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A FINITE-ELEMENT ANALYSIS FOR STEADY AND OSCILLATORY SUPERSONIC FLOWS AROUND COMPLEX CONFIGURATIONS

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ABSTRACT

The problem of small perturbation potential supersonic flow around complex configurations is considered. This problem requires the solution of an integral equation relating the values of the potential on the surface of the body to the values of the normal derivative, which is known from the small perturbation boundary conditions. The surface of the body is divided into small (hyperboloidal quadrilateral) surface elements, Σ_i , which are described in terms of the Cartesian components of the four corner points. The values of the potential (and its normal derivative) within each element is assumed to be constant and equal to its value at the centroid of the element. This yields a set of linear algebraic equations. The coefficients of the equation are given by source and doublet integrals over the surface elements, Σ_i. The results obtained using the above formulation are compared with existing analytical and experimental results.

FOREWORD

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Nomenclature

$\bar{a}_i$	Eq. (2.3a)
a_∞	free-stream speed of sound
b	span
b_{hk}	Eq. (2.1)
C_p	pressure coefficient, $2(p-p_\infty)/\rho_\infty U_\infty^2$
C_ℓ	lift distribution coefficient, $C_{p\ell} - C_{pu}$
$C_{\ell\alpha}$	$(\partial C_\ell/\partial\alpha)_{\alpha=0}$
c	chord
c_{hk}	Eq. (2.23)
E	Eq. (1.22a)
H	Eq. (1.22b)
k	reduced frequency, $\omega\ell/U_\infty$
ℓ	reference length
M	Mach Number, U_∞/a_∞
$\bar{n}$	Eq. (20)
$\bar{N}$	outward normal to surface Σ
NX,NY	number of elements in X,Y directions
P	point of the surface Σ
P_*	control points, (X_*, Y_*, Z_*)
q	Eq. (2.3a)
r	Eq. (1.18)
t	time
T	$a_\infty\beta t/\ell$
x,y,z	space coordinates
X,Y,Z	nondimensional Prandtl-Glauert coordinates
U_∞	free-stream velocity
α	angle of attack

Nomenclature (Continued)

β	$\|1-M^2\|^{1/2}$
Σ	surface surrounding body and wake
Σ_k	surface element
Σ_A	surface of aircraft
Σ_W	surface of wake
τ	thickness ratio
Φ	velocity potential, $U_\infty x + \varphi$
φ	perturbation velocity potential
ϕ	$\varphi / U_\infty \ell$
$\hat{\phi}$	Eq. (1.26)
ϕ_k	values of ϕ, at centroid of element Σ_k
ω	frequency of oscillation
Ω	compressible reduced frequency, $\omega \ell / a_\infty \beta = kM/\beta$
AR	aspect ratio
TR	taper ratio
$\Delta\phi$	$\phi_u - \phi_\ell$
∇	gradient in X,Y,Z space
$\odot$	superproduct, Eq. (1.23)
$\|\tilde{a}\|$	supernorm of a vector, Eq. (1.23)

INTRODUCTION

The evaluation of the aerodynamic loads is important in structural design and flight control of the aircraft. These loads must be evaluated by an iterative process which requires that the mathematical modeling of the problem be general, flexible and efficient. The usual methods for the evaluation of aerodynamic loads is the computational lifting-surface theory. these methods are efficient and flexible but not general enough to consider problem for complex configurations. Some computational methods around complex configurations have already developed. However, they are usually quite cumbersome to use. Furthermore, for oscillatory flow around complex configurations, only techniques based on the doublet-lattice method exist in subsonic range while no method is available in the supersonic ones.

The development of the present method is aimed to overcome the shortages of the existing methods and provide an efficient, general and flexible aerodynamic tool to be used in structural design and flight control of the aircraft.

The present method is mainly based on the theoretical formulations developed by Morino (Refs. 1, 2, and 3). The geometry of the aircraft and wake is approximated by a number of quadrilateral elements described by hyperboloidal surfaces. By applying the Green function method, one obtains a linear equation relating the velocity potential φ , at any point, p_* , in the flow field with the values of φ and its normal derivatives on the surface Σ , surrounding the body and the wake.

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An integral equation is obtained by imposing that the value of the potential at p_* approaches the value of that at point p on the surface if p_* approach p. The value of the velocity potential, φ is assumed to be constant within each element. Then, the integral equation becomes a system of algebraic equations which relates unknowns, φ_k, at the centroids of elements, Σ_k with coefficients evaluated analytically. Once the distribution of the velocity potential is obtained, the pressure distribution as well as the generalized forces are evaluated.

In Section 1, the basic equations are introduced. In Section 2, the numerical formulation is presented. In Section 3 finite values of the integral are considered. Oscillatory flows are discussed in Section 4. In Section 5, numerical results are presented. In Appendices A, B, and C, useful equations are derived.

SECTION I

BASIC EQUATIONS

1.1 Introduction

In this section, basic flow equations are first introduced in Subsection 1.2, while boundary condition of the problem is considered in Subsection 1.3. The Green theorems for steady and oscillatory flow are considered in Subsection 1.4. The integration scheme and the hyperboloidal element are considered in Subsection 1.5. In Subsecion 1.6 the role of the diaphrapms is discussed. Finally in Subsection 1.7 the evaluation of the pressure coefficient and the generalized forces are considered.

1.2 Basic Flow Equation

The flow is assumed to be isentropic, inviscid and initially irrotational such that the flow can be described by the velocity potential Φ . Consider a frame of reference such that the undisturbed flow has a velocity U_∞ in the direction the the positive x axis, so that

$$\Phi = U_\infty x + \varphi \tag{1.1}$$

where φ is the perturbation velocity potential. Then, the linearized equation of potential flow based on the assumption of small perturbation is given by

$$\nabla^2 \varphi - \frac{1}{a_\infty^2} \frac{d^2\varphi}{dt^2} = 0 \tag{1.2}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} \tag{1.3}$$

is the linearized total time derivative.

Consider the nondimensional variables

$$X = x/B\ell \quad Y = y/\ell \quad Z = z/\ell \quad \phi = \varphi/\ell U_\infty \tag{1.4}$$

where

$$B = \sqrt{M^2 - 1} \tag{1.5}$$

and ℓ is a reference length.

