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RELATIVISTIC EFFECTS OF THE ROTATION OF THE EARTH ON REMOTE CLOCK SYNCHRONIZATION

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Abstract

A treatment is given of relativistic clock synchronization effects due to the rotation of the Earth. Unlike other approaches, the point of view of an Earth fixed coordinate system is used which offers insight to many problems. An attempt is made to give the reader an intuitive grasp of the subject as well as to provide formulae for his use. Specific applications to global timekeeping, navigation, VLBI, relativistic clock experiments, and satellite clock synchronization are discussed. The question of whether atomic clocks are ideal clocks is also treated.

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INTRODUCTION

The precision of global timekeeping is approaching the level where one should consider the relativistic effects of the rotation of the Earth on remote clock synchronization. This paper will treat such effects, both mathematically and heuristically, to provide the precise time user with both rigorously derived formulae and, hopefully, an intuitive grasp of the causes underlying these formulae. This paper will also attempt to cover the subject as completely as possible in order to provide a unified reference which will allay the user's qualms about the relevance of some effects as well as allow him to correct for others.

HOW IDEAL ARE ATOMIC CLOCKS?

In analyzing the effects of the Earth's rotation on clock synchronization, we will assume all clocks are ideal clocks. We should, therefore, consider first whether the most accurate clocks available to the precise time user, atomic clocks, deviate significantly from ideal behavior.

Velocity Effects

In atomic clocks, a moving atom is interrogated by in phase electromagnetic fields at two or more points.[6] For various reasons, depending on the device, the first order doppler shift due to atomic motion in the device is cancelled out; only the second order doppler shift due to atomic motion effects the frequency of the clock.[6] This shift is given by:[2]

$$f_c = \sqrt{1 - \frac{v_a^2}{c^2}}\, f_a \tag{1}$$

where f_a is the atomic transition frequency in the atom's rest frame, f_c is the clock's frequency in its rest frame, and v_a is the relative velocity of the atom with respect to the clock. If (1) is true when the clock is in motion as well as at rest, one can ignore the effect of the moving atom, and treat the clock as ideal. (1) has been shown to be invariant with respect to motion of the clock,[2] and so in this respect an atomic clock is indeed an ideal clock. For completeness, the derivation showing (1) is invariant is reproduced in Appendix I.

Acceleration Effects

Only acceleration effects common to all atomic clocks will be considered here. For acceleration effects due to individual clock designs, the reader is referred to the manufacturers' literature.[1] Since the Earth rotates with angular frequency ω, an atomic clock fixed on the Earth will observe its atomic transition frequency from this same rotating frame. In this non-inertial, rotating frame, the atomic transition frequency may appear altered. This would cause the clock to deviate from ideal behavior. The question of ideal behavior, therefore, reduces to the question of whether rotational acceleration will cause the atomic transition frequency to alter.

To determine the rotational effects on an atomic transition frequency, we must determine the effects of rotation on the energy levels of an atomic system. This is accomplished by examining the Hamiltonian of an atomic system in a rotating frame. In a rotating frame, a system's Hamiltonian is given by:[4]

$$H = H_0 - \vec{\omega} \cdot \vec{F}$$

where H_0 is the non-rotating Hamiltonian, $\vec{\omega}$ is the angular frequency vector, and $\vec{F}$ is the total angular momentum. Atomic clocks operate in low magnetic fields where $\vec{F}$ is a good quantum number.[5] Therefore, if $\vec{\omega}$ is parallel to the magnetic field, the energy levels of the atomic system are shifted by:

$$\Delta E_{FM} = -\hbar\omega M$$

where the quantum states are given by the quantum numbers F and M. One can also show that, if $\vec{\omega}$ is perpendicular to $\vec{F}$ (see Appendix II):

$$\Delta E_{FM} = \frac{\hbar\omega^2 M}{2\omega_z}$$

where ω_z is the angular Zeeman frequency. Since atomic frequency standards run on transitions in which M for the initial and final states is zero,[6] there will be no frequency shift in these transitions caused by the rotation of the clock, and thus again an atomic clock behaves like an ideal clock.

EFFECTS OF UNIFORM MOTION ON REMOTE CLOCK SYNCHRONIZATION

Einstein Light Signal Synchronization

Before discussing the more complicated effects of the rotation of the Earth on clock synchronization, it is instructive to consider the effects of uniform motion. The basis for our discussion will be the Lorentz transformations:[2,9]

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (2)$$

$$Y' = Y \qquad Z' = Z$$

where the primed system is moving with velocity v parallel to the x coordinate. These transformations are derived from the principal of the constancy of the velocity of light, and from a definition of clock synchronization (Einstein synchronization) based on a light source emitting pulses an equal distance from the two remote clocks[2] (see Figure 1). From (2), one can see that, to an observer moving with velocity v in the x direction, two remote clocks synchronized by Einstein synchronization will be out of sync by:

$$\Delta t = -\frac{vx}{c^2\sqrt{1 - \frac{v^2}{c^2}}} \qquad (3)$$

where x is the x component of the separation between the clocks.

The significance of (3) is that it is path independent. If one were to set up a "global" network in a flat, special relativistic space with Einstein synchronization, even though, to a moving observer, this network will appear out of sync, it will be out of sync in a self-consistent manner independent of the paths used

