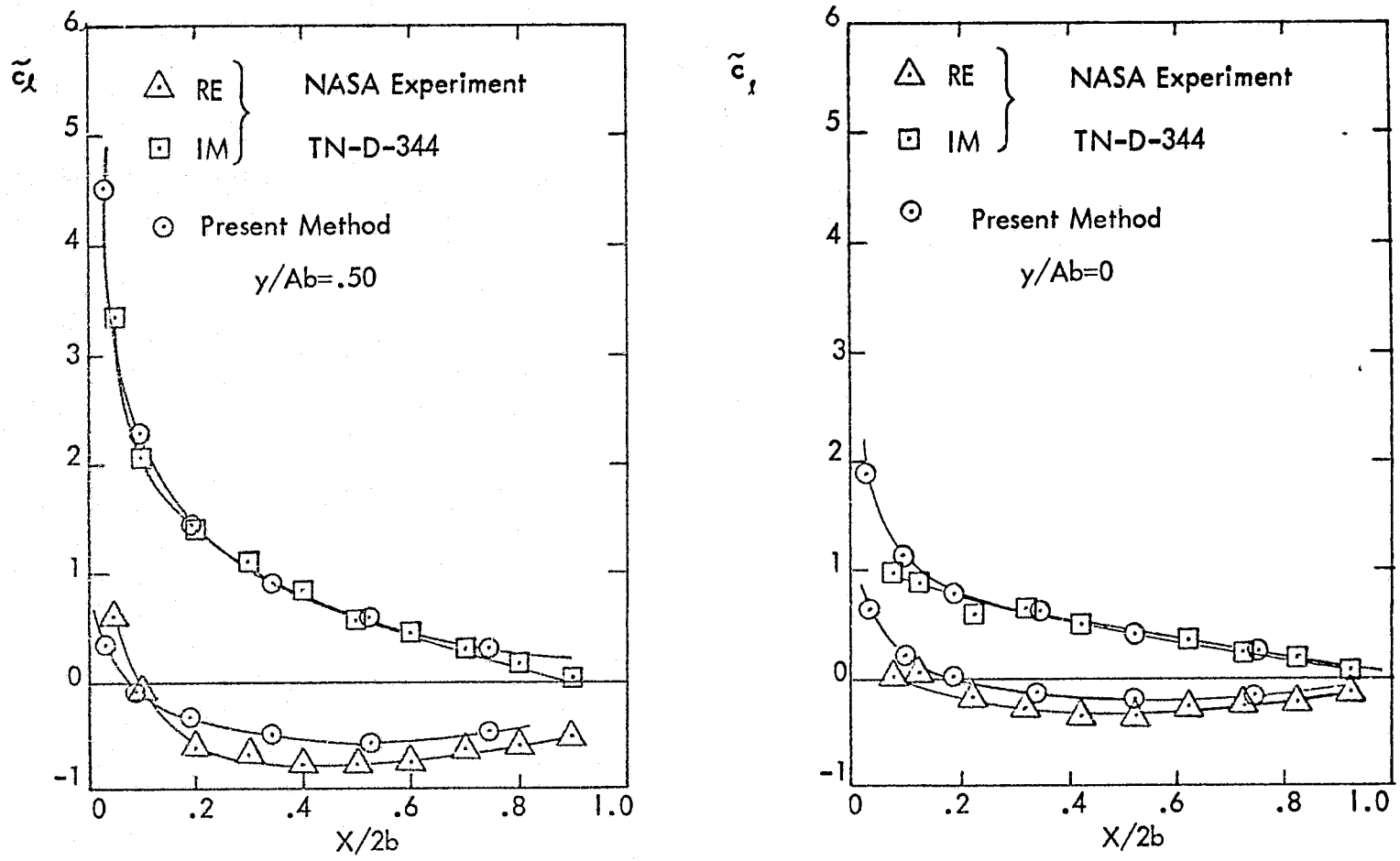

Figure 12. The distribution of lift coefficient, $\tilde{c}_\ell$, for a rectangular wing oscillating in bending mode with $k = \omega c/2U_\infty = .47$, M=.24, AR=3, τ= 0.05, NW =20, L_W=2.5c and NX=NY=7 for comparison with results of Ref. 15

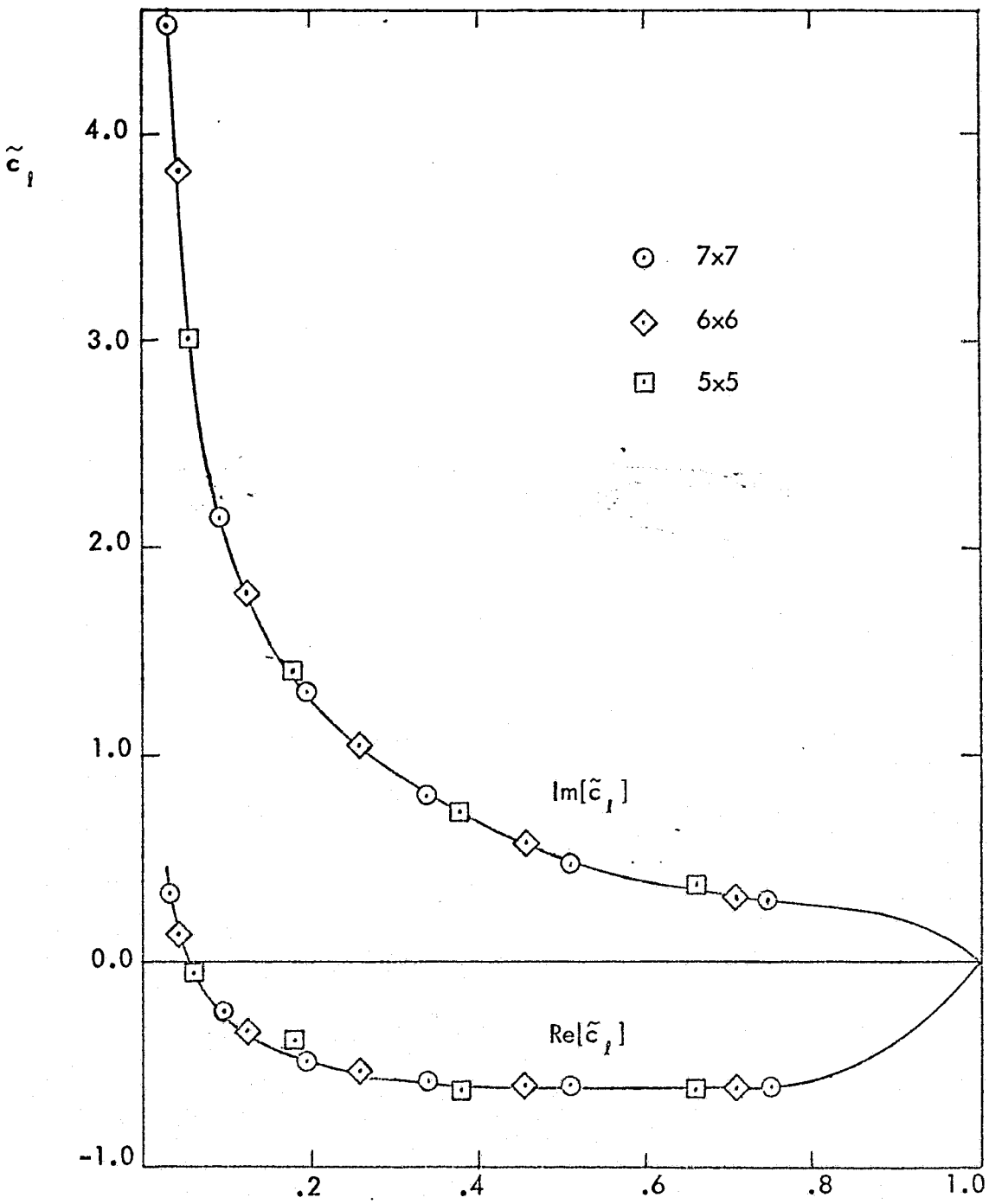

Figure 13a. Analysis of Convergence: The distribution of $\tilde{c}_l$ versus x/c at 2y/b = .05 for a rectangular wing oscillating in bending mode with $k = \omega c/2 U_\infty = .47$, M=.24, AR=3 $\tau = 0.01$, $\alpha = 0^\circ$, $N_w = 30$, $L_w = 3.5c$, NX = NY = 5,6,7.

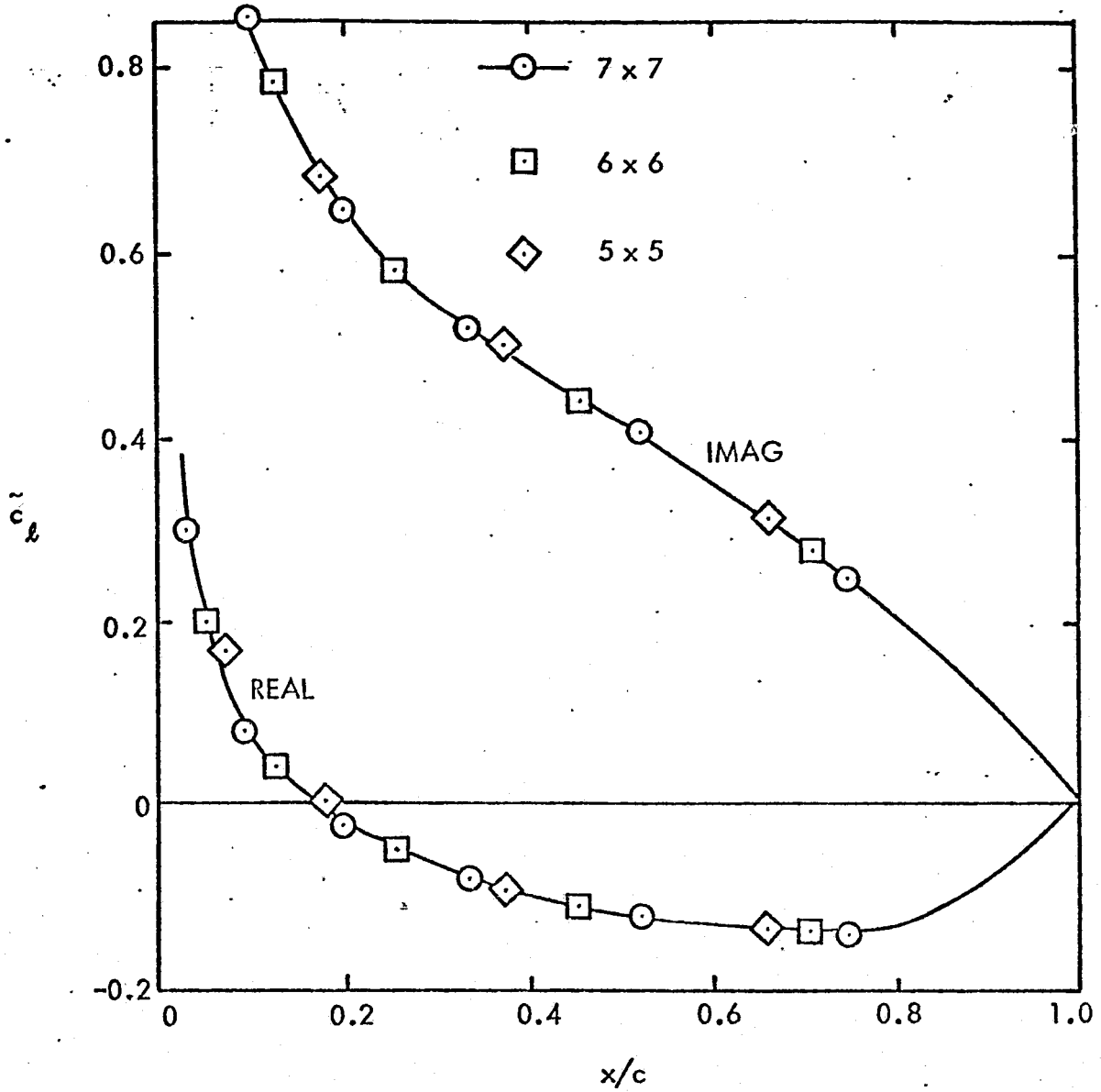

Figure 13b. Analysis of Convergence: Lift Distribution Coefficient, $\tilde{c}_\ell$, Versus x/c, at $2y/b = 0.1328$, for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Subsonic Flow; for AR = 3, $\tau = 0.01$, $M = 0.24$, $\alpha = 0^o$, $K = 0.47$, $N_W = 30$, $L_W = 3.5c$, NX = NY = 5,6,7.

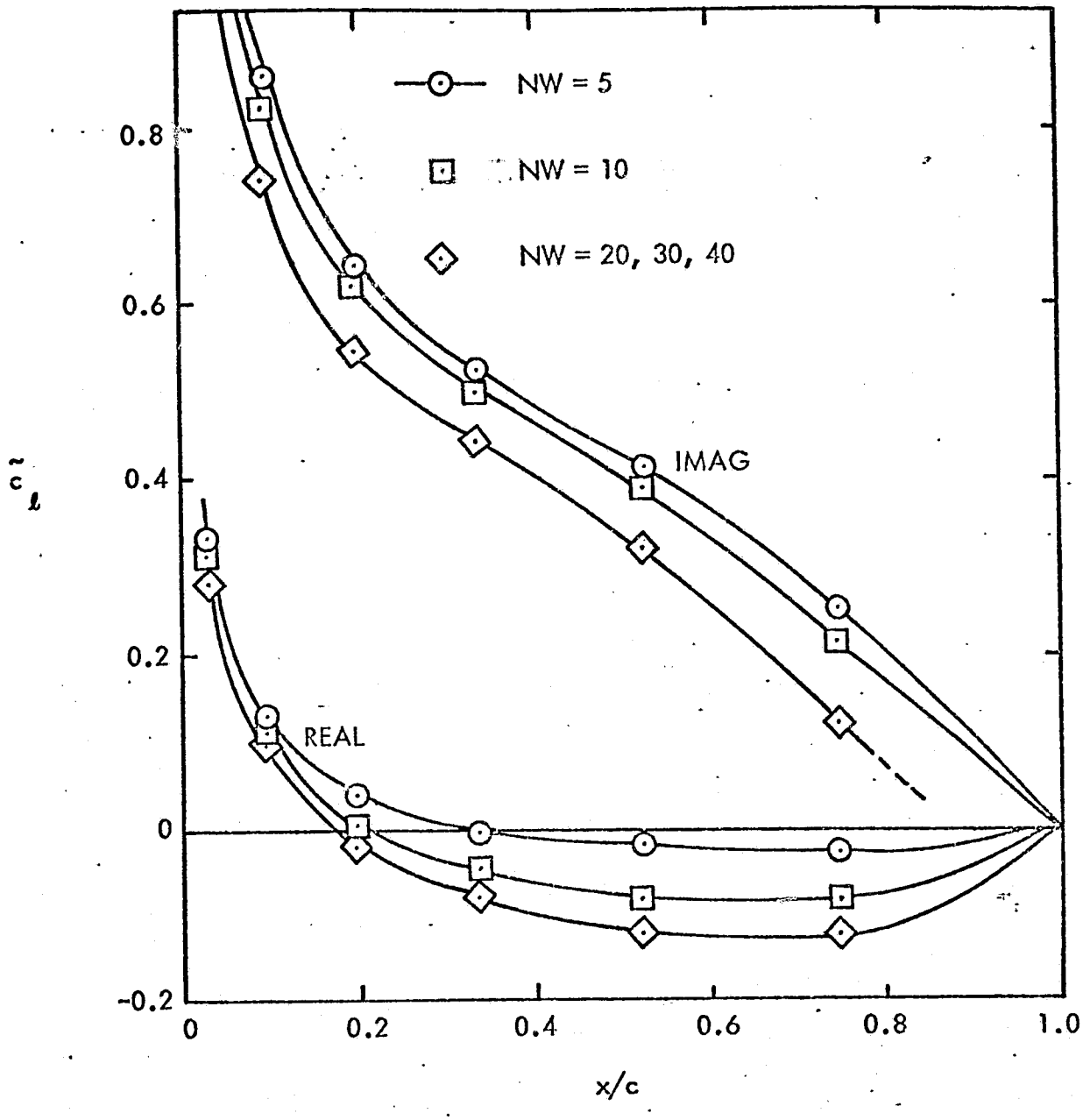

Figure 14. Analysis of Convergence: Lift Distribution Coefficient, $\tilde{c}_\ell$, Versus x/c, at 2y/b = 0.1328 for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Subsonic Flow, for AR = 3, τ = 0.01, M = 0.24, $\alpha = 0^\circ$, K = 0.47, $L_W = 4c$, $N_x = N_y = 7$.

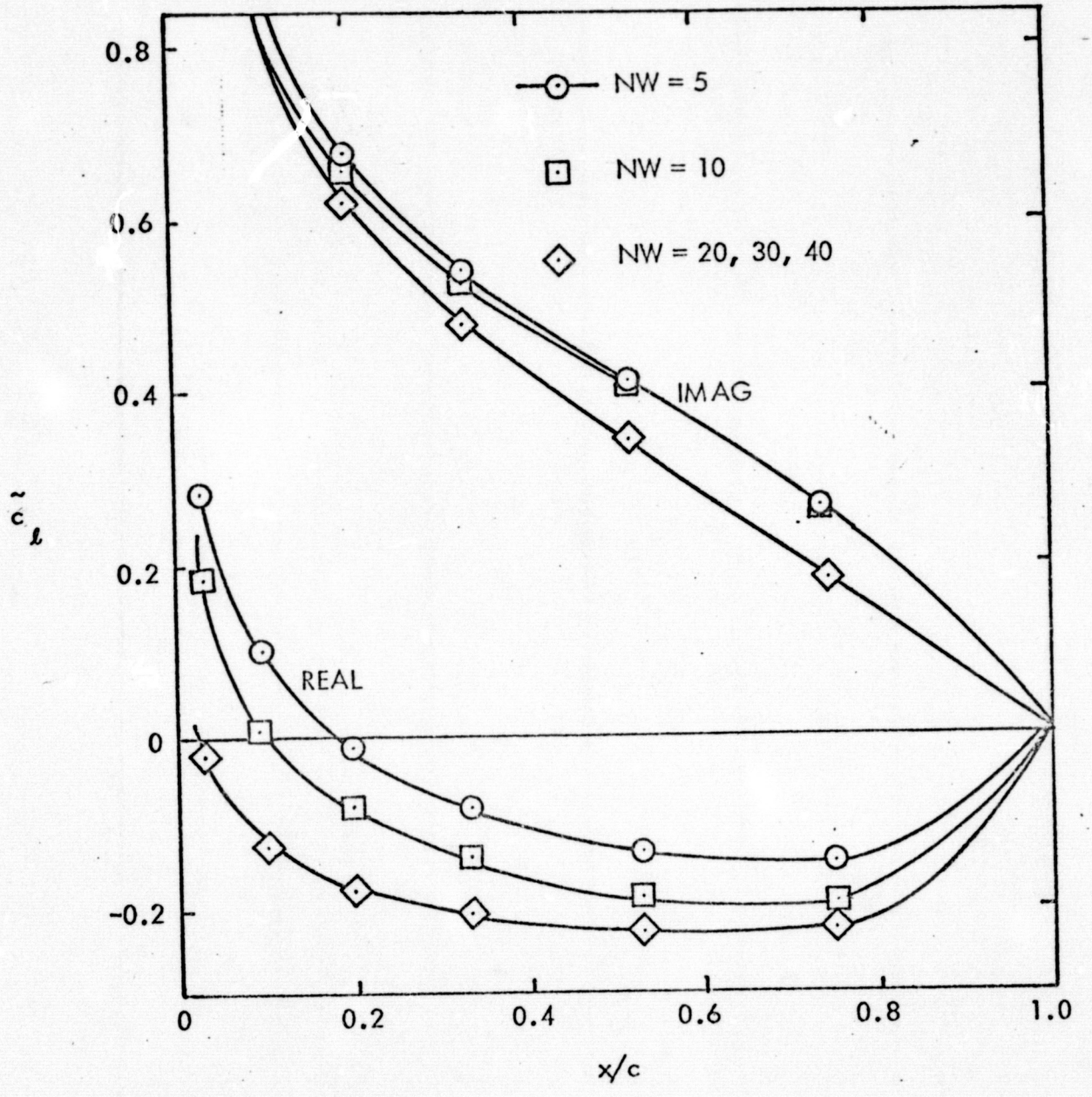

Figure 15. Analysis of Convergence: Lift Distribution Coefficient, $\tilde{c}_\ell$, Versus x/c, at 2y/b = 0.1328, for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Subsonic Flow, for AR = 3, τ = 0.01, M = 0.24, α = 0°, K = 0.47, Δx_W = 0.1, $N_x = N_y = 7$.

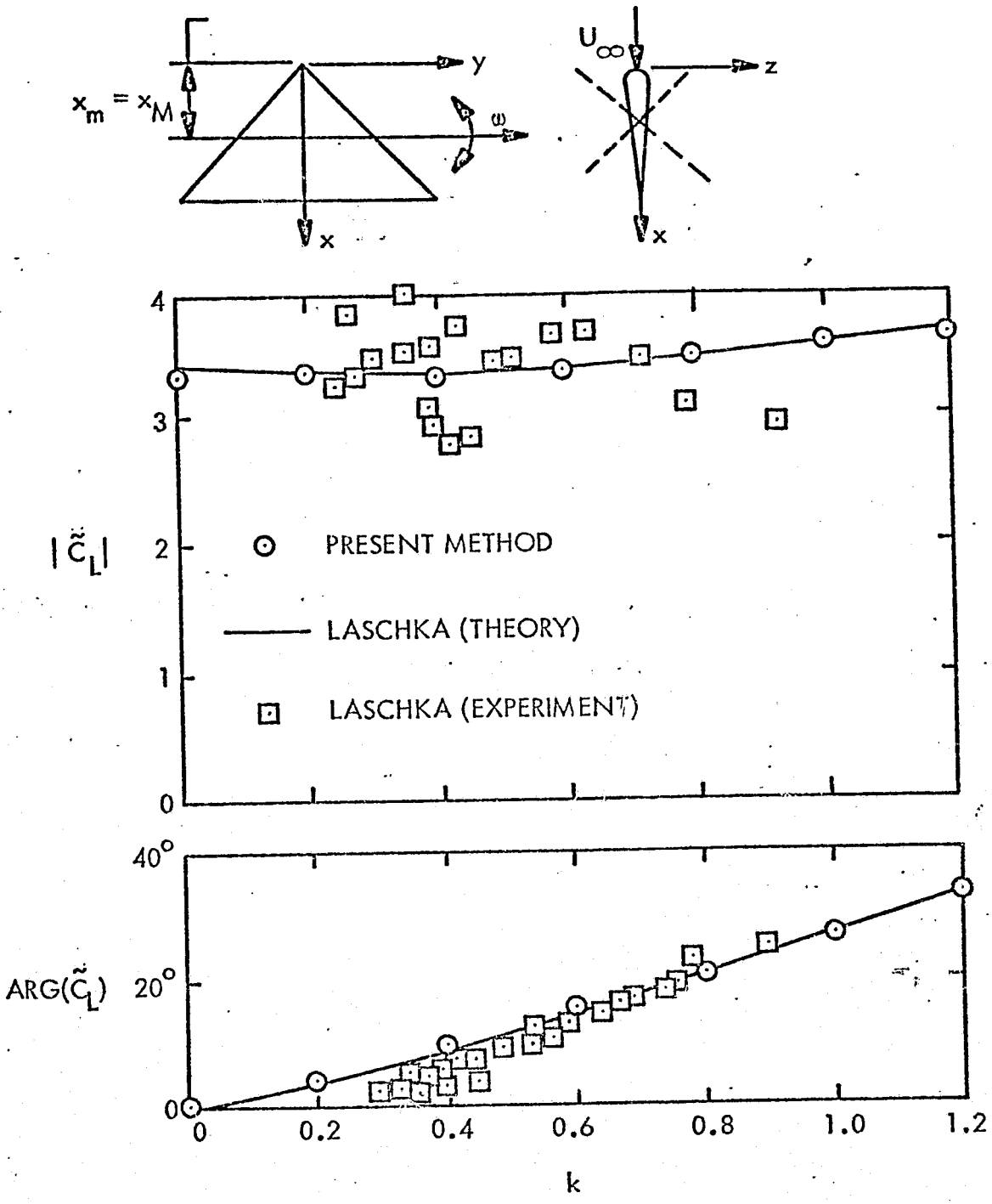

Figure 16a. Lift Coefficient, $\tilde{C}_L$, Versus k, for Delta Wing Oscillating in Pitch, With AR = 4, τ = 0.005, M = 0, N_x = 10, N_y = 6, N_W = 20, L_W/c = 2. Comparison With Results of Reference 18.

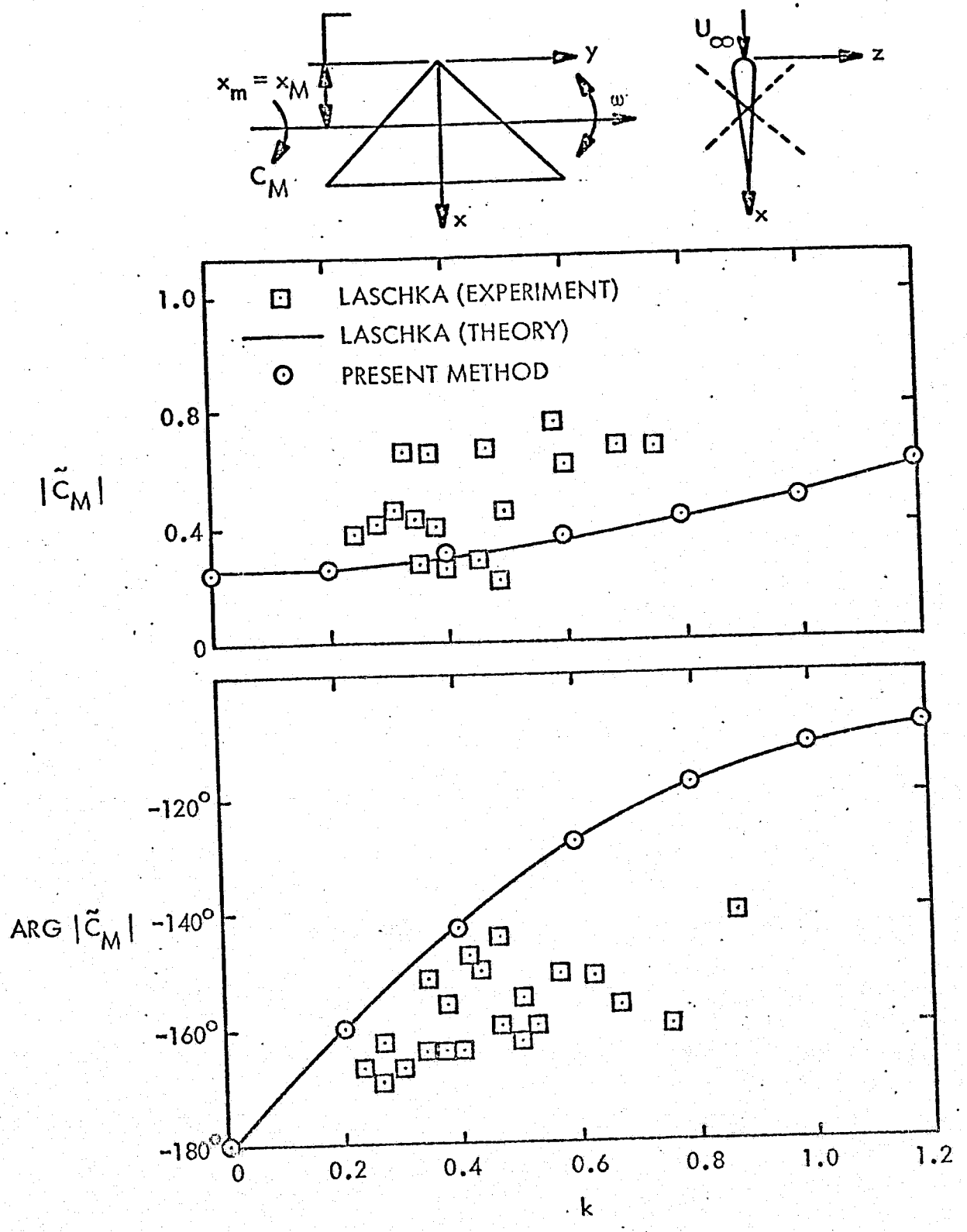

Figure 16b. Moment Coefficient, $\tilde{C}_M$, Versus k, for Delta Wing Oscillating in Pitch With AR = 4, $\tau = 0.005$, M = 0, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W = 2c$. Comparison With Results of Reference 18.

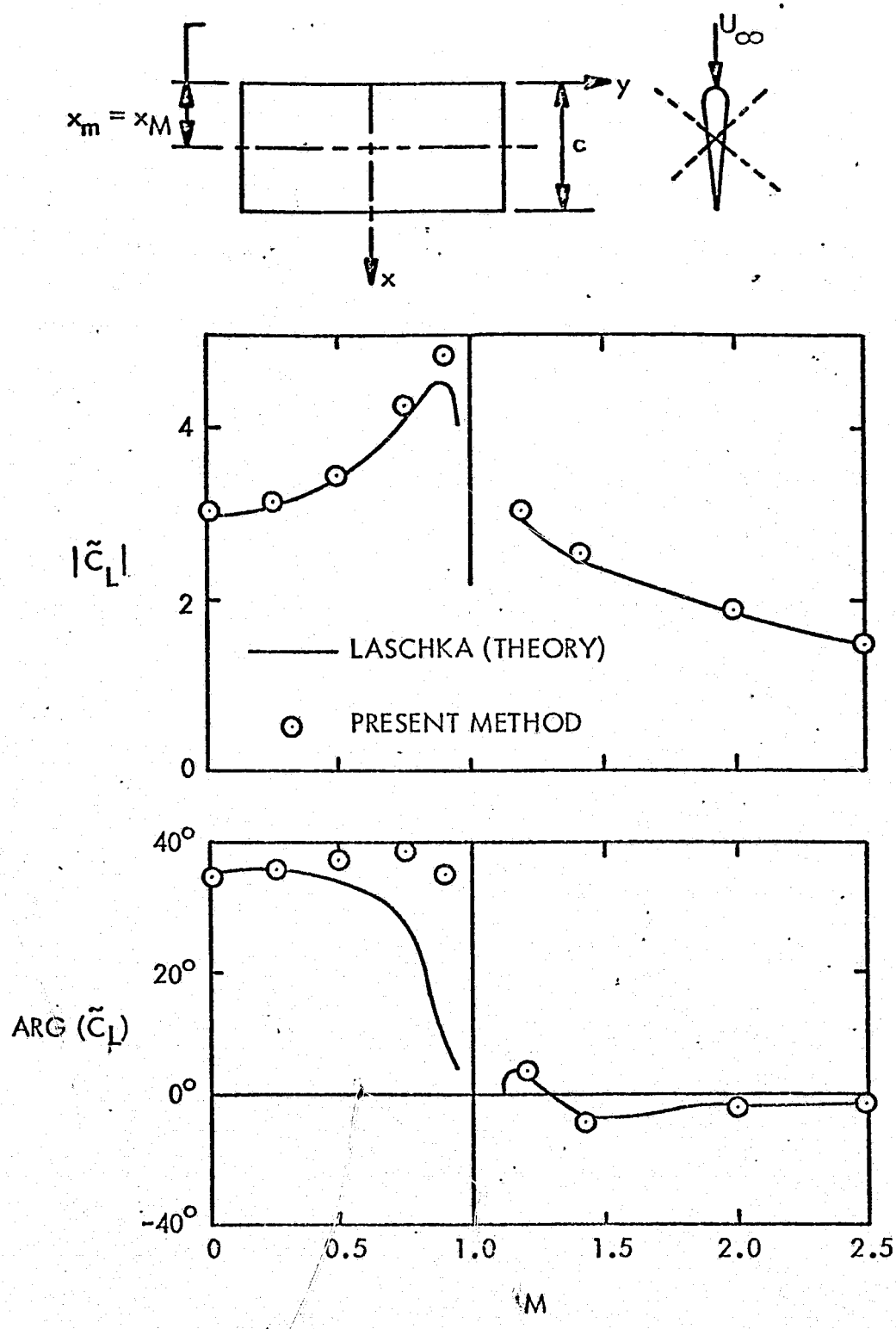

Figure 17a. Lift Coefficient, $\tilde{C}_L$, Versus M, for Rectangular Wing Oscillating in Pitch, With AR = 2, $\tau = 0.001$, k = 1, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W/c = 2$, $N_D = 30$. Comparison With Results of Reference 18.

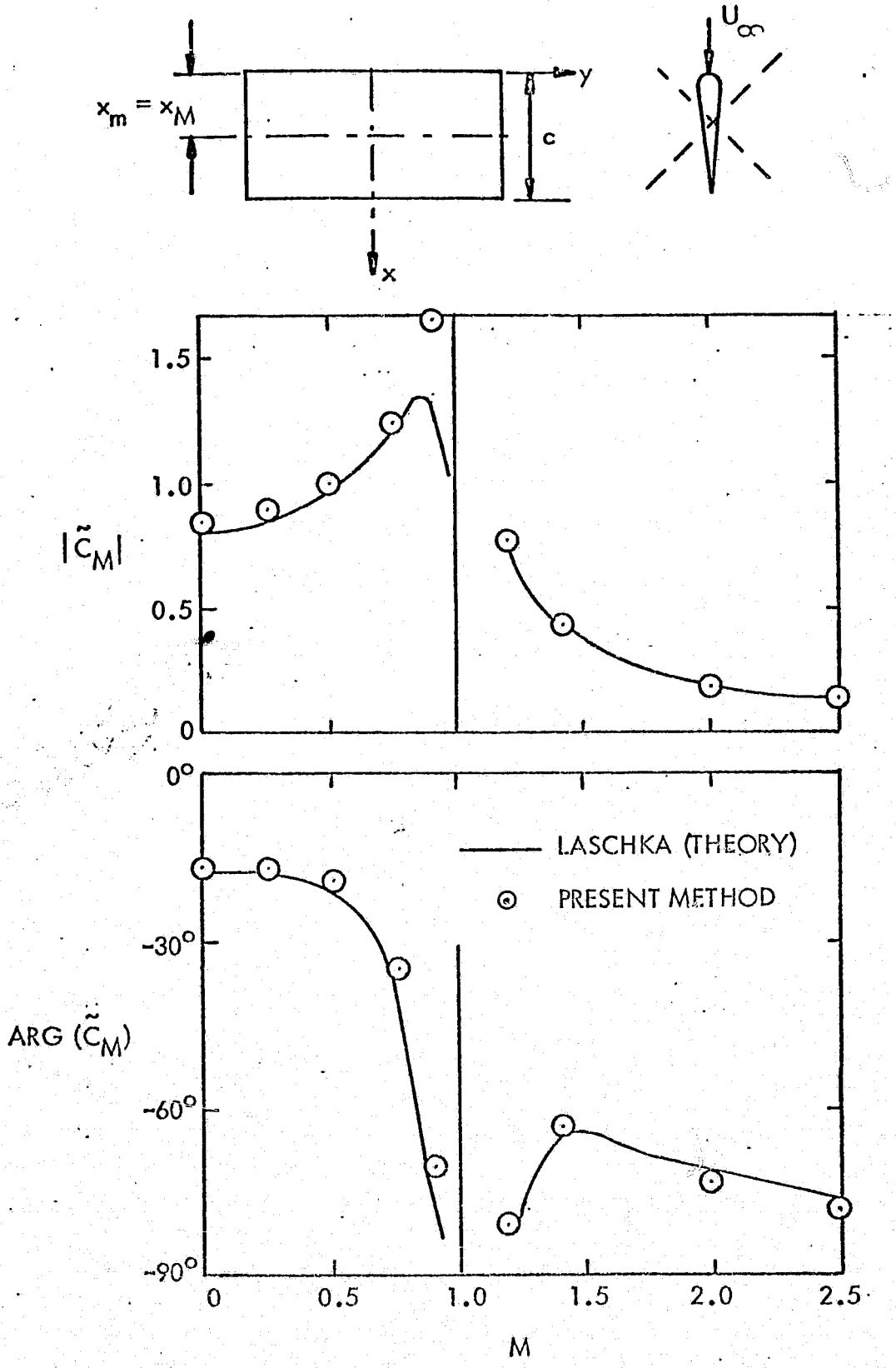

Figure 17b. Moment Coefficient, $\tilde{C}_M$, Versus M, for Rectangular Wing Oscillating in Pitch, for AR = 2, $\tau = 0.001$, k = 1, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W/c = 2$, $N_D = 30$. Comparison With Results of Reference 18.

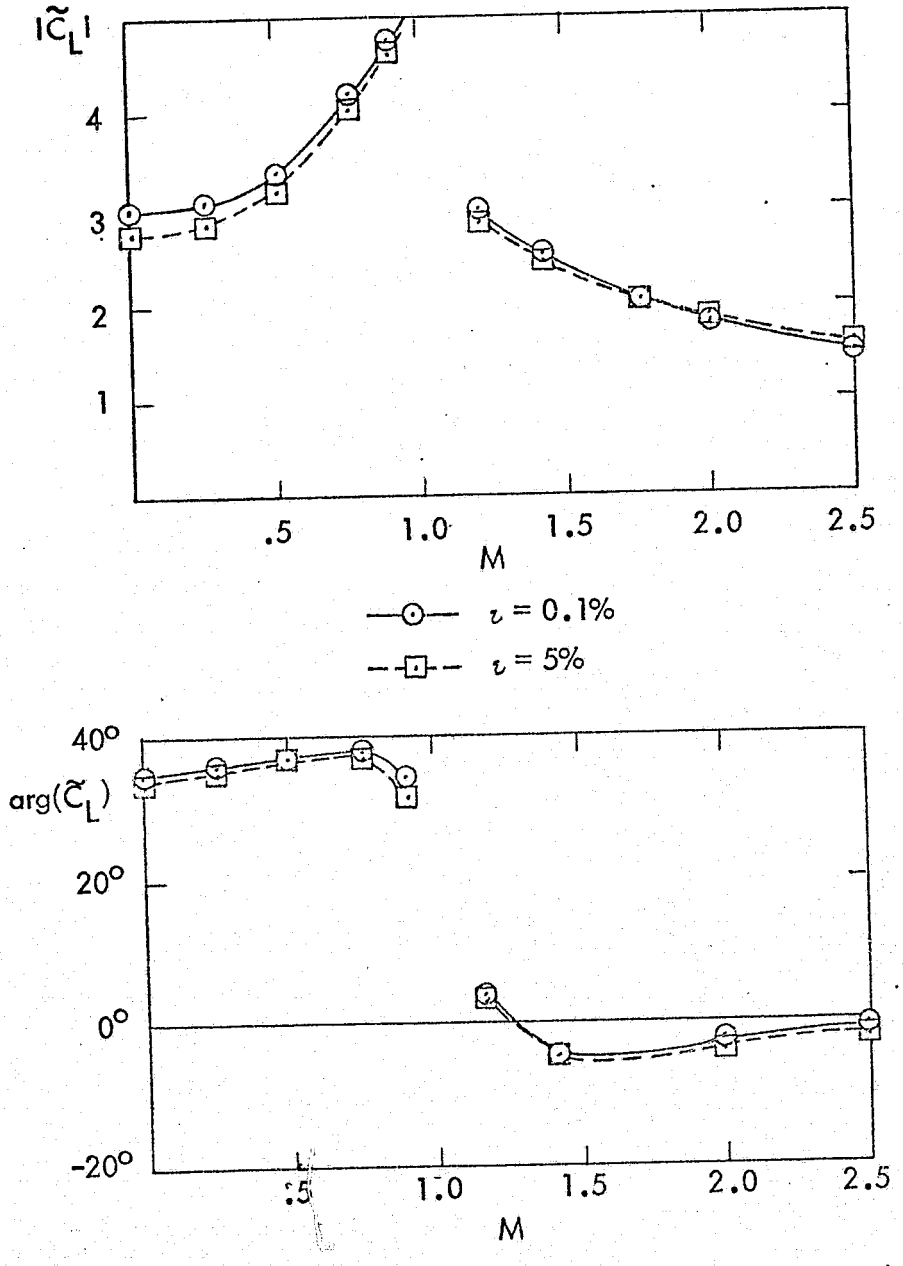

Figure 18a. Thickness effects in oscillatory subsonic and supersonic flows. Results are total lift coefficient, $\tilde{C}_L$, versus M, for a rectangular wing with AR = 2 and $K = \omega c/2U_\infty = 1.0$ oscillating in pitch about axis $X = X_m$.

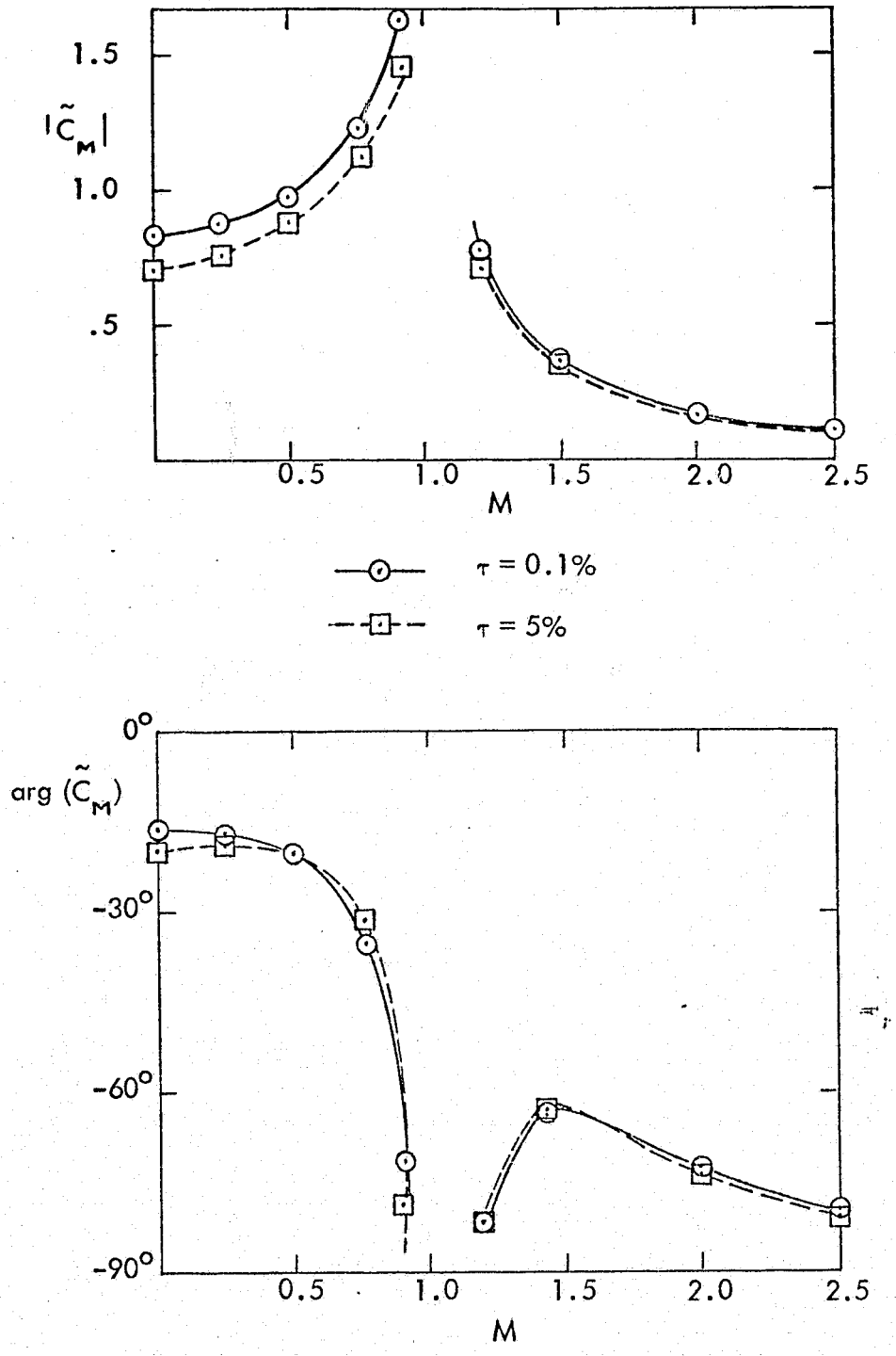

Figure 18b. Thickness Effect in Oscillatory Subsonic and Supersonic Flows. Results are total moment coefficient, $\tilde{C}_M$, versus M, for a rectangular wing with AR = 2 and $k = \omega c/2U_\infty = 1.0$ oscillating in pitch about axis $x = x_m$.