1.3 Boundary Condition

The lifting body considered here has complex configuration and is moving with small vibration with arbitrary mode. Thus the surface of the body is represented in the general form

$$S(x, y, z, t) = 0 \tag{1.6}$$

and the boundary condition on the surface of the body is given by

$$\frac{\partial S}{\partial t} + \nabla \Phi \cdot \nabla S = 0 \tag{1.7}$$

By using Eq. (1.1), Eq. (1.7) becomes

$$\nabla \varphi \cdot \nabla S = -\left(\frac{\partial S}{\partial t} + U_\infty \frac{\partial S}{\partial x} \right) \tag{1.8}$$

Furthermore, because the flow is uniform at infinity, the boundary condition at infinity is given by $\varphi = 0$. For steady subsonic flow, Eq. (1.8) is simplified as

$$\nabla \varphi \cdot \bar{n} = -U_\infty n_x \tag{1.9}$$

where $\bar{n}$ is the surface normal of the aircraft and n_x is the x - component of $\bar{n}$.

In order to use compact vector notation, it is convenient to introduce the concept of the conormal and cogradient

$$\bar{N}^c = N_x \bar{i} - N_Y \bar{j} - N_z \bar{k} \tag{1.10}$$

$$\nabla^c = \frac{\partial}{\partial X}\bar{i} - \frac{\partial}{\partial Y}\bar{j} - \frac{\partial}{\partial Z}\bar{k} \tag{1.11}$$

Furthermore, the conormal derivative is given by

$$\frac{\partial}{\partial N^c} = \bar{N}^c \cdot \nabla = \bar{N} \cdot \nabla^c \tag{1.12}$$

With this notation, Eq (1.9) becomes,

$$\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial S}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial S}{\partial z}\frac{\partial \varphi}{\partial z}\right) + U_\infty \frac{\partial S}{\partial x}$$

$$= \frac{U_\infty}{\ell}\left(\frac{1}{B^2}\frac{\partial S}{\partial X}\frac{\partial \phi}{\partial X} + \frac{\partial S}{\partial Y}\frac{\partial \phi}{\partial Y} + \frac{\partial S}{\partial Z}\frac{\partial \phi}{\partial Z} + \frac{1}{B}\frac{\partial S}{\partial X}\right)$$

$$= \frac{U_\infty}{\ell}\left[\left(-\frac{\partial S}{\partial X}\frac{\partial \phi}{\partial X} + \frac{\partial S}{\partial Y}\frac{\partial \phi}{\partial Y} + \frac{\partial S}{\partial Z}\frac{\partial \phi}{\partial Z}\right) + \left(\frac{1}{B^2}+1\right)\frac{\partial S}{\partial X}\frac{\partial \phi}{\partial X} + \frac{1}{B}\frac{\partial S}{\partial X}\right]$$

$$= \frac{U_\infty}{\ell}\left[\frac{\partial \phi}{\partial N^c} + \frac{M^2}{B^2}N_x\frac{\partial \phi}{\partial X} + \frac{1}{B}N_x\right]|\nabla S| = 0$$

or

$$\frac{\partial \phi}{\partial N^c} = -N_x\left(1 + \frac{M^2}{B}\frac{\partial \phi}{\partial X}\right) \tag{1.13}$$

The second term on the right hand side is negligible. Thus

$$\frac{\partial \phi}{\partial N^c} = -N_x \tag{1.14}$$

In addition to the above notation, it is convenient to introduce a special algebra, called supersonic vector algebra or, <u>super-algebra</u>, to simplify the algebraic manipulation for the supersonic flow theory. Details of this algebra and the proofs of some rules of this algebra are given in Appendix A.

1.4 Green's Theorems

According to Eq. (1.2), the Green function is given by

$$\nabla^2 G - \frac{1}{a_\infty^2}\frac{d^2G}{dt^2} = \delta(x-x_1, y-y_1, z-z_1, t-t_1) \tag{1.15}$$

with

$$G = 0 \tag{1.16}$$

at infinity.

Detail of the derivation of function G is presented in Ref. 1. The results are summerized here. For steady supersonic flow G is given by

$$G = -\frac{H}{2\pi r} \tag{1.17}$$

where H is given by Eq (1.22b) while

$$r = \left\{(x-x_1)^2 - B^2\left[(y-y_1)^2 + (z-z_1)^2\right]\right\}^{1/2} \tag{1.18}$$

For oscillatory supersonic flow G is given by

$$G = -\frac{1}{4\pi r}\left[\delta(t_1 - t + \theta^+) + \delta(t_1 - t + \theta^-)\right] \tag{1.19}$$

where

$$\theta^\pm = \frac{1}{a_\infty B^2}\left[M(x-x_1) \pm r\right] \tag{1.20}$$

By appiying the Green function method and using Eq. (1.17), the linearized equation for the steady supersonic flow can be derived as

$$2\pi E(P_*)\varphi(P_*) = -\oint_\Sigma \frac{\partial\phi}{\partial N^c}\frac{H}{\|R\|}\,d\Sigma + \oint_\Sigma \phi\frac{\partial}{\partial N^c}\left(\frac{H}{\|R\|}\right)d\Sigma \tag{1.21}$$

where

$$\begin{aligned} E &= 1 \quad \text{outside } \Sigma \\ &= 0 \quad \text{inside } \Sigma \end{aligned} \tag{1.22a}$$

and

$$\begin{aligned} H &= 1 \quad \text{for} \quad X_* - X > \left[(Y_* - Y)^2 + (Z_* - Z)^2\right] \\ &= 0 \quad \text{for} \quad X_* - X \le \left[(Y_* - Y)^2 + (Z_* - Z)^2\right] \end{aligned} \tag{1.22b}$$

$\|R\|$ is the supernorm of vector $R = \frac{\bar{r}}{B\ell}$ namely

$$\|R\| = |R \odot R|^{1/2} \tag{1.23}$$

with "super-product", $\odot$, defined by (see Appendix A)

$$\bar{a}\odot\bar{b} = a_x b_x - a_y b_y - a_z b_z \tag{1.24}$$

and the conormal derivative $\frac{\partial}{\partial N^c}$ is given by Eq (1.12).