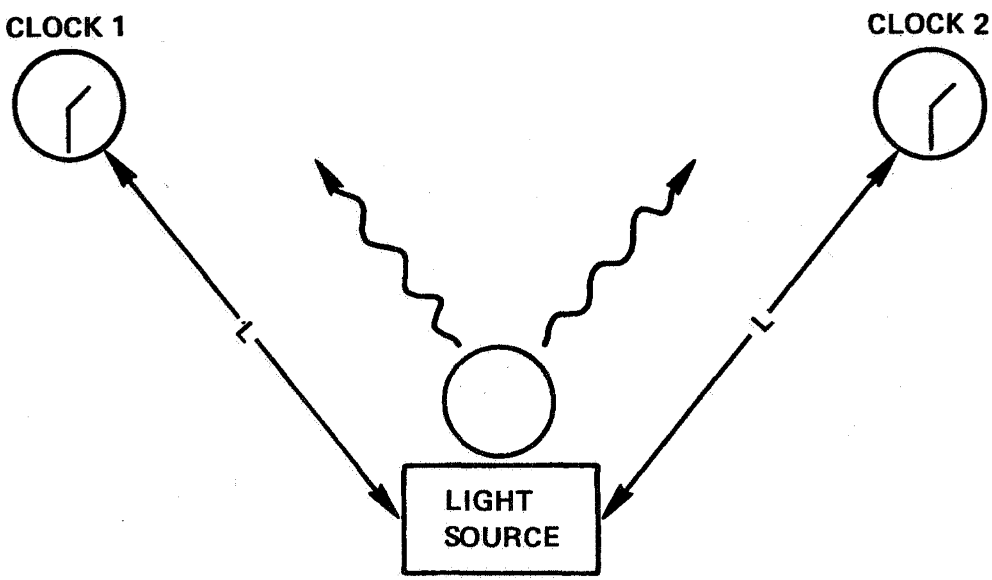

Figure 1. EINSTEIN SYNCHRONIZATION: IF CLOCKS 1 AND 2 RECEIVE LIGHT PULSES AT THE SAME TIME THEY ARE SYNCHRONIZED

for synchronization allowing one to ignore the effect; all the moving observer need do to restore synchronization is to apply (3).

Synchronization by Slowly Moving Clocks

Now let us consider remote clock synchronization by another, often used, method, by slowly moving a clock between the two remote clocks (see Figure 2). In this method, the moving clock, C_M, is synchronized to clock C_1, slowly moved with velocity ϵ along some arbitrary path, P, to clock C_2, and then used to synchronize C_2. Of course while C_M is moving, it will be doppler shifted as given by (1). To first order, its frequency compared with the frequency of C_1 will be given by:

$$\frac{f_M}{f_1} \simeq 1 - \frac{\epsilon^2}{2c^2}$$

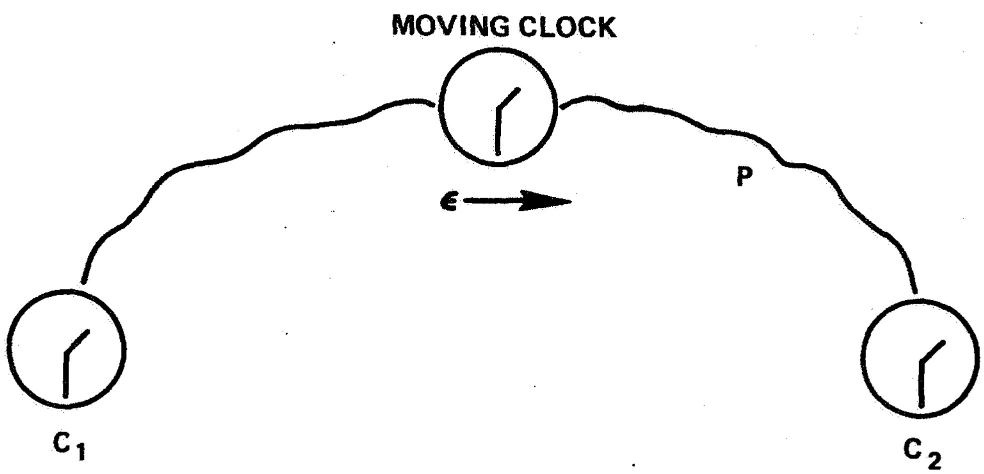

Figure 2. SYNCHRONIZATION BY SLOWLY MOVING CLOCK

which will cause C_M, when it reaches C_2, to differ from C_1 by:

$$\Delta t = -\int_P \frac{\epsilon^2}{2c^2}\, dt_1$$

Since the line element along P is given by:

$$dl = \frac{dt_1}{\epsilon},$$

this becomes:

$$\Delta t = -\int_P \frac{\epsilon}{2C^2}\, dl.$$

which goes to zero as ϵ goes to zero. Thus, in the rest frame of C_1 and C_2, the clocks C_1 and C_2 will be synchronized.

Now let us consider this slow clock synchronization from the point of view of a moving observer (see Figure 3). Let C_2 and C_2 be moving with velocity $\vec{v}$ with respect to this observer whose clock, C_0, reads time t. For simplicity let C_1, C_M, and C_0 be synchronized to zero when C_M leaves C_1.

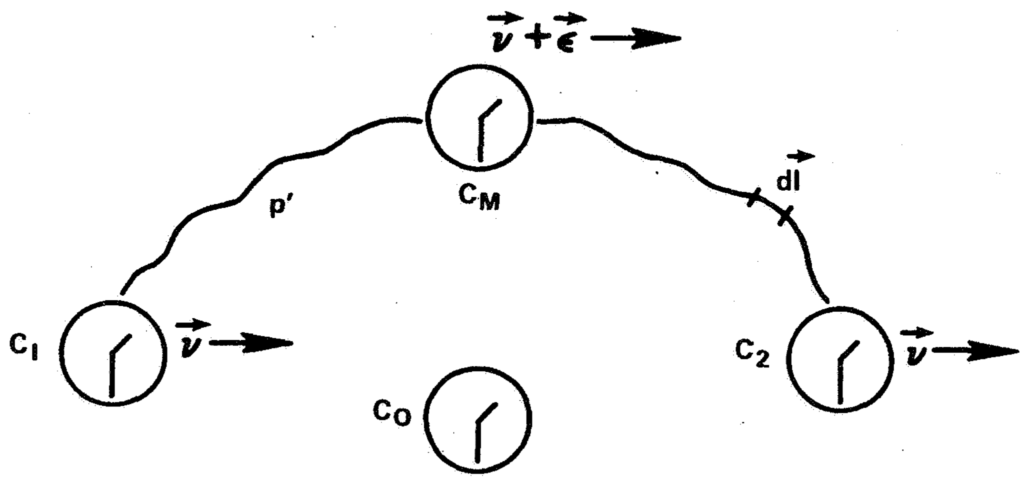

Figure 3. SYNCHRONIZATION IN MOVING FRAME

The frequency of C_M is given in terms of C_0's frequency by (1) (v^2 replaced by $(\vec{v}+\vec{\epsilon})^2$) which to lowest order in ϵ will yield:

$$\frac{f_m}{f_0} = \frac{1 - \vec{\epsilon}\cdot\vec{v}/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This means that now Δt is given by:

$$\Delta t = - \int_{P'} \frac{\vec{\epsilon} \cdot \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} dt$$

But along P', to lowest order in ϵ:

$$d\vec{l} = \vec{\epsilon}\, dt,$$

so to lowest order in ϵ:

$$\Delta t = - \int_{P'} \frac{\vec{v} \cdot d\vec{l}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Again letting ϵ go to zero, we obtain:

$$\Delta t = - \frac{1}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} \int_{P'} \vec{v} \cdot d\vec{l}$$

Since $\vec{v}$ is independent of position, if v is in the x direction, we again obtain (3):

$$\Delta t = - \frac{vx}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} . \qquad (3)$$

so synchronization by slowly moving clocks is equivalent to Einstein light synchronizations in flat, special relativistic space.