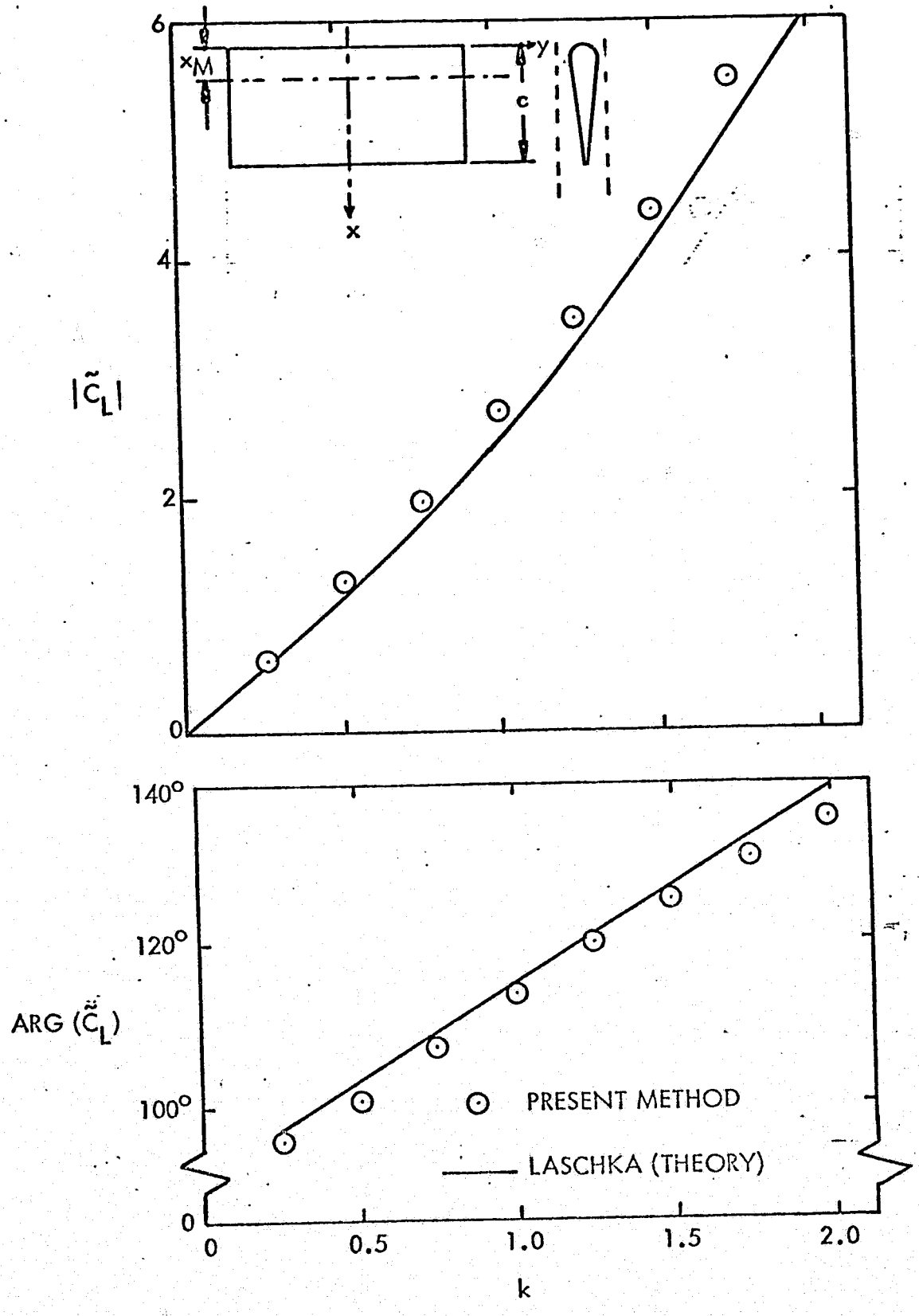

Figure 19a. Lift Coefficient, $\tilde{C}_L$, Versus k, for Rectangular Wing Oscillating in Plunge, With AR = 2, $\tau = 0.001$, M = 0, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W = 2c$. Comparison With Results of Reference 18.

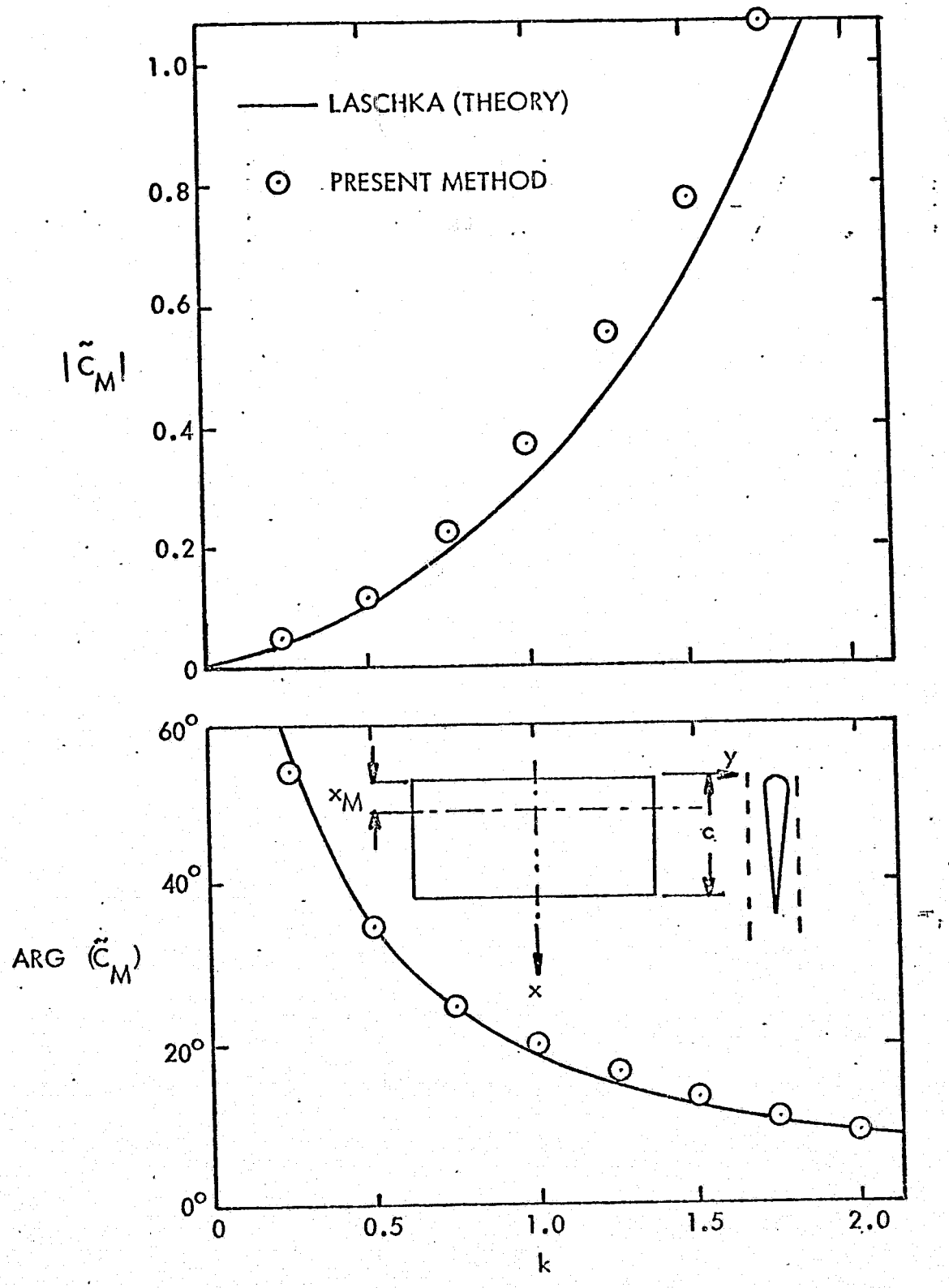

Figure 19b. Moment Coefficient, $\tilde{C}_M$, Versus k for a Rectangular Wing Oscillating in Plunge, With AR = 2, τ = 0.001, M = 0, N_x = 10, N_y = 6, N_W = 20, L_W = 2c. Comparison With Results of Reference 18.

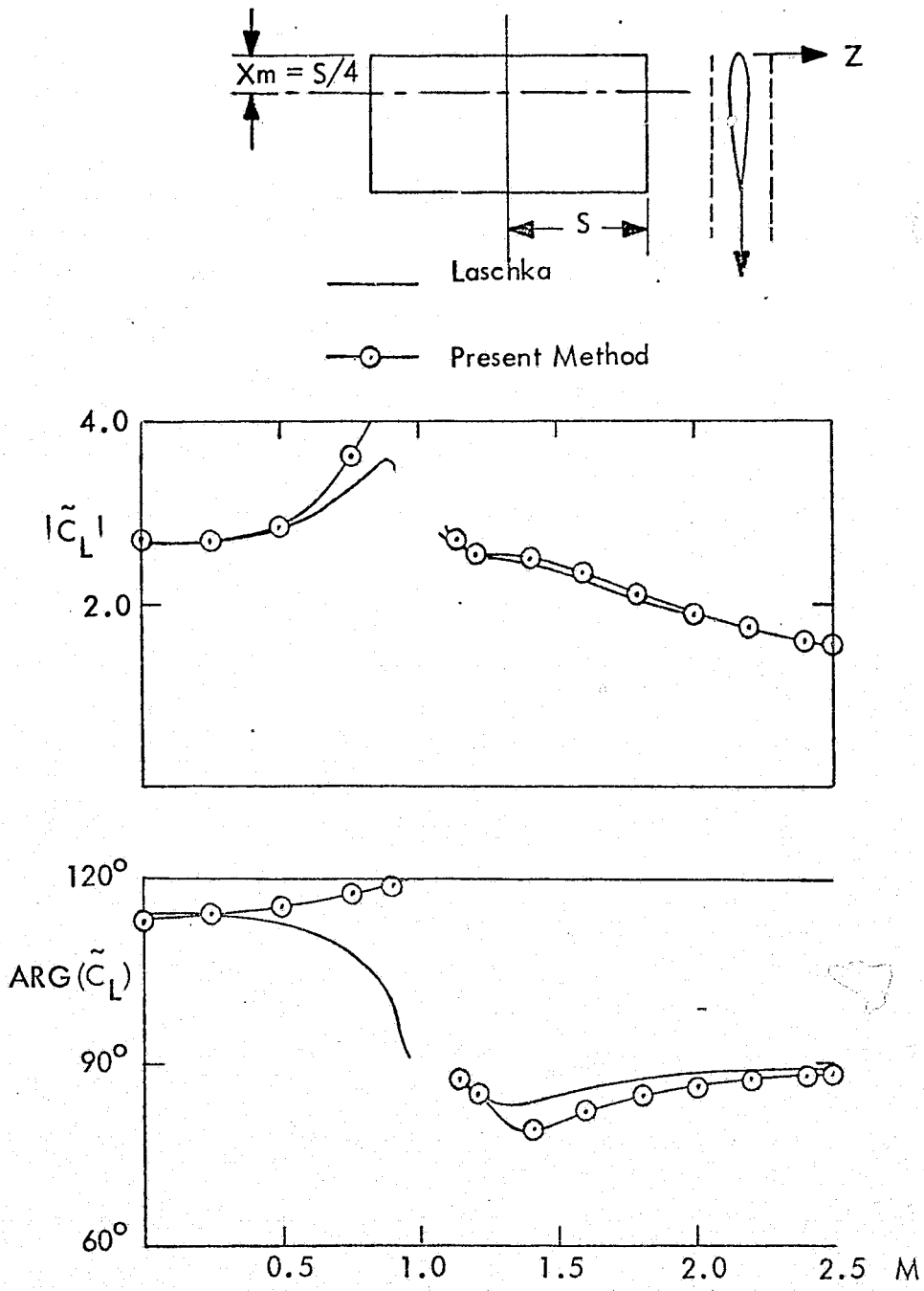

Figure 20a. Total lift coefficient, C_L, versus M, for a rectangular wing with AR = 2, τ = 0.1% and $k = \omega c/2U_\infty$ = 1.0 oscillating in plunge. Results are related to both subsonic and supersonic flows.

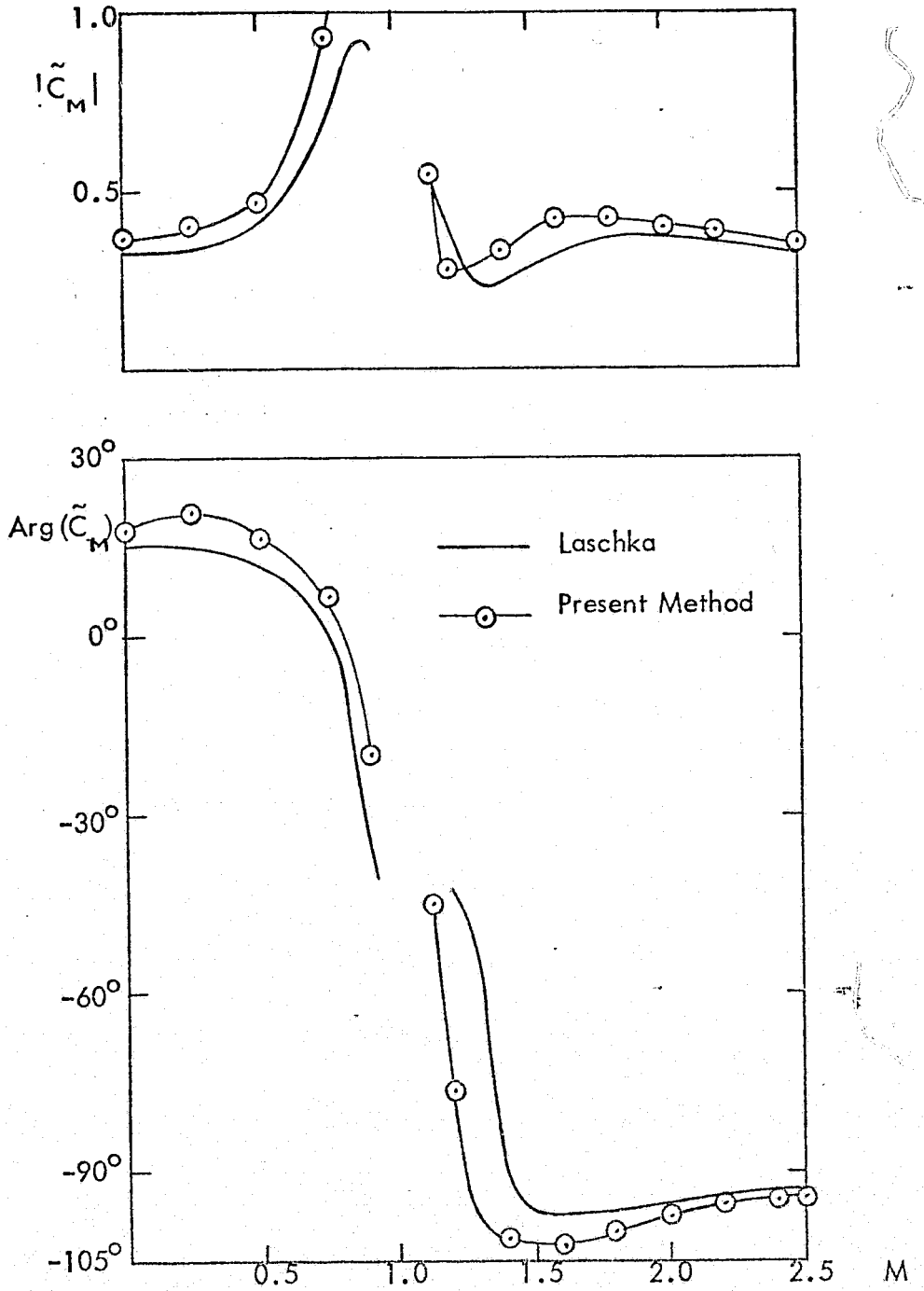

Figure 20b. Total moment coefficient, C_{M_α}, versus M, for a rectangular wing with AR = 2, $\tau = 0.1\%$ and $k = \omega c/2U_\infty$ oscillating in plunge. Results are related to both subsonic and supersonic flows (same problem as Figure 20a).

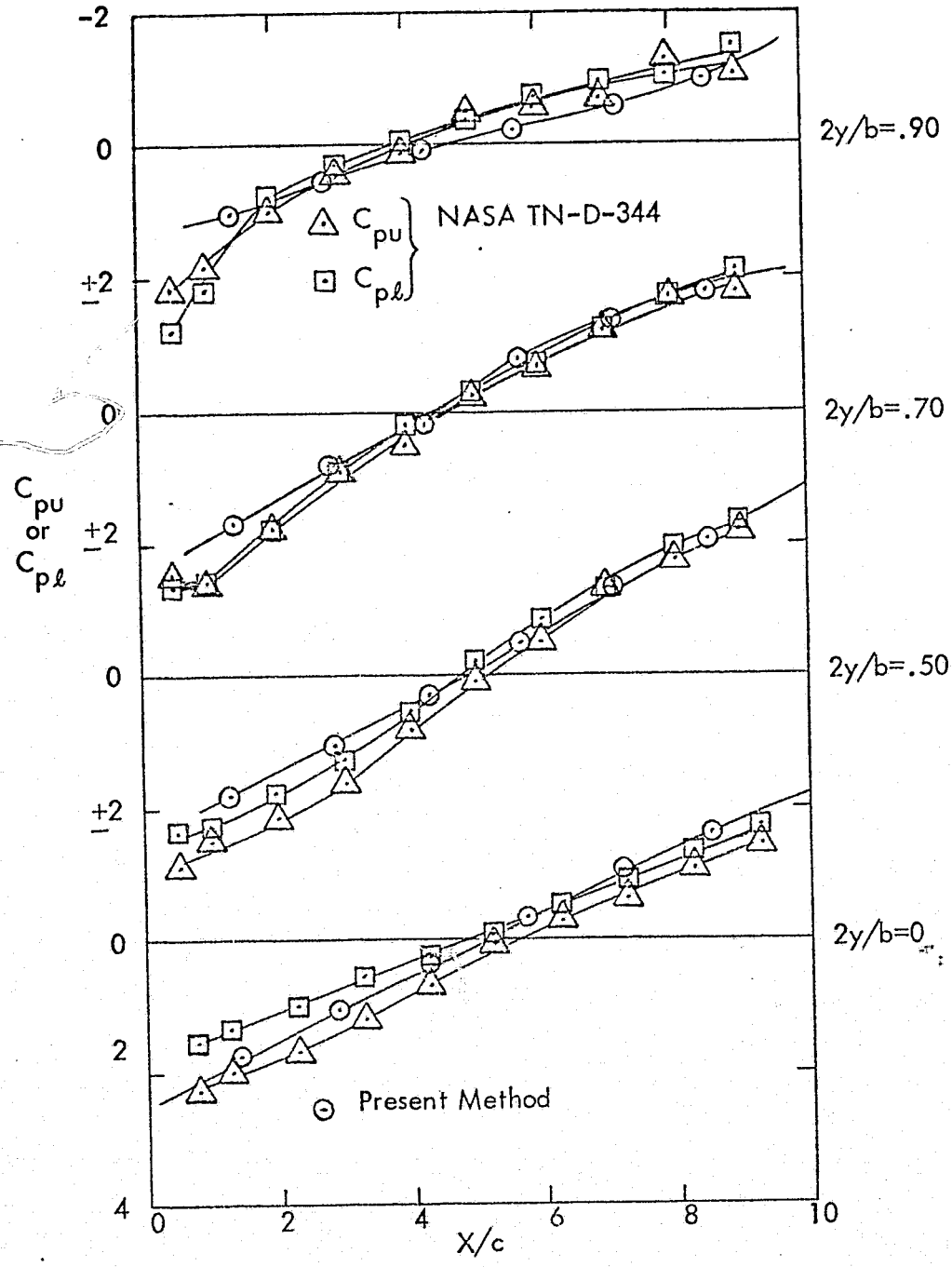

Figure 21. The pressure distribution on a symmetric rectangular wing with AR = 3, τ = 5%, α = 0°, M = 1.3 and NX = NY = 7 for the comparison with results of Ref. 15.

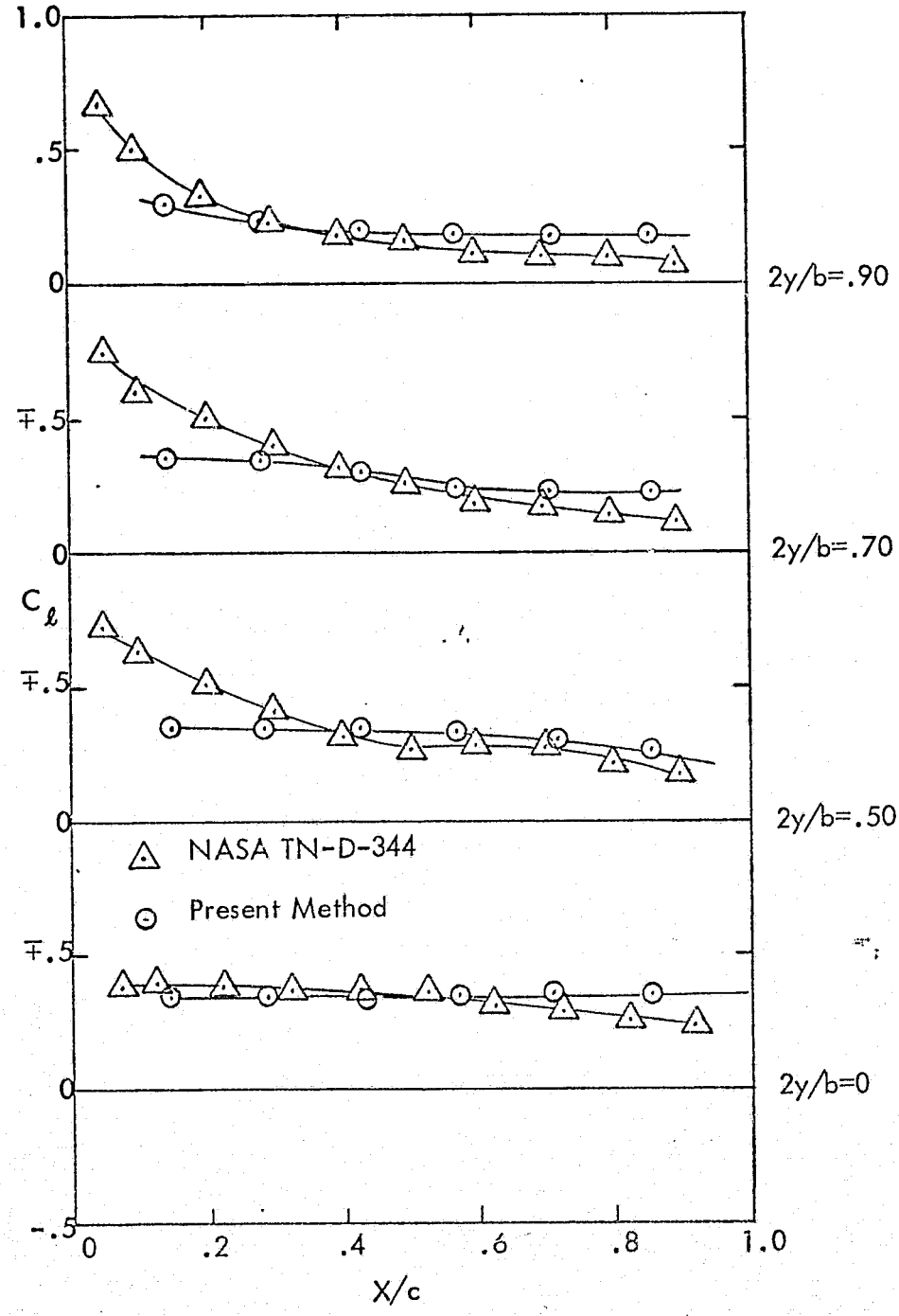

Figure 22a. The lift distribution on symmetric rectangular wing with AR = 3, τ = 5%, α = 5°, M = 1.3 and NX = NY = 7 for the comparison with results of Ref. 15.

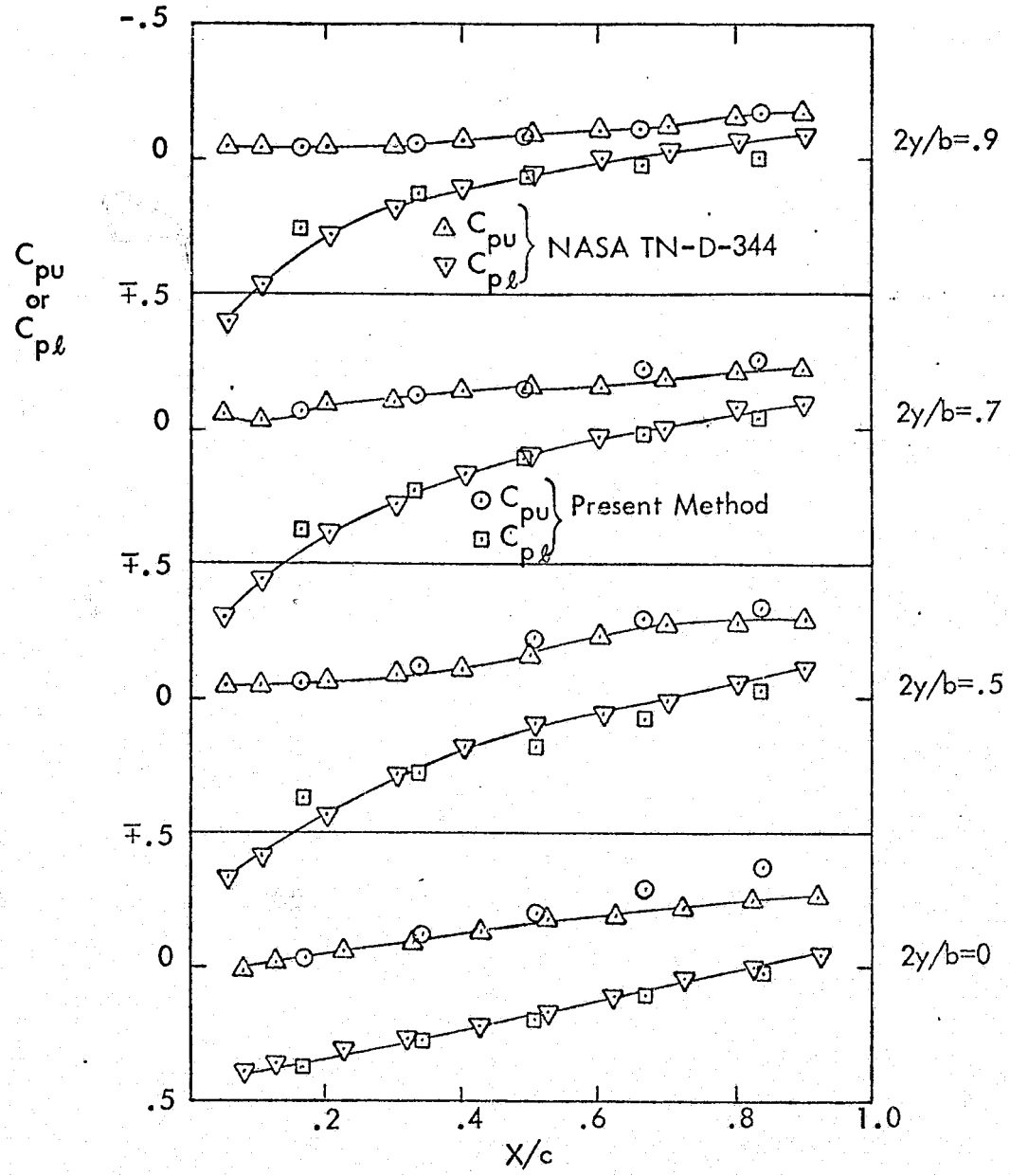

Figure 22b. The distribution of C_p on the upper and lower surfaces of a symmetric rectangular wing with AR = 3, τ = 5%, $\alpha = 5^o$, M = 1.3 and NX = NY = 6 for comparison with results of Ref. 15.

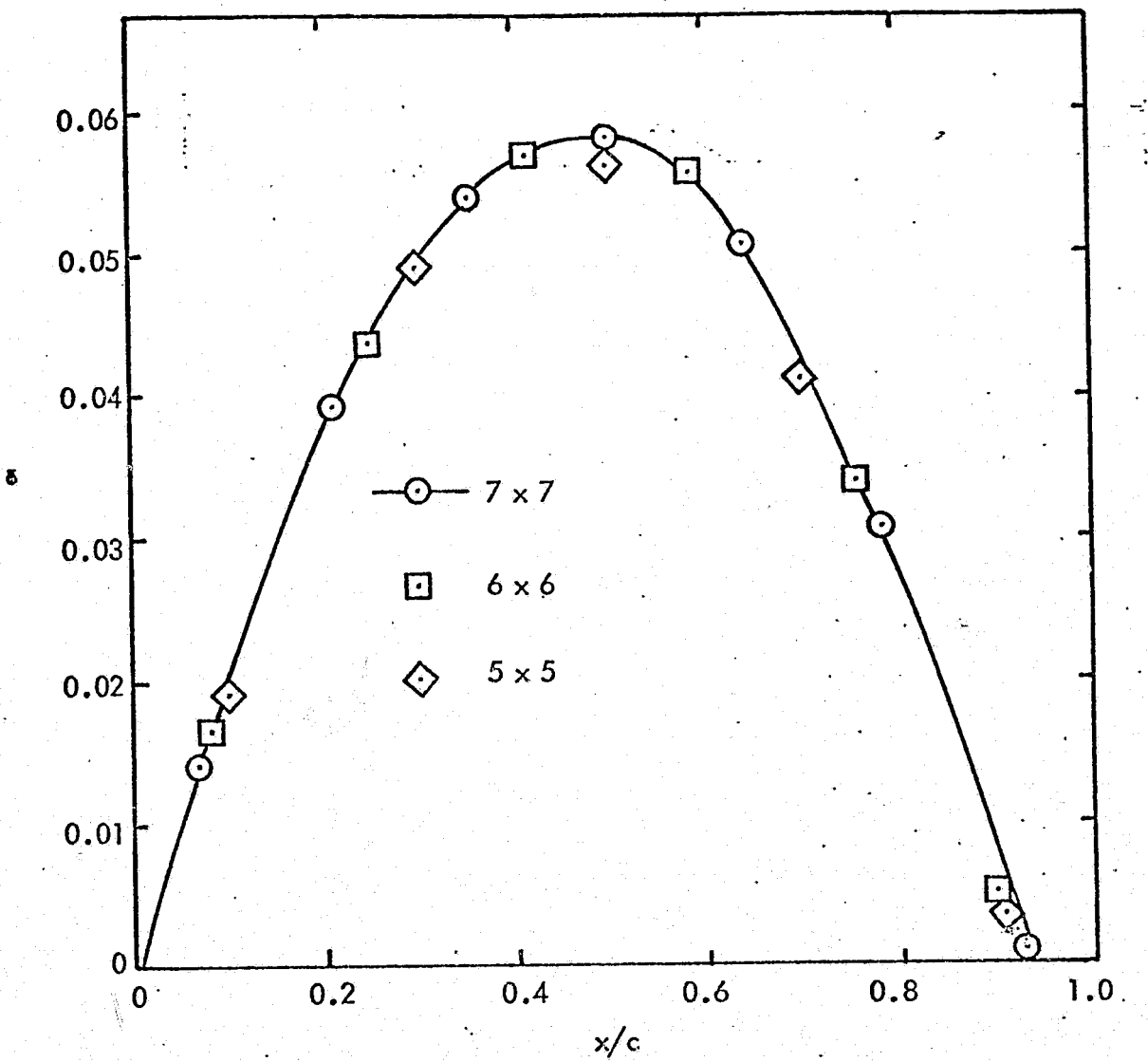

Figure 23. Analysis of Convergence: Potential Distribution, Φ, Versus x/c, at y = 0, for Rectangular Wing With Biconvex Section, in Steady Supersonic Flow, for AR = 3, τ = 0.05, M = 1.3, $\alpha = 0^{o}$, $N_D = 3N_x$, NX = NY = 5,6,7.

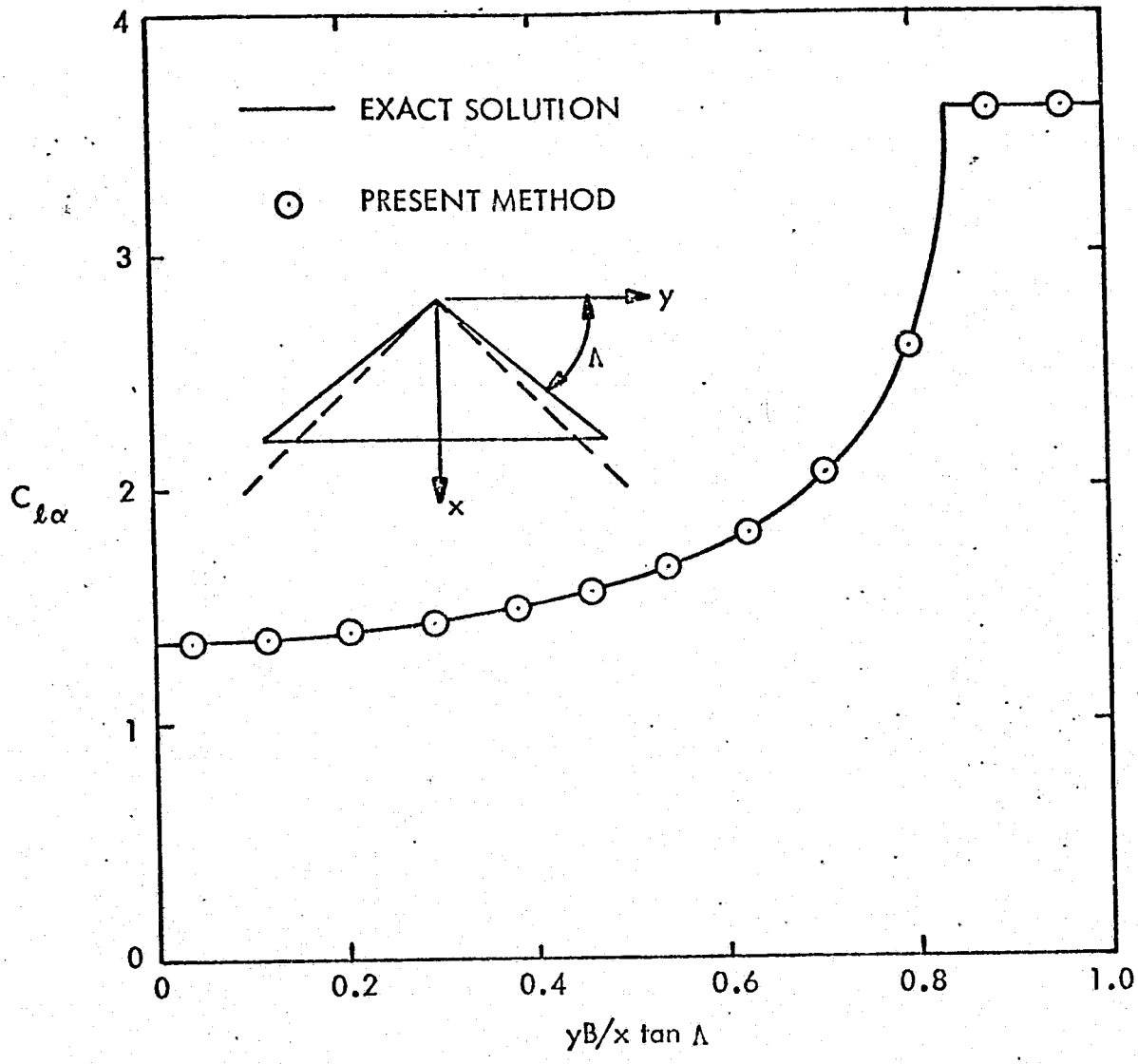

Figure 24. Lift Distribution Coefficient, $C_{\ell\alpha}$, for Delta Wing With Supersonic Leading Edge, in Steady Supersonic Flow, With B/tan Λ = 1.2, $\tau = 0$, $N_x = 8$, $N_y = 12$. Comparison With Exact, Conical-Flow Solution, Reference 20.

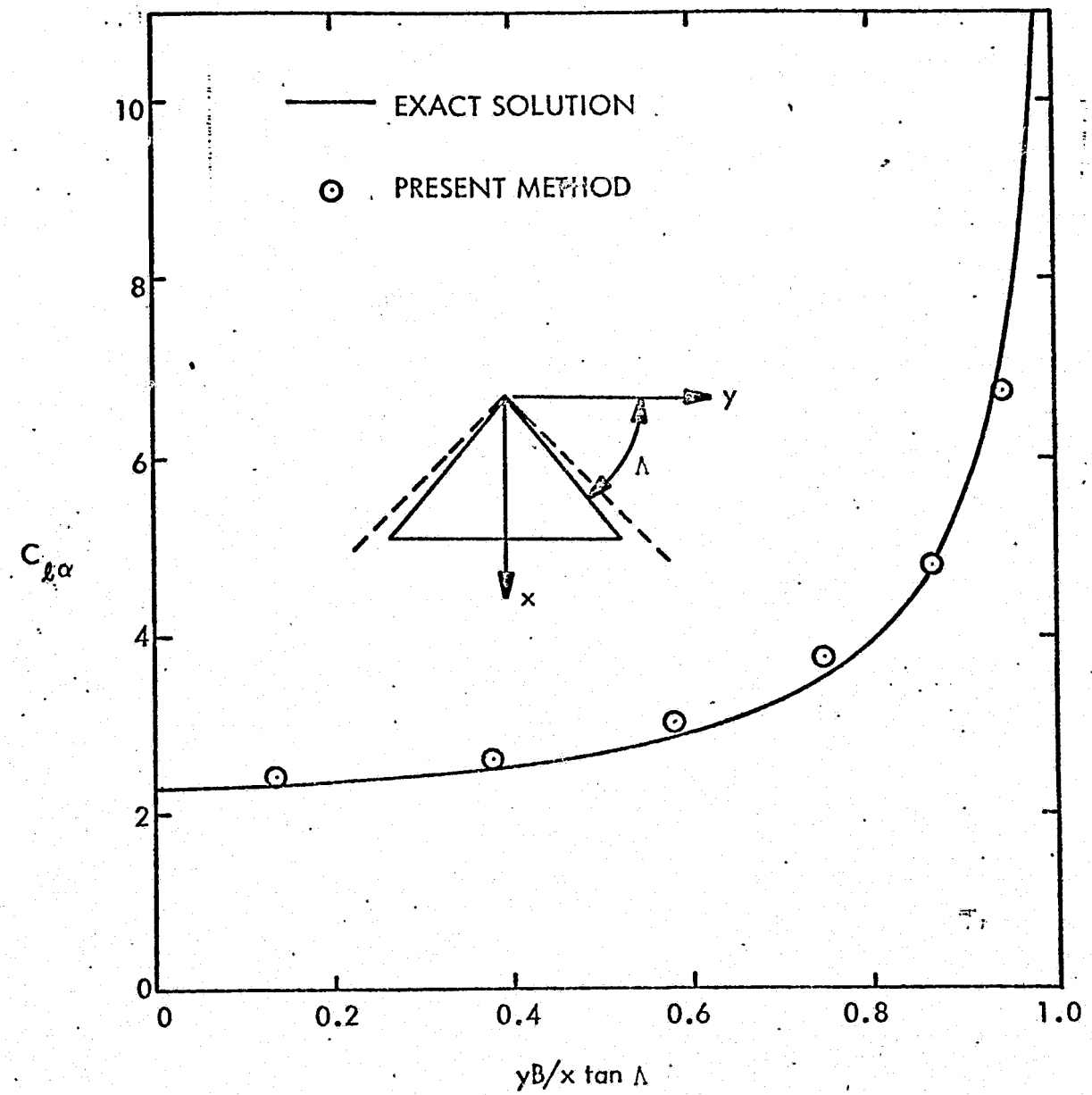

Figure 25. Lift Distribution Coefficient, $C_{\ell\alpha}$, for Delta Wing With Subsonic Leading Edge, in Steady Supersonic Flow, With B/tan Λ = 0.833, π = 0, $N_x = N_y = 7$. Comparison With Exact Conical-Flow Solution, Reference 20.