For oscillatory flow, the Green theorem is given by

$$2\pi E(P_*)\,\hat{\phi}(P_*) = -\oiint \frac{\partial\hat{\phi}}{\partial N^c}\frac{H}{\|R\|}\cos(\Omega\|R\|)\,d\Sigma + \oiint \hat{\phi}\,\frac{\partial}{\partial N^c}\left(\frac{H}{\|R\|}\cos(\Omega\|R\|)\right)d\Sigma \tag{1.25}$$

where $\hat{\phi}$ is given by

$$\hat{\phi} = \phi(X,Y,Z,T)\,e^{-i\Omega(T-MX)} \tag{1.26}$$

1.5 Numerical Procedure

By imposing that the value of the potential at p_* approaches the value of potential at p on the surface Σ , if p_* approach p , the value of $E(p)$ in Eqs (1.21) and (1.25) is found to be 1/2 (see App. F, Ref. 3). Then, an integral equation relating the potential on the surface Σ to its normal derivative is obtained. In order to solve this integral equation the surface, Σ , surrounding the aircraft and wake and diaphragm, if necessary is divided into a number of quadrilateral elements which are approximated by hyperboloidal elements. The general expression of hyperboloidal elements can be written as

$$\bar{p} = \bar{p}_c + \xi\bar{p}_1 + \eta\bar{p}_2 + \xi\eta\bar{p}_3 \quad \begin{pmatrix} -1\le\xi\le 1 \\ -1\le\eta\le 1\end{pmatrix} \tag{1.27}$$

where $\bar{p}_c$, $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_3$ are linear combinations of the four corner vectors, p_{++}, p_{+-}, p_{-+}, and p_{--} (see fig. 1)

By assuming that the values of velocity potential and its normal derivatives within each element, Σ_k, are constant, Eq. (21) for steady supersonic flow reduces to

$$\phi(p) = -\sum_{k=1}^{N} \left(\frac{\partial \phi}{\partial N^c}\right)_k \left[\frac{1}{\pi} \iint_{\Sigma_k} \frac{H}{\|R\|} \, d\Sigma_k\right] + \sum_{k=1}^{N} \phi_k \left[\frac{1}{\pi} \iint_{\Sigma_k} \frac{\partial}{\partial N^c}\left(\frac{H}{\|R\|}\right) d\Sigma_k\right] \quad (1.28)$$

where $\bar{p}$ is any point on the surface Σ, and N is the number of elements on Σ.

Imposing the condition that Eq. (1.28) is satisfied at the centroids, p_h of the elements, Σ_h, a system of algebraic equations is obtained. This yields

$$\left[\delta_{hk} - C_{hk} - W_{hk}\right]\left\{\phi_k\right\} = \left[b_{hk}\right]\left\{\left(\frac{\partial \phi}{\partial N^c}\right)_k\right\}. \quad (1.29)$$

where δ_{hk} is the Kronecker delta, whereas

$$C_{hk} = \left[\frac{1}{\pi} \iint_{\Sigma_k} \frac{\partial}{\partial N^c}\left(\frac{H}{\|R\|}\right) d\Sigma_k\right]_{p_* = p_h} \quad (1.30)$$

$$b_{hk} = \left[-\frac{1}{\pi} \iint_{\Sigma_k} \frac{\partial}{\partial N^c}\left(\frac{H}{\|R\|}\right) d\Sigma_k\right]_{p_* = p_h} \quad (1.31)$$

and the definition of w_{hk} is similar to the one for subsonic flow (see Ref. 1). However, in this report, the wake is not included since only supersonic-trailing-edge configurations are considered here.

Equations for oscillatory supersonic flow are considered in Section 4.

1.6 Diaphragms

There are three categories of wing geometry in supersonic flow. The first ones are wings with supersonic leading edge. For this kind of wings, the element, Σ_h, on the upper (or lower) surface are influenced only by the elements on the surface of the same side, therefore, the integrations in Equations (1.30) and (1.31) are performed over the area on one side only. The second ones are wings with subsonic leading edge. For this kind of wings, diaphragm may or may not be used. For wings of large thickness, results obtained with or without diaphragm are almost the same. However, for wings of small thickness, diaphragms are suggested to be used to avoid the determinant singularity. The third ones are wings with both supersonic and subsonic leading edges, diaphragms have to be used for this kind of wings.

If diaphragms are used, both values of the velocity potential and its normal derivative on the diaphragm elements are unknown. However, two independent integral equations are obtained for each element of the diaphragm, one relates ϕ_D and $(\partial\phi/\partial N^c)_D$ to the upper geometry of the aircraft and whole diaphragm, the other one relates the same quantities to the lower geometry. Therefore, the total number of equations is equal to the total number of unknowns, then a system of algebraic equations can be solved for ϕ_K of wing element and ϕ_D and $(\partial\phi/\partial N^c)_D$ of the diaphragm elements. If the problem is symmetric in the z-direction, then $(\partial\phi/\partial N^c)_D$ is zero while ϕ_D is unknown for each diaphragm element. If the problem is antisymmetric in the z-direction, then ϕ_D is zero while $(\partial\phi/\partial N^c)_D$ is unknown. For these two cases only one independent integral equation can be written for each diaphragm element, and the number of unknown of the diaphragm elements reduces to a half.

1.7 Pressure and Generalized Forces

The pressure on the surface of the body is evaluated from the linearized Bernoulli theorem

$$p - p_\infty = -\rho_\infty\left(\frac{\partial\varphi}{\partial t} + U_\infty\frac{\partial\varphi}{\partial x}\right) \tag{1.32}$$

Then c_p is given by

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = -\frac{2}{U_\infty^2}\left(\frac{\partial\varphi}{\partial t} + U_\infty\frac{\partial\varphi}{\partial x}\right) \tag{1.33}$$

For steady flow by using Eq. (1.4) Eq (1.33) reduces to

$$c_p = -\frac{2}{\beta}\frac{\partial\phi}{\partial X} \tag{1.34}$$

For oscillatory flow, c_p is derived in Section 4 as

$$\tilde{c}_p = c_p e^{-i\Omega T} = -\frac{2}{\beta}\left(\frac{\partial\tilde{\phi}}{\partial X} + i\beta k\tilde{\phi}\right) \tag{1.35}$$

