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The Vortex Street as a Statistical-Mechanical Phenomenon

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Abstract

A simple explanation of the Kármán vortex street is suggested on the basis of the two-temperature canonical distribution for inviscid two-dimensional flows in Navier-Stokes fluids or guiding-center plasmas.

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The Kármán vortex street develops as a final state for a variety of unstable shear flow profiles[1] in two-dimensional Navier-Stokes fluids[2,3] or electrostatic guiding center plasmas.[4] The vortex street is an old and familiar experimental phenomenon[5] which has apparently never been explained theoretically. Though painstaking and accurate numerical solutions of the Navier-Stokes equations have followed the development of initially unstable shear flow profiles into vortex streets (see particularly Zabusky and Deem[6]), there is apparently no simply accessible theoretical picture of why the two rows of oppositely-signed vortices arrange themselves as they do in the characteristically staggered arrangement across the street. The purpose of this letter is to suggest a simple explanation of the vortex street on a statistical-mechanical basis.

Our starting point is a generalized application of the two-temperature canonical distribution of Kraichnan[7,8] for inviscid, two-dimensional, Navier-Stokes flows. The coordinates of the phase space over which this ensemble is defined are the real and imaginary parts of the velocity field Fourier coefficients in a truncated Fourier representation.[9] The canonical distribution varies as $\exp(-\alpha E - \beta\Omega)$, where E and Ω are the mean energy and "enstrophy" (mean square vorticity) densities. α^{-1} and β^{-1} are two temperatures, one for energy and one for enstrophy. The relaxation of arbitrary initial conditions to final states which have properties that are accurately predicted by this ensemble has been given extensive numerical investigation by Seyler et al.[10] Relaxation to this equilibrium state now appears to be beyond dispute.

One of the more interesting predictions of the two-temperature canonical distribution is for the expectation value of the kinetic energy density associated with the $\vec{k}$ th Fourier component:

$$\langle|\vec{u}(\vec{k})|^2\rangle = (\alpha + \beta k^2)^{-1} \quad (1)$$

The angular brackets < > denote an average over the two-temperature canonical distribution. The reciprocal temperatures α and β are in effect Lagrange multipliers whose values are determined by the requirements that Eq. (1) lead to the correct initial values of E and Ω:

$$E = \sum_{\vec{k}} (\alpha + \beta k^2)^{-1}$$
$$\Omega = \sum_{\vec{k}} k^2(\alpha + \beta k^2)^{-1} \quad (2)$$

The interesting thing about Eq. (1) is that qualitatively different spectra and flow patterns[8,10] are associated with different regimes of α and β. Either α or β (but not both) can be negative, depending upon the ratio Ω/E. In particular, for either large k_{max} or low values of Ω/E, $\alpha<0$, $\beta>0$, and $\alpha + \beta k^2$ is only slightly greater than zero at the minimum allowed values of k^2. In these cases, the gross features of the macroscopic turbulent field are dominated by the first few Fourier coefficients (longest wavelength modes) and become relatively insensitive to the higher wave-number contributions. The flow consists of a large pair of counter-rotating vortices that fill up the box inside which the Fourier analysis is performed.[11] This basic pattern had previously been noted in numerical solutions for the finite viscosity case by Deem and Zakusky[12] and Tappert and Hardin,[13] and for the discrete line-vortex representation by Joyce and Montgomery.[14]

The point to be made now is that the two-temperature canonical distribution has a wider applicability than simply to the case of periodic boundary conditions in a square box. For more elaborate geometrical arrangements in which fully developed two-dimensional turbulence is expected, another set of eigenfunctions than simply all the plane waves, $\exp(i\vec{k} \cdot \vec{x})$, may be appropriate. Statistical-mechanical analyses of hydrodynamic situations of a ›re general character using an arbitrary set of basis functions have be suggested by Thompson.[15]

We discuss a situation qualitatively similar to one considered by Zabusky and Deem,[6] though detailed comparisons are not in order (we omit viscosity). An unstable shear flow profile with velocity parallel to the x-axis is assumed to develop into a vortex street between $y = -L_y/2$ and $y = +L_y/2$. The fluid velocity $\vec{u}(\vec{x})$ has only x and y components and is independent of the z coordinate. The vorticity $\vec{\rho}(\vec{x}) = \nabla x \vec{u} = \rho(x,y)\hat{e}_z$ lies normal to the xy plane. The stream function $\phi(x,y)$ obeys $\nabla^2\phi = -\rho$ and is related to the fluid velocity by $\vec{u} = \nabla\phi \times \hat{e}_z$. (Each realization of the ensemble will of course be time-dependent, but we suppress the time arguments.) We assume periodicity in x with a period L_x. We require that there be rigid frictionless walls at $y = \pm L_y/2$, so that $\hat{e}_y \cdot (\nabla \phi \times \hat{e}_z)$ vanishes there. The effect of the boundaries can be replaced, using the method of images, by an infinite periodic array of image vorticity with period $2L_y$ in the y-direction. The vorticity elements outside the basic box of width L_y are chosen so that equal and opposite elements are always connected by straight lines which are perpendicularly bisected by the walls. A situation studied by Zabusky and Deem was for initial velocity $\vec{u} = u_x^o \hat{e}_x$, where $u_x^o = U_o - U_{co} \exp(-y^2/\Delta^2)$, with $L_y \gg 2\Delta$, so that the shear flow velocity was essentially zero at the walls. It is convenient to work in a coordinate system in which the final vortex street is at rest, which implies a non-zero average flow velocity in the x-direction. Since the absolute square of each Fourier coefficient except $\vec{k} = 0$ is invariant to a Galilean transformation anyway, the mean square vorticity distribution can easily be transformed from one coordinate system to another. The $\vec{k} = 0$ component of the fluid velocity is determined strictly by momentum conservation, and is not involved in a discussion of the interactions of the $\vec{k} \neq 0$ modes.