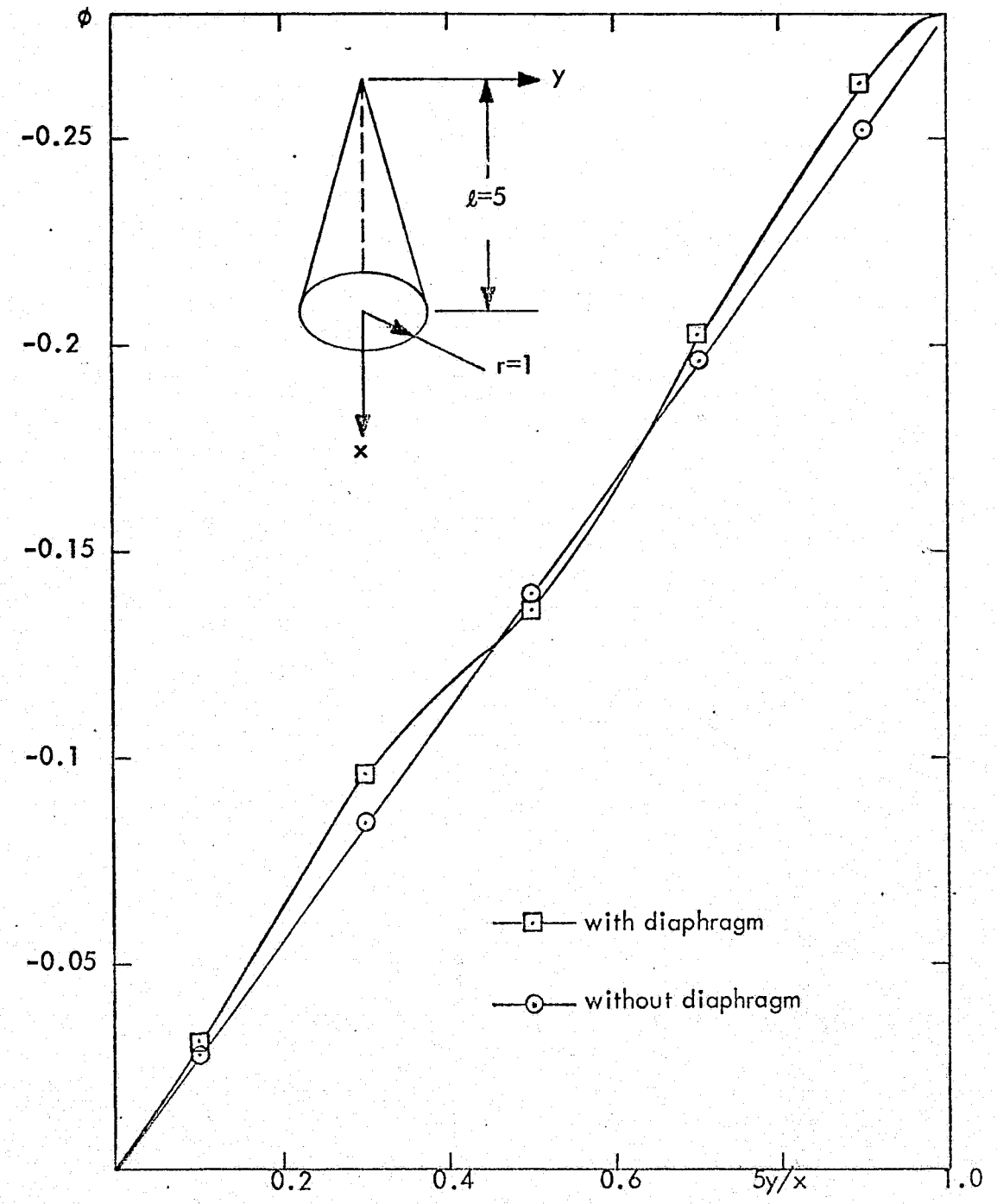

Figure 26. Study of existence of diaphragm; the distribution of velocity potential, ϕ, for a circular cone with subsonic leading edge, $r = 1$, $\ell = 5$, $M = 2.0$ and $NX = 12$, $NY = 5$. Results obtained with and without diaphragm are compared.

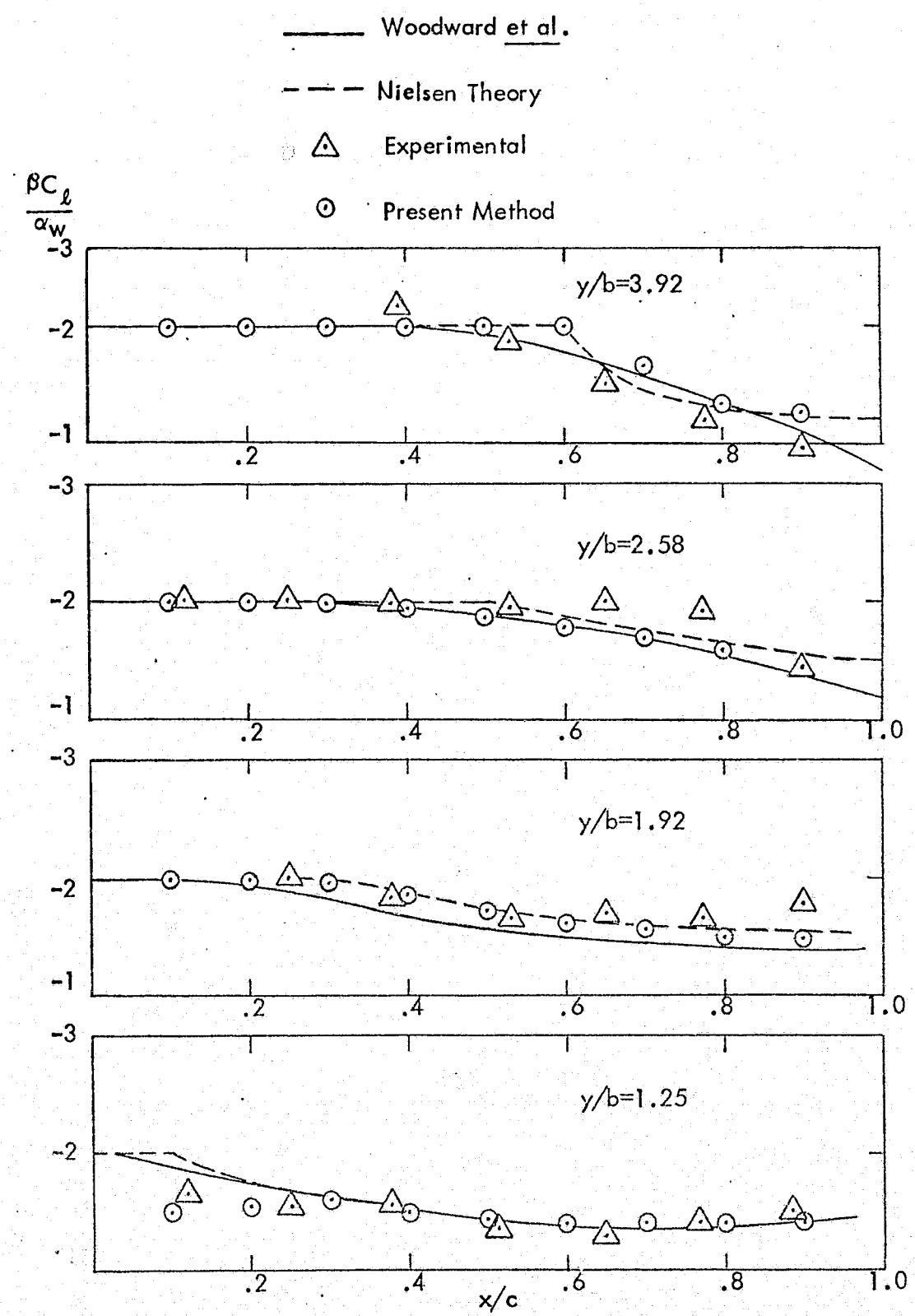

Figure 27a. The distribution of $\beta C_\ell/\alpha_w$ on the wing section for a wing-body configuration (shown in Figure 27c) with M = 1.48, $\alpha_w = 1.92^\circ$ and $\alpha_B = 0^\circ$, for the comparison with results of Ref. 21.

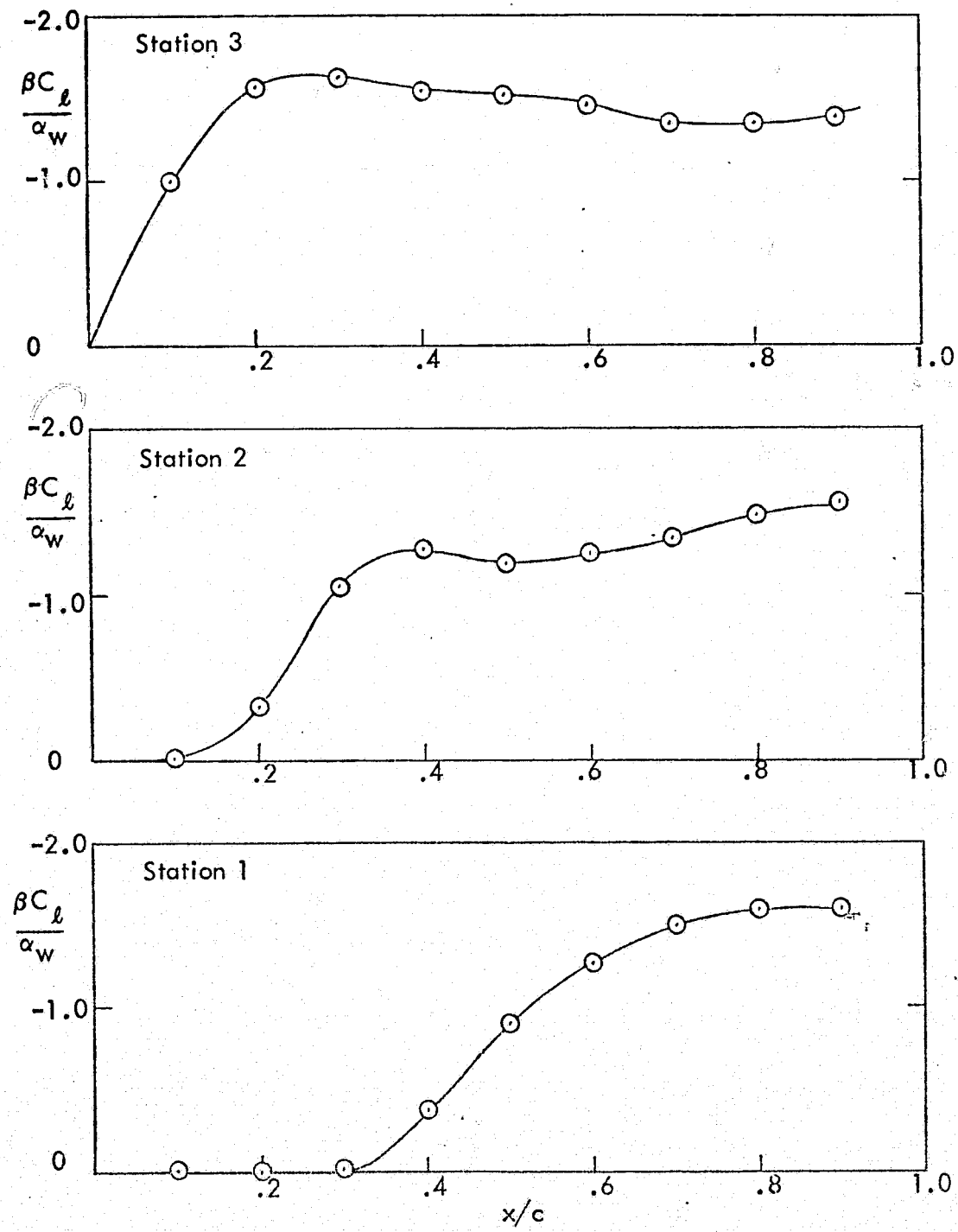

Figure 27b. The distribution of $\beta C_\ell/\alpha_w$ on the fuselage at three circumferential stations for the same problem of Figure 27a.

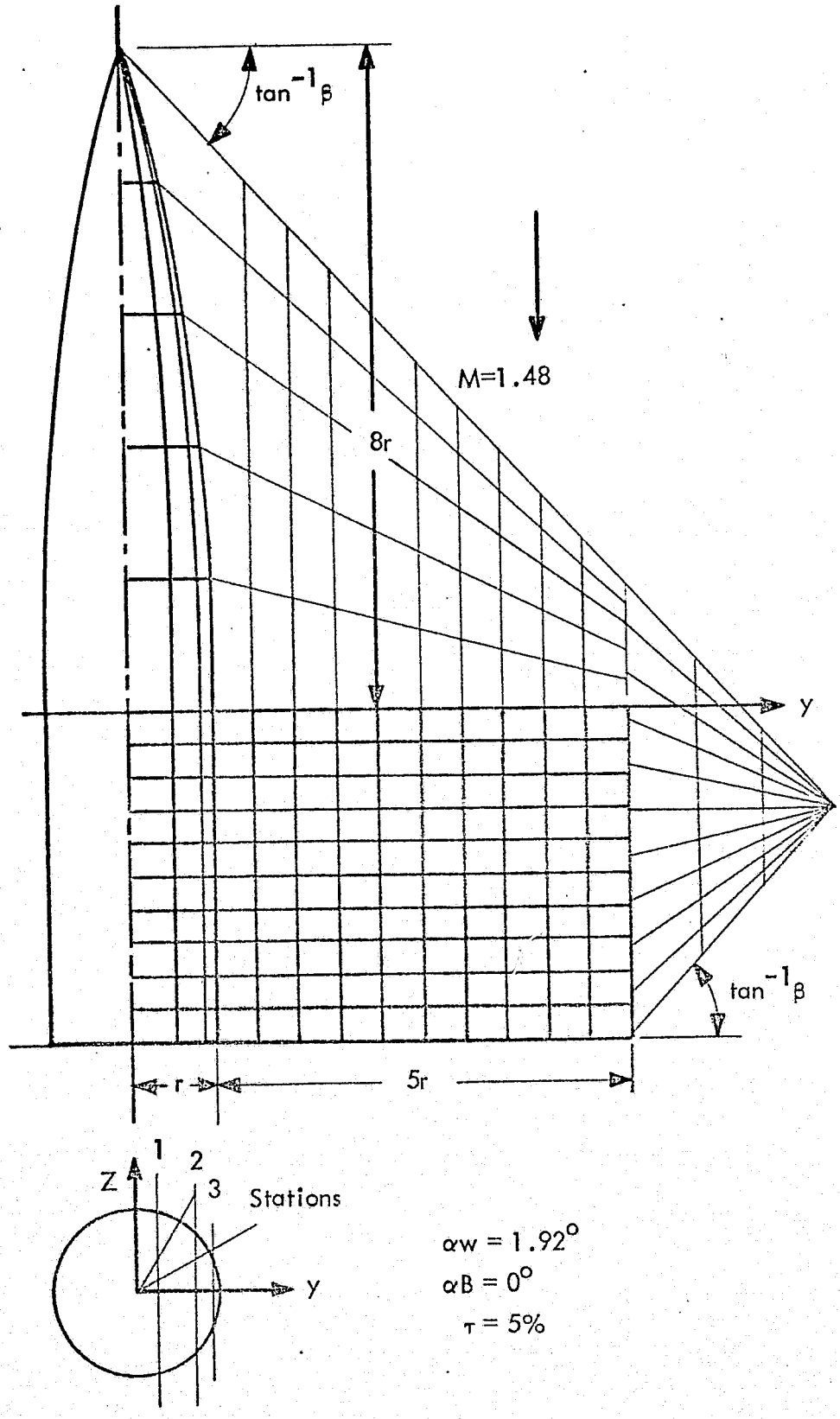

Figure 27c. Wing-body Configuration in Supersonic Flow

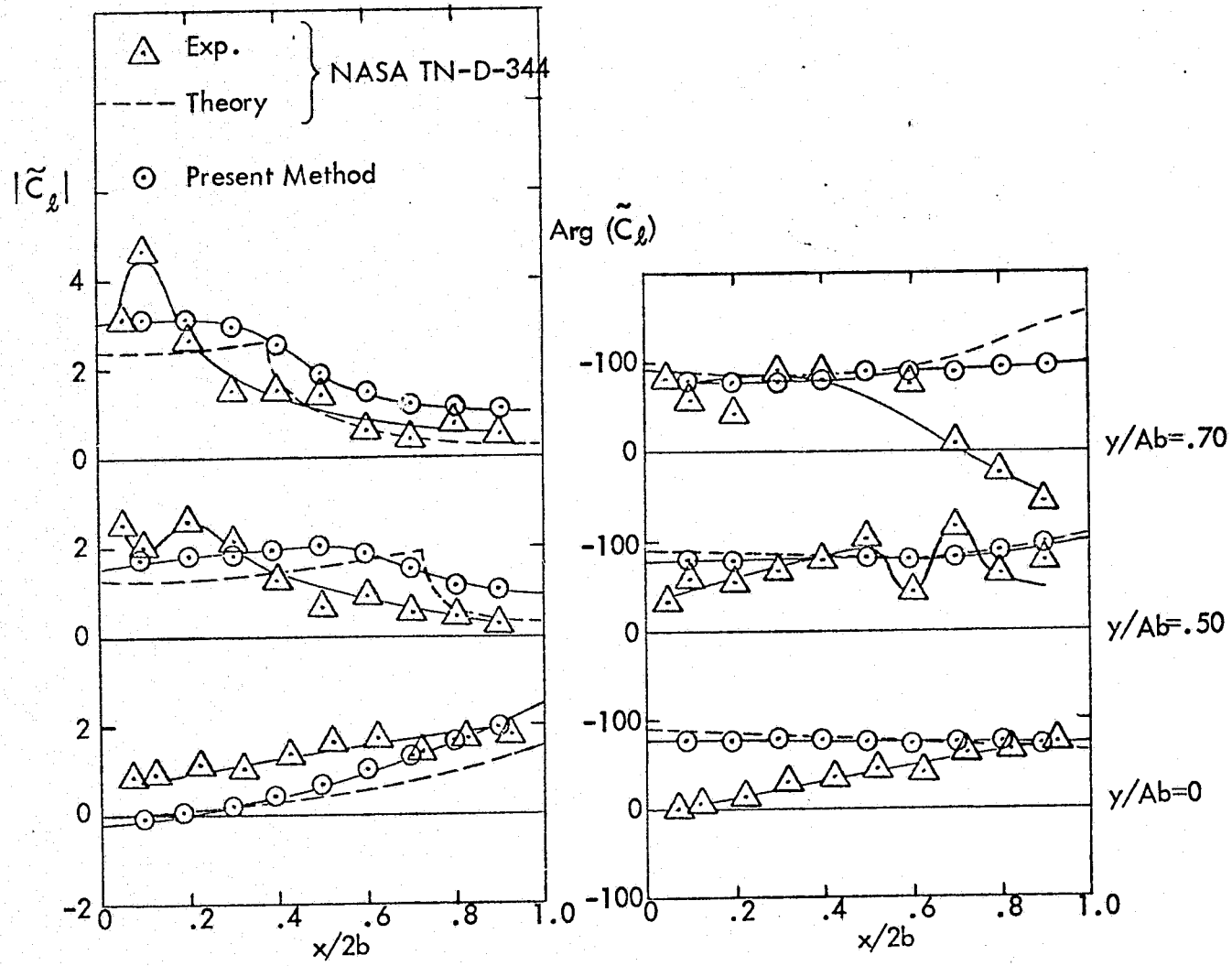

Figure 28. Lift coefficient, $\tilde{C}_l$, for a rectangular wing oscillating in bending mode with $k = \omega c/2U_\infty = 0.1$, $M = 1.3$, $AR = 3$ and $NX = NY = 10$.

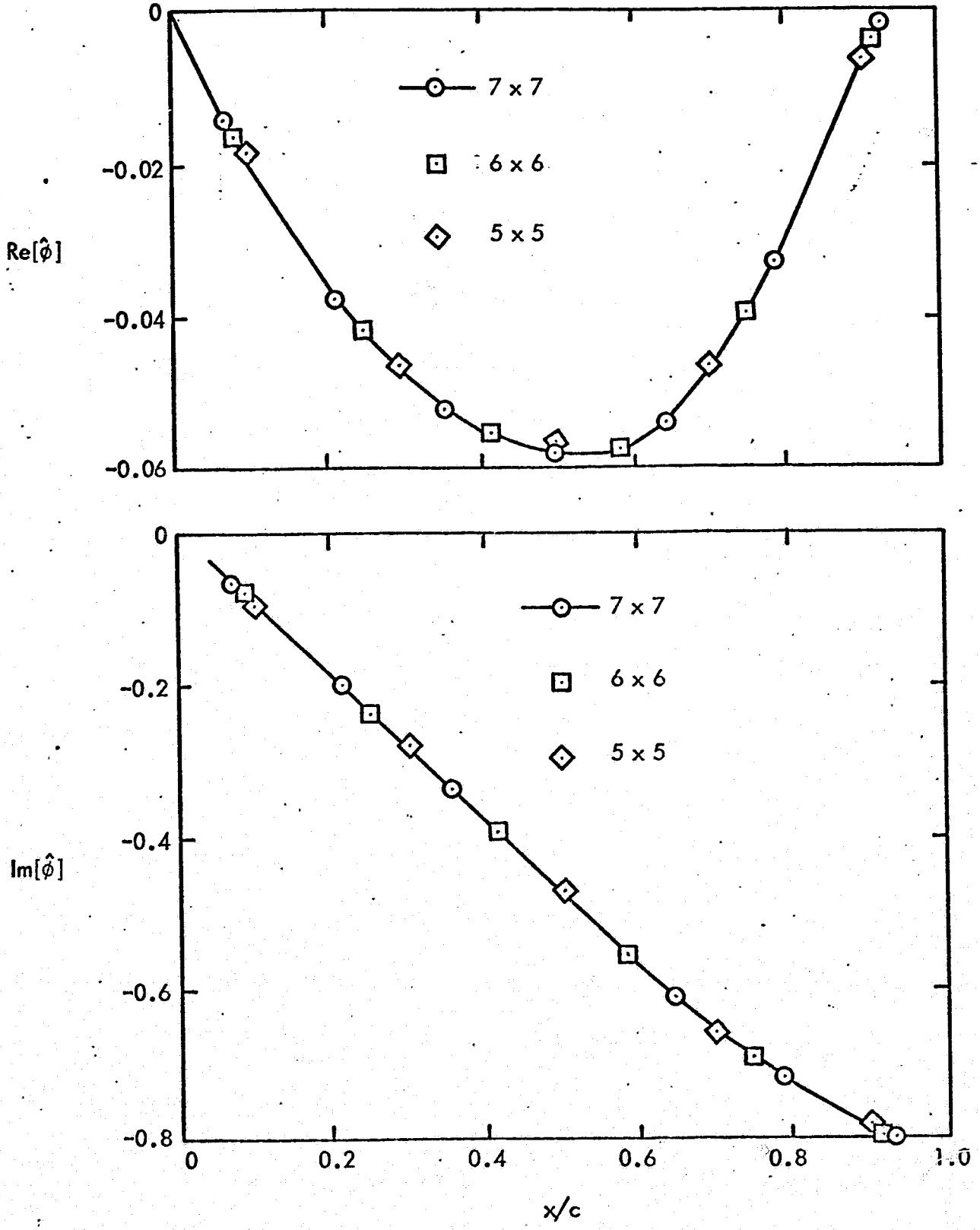

Figure 29 Analysis of Convergence: Distribution of $\hat{\phi} = \tilde{\phi} e^{i\Omega MX}$ Versus x/c, at 2y/b = 0.5, for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Supersonic Flow, for AR = 3, τ = 0.01, M = 1.3, K = 0.1, $\alpha = 0^{\circ}$, $N_D = 3N_x$, NX = NY = 5,6,7.

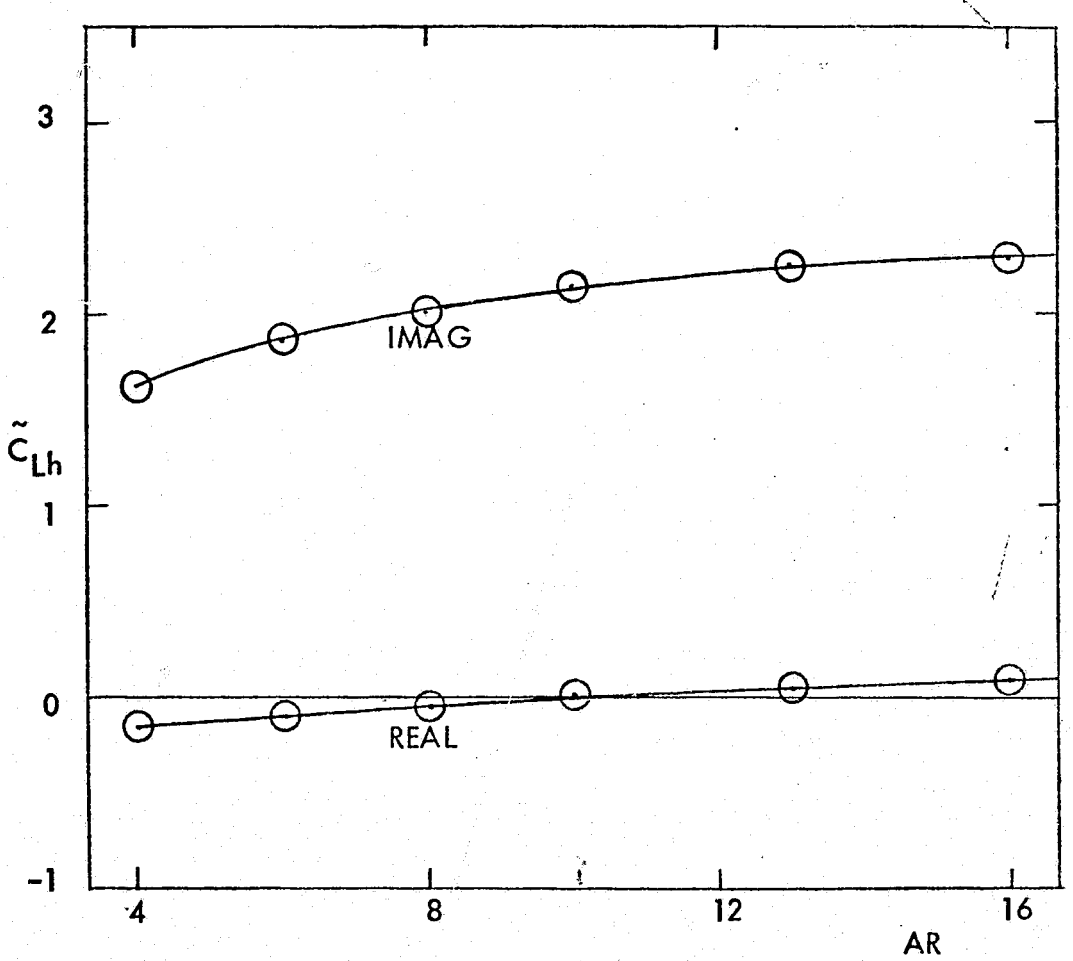

Figure 30a. C_{Lh} as a function of aspect ratio for a rectangular wing with $\tau = 0.1\%$, M = 0 and NX = 8, NY = 10 ($X_{EA} = -0.2C$).

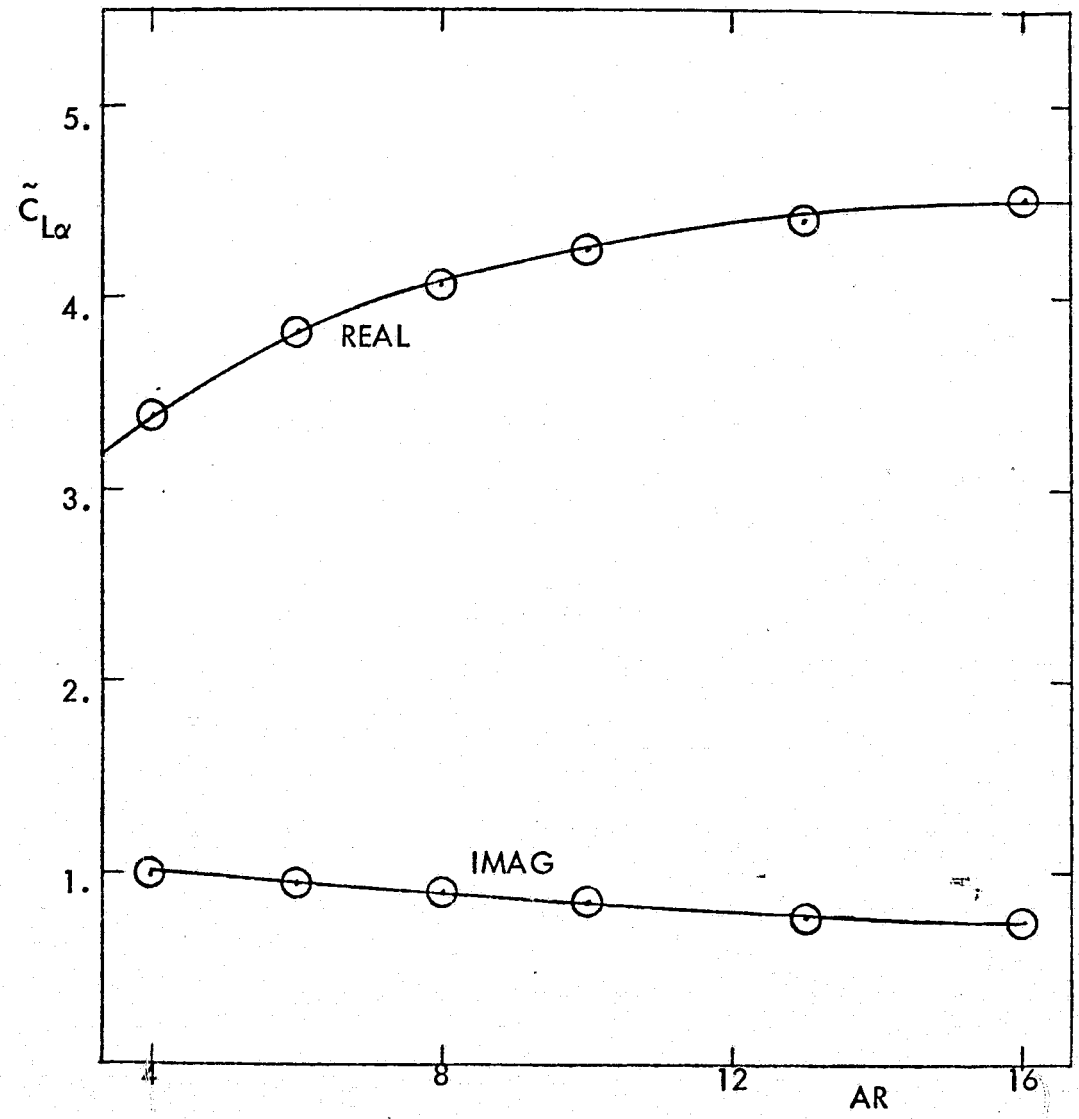

Figure 30b. $C_{L\alpha}$ as a function of aspect ratio for a rectangular wing with τ = .1%, M = 0 and NX = 8, NY = 10 (X_{EA} = -0.2C).

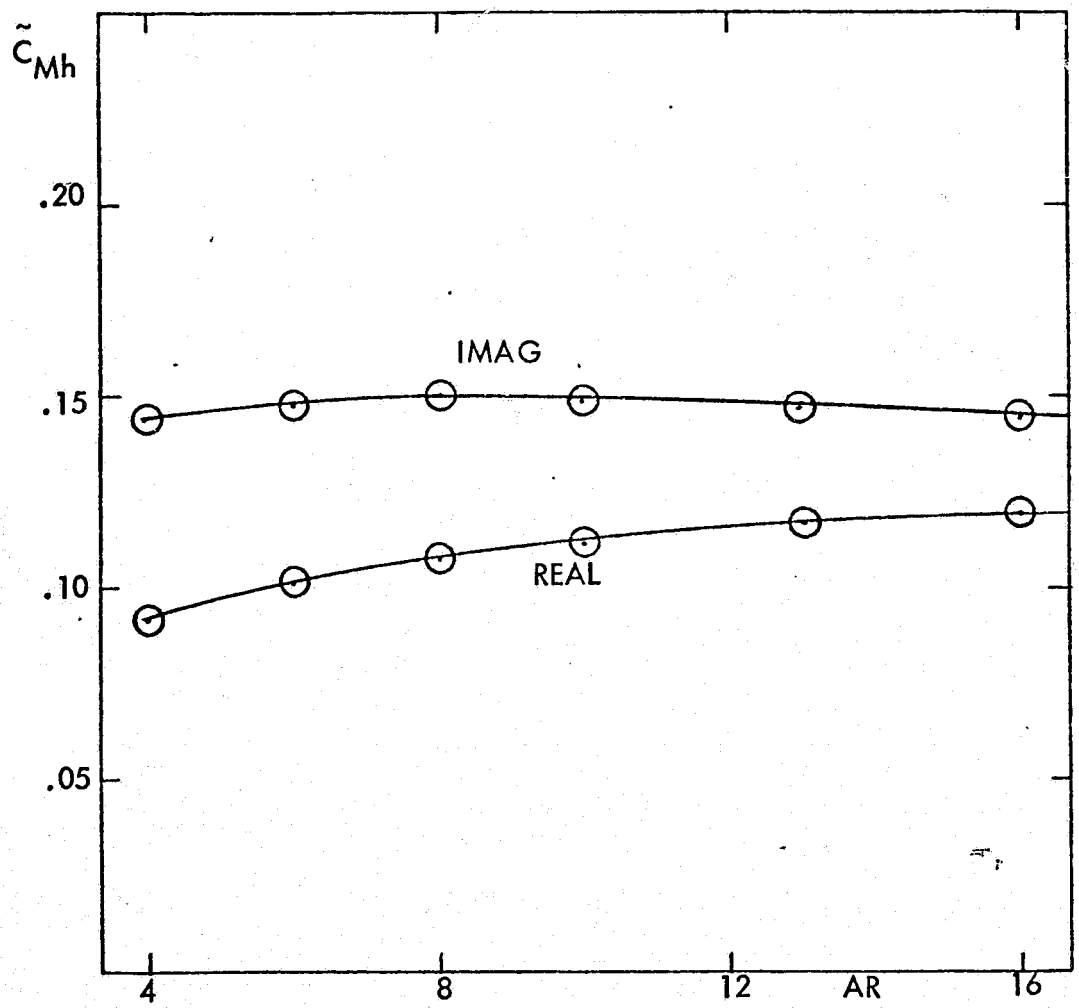

Figure 30c. $\tilde{C}_{Mh}$ as a function of aspect ratio for a rectangular wing with $\tau = 0.1\%$, M = 0 and NX = 8, NY = 10 ($X_{EA} = -0.2C$).

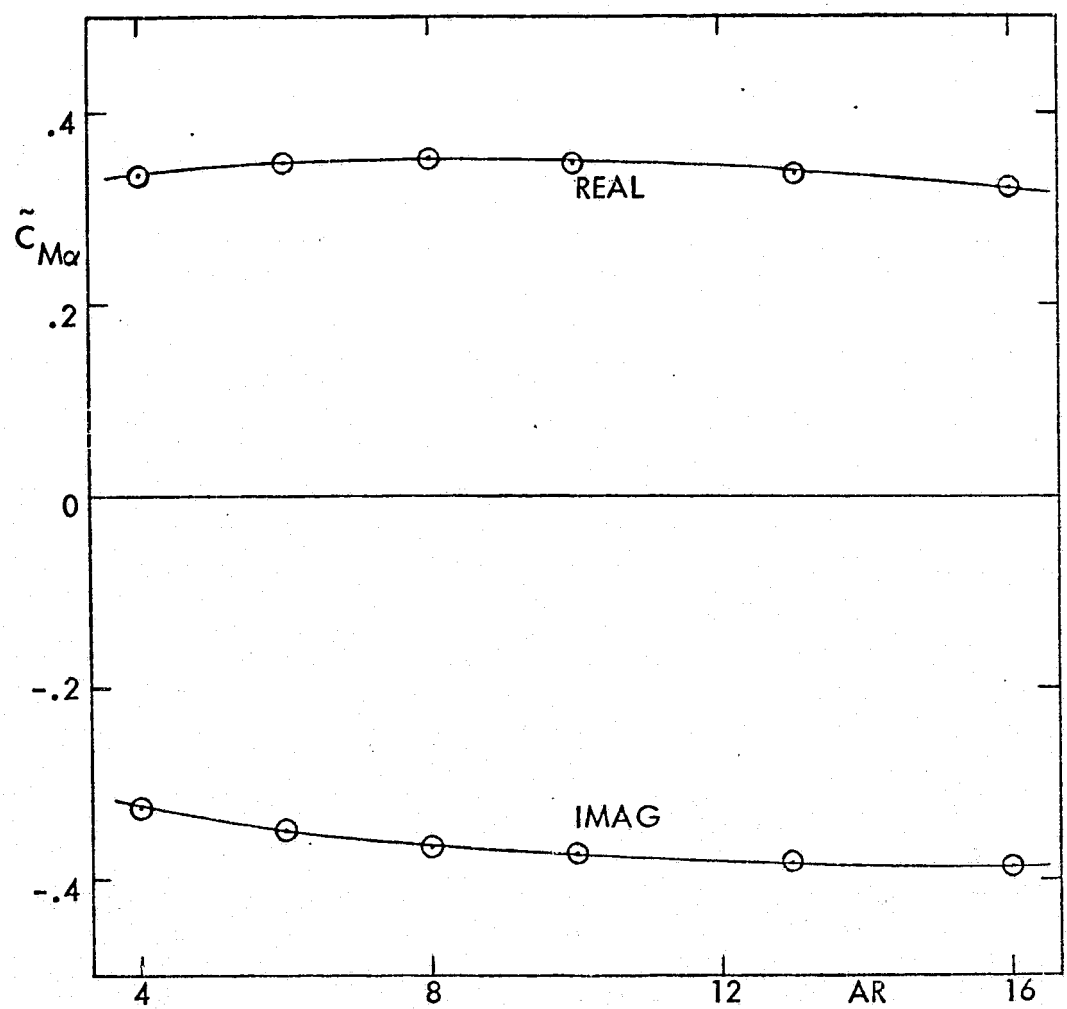

Figure 30d. $\tilde{C}_{M\alpha}$ as a function of aspect ratio for a rectangular wing with $\tau = 0.1\%$, M = 0 and NX = 8, NY = 10 (X_{EA} = -0.2C).

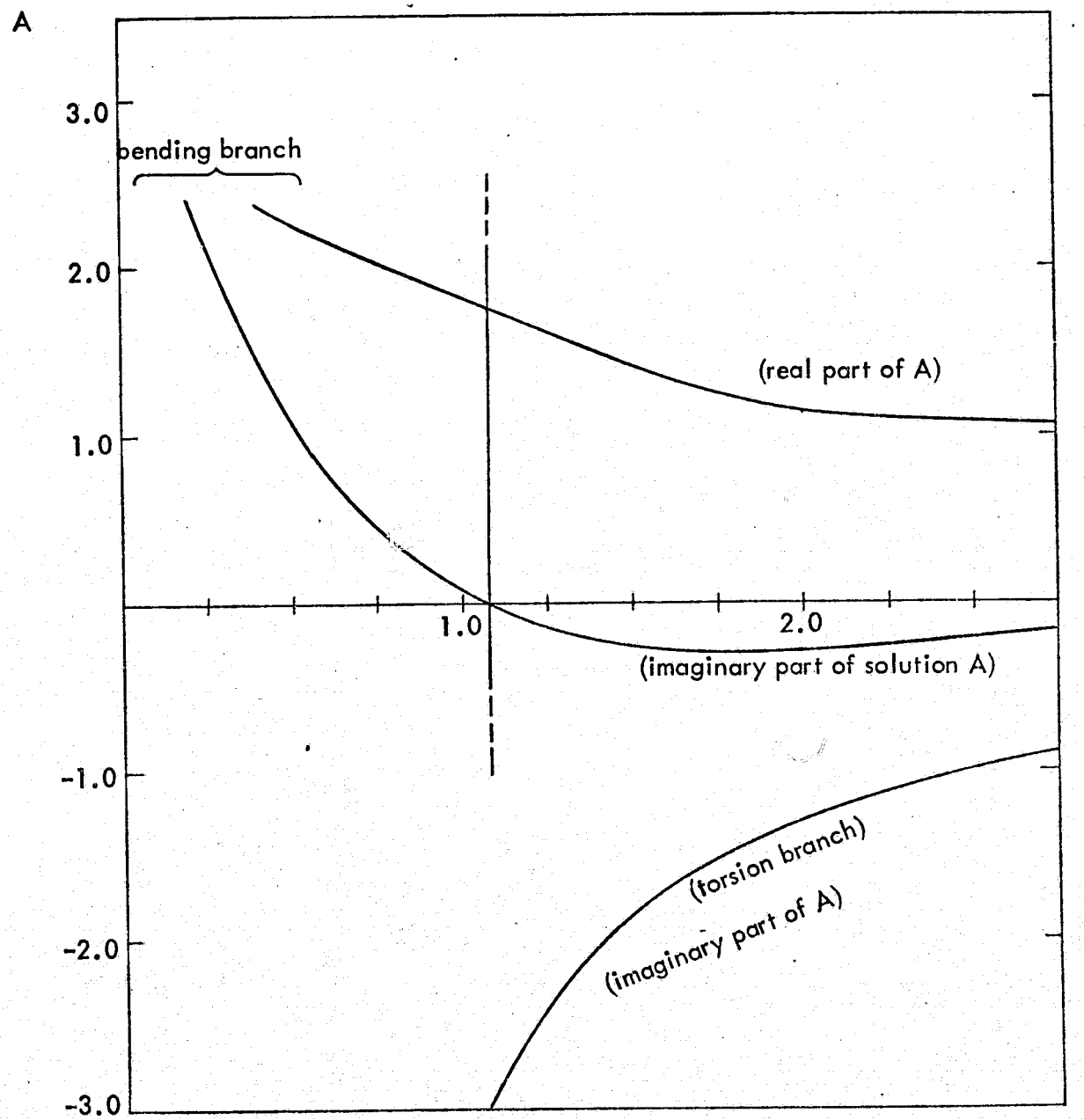

Figure 31. Parameter $A = \omega_\alpha^2/\omega^2$ as a function of reduced frequency, $K = \omega c/2U_\infty$, for a rectangular airfoil with AR = 16, M = 0, $\omega_h/\omega_\alpha = 0.5$, $X_\alpha = 0.2$, $r_\alpha = 0.5$, M = 5 and NX = 8, NY = 10 (X = -0.2C).