The evaluation of the generalized forces is considered as follows. The generalized force is defined as

$$Q_n = \oiint_\Sigma \bar{f}\cdot\bar{U}_n\, d\Sigma \tag{1.36}$$

where $\bar{f}$ is the force acting on the surface of the body and U_n is the vibration mode. For lift, $\bar{U} = \bar{k}$ and thus

$$\bar{f}\cdot\bar{U} = -p\bar{n}\cdot\bar{k} = -p\, n_z \tag{1.37}$$

Therefore

$$L = -\oiint_\Sigma p\, n_z\, d\Sigma = -\oiint_\Sigma p\, dx\, dy \tag{1.38}$$

For pitch moment

$$\begin{aligned}\bar{f}\cdot\bar{U} &= -p\left[-(z - z_m)n_x + (x - x_m)n_z\right] \\ &\sim -p\,(x - x_m)\,n_z\end{aligned} \tag{1.39}$$

since $(z - z_m)\, n_x$ is negligible, therefore

$$M = -\oint\!\!\oint p\,(x - x_m)\, dx\, dy \qquad (1.40)$$

SECTION 2

NUMERICAL FORMULATIONS

2.1 Introduction

By introducing hyperboloidal element, given by Eq. (1.27), the integrals in Eqs. (1.30) and (1.31) are evaluated analytically (See Ref. 3). In Subsections 2.2 and 2.3, solutions for b_{hk} and C_{hk} are shown to be valid for any planar quadrilateral element inside the Mach forecone. For elements outside the Mach forecone, $b_{hk} = C_{hk} = 0$, since all these elements have no influence on the element Σ_h. However, for elements intersected with Mach forecone, singularity problem arise, therefore, solutions have to be considered separately. In Section 3 solutions for this kind of element are considered.

2.2 Source Integral

For element inside the Mach forecone, the solution of b_{hk}, in Eq. (1.31), is given as

$$b_{hk} = -\frac{1}{\pi}\left[I_s(1,1) - I_s(1,-1) - I_s(-1,1) + I_s(-1,-1)\right] \tag{2.1}$$

where

$$I_s(\xi,\eta) = \frac{|\bar{a}_1 \times \bar{a}_2|}{-\|a_1 \times a_2\|^2}\Big\{ \bar{q}\times\bar{a}_1 \odot \bar{a}_2\times\bar{a}_1 \; F_1(\xi,\eta) + \bar{q}\times\bar{a}_2 \odot \bar{a}_1\times\bar{a}_2 \; F_2(\xi,\eta)$$

$$- \bar{q}\cdot\bar{a}_1\times\bar{a}_2 \; \tan_p^{-1}\left(\frac{-\bar{q}\times\bar{a}_1 \odot \bar{q}\times\bar{a}_2}{\|\bar{q}\| \;\; \bar{q}\cdot\bar{a}_1\times\bar{a}_2} \right)\Big\} \tag{2.2}$$

with (see Eq. (1.27))

$$\bar{q} = \bar{P} - \bar{P}_h$$

$$\bar{a}_\alpha = \partial\bar{q}/\partial\xi^\alpha \qquad (\alpha = 1,2)$$

and

$$F_1(\xi,\eta) = \frac{1}{\|\bar{a}_1\|} \ln\left|\frac{\|\bar{q}\|\|\bar{a}_1\| + \bar{q}\odot\bar{a}_1}{\|\bar{q}\times\bar{a}_1\|}\right| \qquad (\bar{a}_1\odot\bar{a}_1 > 0)$$

$$= \|\bar{q}\| / \bar{q}\odot\bar{a}_1 \qquad (\bar{a}_1\odot\bar{a}_1 = 0) \qquad (2.3a)$$

$$= -\frac{1}{\|\bar{a}_1\|} \sin^{-1}\left(\frac{\bar{q}\odot\bar{a}_1}{\|\bar{q}\times\bar{a}_1\|}\right) \qquad (\bar{a}_1\odot\bar{a}_1 < 0)$$

$$F_2(\xi,\eta) = \frac{1}{\|\bar{a}_2\|} \ln\left|\frac{\|\bar{q}\|\|\bar{a}_2\| + \bar{q}\odot\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right| \qquad (\bar{a}_2\odot\bar{a}_2 > 0)$$

$$= \|\bar{q}\| / \bar{q}\odot\bar{a}_2 \qquad (\bar{a}_2\odot\bar{a}_2 = 0)$$

$$= -\frac{1}{\|\bar{a}_2\|} \sin^{-1}\left(\frac{\bar{q}\odot\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right) \qquad (\bar{a}_2\odot\bar{a}_2 < 0) \qquad (2.3b)$$

Note that Eq. (2.2) may be rewritten as

$$I_s = \frac{1}{-\|\bar{n}\|^2}\Big\{-\bar{q}\times\bar{a}_1\odot\bar{n}\; F_1(\xi,\eta) + \bar{q}\times\bar{a}_2\odot\bar{n}\; F_2(\xi,\eta)$$
$$- \bar{q}\cdot\bar{n}\, \tan_p^{-1}\left(\frac{-\bar{q}\times\bar{a}_1\odot\bar{q}\times\bar{a}_2}{\|\bar{q}\|\;\; \bar{q}\cdot\bar{a}_1\times\bar{a}_2}\right) \qquad (2.4)$$

where

$$\bar{n} = \bar{a}_1\times\bar{a}_2 / |\bar{a}_1\times\bar{a}_2| \qquad (2.5)$$

is the unit normal.

The following is to prove that Eq. (2.4) is valid for any quadrilateral planar element. For a planar element, the unit normal $\bar{n}$ is independent of ξ and η,

$$\frac{\partial\bar{n}}{\partial\xi} = \frac{\partial\bar{n}}{\partial\eta} = 0 \qquad (2.6)$$

Note also

$$\frac{\partial\bar{a}_1}{\partial\xi} = \frac{\partial\bar{a}_2}{\partial\eta} = 0 \qquad (2.7)$$

and

$$\frac{\partial\bar{a}_1}{\partial\eta} = \frac{\partial\bar{a}_2}{\partial\xi} = \bar{p}_3 \qquad (2.8)$$

Hence

$$\frac{\partial}{\partial \xi}\left(\bar{q} \times \bar{a}_1 \odot \bar{n}\right) = \bar{a}_1 \times \bar{a}_1 \odot \bar{n} = 0 \tag{2.9}$$

$$\frac{\partial}{\partial \eta}\left(\bar{q} \times \bar{a}_2 \odot \bar{n}\right) = \bar{a}_2 \times \bar{a}_2 \odot \bar{n} = 0 \tag{2.10}$$

Furthermore, as shown in Appendix B, (see Eqs. P.5, B.6 and B.7)

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta}\left\{\frac{1}{\|\bar{a}_2\|} \ln\left|\frac{\sqrt{\bar{q} \odot \bar{q}}\sqrt{\bar{q} \odot \bar{a}_2} + \bar{q} \odot \bar{a}_2}{\|\bar{q} \times \bar{a}_2\|}\right|\right\} = \frac{1}{\|\bar{q}\|} \qquad \bar{Q}_2 \odot \bar{a}_2 > 0$$