The most general representation of the stream function under the above periodicities is (again suppressing the time arguments and omitting the part associated with the uniform translation):

$$\phi(x,y) = \sum_{n_x, n_y} [A_x(n_x,n_y) \sin k_x x + B_x(n_x,n_y) \cos k_x x] [A_y(n_x,n_y) \sin k_y y + B_y(n_x,n_y) \cos k_y y] \tag{3}$$

where $k_x = 2\pi\, n_x/L_x$, $n_x = 0, 1, 2, 3, \ldots$, and $k_y = 2\pi\, n_y/2\, L_y$, $n_y = 0, 1, 2, 3, \ldots$. However, the geometrical requirements of the situation demand that certain of the A and B coefficients vanish. First, $(n_x,n_y) = (0,0)$ does not become involved in the dynamics and may be omitted.

More importantly, the condition that $\hat{e}_y \cdot (\hat{e}_z \times \nabla\phi) = 0$ at $y = \pm L_y/2$ demands that for $n_x \neq 0$, $A_y(n_x,n_y) = 0$ for n_y odd or zero, and that for $n_x \neq 0$, $B_y(n_x,n_y) = 0$ for n_y even or zero. For $n_x = 0$, the presence of the walls imposes no additional condition, but the physical requirement that the average x velocity at the upper and lower boundaries of the box be equal requires that $B_y(0,1) = 0$.

For the present situation, the prediction of the mean modal energy given by Eq. (1) is still correct in the frame with zero total momentum, as long as one keeps in mind that the space of eigenfunctions has been somewhat pruned. For large values of E/Ω (and since in the Zabusky-Deem example, this ratio is just Δ^2 as long as $\Delta \ll L_y$, this can be as large as we like), we are in the regime $\alpha<0$, $\beta>0$, and $\alpha + \beta\, k^2_{min}$ very small and positive, so that only the lowest k-modes dominate. Since $\vec{k} \neq 0$ for the modes described by Eq. (1), the expectation values are the same in any frame, including the one in which the contours of constant vorticity are expected not to translate. The problem reduces itself o that of enumerating the allowed modes with non-vanishing Fourier coefficients and the lowest values of $k_x^2 + k_y^2$. Since the $(n_x,n_y) = (1,0)$ is ruled out, the first two modes clearly are $(n_x,n_y) = (0,1)$ and $(1,1)$. Since $B_y(0,1) = 0$, only the $\sin \pi y/L_y$ term is permitted. To the extent that $4\beta k_y^2 \gg \alpha + \beta k_y^2 > 0$, values of $n_y > 1$ need not be considered. The energy associated with the (1,1) mode is comparable to that of (0,1) to the extent that βk_x^2 is of the order of $\alpha + \beta\, k_y^2$ (this implies $k_y^2 \gg k_x^2$, or $L_x \gg L_y$). The energy of the (1,2) mode is not totally negligible compared to that of the (0,1) mode under these circumstances (a factor of about 1/6 follows from assuming $\alpha + \beta\, k_y^2 \approx \beta\, k_x^2$), but for purposes of determining the qualitative behavior, significant insight can be gained by estimating the flow field solely on the basis of the contributions from the (0,1) and (1,1) modes plus the uniform translation associated with the (0,0) mode. Zabusky and Deem found in fact that the significant fraction of the energy was contained in their first three modes.[16]

Irrelevant phase factors, not provided by the canonical distribution, are associated with each realization of the ensemble and must be assumed in order to draw typical flow fields. For illustrative purposes, there is no reason to assume they are anything but zero. For a first approximation, then, we may assume for ϕ the approximate form (omitting the uniform translation):

$$\phi(x,y) \cong a \sin(\pi y/L_y) + b \cos(\pi y/L_y) \cos(2\pi x/L_x) , \tag{4}$$

where b/a is a number comparable to, but somewhat less than, unity.

Figure 1 is a schematic representation of the way the maxima and minima of the simple approximate form (4) for ϕ combine and interfere to produce the vortex street configuration with periodicity length L_x in the x-direction. Addition of higher terms of the series will modify the details of the contours of constant ϕ, but cannot change the gross configuration. It is therefore suggested that the two-temperature canonical distribution explains in an exceedingly simple way the basic structure of the Kármán vortex street.

The intention is to carry out future numerical inviscid solutions to the 2D Navier-Stokes equations in the present geometry, watch the vortex street develop, and attempt a detailed comparison of the time-averaged modal energies with the spectral predictions of Eq. (1).

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FOOTNOTES

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FIGURE CAPTION

Fig. 1 Schematic arrangement of the maxima and minima of the stream function (the uniform translation omitted) and/or the vorticity. The upper panel is for the first term in Eq. (4), the middle panel for the second, and the bottom panel for the sum. Without loss of generality we may assume $a>b>0$. The values chosen, for illustrative purposes only, are $a = 1.0$, $b = 0.8$.