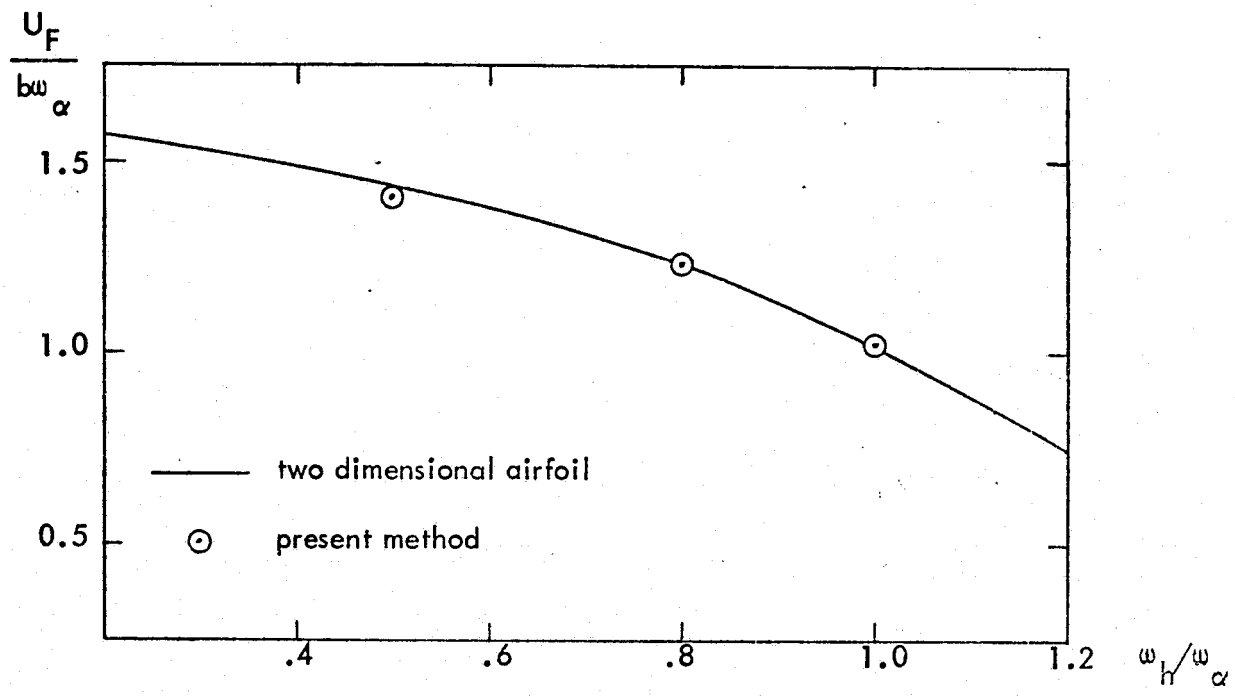

Figure 32a. Flutter speed as a function of ω_h/ω_α for a rectangular wing with AR = 16, M = 0, τ = 0.1%, μ = 5, X_α = 0.2, r_α = 0.5, and NX = 8, NY = 10. Results are compared with exact solution given by two dimensional airfoil theory (Ref. 24) (X_{EA} = -0.2C).

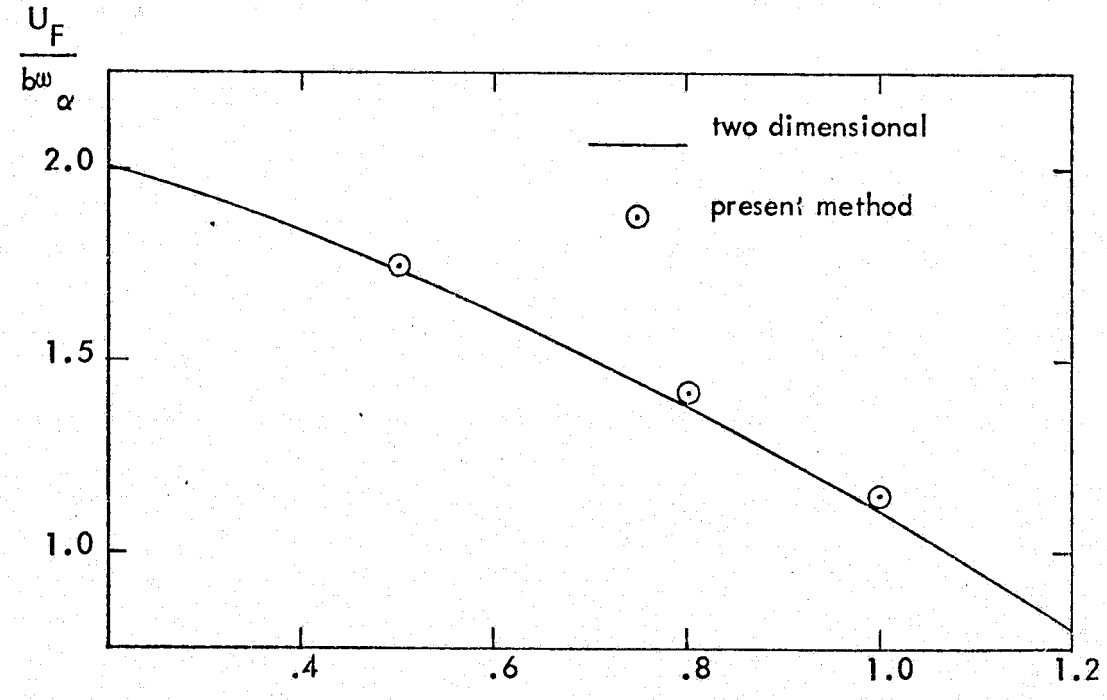

Figure 32b. Flutter speed as a function of ω_h/ω_α for a rectangular wing with AR = 16, M = 0, τ = 0.1%, μ = 10, X_α = 0.2, r_α = 0.5 and NX = 8, NY = 10. Results are compared with exact solution given by two dimensional airfoil theory (Ref. 24)(X_{EA} = -0.2C).

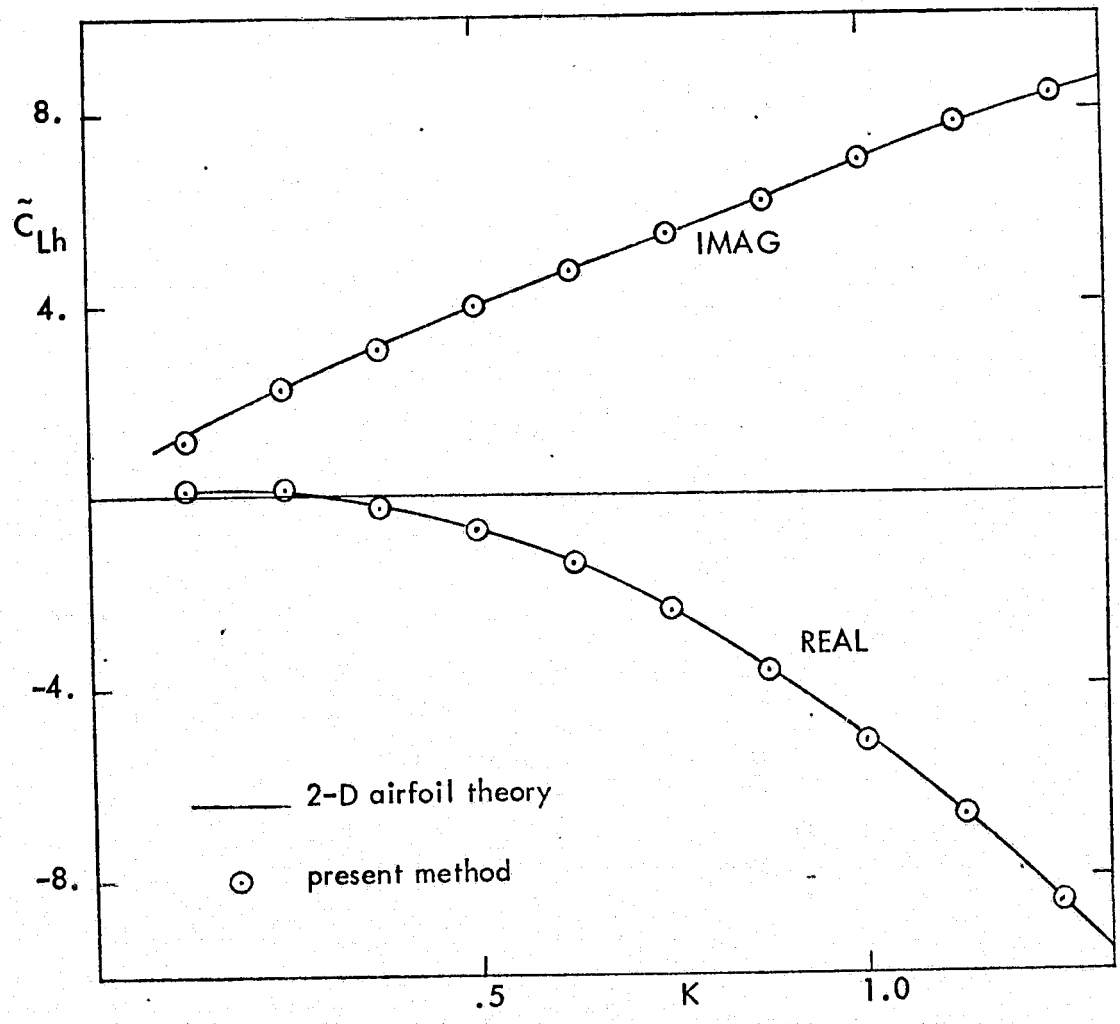

Figure 33a. $\tilde{C}_{Lh}$ as a function of the reduced frequency, $k = \omega c/2U_\infty$, for a rectangular wing with AR = 16, τ = 0.1%, M = 0 and NX = 8, NY = 10. Comparison with the exact solution given by 2-D airfoil theory (Ref. 24) (X_{EA} = -0.2C).

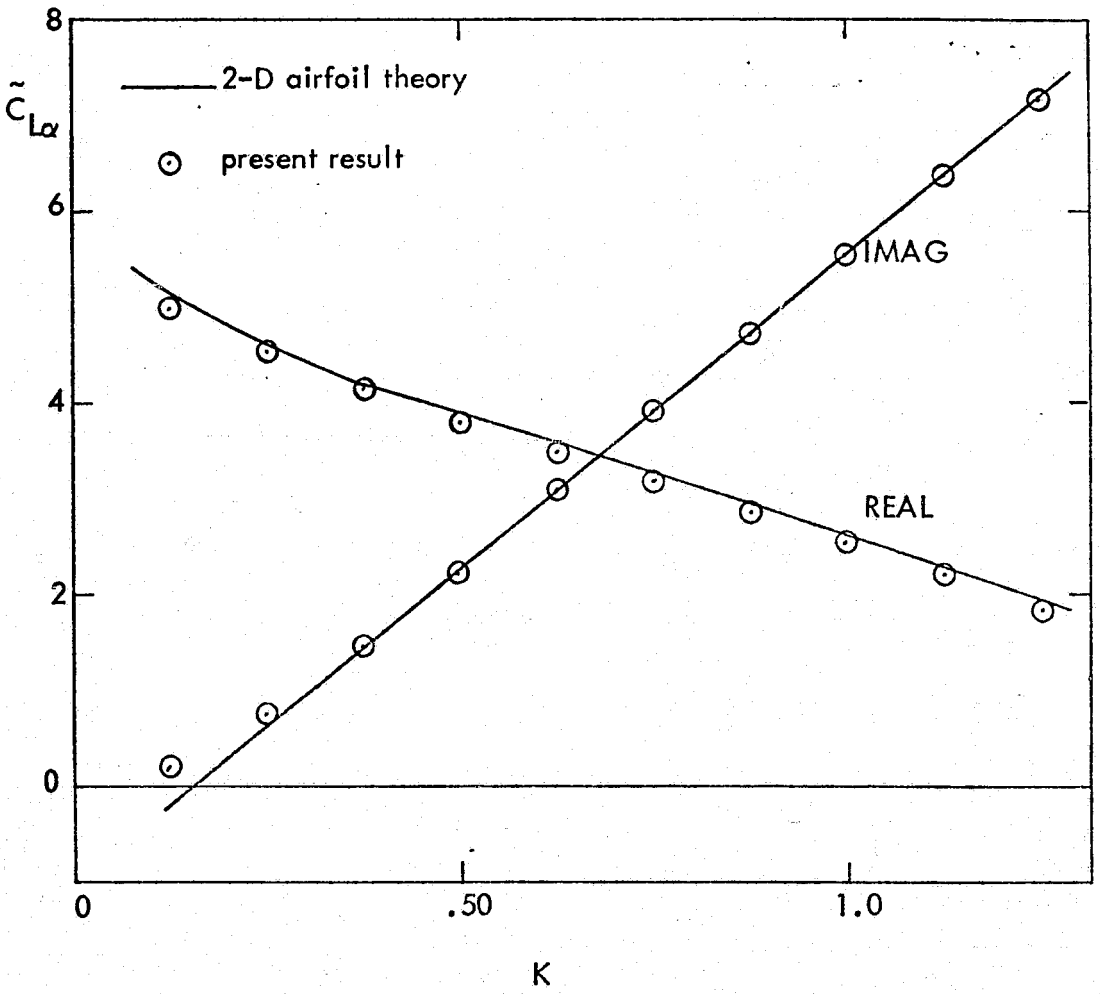

Figure 33b. $\tilde{C}_{L\alpha}$ as a function of the reduced frequency, $k = \omega c/2U_\infty$ for a rectangular wing with AR = 16, τ = 0.1%, M = 0 and NX = 8, NY = 10. Comparison with the exact solution given by 2-D airfoil theory (ref. 24) (X_{EA} = -0.2C).

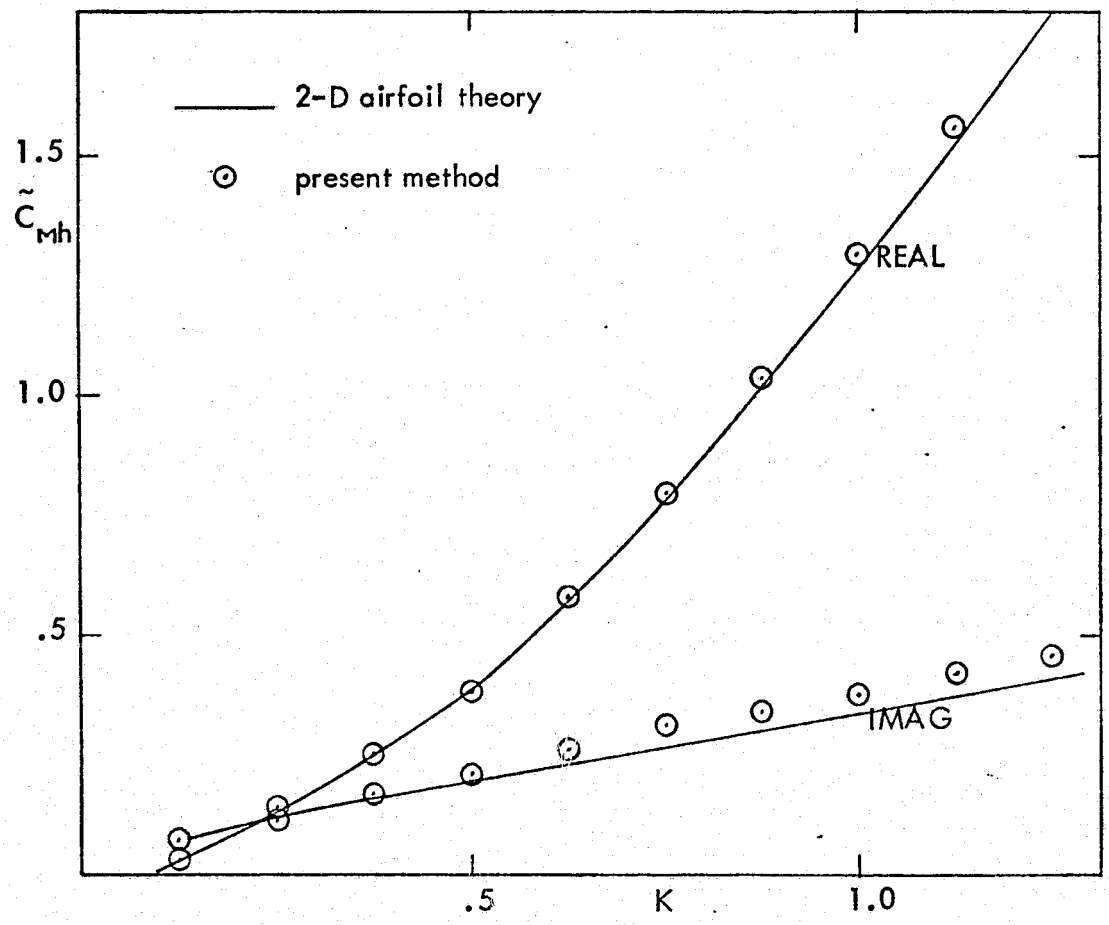

Figure 33c. $\tilde{C}_{Mh}$ as a function of reduced frequency, $k = \omega c/2U_\infty$ for a rectangular wing with AR = 16, τ = 0.1%, M = 0 and NX = 8, NY = 10. Comparison with the exact solution given by 2-D airfoil theory (Ref. 24) (X_{EA} = -0.2C).

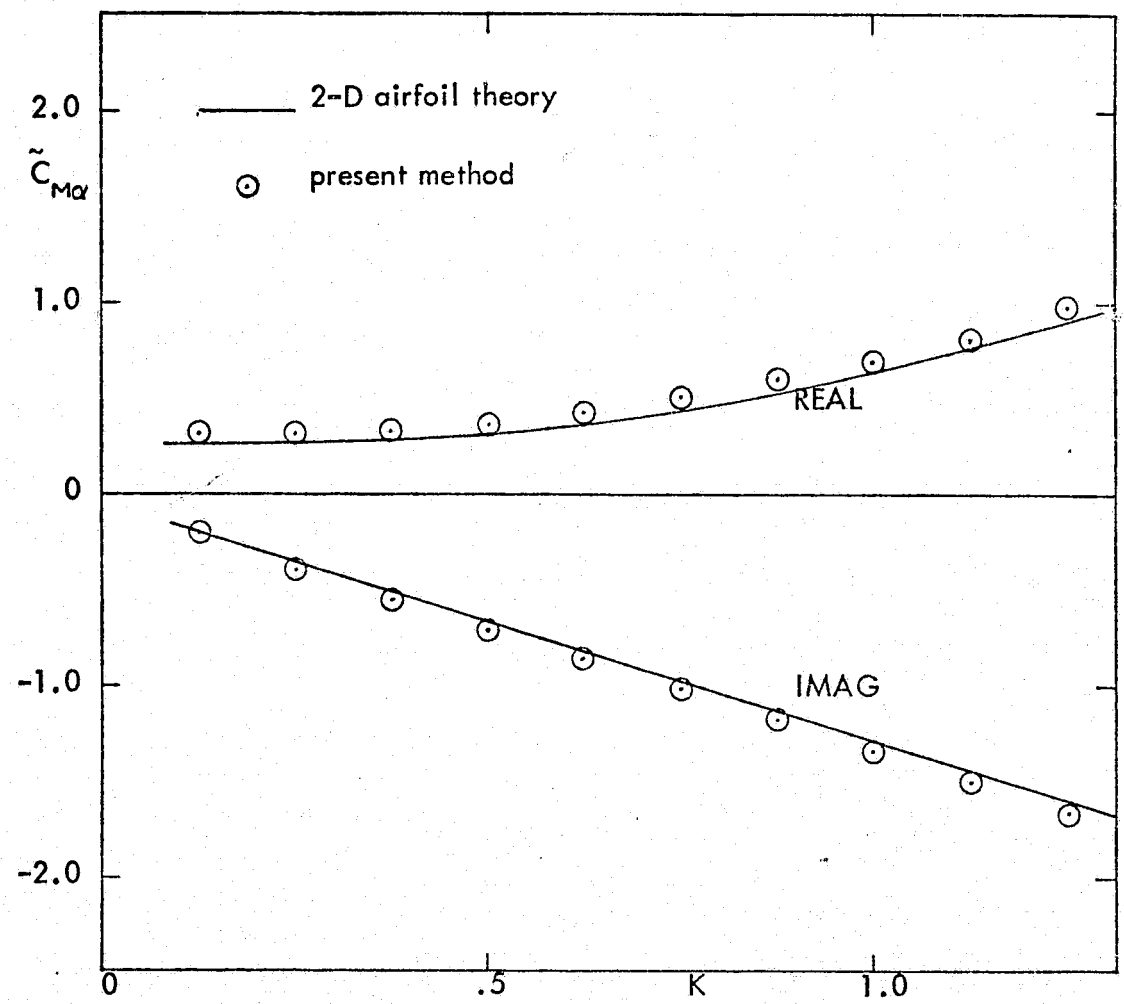

Figure 33d. $\tilde{C}_{M\alpha}$ as a function of reduced frequency, $k = \omega c/2U_\infty$ for a rectangular wing with AR = 16, τ = 0.1%, M = 0 and NX = 8, NY = 10. Comparison with the exact solution given by 2-D airfoil theory (Ref. 24) (X_{EA} = -0.2C).

APPENDIX A

FORMULATIONS FOR STEADY SUBSONIC FLOW

A.1 INTRODUCTION

The derivation of Equations (3.10) and (3.11) is presented in Reference 6. In this appendix it will be shown by differentiation that doublet and source integrals in Equations (3.10) and (3.11) are valid for any planar quadrilateral element. In Subsection A.2, basic equations for hyperboloidal surface are introduced. In Subsection A.3, the doublet integral in Equation (3.10) is shown to be valid for any planar quadrilateral element. In Subsection A.4, the source integral in Equation (3.11) is shown to be valid for any planar quadrilateral element. In addition, the wake term in Equation (3.16) is derived in Subsection A.5.

A.2 BASIC EQUATION

According to Equation (3.15)

$$\bar{n} = \bar{a}_1 \times \bar{a}_2 / |\bar{a}_1 \times \bar{a}_2| \tag{A.1}$$

is independent of ξ and η, therefore,

$$\frac{\partial \bar{n}}{\partial \xi} = \frac{\partial \bar{n}}{\partial \eta} = 0 \tag{A.2}$$

Also, it is noted that

$$\frac{\partial \bar{a}_1}{\partial \xi} = 0 \tag{A.3}$$

$$\frac{\partial \bar{a}_2}{\partial \eta} = 0 \tag{A.4}$$

$$\frac{\partial}{\partial \xi}(\bar{q}\cdot\bar{n}) = \bar{a}_1 \cdot \bar{n} = 0 \tag{A.5}$$

$$\frac{\partial}{\partial \eta}(\bar{q}\cdot\bar{n}) = \bar{a}_2 \cdot \bar{n} = 0 \tag{A.6}$$

$$\frac{\partial \bar{a}_1}{\partial \eta} = \frac{\partial \bar{a}_2}{\partial \xi} = \bar{p}_3 \tag{A.7}$$

For hyperboloidal elements with $\bar{a}_1$ and $\bar{a}_2$ defined by Equations (3.12), (3.13), and (3.14) with

$$d\Sigma = |\bar{a}_1 \times \bar{a}_2| \, d\xi \, d\eta \tag{A.8}$$

the doublet and source integrals in Equations (3.4) and (3.5) can be rewritten as

$$C_{hk} = \frac{1}{2\pi} \int_{-1}^{1}\int_{-1}^{1} \frac{\partial}{\partial N} \left(\frac{1}{\sqrt{\bar{q}\cdot\bar{q}}} \right) |\bar{a}_1 \times \bar{a}_2| \, d\xi \, d\eta$$

$$= -\frac{1}{2\pi} \int_{-1}^{1}\int_{-1}^{1} \frac{\bar{a}_1 \times \bar{a}_2 \cdot \bar{q}}{(\bar{q}\cdot\bar{q})^{3/2}} \, d\xi \, d\eta \tag{A.9}$$

and

$$b_{hk} = \frac{1}{2\pi} \int_{-1}^{1}\int_{-1}^{1} \frac{1}{\sqrt{\bar{q}\cdot\bar{q}}} \, |\bar{a}_1 \times \bar{a}_2| \, d\xi \, d\eta \tag{A.10}$$

since

$$\frac{\partial}{\partial N} \left(\frac{1}{\sqrt{\bar{q}\cdot\bar{q}}} \right) |\bar{a}_1 \times \bar{a}_2|$$

$$= -\frac{|\bar{a}_1 \times \bar{a}_2|}{(\bar{q}\cdot\bar{q})^{3/2}} \left(\frac{\partial}{\partial N} \frac{1}{2} \bar{q}\cdot\bar{q} \right) = -\frac{|\bar{a}_1 \times \bar{a}_2|}{(\bar{q}\cdot\bar{q})^{3/2}} \left(\frac{\partial \bar{q}}{\partial N} \cdot \bar{q} \right)$$

$$= -\frac{|\bar{a}_1 \times \bar{a}_2|}{(\bar{q}\cdot\bar{q})^{3/2}} \left[\frac{\partial}{\partial X} (X - X_*) N_x \bar{i} + \frac{\partial}{\partial y} (Y - Y_*) N_Y \bar{j} + \frac{\partial}{\partial z} (Z - Z_*) N_z \bar{k} \right] \cdot \bar{q}$$

$$= -\frac{|\bar{a}_1 \times \bar{a}_2|}{(\bar{q}\cdot\bar{q})^{3/2}} \, \bar{N}\cdot\bar{q} = -\frac{\bar{a}_1 \times \bar{a}_2 \cdot \bar{q}}{(\bar{q}\cdot\bar{q})^{3/2}} \tag{A.11}$$

A.3 DOUBLET INTEGRAL

The following proves that Equation (3.10) is valid for any quadrilateral element. By using Equations (A.3) to (A.7) one obtains:

$$\frac{\partial}{\partial \eta} \tan_p^{-1} \frac{-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)} = \frac{1}{1 + \left[\frac{(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)} \right]^2} \times$$

$$\times \left\{ \left[(\bar{a}_2 \times \bar{a}_1) \cdot (\bar{q}_f \times \bar{a}_2) + (\bar{q}_f \times \bar{p}_3)(\bar{q}_f \times \bar{a}_2) + (\bar{q}_f \times \bar{a}_1)(\bar{a}_2 \times \bar{a}_1) \right] \frac{1}{|\bar{q}_f|\,|\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)} \right.$$

$$\left. + \frac{(\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)}{-(\bar{q}_f \cdot \bar{q}_f)(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)^2} \left[\frac{\bar{a}_2 \cdot \bar{q}_f}{\sqrt{\bar{q}_f \cdot \bar{q}_f}} \; \bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2 + \sqrt{\bar{q}_f \cdot \bar{q}_f}\,(\bar{a}_2 \cdot \bar{a}_1 \times \bar{a}_2 + \bar{q}_f \cdot \bar{p}_3 \times \bar{a}_2) \right] \right\}$$

$$= \frac{\gamma^2 (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)^2}{\gamma^2 (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)^2 + \left[(\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2) \right]^2} \times \frac{1}{\gamma^3 (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)} \times \left\{ \left[(\bar{a}_2 \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2) + (\bar{q}_f \times \bar{p}_3)\cdot(\bar{q}_f \times \bar{a}_2) \right] \times \right.$$

$$\left. (\bar{q}_f \cdot \bar{q}_f)(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) - (\bar{q}_f \times \bar{a}_1)(\bar{q}_f \times \bar{a}_2) \left[(\bar{a}_2 \cdot \bar{q}_f)(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) + \bar{q}_f \cdot \bar{q}_f (\bar{q}_f \cdot \bar{p}_3 \times \bar{a}_2) \right] \right\}$$

$$= \frac{1}{\gamma} \frac{1}{\left[(\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2) \right]^2 + \gamma^2 (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)^2} \times \left\{ \left[(\bar{a}_2 \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{q}_f \cdot \bar{q}_f) - (\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{a}_2 \cdot \bar{q}_f) \right] \times \right.$$

$$\left. (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) + \left[(\bar{q}_f \times \bar{p}_3)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) - (\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{q}_f \cdot \bar{p}_3 \times \bar{a}_2) \right] \bar{q}_f \cdot \bar{q}_f \right\} \quad \text{(A.12)}$$

Considering terms inside the bracket yields

$$\left[(\bar{a}_2 \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{q}_f \cdot \bar{q}_f) - (\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{a}_2 \cdot \bar{q}_f) \right] (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) =$$

$$+ \left[(\bar{q}_f \times \bar{p}_3)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) - (\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)(\bar{q}_f \cdot \bar{p}_3 \times \bar{a}_2) \right] (\bar{q}_f \cdot \bar{q}_f)$$

$$= \left\{ \left[(\bar{a}_2 \cdot \bar{q}_f)(\bar{a}_1 \cdot \bar{a}_2) - (\bar{a}_2 \cdot \bar{a}_2)(\bar{a}_1 \cdot \bar{q}_f) \right] (\bar{q}_f \cdot \bar{q}_f) - \left[(\bar{q}_f \cdot \bar{q}_f)(\bar{a}_1 \cdot \bar{a}_2) - (\bar{q}_f \cdot \bar{a}_2)(\bar{q}_f \cdot \bar{a}_1) \right] \right.$$

$$\left. (\bar{a}_2 \cdot \bar{q}_f) \right\} (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) + \left\{ \left[(\bar{q}_f \cdot \bar{q}_f)(\bar{p}_3 \cdot \bar{a}_2) - (\bar{q}_f \cdot \bar{a}_2)(\bar{q}_f \cdot \bar{p}_3) \right] \right.$$

$$\left. (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2) - \left[(\bar{q}_f \cdot \bar{q}_f)(\bar{a}_1 \cdot \bar{a}_2) - (\bar{q}_f \cdot \bar{a}_2)(\bar{q}_f \cdot \bar{a}_1) \right] (\bar{q}_f \cdot \bar{p}_3 \times \bar{a}_2) \right\} (\bar{q}_f \cdot \bar{q}_f)$$

$$= -(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\left[(\bar{q}\cdot\bar{q})(\bar{a}_2\cdot\bar{a}_2)-(\bar{q}\cdot\bar{a}_2)^2\right]+\bar{q}\cdot\bar{q}\left\{(\bar{q}\cdot\bar{q})\times\right.$$

$$\left.\left[(\bar{a}_2\cdot\bar{\rho}_3)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{a}_1\cdot\bar{a}_2)(\bar{q}\cdot\bar{\rho}_3\times\bar{a}_2)\right]-(\bar{q}\cdot\bar{a}_2)\left[(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)(\bar{q}\cdot\bar{\rho}_3)-(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{\rho}_3\times\bar{a}_2)\right]\right\}$$

$$= -|\bar{q}\times\bar{a}_2|^2\left[(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{\rho}_3)\right] \qquad \text{(A.13)}$$

since (note the change of the order in the triple product)

$$(\bar{q}\cdot\bar{q})\left[(\bar{a}_2\cdot\bar{\rho}_3)(\bar{q}\times\bar{a}_2\cdot\bar{a}_1)-(\bar{a}_1\cdot\bar{a}_2)(\bar{q}\times\bar{a}_2\cdot\bar{\rho}_3)\right]$$

$$-(\bar{q}\cdot\bar{a}_2)\left[(\bar{q}\cdot\bar{\rho}_3)(\bar{q}\times\bar{a}_2\cdot\bar{a}_1)-(\bar{q}\cdot\bar{a}_1)(\bar{q}\times\bar{a}_2\cdot\bar{\rho}_3)\right]$$

$$=(\bar{q}\cdot\bar{q})\left[\bar{a}_2\times(\bar{q}\times\bar{a}_2)\right]\cdot(\bar{\rho}_3\times\bar{a}_1)-(\bar{q}\cdot\bar{a}_2)\left[\bar{q}\times(\bar{q}\times\bar{a}_2)\right]\cdot(\bar{\rho}_3\times\bar{a}_1)$$

$$=(\bar{q}\cdot\bar{q})\left[(\bar{a}_2\cdot\bar{a}_2)(\bar{q}\cdot\bar{\rho}_3\times\bar{a}_1)-(\bar{a}_2\cdot\bar{q})(\bar{a}_2\cdot\bar{\rho}_3\times\bar{a}_1)\right]$$

$$-(\bar{q}\cdot\bar{a}_2)\left[(\bar{q}\cdot\bar{a}_2)(\bar{q}\cdot\bar{\rho}_3\times\bar{a}_1)-(\bar{q}\cdot\bar{q})(\bar{a}_2\cdot\bar{\rho}_3\times\bar{a}_1)\right]$$

$$=\left[(\bar{q}\cdot\bar{q})(\bar{a}_2\cdot\bar{a}_1)-(\bar{q}\cdot\bar{a}_2)^2\right](\bar{q}\cdot\bar{\rho}_3\times\bar{a}_1)$$

$$=|\bar{q}\times\bar{a}_2|^2\,\bar{q}\cdot\bar{\rho}_3\times\bar{a}_1 \qquad \text{(A.14)}$$

Finally, by using Equation (D.2) and combining Equations (A.12) and (A.13) one obtains

$$\frac{\partial}{\partial\eta}\tan^{-1}\rho\;\frac{-(\bar{q}\times\bar{a}_1)(\bar{q}\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}$$

$$=\frac{1}{\sqrt{\bar{q}\cdot\bar{q}}}\;\frac{1}{|\bar{q}\times\bar{a}_2|^2|\bar{q}\times\bar{a}_2|^2}|\bar{q}\times\bar{a}_2|^2\left[(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{\rho}_3)\right]$$

$$=\frac{1}{\sqrt{\bar{q}\cdot\bar{q}}}\;\frac{(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{\rho}_3)}{|\bar{q}\times\bar{a}_1|^2} \qquad \text{(A.15)}$$

Finally, by using Equations (A.3), (A.7) and

$$\frac{\partial}{\partial \xi}(\bar{q}\times\bar{a}_1) = \bar{a}_1\times\bar{a}_1 = 0 \tag{A.16}$$

one obtains,

$$2\pi\frac{\partial^2 I_p}{\partial\xi\partial\eta} = \frac{\partial^2}{\partial\xi\partial\eta}\tan^{-1}\frac{-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2}$$

$$= \frac{\partial}{\partial\xi}\left[\frac{(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)-(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\;|\bar{q}\times\bar{a}_1|^2}\right]$$

$$= \frac{1}{|\bar{q}\times\bar{a}_1|^2}\left\{-\frac{\bar{q}\,\bar{a}_1}{(\bar{q}\cdot\bar{q})^{3/2}}\left[(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)-(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\right]+\right.$$

$$+\frac{1}{\sqrt{\bar{q}\cdot\bar{q}}}\left[2(\bar{a}_1\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)+(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_1\times\bar{p}_3)\right.$$

$$\left.\left.-(\bar{a}_1\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\cdot\bar{a}_1)(\bar{a}_1\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)\right]\right\}$$

$$= \frac{1}{(\bar{q}\cdot\bar{q})^{3/2}|\bar{q}\times\bar{a}_1|^2}\left[(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1)-(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_1)\right](\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$= \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{(\bar{q}\cdot\bar{q})^{3/2}} \tag{A.18}$$

This proves that Equation (3.10) provides an analytical solutions of the doublet integral, given by Equation (A.9), for any hyperboloidal element.

A.4 SOURCE INTEGRAL

Use Equation (A.3) and note that

$$\frac{\partial}{\partial\xi}(\bar{q}\times\bar{a}_1\cdot\bar{n}) = \bar{a}_1\times\bar{a}_1\cdot\bar{n} = 0 \tag{A.19}$$

and

$$\frac{\partial}{\partial \xi} \ln \left(|\bar{a}_1||\bar{q}| + \bar{q}\cdot\bar{a}_1 \right)$$

$$= \frac{1}{|\bar{a}_1||\bar{q}| + \bar{q}\cdot\bar{a}_1} \left\{ \frac{\bar{q}\cdot\bar{a}_1}{\sqrt{\bar{q}\cdot\bar{q}}} |\bar{a}_1| + \bar{a}_1\cdot\bar{a}_1 \right\} \frac{\sqrt{\bar{a}_1\cdot\bar{a}_1}}{\sqrt{\bar{q}\cdot\bar{q}}} \tag{A.20}$$

Hence,

$$\frac{\partial^2}{\partial\xi\partial\eta} \left[\bar{q}\times\bar{a}_1\cdot\bar{n} \; \frac{1}{|\bar{a}_1|} \ln \left(|\bar{a}_1||\bar{q}| + \bar{q}\cdot\bar{a}_1 \right) \right] = \frac{\partial}{\partial\eta} \left(\frac{\bar{q}\times\bar{a}_1\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} \right)$$

$$\frac{\bar{a}_2\times\bar{a}_1\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} + \frac{\bar{q}\times\bar{p}_3\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} - \bar{q}\times\bar{a}\cdot\bar{n} \; \frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}|^3} \tag{A.21}$$

Similarly

$$\frac{\partial^2}{\partial\xi\partial\eta} \left[\bar{q}\times\bar{a}_2\cdot\bar{n} \; \frac{1}{|\bar{a}_2|} \ln \left(|\bar{a}_2||\bar{q}| + \bar{q}\cdot\bar{a}_2 \right) \right] = \frac{\partial}{\partial\xi} \left(\frac{\bar{q}\times\bar{a}_2\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} \right)$$

$$= \frac{\bar{a}_1\times\bar{a}_2\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} + \frac{\bar{q}\times\bar{p}_3\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} - \bar{q}\times\bar{a}_2\cdot\bar{n} \; \frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|} \tag{A.22}$$

Furthermore, noting that

$$\frac{\partial}{\partial\xi} \left(\bar{q}\cdot\bar{n} \right) = \bar{q} \tag{A.23}$$

and combining Equation (A.14) with Equations (A.5) and (A.6) yields

$$\frac{\partial^2}{\partial\xi\partial\eta} \left(\bar{q}\cdot\bar{n} \; \tan^{-1} P \; \frac{-(\bar{q}\times\bar{a}_1)(\bar{q}\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}} \; \bar{q}\cdot\bar{a}_1\times\bar{a}_2} \right)$$

$$= -\bar{q}\cdot\bar{n} \; \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{(\bar{q}\cdot\bar{q})^{3/2}} \tag{A.24}$$

Finally combining Equations (A.21), (A.22) and (A.24) yields

$$\frac{2\pi}{\bar{n}\cdot\bar{i}_x}\quad\frac{\partial^2 I_S}{\partial\xi\,\partial\eta}$$

$$= \frac{|\bar{n}\times\bar{a}_2|}{|\bar{q}|} + (\bar{q}\cdot\bar{a}_1\times\bar{n})\,\frac{\bar{q}\,\bar{a}_2}{|\bar{q}|^3} + \frac{|\bar{a}_1\times\bar{a}_2|}{|\bar{q}|} - (\bar{q}\cdot\bar{a}_2\times\bar{n})\,\frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|^3} - \frac{(\bar{q}\cdot\bar{n})\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}|^3}$$

$$= 2\,\frac{|\bar{a}_1\times\bar{a}_2|}{|\bar{q}|} + \frac{1}{|\bar{q}|^3}\left[(-\bar{q}\times\bar{n}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_2) + (\bar{q}\times\bar{n}\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1) - (\bar{q}\cdot\bar{n})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\right]$$

$$= 2\,\frac{|\bar{a}_1\times\bar{a}_2|}{|\bar{q}|} + \frac{1}{|\bar{q}|^3}\left[-(\bar{q}\times\bar{n})\times\bar{q}\cdot(\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{n})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\right]$$

$$= 2\,\frac{|\bar{a}_1\times\bar{a}_2|}{|\bar{q}|} + \frac{1}{|\bar{q}|^3}\left[(\bar{q}\cdot\bar{n})(\bar{q}\cdot\bar{a}_1\cdot\bar{a}_2) - (\bar{q}\cdot\bar{q})(\bar{n}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{n})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\right]$$

$$= 2\,\frac{|\bar{a}_1\times\bar{a}_2|}{|\bar{q}|} - \frac{|\bar{n}\cdot\bar{a}_1\times\bar{a}_2|}{|\bar{q}|} = \frac{|\bar{a}_1\times\bar{a}_2|}{|\bar{q}|} \qquad \text{(A.25)}$$

This proves that Equation (3.11) provides an analytical solution of source integral, given by Equation (A.10), for any quadrilateral planar element.

A.5 WAKE COEFFICIENT

Consider Figure 2b, and assume that the wake element is truncated such that

$$\bar{a}_1 = \bar{p}_1 = \chi\bar{u} = \chi\bar{i} \qquad \text{(A.26)}$$

By letting χ go to infinity, wake coefficient can be obtained from I_D in Equation (3.10). Note that (see Figure 2b)

$$\bar{p}_0 - \bar{p}_1 = (\bar{p}_+ + \bar{p}_-)/2 = \bar{p}_m \qquad \text{(A.27)}$$

$$\bar{p}_2 = \frac{\bar{p}_+ - \bar{p}_-}{2} = \bar{p}_J = \bar{a}_2 \qquad \text{(A.28)}$$

$$\bar{p}_3 = 0 \qquad \text{(A.29)}$$

It is convenient to separate the contribution from the trailing edge ($\xi = -1$) and the edge that goes to infinity ($\xi = 1$):

$$I_w = \mp \frac{S}{2\pi} \left[J_w(1,\eta) - J_w(-1,\eta) \right]_{\eta=-1}^{\eta=1} \tag{A.30}$$

where (note that $\bar{q}_f = \bar{\rho}_m + (1+\xi)\chi\bar{i} + \eta\,\bar{\rho}_d$)

$$S = \text{sign}\left(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2 \right) = \text{sign}\left(\bar{\rho}_{md} \cdot \bar{i} \times \bar{\rho}_d \right) \tag{A.31}$$

with

$$\bar{\rho}_{md} = \bar{\rho}_m + \eta\,\bar{\rho}_d \tag{A.32}$$

while (note that $\bar{q}_f(\xi=1) = \bar{\rho}_{md} + 2\chi\bar{i}$)

$$J_w(1,\eta) = \lim_{\chi\to\infty} \tan_\rho^{-1} \left(\frac{-(\bar{q}_f \times \bar{a}_1)\cdot(\bar{q}_f \times \bar{a}_2)}{|\bar{q}_f|\,|\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2|} \right)_{\xi=1}$$

$$= \lim_{\chi\to\infty} \tan_\rho^{-1} \left(\frac{-(\bar{q}_f \times \bar{i})\cdot(\bar{q}_f \times \bar{\rho}_d)}{|\bar{q}_f|\,(\bar{q}_f \cdot \bar{i} \times \bar{\rho}_d)} \right)_{\xi=1}$$

$$= \lim_{\chi\to\infty} \tan_\rho^{-1} \frac{-\left[(\bar{\rho}_m + \eta\,\bar{\rho}_d) \times \bar{i}\right]\cdot\left[(\rho_m + 2\chi\bar{i}) \times \bar{\rho}_d\right]}{\left[(\bar{\rho}_m + \eta\,\bar{\rho}_d + 2\,\bar{i})\cdot(\bar{\rho}_m + \eta\,\bar{\rho}_d + 2\chi\,\bar{i})\right]^{1/2} |\bar{\rho}_m \cdot \bar{i} \times \bar{\rho}_d|}$$

$$= \tan_\rho^{-1} \frac{-(\bar{\rho}_{md} \times \bar{i})\cdot(\bar{i} \times \bar{\rho}_d)}{|\bar{\rho}_{md} \cdot \bar{i} \times \bar{\rho}_d|}$$

$$= \tan_\rho^{-1} \frac{\bar{\rho}_{md} \cdot \bar{\rho}_d - (\bar{\rho}_{md} \cdot \bar{i})(\bar{\rho}_d \cdot \bar{i})}{|\bar{\rho}_{md} \cdot \bar{i} \times \bar{\rho}_d|} \tag{A.33}$$

and similarly (note that $\bar{q}_f(\xi=-1) = \bar{\rho}_m + \eta\,\bar{\rho}_d = \bar{\rho}_{md}$)

$$J_w(-1,\eta) = \tan_p^{-1}\left\{-\frac{(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{|\bar{q}||\bar{q}\cdot\bar{a}_1\times\bar{a}_2|}\right\}_{\xi=-1}$$

$$= \tan_p^{-1}\left\{-\frac{(\bar{q}\times\bar{\imath})\cdot(\bar{q}\times\bar{P}_d)}{|\bar{q}||\bar{q}\cdot\bar{\imath}\times\bar{P}_d|}\right\}_{\xi=-1}$$

$$= \tan_p^{-1}\left\{\frac{-(\bar{P}_{md}\times\bar{\imath})\cdot(\bar{P}_{md}\times\bar{P}_d)}{|\bar{P}_{md}||\bar{P}_{md}\cdot\bar{\imath}\times\bar{P}_d|}\right\}$$

$$= \tan_p^{-1}\left\{-\frac{(\bar{P}_{md}\cdot\bar{P}_{md})(\bar{\imath}\cdot\bar{P}_d)+(\bar{P}_{md}\cdot\bar{P}_d)(\bar{P}_{md}\cdot\bar{\imath})}{|\bar{P}_{md}||\bar{P}_{md}\cdot\bar{\imath}\times\bar{P}_d|}\right\} \qquad \text{(A.34)}$$

APPENDIX B

FORMULATIONS FOR STEADY SUPERSONIC FLOW

B.1 INTRODUCTION

The derivation of Equations (5.8) and (5.9) is presented in Reference 7. In Subsections B.2 and B.3 it will be shown that these equations are valid for any quadrilateral planar element. In Subsection B.4, formulations of the doublet and source integral of the element intersected with the Mach forecone are considered.