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{\sqrt{\bar{q} \odot \bar{q}}}{\bar{q} \odot \bar{a}_2}\right) = \frac{1}{\|\bar{q}\|} \qquad \bar{Q}_2 \odot \bar{a}_2 = 0$$

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta}\left[\frac{-1}{\|\bar{a}_2\|} \sin^{-1}\left(\frac{\bar{q} \odot \bar{a}_2}{\|\bar{q} \times \bar{a}_2\|}\right)\right] = \frac{1}{\|\bar{q}\|} \qquad \bar{Q}_2 \odot \bar{a}_2 < 0$$

or

$$\frac{\partial F_2}{\partial \eta} = \frac{1}{\|\bar{q}\|} \qquad \bar{Q}_2 \odot \bar{a}_2 \gtreqless 0 \tag{2.12}$$

Similarly

$$\frac{\partial F_1}{\partial \xi} = \frac{1}{\|\bar{q}\|} \qquad \bar{Q}_1 \odot \bar{a}_1 \gtreqless 0 \tag{2.13}$$

Using Eqs. (2.8), (2.10) and (2.12) one obtains

$$\frac{\partial^2}{\partial \xi \partial \eta}\left\{\bar{q} \times \bar{a}_2 \odot \bar{n} \;\; F_2(\xi,\eta)\right\} = \frac{\partial}{\partial \xi}\left(\bar{q} \times \bar{a}_2 \odot \bar{n} \frac{1}{\sqrt{\bar{q} \odot \bar{q}}}\right)$$

$$= \frac{\partial}{\partial \xi}\left(\bar{q} \times \bar{a}_2 \odot \bar{n} \;\; \frac{1}{\sqrt{\bar{q} \odot \bar{q}}}\right) \tag{2.14a}$$

or

$$\frac{\partial^2}{\partial \xi \partial \eta}\left\{\bar{q} \times \bar{a}_2 \odot \bar{n} \;\; F_2(\xi,\eta)\right\}$$

$$= \frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}\|} + \frac{\bar{q} \times \bar{P}_1 \odot \bar{n}}{\|\bar{q}\|} - \bar{q} \times \bar{a}_2 \odot \bar{n} \frac{\bar{q} \odot \bar{a}_1}{(\bar{q} \odot \bar{q})^{3/2}} \tag{2.14b}$$

Similarly,

$$\frac{\partial^2}{\partial\xi\partial\eta}\left\{\bar q\times\bar a_1\circ\bar n\,F_1(\xi,\eta)\right\}=\frac{\partial}{\partial\eta}\left(\bar q\times\bar a_1\circ\bar n\,\frac{1}{\sqrt{\bar q\circ\bar q}}\right)\tag{2.15a}$$

or

$$\frac{\partial^2}{\partial\xi\partial\eta}\left\{\bar q\times\bar a_1\circ\bar n\,F_2(\xi,\eta)\right\}=\frac{\bar a_2\times\bar a_1\circ\bar n}{\|\bar q\|}+\frac{\bar q\times\bar p_3\circ\bar n}{\|\bar q\|}-\bar q\times\bar a_1\circ\bar n\,\frac{\bar q\circ\bar a_2}{\|q\|^3}\tag{2.15b}$$

In addition as shown in Appendix C,

$$\frac{\partial^2}{\partial\xi\partial\eta}\left\{|\bar q\cdot\bar n|\tan^{-1}\left(\frac{-\bar q\times\bar a_1\circ\bar q\times\bar a_2}{\|\bar q\|\,|\bar q\cdot\bar a_1\times\bar a_2|}\right)\right\}=\bar q\cdot\bar n\,\frac{\bar q\cdot\bar a_1\times\bar a_2}{\|\bar q\|^3}\tag{2.16}$$

Finally, combining Eqs. (2.4), (2.14b), (2.15b) and (2.16), and noting that $\|n\|^2=-\bar n\circ\bar n$ yields

$$\begin{aligned}\frac{\partial^2 I_s}{\partial\xi\partial\eta}=\frac{1}{\bar n\circ\bar n}\Bigg\{&-\left(\frac{\bar a_2\times\bar a_1\circ\bar n}{\|\bar q\|}+\cancel{\frac{\bar q\times\bar p_3\circ\bar n}{\|\bar q\|}}-\bar q\times\bar a_1\circ\bar n\,\frac{\bar q\circ\bar a_2}{(\bar q\circ\bar q)^{3/2}}\right)\\&+\left(\frac{\bar a_1\times\bar a_2\circ\bar n}{\|\bar q\|}+\cancel{\frac{\bar q\times\bar p_3\circ\bar n}{\|\bar q\|}}-\bar q\times\bar a_2\circ\bar n\,\frac{\bar q\circ\bar a_1}{(\bar q\circ\bar q)^{3/2}}\right)\\&+\bar q\cdot\bar n\,(\bar q\cdot\bar a_1\times\bar a_2)/\|\bar q\|^3\Bigg\}\end{aligned}\tag{2.17}$$

or

$$\begin{aligned}\frac{\partial^2 I_s}{\partial\xi\partial\eta}=\frac{1}{\bar n\circ\bar n}\Bigg\{&2\,|\bar a_1\times\bar a_2|\,\frac{\bar n\circ\bar n}{\|\bar q\|}+\\&\frac{1}{\|\bar q\|^3}\Big[(\bar q\times\bar a_1\circ\bar n)(\bar q\circ\bar a_2)-(\bar q\times\bar a_2\circ\bar n)\bar q\circ\bar a_1+\bar q\cdot\bar n\,\bar q\cdot\bar a_1\times\bar a_2\Big]\Bigg\}\end{aligned}\tag{2.18}$$

According to the Second Super-rule (Eq. A.11 with $\bar a=\bar q$, $\bar b=\bar a_1$, $\bar c=\bar a_2$)

$$- |\bar{a}_1 \times \bar{a}_2| \left(\bar{q} \times \bar{a}_1 \circ \bar{n} \; \bar{q} \circ \bar{a}_2 - \bar{q} \times \bar{a}_2 \circ \bar{n} \; \bar{q} \circ \bar{a}_1 - \bar{q} \cdot \bar{n} \; \bar{q} \cdot \bar{a}_1 \times \bar{a}_2 \right)$$

$$= (\bar{q} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \, \bar{q} \circ \bar{a}_2 + (\bar{q} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \, \bar{q} \circ \bar{a}_1 + (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2$$

$$= \bar{q} \circ \bar{q} \;\; \bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2 - (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2$$

$$= \bar{q} \circ \bar{q} \;\; \bar{n} \circ \bar{n} \; |\bar{a}_1 \times \bar{a}_2|^2 \tag{2.19}$$