CLOCK SYNCHRONIZATION ON THE ROTATING EARTH

For the proper treatment of clock synchronization on the rotating Earth, both the presence of gravitational fields and the non-uniformity of the motion necessitate the use of general relativity. Both the behavior of clocks and the propagation of light signals, in general relativity, is completely defined by the proper time line element metrically decomposed in terms of a suitable coordinate system.[9,10,11] Thus to describe clock synchronization on the rotating Earth, all we need derive is the proper time line element given in terms of Earth fixed coordinates.

For our starting point in deriving the proper time line element, we shall use the Schwarzchild line element for a point gravitational source in non-rotating spherical coordinates (r, θ, ϕ'):

$$d\tau^2 = \left(1 + \frac{2U}{c^2}\right) dt^2 - \frac{1}{c^2} r^2 d\theta^2 - \frac{1}{c^2} r^2 \sin^2 \theta d\phi'^2 - (c^2 + 2U)^{-1} dr^2 \tag{4}$$

where:

$$U = -\frac{GM_{Earth}}{r}$$

This, to accuracy sufficient for our purposes, will properly describe the effects of the Earth's gravitation. To go to Earth fixed coordinates, we use the transformation:

$$\phi = \phi' - \omega t$$

to obtain the desired form of the proper time line element:

$$d\tau^2 = \left(1 + \frac{2U_T}{c^2}\right) dt^2 - \frac{1}{c^2} r^2 d\theta^2 - \frac{r^2 \sin^2 \theta}{c^2} (d\phi^2 + 2\omega d\phi dt) - (c^2 + 2U)^{-1} dr^2 \tag{5}$$

where, U_T, the total gravitational potential in the rotating frame, is:

$$U_T = U - \frac{1}{2} r^2 \omega^2 \sin^2 \theta$$

(Note that U_T contains the centrifugal potential).

Synchronization by Slowly Moving Clocks

Consider, now, the consequences of (5) for synchronization by slowly moving clocks. For a clock moving along a differential path (dt, dr, $d\theta$, $d\phi$), in the limit where

$$\left(\frac{dr}{dt}, \frac{d\theta}{dt}, \frac{d\phi}{dt}\right)$$

goes to zero, to lowest order, (5) yields:

$$d\tau = \left(1 + \frac{u_T}{c^2}\right) dt - \frac{1}{c^2} r^2\omega \sin^2 \theta d\phi \tag{6}$$

For a finite path, P, this becomes:

$$\Delta\tau = \int_P \frac{U_T}{c^2} dt - \frac{\omega}{c^2} \int_P r^2 \sin^2 \theta d\phi \tag{7}$$

where $\Delta\tau$ is the difference between the slowly moving clock and a coordinate clock.

Equation (7) has two terms both of which are path dependent. The first term is the usual gravitational red shift term which has been described elsewhere[12,13] except that, in this case, the centrifugal potential is included as part of the gravitational potential. The second term is analogous to (3) in the uniform motion case except that, now, the time difference accrued by the slowly moving clock is path dependent. To see this more clearly, consider the following heuristic derivation.

Let us set up a non-rotating system of clocks to view our Earth clocks as shown in figure 4. These non-rotating clocks are all placed at the same

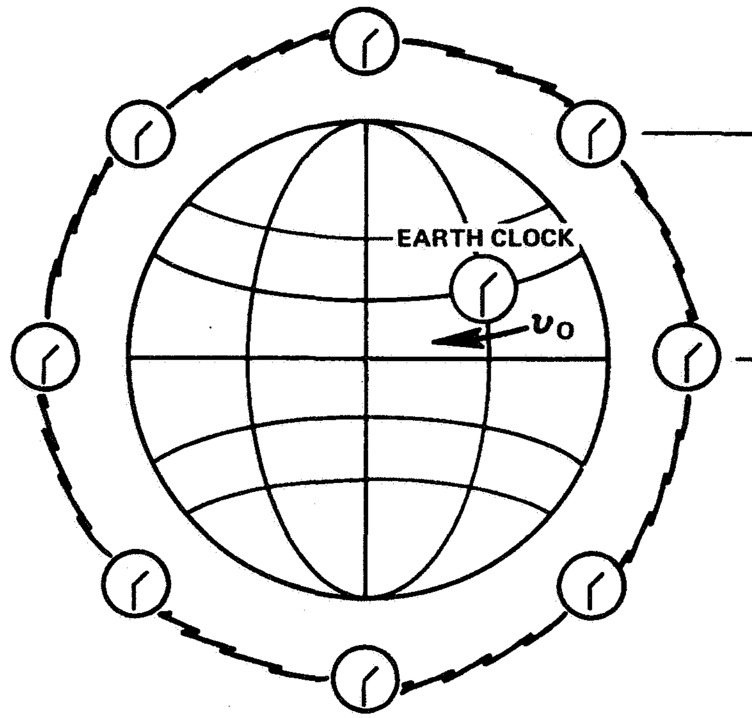

Figure 4. SYSTEM TO MEASURE ROTATIONAL EFFECT

gravitational potential, so they all run at the same rate, and can be synchronized to a clock at the north pole. If our clock system is placed at the same gravitational potential as a slowly moving clock on Earth, one of our system clocks can locally view the slowly moving clock, so special relativity can be used.[11] The local system clock sees the Earth clock doppler shifted by:

$$\frac{\Delta f}{f} \simeq -\frac{v_0^2}{2c^2} = -\frac{r^2\omega^2 \sin^2\theta}{2c^2},$$

and also sees the synchronization error given by (3) for a slowly moving clock:

$$d\tau_M \simeq -\frac{\vec{v}_0 \cdot d\vec{l}}{c^2} = -\frac{r^2\omega \sin^2\theta}{c^2}\, d\phi$$

Using this and the fact that our system of clocks will be red shifted from a coordinate clock by:

$$\frac{\Delta f}{f} = \frac{U}{c^2},$$

we obtain for the slowly moving clock on Earth:

$$d\tau_M \simeq \left(1 + \frac{U - \frac{1}{2} r^2\omega^2 \sin^2\theta}{c^2}\right) dt$$

$$- \frac{r^2 \omega \sin^2 \theta d\phi}{c^2}$$

where dt is the coordinate time interval. This equation is precisely that given by (6).