For hyperboloidal elements with $\bar{q}$, $\bar{a}_1$, $\bar{a}_2$, and $d\Sigma$ defined by Equations (3.12), (3.13), and (3.14) and (A.8), the doublet and source integral in Equations (4.7a) and (4.7b) can be rewritten as

$$C_{hK} = \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{\partial}{\partial N^c}\left(\frac{H}{\|\bar{q}\|}\right)|\bar{a}_1 \times \bar{a}_2|\, d\xi\, d\eta$$

$$= \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\|\bar{q}\|^3} H\, d\xi\, d\eta \qquad \text{(B.1)}$$

and

$$b_{hK} = -\frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{H}{\|\bar{q}\|}|\bar{a}_1 \times \bar{a}_2|\, d\xi\, d\eta \qquad \text{(B.2)}$$

Since

$$\frac{\partial}{\partial N^c}\left(\frac{H}{\|\bar{q}\|}\right)|\bar{a}_1\times\bar{a}_2| = -\frac{|\bar{a}_1\times\bar{a}_2|}{(\bar{q}\odot\bar{q})^{3/2}}\left(\frac{\partial\bar{q}}{\partial N^c}\odot\bar{q}\right)$$

$$= -\frac{|\bar{a}_1\times\bar{a}_2|}{(\bar{q}\odot\bar{q})^{3/2}}\left[N_X\frac{\partial}{\partial X}(X-X_*)\bar{i} - N_Y\frac{\partial}{\partial Y}(Y-Y_*)\bar{j} - N_Z\frac{\partial}{\partial Z}(Z-Z_*)\bar{k}\right]\odot\bar{q}$$

$$= -\frac{|\bar{a}_1\times\bar{a}_2|}{(\bar{q}\odot\bar{q})^{3/2}}\,\bar{N}^c\odot\bar{q} = \frac{\bar{a}_1\times\bar{a}_2\cdot\bar{q}}{(\bar{q}\odot\bar{q})^{3/2}} \qquad \text{(B.3)}$$

B.2 DOUBLET INTEGRAL

Following is the proof that Equation (5.8) is valid for any quadrilateral element within the Mach forecone. By using Equations (A.3), (A.4) and

$$\frac{\partial \bar{a}_1}{\partial \eta} = \frac{\partial \bar{a}_2}{\partial \xi} = \bar{\beta}_3 \tag{B.4}$$

one obtains

$$\frac{\partial}{\partial \eta} \tan_p^{-1} \frac{-\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2}{\sqrt{\bar{q}_\beta \odot \bar{q}_\beta}\ \bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2} = \frac{1}{1+\left(\frac{\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2}{\sqrt{\bar{q}_\beta \odot \bar{q}_\beta}\ \bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2}\right)^2} \left\{ (\bar{a}_2 \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2 + \bar{q}_\beta \times \bar{\beta}_3 \odot \bar{q}_\beta \times \bar{a}_2 \right.$$

$$\left. + \bar{q}_\beta \times \bar{a}_1 \odot \bar{a}_2 \times \bar{a}_2 \right] \frac{1}{\|\bar{q}_\beta\|\, \bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2} - \frac{\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2}{\bar{q}_\beta \odot \bar{q}_\beta (\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2)^2} \left[\frac{\bar{a}_2 \odot \bar{q}_\beta}{\sqrt{\bar{q}_\beta \odot \bar{q}_\beta}} (\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2) + \sqrt{\bar{q}_\beta \odot \bar{q}_\beta} (\bar{a}_2 \cdot \bar{a}_1 \times \bar{a}_2 + \bar{q}_\beta \cdot \bar{\beta}_3 \times \bar{a}_2) \right] \right\}$$

$$= -\frac{\bar{q}_\beta \odot \bar{q}_\beta\, (\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2)^2}{(\bar{q}_\beta \odot \bar{q}_\beta)(\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2)} \frac{1}{\|\bar{q}_\beta\|^3 (\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2)} \left\{ (\bar{q}_\beta \odot \bar{q}_\beta)(\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2) \left[\bar{a}_2 \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{\beta}_2 + \right. \right.$$

$$\left. \left. + \bar{q}_\beta \times \bar{\beta}_3 \odot \bar{q}_\beta \times \bar{a}_3 \right] - (\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2) \left[(\bar{a}_2 \odot \bar{q}_\beta)(\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2) + (\bar{q}_\beta \odot \bar{q}_\beta)(\bar{q}_\beta \cdot \bar{\beta}_3 \times \bar{a}_2) \right] \right\}$$

$$= -\frac{1}{\sqrt{\bar{q}_\beta \odot \bar{q}_\beta}} \frac{1}{(\bar{q}_\beta \odot \bar{q}_\beta)(\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2)^2} \left\{ (\bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2 \left[(\bar{a}_2 \times \bar{a}_1 \odot \bar{q}_\beta \times a_2)\, \bar{q}_\beta \odot \bar{q}_\beta - (\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2) \right. \right.$$

$$\left. \left. \bar{a}_2 \odot \bar{q}_\beta \right] + \bar{q}_\beta \odot \bar{q}_\beta \left[(\bar{q}_\beta \times \bar{\beta}_3 \odot \bar{q}_\beta \times \bar{a}_2)\, \bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2 - (\bar{q}_\beta \times \bar{a}_1 \odot \bar{q}_\beta \times \bar{a}_2)\, \bar{q}_\beta \cdot \bar{\beta}_3 \times \bar{a}_2 \right] \right\} \tag{B.5}$$

Furthermore,

$$\left[(\bar{a}_2\times\bar{a}_1\circ\bar{q}\times\bar{a}_2)\,\bar{q}\circ\bar{q}-(\bar{q}\times\bar{a}_1\circ\bar{q}\times\bar{a}_2)\,\bar{a}_2\circ\bar{q}\right](\bar{q}\cdot\bar{a}_1\times\bar{a}_2)+$$

$$+\left[(\bar{q}\times\bar{p}_3\circ\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\times\bar{a}_1\circ\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{p}_3\times\bar{a}_2)\right]\bar{q}\circ\bar{q}$$

$$\equiv\left\{\left[(\bar{a}_2\circ\bar{q})(\bar{a}_1\times\bar{a}_2)-(\bar{a}_2\circ\bar{a}_2)(\bar{a}_1\circ\bar{q})\right]\bar{q}\circ\bar{q}-\left[(\bar{q}\circ\bar{q})(\bar{a}_1\times\bar{a}_2)-(\bar{q}\circ\bar{a}_2)(\bar{q}\circ\bar{a}_2)\right]\bar{a}_2\circ\bar{q}\right\}\bar{q}\cdot\bar{a}_1\times\bar{a}_2$$

$$+\left\{\left[(\bar{q}\circ\bar{q})(\bar{p}_3\circ\bar{a}_2)-(\bar{q}\circ\bar{a}_2)(\bar{q}\circ\bar{p}_3)\right]\bar{q}\cdot\bar{a}_1\times\bar{a}_2-\left[(\bar{q}\circ\bar{q})(\bar{a}_1\circ\bar{a}_2)-(\bar{q}\circ\bar{a}_2)(\bar{q}\circ\bar{a}_1)\right]\bar{q}\cdot\bar{p}_3\times\bar{a}_2\right\}\bar{q}\circ\bar{q}$$

$$=-\left\{(\bar{q}\circ\bar{q})(\bar{a}_2\circ\bar{a}_2)-(\bar{q}\circ\bar{a}_2)^2\right\}(\bar{q}\circ\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)+\left\{(\bar{q}\circ\bar{q})\left[\bar{a}_2\circ\bar{p}_3\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2\right.\right.$$

$$\left.\left.-(\bar{a}_1\circ\bar{a}_2)(\bar{q}\cdot\bar{p}_3\times\bar{a}_2)\right]-(\bar{q}\circ\bar{a}_2)\left[(\bar{q}\circ\bar{p}_3)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\circ\bar{a}_1)(\bar{q}\cdot\bar{p}_3\times\bar{a}_2)\right]\right\}$$

$$=-\left[(\bar{q}\circ\bar{q})\,\bar{a}_2\circ\bar{a}_2-(\bar{q}\circ\bar{a}_2)^2\right](\bar{q}\circ\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)+(\bar{q}\times\bar{a}_2\circ\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)(\bar{q}\circ\bar{q})$$

$$=\|\bar{q}\times\bar{a}_2\|^2\left[(\bar{q}\circ\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\circ\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)\right] \tag{B.6}$$

since

$$(\bar{q}\circ\bar{q})(\bar{a}_2\circ\bar{a}_2)-(\bar{q}\circ\bar{a}_2)^2\equiv(\bar{q}\times\bar{a}_2)\circ(\bar{q}\times\bar{a}_2)=-\|\bar{q}\times\bar{a}_2\|^2 \tag{B.7}$$

Finally, combining Equation (B.5) with (B.6) yields

$$\frac{\partial I_p}{\partial\eta}\quad\frac{\|\bar{q}\times\bar{a}_2\|^2\left[(\bar{q}\circ\bar{a}_1)\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2-(\bar{q}\circ\bar{q})\,\bar{q}\cdot\bar{a}_1\times\bar{p}_3\right]}{\|\bar{q}\times\bar{a}_1\|^2\,\|\bar{q}\times\bar{a}_2\|^2\,\|\bar{q}\|}$$

$$=\frac{1}{\|\bar{q}\times\bar{a}_1\|^2\,\|\bar{q}\|}\left[(\bar{q}\circ\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\circ\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)\right] \tag{B.8}$$

Next, consider the second mixed derivative, since

$$\frac{\partial}{\partial\xi}(\bar{q}\times\bar{a}_1)=\frac{\partial}{\partial\xi}\left[(\bar{p}_0+\eta\,\bar{p}_2)\times(\bar{p}_1+\eta\,\bar{p}_3)\right]\equiv 0$$

Therefore,

$$\frac{\partial^2 I_p}{\partial\xi\,\partial\eta} = \frac{1}{\|\bar{q}\times\bar{a}_1\|^2}\frac{\partial}{\partial\xi}\left\{\frac{(\bar{q}\odot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)-(\bar{q}\odot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}{\sqrt{\bar{q}\odot\bar{q}}}\right\}$$

$$= \frac{1}{\|\bar{q}\times\bar{a}_1\|^2}\left\{-\frac{\bar{q}\odot\bar{a}_1}{(\bar{q}\odot\bar{q})^{3/2}}\left[(\bar{q}\odot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)-(\bar{q}\odot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\right]+\frac{1}{\sqrt{\bar{q}\odot\bar{q}}}\left[2(\bar{q}\odot\bar{a}_1)\bar{q}\cdot\bar{a}_1\times\bar{p}_3\right.\right.$$

$$\left.\left.+(\bar{q}\odot\bar{q})\bar{a}_1\cdot\bar{a}_1\times\bar{p}_3-(\bar{a}_1\odot\bar{a}_1)\bar{q}\cdot\bar{a}_1\times\bar{a}_2-(\bar{q}\odot\bar{a}_1)(\bar{a}_1\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\odot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)\right]\right\}$$

$$= \frac{1}{\|\bar{q}\|^3\|\bar{q}\times\bar{a}_1\|^2}\left[(\bar{q}\odot\bar{a}_1)^2-(\bar{q}\odot\bar{q})(\bar{a}_1\odot\bar{a}_1)\right]\bar{q}\cdot\bar{a}_1\times\bar{a}_2$$

$$= \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\|\bar{q}\|^3} \qquad \text{(B.9)}$$

Thus, prove that Equation (5.8) provides an analytical solution of the doublet integral given by Equation (B.1) for any quadrilateral element.

B.3 SOURCE INTEGRAL

Following is the proof that Equation (5.19) is valid for any quadrilateral element within the Mach forecone. Noting that, as shown in Appendix D,

$$\frac{\partial F_1}{\partial\xi} = \frac{\partial F_2}{\partial\eta} = \frac{1}{\|\bar{q}\|} \qquad \text{(B.10)}$$

and

$$\frac{\partial}{\partial\xi}(\bar{q}\times\bar{a}_1\odot\bar{n}) = \bar{a}_1\times\bar{a}_1\odot\bar{n} = 0 \qquad \text{(B.11a)}$$

$$\frac{\partial}{\partial\eta}(\bar{q}\times\bar{a}_2\odot\bar{n}) = \bar{a}_2\times\bar{a}_2\odot\bar{n} = 0 \qquad \text{(B.11b)}$$

yields

$$\frac{\partial^2}{\partial\xi\,\partial\eta}\left\{\bar{q}\times\bar{a}_1\odot\bar{n}\,F_1(\xi,\eta)\right\} =$$

$$= \frac{\partial}{\partial \eta}\left(\bar{q}_f \times \bar{a}_1 \odot \bar{n}\ \frac{1}{\sqrt{\bar{q}_f \odot \bar{q}_f}}\right) \tag{B.12a}$$

$$= \frac{\bar{a}_2 \times \bar{a}_1 \odot \bar{n}}{\|\bar{q}_f\|} + \frac{\bar{q}_f \times \bar{p}_3 \odot \bar{n}}{\|\bar{q}_f\|} - \bar{q}_f \times \bar{a}_1 \odot \bar{n}\ \frac{\bar{q}_f \odot \bar{q}_2}{(\bar{q}_f \odot \bar{q}_f)^{3/2}} \tag{B.12b}$$

Similarly,

$$\frac{\partial^2}{\partial \xi \partial \eta}\left\{\bar{q}_f \times \bar{a}_2 \odot \bar{n}\ F_2(\xi, \eta)\right\} = \frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}_f\|} + \frac{\bar{q}_f \times \bar{p}_3 \odot \bar{n}}{\|\bar{q}_f\|} - \bar{q}_f \times \bar{a}_2 \odot \bar{n}\ \frac{\bar{q}_f \odot \bar{a}_1}{(\bar{q}_f \odot \bar{q}_f)^{3/2}} \tag{B.13}$$

In addition, from Equation (B.9)

$$\frac{\partial^2}{\partial \xi \partial \eta}\left\{\bar{q}_f \cdot \bar{n}\ \tan_p^{-1}\left(\frac{-\bar{q}_f \times \bar{a}_1 \odot \bar{q}_f \times \bar{a}_2}{\|\bar{q}_f\|\ \bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2}\right)\right\} = \bar{q}_f \cdot \bar{n}\ \frac{\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{q}_f\|^3} \tag{B.14}$$

Finally, combining Equations (4.10), (B.12b), (B.13) and (B.14) and noting that $\|\bar{n}\| = -\bar{n} \odot \bar{n}$, yields

$$\frac{\partial^2 I_s}{\partial \xi \partial \eta} = \frac{1}{\pi}\ \frac{1}{\bar{n} \odot \bar{n}}\left\{-\left[\frac{\bar{a}_2 \times \bar{a}_1 \odot \bar{n}}{\|\bar{q}_f\|} + \frac{\bar{q}_f \times \bar{p}_3 \odot \bar{n}}{\|\bar{q}_f\|} - \bar{q}_f \times \bar{a}_1 \odot \bar{n}\ \frac{\bar{q}_f \odot \bar{a}_2}{(\bar{q}_f \odot \bar{q}_f)^{3/2}}\right]\right.$$

$$\left. + \left[\frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}_f\|} + \frac{\bar{q}_f \times \bar{p}_3 \odot \bar{n}}{\|\bar{q}_f\|} - (\bar{q}_f \times \bar{a}_2 \odot \bar{n})\ \frac{\bar{q}_f \odot \bar{a}_1}{(\bar{q}_f \odot \bar{q}_f)^{3/2}}\right] - \bar{q}_f \cdot \bar{n}\ (\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)\frac{1}{\|\bar{q}_f\|^3}\right\} \tag{B.15a}$$

$$= \frac{1}{\bar{n} \odot \bar{n}}\ \frac{1}{\pi}\left\{2\left|\bar{a}_1 \times \bar{a}_2\right|\frac{\bar{n} \odot \bar{n}}{\|\bar{q}_f\|} + \frac{1}{\|\bar{q}_f\|^3}\left[(\bar{q}_f \times \bar{a}_1 \odot \bar{n})(\bar{q}_f \odot \bar{a}_2) - (\bar{q}_f \times \bar{a}_2 \odot \bar{n})(\bar{q}_f \odot \bar{a}_1) - (\bar{q}_f \cdot \bar{n})(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)\right]\right\} \tag{B.15b}$$

According to Equation (D.7) for

$$-\left|\bar{a}_1 \times \bar{a}_2\right|\left[(\bar{q}_f \times \bar{a}_1 \odot \bar{n})\bar{q}_f \odot \bar{a}_2 - (\bar{q}_f \times \bar{a}_2 \odot \bar{n})\ \bar{q}_f \odot \bar{a}_1 - \bar{q}_f \cdot \bar{n}\,(\bar{q}_f \cdot \bar{a}_1 \times \bar{a}_2)\right]$$

$$=(\bar{q}\times\bar{a}_1\circ\bar{a}_2\times\bar{a}_1)\,\bar{q}\circ\bar{a}_2+(\bar{q}\times\bar{a}_2\circ\bar{a}_1\times\bar{a}_2)(\bar{q}\circ\bar{a}_1)+(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2$$

$$=(\bar{q}\circ\bar{q})(\bar{a}_1\times\bar{a}_2\circ\bar{a}_1\times\bar{a}_2)-(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2+(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2$$

$$=\bar{q}\circ\bar{q}\;\;\bar{n}\circ\bar{n}\;|\bar{a}_1\times\bar{a}_2| \qquad \text{(B.16)}$$

By using Equation (B.16), Equation (B.15) reduces to

$$\frac{\partial^2 I_S}{\partial\xi\,\partial\eta}=\frac{1}{\bar{n}\circ\bar{n}}\left\{2\bar{n}\circ\bar{n}\;\frac{|\bar{a}_1\times\bar{a}_2|}{\|\bar{q}\|}-\bar{q}\circ\bar{q}\;\bar{n}\circ\bar{n}\;\frac{|\bar{a}_1\times\bar{a}_2|}{\|\bar{q}\|^3}\right\}$$

$$=\frac{1}{\bar{n}\circ\bar{n}}\left\{\bar{n}\circ\bar{n}\;\frac{|\bar{a}_1\times\bar{a}_2|}{\|\bar{q}\|}\right\}$$

$$=\frac{|\bar{a}_1\times\bar{a}_2|}{\|\bar{q}\|} \qquad \text{(B.17)}$$

This proves that Equation (5.9) provides an analytical solution of the integral in Equation (5.5) for any quadrilateral element.

B.4 FINITE PARTS OF INTEGRALS

In order to extend the results to a planar quadrilateral element intersected with the Mach forecone, the finite part of the following special integral is investigated.

$$I=\int_{-\infty}^{a}\frac{\partial}{\partial x}\left(\frac{g(x)}{\sqrt{x}}H(x)\right)dx \qquad \text{(B.18)}$$

where g is a regular function and H is the Heaviside function, Equation (B.18) can be rewritten as

$$I=\lim_{\varepsilon\to 0}\int_{-\infty}^{a}\frac{\partial}{\partial x}\left[\frac{g}{\sqrt{x}}H(x-\varepsilon)\right]dx$$

$$= \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{a} \frac{\partial}{\partial x} \left(\frac{g}{\sqrt{x}} \right) H \, dx + \int_{-\infty}^{a} \frac{g}{\sqrt{x}} \delta(x-\varepsilon) \, dx \right]$$

$$= \lim_{\varepsilon \to 0} \left[\int_{\varepsilon}^{a} \frac{\partial}{\partial x} \left(\frac{g}{\sqrt{x}} \right) dx + \frac{g(\varepsilon)}{\sqrt{\varepsilon}} \right]$$

$$= \frac{g(a)}{\sqrt{a}} - \frac{g(\varepsilon)}{\sqrt{\varepsilon}} + \frac{g(\varepsilon)}{\sqrt{\varepsilon}} = \frac{g(a)}{\sqrt{a}} \qquad \text{(B.19)}$$

This shows that the singular contribution disappears and should not be taken into account.

Thus, consider the source integral given in Equation (B.9). According to Equations (B.15a) and (B.17), the integran of Equation (B.9), $\frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|}$, can be written as

$$\frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|} H = \frac{1}{\bar{n} \odot \bar{n}} \left\{ - \left[\frac{\bar{a}_2 \times \bar{a}_1 \odot \bar{n}}{\|\bar{q}\|} - \frac{\bar{q} \times \bar{\beta}_3 \odot \bar{n}}{\|\bar{q}\|} - \bar{q} \times \bar{a}_1 \odot \bar{n} \frac{\bar{q} \odot \bar{a}_2}{\|\bar{q}\|^3} \right] H \right.$$

$$+ \left[\frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}\|} - \frac{\bar{q} \times \bar{\beta}_3 \odot \bar{n}}{\|\bar{q}\|} - \bar{q} \times \bar{a}_2 \odot \bar{n} \frac{\bar{q} \odot \bar{a}_1}{\|\bar{q}\|^3} \right] H$$

$$\left. - \left[\bar{q} \cdot \bar{n} \frac{\bar{q} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{q}\|} \right] H \right\} \qquad \text{(B.20)}$$

therefore, Equation (B.9) can be rewritten as

$$b_{hK} = \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{H}{\|\bar{q}\|} |\bar{a}_1 \times \bar{a}_2| \, d\xi \, d\eta = \frac{1}{\pi} \frac{1}{\bar{n} \odot \bar{n}} (S_1 + S_2 + S_3) \qquad \text{(B.21)}$$

$$S_1 = I_{s_1}(1,1) - I_{s_1}(1,-1) - I_{s_1}(-1,1) + I_{s_1}(-1,-1) \qquad \text{(B.22a)}$$

$$= - \int_{-1}^{1} \int_{-1}^{1} H \left[\frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}\|} + \frac{\bar{q} \times \bar{\beta}_3 \odot \bar{n}}{\|\bar{q}\|^3} - \bar{q} \times \bar{a}_1 \odot \bar{n} \frac{\bar{q} \odot \bar{a}_2}{\|\bar{q}\|^3} \right] d\xi \, d\eta \qquad \text{(B.22b)}$$

$$S_2 = I_{s2}(1,1) - I_{s2}(1,-1) - I_{s2}(-1,1) + I_{s2}(-1,-1) \tag{B.23a}$$

$$= \int_{-1}^{1}\int_{-1}^{1} H\left[\frac{\bar{a}_1 \times \bar{a}_2 \circ \bar{n}}{\|\bar{q}\|} - \frac{\bar{q} \times \bar{a}_2 \circ \bar{n}}{\|\bar{q}\|} - \bar{q} \times \bar{a}_2 \circ \bar{n} \frac{\bar{q} \circ \bar{a}_1}{\|\bar{q}\|^3}\right] d\xi\, d\eta \tag{B.23b}$$

$$S_3 = I_{s3}(1,1) - I_{s3}(1,-1) - I_{s3}(-1,1) + I_{s3}(-1,-1) \tag{B.24a}$$

$$= -\int_{-1}^{1}\int_{-1}^{1} H\, \bar{q} \cdot \bar{n}\, \frac{\bar{q} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{q}\|} d\xi\, d\eta \tag{B.24b}$$

First, according to Equation (B.12b), Equation (B.22b) can be rewritten as

$$S_1 = -\int_{-1}^{1} d\xi \int_{-1}^{1} \frac{\partial}{\partial \eta}\left[H(\bar{q} \times \bar{a}_1 \circ \bar{n}) \frac{1}{\sqrt{\bar{q} \circ \bar{q}}}\right] d\eta \tag{B.25}$$

Note that $\sqrt{\bar{q} \circ \bar{q}}$ can be expressed as $h(\eta)\sqrt{\eta - \eta_0}$, where $h(\eta)$ is a regular function and η_0 is defined such that $\bar{q} \circ \bar{q}$. Compared with Equation (B.18), it can be concluded that the portions along the intersection line of the element with the Mach forecone yield no contributions to the integral S_1. Therefore, in Equation (B.22a)

$$I_{s1}(1,1) - I_{s1}(-1,1) = 0 \tag{B.26}$$

or

$$I_{s1}(1,-1) - I_{s1}(-1,-1) = 0 \tag{B.27}$$

if the edge of $\eta = 1$ or $\eta = -1$, respectively, is completely outside the Mach forecone. Otherwise, I_{S1} is given by Equation (5.10) if the corner point is inside the Mach forecone, or

$$\begin{aligned} I_{s1}(\xi, \eta^*) &= 0 && ; \ \bar{a}_1 \circ \bar{a}_1 \geq 0 \\ &= (\bar{q} \times \bar{a}_1 \cdot \bar{n})\, \mathrm{sign}(\bar{q} \circ \bar{a}_1)/\|\bar{a}_1\| && ; \ \bar{a}_1 \circ \bar{a}_1 < 0 \end{aligned} \tag{B.28}$$

if the corner point is outside the Mach forecone. Similarly, Equation (B.23b) can be rewritten as

$$S_2 = \int_{-1}^{1} d\eta \int_{-1}^{1} \frac{\partial}{\partial \eta} \left(H(\bar{q}_\beta \times \bar{a}_2 \circ \bar{n}) \frac{1}{\sqrt{\bar{q}_\beta \circ \bar{q}_\beta}} \right] d\xi \tag{B.29}$$

therefore.

$$I_{S2}(1,1) - I_{S2}(1,-1) = 0 \tag{B.30}$$

or

$$I_{S2}(-1,1) - I_{S2}(-1,-1) = 0 \tag{B.31}$$

if the edge of $\xi = 1$, or $\xi = -1$, respectively, is completely outside the Mach forecone. Otherwise, I_{S2} is given by Equation (5.11) if the corner point is inside the Mach forecone, or

$$\begin{aligned} I_{S2}(\xi^*, \eta) &= 0 && ; \bar{a}_2 \circ \bar{a}_2 \geq 0 \\ &= -(\bar{q}_\beta \times \bar{a}_2 \cdot \bar{n}) \operatorname{sign}(\bar{q}_\beta \circ \bar{a}_2) / \|\bar{a}_2\| && ; \bar{a}_2 \circ \bar{a}_2 < 0 \end{aligned} \tag{B.32}$$

if the corner point is outside the Mach forecone. Finally, considering Equation (D.10), Equations (B.24b) and (B.7) can be rewritten as

$$S_3 = -\bar{q}_\beta \cdot \bar{n} \int_{-1}^{1} d\xi \int_{-1}^{1} \frac{\partial}{\partial \eta} \left(\frac{H}{\|\bar{q}_\beta \times \bar{a}_1\|^2} (\bar{q}_\beta \circ \bar{a}_1 \; \bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2 - \bar{q}_\beta \circ \bar{q}_\beta \; \bar{q}_\beta \cdot \bar{a}_1 \times \bar{\beta}_3) \frac{1}{\|\bar{q}_\beta\|} \right] d\eta \tag{B.33}$$

and

$$C_{hK} = \frac{1}{\pi} \int_{-1}^{1} d\xi \int_{-1}^{1} \frac{\partial}{\partial \eta} \left(\frac{H}{\|\bar{q}_\beta \times \bar{a}_1\|^2} (\bar{q}_\beta \circ \bar{a}_1 \; \bar{q}_\beta \cdot \bar{a}_1 \times \bar{a}_2 - \bar{q}_\beta \circ \bar{q}_\beta \; \bar{q}_\beta \cdot \bar{a}_1 \times \bar{\beta}_3) \frac{1}{\|\bar{q}_\beta\|} \right] d\eta \tag{B.34}$$

with the same reason given above, one obtains

$$I_{S3}(1,1) - I_{S3}(-1,1) = 0 \tag{B.35}$$

$$I_D(1,1) - I_D(-1,1) = 0 \tag{B.36}$$

or

$$I_{S3}(1,-1) - I_{S3}(-1,-1) = 0 \tag{B.37}$$

$$I_D(1,-1) - I_D(-1,-1) = 0 \tag{B.38}$$

if the edge of $\eta=1$ or $\eta=-1$, respectively, is completely outside the Mach forecone. Otherwise, I_{S3} and I_D are given by Equations (5.12) and (5.8), if the corner point is inside the Mach forecone, or

$$I_{S3}(\xi,\eta^*) = (\bar{q}_\beta \cdot \bar{n}) \frac{\pi}{2} \text{sign}\left[(\bar{q}_\beta \odot \bar{a}_1)(\bar{q}_\beta \odot \bar{a}_2)(\bar{q}_\beta \cdot \bar{n})\right] \tag{B.39}$$

$$I_D(\xi,\eta^*) = -\frac{1}{2} \text{sign}\left[(\bar{q}_\beta \odot \bar{a}_1)(\bar{q}_\beta \odot \bar{a}_2)(\bar{q}_\beta \cdot \bar{n})\right] \tag{B.40}$$

Note that the values of ξ^* and η^* in Equations (B.32), (B.28), (B.39) and (B.40) are evaluated such that $\bar{q}_\beta \odot \bar{q}_\beta = 0$.

B.5 INTEGRAL EQUATIONS FOR DIAPHRAGM-ATTACHED CONFIGURATION

As mentioned in Subsection 5.3, if diaphragms are used, both values of the velocity potential and its normal derivative, ϕ_D and ψ_D, of the diaphragm element are unknowns while two integral equations are obtained for each diaphragm element. Therefore, a system of equations, instead of Equation (5.3), may be obtained as follows.

$$[\delta_{hK}' - a_{hK}] \begin{Bmatrix} \underline{\varphi_K^u} \\ \underline{\varphi_K^\ell} \\ \underline{\varphi_D} \\ \psi_D \end{Bmatrix} = [b_{hK}'] \begin{Bmatrix} \underline{\psi_K^u} \\ \underline{\psi_K^\ell} \\ \underline{0} \\ 0 \end{Bmatrix} \tag{B.41}$$

where $\underset{\sim}{\varphi_K^u}$ and $\underset{\sim}{\psi_K^u}$ (or $\underset{\sim}{\varphi_K^\ell}$ and $\underset{\sim}{\psi_K^\ell}$) are the velocity potential and its normal at the centroid of the element on the upper (or lower) surface of the aircraft, while $\underset{\sim}{\varphi_D}$ and $\underset{\sim}{\psi_D}$ are those quantities for the diaphragm element, and δ_{hK}' is given by

$$[\delta'_{hK}] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \tag{B.42}$$

$\underset{\sim}{a}_{hK}$ and $\underset{\sim}{b}_{hK}$ are given by

$$[a_{hK}] = \begin{bmatrix} C^u_{am} & 0 & C^u_{an} & b^u_{an} \\ 0 & C^\ell_{am} & C^\ell_{an} & -b^\ell_{an} \\ C^u_{dm} & 0 & C^u_{dn} & b^u_{dn} \\ 0 & C^\ell_{dm} & C^\ell_{dn} & -b^\ell_{dn} \end{bmatrix} \tag{B.43}$$

and

$$[b'_{hK}] = \begin{bmatrix} b^u_{am} & 0 & 0 & 0 \\ 0 & b^\ell_{am} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{B.44}$$

where the superscript u (or ℓ) reminds that both the control point and dummy element are on the upper (or lower) surface and first subscript a (ord) reminds that the control point is on the surface of aircraft (or diaphragm) second subscript m (or n) reminds that the dummy element is on the surface of aircraft (or diaphragm). For example, $\underset{\sim}{C}^u_{an}$ represents the submatrix formed by the doublet integrals, given by Equation (5.4), with control point on the upper surface of aircraft,

and dummy element on the upper surface of diaphragm. $\underset{\sim}{b}^{\ell}_{dm}$ represents the submatrix formed by the source integrals, given by Equation (5.5) with control point on the lower surface of diaphragm and dummy element on the lower surface of aircraft.

Furthermore, if the problem is symmetric, the normal derivative of the velocity potential, ψ_D, on the diaphragm is zero, while the velocity potential, φ_D, is unknown,

$$\underset{\sim}{\varphi}^{u}_{K} = \underset{\sim}{\varphi}^{\ell}_{K}, \qquad \underset{\sim}{\psi}^{u}_{K} = \underset{\sim}{\psi}^{\ell}_{K}$$

$$\underset{\sim}{C}^{u}_{am} = \underset{\sim}{C}^{\ell}_{am}, \qquad \underset{\sim}{C}^{u}_{an} = \underset{\sim}{C}^{\ell}_{an}$$

$$\underset{\sim}{C}^{u}_{dm} = \underset{\sim}{C}^{\ell}_{dm}, \qquad \underset{\sim}{C}^{u}_{dn} = \underset{\sim}{C}^{\ell}_{dn}$$

$$\underset{\sim}{b}^{u}_{am} = \underset{\sim}{b}^{\ell}_{am}, \qquad \underset{\sim}{b}^{u}_{an} = \underset{\sim}{b}^{\ell}_{an}$$

$$\underset{\sim}{b}^{u}_{dm} = \underset{\sim}{b}^{\ell}_{dm}, \qquad \underset{\sim}{b}^{u}_{dn} = \underset{\sim}{b}^{\ell}_{dn} \tag{B.45}$$

therefore, Equation (B.41) reduced to

$$[\delta_{hK} - a_{hK}] \left\{ -\frac{\varphi^{u}_{K}}{\varphi_D} \right\} = [b_{hK}] \left\{ -\frac{\chi^{u}_{K}}{0} \right\} \tag{B.46}$$

with

$$[a_{hK}] = \left[\begin{array}{c|c} C^{u}_{am} & C^{u}_{an} \\ \hline C^{u}_{dn} & C^{u}_{dn} \end{array} \right] \tag{B.47}$$

and

$$[b_{hK}] = \left[\begin{array}{c|c} b^{u}_{am} & 0 \\ \hline 0 & 0 \end{array} \right] \tag{B.48}$$

For the case that the problem is antisymmetric, φ_D, is zero while ψ_D is unknown. Then Equation (B.41) reduced to Equation (B.46) with $[b_{hk}]$ given by (B.48) and $[a_{hk}]$ given by

$$[a_{hK}] = \left[\begin{array}{c|c} C^{u}_{am} & b^{u}_{an} \\ \hline C^{u}_{dm} & b^{u}_{dn} \end{array}\right] \qquad (B.49)$$

APPENDIX C

FORMULATIONS FOR OSCILLATORY FLOW

C.1 BOUNDARY CONDITION

The boundary condition for oscillatory flow is given by

$$\nabla_{xyz} S \cdot \nabla_{xyz} \varphi = -\frac{\partial S}{\partial t} - U_\infty \frac{\partial S}{\partial x} \tag{C.1}$$

C.1.1 Subsonic Flow

By introducing variables X, Y, Z, T, Ω and ϕ given by Equations (2.2) and (3.3), and noting that $\beta^2 = 1 - M^2$ Equation (C.1) yields

$$\nabla_{XYZ} S \cdot \nabla_{XYZ} \phi + \frac{\beta}{M} \frac{\partial S}{\partial T} + \frac{1}{\beta} \frac{\partial S}{\partial X} + \frac{M^2}{\beta^2} \frac{\partial S}{\partial X} \frac{\partial \phi}{\partial X} = 0 \tag{C.2}$$

The motion of the surface is assumed to consist of small harmonic oscillations around a rest configuration that is

$$S = S_0(X, Y, Z) + \tilde{S}(X, Y, Z)\, e^{i\Omega T} \tag{C.3}$$

and

$$\phi = \phi_0(X, Y, Z) + \tilde{\phi}(X, Y, Z)\, e^{i\Omega T} \tag{C.4}$$

Substituting Equations (C.3) and (C.4) into Equation (C.2) yields

$$\begin{aligned} &\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi_0 + (\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \tilde{\phi} + \nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \phi_0) e^{i\Omega T} + \\ &+ (\nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \tilde{\phi}) e^{i2\Omega T} + \frac{\beta}{M} i\Omega \tilde{S} e^{i\Omega T} + \frac{1}{\beta} \left(\frac{\partial S_0}{\partial X} + \frac{\partial \tilde{S}}{\partial X} e^{i\Omega T} \right) + \\ &+ \frac{M^2}{\beta^2} \left[\frac{\partial S_0}{\partial X} \frac{\partial \phi_0}{\partial X} + \left(\frac{\partial S_0}{\partial X} \frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \phi_0}{\partial X} \right) e^{i\Omega T} + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \tilde{\phi}}{\partial X} e^{2i\Omega T} \right] = 0 \end{aligned} \tag{C.5}$$

Small terms can be dropped out by investigating the order of each quantity. Assuming*

*For instance; if $z = \epsilon z_0(x, y) + \epsilon^2 \tilde{z}(x, y) e^{i\Omega T}$, then $S_0 = z - \epsilon z_0(x, y)$, $\tilde{S} = \epsilon^2 z(x, y)$.

$$S_o = O(1) \tag{C.6}$$

$$\frac{\partial S_o}{\partial X} = O(\varepsilon) \tag{C.7}$$

with

$$|\nabla_{XYZ} S_o| = O(1) \tag{C.8}$$

and

$$\tilde{S} = O(\varepsilon^2) \tag{C.9}$$

$$\frac{\partial \tilde{S}}{\partial X} = O(\varepsilon^2) \tag{C.10}$$

$$|\nabla_{XYZ} \tilde{S}| = O(\varepsilon^2) \tag{C.11}$$

Also, [see, for instance, Equations (C.16) and (C.17)], it is noted that

$$|\nabla_{XYZ} \phi_o| = O(\varepsilon) \tag{C.12}$$

$$|\nabla_{XYZ} \tilde{\phi}| = O(\varepsilon^2) \tag{C.13}$$

Investigating the order of each term of Equation (C.5) by Equations (C.6) to (C.13) and ignoring the terms which contain $e^{i2\Omega T}$ (of order ε^4), one obtains the stationary and the oscillatory boundary conditions which are given as

$$\nabla_{XYZ} S_o \cdot \nabla_{XYZ} \phi_o + \frac{1}{\beta} \frac{\partial S_o}{\partial X} + \frac{M^2}{\beta^2} \frac{\partial S_o}{\partial X} \frac{\partial \phi_o}{\partial X} = 0 \tag{C.14}$$

and

$$\nabla_{XYZ} S_o \cdot \nabla_{XYZ} \tilde{\phi} + \nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \phi_o + \frac{\beta}{M} i\Omega \tilde{S}$$

$$+ \frac{1}{\beta} \frac{\partial \tilde{S}}{\partial X} + \frac{M^2}{\beta^2} \left(\frac{\partial S_o}{\partial X} \frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \phi_o}{\partial X} \right) = 0 \tag{C.15}$$

Neglecting terms of order ε^2 in Equation (C.14) and terms of order ε^3 in Equation (C.15), one obtains

$$\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi_0 = -\frac{1}{\beta} \frac{\partial S_0}{\partial X} \tag{C.16}$$

$$\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi = -\left(i \frac{\beta}{M} \Omega \tilde{S} + \frac{1}{\beta} \frac{\partial \tilde{S}}{\partial X} \right) \tag{C.17}$$

In particular for*

$$S = \pm \frac{1}{\ell} \left[Z - Z_0(X,Y) - \tilde{Z}(X,Y) e^{i\omega t} \right] \tag{C.18}$$

(where the upper or lower sign holds on the upper or lower surface); one obtains

$$S_0 = \pm \frac{1}{\ell} \left[Z - Z_0(X,Y) \right] \tag{C.19}$$

$$\tilde{S} = \mp \frac{1}{\ell} \tilde{Z}(X,Y) \tag{C.20}$$

$$\frac{1}{|\nabla_{XYZ} S_0|} = \frac{\left| \frac{\partial S_0}{\partial Z} \right|}{|\nabla_{XYZ} S_0|} = |Z_Z| = \pm N_Z \tag{C.21}$$

Therefore,

$$\frac{\partial \phi}{\partial N} = \frac{\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi}{|\nabla_{XYZ} S_0|} = N_Z \left(i K \frac{\tilde{Z}}{\ell} + \frac{\partial \tilde{Z}}{\partial X} \right) \tag{C.22}$$

where

$$K = \beta \Omega / M = \omega \ell / U_\infty \tag{C.23}$$

By using Equation (2.31), Equation (C.22) yields

$$\frac{\partial \hat{\phi}}{\partial N} = N_Z \left(i K \tilde{Z} + \frac{1}{\beta} \frac{\partial \tilde{Z}}{\partial X} \right) e^{-i \Omega M X} \tag{C.24}$$

Equation (C.20) gives the value of $\frac{\partial \hat{\phi}}{\partial N}$ to be used in Equation (4.3).

---

*Note that z_0 is of order ε and z is of order ε^2.

C.1.2 Supersonic Flow

By introducing variable x, y, z, T, Ω and ϕ given by Equations (2.2) and (3.3), and noting that $\beta^2 = M^2 - 1$, Equation (C.1) yields

$$\nabla_{XYZ} S \cdot \nabla^c_{XYZ} \phi + \frac{\beta}{M} \frac{\partial S}{\partial T} + \frac{1}{\beta} \frac{\partial S}{\partial X} + \frac{M^2}{\beta^2} \frac{\partial S}{\partial X} \frac{\partial \phi}{\partial X} = 0 \qquad \text{(C.25)}$$

with ∇^c_{xyz} defined by

$$\nabla^c_{XYZ} = -\frac{\partial}{\partial X} \bar{i} + \frac{\partial}{\partial Y} \bar{j} + \frac{\partial}{\partial Z} \bar{k} \qquad \text{(C.26)}$$

By replacing ∇_{xyz} with ∇^c_{xyz}, $\nabla_{xyz}\phi_0$ with $\nabla^c_{xyz}\phi_0$ and $\nabla_{xyz}\tilde{\phi}$ with $\nabla^c_{xyz}\tilde{\phi}$, Equations (C.5) to (C.22) are valid for supersonic flow. Therefore, the boundary condition for supersonic flow can be rewritten as

$$\frac{\partial \phi}{\partial N^c} = N_z \left(i K \tilde{z} + \frac{\partial \tilde{z}}{\partial x} \right) \qquad \text{(C.27)}$$

By using Equation (2.39), Equation (C.27) yields

$$\frac{\partial \hat{\phi}}{\partial N^c} = N_z \left(i K \tilde{z} + \frac{1}{\beta} \frac{\partial \tilde{z}}{\partial x} \right) e^{i\Omega M x} \qquad \text{(C.28)}$$

Equation (C.28) gives the value of $\frac{\partial \hat{\phi}}{\partial N^c}$ to be used in Equation (6.3).