Hence

$$\frac{\partial^2 I_s}{\partial \xi \partial \eta} = \frac{1}{\bar{n} \circ \bar{n}} \left\{ 2 \, \bar{n} \circ \bar{n} \, \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|} - \bar{q} \circ \bar{q} \, \bar{n} \circ \bar{n} \, \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|^3} \right\}$$

$$= \frac{1}{\bar{n} \circ \bar{n}} \left\{ \bar{n} \circ \bar{n} \, \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|} \right\}$$

$$= \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|} \tag{2.20}$$

Substituting Eq. (2.19) into Eq. (1.52) and noting that H = 1 for the element internal to the Mach forecone, yields

$$-\pi b_{hk} = \int_{-1}^{1} \int_{-1}^{1} \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|} \, d\xi \, d\eta = \int_{-1}^{1} \int_{-1}^{1} \frac{\partial I_s}{\partial \xi \partial \eta} \, d\xi \, d\eta \tag{2.21}$$

therefore,

$$-\pi b_{hk} = I_s(1,1) - I_s(1,-1) - I_s(-1,1) + I_s(-1,1) \tag{2.22}$$

with I_s given by Eq. (2.2).

2.3 Doublet Integral

Consider the doublet integral in Eq. (1.30) which can be rewritten as

$$C_{hk} = \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \bar{N} \circ \bar{\nabla} \left(\frac{H}{\|\bar{q}\|} \right) |a_1 \times a_2| \, d\xi \, d\eta \tag{2.23}$$

Note that H = 1, for the element internal to Mach forecone and

$$(\bar{N}\odot\bar{\nabla})\,\bar{q} = \left(N_x \frac{\partial}{\partial X} - N_y \frac{\partial}{\partial Y} - N_z \frac{\partial}{\partial Z}\right)\begin{Bmatrix} X - X_h \\ Y - Y_h \\ Z - Z_h \end{Bmatrix}$$

$$= \begin{Bmatrix} N_x \\ -N_y \\ -N_z \end{Bmatrix} = \bar{N}^c \tag{2.24}$$

Therefore

$$(\bar{N}\odot\bar{\nabla})\,\frac{1}{\sqrt{\bar{q}\odot\bar{q}}} = \frac{1}{(\bar{q}\odot\bar{q})^{3/2}}\left[(\bar{N}\odot\bar{\nabla})\bar{q}\right]\odot\bar{q}$$

$$= \frac{1}{(\bar{q}\odot\bar{q})^{3/2}}\,(\bar{N}^c\odot\bar{q})$$

$$= \frac{1}{(\bar{q}\odot\bar{q})^{3/2}}\,\bar{N}\cdot\bar{q} \tag{2.25}$$

Thus

$$C_{hk} = \frac{1}{\pi}\left[I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1)\right] \tag{2.26}$$

where

$$\frac{\partial^2 I_D}{\partial\xi\partial\eta} = \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\|\bar{q}\|^3} \tag{2.27}$$

In Appendix C, it is proved that

$$I_D = \tan_r^{-1}\left(\frac{-\bar{q} \times \bar{a}_1 \odot \bar{q} \times \bar{a}_2}{\|\bar{q}\| \; \bar{q} \cdot \bar{a}_1 \times \bar{a}_2} \right) \tag{2.28}$$

FINITE PART OF INTEGRALS

3.1 Introduction

In order to extend the solution given in the previous section, the finite part of integrals are investigated in this section and the solutions for b_{hk} and c_{hk} for the elements intersected with the Mach forecone are considered. In Subsection 3.2, a general integral function is considered. In Subsection 3.3, the source integral is considered. In Subsection 3.4, the doublet integral is considered.

3.2 A General Integral Function

Consider the following integral equation

$$I = \int_{-\infty}^{a} \frac{\partial}{\partial x}\left(\frac{g(x)}{\sqrt{x}} H(x)\right) dx \tag{3.1}$$

where g (x) is a regular function and H(x) is the Heaviside function.

It is observed that as x approaches zero, the integrand in Eq. (3.1) become singular. However it can be proved as follows that this singularity is avoided after the integration is carried out. Let ϵ be an infinitesimal quantity, then Eq. (3.1) can be rewritten as

$$I = \lim_{\epsilon \to 0} \int_{-\infty}^{a} \frac{\partial}{\partial x}\left[\frac{g}{\sqrt{x}} H(x-\epsilon)\right] dx$$

$$= \lim_{\epsilon \to 0}\left[\int_{-\infty}^{a} \frac{\partial}{\partial x}\left(\frac{g}{\sqrt{x}}\right) H \; dx + \int_{-\infty}^{a} \frac{g}{\sqrt{x}}\, \delta(x-\epsilon)\, dx\right]$$

$$= \lim_{\epsilon \to 0}\left[\int_{\epsilon}^{a} \frac{\partial}{\partial x}\left(\frac{g}{\sqrt{x}}\right) dx + \frac{g(\epsilon)}{\sqrt{\epsilon}}\right]$$

$$= \frac{g(a)}{\sqrt{a}} - \frac{g(\epsilon)}{\sqrt{\epsilon}} + \frac{g(\epsilon)}{\sqrt{\epsilon}}$$

$$= \frac{g(a)}{\sqrt{a}} \tag{3.2}$$

Therefore, the singularity contribution disappears and shou?? not be taken into account.