Light Synchronization

For the general relativistic case, the dependence of proper lengths on the coordinates complicates the definition of Einstein light synchronization. To simplify analysis, therefore, let us redefine light synchronization as shown in Figure 5. In this new definition, clock C_1 sends a light pulse to C_2, and records the time he sends it. When C_2 receives the light pulse, he immediately returns another pulse over the same path, P, and records the time of arrival of the first pulse. C_1 now receives the second pulse, and measures the time difference between the transmission of the first pulse and the reception of the second pulse, Δt. From this time, C_1 determines the propagation time, $\Delta t/2$, which C_2 can transmit to C_2 along with the time C_1 sent the first pulse. This enables C_2 to synchronize to C_1.

In order to analyze the relativistic effects of rotation on this form of synchronization, all we need use is the line element (5) and the fact that for light propagation (in vacuo):

$$d\tau = 0$$

These together with a path, P, parametrically described by $(r(\lambda), \theta(\lambda), \phi(\lambda))$ defines light propagation in terms of the quadratic equation:

$$Adt^2 + B\left(\frac{d\phi}{d\lambda}\right) dt d\lambda + C\left(\frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda}\right) d\lambda^2 = 0$$

Solving for dt, one obtains:

$$dt = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} d\lambda$$

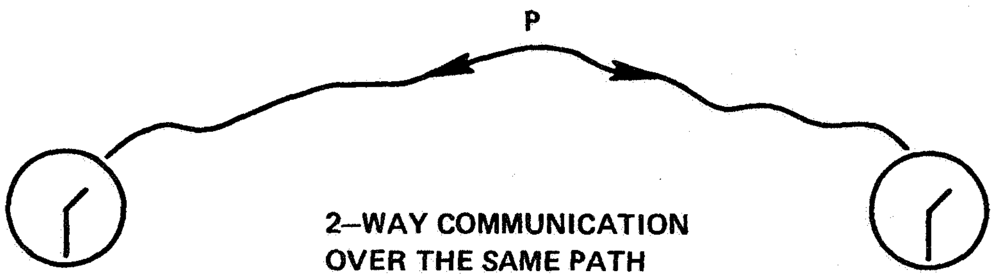

Figure 5. NEW LIGHT SYNCHRONIZATION DEFINITION

which for a finite path becomes:

$$t = \int_{\lambda_1}^{\lambda_2} \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \, d\lambda \tag{8}$$

Notice that there is an ambiguity of sign in (8). This occurs because, along P, light can propagate in two directions, from $P(\lambda_1)$ to $P(\lambda_2)$, and vice versa. The presence of the B term causes the propagation time for light traveling in opposite directions to differ by:

$$\Delta t_p = 2 \int_{\lambda_1}^{\lambda_2} \frac{-B}{2A} \, d\lambda$$

or

$$\Delta t_p = -2\omega \int_p \frac{r^2 \sin^2 \theta}{c^2 + 2U_T} \, d\phi$$

But from our light synchronization definition, this will introduce a synchronization error of:

$$\Delta t = \frac{\Delta t_p}{2}$$

or:

$$\Delta t = -\omega \int_p \frac{r^2 \sin^2 \theta}{c^2 + 2U_T} d\phi \tag{9}$$

where Δt is defined in terms of a coordinate clock. To lowest order, (9) reduces to:

$$\Delta t = -\frac{\omega}{c^2} \int_p r^2 \sin^2 \theta d\phi \tag{10}$$

which is the same as the second term in (7).

In order to obtain some insight into the reasons for the time difference given by (10), consider the following heuristic derivation from the point of view of a non-rotating frame as shown in Figure 6. In Figure 6, two clocks, C_1 and C_2, rotating with the Earth and separated by a small distance L, light synchronize along a straight line. In the time, t, it takes for the light to travel from C_1 to C_2, C_2 will move:

$$\Delta\vec{L} = \vec{v}_0 t = \hat{\phi} \frac{\omega r L}{c} \sin \theta$$

where $L = ct$ has been used. This will introduce a change in path given approximately by:

$$\Delta L \simeq \frac{\vec{L} \cdot \Delta\vec{L}}{L}$$

or:

$$\Delta L = -\frac{\omega r^2}{c} \sin^2 \theta \Delta\phi$$

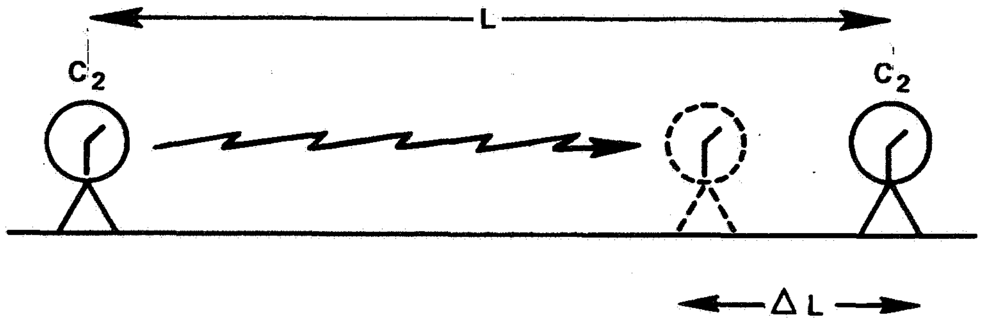

Figure 6. ROTATIONAL ERROR FOR LIGHT SYNCHRONIZATION

which corresponds to a change propagation time:

$$\Delta t = -\frac{\omega r^2}{c^2} \sin^2 \theta \Delta\phi \qquad (11)$$

For a light signal going from C_2 to C_1, we would, similarly, obtain a change in propagation time as described by (11), but with the opposite sign. As viewed by a coordinate clock, therefore, C_1 and C_2 would be out of synchronization as given by (11). For a general light path, P, we can break the path down into N infinitesimal straight line segments, and repeatedly use (11) to obtain:

$$\Delta t = -\frac{\omega}{c^2} \int_P r^2 \sin^2 \theta d\phi$$

which is the same as (10).