C.2 PRESSURE COEFFICIENT

The pressure coefficient is given by the linear Bernoulli theorem, Equation (2.41), as

$$C_p = -\frac{2}{U_\infty^2} \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} \right)$$

$$= -2 \left(\frac{\beta}{M} \frac{\partial \phi}{\partial T} + \frac{1}{\beta} \frac{\partial \phi}{\partial X} \right) \qquad \text{(C.29)}$$

For oscillatory flow, setting

$$\phi = \tilde{\phi}\, e^{i\Omega\tau} \tag{C.30}$$

$$C_p = \tilde{C}_p\, e^{i\Omega\tau}$$

one obtains

$$\tilde{C}_p = -2\left(\frac{\beta}{M} i\Omega\tilde{\phi} + \frac{1}{\beta}\frac{\partial\tilde{\phi}}{\partial x}\right)$$

$$= -\frac{2}{\beta}\left(\frac{\partial\tilde{\phi}}{\partial x} + i\kappa\beta\tilde{\phi}\right) \tag{C.31}$$

C.2.1 Subsonic Flow

By using Equation (2.31)

$$\tilde{\phi} = \tilde{\phi}\, e^{i\Omega M x} \tag{C.32}$$

C_p for oscillatory subsonic flow can be rewritten from Equation (C.31) as

$$\tilde{C}_p = -\frac{2}{\beta} e^{-i\kappa\beta x}\frac{\partial}{\partial x}\left(\tilde{\phi}\, e^{i\kappa\beta x}\right)$$

$$= -\frac{2}{\beta} e^{-i\kappa\beta x}\frac{\partial}{\partial x}\left(\hat{\phi}\, e^{i\kappa x/\beta}\right) \tag{C.33}$$

C.2.2 Supersonic Flow

By using Equation (2.39)

$$\tilde{\phi} = \hat{\phi}\, e^{-i\Omega M x} \tag{C.34}$$

C_p for oscillatory supersonic flow can be rewritten from Equation (C.31) as

$$\tilde{C}_p = -\frac{2}{\beta} e^{-i\kappa\beta x}\frac{\partial}{\partial x}\left(\tilde{\phi}\, e^{i\kappa\beta x}\right)$$

$$= -\frac{2}{\beta} e^{-i\kappa\beta x}\frac{\partial}{\partial x}\left(\hat{\phi}\, e^{-i\kappa x/\beta}\right) \tag{C.35}$$

APPENDIX D

USEFUL EQUATIONS

D.1 INTRODUCTION

In deriving the formulations presented in previous sections, several useful equations are applied. They are proven in this appendix.

For the subsonic flow theory, the following two equations are used.

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})=(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d})-(\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c}) \tag{D.1}$$

$$(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)+\left[(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)\right]^2\equiv|\bar{q}\times\bar{a}_1|^2|\bar{q}\times\bar{a}_2|^2 \tag{D.2}$$

For the supersonic flow theory, the following five equations are used.

$$(\bar{a}\times\bar{b})\odot(\bar{c}\times\bar{d})=(\bar{a}\odot\bar{c})(\bar{b}\odot\bar{d})-(\bar{a}\odot\bar{d})(\bar{b}\odot\bar{c}) \tag{D.3}$$

$$(\bar{q}\odot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)+\left[(\bar{q}\times\bar{a}_1)\odot(\bar{q}\times\bar{a}_2)\right]^2=\|\bar{q}\times\bar{a}_1\|^2\|\bar{q}\times\bar{a}_2\|^2 \tag{D.4}$$

$$\frac{\partial F_1}{\partial\xi}=\frac{1}{\|\bar{q}\|} \tag{D.5}$$

$$\frac{\partial F_2}{\partial\eta}\quad\frac{1}{\|\bar{q}\|} \tag{D.6}$$

$$(\bar{a}\odot\bar{a})(\bar{b}\times\bar{c})\odot(\bar{b}\times\bar{c})-(\bar{a}\cdot\bar{b}\times\bar{c})^2$$
$$=(\bar{a}\odot\bar{c})(\bar{b}\times\bar{c})\odot(\bar{b}\times\bar{a})+\bar{a}\odot\bar{b}\,(\bar{c}\times\bar{b})\odot(\bar{c}\times\bar{a}) \tag{D.7}$$

D.2 PROOF OF EQUATION (D.1)

$$(\bar{A}\cdot\bar{B})(\bar{C}\cdot\bar{D})-(\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{D})$$

$$=(A_xB_x+A_yB_y+A_zB_z)(C_xD_x+C_yD_y+C_zD_z)$$

$$-(A_xC_x+A_yC_y+A_zC_z)(B_xD_x+B_yD_y+B_zD_z)$$

$$=A_xB_xC_xD_x+A_yB_yC_yD_y+A_zB_zC_zD_z+A_xB_xC_yD_y+A_xB_xC_zD_z$$

$$+ A_Y B_Y C_X D_X + A_Z B_Z C_X D_X + A_Y B_Y C_Z D_Z + A_Z B_Z C_Y D_Y - A_X C_X B_X D_X$$

$$- A_Y C_Y B_Y D_Y - A_Z C_Z B_Z D_Z - A_X C_X B_Y D_Y - A_X C_X B_Z D_Z - A_Y C_Y B_X D_X$$

$$- A_Z C_Z B_X D_X - A_Y C_Y B_Z D_Z - A_Z C_Z B_X D_Y$$

while

$$(\bar{A} \times \bar{D}) \cdot (\bar{B} \times \bar{C})$$

$$= (A_Y D_Z - A_Z D_Y)(B_Y C_Z - B_Z C_Y) + (A_Z D_X - A_X D_Z)(B_Z C_X - B_X C_Z) + (A_X D_Y - A_Y D_X)(B_X C_Y - B_Y C_X)$$

$$= A_Y B_Y C_Z D_Z - A_Y B_Z C_Y D_Z - (A_Z B_Y C_Z D_Y - A_Z B_Z C_Y D_Y)$$

$$+ A_Z B_Z C_X D_X + A_X B_X C_Z D_Z - (A_Z C_Z B_X D_X + A_X C_X B_Z D_Z)$$

$$+ A_X B_X C_Y D_Y + A_Y B_Y C_X D_X - (A_Y C_Y B_X D_X + A_X C_X B_Y D_Y) \qquad (D.8)$$

D.3 PROOF OF EQUATION (D.2)

Following is the proof of Equation (D.2):

$$|\bar{q} \times \bar{a}_1|^2 |\bar{q} \times \bar{a}_2|^2 - \left\{ (\bar{q} \cdot \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 + [(\bar{q} \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2)]^2 \right\}$$

$$= [(\bar{q} \cdot \bar{q})(\bar{a}_1 \cdot \bar{a}_1) - (\bar{q} \cdot \bar{a}_1)^2][(\bar{q} \cdot \bar{q})(\bar{a}_2 \cdot \bar{a}_2) - (\bar{q} \cdot \bar{a}_2)^2] - (\bar{q} \cdot \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2$$

$$- [(\bar{q} \cdot \bar{q})(\bar{a}_1 \cdot \bar{a}_2) - (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_2)]^2$$

$$= (\bar{q} \cdot \bar{q})^2 (\bar{a}_1 \cdot \bar{a}_1)(\bar{a}_2 \cdot \bar{a}_2) + (\bar{q} \cdot \bar{a}_1)^2 (\bar{q} \cdot \bar{a}_2)^2 - (\bar{q} \cdot \bar{q})[(\bar{a}_1 \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_2)^2 + (\bar{a}_2 \cdot \bar{a}_2)(\bar{q} \cdot \bar{a}_1)^2]$$

$$- (\bar{q} \cdot \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 - (\bar{q} \cdot \bar{q})^2 (\bar{a}_1 \cdot \bar{a}_2)^2 - (\bar{q} \cdot \bar{a}_1)^2 (\bar{q} \cdot \bar{a}_2)^2 + 2(\bar{q} \cdot \bar{q})(\bar{a}_1 \cdot \bar{a}_2)(\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_2)$$

$$= (\bar{q} \cdot \bar{q}) \left\{ (\bar{q} \cdot \bar{q})[(\bar{a}_1 \cdot \bar{a}_1)(\bar{a}_2 \cdot \bar{a}_2) - (\bar{a}_1 \cdot \bar{a}_2)^2] - (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 - [(\bar{a}_1 \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_2)^2 + \right.$$

$$\left. + (\bar{a}_2 \cdot \bar{a}_2)(\bar{q} \cdot \bar{a}_1)^2 - 2(\bar{a}_1 \cdot \bar{a}_2)(\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_2)] \right\}$$

$$= (\bar{q}\cdot\bar{q})\left\{(\bar{q}\cdot\bar{q})(\bar{a}_1\times\bar{a}_2)\cdot(\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2 - |\bar{a}_1(\bar{q}\cdot\bar{a}_2) - \bar{a}_2(\bar{q}\cdot\bar{a}_1)|^2\right\}$$

$$= (\bar{q}\cdot\bar{q})\left\{|\bar{q}\times(\bar{a}_1\times\bar{a}_2)|^2 - |\bar{a}_1(\bar{q}\cdot\bar{a}_2) - \bar{a}_2(\bar{q}\cdot\bar{a}_2)|^2\right\}$$

$$= 0 \qquad \text{(D.9)}$$

since, by using Equation (D.1),

$$|\bar{q}\times(\bar{a}_1\times\bar{a}_2)|^2 = [\bar{q}\times(\bar{a}_1\times\bar{a}_2)]\cdot[\bar{q}\times(\bar{a}_1\times\bar{a}_2)]$$

$$= (\bar{q}\cdot\bar{q})(\bar{a}_1\times\bar{a}_2)\cdot(\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2 \qquad \text{(D.10)}$$

and by using $\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$, yields

$$\bar{q}\times(\bar{a}_1\times\bar{a}_2) = \bar{a}_1(\bar{q}\cdot\bar{a}_1) - \bar{a}_2(\bar{q}\cdot\bar{a}_1) \qquad \text{(D.11)}$$

D.4 PROOF OF EQUATION (D.3)

Note that

$$\begin{aligned}
&(\bar{a}\times\bar{b})\circ(\bar{c}\times\bar{d}) \\
&= (a_Y b_Z - a_Z b_Y)(C_Y d_Z - C_Z d_Y) - (C_Z b_X - a_X b_Z)(C_Z d_X - C_X d_Z) \\
&\quad - (a_X b_Y - a_Y b_X)(C_X d_Y - C_Y d_X) \\
&= a_Y b_Z C_Y d_Z + a_Z b_Y C_Z d_Y - a_Y b_Z C_Z d_Y - a_Z b_Y C_Y d_Z \\
&\quad - a_Z b_X C_Z d_X - a_X b_Z C_X d_Z + a_Z b_X C_X d_Z + a_X b_Z C_Z d_X \\
&\quad - a_X b_Y C_X d_Y - a_Y b_X C_Y d_X + a_X b_Y C_Y d_X + a_Y b_X C_X d_Y
\end{aligned} \qquad \text{(D.12)}$$

while

$$(\bar{a}\circ\bar{c})(\bar{b}\circ\bar{d}) - (\bar{a}\circ\bar{d})(\bar{b}\circ\bar{c})$$

$$= (a_X C_X - a_Y C_Y - a_Z C_Z)(b_X d_X - b_Y d_Y - b_Z d_Z)$$

$$-(a_X d_X - a_Y d_Y - a_Z d_Z)(b_X C_X - b_Y C_Y - b_Z C_Z)$$

$$= a_X C_X (b_X d_X - b_Y d_Y - b_Z d_Z) - a_Y C_Y (b_X d_X - b_Y d_Y - b_Z d_Z) -$$

$$- a_Z C_Z (b_X d_X - b_Y d_Y - b_Z d_Z) - a_X d_X (b_X C_X - b_Y C_Y - b_Z C_Z) +$$

$$+a_y d_y (b_x c_x - b_y c_y - b_z c_z) + a_z d_z (b_x c_x - b_y c_y - b_z c_z)$$
$$= a_y b_z c_y d_z + a_z b_y c_z d_y - a_y b_z c_z d_y - a_z b_y c_y d_z - a_z b_x c_z d_x - a_x b_z c_z d_z$$
$$+a_z b_x c_x d_z - a_x b_y c_y d_y - a_y b_x c_y d_x + a_x b_y c_y d_x + a_y b_x c_x d_y + a_x b_z c_z d_x \qquad \text{(D.13)}$$

D.5 PROOF OF EQUATION (D.4)

$$\bar{q} \circ \bar{q} \, (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2)^2$$

$$= (\bar{q} \circ \bar{q}) \left[(\bar{q} \circ \bar{q}) (\bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) - (\bar{q} \circ \bar{a}_2)(\bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{q}) - (\bar{q} \circ \bar{a}_1)(\bar{a}_2 \times \bar{a}_1 \circ \bar{a}_2 \times \bar{q}) \right]$$

$$+ \left[(\bar{q} \circ \bar{q})(\bar{a}_1 \circ \bar{a}_2) - (\bar{q} \circ \bar{a}_1)(\bar{q} \circ \bar{a}_2) \right]^2$$

$$= (\bar{q} \circ \bar{q})^2 \left[(\bar{a}_1 \circ \bar{a}_2)(\bar{a}_2 \circ \bar{a}_2) - (\bar{a}_1 \circ \bar{a}_2)^2 \right] - (\bar{q} \circ \bar{q})(\bar{q} \circ \bar{a}_2) \left[(\bar{a}_1 \circ \bar{a}_1)(\bar{a}_2 \circ \bar{q}) - (\bar{a}_1 \circ \bar{q})(\bar{a}_2 \circ \bar{a}_1) \right]$$

$$- (\bar{q} \circ \bar{q})(\bar{q} \circ \bar{a}_1) \left[(\bar{a}_2 \circ \bar{a}_2)(\bar{a}_1 \circ \bar{q}) - (\bar{a}_2 \circ \bar{q})(\bar{a}_1 \circ \bar{a}_2) \right] + (\bar{q} \circ \bar{q})^2 (\bar{a}_1 \circ \bar{a}_2)^2 -$$

$$- 2 (\bar{q} \circ \bar{q})(\bar{a}_1 \circ \bar{a}_2)(\bar{q} \circ \bar{a}_2)(\bar{q} \circ \bar{a}_2) + (\bar{q} \circ \bar{a}_1)^2 (\bar{q} \circ \bar{a}_2)^2$$

$$= \left[(\bar{q} \circ \bar{q})(\bar{a}_1 \circ \bar{a}_1) - (\bar{q} \circ \bar{a}_1)^2 \right] \left[(\bar{q} \circ \bar{q})(\bar{a}_2 \circ \bar{a}_2) - (\bar{q} \circ \bar{a}_2)^2 \right]$$

$$= (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_1) = (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_1)(\bar{q} \times \bar{a}_2 \circ \bar{q} \times \bar{a}_2)$$

$$= \| \bar{q} \times \bar{a}_1 \|^2 \, \| \bar{q} \times \bar{a}_2 \|^2$$

D.6 PROOF OF EQUATIONS (D.5) and (D.6)

Proof of Equation (D.6) is shown below. Proof of Equation (D.5) is similar; therefore, it is omitted. There are three different cases of F_2. Considering the first case $\bar{a}_2 \circ \bar{a}_2 > 0$, and noting that

$$\frac{\partial}{\partial \eta} \| \bar{a}_2 \| = 0 \qquad \text{(D.14)}$$

and

$$\frac{\partial}{\partial \eta}\left(\bar{q}\times\bar{a}_2 \odot \bar{q}\times\bar{a}_2\right)$$

$$= 2\left(\frac{\partial \bar{q}}{\partial \eta}\times\bar{a}_2\right)\odot\left(\bar{q}\times\bar{a}_2\right)$$

$$= 2\left(\bar{a}_2\times\bar{a}_2\right)\odot\left(\bar{q}\times\bar{a}_2\right) = 0 \qquad \text{(D.15)}$$

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta}\left[\frac{1}{\|\bar{a}_2\|}\ln\left|\frac{\|\bar{q}\|\,\|\bar{a}_2\| + \bar{q}\odot\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right|\right]$$

$$= \frac{1}{\|\bar{a}_2\|}\left[\frac{\partial}{\partial \eta}\ln\left|\|\bar{q}\|\,\|\bar{a}_2\| + \bar{q}\odot\bar{a}_2\right| - \frac{\partial}{\partial \eta}\ln\|\bar{q}\times\bar{a}_2\|\right]$$

$$= \frac{1}{\|\bar{a}_2\|}\,\frac{1}{\|\bar{q}\|\,\|\bar{a}_2\| + \bar{q}\odot\bar{a}_2}\left(\frac{\bar{q}\odot\bar{a}_2}{\sqrt{\bar{q}\odot\bar{q}}}\sqrt{\bar{a}_2\odot\bar{a}_2} + \bar{a}_2\odot\bar{a}_2\right)$$

$$= \frac{1}{\|\bar{a}_2\|}\,\frac{1}{\|\bar{q}\|\,\|\bar{a}_2\| + \bar{q}\odot\bar{a}_2}\left(\bar{q}\odot\bar{a}_2\,\|\bar{a}_2\| + \|\bar{q}\|\,\|\bar{a}_2\|^2\right)\frac{1}{\|\bar{q}\|}$$

$$= \frac{1}{\|\bar{q}\|} \qquad \text{(D.16)}$$

Consider the second case, $\bar{a}_2 \odot \bar{a}_2 = 0$

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta}\left[\frac{\|\bar{q}\|}{\bar{q}\odot\bar{a}_2}\right]$$

$$= \frac{1}{\bar{q}\odot\bar{a}_2}\,\frac{\partial}{\partial \eta}\|\bar{q}\|$$

$$= \frac{1}{\bar{q}\odot\bar{a}_2}\,\frac{\bar{q}\odot\bar{a}_2}{\sqrt{\bar{q}\odot\bar{q}}} = \frac{1}{\|\bar{q}\|} \qquad \text{(D.17)}$$

since

$$\frac{\partial}{\partial \eta}\left(\bar{q}\odot\bar{a}_2\right) = \left(\frac{\partial}{\partial \eta}\bar{q}\right)\odot\bar{a}_2 = \bar{a}_2\odot\bar{a}_2 = 0 \qquad \text{(D.18)}$$

Consider the third case, $\bar{a}_2 \odot \bar{a}_2 < 0$

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta}\left[-\frac{1}{\|\bar{a}_2\|}\sin^{-1}\left(\frac{\bar{q}\odot\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right)\right] = -\frac{1}{\|\bar{a}_2\|}\frac{\partial}{\partial \eta}\sin^{-1}\left(\frac{\bar{q}\odot\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right)$$

$$= -\frac{1}{\|\bar{a}_2\|}\;\frac{1}{\sqrt{1-\left(\frac{\bar{q}\odot\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right)^2}}\;\frac{\bar{a}_2\odot\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}$$

$$= \frac{1}{\|\bar{a}_2\|}\;\frac{\|\bar{q}\times\bar{a}_2\|}{\sqrt{-\bar{q}\times\bar{a}_2\odot\bar{q}\times\bar{a}_2-(\bar{q}\odot\bar{a}_2)^2}}\;\frac{\|\bar{a}_2\|^2}{\|\bar{q}\times\bar{a}_2\|}$$

$$= \frac{\|\bar{a}_2\|}{\sqrt{(\bar{q}\odot\bar{q})(-\bar{a}_2\odot\bar{a}_2)}} = \frac{1}{\|\bar{q}\|} \qquad \text{(D.19)}$$

Combining (B.5), (B.6), and (B.7), yields

$$\frac{\partial F_2}{\partial \eta} = \frac{1}{\|\bar{q}\|} \qquad \bar{a}_2\odot\bar{a}_2 \gtreqless 0 \qquad \text{(D.20)}$$

Similarly,

$$\frac{\partial F_1}{\partial \xi} = \frac{1}{\|\bar{q}\|} \qquad \bar{a}_1\odot\bar{a}_1 \gtreqless 0 \qquad \text{(D.21)}$$

D.7 PROOF OF EQUATION D.7

Consider the regular vector algebra rule

$$\bar{a}\times(\bar{b}\times\bar{c}) = \bar{b}(\bar{a}\cdot\bar{c}) - \bar{c}(\bar{a}\cdot\bar{b}) \qquad \text{(D.22)}$$

and note

$$\bar{a}\odot\bar{b} = \bar{a}^c\cdot\bar{b} = \bar{a}\cdot\bar{b}^c$$

One obtains

$$\bar{a}^c \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a}^c \cdot \bar{c}) - \bar{c}(\bar{a}^c \cdot \bar{b})$$

$$= \bar{b}(\bar{a} \circ \bar{c}) - \bar{c}(\bar{a} \circ \bar{b}) \qquad \text{(D.23)}$$

On the other hand, according to Equations (D.3) and (D.22)

$$\bar{a}^c \times (\bar{b} \times \bar{c}) \circ \bar{a}^c \times (\bar{b} \times \bar{c})$$

$$= \bar{a}^c \circ \bar{a}^c \, (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - [\bar{a}^c \circ (\bar{b} \times \bar{c})]^2$$

$$= \bar{a} \circ \bar{a} \, (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b} \times \bar{c})^2$$

while, according to Equations (D.3) and (D.22)

$$[\bar{b}(\bar{a} \circ \bar{c}) - \bar{c}(\bar{a} \circ \bar{b})] \circ [\bar{b}(\bar{a} \circ \bar{c}) - \bar{c}(\bar{a} \circ \bar{b})]$$

$$= \bar{b} \circ \bar{b} \, (\bar{a} \circ \bar{c})^2 - 2\bar{b} \circ \bar{c} \; \bar{a} \circ \bar{c} \; \bar{a} \circ \bar{b} + \bar{c} \circ \bar{c} \, (\bar{a} \circ \bar{b})^2$$

$$= \bar{a} \circ \bar{c} \, (\bar{b} \times \bar{c} \circ \bar{b} \times \bar{a}) + \bar{a} \circ \bar{b} \, (\bar{c} \times \bar{b} \circ \bar{c} \times \bar{a}) \qquad \text{(D.24)}$$

Combining Equations (D.22), (D.23) and (D.24) yields

$$\bar{a} \circ \bar{a} \, (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b} \times \bar{c})^2$$

$$= [\bar{a}^c \times (\bar{b} \times \bar{c})] \circ [\bar{a}^c \times (\bar{b} \times \bar{c})]$$

$$= [\bar{b}(\bar{a} \circ \bar{c}) - \bar{c}(\bar{a} \circ \bar{b})] \circ [\bar{b}(\bar{a} \circ \bar{c}) - \bar{c}(\bar{a} \circ \bar{b})]$$

$$= \bar{a} \circ \bar{c} \, (\bar{b} \times \bar{c} \circ \bar{b} \times \bar{a}) + \bar{a} \circ \bar{b} \, (\bar{c} \times \bar{b} \circ \bar{c} \times \bar{a}) \qquad \text{(D.25)}$$