3.3 Finite Part of Source Integral

By combining Eq. (2.17) with Eq. (2.20), one obtains

$$\frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{q}\|} = \frac{1}{\bar{n} \odot \bar{n}} \Big\{ - \Big(\frac{\bar{a}_2 \times \bar{a}_1 \odot \bar{n}}{\|\bar{q}\|} + \frac{\bar{q} \times \bar{P}_3 \odot \bar{n}}{\|\bar{q}\|} - \bar{q} \times \bar{a}_1 \odot \bar{n} \frac{\bar{q} \odot \bar{a}_1}{(\bar{q} \odot \bar{q})^{3/2}} \Big)$$

$$+ \Big(\frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}\|} + \frac{\bar{q} \times \bar{P}_3 \odot \bar{n}}{\|\bar{q}\|} - \bar{q} \times \bar{a}_2 \odot \bar{n} \frac{\bar{q} \odot \bar{a}_2}{(\bar{q} \odot \bar{q})^{3/2}} \Big)$$

$$+ \bar{q} \cdot \bar{n} \, (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2) \frac{1}{\|\bar{q}\|^3} \Big\} \tag{3.3}$$

Therefore, the source integral, b_{hk}, can be separated into three integrals, i.e.

$$b_{hk} = \frac{1}{\pi} \int_{-1}^{1}\int_{-1}^{1} \frac{H}{\|\bar{q}\|} |\bar{a}_1 \times \bar{a}_2| \, d\xi \, d\eta = \frac{1}{\pi} \frac{1}{\bar{n} \odot \bar{n}} (S_1 + S_2 + S_3) \tag{3.4}$$

where

$$S_1 = -\int_{-1}^{1}\int_{-1}^{1} H \Big[-\frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}\|} + \frac{\bar{q} \times \bar{P}_3 \odot \bar{n}}{\|\bar{q}\|^3} - \bar{q} \times \bar{a}_2 \odot \bar{n} \frac{\bar{q} \odot \bar{a}_2}{\|\bar{q}\|^3} \Big] d\xi \, d\eta \tag{3.5}$$

$$S_2 = \int_{-1}^{1}\int_{-1}^{1} H \Big[\frac{\bar{a}_1 \times \bar{a}_2 \odot \bar{n}}{\|\bar{q}\|} + \frac{\bar{q} \times \bar{P}_3 \odot \bar{n}}{\|\bar{q}\|} - \bar{q} \times \bar{a}_2 \odot \bar{n} \frac{\bar{q} \odot a_1}{\|\bar{q}\|^3} \Big] d\xi \, d\eta \tag{3.6}$$

and

$$S_3 = \int_{-1}^{1}\int_{-1}^{1} H\left[\bar{q}\cdot\bar{n} \quad \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\|\bar{q}\|}\right] d\xi\, d\eta \tag{3.7}$$

Consider the first integral in Eq. (3.5) and assume that I_{s1} is the solution of S_1 through the double integrations. Then

$$S_1 = I_{s1}(1,1) - I_{s1}(1,-1) - I_{s1}(-1,1) + I_{s1}(-1,-1) \tag{3.8}$$

By using Eq. (2.15), Eq. (3.5) can be rewritten as

$$S_1 = -\int_{-1}^{1} d\xi \int_{-1}^{1} \frac{\partial}{\partial\eta}\left[H(\bar{q}\times\bar{a}_1\odot\bar{n})\frac{1}{\sqrt{\bar{q}\odot\bar{q}}}\right] d\eta \tag{3.9}$$

Note that $\sqrt{\bar{q}\odot\bar{q}}$ can be expressed as $h(\eta)\sqrt{\eta-\eta_0}$ where $\bar{q}\times\bar{a}_1\odot\bar{n}/h(\eta)$ is a regular function and η_0 is defined such that $\bar{q}\odot\bar{q}=0$. Compared with Eq. (3.2), it can be concluded that after the first integration, solution along the intersection line of the element with the Mach forecone yields no contribution, i.e.

$$I_{s1}(1,1) - I_{s1}(-1,1) = 0 \tag{3.10}$$

or

$$I_{s1}(1,-1) - I_{s1}(-1,-1) = 0 \tag{3.11}$$

if the edges of $\eta = 1$, or $\eta = -1$ is completely outside the Mach forecone. Otherwise, I_{s1} is given by the first term of Eq. (2.2) if the corner point is inside the Mach forecone, or

$$\begin{aligned} I_{s1}(\xi,\eta^*) &= 0 & \bar{a}_1\odot\bar{a}_1 \geq 0 \\ &= (\bar{q}\times\bar{a}_1\cdot\bar{n})\,\mathrm{sign}(\bar{q}\odot\bar{a}_1)/\|\bar{a}_1\| & \bar{a}_1\odot\bar{a}_1 < 0 \end{aligned} \tag{3.12}$$

if the corner point is outside the Mach forecone.

Similarly, by using Eq. (2.14a), Eq. (3.6) can be rewritten as

$$S_2 = \int_{-1}^{1} d\eta \int_{-1}^{1} \frac{\partial}{\partial \xi} \left[H(\bar{q} \times \bar{a}_2 \odot \bar{n}) \frac{1}{\sqrt{\bar{q} \odot \bar{q}}} \right] d\xi \tag{3.13}$$

For the same reason given for S_1 . if S_2 is expressed as

$$S_2 = I_{s2}(1,1) - I_{s2}(1,-1) - I_{s2}(-1,1) + I_{s2}(-1,-1) \tag{3.14}$$

the solution of S_1 is given as follows:

$$I_{s2}(1,1) - I_{s2}(1,-1) = 0 \tag{3.15}$$

or

$$I_{s2}(-1,1) - I_{s2}(-1,-1) = 0 \tag{3.16}$$

if the edge of $\xi = 1$, or $\xi = -1$,is completely outside the Mach forecone. Otherwise, I_{s2} is given by the second term of Eq. (2.2) if the corner point is inside the Mach forecone, or

$$\begin{aligned} I_{s2}\ (\xi^*, \eta) &= 0 & ;\ \bar{a}_2 \odot \bar{a}_2 \geq 0 \\ &= -(\bar{q} \times \bar{a}_2 \cdot \bar{n}) \, sign(\bar{q} \odot \bar{a}_2) / \|\bar{a}_2\| & ;\ \bar{a}_2 \odot \bar{a}_2 < 0 \end{aligned} \tag{3.17}$$

if the corner point is outside the Mach forecone.

Note that the values of ξ^* and η^* are evaluated such that $\bar{q} \odot \bar{q} = 0$. The solution of S_3 is considered together with the solution of c_{hk} in the following subsection.