Rotational Frequency Effects

Equation 9 shows that a clock on Earth has its frequency shifted from a coordinate clock by:

$$\frac{\Delta f}{f} = \frac{U_T}{c^2}$$

where:

$$U_T = U - \frac{1}{2} r^2\omega^2 \sin^2 \theta$$

This is different from other formulations[15, 12] in the inclusion of the centrifugal potential as part of the gravitational frequency shift term. This centrifugal effect has even been ignored entirely by some authors.[12] Inclusion of the centrifugal term in one form or another is important to obtain the proper operating frequency since the centrifugal term contributes a fractional frequency difference of 1.2×10^{-12} between a clock at a pole and one at the equator.

Because of this centrifugal term, however, on the surface of the Earth, one can ignore variations in clocks caused by the gravitational shift. If the Earth's surface was a rigidly rotating fluid, the surface of the Earth would be defined by:

$$U_T = \text{constant}$$

since a static fluid cannot maintain shear stresses. But sea level by definition is the surface of a static fluid; therefore, all clocks at sea level run at the same frequency. This means that, so far as the gravitational red shift is concerned, to obtain a consistent system of clocks, all one need do is to correct the frequency of a clock for deviations from sea level (at the fractional rate of 1.09×10^{-13} per km near sea level).

The fact that clocks at sea level all run at the same rate, however, is of little comfort if the user has a rigidly mounted clock, and sea level changes as a function of time. Tidal forces due to the Sun and the Moon cause such a time dependent change of sea level. As shown in Appendix III, these forces lead to frequency shifts given by:

$$\left.\frac{\Delta f}{f}\right|_{\text{Sun}} = -2.69 \times 10^{-17} \cos^2(\omega t)$$

$$\left.\frac{\Delta f}{f}\right|_{Moon} = - 5.85 \times 10^{-17} \cos^2(\omega t)$$

APPLICATIONS

Global Timing Networks

We have shown that for remote synchronization on the Earth, there is a path dependent synchronization error given by (7) for slow clock synchronization and (9) for light synchronization. This path dependent effect can cause discrepancies of as much as 0.2 μsec for differing synchronization paths. (The maximum effect occurs in synchronizing two clocks on opposite sides of the Earth on the Equator along paths going in opposite directions around the equator.) To set up a self-consistent timing network, one must either set up the synchronization network as shown in figure 7 or make corrections for (7) or (9) in all remote synchronization. For future reference, let us call this form of synchronization coordinate synchronization, and call synchronization in which (7) or (9) are not corrected for link synchronization.

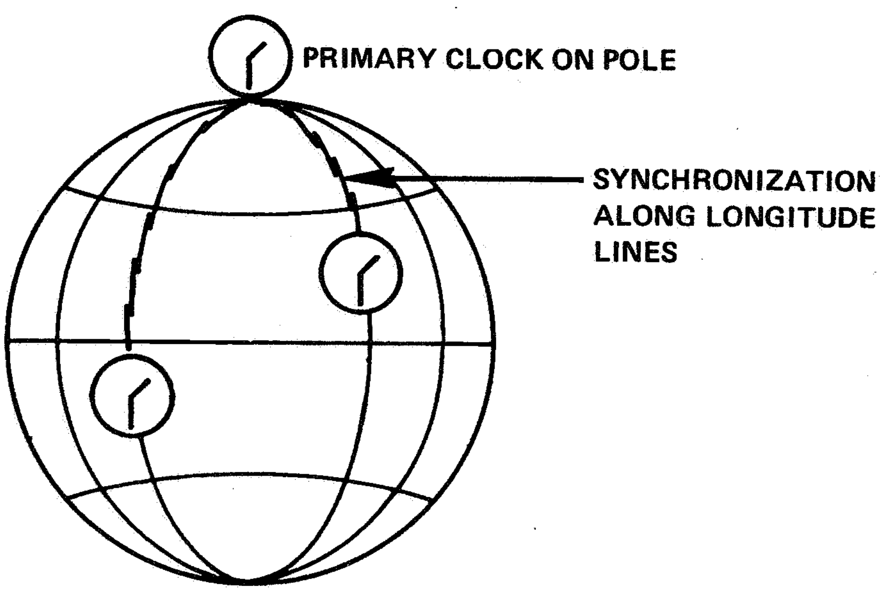

Figure 7. COORDINATE SYNCHRONIZATION NETWORK

Since we have also shown that clocks at sea level all run at the same rate, to obtain coordinate time from a sea-level, coordinate synchronized global timing network, all we need do is offset the network's frequency by:

$$\frac{\Delta f}{f} = -\frac{U_P}{c^2}$$

where U_P is the Newtonian gravitational potential at the pole.

Relativistic Timing Experiments

There are two basic groups of relativistic timing experiments involving moving clocks, those that involve only two clocks, and those that involve monitoring the moving clock with remotely synchronized clocks. The first type of experiment is detailed in Figure 8. In this type of experiment, a moving clock, C_M, is compared with a clock on the Earth, C_0, before and after C_M makes a trip around a closed path, P. One can see from (7) that, even for a slowly moving clock always in the same gravitational potential, there will be different results depending on the path.