APPENDIX E

LIST OF COMPUTER PROGRAM SOSSA ACTS

MAIN PROGRAM

```
      COMPLEX AA,BB,SOURCA,SOURCB,VELPOT
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
C
      DIMENSION YK(3,16,16,5),PC(3,100),YPP(3,100),YPM(3,100),YMP(3,1
     1),YMM(3,100),KWAKE(100),SOURCA(100),SOURCB(100),AA(10000),VELPOT(
     1100),BB(10000)
C
      EQUIVALENCE (YK(1,1,1,1),AA(1))
C
      NREAD=5
      NWRITE=6
      NXMAX=15
      NYMAX=15
      NTMAX=100
      NXMAXP=NXMAX+1
      NYMAXP=NYMAX+1
C
      READ(NREAD,2)NCASE
2     FORMAT(1X,I2)
      DO 999 ICASE=1,NCASE
      WRITE(NWRITE,1)
1     FORMAT('1'//////)
      NLOOP=1
      CALL COODPT(NTOTAL,NXMAXP,NYMAXP,YK)
      CALL CHECK(NTOTAL,NXMAX,NYMAX,NTMAX)
      NT2S=NTOTAL**2
      CALL GEOMET(NTOTAL,NXMAXP,NYMAXP,YK,YPP,YPM,YMP,YMM,KWAKE)
      CALL VEC123(NTOTAL,PC,YPP,YPM,YMP,YMM)
      CALL COEFF(NTOTAL,PC,YPP,YPM,YMP,YMM,KWAKE,NT2S,AA,SOURCA,SOURCB)
150   CONTINUE
      DO 160 NNN=1,NT2S
160   BB(NNN)=AA(NNN)
      CALL SOLUTN(NTOTAL,NT2S,BB,SOURCA,VELPOT)
      CALL COEFPR(NTOTAL,PC,YPP,YPM,YMP,YMM,SOURCA,VELPOT)
      IF(IZZZ.LT.500)GO TO 999
      IF(NLOOP.EQ.2)GO TO 999
      WRITE(NWRITE,1)
      DO 200 I=1,NTOTAL
200   SOURCA(I)=SOURCB(I)
      NLOOP=2
      GO TO 150
999   CONTINUE
      STOP
      END
```

```
C
      SUBROUTINE COODPT(NTOTAL,NXMAXP,NYMAXP,YK)
      DOUBLE PRECISION TODAY
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ10/KWAKES(20),NWAKE,KXINCR
      COMMON/ZZZ11/ALFA,ALFABC,REDFRE,R,XLEZ,XTEZ,XNOSE,XTAIL,KCM
      COMMON/ZZZ12/NSFX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(20)
      COMMON/ZZZ13/DCHORD(10),CHAXIS(10)
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
      DIMENSION ACHORD(20),WNPP(2),WNPM(2),WNMP(2),WNMM(2)
      DIMENSION YK(3,NXMAXP,NYMAXP,5),GG(20),KSFACE(20)
      DIMENSION DFPP(2),DFPM(2),DFMP(2),DFMM(2)
C
      NSFX=12
C     CALL DATE(TODAY)
C     WRITE(NWRITE,111)TODAY
111   FORMAT(///1X,'     DATE : ',A8//)
      DO 19 I=1,3
      READ(NREAD,10)  GG
19    WRITE(NWRITE,20)GG
10    FORMAT(20A4)
20    FORMAT(1X,20A4)
      READ(NREAD,18)  GG(1),GG(2),NS,KSYMMY,KSYMMZ
      WRITE(NWRITE,28)GG(1),GG(2),NS,KSYMMY,KSYMMZ
      READ(NREAD,18)  GG(1),GG(2),NWING,NBODY1,NBODY2,NBODY3,NDIAF1
     1,NDIAF2,NDIAF3
      WRITE(NWRITE,28)GG(1),GG(2),NWING,NBODY1,NBODY2,NBODY3,NDIAF1
     1,NDIAF2,NDIAF3
18    FORMAT(2A4,10I5)
28    FORMAT(1X,2A4,10I5)
      READ(NREAD,12)  GG(1),GG(2),UMACH,REDFRE
      WRITE(NWRITE,22)GG(1),GG(2),UMACH,REDFRE
      READ(NREAD,13)  GG(1),GG(2),ALFA,ALFABC,IZZZ,KCM,MZLOOP
      WRITE(NWRITE,23)GG(1),GG(2),ALFA,ALFABC,IZZZ,KCM,MZLOOP
12    FORMAT(2A4,3F8.3,3E12.5)
22    FORMAT(1X,2A4,3F8.3,3E12.5)
13    FORMAT(2A4,2F8.3,3I5)
23    FORMAT(1X,2A4,2F8.3,3I5)
      READ(NREAD,12)  GG(1),GG(2),SPAN,XLEZ,XTEZ,TANGLE,TANGTE,TAU
      WRITE(NWRITE,22)GG(1),GG(2),SPAN,XLEZ,XTEZ,TANGLE,TANGTE,TAU
      READ(NREAD,12)  GG(1),GG(2),XNOSE,XTAIL,R
      WRITE(NWRITE,22)GG(1),GG(2),XNOSE,XTAIL,R
15    FORMAT(2A4,8F8.3)
25    FORMAT(1X,2A4,8F8.3)
      IF(UMACH.GT.1..OR.REDFRE.EQ.0.OR.NDIAF1.NE.0.OR.NDIAF3.NE.0)
     1GO TO 30
      READ(NREAD,18)  GG(1),GG(2),NWAKE,KXINCR
      WRITE(NWRITE,28)GG(1),GG(2),NWAKE,KXINCR
30    CONTINUE
      READ(NREAD,18)  GG(1),GG(2),(KPRINT(K),K=1,10)
      WRITE(NWRITE,28)GG(1),GG(2),(KPRINT(K),K=1,10)
C
      DO 21 ISFIX=1,NSFX
21    KSFACE(ISFIX)=0
```

```
C
      REFLEN=1.
      CHORD=XTEZ-XLEZ
      BETA=SQRT(ABS(UMACH**2-1.))
      KS=0
      NSBODY=0
      NSBTOT=0
C
C
C     ----------WING--------------------------
C
      IF(NWING.EQ.0)GO TO 199
      DO 198 IWING=1,NWING
      KS=KS+1
      NSBODY=NSBODY+1
      ISFIX=0+IWING
      READ(NREAD,18)  GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KWNELE,KWNSHP,KWNTYP
      WRITE(NWRITE,28)GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KWNELE,KWNSHP,KWNTYP
      NXY(KS)=NX(KS)*NY(KS)
      KSFACE(ISFIX)=KS
      ISFACE(KS)=ISFIX
      NT(KS)=NSBTOT
      NSBTOT=NSBTOT+NXY(KS)
C
      IF(KWNTYP.EQ.1)GO TO 14
      READ(NREAD,15)  GG(1),GG(2),WNPP,WNPM,WNMP,WNMM
      WRITE(NWRITE,25)GG(1),GG(2),WNPP,WNPM,WNMP,WNMM
14    CONTINUE
      IF(ISFIX.EQ.1)SIGNZ=+1.
      IF(ISFIX.EQ.2)SIGNZ=-1.
C     WRITE(NWRITE,117)ISFIX,XLEZ,XTEZ
117   FORMAT(//2X,'PART ',I2,5X,'XINIT=',F6.1,5X,'XFIN=',F6.1)
      WINGLN=SPAN-2.*R
      WINGWD=WINGLN/2.
      RLE=R
      RTE=R
      DXX=1./NX(KS)
      DYY=1./NY(KS)
      NXP=NX(KS)+1
      NYP=NY(KS)+1
      DO 118 IX=1,NXP
      DO 118 IY=1,NYP
      XX=(IX-1)*DXX
      YY=(IY-1)*DYY
      GO TO (1131,1132,1133,1134),KWNELE
1131  CONTINUE
      CSI=XX
      ETA=YY
      GO TO 113
1132  CONTINUE
      CSI=XX*XX
      ETA=1.-(1.-YY)**2
      GO TO 113
1133  CONTINUE
      CSI=XX
      ETA=1.-(1.-YY)**2
```

```
      GO TO 113
1134  CONTINUE
      CSI=XX*XX
      ETA=YY
113   CONTINUE
C     IF(KWNTYP.NE.1)GO TO 1139
      Y=WINGWD*ETA+R
      XLE=XLEZ+TANGLE*(Y-RLE)
      XTE=XTEZ+TANGTE*(Y-RTE)
      DCHORD(IY)=XTE-XLE
      CHAXIS(IY)=XLE+DCHORD(IY)*0.5
      X=XLE+(XTE-XLE)*CSI
C
1139  CONTINUE
      IF(KWNTYP.NE.2)GO TO 1149
      X1=(WNPP(1)-WNMP(1))*CSI+WNMP(1)
      Y1=(WNPP(2)-WNMP(2))*CSI+WNMP(2)
      X2=(WNPM(1)-WNMM(1))*CSI+WNMM(1)
      Y2=(WNPM(2)-WNMM(2))*CSI+WNMM(2)
      X3=(WNPP(1)-WNPM(1))*ETA+WNPM(1)
      Y3=(WNPP(2)-WNPM(2))*ETA+WNPM(2)
      X4=(WNMP(1)-WNMM(1))*ETA+WNMM(1)
      Y4=(WNMP(2)-WNMM(2))*ETA+WNMM(2)
      DET=(X1-X2)*(Y3-Y4)-(X3-X4)*(Y1-Y2)
      IF(DET.EQ.0.)GO TO 1147
      X=((X1-X2)*(X4*Y3-X3*Y4)-(X3-X4)*(X2*Y1-X1*Y2))/DET
      Y=((Y1-Y2)*(X4*Y3-X3*Y4)-(Y3-Y4)*(X2*Y1-X1*Y2))/DET
      GO TO 1149
1147  CONTINUE
      X=X4
      Y=Y4
1149  CONTINUE
C
      GO TO (1151,1152,1153),KWNSHP
1151  CONTINUE
C
C     FOR CIRCULAR BICONVEX
C
      XC=0.5*(XLE+XTE)
      PL=0.5*(XTE-XLE)
      H=TAU*PL
      ATAU=2.*H
      ETACR=1.-ATAU/SPAN
      XXC=X-XC
      Z=0.
      IF(H.EQ.0..OR.IY.EQ.NYP.OR.IX.EQ.1.OR.IX.EQ.NXP)GO TO 115
      Z=SIGNZ*(SQRT(((PL*PL+H*H)/(2.*H))**2-XXC**2)-(PL*PL-H*H)/(2.*H))
      IF(ETA.GE.ETACR)Z=Z*SQRT(1.-((ETA-ETACR)*SPAN/ATAU)**2)
      GO TO 115
1152  CONTINUE
      TAUBAR=TAU*.75*SQRT(3.)*(XTEZ-XLEZ)
      Z=SIGNZ*TAUBAR*SQRT(CSI)*(1.-CSI)*SQRT(1.-ETA**2)
      GO TO 115
1153  CONTINUE
      Z=SIGNZ*4.*TAU*(XTEZ-XLEZ)*CSI*(1.-CSI)*SQRT(1.-ETA**2)
115   CONTINUE
C
      YK(1,IX,IY,KS)=X
```

```
      YK(2,IX,IY,KS)=Y
      YK(3,IX,IY,KS)=Z
118   CONTINUE
      KWAKES(KS)=1
198   CONTINUE
199   CONTINUE
C
C     ----------NOSE-------------------------
C
      IF(NBODY2.EQ.0)GO TO 599
      DO 598 IBODY2=1,NBODY2
      KS=KS+1
      NSBODY=NSBODY+1
      ISFIX=4+IBODY2
      READ(NREAD,18)  GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KNSELE,KNSSHP,KNSTYP
      WRITE(NWRITE,28)GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KNSELE,KNSSHP,KNSTYP
      NXY(KS)=NX(KS)*NY(KS)
      KSFACE(ISFIX)=KS
      ISFACE(KS)=ISFIX
      NT(KS)=NSBTOT
      NSBTOT=NSBTOT+NXY(KS)
      IF(ISFIX.EQ.5)SIGNZ=+1
      IF(ISFIX.EQ.6)SIGNZ=-1
      NXP=NX(KS)+1
      NYP=NY(KS)+1
      XINIT=XLEZ-XNOSE
      XFIN=XLEZ
C     WRITE(NWRITE,117)ISFIX,XINIT,XFIN
      DXX=1./NX(KS)
      XZERO=XLEZ
      RR=R
      SLOPE=XNOSE/R
      CONST=1.5
      TINCR=0.5*3.14159/NY(KS)
      DO 550 JX=1,NXP
      XX=(IX-1)*DXX
      IF(KNSELE.EQ.1)CSI=XX
      IF(KNSELE.EQ.2)CSI=XX*XX
      X=XINIT+(XFIN-XINIT)*CSI
      IX=JX
      IF(KNSELE.NE.3)GO TO 540
      IX=NXP-JX+1
      DX=2.*BETA*CONST*RR/(BETA/SLOPE+1.)
      X=XZERO-DX
      XZERO=XZERO-DX
      IF(IX.EQ.1)X=-XNOSE
540   CONTINUE
      IF(KNSSHP.EQ.1)RR=R*SQRT(CSI)
      IF(KNSSHP.EQ.2)RR=R*(1.0001-((X+0.01*XNOSE)/XNOSE)**2)
      IF(KNSSHP.EQ.3)RR=R*(1.-(X/XNOSE)**2)
      IF(KNSSHP.EQ.4)RR=R*(1.-ABS(X)/XNOSE)
      IF(KNSSHP.EQ.5)RR=R*(1.005-ABS(X+0.005*XNOSE)/XNOSE)
      DO 550 IY=1,NYP
      THET=(IY-1)*TINCR
      THETA=(+0.5*3.14159-THET)*SIGNZ
      ETA=COS(THETA)
```

```
      YY=RR*ETA
      GO TO (541,542,543),KNSTYP
541   CONTINUE
      ZZ=RR*SIN(THETA)
      GO TO 544
542   CONTINUE
      ZZ=0.01*RR*SQRT(1.-ETA)*4.0
      GO TO 544
543   CONTINUE
      ZZ=0.
544   CONTINUE
      YK(1,IX,IY,KS)=X
      YK(2,IX,IY,KS)=YY
      YK(3,IX,IY,KS)=ZZ
550   CONTINUE
      KWAKES(KS)=0
598   CONTINUE
599   CONTINUE
C
C     ----------WING-DIAPHRAGM----------------
C
      IF(NDIAF1.EQ.0)GO TO 999
      DO 998 IDIAF1=1,NDIAF1
      KS=KS+1
      ISFIX=8+IDIAF1
      READ(NREAD,18)  GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KDFELE,KDFSHP
      WRITE(NWRITE,28)GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KDFELE,KDFSHP
      NXY(KS)=NX(KS)*NY(KS)
      KSFACE(ISFIX)=KS
      ISFACE(KS)=ISFIX
      NT(KS)=NSBTOT
      NSBTOT=NSBTOT+NXY(KS)
      DYY=1./NY(KS)
      NYP=NY(KS)+1
      NXP=NX(KS)+1
      XTIPC=0.5*(XTEZ+WINGWD*TANGTE+XLEZ+WINGWD*TANGLE)
      DXTIP=1.0*(XTEZ+WINGWD*TANGTE-XLEZ-WINGWD*TANGLE)
      XDIAC=XTIPC
      DXDIA=DXTIP
      IF(NBODY2.EQ.0)GO TO 992
      DXDIAF=XLEZ-XNOSE+BETA*(R+WINGWD)
      XTIPLE=XLEZ+WINGWD*TANGLE
      IF((XTIPLE-DXDIAF).LE.0.)GO TO 992
      XDIAC=0.5*(XTEZ+WINGWD*TANGTE+DXDIAF)
      DXDIA=1.0*(XTEZ+WINGWD*TANGTE-DXDIAF)
992   CONTINUE
C
      IF(NBODY2.EQ.0.AND.NDIAF2.NE.0)GO TO 9909
      GO TO 9910
9909  DXDIA=XTEZ+WINGWD*TANGTE-XLEZ-WINGWD*BETA
      XDIAC=XTEZ+WINGWD*TANGTE-DXDIA*0.5
9910  CONTINUE
      DIAFLN=DXDIA/(2.*BETA)
      DO 990 IY=1,NYP
      YY=(IY-1)*DYY
      IF(KDFELE.EQ.1)ETA=YY**1
```

```
      IF(KDFELE.EQ.2)ETA=YY**2
      IF(KDFELE.EQ.3)ETA=YY**3
      YYY=DIAFLN*ETA
      ACHORD(IY)=(DIAFLN-YYY)*2.*BETA*1.005
      IF(IY.EQ.1.AND.DXDIA.EQ.DXTIP)ACHORD(IY)=DXTIP
      DXX=ACHORD(IY)/NX(KS)
      Y=YYY+WINGWD+R
      IF(IY.EQ.NYP)Y=Y*1.002
      DO 990 IX=1,NXP
      X=(IX-1)*DXX+XDIAC-ACHORD(IY)/2.
      IF(IY.EQ.1.AND.IX.EQ.NXP)X=XTEZ+WINGWD*TANGTE
      YK(1,IX,IY,KS)=X
      YK(2,IX,IY,KS)=Y
      YK(3,IX,IY,KS)=0.
990   CONTINUE
      KWAKES(KS)=0
998   CONTINUE
999   CONTINUE
C
C
C     ---------- TRIANGULAR DIAPHRAGM ------------
C
      IF(NDIAF3.EQ.0)GO TO 1399
      DO 1398 IDIAF3=1,NDIAF3
      KS=KS+1
      ISFIX=12+IDIAF3
      READ(NREAD,18)  GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KDFELE,KDFSHP
      WRITE(NWRITE,28)GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
     1                KDFELE,KDFSHP
      NXY(KS)=NX(KS)*NY(KS)
      KSFACE(ISFIX)=KS
      ISFACE(KS)=ISFIX
      NT(KS)=NSBTOT
      NSBTOT=NSBTOT+NXY(KS)
      DYY=1./NY(KS)
      NYP=NY(KS)+1
      NXP=NX(KS)+1
      READ(NREAD,15)  GG(1),GG(2),DFPP,DFPM,DFMP,DFMM
      WRITE(NWRITE,15)GG(1),GG(2),DFPP,DFPM,DFMP,DFMM
      DXX=1./NX(KS)
      DO 1396 IX=1,NXP
      DO 1396 IY=1,NYP
      XX=(IX-1)*DXX
      YY=(IY-1)*DYY
      GO TO (1302,1303),KDFELE
1302  ETA=YY
      GO TO 1304
1303  ETA=YY**2
1304  CSI=XX.
      X1=(DFPP(1)-DFMP(1))*CSI+DFMP(1)
      Y1=(DFPP(2)-DFMP(2))*CSI+DFMP(2)
      X2=(DFPM(1)-DFMM(1))*CSI+DFMM(1)
      Y2=(DFPM(2)-DFMM(2))*CSI+DFMM(2)
      X3=(DFPP(1)-DFPM(1))*ETA+DFPM(1)
      Y3=(DFPP(2)-DFPM(2))*ETA+DFPM(2)
      X4=(DFMP(1)-DFMM(1))*ETA+DFMM(1)
      Y4=(DFMP(2)-DFMM(2))*ETA+DFMM(2)
```

```
      DET=(X1-X2)*(Y3-Y4)-(X3-X4)*(Y1-Y2)
      IF(DET.EQ.0.)GO TO 1347
      X=((X1-X2)*(X4*Y3-X3*Y4)-(X3-X4)*(X2*Y1-X1*Y2))/DET
      Y=((Y1-Y2)*(X4*Y3-X3*Y4)-(Y3-Y4)*(X2*Y1-X1*Y2))/DET
      GO TO 1349
1347  CONTINUE
      X=X4
      Y=Y4
1349  CONTINUE
      YK(1,IX,IY,KS)=X
      YK(2,IX,IY,KS)=Y
      YK(3,IX,IY,KS)=0.
1396  CONTINUE
      KWAKES(KS)=0
1398  CONTINUE
1399  CONTINUE
      WRITE(NWRITE,31)KWNELE,KNSELE,KBDELE,KTLELE,KWNSHP,KNSSHP,KBDSHP
     1                ,KTLSHP,KDFELE
31    FORMAT(//5X,'WING',3X,'NOSE',3X,'BODY',3X,'TAIL',3X,'DIAPH'/417/
     1       417,1X,17/)
C
C
C
      IF(KS.NE.NS)CALL DEBUG(1020)
      IF(NSBODY.LT.NS)KKDIAF=1
      IF(KSYMMY.EQ.0)NSYMMY=1
      IF(KSYMMY.NE.0)NSYMMY=2
      IF(KSYMMZ.EQ.0)NSYMMZ=1
      IF(KSYMMZ.NE.0)NSYMMZ=2
      IF(KSYMMZ.NE.0.AND.KKDIAF.EQ.1)NSYMMZ=1
      IF(MZLOOP.EQ.1)NSYMMZ=1
C
      NTOTAL=NSBTOT
C
      RETURN
      END



C
      SUBROUTINE CHECK(NTOTAL,NXMAX,NYMAX,NTMAX)
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ12/NSFX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(20)
C
      DO 100 IS=1,NS
      IF(NX(IS).GT.NXMAX)CALL DEBUG(1001)
      IF(NY(IS).GT.NYMAX)CALL DEBUG(1002)
      IF(NTOTAL.GT.NTMAX)CALL DEBUG(1003)
100   CONTINUE
      IF(ALFA.NE.0..AND.(IZZZ.EQ.12.OR.IZZZ.EQ.13.OR.IZZZ.EQ.14))
     1CALL DEBUG(1004)
      IF(KSYMMZ.NE.-1.AND.(IZZZ.EQ.13.OR.IZZZ.EQ.14))
     1CALL DEBUG(1005)
      RETURN
      END
```

```
C
      SUBROUTINE GEOMET(NTOTAL,NXMAXP,NYMAXP,YK,YPP,YPM,YMP,YMM,KWAKE)
C
C     THIS SUBROUTINE IS FOR QUADRILATERAL ELEMENTS
C
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ10/KWAKES(20)
      COMMON/ZZZ12/NSFX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(20)
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
      DIMENSION YK(3,NXMAXP,NYMAXP,5),YPP(3,NTOTAL),YPM(3,NTOTAL),
     1YMP(3,NTOTAL),YMM(3,NTOTAL),KWAKE(NTOTAL)
C
      DO 1999 IS=1,NS
      ISFIX=ISFACE(IS)
      NXX=NX(IS)
      NYY=NY(IS)
      DO 999 IX=1,NXX
      DO 999 IY=1,NYY
C
C
      IND=IX+NX(IS)*(IY-1)+NT(IS)
      IF(KNORML(IS).EQ.-1)GO TO 906
C
C     --      -+
C     +-      ++
C
      IXMM=IX
      IXPM=IX+1
      IXPP=IX+1
      IXMP=IX
      IYMM=IY
      IYPM=IY
      IYPP=IY+1
      IYMP=IY+1
C
      GO TO 907
906   CONTINUE
C
C     -+      --
C     ++      +-
C
      IXMM=IX
      IXMP=IX
      IXPP=IX+1
      IXPM=IX+1
      IYMM=IY+1
      IYMP=IY
      IYPP=IY
      IYPM=IY+1
907   CONTINUE
C
      DO 908 K=1,3
      YPP(K,IND)=YK(K,IXPP,IYPP,IS)
      YPM(K,IND)=YK(K,IXPM,IYPM,IS)
      YMP(K,IND)=YK(K,IXMP,IYMP,IS)
```

```
      YMM(K,IND)=YK(K,IXMM,IYMM,IS)
908   CONTINUE
C
      KWAKE(IND)=0
      IF(KWAKES(IS).EQ.1.AND.IX.EQ.NXX)KWAKE(IND)=1
999   CONTINUE
1999  CONTINUE
      IF(KSYMMY.NE.0)GO TO 701
      NTOTAL=2*NTOTAL
      NS=2*NS
      NTHALF=NTOTAL/2
      NSHALF=NS/2
      DO 2990 IS=1,NSHALF
      JS=IS+NSHALF
      KNORML(JS)=KNORML(IS)
      KDIAF(JS)=KDIAF(IS)
      NT(JS)=NT(IS)+NTHALF
      NXY(JS)=NXY(IS)
      NX(JS)=NX(IS)
      NY(JS)=NY(IS)
      ISFACE(JS)=ISFACE(IS)+100
2990  CONTINUE
      DO 300 IR=1,NTHALF
      IL=IR+NTHALF
      DO 299 K=1,3
      GO TO (2991,2992,2991),K
2991  CONTINUE
      YPP(K,IL)=+YPM(K,IR)
      YMP(K,IL)=+YMM(K,IR)
      YPM(K,IL)=+YPP(K,IR)
      YMM(K,IL)=+YMP(K,IR)
      GO TO 299
2992  CONTINUE
      YPP(K,IL)=-YPM(K,IR)
      YMP(K,IL)=-YMM(K,IR)
      YPM(K,IL)=-YPP(K,IR)
      YMM(K,IL)=-YMP(K,IR)
299   CONTINUE
      KWAKE(IL)=KWAKE(IR)
300   CONTINUE
701   CONTINUE
      IF(KPRINT(1).EQ.1)
     1CALL PRINTA(NTOTAL,NXMAXP,NYMAXP,YK,PC,YPP,YPM,YMP,YMM,1)
      IF(KPRINT(2).EQ.1)
     1CALL PRINTA(NTOTAL,NXMAXP,NYMAXP,YK,PC,YPP,YPM,YMP,YMM,2)
      IF(KPRINT(3).EQ.1)
     1CALL PRINTA(NTOTAL,NXMAXP,NYMAXP,YK,PC,YPP,YPM,YMP,YMM,3)
      RETURN
      END
```

```
      SUBROUTINE VEC123(NTOTAL,PC,YPP,YPM,YMP,YMM)
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ11/ALFA,ALFABC,REDFRE,BODYR,XLEZ,XTEZ,XNOSE,XTAIL
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
      DIMENSION YPP(3,NTOTAL),YPM(3,NTOTAL),YMP(3,NTOTAL),YMM(3,NTOTAL)
      DIMENSION PC(3,NTOTAL)
C
      ALFAR=ALFA*3.14159/180.
      SINALF=SIN(ALFAR)
      COSALF=COS(ALFAR)
      BETA=SQRT(ABS(1.-UMACH*UMACH))
      DO 100 IND=1,NTOTAL
      YPP1=YPP(1,IND)
      YPP3=YPP(3,IND)
      YPP(1,IND)=(YPP1*COSALF+YPP3*SINALF)/BETA
      YPP(3,IND)=-YPP1*SINALF+YPP3*COSALF
      YPM1=YPM(1,IND)
      YPM3=YPM(3,IND)
      YPM(1,IND)=(YPM1*COSALF+YPM3*SINALF)/BETA
      YPM(3,IND)=-YPM1*SINALF+YPM3*COSALF
      YMP1=YMP(1,IND)
      YMP3=YMP(3,IND)
      YMP(1,IND)=(YMP1*COSALF+YMP3*SINALF)/BETA
      YMP(3,IND)=-YMP1*SINALF+YMP3*COSALF
      YMM1=YMM(1,IND)
      YMM3=YMM(3,IND)
      YMM(1,IND)=(YMM1*COSALF+YMM3*SINALF)/BETA
      YMM(3,IND)=-YMM1*SINALF+YMM3*COSALF
100   CONTINUE
      DO 200 IND=1,NTOTAL
      DO 199 K=1,3
      PC(K,IND)=(YPP(K,IND)+YPM(K,IND)+YMP(K,IND)+YMM(K,IND))/4.
C     P1(K,IND)=(YPP(K,IND)+YPM(K,IND)-YMP(K,IND)-YMM(K,IND))/4.
C     P2(K,IND)=(YPP(K,IND)-YPM(K,IND)+YMP(K,IND)-YMM(K,IND))/4.
C     P3(K,IND)=(YPP(K,IND)-YPM(K,IND)-YMP(K,IND)+YMM(K,IND))/4.
199   CONTINUE
200   CONTINUE
      IF(KPRINT(4).EQ.1)
     1CALL PRINTA(NTOTAL,NXMAXP,NYMAXP,YK,PC,YPP,YPM,YMP,YMM,4)
      RETURN
      END



C
      SUBROUTINE DEBUG(K)
C
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
      WRITE(NWRITE,1)K
1     FORMAT(/2X,'  ERROR CODE =',I6/)
      CALL EXIT
      END
```

```
C
      SUBROUTINE PRINTA(NTOTAL,NXP,NYP,YK,PC,YPP,YPM,YMP,YMM,NPRINT)
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ11/ALFA,ALFABC,REDFRE,R,XLEZ,XTEZ,XNOSE,XTAIL
      COMMON/ZZZ12/NSFX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(2C
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
      DIMENSION YK(3,NXP,NYP,5),PC(3,NTOTAL),YPP(3,NTOTAL),YPM(3,NTOTAL
     1),YMP(3,NTOTAL),YMM(3,NTOTAL)
C
      NY4=4*(NY(1)-1)
      GO TO(1,2,3,4),NPRINT
1     CONTINUE
      WRITE(NWRITE,110)
110   FORMAT(//2X,'SPECIFICATIONS OF THE PROBLEM'/)
      DO 112 II=1,NS
      IF(NXY(II).EQ.0)GO TO 112
      WRITE(NWRITE,113)ISFACE(II)
      WRITE(NWRITE,114)NX(II),NY(II)
112   CONTINUE
113   FORMAT(/2X,'FOR PART ',I2)
      WRITE(NWRITE,115)NTOTAL,KSYMMY,KSYMMZ,REFLEN,SPAN,TAU,
     1ALFA,ALFABC,UMACH,IZZZ
114   FORMAT(2X,'NX=',I2/2X,'NY=',I2/)
115   FORMAT(/2X,'NTOTAL=',I3//2X,'KSYMMY=',I2/2X,'KSYMMZ=',I2//
     12X,'REFERENCE LENGTH=',F6.2/2X,'SPAN =',F6.2/
     12X,'WING THICKNESS  =',F9.5//2X,'ALFA=',F7.3/2X,'ALFABC=',F7.3//
     12X,'MACH NUMBER =',F7.3//2X,'IZZZ=',I5//)
      WRITE(NWRITE,116)TANGLE,TANGTE,CHORD,REDFRE,R
116   FORMAT(2X,'TANGLE=',F6.2/2X,'TANGTE=',F6.2//2X,'CHORD =',F6.
     12//2X,'REDFRE=',F8.4/2X,'RADIUS=',F6.2)
      RETURN
2     CONTINUE
      DO 220 IS=1,NS
      ISFIX=ISFACE(IS)
      NXX=NX(IS)+1
      NYY=NY(IS)+1
      WRITE(NWRITE,222)
222   FORMAT(//4X,'X',10X,'Y',10X,'Z'//)
      WRITE(NWRITE,221)(((YK(J,IX,IY,IS),J=1,3),IX=1,NXX),IY=1,NYY)
221   FORMAT(1X,F10.5,1X,F10.5,1X,F10.5)
220   CONTINUE
      RETURN
3     CONTINUE
      WRITE(NWRITE,332)
      DO 330 I=1,NTOTAL
      WRITE(NWRITE,331)I,(YPP(K,I),K=1,3),(YPM(K,I),K=1,3),(YMP(K,I),
     1          K=1,3),(YMM(K,I),K=1,3)
331   FORMAT(1X,I3,3X,3F9.5,3X,3F9.5,3X,3F9.5,3X,3F9.5)
332   FORMAT(//2X,'IND',16X,'PP',28X,'PM',28X,'MP',28X,'MM')
330   CONTINUE
      RETURN
4     CONTINUE
      WRITE(NWRITE,440)
440   FORMAT(////2X,'IND',4X,'XPC',7X,'YPC',7X,'ZPC')
      DO 444 I=1,NTOTAL
444   WRITE(NWRITE,445)I,(PC(K,I),K=1,3)
445   FORMAT(1X,I3,12F10.5)
      RETURN
      END
```

```
C
      SUBROUTINE COEFPR(NTOTAL,PC,YPP,YPM,YMP,YMM,SOURCE,VELPOT)
      COMPLEX SOURCE,VELPOT,UNIMAG,AA,CL,CM,ACL,ACM,VELPTE
      COMPLEX VELPTX,PHI1,PHI2,PHI3,COEF1,COEF2,COEF3
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ11/ALFA,ALFABC,REDFRE,BODYR,XLEZ,XTEZ,XNOSE,XTAIL,KCM
      COMMON/ZZZ12/NSFX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(2C
      COMMON/ZZZ13/DCHORD(10),CHAXIS(10)
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
C
      DIMENSION YPP(3,NTOTAL),YPM(3,NTOTAL),YMP(3,NTOTAL),YMM(3,NTOTAL)
      DIMENSION SOURCE(NTOTAL),VELPOT(NTOTAL),PC(3,NTOTAL)
      DIMENSION ACL(10),ACM(10),VELPTE(10),AA(1)
C
      BETA2S=ABS(1.-UMACH**2)
      UNIMAG=(0.,1.)
      BETA=SQRT(BETA2S)
C
C
      IF(UMACH.GT.1)SGNEXP=-1.
      IF(UMACH.LT.1)SGNEXP=+1.
C
      IF(KCM.EQ.11)AXISMX=XLEZ+(XTEZ-XLEZ)*0.15
      IF(KCM.EQ.12)AXISMX=XLEZ+(XTEZ-XLEZ)*0.20
      IF(KCM.EQ.13)AXISMX=XLEZ+(XTEZ-XLEZ)*0.25
      IF(KCM.EQ.14)AXISMX=XLEZ+(XTEZ-XLEZ)*0.30
      IF(KCM.EQ.15)AXISMX=XLEZ+(XTEZ-XLEZ)*0.325
      IF(KCM.EQ.16)AXISMX=XLEZ+(XTEZ-XLEZ)*0.350
      IF(KCM.EQ.17)AXISMX=XLEZ+(XTEZ-XLEZ)*0.1875
      IF(KCM.EQ.18)AXISMX=XLEZ+(XTEZ-XLEZ)*0.4500
C
      DO 600 IND=1,NTOTAL
      VELPOT(IND)=SOURCE(IND)*CEXP(SGNEXP*UNIMAG*REDFRE*UMACH**2
     1                                          *PC(1,IND)/BETA)
600   CONTINUE
C
      IF(KCM.EQ.0)GO TO 300
C
      IF(KSYMMZ.EQ.0)NZLOOP=2
      IF(KSYMMZ.NE.0)NZLOOP=1
C
      NYLOOP=NY(1)
C
      NX1=NX(1)
      NY1=NY(1)
      NXY1=NX1*NY1
      DO 200 METHOD=1,2
      DO 170 IY=1,NYLOOP
      DO 170 IZ=1,NZLOOP
      ITEM0=NX1*IY
      ITEM1=ITEM0-1
      ITEM2=ITEM0-2
      KY=IY+NY1*(IZ-1)
      X1=PC(1,ITEM0)
```

```
      X2=PC(1,ITEM1)
      X3=PC(1,ITEM2)
      XTE=0.5*(YPP(1,ITEMO)+YPM(1,ITEMO))
      DET=(X1-X2)*(X2**2-X3**2)-(X2-X3)*(X1**2-X2**2)
      GO TO (153,152),METHOD
152   PHI1=VELPOT(ITEMO)*CEXP(UNIMAG*REDFRE*BETA*PC(1,ITEMO))
      PHI2=VELPOT(ITEM1)*CEXP(UNIMAG*REDFRE*BETA*PC(1,ITEM1))
      PHI3=VELPOT(ITEM2)*CEXP(UNIMAG*REDFRE*BETA*PC(1,ITEM2))
      GO TO 154
153   PHI1=VELPOT(ITEMO)
      PHI2=VELPOT(ITEM1)
      PHI3=VELPOT(ITEM2)
154   CONTINUE
      COEF1=(+(PHI1-PHI2)*(X2**2-X3**2)-(PHI2-PHI3)*(X1**2-X2**2))/DET
      COEF2=(-(PHI1-PHI2)*(X2-X3)+(PHI2-PHI3)*(X1-X2))/DET
      COEF3=PHI1-COEF1*X1-COEF2*X1**2
      VELPTE(KY)=COEF3+COEF1*XTE+COEF2*XTE**2
C     WRITE(NWRITE,181)PHI3,PHI2,PHI1,VELPTE(KY)
      IF(METHOD.EQ.2)
     1VELPTE(KY)=VELPTE(KY)*CEXP(-UNIMAG*REDFRE*BETA*XTE)
C     WRITE(NWRITE,181)VELPTE(KY)
170   CONTINUE
C
      CL=0.
      CM=0.
      AREAXY=0.
      ACHORD=0.
      DO 190 IY=1,NYLOOP
      ACL(IY)=0.
190   ACM(IY)=0.
      DO 199 IX=1,NX1
      DO 199 IY=1,NYLOOP
      DO 199 IZ=1,NZLOOP
      I=IX+NX1*(IY-1)+NXY1*(IZ-1)
C
C     TWO EDGES OF EACH ELEMENT ARE ASSUMED TO BE PARALLEL TO X-AXIS
C
      DX=0.5*(YPP(1,I)+YPM(1,I)-YMP(1,I)-YMM(1,I))*BETA
      DY=0.5*(YPP(2,I)+YMP(2,I)-YPM(2,I)-YMM(2,I))
      DXY=DX*DY
      AREAXY=AREAXY+DXY
      IF(KSYMMZ.EQ.-1)COEFFT=4.
      IF(KSYMMZ.EQ.+0)COEFFT=2.*(3.-IZ*2)
      IF(KSYMMZ.EQ.+1)COEFFT=0.
C
      IF(KCM.EQ.1)AXISMX=XLEZ+(XTEZ-XLEZ)*0.00
      IF(KCM.EQ.2)AXISMX=XLEZ+(XTEZ-XLEZ)*0.50
      IF(KCM.EQ.3)AXISMX=0.5*(CHAXIS(IY)+CHAXIS(IY+1))
      IF(KCM.EQ.4)AXISMX=XLEZ+(XTEZ-XLEZ)*0.25
      ACL(IY)=ACL(IY)+COEFFT*VELPOT(I)*UNIMAG*REDFRE*DXY
      ACM(IY)=ACM(IY)+COEFFT*VELPOT(I)*(1.-UNIMAG*REDFRE*(BETA*PC(1,I)
     1                                                  -AXISMX))*DXY
      IF(IX.NE.NX1)GO TO 199
      ACHORD=ACHORD+(0.5*(DCHORD(IY)+DCHORD(IY+1)))**2*DY
      ACL(IY)=ACL(IY)+COEFFT*VELPTE(IY)*DY
      ACM(IY)=ACM(IY)-COEFFT*VELPTE(IY)*(0.5*(YPP(1,I)+YPM(1,I))*
     1                                                  BETA-AXISMX)*DY
199   CONTINUE
```

```
      ACHORD=ACHORD/AREAXY
      WRITE(NWRITE,182)AREAXY,ACHORD,AXISMX,IZZZ,KCM
182   FORMAT(///5X,3F9.5,5X,5I6)
181   FORMAT(5X,2F9.5,3X,2F9.5,3X,2F9.5,3X,2F9.5,3X,2F9.5)
      WRITE(NWRITE,101)
101   FORMAT(//)
      DO 195 IY=1,NYLOOP
      WRITE(NWRITE,102)ACL(IY),ACM(IY)
      ABSCL=CABS(ACL(IY))
      ABSCM=CABS(ACM(IY))
      REALCL=REAL(ACL(IY))
      REALCM=REAL(ACM(IY))
      AIMGCL=AIMAG(ACL(IY))
      AIMGCM=AIMAG(ACM(IY))
      FSAGCL=ATAN2(AIMGCL,REALCL)*180./3.14159
      FSAGCM=ATAN2(AIMGCM,REALCM)*180./3.14159
      WRITE(NWRITE,102)ABSCL,FSAGCL,ABSCM,FSAGCM
102   FORMAT(5X,2E15.5,4X,2E15.5)
      WRITE(NWRITE,104)
104   FORMAT(/)
      CL=CL+ACL(IY)/AREAXY
      CM=CM+ACM(IY)/(AREAXY*ACHORD)
195   CONTINUE
C
      WRITE(NWRITE,107)CL,CM,REDFRE
107   FORMAT(///5X,'TOTAL  LIFT  COEFFICIENT = ',2E15.5//5X,
     1'TOTAL MOMENT COEFFICIENT = ',2E15.5,5X,'FREQUENCY =',F7.3//)
      ABSCL=CABS(CL)
      ABSCM=CABS(CM)
      REALCL=REAL(CL)
      REALCM=REAL(CM)
      AIMGCL=AIMAG(CL)
      AIMGCM=AIMAG(CM)
      FSAGCL=ATAN2(AIMGCL,REALCL)*180./3.14159
      FSAGCM=ATAN2(AIMGCM,REALCM)*180./3.14159
      WRITE(NWRITE,107)ABSCL,FSAGCL,ABSCM,FSAGCM,REDFRE
200   CONTINUE
300   CONTINUE
C
      DO 400 IS=1,NS
      NXX=NX(IS)
      NYY=NY(IS)
      DO 10 IY=1,NYY
      DO 10 IX=1,NXX
      IF(IX.EQ.NXX)GO TO 10
      I=IX+NX(IS)*(IY-1)+NT(IS)
      J=I+1
      DDX=PC(1,J)-PC(1,I)
      XPCM=0.5*(PC(1,J)+PC(1,I))
      SOURCE(I)=-2.*CEXP(-UNIMAG*REDFRE*BETA*XPCM)*
     1   (SOURCE(J)*CEXP(SGNEXP*UNIMAG*REDFRE*PC(1,J)/BETA)
     1   -SOURCE(I)*CEXP(SGNEXP*UNIMAG*REDFRE*PC(1,I)/BETA))/
     1   (DDX*BETA)
C     SOURCE(I)=-2.*((VELPOT(J)-VELPOT(I))/(BETA*DDX)+0.5*UNIMAG*REDFRE
C    1                *(VELPOT(J)+VELPOT(I)))
10    CONTINUE
400   CONTINUE
C
```

```
      IF(KPRINT(8).EQ.1) CALL PRINTB(NTOTAL,1,AA,SOURCE,VELPOT,4)
      IF(KPRINT(9).EQ.1) CALL PRINTB(NTOTAL,1,AA,SOURCE,VELPOT,5)
      IF(KPRINT(10).EQ.1)CALL PRINTB(NTOTAL,1,AA,SOURCE,VELPOT,6)
C
      RETURN
      END




C
      SUBROUTINE SOLUTN(NTOTAL,NT2S,AA,SOURCE,VELPOT)
C
      COMPLEX SOURCE,AA
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
      DIMENSION AA(NT2S),SOURCE(NTOTAL),VELPOT(NTOTAL)
C
      IF(KPRINT(5).EQ.1)CALL PRINTB(NTOTAL,NT2S,AA,SOURCE,VELPOT,1)
      IF(KPRINT(6).EQ.1)CALL PRINTB(NTOTAL,NT2S,AA,SOURCE,VELPOT,2)
      TOL=0.001
      CALL CCGELG(SOURCE,AA,NTOTAL,1,TOL,IER)
C
      IF(IER.NE.0)WRITE(NWRITE,101)IER
C     IF(IER.NE.0)CALL DEBUG(1001)
101   FORMAT(////2X,'----------   IER = ',I4,'   ----------'///)
      IF(KPRINT(7).EQ.1)CALL PRINTB(NTOTAL,NT2S,AA,SOURCE,VELPOT,3)
      RETURN
      END
```

```
C
      SUBROUTINE PRINTB(NTOTAL,NT2S,AA,SOURCE,VELPOT,NPRINT)
      COMPLEX SOURCE,VELPOT,AA,UNIMAG
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ11/ALFA,ALFABC,REDFRE,BODYR,XLEZ,XTEZ,XNOSE,XTAIL,KCM
      COMMON/ZZZ12/NSFX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(20)
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
C
      DIMENSION SOURCE(NTOTAL),VELPOT(NTOTAL),AA(NT2S)
      DIMENSION ABSVAL(100),FASEAN(100)
      NY4=4*(NY(1)-1)
      GO TO(1,2,3,4,5,6),NPRINT
1     CONTINUE
      WRITE(NWRITE,110)
110   FORMAT(///2X,'DISTRIBUTION OF AA(I,J)'/)
      DO 11 I=1,NTOTAL
      WRITE(NWRITE,111)I
      N1=I
      N2=NTOTAL*NTOTAL
111   FORMAT(2X,'INDEX=',I2)
      WRITE(NWRITE,112)(AA(K),K=N1,N2,NTOTAL)
112   FORMAT(1X,8E15.6)
11    CONTINUE
      RETURN
2     CONTINUE
      WRITE(NWRITE,221)
221   FORMAT(///2X,'THE DISTRIBUTION OF SOURCE')
225   CONTINUE
      NSBTOT=0
      DO 229 IS=1,NS
      IF(NXY(IS).EQ.0)GO TO 229
      WRITE(NWRITE,223)ISFACE(IS)
223   FORMAT(//5X,'FOR PART ',I3)
      IND=NSBTOT
      NSBTOT=NSBTOT+NXY(IS)
      IFIN=NSBTOT
      NXX=NX(IS)
      DO 226 IX=1,NXX
      WRITE(NWRITE,228)
      IF(NPRINT.GE.4.AND.(IX.EQ.NXX))GO TO 226
      IND=IND+1
      WRITE(NWRITE,227)(SOURCE(KK),KK=IND,IFIN,NXX)
226   CONTINUE
227   FORMAT(1X,8E15.5)
228   FORMAT(/)
229   CONTINUE
      RETURN
3     CONTINUE
      WRITE(NWRITE,330)
330   FORMAT(///2X,'THE DISTRIBUTION OF THE VELOCITY POTENTIAL')
      GO TO 225
4     CONTINUE
      WRITE(NWRITE,440)
440   FORMAT(///2X,'THE DISTRIBUTION OF CP')
      GO TO 225
5     CONTINUE
      WRITE(NWRITE,550)
```

```
550   FORMAT(///2X,'DISTRIBUTION OF ABSOLUTE VALUES AND PHASE-ANGLES '
     1,'OF CP')
      DO 555 KK=1,NTOTAL
      ABSVAL(KK)=CABS(SOURCE(KK))
      REALCP=REAL(SOURCE(KK))
      AIMGCP=AIMAG(SOURCE(KK))
      FASEAN(KK)=90.
      IF(REALCP.NE.0.)FASEAN(KK)=ATAN2(AIMGCP,REALCP)*180./3.14159
555   CONTINUE
551   CONTINUE
      NSBTOT=0
      DO 559 IS=1,NS
      IF(NXY(IS).EQ.0)GO TO 559
      WRITE(NWRITE,553)ISFACE(IS)
553   FORMAT(//5X,'FOR PART ',I2)
      IND=NSBTOT
      NSBTOT=NSBTOT+NXY(IS)
      IFIN=NSBTOT
      NXX=NX(IS)
      DO 556 IX=1,NXX
      WRITE(NWRITE,558)
      IF(NPRINT.GE.4.AND.(IX.EQ.NXX))GO TO 556
      IND=IND+1
      WRITE(NWRITE,557)(ABSVAL(KK),KK=IND,IFIN,NXX)
      WRITE(NWRITE,557)(FASEAN(KK),KK=IND,IFIN,NXX)
556   CONTINUE
557   FORMAT(1X,8E15.5)
558   FORMAT(/)
559   CONTINUE
      RETURN
6     CONTINUE
      WRITE(NWRITE,601)
601   FORMAT(///2X,'THE DISTRIBUTION OF CORRECTED VELOCITY POTENTIAL'/)
      DO 600 I=1,NTOTAL
600   SOURCE(I)=VELPOT(I)
      GO TO 225
      END
```

```
C
C
      SUBROUTINE CCGELG(R,A,M,N,EPS,IER)
C
C
      COMPLEX WAKE,SO,R,PIVI,TB,A
      DIMENSION A(1),R(1)
      IF(M)23,23,1
C
C     SEARCH FOR GREATEST ELEMENT IN MATRIX A
    1 IER=0
      PIV=0.
      MM=M*M
      NM=N*M
      DO 3 L=1,MM
      TBB=CABS(A(L))
      IF(TBB-PIV)3,3,2
    2 PIV=TBB
      I=L
    3 CONTINUE
      TOL=EPS*PIV
C     A(I) IS PIVOT ELEMENT. PIV CONTAINS THE ABSOLUTE VALUE OF A(I).
C
C
C     START ELIMINATION LOOP
      LST=1
      DO 17 K=1,M
C
C     TEST ON SINGULARITY
      IF(PIV)23,23,4
    4 IF(IER)7,5,7
    5 IF(PIV-TOL)6,6,7
    6 IER=K-1
    7 PIVI=1./A(I)
      J=(I-1)/M
      I=I-J*M-K
      J=J+1-K
C     I+K IS ROW-INDEX, J+K COLUMN-INDEX OF PIVOT ELEMENT
C
C     PIVOT ROW REDUCTION AND ROW INTERCHANGE IN RIGHT HAND SIDE R
      DO 8 L=K,NM,M
      LL=L+I
      TB=PIVI*R(LL)
      R(LL)=R(L)
    8 R(L)=TB
C
C     IS ELIMINATION TERMINATED
      IF(K-M)9,18,18
C
C     COLUMN INTERCHANGE IN MATRIX A
    9 LEND=LST+M-K
      IF(J)12,12,10
   10 II=J*M
      DO 11 L=LST,LEND
      TB=A(L)
      LL=L+II
      A(L)=A(LL)
   11 A(LL)=TB
```

```
C
C       ROW INTERCHANGE AND PIVOT ROW REDUCTION IN MATRIX A
   12   DO 13 L=LST,MM,M
        LL=L+I
        TB=PIVI*A(LL)
        A(LL)=A(L)
   13   A(L)=TB
C
C       SAVE COLUMN INTERCHANGE INFORMATION
        A(LST)=J
C
C       ELEMENT REDUCTION AND NEXT PIVOT SEARCH
        PIV=0.
        LST=LST+1
        J=0
        DO 16 II=LST,LEND
        PIVI=-A(II)
        IST=II+M
        J=J+1
        DO 15 L=IST,MM,M
        LL=L-J
        A(L)=A(L)+PIVI*A(LL)
        TBB=CABS(A(L))
        IF(TBB-PIV)15,15,14
   14   PIV=TBB
        I=L
   15   CONTINUE
        DO 16 L=K,NM,M
        LL=L+J
   16   R(LL)=R(LL)+PIVI*R(L)
   17   LST=LST+M
C       END OF ELIMINATION LOOP
C
C
C       BACK SUBSTITUTION AND BACK INTERCHANGE
   18   IF(M-1)23,22,19
   19   IST=MM+M
        LST=M+1
        DO 21 I=2,M
        II=LST-I
        IST=IST-LST
        L=IST-M
        L=A(L)+.5
        DO 21 J=II,NM,M
        TB=R(J)
        LL=J
        DO 20 K=IST,MM,M
        LL=LL+1
   20   TB=TB-A(K)*R(LL)
        K=J+L
        R(J)=R(K)
   21   R(K)=TB
   22   RETURN
C
C
C       ERROR RETURN
   23   IER=-1
        RETURN
        END
```

```
C
      SUBROUTINE COEFF(NTOTAL,PC,YPP,YPM,YMP,YMM,KWAKE,NT2S,AA,SOURCE,
     1SCTRAN)
C
C     THIS SUBPROGRAM IS TO CALCULATE DOUBLET AND SOURCE FOR          SUBSONIC
C     FLOW
C
      COMPLEX SOURCE,SCTRAN,AA,UNIMAG,XNUNST,CMPXDB,CMPXSC,UIOMER        UNST
      COMPLEX XNTRAN
C
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ10/KWAKES(20),NWAKE,KXINCR
      COMMON/ZZZ11/ALFA,ALFABC,REDFRE,BODYR,XLEZ,XTEZ,XNOSE,XTAIL,KCM
      COMMON/ZZZ12/NSFX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(20)
      COMMON/ZZZ13/ACHORD(10),XCHORD(10)
      COMMON/ZZZ88/QV(3),QQ,I,J,ISYMMY,ISYMMZ
      COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
      DIMENSION YPP(3,NTOTAL),YPM(3,NTOTAL),YMP(3,NTOTAL),YMM(3,NTOTAL),
     1PC(3,NTOTAL),SOURCE(NTOTAL),SCTRAN(NTOTAL),KWAKE(NTOTAL),AA(NT2S)
      DIMENSION A1(3,2),A2(3,2),A1A1(2),A2A2(2),QCRA1(3),QCRA2(3)
      DIMENSION A1V(3),A2V(3),YNORM(3),UN(3),SN(3),SNUN(3),A1A2(2,2)
      DIMENSION PZ(3),Q(3,4),QMCRA1(3,2),QMCRA2(3,2),AA1(3,2),AA2(3,2)
      DIMENSION VMM(3),VDD(3),VMD(3),YTEPZ(3),YTEMZ(3),SIGNPT(3)
      DIMENSION AVA1(3),AVA2(3),A1CRA2(4),QM(3,4)
      DIMENSION WPP(3),WPM(3),WMP(3),WMM(3),WPC(3)
C
      DOTPRO(X1,Y1,Z1,X2,Y2,Z2)=X1*X2+Y1*Y2+Z1*Z2
C
      PROMIX(XX1,YY1,ZZ1,XX2,YY2,ZZ2,XX3,YY3,ZZ3)=(YY2*ZZ3-YY3*ZZ2)*XX1
     1                    -(XX2*ZZ3-XX3*ZZ2)*YY1+(XX2*YY3-XX3*YY2)*ZZ1
C
      IF(UMACH.GT.1.)CALL DEBUG(400)
      BETA=SQRT(1.-UMACH**2)
      WRITE(NWRITE,2)NSBODY,NS,(NT(I),I=1,NS),(KNORML(J),J=1,NS),(KDIAF
     1(K),K=1,NS),(ISFACE(II),II=1,NS),(NXY(KK),KK=1,NS),NSYMMY,NSYMMZ
2     FORMAT(3X,25I5)
C
      IF(IZZZ.GE.100.AND.IZZZ.LT.200)WRITE(NWRITE,9911)                  UNST
      IF(IZZZ.GE.200.AND.IZZZ.LT.300)WRITE(NWRITE,9922)                  UNST
      IF(IZZZ.GE.300.AND.IZZZ.LT.400)WRITE(NWRITE,9933)                  UNST
9911  FORMAT(//2X,'------ OSCILLATION IN FIRST BENDING MODE ------'//)   UNST
9922  FORMAT(//2X,'------ OSCILLATION IN TRANSLATION -------------'//)   UNST
9933  FORMAT(//2X,'------ OSCILLATION IN PITCH -------------------'//)   UNST
      IF(IZZZ.EQ.301)XPITCH=XLEZ+(XTEZ-XLEZ)*0.00                        UNST
      IF(IZZZ.EQ.302)XPITCH=XLEZ+(XTEZ-XLEZ)*0.50                        UNST
      IF(KCM.EQ.11)XPITCH=XLEZ+(XTEZ-XLEZ)*0.15                          UNST
      IF(KCM.EQ.12)XPITCH=XLEZ+(XTEZ-XLEZ)*0.20                          UNST
      IF(KCM.EQ.13)XPITCH=XLEZ+(XTEZ-XLEZ)*0.25                          UNST
      IF(KCM.EQ.14)XPITCH=XLEZ+(XTEZ-XLEZ)*0.30                          UNST
      IF(KCM.EQ.15)XPITCH=XLEZ+(XTEZ-XLEZ)*0.325                         UNST
      IF(KCM.EQ.16)XPITCH=XLEZ+(XTEZ-XLEZ)*0.350                         UNST
      IF(KCM.EQ.17)XPITCH=XLEZ+(XTEZ-XLEZ)*0.1875                        UNST
      IF(KCM.EQ.18)XPITCH=XLEZ+(XTEZ-XLEZ)*0.4500                        UNST
C
```

```
      UNIMAG=(0.,1.)                                                    UNST
      OMEGA=REDFRE*UMACH/BETA                                           UNST
C
      ALFRBC=ALFABC*3.14159/180.
      SINABC=SIN(ALFRBC)
      COSABC=COS(ALFRBC)
C
      DO 6 NN=1,NT2S
6     AA(NN)=0.
C
C     INSERT THE KRONECKER DELTA (SWITCH FOR THE LOWER SIDES OF DIAPHRAGMS)
C
      DO 9 IS=1,NS
      NXYS=NXY(IS)
      DO 9 II=1,NXYS
      I=II+NT(IS)
      J=I
      NNN=I+(J-1)*NTOTAL
      AA(NNN)=1.
9     CONTINUE
C
      DO 10 I=1,NTOTAL
      SOURCE(I)=0.
      SCTRAN(I)=0.
10    CONTINUE
C
      SIGNPT(1)=1.0
C                                                                       SUB
      CONST=+0.5/3.14159
C
      DO 180 JFACE=1,NS
      JSBTOT=NXY(JFACE)
      DO 180 JJ=1,JSBTOT
      J=NT(JFACE)+JJ
C
      DO 1002 K=1,3
      AA1(K,1)=0.5*(YPP(K,J)-YMP(K,J))
      AA1(K,2)=0.5*(YPM(K,J)-YMM(K,J))
      AA2(K,1)=0.5*(YPP(K,J)-YPM(K,J))
      AA2(K,2)=0.5*(YMP(K,J)-YMM(K,J))
      A1(K,1)=AA1(K,1)
      A1(K,2)=AA1(K,2)
      A2(K,1)=AA2(K,1)
      A2(K,2)=AA2(K,2)
1002  CONTINUE
      DO 1003 L=1,2
      A1A1(L)=DOTPRO(A1(1,L),A1(2,L),A1(3,L),A1(1,L),A1(2,L),A1(3,L))
1003  A2A2(L)=DOTPRO(A2(1,L),A2(2,L),A2(3,L),A2(1,L),A2(2,L),A2(3,L))
C
      DO 1004 K=1,3
      IF(A1A1(1).EQ.0.)A1(K,1)=A1(K,2)
      IF(A1A1(2).EQ.0.)A1(K,2)=A1(K,1)
      IF(A2A2(1).EQ.0.)A2(K,1)=A2(K,2)
      IF(A2A2(2).EQ.0.)A2(K,2)=A2(K,1)
1004  CONTINUE
C
      DO 1006 L=1,2
      A1A1(L)=DOTPRO(A1(1,L),A1(2,L),A1(3,L),A1(1,L),A1(2,L),A1(3,L))
```

```
1006  A2A2(L)=DOTPRO(A2(1,L),A2(2,L),A2(3,L),A2(1,L),A2(2,L),A2(3,L))
C
      DO 1007 L=1,2
      DO 1007 M=1,2
1007  A1A2(L,M)=DOTPRO(A1(1,L),A1(2,L),A1(3,L),A2(1,M),A2(2,M),A2(3,M))
C
      A1CRA2(1)=SQRT(A1A1(2)*A2A2(1)-A1A2(2,1)**2)
      A1CRA2(2)=SQRT(A1A1(1)*A2A2(1)-A1A2(1,1)**2)
      A1CRA2(3)=SQRT(A1A1(1)*A2A2(2)-A1A2(1,2)**2)
      A1CRA2(4)=SQRT(A1A1(2)*A2A2(2)-A1A2(2,2)**2)
C
      DO 1008 K=1,3
      AVA1(K)=0.5*(A1(K,1)+A1(K,2))
1008  AVA2(K)=0.5*(A2(K,1)+A2(K,2))
C
      YNORM(1)=AVA1(2)*AVA2(3)-AVA1(3)*AVA2(2)
      YNORM(2)=AVA1(3)*AVA2(1)-AVA1(1)*AVA2(3)
      YNORM(3)=AVA1(1)*AVA2(2)-AVA1(2)*AVA2(1)
C
      SN(1)=YNORM(1)/BETA
      SN(2)=YNORM(2)
      SN(3)=YNORM(3)
      ASN=SQRT(SN(1)**2+SN(2)**2+SN(3)**2)
      AYN=SQRT(YNORM(1)**2+YNORM(2)**2+YNORM(3)**2)
      DO 1010 K=1,3
      UN(K)=YNORM(K)/AYN
      SNUN(K)=SN(K)/ASN
1010  CONTINUE
C
C
      DO 170 IFACE=1,NS
C
      ISBTOT=NXY(IFACE)
      DO 160 II=1,ISBTOT
      I=NT(IFACE)+II
C
      DO 160 ISYMMY=1,NSYMMY
      DO 160 ISYMMZ=1,NSYMMZ
      SIGNPT(2)=3.-2*ISYMMY
      SIGNPT(3)=3.-2*ISYMMZ
C
      DO 1102 K=1,3
1102  PZ(K)=PC(K,J)-PC(K,I)*SIGNPT(K)
      QDOTUN=DOTPRO(UN(1),UN(2),UN(3),PZ(1),PZ(2),PZ(3))
C
      DO 1110 K=1,3
      Q(K,1)=YPM(K,J)-PC(K,I)*SIGNPT(K)
      Q(K,2)=YPP(K,J)-PC(K,I)*SIGNPT(K)
      Q(K,3)=YMP(K,J)-PC(K,I)*SIGNPT(K)
      Q(K,4)=YMM(K,J)-PC(K,I)*SIGNPT(K)
1110  CONTINUE
      DO 1111 K=1,3
      QM(K,1)=0.5*(Q(K,1)+Q(K,2))
      QM(K,2)=0.5*(Q(K,2)+Q(K,3))
      QM(K,3)=0.5*(Q(K,3)+Q(K,4))
      QM(K,4)=0.5*(Q(K,4)+Q(K,1))
1111  CONTINUE
C
```

```
      DO 1112 K=1,3
      KP1=K+1
      KP2=K+2
      IF(KP1.GT.3)KP1=KP1-3
      IF(KP2.GT.3)KP2=KP2-3
      QMCRA1(K,1)=QM(KP1,2)*A1(KP2,1)-QM(KP2,2)*A1(KP1,1)
      QMCRA1(K,2)=QM(KP1,4)*A1(KP2,2)-QM(KP2,4)*A1(KP1,2)
      QMCRA2(K,1)=QM(KP1,1)*A2(KP2,1)-QM(KP2,1)*A2(KP1,1)
      QMCRA2(K,2)=QM(KP1,3)*A2(KP2,2)-QM(KP2,3)*A2(KP1,2)
1112  CONTINUE
C
      RC=SQRT(DOTPRO(PZ(1),PZ(2),PZ(3),PZ(1),PZ(2),PZ(3)))          UNST
      UIOMER=UNIMAG*OMEGA*RC                                         UNST
      CMPXDB=CEXP(-UIOMER)*(1.+UIOMER)                               UNST
      CMPXSC=CEXP(-UIOMER)*CEXP(-UNIMAG*OMEGA*UMACH*PC(1,J))         UNST
C
      DO 155 ICORNR=1,4
      GO TO (5502,5504,5506,5508),ICORNR
5502  CONTINUE
      SIGN12=-1.
      ICSI=1
      IETA=2
      GO TO 5510
5504  CONTINUE
      SIGN12=+1.
      ICSI=1
      IETA=1
      GO TO 5510
5506  CONTINUE
      SIGN12=-1.
      ICSI=2
      IETA=1
      GO TO 5510
5508  CONTINUE
      SIGN12=+1.
      ICSI=2
      IETA=2
5510  CONTINUE
C
      DO 5520 K=1,3
      QV(K)=Q(K,ICORNR)
      A1V(K)=A1(K,IETA)
      A2V(K)=A2(K,ICSI)
      QCRA1(K)=QMCRA1(K,IETA)
      QCRA2(K)=QMCRA2(K,ICSI)
5520  CONTINUE
C
      QQ=SQRT(DOTPRO(QV(1),QV(2),QV(3),QV(1),QV(2),QV(3)))           SUB
C
      CALL LOG(ICORNR,A1V,QCRA1,ALOG1,1)
C
      CALL LOG(ICORNR,A2V,QCRA2,ALOG2,2)
C
      TANP=0.
      IF(QDOTUN.EQ.0.)GO TO 5550
      HNUMER=-DOTPRO(QCRA1(1),QCRA1(2),QCRA1(3),QCRA2(1),QCRA2(2),
     1                                                   QCRA2(3))  SUB
      DENOM=QQ*QDOTUN*A1CRA2(ICORNR)
```

```
      IF(DENOM.NE.0.)TANP=ATANP(HNUMER,DENOM)
5550  CONTINUE
C
      COEFF1=DOTPRO(UN(1),UN(2),UN(3),QCRA1(1),QCRA1(2),QCRA1(3))          SUB
      COEFF2=DOTPRO(UN(1),UN(2),UN(3),QCRA2(1),QCRA2(2),QCRA2(3))          SUB
C
      SRCINT=+CONST*SIGN12*                                                SUB
     1       (-COEFF1*ALOG1+COEFF2*ALOG2-QDOTUN*TANP)
C
      DBTINT=-CONST*SIGN12*TANP
C
      SGNINT=1.
      IF(ISYMMY.EQ.2)SGNINT=SGNINT*KSYMMY
      IF(ISYMMZ.EQ.2)SGNINT=SGNINT*KSYMMZ
C
      NNN=I+(J-1)*NTOTAL
C
      AA(NNN)=AA(NNN)-SGNINT*DBTINT*CMPXDB                                 UNST
C
      ARG=ABS((PC(2,J)-BODYR)/(0.5*SPAN-BODYR))
      IF(IZZZ.EQ.101)XNUNST=-UNIMAG*SNUN(3)*                               UNST
     1       (0.18043*ARG+1.70255*ARG**2-1.13688*ARG**3+0.25387*ARG**4)    UNST
      IF(IZZZ.GE.200.AND.IZZZ.LT.300)XNUNST=+UNIMAG*REDFRE*SNUN(3)         UNST
      IF(IZZZ.GE.300.AND.IZZZ.LT.400)                                      UNST
     1XNUNST=+(UNIMAG*(PC(1,J)*BETA-XPITCH)*REDFRE+1.0)*SNUN(3)            UNST
C
C     FOLLOWING IS FOR FLUTTER
      IF(IZZZ.LT.500)GO TO 499
      XNUNST=+(UNIMAG*(PC(1,J)*BETA-XPITCH)*REDFRE+1.0)*SNUN(3)
      XNTRAN=+UNIMAG*REDFRE*SNUN(3)
      SCTRAN(I)=SCTRAN(I)+SGNINT*SRCINT*XNTRAN*CMPXSC
499   CONTINUE
C
      SOURCE(I)=SOURCE(I)+SGNINT*SRCINT*XNUNST*CMPXSC                      UNST
C
150   CONTINUE
      IF(I.GT.0)GO TO 155
      IF(J.GT.5)GO TO 155
      WRITE(NWRITE,660)I,J,ISYMMY,ISYMMZ,ICORNR,DBTINT,SRCINT,COEFF1,
     1ALOG1,COEFF2,ALOG2,SOURCE(I)
660   FORMAT(1X,5I4,8E13.4)
155   CONTINUE
160   CONTINUE
170   CONTINUE
180   CONTINUE
C
C     CALL TIME(10)
C
C     FOLLOWING IS THE EVALUATION OF WAKE COEFFICIENTS
C
      FACTOR=1.0
      IXYWAK=0
C
      IF(KXINCR.EQ.20)XINCRZ=2.0*CHORD/NWAKE
      IF(KXINCR.EQ.21)XINCRZ=2.5*CHORD/NWAKE
      IF(KXINCR.EQ.22)XINCRZ=3.0*CHORD/NWAKE
      IF(KXINCR.EQ.23)XINCRZ=4.0*CHORD/NWAKE
      IF(KXINCR.EQ.101)XINCRZ=0.04*CHORD/FACTOR
```

```
      IF(KXINCR.EQ.102)XINCRZ=0.06*CHORD/FACTOR
      IF(KXINCR.EQ.103)XINCRZ=0.08*CHORD/FACTOR
      IF(KXINCR.EQ.104)XINCRZ=0.10*CHORD/FACTOR
C
      WRITE(6,188)KXINCR,NWAKE,XINCRZ
188   FORMAT(//5X,2I7,F14.4//)
C
      XINCR=XINCRZ
      DO 999 IXWAKE=1,NWAKE
      XINCR=XINCR*FACTOR
      XWAKE1=XWAKE2
      XWAKE2=XWAKE1+XINCR
C
      DO 888 J=1,NTOTAL
      IF(KWAKE(J).EQ.0)GO TO 888
C
      WPP(1)=YPP(1,J)+XWAKE2
      WPP(2)=YPP(2,J)
      WPP(3)=YPP(3,J)
      WPM(1)=YPM(1,J)+XWAKE2
      WPM(2)=YPM(2,J)
      WPM(3)=YPM(3,J)
      WMP(1)=YPP(1,J)+XWAKE1
      WMP(2)=YPP(2,J)
      WMP(3)=YPP(3,J)
      WMM(1)=YPM(1,J)+XWAKE1
      WMM(2)=YPM(2,J)
      WMM(3)=YPM(3,J)
C
      DO 770 K=1,3
      WPC(K)=(WPP(K)+WPM(K)+WMP(K)+WMM(K))/4.
      A1V(K)=0.5*(WPP(K)-WMP(K))
      A2V(K)=0.5*(WMP(K)-WMM(K))
770   CONTINUE
C
      A1VA1V=DOTPRO(A1V(1),A1V(2),A1V(3),A1V(1),A1V(2),A1V(3))
      A2VA2V=DOTPRO(A2V(1),A2V(2),A2V(3),A2V(1),A2V(2),A2V(3))
      A1VA2V=DOTPRO(A1V(1),A1V(2),A1V(3),A2V(1),A2V(2),A2V(3))
C
      A1CSA2=SQRT(A1VA1V*A2VA2V-A1VA2V**2)
C
      YNORM(1)=A1V(2)*A2V(3)-A1V(3)*A2V(2)
      YNORM(2)=A1V(3)*A2V(1)-A1V(1)*A2V(3)
      YNORM(3)=A1V(1)*A2V(2)-A1V(2)*A2V(1)
      ABNORM=SQRT(YNORM(1)**2+YNORM(2)**2+YNORM(3)**2)
      DO 6601 K=1,3
      UN(K)=YNORM(K)/ABNORM
6601  CONTINUE
C
      DO 777 I=1,NTOTAL
      NNN=I+(J-1)*NTOTAL
C
      DO 777 ISYMMY=1,NSYMMY
      DO 777 ISYMMZ=1,NSYMMZ
      SIGNPT(2)=3.-2*ISYMMY
      SIGNPT(3)=3.-2*ISYMMZ
C
      DO 6602 K=1,3
```

```
6602  PZ(K)=WPC(K)-PC(K,I)*SIGNPT(K)
      QDOTUN=DOTPRO(UN(1),UN(2),UN(3),PZ(1),PZ(2),PZ(3))
C
      DO 6610 K=1,3
      Q(K,1)=WPM(K)-PC(K,I)*SIGNPT(K)
      Q(K,2)=WPP(K)-PC(K,I)*SIGNPT(K)
      Q(K,3)=WMP(K)-PC(K,I)*SIGNPT(K)
      Q(K,4)=WMM(K)-PC(K,I)*SIGNPT(K)
6610  CONTINUE
      DO 6611 K=1,3
      QM(K,1)=0.5*(Q(K,1)+Q(K,2))
      QM(K,2)=0.5*(Q(K,2)+Q(K,3))
      QM(K,3)=0.5*(Q(K,3)+Q(K,4))
      QM(K,4)=0.5*(Q(K,4)+Q(K,1))
6611  CONTINUE
C
      DO 6612 K=1,3
      KP1=K+1
      KP2=K+2
      IF(KP1.GT.3)KP1=KP1-3
      IF(KP2.GT.3)KP2=KP2-3
      QMCRA1(K,1)=QM(KP1,2)*A1V(KP2)-QM(KP2,2)*A1V(KP1)
      QMCRA1(K,2)=QM(KP1,4)*A1V(KP2)-QM(KP2,4)*A1V(KP1)
      QMCRA2(K,1)=QM(KP1,1)*A2V(KP2)-QM(KP2,1)*A2V(KP1)
      QMCRA2(K,2)=QM(KP1,3)*A2V(KP2)-QM(KP2,3)*A2V(KP1)
6612  CONTINUE
C
      RC=SQRT(DOTPRO(PZ(1),PZ(2),PZ(3),PZ(1),PZ(2),PZ(3)))                  UNS
      WAKE=0.
C
      DO 666 ICORNR=1,4
      GO TO (7702,7704,7706,7708),ICORNR
7702  CONTINUE
      SIGN12=-1.
      ICSI=1
      IETA=2
      GO TO 7710
7704  CONTINUE
      SIGN12=+1.
      ICSI=1
      IETA=1
      GO TO 7710
7706  CONTINUE
      SIGN12=-1.
      ICSI=2
      IETA=1
      GO TO 7710
7708  CONTINUE
      SIGN12=+1.
      ICSI=2
      IETA=2
7710  CONTINUE
C
      DO 7720 K=1,3
      QV(K)=Q(K,ICORNR)
      QCRA1(K)=QMCRA1(K,IETA)
      QCRA2(K)=QMCRA2(K,ICSI)
7720  CONTINUE
```

```
C
      QQ=SQRT(DOTPRO(QV(1),QV(2),QV(3),QV(1),QV(2),QV(3)))                SUB
C
      TANP=0.
      IF(QDOTUN.EQ.0.)GO TO 7750
      HNUMER=-DOTPRO(QCRA1(1),QCRA1(2),QCRA1(3),QCRA2(1),QCRA2(2),
     1                                                    QCRA2(3))      SUB
      DENOM=QQ*QDOTUN*A1CSA2
      IF(DENOM.NE.0.)TANP=ATANP(HNUMER,DENOM)
7750  CONTINUE
C
      WAKE=WAKE-SIGN12*TANP*CONST
C
666   CONTINUE
C
      IXYWAK=IXYWAK+1
      IF(IXYWAK.EQ.1)WRITE(6,2244)I,J,ISYMMY,ISYMMZ,WAKE
2244  FORMAT(4I5,E16.5)
C
C
      SGNINT=1.
      IF(ISYMMY.EQ.2)SGNINT=SGNINT*KSYMMY
      IF(ISYMMZ.EQ.2)SGNINT=SGNINT*KSYMMZ
C
      XXX=WPC(1)-PC(1,J)
C
      AA(NNN)=AA(NNN)-WAKE*SGNINT*(1.+UNIMAG*OMEGA*RC)*
     1                CEXP(-UNIMAG*REDFRE*(XXX+UMACH*RC)/BETA)
777   CONTINUE
888   CONTINUE
999   CONTINUE
      RETURN
      END
```

```
C
C
      SUBROUTINE LOG(ICORNR,AV,QCRAV,ALOG,ICOOR)
      COMMON/ZZZ88/QV(3),QQ,I,J,ISYMMY,ISYMMZ
      DIMENSION AV(3),QCRAV(3)
C
      DOTPRO(X1,Y1,Z1,X2,Y2,Z2)=X1*X2+Y1*Y2+Z1*Z2
C
      AVAV=DOTPRO(AV(1),AV(2),AV(3),AV(1),AV(2),AV(3))
      QQAV=DOTPRO(QV(1),QV(2),QV(3),AV(1),AV(2),AV(3))
      QXA=SQRT(DOTPRO(QCRAV(1),QCRAV(2),QCRAV(3),
     1                QCRAV(1),QCRAV(2),QCRAV(3)))
      ALOG=ASINH(QQAV/QXA)/SQRT(AVAV)
      RETURN
      END
C
      FUNCTION ASINH(X)
      ABX=ABS(X)
      XSQ=X*X
      IF(ABX.LE.0.001)ASINH=X*(1.-0.1666666*XSQ+.075*XSQ*XSQ)
      IF(ABX.GT.0.001)ASINH=(X/ABX)*ALOG(ABX+SQRT(1.+XSQ))
      RETURN
      END
C
C
      FUNCTION ATANP(HNUMER,DENOM)
      ADENOM=ABS(DENOM)
      ATANP=ATAN2(HNUMER,ADENOM)
      IF(DENOM.LT.0.)ATANP=-ATANP
      RETURN
      END
```

```
      SUBROUTINE COEFF(NTOTAL,PC,YPP,YPM,YMP,YMM,SUMDB,NT2S,AA,SOURCE,
     1SCTRAN)
C
C     THIS SUBPROGRAM IS TO CALCULATE DOUBLET
C     AND SOURCE FOR UNSTEADY SUPERSONIC FLOW
C
      COMPLEX SOURCE,SCTRAN,AA,UNIMAG,XNUNST,XNTRAN,CMPXDB,CMPXSC          UNST
      COMPLEX COMEGA,SINHU,COSHU
C
      COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYMMY,KSYMMZ,NSYMMY,NSYMMZ
      COMMON/ZZZ2/TAU,SPAN,XX,YY,CHORD,IZZZ,UMACH,REFLEN
      COMMON/ZZZ10/KWAKES(20)
      COMMON/ZZZ11/ALFA,ALFABC,BRED,CRED,BODYR,XLEZ,XTEZ,XNOSE,XTAIL,CTR
      COMMON/ZZZ12/NSEX,NSBODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(20)
      COMMON/ZZZ66/NKCOF,KORINT(4),MDEBUG(4,2),OM(3,4)
      COMMON/ZZZ77/QV(3),QQ,KODE(5),SIGNV1(4,2),SIGNV2(4,2)
      COMMON/ZZZ88/I,J,ISYMMY,ISYMMZ
      COMMON/ZZZ99/KPRINT(20)
      DIMENSION YPP(3,NTOTAL),YPM(3,NTOTAL),YMP(3,NTOTAL),YMM(3,NTOTAL),
     1PC(3,NTOTAL),SOURCE(NTOTAL),SCTRAN(NTOTAL),SUMDB(NTOTAL),AA(NT2S)
      DIMENSION A1(3,2),A2(3,2),A1A1(2),A1A2(2),QCRA1(3),QCRA2(3)
      DIMENSION A1CRA2(4),A2A1(3),A2A2(3),SIGNRT(3)
      DIMENSION AVA1(3),AVA2(3),QSUBC(4),KODES1(4),KODES2(4),KODES3(4)
      DIMENSION A1V(3),A2V(3),YNORM(3),UN(3),SN(3),SNUN(3),A1A2(2,2)
      DIMENSION PZ(3),Q(3,4),QMCRA1(3,2),QMCRA2(3,2),AA1(3,2),AA2(3,2)
C
C
      DOTPRD(X1,Y1,Z1,X2,Y2,Z2)=X1*X2+Y1*Y2+Z1*Z2
C
      SUBPRD(XX1,YY1,ZZ1,XX2,YY2,ZZ2)=XX1*XX2-YY1*YY2-ZZ1*ZZ2
C
C
      IF(UMACH.EQ.1.)UMACH=SQRT(2.)
      IF(UMACH.LT.1.)CALL DEBUG(500)
      BETA=SQRT(UMACH**2-1.)
      WRITE(6,2)NSBODY,NS,(NT(I),I=1,NS),(KNORML(J),J=1,NS),(KDIAF(K),
     1K=1,NS),(ISFACE(II),II=1,NS),(NXY(KK),KK=1,NS),NSYMMY,NSYMMZ
2     FORMAT(3X,2515)
C                                                                          UNST
      IF(IZZZ.GE.100.AND.IZZZ.LT.200)WRITE(6,9911)                         UNST
      IF(IZZZ.GE.200.AND.IZZZ.LT.300)WRITE(6,9922)                         UNST
      IF(IZZZ.GE.300.AND.IZZZ.LT.400)WRITE(6,9933)                         UNST
9911  FORMAT(//2X,'------ OSCILLATION IN FIRST BENDING MODE ------'//)    UNST
9922  FORMAT(//2X,'------ OSCILLATION IN TRANSLATION ----------'//)       UNST
9933  FORMAT(//2X,'------ OSCILLATION IN PITCH ----------'//)             UNST
      XPITCH=XLEZ+(XTEZ-XLEZ)*CTR                                          UNST
      IF(IZZZ.EQ.301)XPITCH=XLEZ+(XTEZ-XLEZ)*0.00                          UNST
      IF(IZZZ.EQ.302)XPITCH=XLEZ+(XTEZ-XLEZ)*0.5)                          UNST
CD    IF(KOM.EQ.11)XPITCH=XLEZ+(XTEZ-XLEZ)*0.15                            UNST
CD    IF(KOM.EQ.12)XPITCH=XLEZ+(XTEZ-XLEZ)*0.20
```

```
CD      IF(KCM.EQ.13)XPITCH=XLEZ+(XTEZ-XLEZ)*0.25                        UNST
CD      IF(KCM.EQ.14)XPITCH=XLEZ+(XTEZ-XLEZ)*0.30                        UNST
CD      IF(KCM.EQ.15)XPITCH=XLEZ+(XTEZ-XLEZ)*0.325                       UNST
CD      IF(KCM.EQ.16)XPITCH=XLEZ+(XTEZ-XLEZ)*0.350                       UNST
CD      IF(KCM.EQ.17)XPITCH=XLEZ+(XTEZ-XLEZ)*0.1875                      UNST
CD      IF(KCM.EQ.18)XPITCH=XLEZ+(XTEZ-XLEZ)*0.4500                      UNST
C
        UNIMAG=(0.,1.)                                                   UNST
        COMEGA=(RRED+UNIMAG*CRED)*UMACH/BETA
C
        ALERBC=ALFABC*3.14159/180.
        SINABC=SIN(ALERBC)
        COSABC=COS(ALERBC)
C
        DO 6 NN=1,NTZS
6       AA(NN)=0.
C
C       INSERT THE KRONECKER DELTA (SWITCH FOR THE LOWER SIDES OF DIAPHRAGMS)
C
        DO 9 IS=1,NS
        NXYS=NXY(IS)
        DO 9 II=1,NXYS
        I=II+NT(IS)
        J=I
C
C       LOWER SIDES OF DIAPHRAGMS
C
        IF((KNORML(IS).EQ.-1).AND.(IS.GT.NSBODY))J=I-NXY(IS)
        NNN=I+(J-1)*NTOTAL
        AA(NNN)=1.
9       CONTINUE
C
        CONST=-1./3.14159
C
        DO 10 I=1,NTOTAL
        SUMDB(I)=0.
        SOURCE(I)=0.
10      SCTRAN(I)=0.
C
        SIGNPT(1)=1.0
C
        DO 180 JFACE=1,NS
        JSBTOT=NXY(JFACE)
        DO 180 JJ=1,JSBTOT
        J=NT(JFACE)+JJ
C
        DO 1002 K=1,3
        AA1(K,1)=0.5*(YPP(K,J)-YMP(K,J))
        AA1(K,2)=0.5*(YPM(K,J)-YMM(K,J))
        AA2(K,1)=0.5*(YPP(K,J)-YPM(K,J))
        AA2(K,2)=0.5*(YMP(K,J)-YMM(K,J))
        A1(K,1)=AA1(K,1)
        A1(K,2)=AA1(K,2)
        A2(K,1)=AA2(K,1)
```