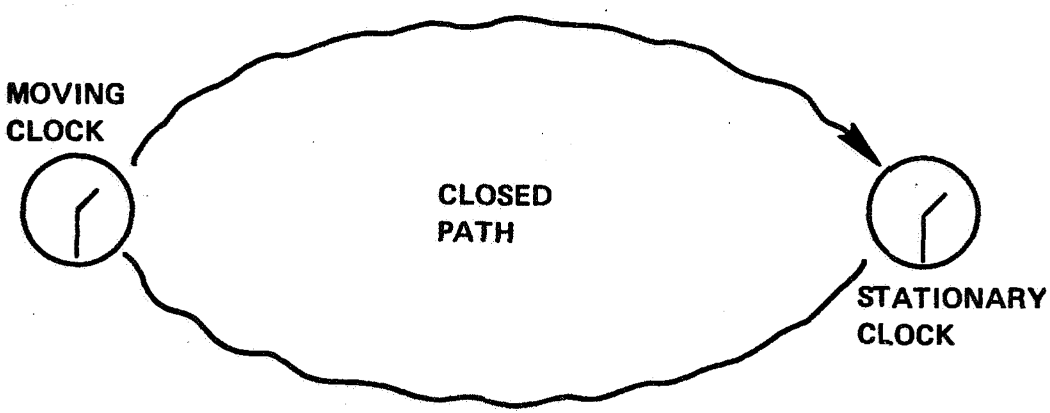

Figure 8. TWO CLOCK EXPERIMENT

The Hafele-Keating experiment[15] is a classic example of this type of experiment. In this experiment, sets of clocks which traveled around the world equatorially in opposite directions were compared with the U.S.N.O. Master Clock before and after each trip. Different results were obtained for the westerly versus easterly moving clocks. Qualitatively, this can be seen as a result of the part of (7) given by:

$$\Delta t = -\frac{\omega}{c^2} \int_P r^2 \sin^2 \theta d\phi$$

which for the same P will give Δt's of opposite sign depending on the sign of $d\phi$.

The second type of experiment involves remotely synchronized clocks as shown in Figure 9. In this type of experiment, a clock, C_M, is moved along a path, P, between two remotely synchronized clocks, C_1 and C_2, and compared with them. Here the results depend on which type of synchronization is used. For example, if we let the synchronization path, C_M's path, C_1, and C_2 all be along the same latitude, we let v_M, C_M's ground velocity, be constant, and we keep C_M at ground level, for coordinate synchronization, we obtain:

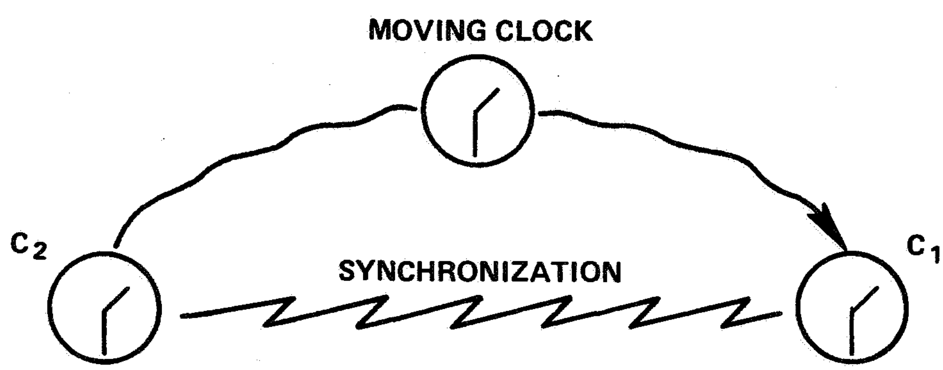

Figure 9. EXPERIMENT INVOLVING REMOTE SYNCHRONIZATION

$$\frac{\tau_M - \tau_2}{\tau_1} = -\frac{1}{2c^2} [v_e^2 + v_M^2 + 2\vec{v}_e \cdot \vec{v}_M] \tag{13}$$

where $\vec{v}_e$ is the rotational velocity of the Earth at the latitude chosen. Notice this result depends on the direction of $\vec{v}_M$ with respect to $\vec{v}_e$ just as the Hafele-Keating experiment does. If we use link synchronization for the same experiment, we obtain:

$$\frac{\tau_M - \tau_2}{\tau_1} = -\frac{1}{2c^2} [v_e^2 + v_M^2] \tag{14}$$

which is independent of the direction of $\vec{v}_M$ and $\vec{v}_e$. This is because the synchronization error between C_1 and C_2:

$$\Delta t \simeq \frac{\vec{v}_e \cdot \vec{x}}{c^2} = \frac{\vec{v}_e \cdot \vec{v}_M}{c^2} \tau_1, \tag{15}$$

just cancels the cross terms in (13).

For the two examples just given, the link synchronization case is the only one which has a special relativistic analogue; there are no cross terms just as would be true in special relativity. However if one tries to extend this analogue to the Hafele-Keating experiment, one gets into trouble. Treating the Hafele-Keating experiment special relativistically, leads one to the absurd consequence that the stationary clock is out of synchronization with itself. This occurs because a special relativistic treatment implies a flat space in which it would be impossible to return to the same clock by continuously moving in the same direction; the Hafele-Keating experiment, as well as all experiments of the first type, have no special relativistic analogue!

Global Radio Navigation

Radio navigation systems such as Loran-C and Omega utilize precise timing to determine the user's position.[14,16] By measuring the propagation delay for timing signals broadcast from fixed system transmitters with a portable clock, the user can determine his distance to the system transmitters, and thus determine his position. Rotational synchronization errors can introduce errors in the navigation system through two sources. First, if the fixed transmitters use link synchronization, timing errors can be introduced. These errors can be removed by using coordinate synchronization. Second, since the user is generally moving around, his clock will develop a cumulative synchronization error given by (7). This error can only be removed by a continuous path dependent correction of the users clock. This error, however, would be typically less than 0.1μ s if the user coordinately resynchronized his clock every time he traveled half way around the Earth, and so would lead to a navigation error of less than 100 ft.

For ultra precise navigation, the user could use the following method which does not rely on a precise onboard clock. In this method, the user monitors three fixed stations simultaneously. From the measured propagation delays he could then solve for three unknowns, his coordinates θ , and ϕ, and his clock error, Δ t.

Very Long Baseline Interferometry

Very long baseline interferometry has been suggested for both purposes of remote time synchronization,[14] and navigation.[17] The basic technique is outlined in Figure 10. Two remote stations on Earth, A and B, monitor a stellar radio source, record the results with timing marks from their local clocks, and later cross-correlate their results to determine Δ t. For the purposes of remote time synchronization, θ and s are known, and Δ t is used to synchronize clocks A and B. Since a non-rotating radio source is used, clocks A and B will be coordinately synchronized. For navigation, clocks A and B are synchronized, and Δt is used, together with a knowledge of θ, to determine s. If A and B are link synchronized, there will be a timing error $\Delta t'$ given by (7) or (12) which will produce an error in s:

$$\delta_s = \frac{c\Delta t'}{\sin\theta}$$

With Δ t typically less than $0.1\ \mu$s, typically:

$$\delta s < \frac{30\text{m}}{\sin\theta}$$

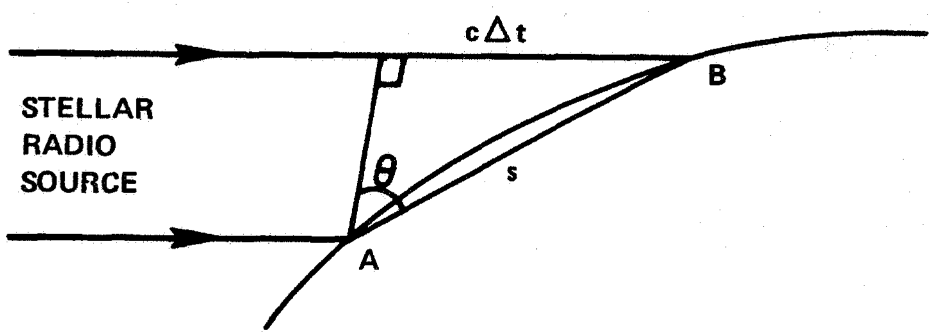

Figure 10. V L B I

Satellite Clock Synchronization

The most accurate form of remote synchronization by satellites is in the two way timing mode[14] as shown in Figure 11. In this mode radio signals are bounced or transponded off the satellite in both directions. Time synchronization is determined by the light synchronization method outlined in the previous section, so there is a synchronization error with this method given by (9). If uncorrected, this would lead to synchronization errors typically on the order of 0.1 μs or less.

For satellites carrying an onboard clock, (7) seems to indicate that there would be a frequency shift given by:

$$\frac{\Delta f}{f} = -\frac{\omega}{c^2} r^2 \sin^2\theta \frac{d\phi}{dt} \qquad (12)$$

This is not true because the satellite's finite velocity introduces other terms from (5) which cancel (12). To see this, consider the satellite from a non-rotating frame as viewed by a clock at the north pole where there are no

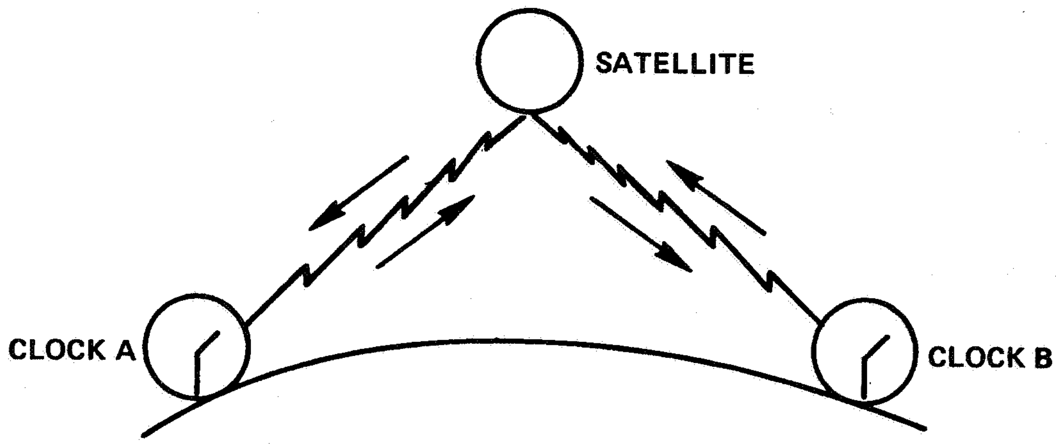

Figure II. SATELLITE TWO-WAY TIMING

rotational effects. The frequency difference between the pole clock and the satellite clock is given by:

$$\frac{\Delta f}{f} = \frac{U_s - U_P}{c^2} - \frac{v_T^2}{2c^2}$$

where U_s and U_P are the gravitational potentials at the satellite and at the pole respectively, and $\vec{v}_T$ is the satellite's velocity relative to the non-rotating frame. For a circular orbit, v_T is a constant, and so $\Delta f/f$ would just be a constant. Since for coordinate synchronization, all Earth clocks would be synchronized to the pole clock, $\Delta f/f$ would be a constant with respect to them also, precluding the possibility of any terms of the form of (12).

The difference between a moving satellite carrying a clock, and a moving ship or airplane carrying a clock is that the motion of the ship or airplane is simple (nearly uniform) with respect to the Earth, but the motion of the satellite is simple with respect to a non-rotating frame; as viewed from a non-rotating frame, the ship or airplane's motion is directly affected by the rotation of the Earth; whereas the satellite's motion, one of free fall, is not influenced by the rotation of the Earth. Formally both a rotating frame and a non-rotating frame are equally correct for analyzing relativistic problems; the choice between them is a subjective matter governed by simplifications or clarifications one frame or the other will bring to the solution of the particular problem of interest.

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APPENDIX I

An idealized picture of a moving atomic clock is shown in Figure I-1. In a moving observer's frame, the atom whose transition is being monitored is moving with velocity v_T. As the atom passes two clocks, C_1 and C_2, which are synchronized in their rest frame, and which are moving with velocity v_C, the atom's clock, C_a, is interrogated by C_1 and C_2. When the clock is at rest, C_1 and C_2 see C_a doppler shifted by (1). We must prove that this is also true to our moving observer.