C-3

```
      A2(K,2)=AA2(K,2)
1002  CONTINUE
      DO 1003 L=1,2
      A1A1(L)=DOTPRD(A1(1,L),A1(2,L),A1(3,L),A1(1,L),A1(2,L),A1(3,L))
1003  A2A2(L)=DOTPRD(A2(1,L),A2(2,L),A2(3,L),A2(1,L),A2(2,L),A2(3,L))
C
      DO 1004 K=1,3
      IF(A1A1(1).EQ.0.)A1(K,1)=A1(K,2)
      IF(A1A1(2).EQ.0.)A1(K,2)=A1(K,1)
      IF(A2A2(1).EQ.0.)A2(K,1)=A2(K,2)
      IF(A2A2(2).EQ.0.)A2(K,2)=A2(K,1)
1004  CONTINUE
C
      DO 1006 L=1,2
      A1A1(L)=DOTPRD(A1(1,L),A1(2,L),A1(3,L),A1(1,L),A1(2,L),A1(3,L))
1006  A2A2(L)=DOTPRD(A2(1,L),A2(2,L),A2(3,L),A2(1,L),A2(2,L),A2(3,L))
C
      DO 1007 L=1,2
      DO 1007 M=1,2
1007  A1A2(L,M)=DOTPRD(A1(1,L),A1(2,L),A1(3,L),A2(1,M),A2(2,M),A2(3,M))
C
      A1CRA2(1)=SQRT(A1A1(2)*A2A2(1)-A1A2(2,1)**2)
      A1CRA2(2)=SQRT(A1A1(1)*A2A2(1)-A1A2(1,1)**2)
      A1CRA2(3)=SQRT(A1A1(1)*A2A2(2)-A1A2(1,2)**2)
      A1CRA2(4)=SQRT(A1A1(2)*A2A2(2)-A1A2(2,2)**2)
C
      DO 1008 K=1,3
      AVA1(K)=0.5*(A1(K,1)+A1(K,2))
1008  AVA2(K)=0.5*(A2(K,1)+A2(K,2))
C
      YNORM(1)=AVA1(2)*AVA2(3)-AVA1(3)*AVA2(2)
      YNORM(2)=AVA1(3)*AVA2(1)-AVA1(1)*AVA2(3)
      YNORM(3)=AVA1(1)*AVA2(2)-AVA1(2)*AVA2(1)
C
      SN(1)=YNORM(1)/BETA
      SN(2)=YNORM(2)
      SN(3)=YNORM(3)
      ASN=SQRT(SN(1)**2+SN(2)**2+SN(3)**2)
      AYN=SQRT(YNORM(1)**2+YNORM(2)**2+YNORM(3)**2)
      DO 1010 K=1,3
      UN(K)=YNORM(K)/AYN
      SNUN(K)=SN(K)/ASN
1010  CONTINUE
C
      SUNONO=SUPPRD(UN(1),UN(2),UN(3),UN(1),UN(2),UN(3))
C
      DO 170 IFACE=1,NS
C
      IF(KDIAF(IFACE).NE.KDIAF(JFACE).AND.NSBODY.LT.NS)GO TO 170
C
      ISBTOT=NXY(IFACE)
      DO 160 II=1,ISBTOT
      I=NT(IFACE)+II
C
```

```
      DO 160 ISYMMY=1,NSYMMY
      DO 160 ISYMMZ=1,NSYMMZ
      SIGNPT(2)=3.-2*ISYMMY
      SIGNPT(3)=3.-2*ISYMMZ
C
      DO 1101 ICORNR=1,4
      DO 1101 ICOOR=1,2
      SIGNV1(ICORNR,ICOOR)=0.
1101  SIGNV2(ICORNR,ICOOR)=0.
C
      DO 1102 K=1,3
1102  PZ(K)=PC(K,J)-PC(K,I)*SIGNPT(K)
      PZDOPZ=DOTPRO(PZ(1),PZ(2),PZ(3),PZ(1),PZ(2),PZ(3))
      QDOTUN=DOTPRO(UN(1),UN(2),UN(3),PZ(1),PZ(2),PZ(3))
      IF(I.EQ.J)GO TO 1105
      IF((ABS(QDOTUN)/SQRT(PZDOPZ)).LT.1.0E-4)QDOTUN=0.
1105  CONTINUE
C
      DO 1110 K=1,3
      Q(K,1)=YPM(K,J)-PC(K,I)*SIGNPT(K)
      Q(K,2)=YPP(K,J)-PC(K,I)*SIGNPT(K)
      Q(K,3)=YMP(K,J)-PC(K,I)*SIGNPT(K)
      Q(K,4)=YMM(K,J)-PC(K,I)*SIGNPT(K)
1110  CONTINUE
      DO 1111 K=1,3
      QM(K,1)=0.5*(Q(K,1)+Q(K,2))
      QM(K,2)=0.5*(Q(K,2)+Q(K,3))
      QM(K,3)=0.5*(Q(K,3)+Q(K,4))
      QM(K,4)=0.5*(Q(K,4)+Q(K,1))
1111  CONTINUE
C
      DO 1112 K=1,3
      KP1=K+1
      KP2=K+2
      IF(KP1.GT.3)KP1=KP1-3
      IF(KP2.GT.3)KP2=KP2-3
      QMCRA1(K,1)=QM(KP1,2)*A1(KP2,1)-QM(KP2,2)*A1(KP1,1)
      QMCRA1(K,2)=QM(KP1,4)*A1(KP2,2)-QM(KP2,4)*A1(KP1,2)
      QMCRA2(K,1)=QM(KP1,1)*A2(KP2,1)-QM(KP2,1)*A2(KP1,1)
      QMCRA2(K,2)=QM(KP1,3)*A2(KP2,2)-QM(KP2,3)*A2(KP1,2)
1112  CONTINUE
C
      DO 1120 ICORNR=1,4
      DO 1120 ICOOR=1,2
1120  MDEBUG(ICORNR,ICOOR)=0
C
      MKODE=0
      DO 1130 ICORNR=1,4
      KODE(ICORNR)=0
      KDBINT(ICORNR)=0
      KODES1(ICORNR)=0
      KODES2(ICORNR)=0
      KODES3(ICORNR)=0
      QSUPO(ICORNR)=SUPPRC(Q(1,ICORNR),Q(2,ICORNR),Q(3,ICORNR),
```

```
     1                        Q(1,ICORNP),Q(2,ICORNR),Q(3,ICORNR))
      IF(QSUPQ(ICORNP).LE.0.OR.Q(1,ICORNR).GE.0)KODE(ICORNP)=1
      MKODE=MKODE+KODE(ICORNR)
1130  CONTINUE
C
      IF(MKODE.EQ.0)GO TO 2299
C
C     FOLLOWING IS TO CHECK THE DOUBLE INTERSECTIONS ALONG ETA DIRECTION AND
C     TO EVALUATE THE CRITICAL VECTORS OF Q , A1 AND SIGNV1(I,2) AT ANY CORN
C     POINT OF WHICH THE ELEMENT IS PARTIALLY OUTSIDE THE MACH FORECONE
C
      NKODE=0
C
      DO 2220 IA2=1,2
      ISIDE=(IA2-1)*2+3
      IF(KODE(ISIDE).EQ.0.AND.KODE(ISIDE+1).EQ.0)GO TO 2220
      DO 2210 K=1,3
2210  AAA2(K)=AA2(K,IA2)
C
      CALL CRVECT(ISIDE,AAA2,AA1,2)
C
2220  CONTINUE
C
C     FOLLOWING IS TO CHECK THE DOUBLE INTERSECTIONS ALONG CSI DIRECTION AND
C     TO EVALUATE THE CRITICAL VECTORS OF Q , A2 AND SIGNV1(I,1)
C
      KODE(5)=KODE(1)
      DO 2240 IA1=1,2
      ISIDE=(IA1-1)*2+2
      IF(KODE(ISIDE).EQ.0.AND.KODE(ISIDE+1).EQ.0)GO TO 2240
      DO 2230 K=1,3
2230  AAA1(K)=AA1(K,IA1)
C
      CALL CRVECT(ISIDE,AAA1,AA2,1)
C
2240  CONTINUE
2299  CONTINUE
C
      IF(MKODE.EQ.4.AND.NKODE.EQ.0)GO TO 160
C
      IF(KODE(1).EQ.0.OR.KODE(2).EQ.0.OR.KOBINT(1).EQ.1)GO TO 3310
      KODES2(1)=1
      KODES2(2)=1
3310  IF(KODE(3).EQ.0.OR.KODE(4).EQ.0.OR.KOBINT(3).EQ.1)GO TO 3320
      KODES2(3)=1
      KODES2(4)=1
3320  IF(KODE(1).EQ.0.OR.KODE(4).EQ.0.OR.KOBINT(4).EQ.1)GO TO 3330
      KODES1(1)=1
      KODES1(4)=1
3330  IF(KODE(2).EQ.0.OR.KODE(3).EQ.0.OR.KOBINT(2).EQ.1)GO TO 3340
      KODES1(2)=1
      KODES1(3)=1
3340  IF(KODE(1).EQ.1.AND.KODE(2).EQ.0)KODES3(1)=1
      IF(KODE(2).EQ.1.AND.KODE(1).EQ.0)KODES3(2)=1
```

```
      IF(KODE(3).EQ.1.AND.KODE(4).EQ.0)KODES3(3)=1
      IF(KODE(4).EQ.1.AND.KODE(3).EQ.0)KODES3(4)=1
      IF(KDBINT(1).NE.1)GO TO 3350
      KODES3(1)=1
      KODES3(2)=1
3350  IF(KDBINT(3).NE.1)GO TO 3360
      KODES3(3)=1
      KODES3(4)=1
3360  CONTINUE
C
      RC=0.                                                          UNST
      RZSPPZ=SUPPRO(PZ(1),PZ(2),PZ(3),PZ(1),PZ(2),PZ(3))             UNST
      IF(RZSPPZ.GT.0.)RC=SQRT(RZSPPZ)                                UNST
      COSHU=(CEXP(COMEGA*RC)+CEXP(-COMEGA*RC))*0.5
      SINHU=(CEXP(COMEGA*RC)-CEXP(-COMEGA*RC))*0.5
      CMPXDB=COSHU-COMEGA*RC*SINHU
      CMPXSC=COSHU*CEXP(COMEGA*UMACH*RC(1,J))
C
      DO 155 ICORNR=1,4
      GO TO (5502,5504,5506,5508),ICORNR
5502  CONTINUE
      SIGN12=-1.
      ICSI=1
      IETA=2
      GO TO 5510
5504  CONTINUE
      SIGN12=+1.
      ICSI=1
      IETA=1
      GO TO 5510
5506  CONTINUE
      SIGN12=-1.
      ICSI=2
      IETA=1
      GO TO 5510
5508  CONTINUE
      SIGN12=+1.
      ICSI=2
      IETA=2
5510  CONTINUE
C
      DO 5520 K=1,3
      QV(K)=Q(K,ICORNR)
      A1V(K)=A1(K,IETA)
      A2V(K)=A2(K,ICSI)
      QCRA1(K)=QMCRA1(K,IETA)
      QCRA2(K)=QMCRA2(K,ICSI)
5520  CONTINUE
C
      QQ=SQRT(ABS(QSUPQ(ICORNR)))
C
      ALOG1=0.
      IF(KODES1(ICORNR).EQ.1)GO TO 553)
      CALL LOG(ICORNR,A1V,QCRA1,ALOG1,1)
```

```
5530  CONTINUE
C
      ALOG2=0.
      IF(KODES2(ICORNR).EQ.1)GO TO 5540
      CALL LOG(ICORNR,A2V,QCRA2,ALOG2,2)
5540  CONTINUE
C
      TANP=0.
      IF(QDOTUN.EQ.0.)GO TO 5550
      IF(KODE(ICORNR).EQ.1)GO TO 5549
      HNUMER=-SUPPRO(QCRA1(1),QCRA1(2),QCRA1(3),QCRA2(1),QCRA2(2),
     1                                                    QCRA2(3))
      DENOM=QQ*QDOTUN*A1CRA2(ICORNR)
      IF(DENOM.NE.0.)TANP=ATANP(HNUMER,DENOM)
      GO TO 5550
5549  CONTINUE
      IF(KODES2(ICORNR).EQ.0)GO TO 5550
      IF(MDEBUG(ICORNR,2).EQ.0)CALL DEBUG(505)
      SGNARG=SIGNV1(ICORNR,2)*SIGNV2(ICORNR,2)*QDOTUN
      IF(SGNARG.LT.0.)SGNTAN=-1.
      IF(SGNARG.EQ.0.)SGNTAN=+0.
      IF(SGNARG.GT.0.)SGNTAN=+1.
      TANP=SGNTAN*1.570795
5550  CONTINUE
C
      COEFF1=SUPPRO(UN(1),UN(2),UN(3),QCRA1(1),QCRA1(2),QCRA1(3))
      COEFF2=SUPPRO(UN(1),UN(2),UN(3),QCRA2(1),QCRA2(2),QCRA2(3))
C
      SRCINT=-CONST*SIGN12/SUNORO*
     1       (-COEFF1*ALOG1+COEFF2*ALOG2-QDOTUN*TANP)
C
      DBTINT=-CONST*SIGN12*TANP
C
      SGNINT=1.
      IF(ISYMMY.EQ.2)SGNINT=SGNINT*KSYMMY
      IF(ISYMMZ.EQ.2)SGNINT=SGNINT*KSYMMZ
C
      IF(JFACE.GT.NSBODY)GO TO 548
C
C     AIRCRAFT
C
      NNN=I+(J-1)*NTOTAL
C
      AA(NNN)=AA(NNN)-SGNINT*DBTINT*CMPXDB                              UNST
C
      ARG=ABS((PC(2,J)-BODYR)/(0.5*SPAN-BODYR))
      IF(IZZZ.EQ.101)XNUNST=-(RRED+UNIMAG*CRED)*SNUN(3)*
     1       (0.78043*ARG+0.70255*ARG**2-1.13638*ARG**3+0.25287*ARG**4) UNST
      IF(IZZZ.GE.200.AND.IZZZ.LT.300)XNUNST=+(RRED+UNIMAG*CRED)*SNUN(3)
      IF(IZZZ.GE.300.AND.IZZZ.LT.400)                                   UNST
     1XNUNST=+((RRED+UNIMAG*CRED)*(PC(1,J)*BETA-XPITCH)+1.0)*SNUN(3)
C
C     FOLLOWING IS FOR FLUTTER
      IF(IZZZ.LT.500)GO TO 499
```

```
      XNUNST=+((RRED+UNIMAG*CRED)*(PC(I,J)*BETA-XPITCH)+1.0)*SNUN(3)
      XNTRAN=+(RRED+UNIMAG*CRED)*SNUN(3)
      SCTRAN(I)=SCTRAN(I)+SGNINT*SRCINT*XNTRAN*CMPXSC
499   CONTINUE
C
      SOURCE(I)=SOURCE(I)+SGNINT*SRCINT*XNUNST*CMPXSC                    UNST
C
      GO TO 150
148   CONTINUE
C
C     DIAPHRAGM
C
      IF(KSYMMZ.NE.0)GO TO 149
      JPHI=J
      JCHI=J
      IF(KDIAF(IFACE).EQ.-1)JPHI=J-NXY(JFACE)
      IF(KDIAF(IFACE).EQ.+1)JCHI=J+NXY(JFACE)
      NPHI=I+(JPHI-1)*NTOTAL
      NCHI=I+(JCHI-1)*NTOTAL
C
      AA(NPHI)=AA(NPHI)-SGNINT*DBTINT*CMPXDB                             UNST
      AA(NCHI)=AA(NCHI)-SGNINT*SRCINT*KDIAF(IFACE)                       UNST
C
      GO TO 150
149   IF(KSYMMZ.EQ.-1)GO TO 1491
C
C     SYMMETRIC : CHI=0
C
      NPHI=I+(J-1)*NTOTAL
      AA(NPHI)=AA(NPHI)-SGNINT*DBTINT*CMPXDB                             UNST
      GO TO 150
1491  CONTINUE
C
C     ANTISYMMETRIC : PHI=0
C
      NCHI=I+(J-1)*NTOTAL
      AA(NCHI)=AA(NCHI)-SGNINT*SRCINT*CMPXSC                             UNST
150   CONTINUE
      IF(I.GT.0)GO TO 155
C     IF(MKODE.EQ.0)GO TO 155
      WRITE(6,66)I,J,ISYMMY,ISYMMZ,ICORNR,KODE(ICORNR),SRCINT,DBTINT,
     1ALOG1,COEFF1,ALOG2,COEFF2
66    FORMAT(1X,6I4,7E15.6)
155   CONTINUE
160   CONTINUE
170   CONTINUE
180   CONTINUE
C
C
      RETURN
      END
```

```
C
      SUBROUTINE CRVECT(ISIDE,V1,V2,ICOOR)
      COMMON/ZZZ66/NKODE,KOBINT(4),MDEBUG(4,2),QM(3,4)
      COMMON/ZZZ77/QV(3),QO,KODE(5),SIGNV1(4,2),SIGNV2(4,2)
      COMMON/ZZZ88/I,J,ISYMMY,ISYMMZ
      DIMENSION GCOOR(2),CRV2(3),QQC(3),QQCX(2),V1(3),V2(3,2)
      SPV1V1=V1(1)**2-V1(2)**2-V1(3)**2
      IF(SPV1V1.EQ.0.)GO TO 1999
      SPQZQZ=QM(1,ISIDE)**2-QM(2,ISIDE)**2-QM(3,ISIDE)**2
      SPV1QZ=V1(1)*QM(1,ISIDE)-V1(2)*QM(2,ISIDE)-V1(3)*QM(3,ISIDE)
      DISCRM=SPV1QZ**2-SPQZQZ*SPV1V1
      IF(DISCRM.LE.0.)GO TO 1999
      AVCOOR=-SPV1QZ/SPV1V1
      XAVERG=QM(1,ISIDE)+AVCOOR*V1(1)
      SQDIS=SQRT(DISCRM)/ABS(SPV1V1)
      GCOOR(1)=AVCOOR+SQDIS
      GCOOR(2)=AVCOOR-SQDIS
      QQCX(1)=QM(1,ISIDE)+GCOOR(1)*V1(1)
      QQCX(2)=QM(1,ISIDE)+GCOOR(2)*V1(1)
      IF((QQCX(1)*QQCX(2)).GE.0.)GO TO 100
      GCOOR(1)=AVCOOR-SQDIS
      GCOOR(2)=AVCOOR+SQDIS
      QQCX(1)=QM(1,ISIDE)+GCOOR(1)*V1(1)
      QQCX(2)=QM(1,ISIDE)+GCOOR(2)*V1(1)
100   CONTINUE
      DO 300 IEND=1,2
      IF(GCOOR(IEND).GT.1..OR.GCOOR(IEND).LT.-1..OR.QQCX(IEND).GT.0.)
     1GO TO 300
      IF(ISIDE.EQ.1)ICORNR=ISIDE+(2-IEND)
      IF(ISIDE.EQ.2)ICORNR=ISIDE+(IEND-1)
      IF(ISIDE.EQ.3)ICORNR=ISIDE+(IEND-1)
      IF(ISIDE.EQ.4)ICORNR=ISIDE+(IEND-2)*3
      IF(MDEBUG(ICORNR,ICOOR).EQ.1)CALL DEBUG(510)
      IF(KODE(ICORNR).EQ.0)GO TO 300
      DO 200 K=1,3
      CRV2(K)=0.5*(V2(K,1)+V2(K,2))+0.5*GCOOR(IEND)*(V2(K,1)-V2(K,2))
200   QQC(K)=QM(K,ISIDE)+GCOOR(IEND)*V1(K)
      QSUPV1=QQC(1)*V1(1)-QQC(2)*V1(2)-QQC(3)*V1(3)
      QSUPV2=QQC(1)*CRV2(1)-QQC(2)*CRV2(2)-QQC(3)*CRV2(3)
      SIGNV1(ICORNR,ICOOR)=QSUPV1/ABS(QSUPV1)
      SIGNV2(ICORNR,ICOOR)=QSUPV2/ABS(QSUPV2)
      MDEBUG(ICORNR,ICOOR)=1
300   CONTINUE
      IF(KODE(ISIDE).EQ.0.OR.KODE(ISIDE+1).EQ.0)GO TO 1999
      IF(AVCOOR.GT.1..OR.AVCOOR.LT.-1..OR.XAVERG.GE.0..OR.SPQZQZ.LE.0.)
     1GO TO 1999
      KOBINT(ISIDE)=1
      NKODE=1
1999  CONTINUE
      RETURN
      END
```

```
C
      SUBROUTINE LOG(ICORNR,AV,QCRAV,ALOG,ICOCR)
      COMMON/ZZZZZ/QV(3),QQ,KODE(5),SIGNV1(4,2),SIGNV2(4,2)
      DIMENSION AV(3),QCRAV(3)
C
      SUPPRO(X1,Y1,Z1,X2,Y2,Z2)=X1*X2-Y1*Y2-Z1*Z2
C
      AVSPAV=SUPPRO(AV(1),AV(2),AV(3),AV(1),AV(2),AV(3))
      AVNORM=0.
      IF(AVSPAV.NE.0.)AVNORM=SQRT(ABS(AVSPAV))
      IF(KODE(ICORNR).EQ.1)GO TO 10
      QSUPAV=SUPPRO(QV(1),QV(2),QV(3),AV(1),AV(2),AV(3))
      QCRASP=SUPPRO(QCRAV(1),QCRAV(2),QCRAV(3),QCRAV(1),QCRAV(2),
     1                                                  QCRAV(3))
      IF(QCRASP.GE.0.)CALL DEBUG(520)
      QXA=SQRT(-QCRASP)
      IF(AVSPAV.GT.0.)ALOG=ASINH(QQ*AVNORM/QXA)/AVNORM
      IF(QSUPAV.LT.0.)ALOG=-ALOG
      IF(AVSPAV.EQ.0.)ALOG=QQ/QSUPAV
      IF(AVSPAV.LT.0.)
     1ALOG=-ARSIN(QSUPAV/SQRT(QSUPAV*QSUPAV-QQ*QQ*AVSPAV))/AVNORM
      RETURN
10    CONTINUE
      IF(AVSPAV.GE.0.)ALOG=0.
      IF(AVSPAV.LT.0.)ALOG=-SIGNV1(ICORNR,ICOCR)*1.570795/AVNORM
      RETURN
      END
```