Let C_1, C_a, and the observer's clock, C_0, all be synchronized to zero when C_a passes C_1. Using the Lorentz transformations given by (2), when C_a passes C_a, in terms of observers time, t_0, C_a and C_2 will read:

$$t_a = \frac{t_0 - v_T x_0}{\sqrt{1 - \frac{v_T^2}{c^2}}}$$

$$t_2 = \frac{t_0 - v_c x_0}{\sqrt{1 - \frac{v_C^2}{c^2}}} \qquad \text{(I-1)}$$

where x_0 is the position of C_2 when C_a passes C_2(C_1 at $x = 0$ when C_a passes C_1). Using the fact that:

$$x_0 = v_T t_0$$

and (I-1), we obtain:

$$\frac{t_a}{t_2} = \frac{(1 - v_c v_T)}{\sqrt{1 - \frac{v_c^2}{c^2}}} \quad \frac{\sqrt{1 - \frac{v_T^2}{c^2}}}{(1 - v_T^2)} \qquad \text{(I-2)}$$

But in terms of v_a and v_c, where v_a is the velocity of C_a with respect to C_1 and C_2, v_T is:[2]

$$v_T = \frac{v_a + v_c}{1 + \frac{v_a v_c}{c^2}}$$

Using this and (I-2), after some algebraic manipulation, we obtain:

$$\frac{t_a}{t_2} = \frac{f_c}{f_a} = \sqrt{1 - \frac{v_a^2}{c^2}}$$

which is the same as (1).

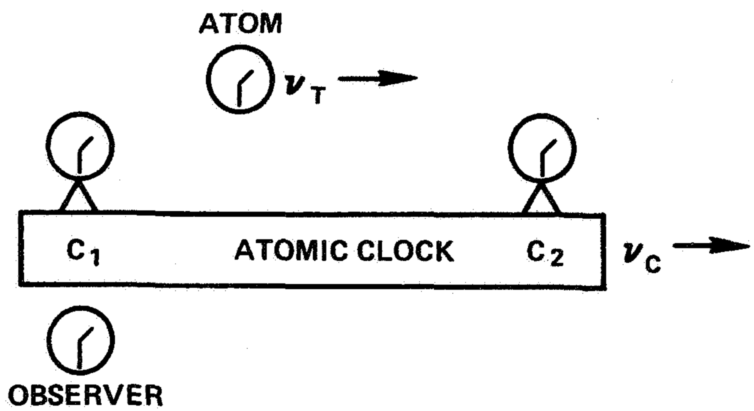

Figure I–I. IDEALIZED ATOMIC CLOCK

APPENDIX II

If $\vec{\omega}$ is perpendicular to $\vec{F}$, the Hamiltonian becomes:[7]

$$H = H_0 + \Delta H$$

where:

$$\Delta H = -\frac{\hbar\omega}{2}(F_+ + F_-)$$

and:

$$F_\pm = F_x \pm iFy$$

Since:[7]

$$\langle FM \,|(F_+ + F_-)|\, FM\pm 1\rangle = \sqrt{F(F+1) - M(M\pm 1)} \qquad \text{(II-1)}$$

and all other matrix elements are zero, there is no first order perturbation contribution to the energy levels.[8] The next highest perturbation contribution to the energy levels is the second order term:[8]

$$\Delta E_{FM} = \sum_{M' \neq M} \frac{\langle FM\,|\Delta H|\,FM'\rangle \langle FM'\,|\Delta H|\,FM\rangle}{E^{(0)}_{FM} - E^{(0)}_{FM'}}$$

using this and (I-1), one obtains:

$$\Delta E_{FM} = \frac{\hbar\omega^2}{2\omega z} M$$

where ω_z is the angular Zeeman frequency:

$$h\omega_z = E^{(0)}_{FM+1} - E^{(0)}_{FM}$$

APPENDIX III

To determine the effects of the tidal forces due to the Sun and the Moon on the frequency of a clock, consider the following derivation outlined in Figure III-1. At a point on the Earth, two sets of potentials caused by the Sun or Moon are at work, the gravitational potential:

$$-\frac{GM}{R}$$

where M, and R are the mass and the distance respectively to the Sun or the Moon, and the accelerational potential caused by the Earth's motion around the center of mass of the Earth-Sun or Earth-Moon system:

$$-\frac{1}{2}R'^2\omega_0^2$$

where R' is the distance to the center of mass, and ω_0 is the angular velocity of the Earth revolving around the Sun or Moon. The total potential from these effects, then, is:

$$U_T = -\frac{1}{2}R'^2\omega_0^2 - \frac{GM}{R}$$

Expanding in a power series about the center of the Earth (center of mass), and noting that, at the center of the Earth, the gravitational and accelerational forces must cancel, one obtains, to lowest order, a varying term given by:

$$\delta U_T = -\frac{3}{2}\frac{GM}{R^3}r^2\cos^2\omega t$$

where R_0 is the distance of the center of the Earth to the Sun or Moon, and r is the radius of the Earth. This yields a frequency shift:

$$\frac{\Delta f}{f} = -\frac{3GMr^2}{2c^2R_0^3}\cos^2\omega t$$

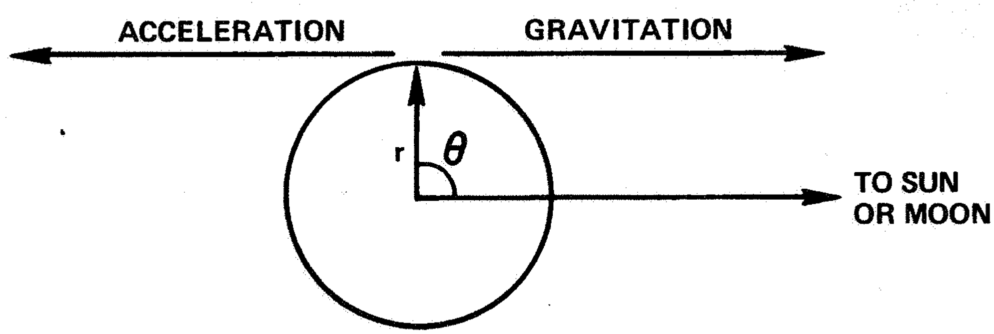

Figure III–I. EFFECT OF TIDAL FORCES

Substituting the relevant quantities for the Sun and the Moon, one obtains:

$$\left.\frac{\Delta f}{f}\right|_{sun} = - 2.69 \times 10^{-17} \cos^2(\omega t)$$

$$\left.\frac{\Delta f}{f}\right|_{moon} = - 5.85 \times 10^{-17} \cos^2(\omega t)$$