3.4 Finite Part of the Doublet Integral

The doublet integral is given by Eq. (2.26) as

$$C_{hk} = \frac{1}{\pi} \left[I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1) \right] \tag{3.18}$$

one can also express S_3 as

$$S_3 = I_{s3}(1,1) - I_{s3}(1,-1) - I_{s3}(-1,1) + I_{s3}(-1,-1) \tag{3.19}$$

In Appendix C, it is proved that (Eq. C.15)

$$\frac{\partial I_D}{\partial \eta} = \frac{1}{\|\bar{q}\times\bar{a}_1\|\,\|\bar{q}\|}\left(\bar{q}\odot\bar{a}_1\ \ \bar{q}\cdot\bar{a}_1\times\bar{a}_2 - \bar{q}\odot\bar{q}\ \ \bar{q}\cdot\bar{a}_1\times\bar{P}_3\right) \tag{3.20}$$

Therefore, the doublet integral and S_3 can always be expressed as

$$C_{hk} = \frac{1}{\pi}\int_{-1}^{1} d\xi \int_{-1}^{1} \frac{\partial}{\partial\eta}\left[\frac{H}{\|\bar{q}\times\bar{a}_1\|^2}\left(\bar{q}\odot\bar{a}_1\ \ \bar{q}\cdot\bar{a}_1\times\bar{a}_2 - \bar{q}\odot\bar{q}\ \ \bar{q}\times\bar{a}_1\times\bar{P}_3\right)\cdot\frac{1}{\|\bar{q}\|}\right] d\eta \tag{3.21}$$

and

$$S_3 = -\bar{q}\cdot\bar{n}\int_{-1}^{1} d\xi \int_{-1}^{1} \frac{\partial}{\partial\eta}\left[\frac{H}{\|\bar{q}\times\bar{a}_1\|^2}\left(\bar{q}\odot\bar{a}_1,\ \bar{q}\cdot\bar{a}_1\times\bar{a}_2 - \bar{q}\odot\bar{q}\ \ \bar{q}\times\bar{a}_1\times\bar{P}_3\right)\frac{1}{\|\bar{q}\|}\right] d\eta \tag{3.22}$$

Both integrands of the above equations are of the form of the integral in Eq. (3.1). For the same reason given above, one obtains

$$I_{S3}(1,1) - I_{S3}(-1,1) = 0 \tag{3.23}$$

$$I_D(1,1) - I_D(-1,1) = 0 \tag{3.24}$$

or

$$I_{S3}(1,-1) - I_{S3}(-1,-1) = 0 \tag{3.25}$$

$$I_D(1,-1) - I_D(-1,-1) = 0 \tag{3.26}$$

if the edge of $\eta = 1$ or $\eta = -1$ is completely outside of the Mach forecone. Otherwise, I_D and I_{S3} are given by Eq. (2.27) and the third term of Eq. (2.2) if the corner point is inside the Mach forecone, or

$$I_{S3}(\xi,\eta^*) = (\bar{q}\cdot\bar{n})\frac{\pi}{2}\,\mathrm{sgn}\left[(\bar{q}\odot\bar{a}_1)(\bar{q}\odot\bar{a}_2)(\bar{q}\cdot\bar{n})\right] \tag{3.27}$$

$$I_D(\xi,\eta^*) = -\frac{1}{2}\,\mathrm{sign}\left[(\bar{q}\odot\bar{a}_1)(\bar{q}\odot\bar{a}_2)(\bar{q}\cdot\bar{n})\right] \tag{3.28}$$

As mentioned before, the value of η^* is evaluated such that

$$\bar{q}\odot\bar{q} = 0 \quad .$$

SECTION 4

SUPERSONIC OSCILLATORY FLOW

4.1 Integral Equation

In this Section it is shown how the results obtained in the preceeding Sections can be extended to supersonic oscillatory flow. Introducing the variables

$$X = \frac{x}{B\ell} \quad Y = \frac{y}{\ell} \quad Z = \frac{z}{\ell} \quad T = \frac{B a_\infty t}{\ell} \quad \Omega = \frac{\omega \ell}{B a_\infty} \tag{4.1}$$

and the complex potential $\hat{\phi}$ such that

$$\varphi(x,y,z,t) = U_\infty \ell \, \hat{\phi}(X,Y,Z) \, e^{i\Omega(T - MX)} \tag{4.2}$$

the integral equation for the subsonic oscillatory flow is given by

$$\pi \hat{\phi} = \iint_\Sigma \left[\frac{\partial \hat{\phi}}{\partial N^c} \frac{H}{\|\underset{\sim}{q}\|} \cos(\Omega \|\underset{\sim}{q}\|) + \hat{\phi} \frac{\partial}{\partial N^c} \left(\frac{H}{\|\underset{\sim}{q}\|} \cos(\Omega \|\underset{\sim}{q}\|) \right) \right] d\Sigma \tag{4.3}$$

where Σ surrounds body and wake.

4.2 Boundary Condition

The boundary condition is given by

$$\nabla_{xyz} S \cdot \nabla_{xyz} \varphi = - \frac{\partial S}{\partial t} - U_\infty \frac{\partial S}{\partial x} \tag{4.4}$$

or

$$-\nabla_{XYZ} S \odot \nabla_{XYZ} \phi + \frac{B}{M}\frac{\partial S}{\partial T} + \frac{1}{B}\frac{\partial S}{\partial X} + \frac{M^2}{B^2}\frac{\partial S}{\partial X}\frac{\partial \phi}{\partial X} = 0 \tag{4.5}$$

where φ and ϕ are such that

$$\Phi = U_\infty x + \varphi = U_\infty \ell (\beta X + \phi) \tag{4.6}$$

Next, assume that the motion of the surface consists of small harmonic oscillations around a rest configuration, that is

$$S = S_0(X, Y, Z) + \tilde{S}(X, Y, Z)\, e^{i\Omega T} \tag{4.7}$$

Then, setting

$$\phi = \phi_0(X, Y, Z) + \tilde{\phi}(X, Y, Z)\, e^{i\Omega T} \tag{4.8}$$

one obtains

$$\begin{aligned}
&-\nabla_{XYZ} S_0 \odot \nabla_{XYZ} \phi_0 - \left(\nabla_{XYZ} S_0 \odot \nabla_{XYZ} \tilde{\phi} + \nabla_{XYZ} \tilde{S} \odot \nabla_{XYZ} \phi_0\right) e^{i\Omega T} \\
&-\left(\nabla_{XYZ} \tilde{S} \odot \nabla_{XYZ} \tilde{\phi}\right) e^{i2\Omega T} + \frac{B}{M}\, i\Omega \tilde{S}\, e^{i\Omega T} \\
&+ \frac{1}{B}\left(\frac{\partial S_0}{\partial X} + \frac{\partial \tilde{S}}{\partial X}\, e^{i\Omega T}\right) \\
&+ \frac{M^2}{B^2}\left[\frac{\partial S_0}{\partial X}\frac{\partial \phi_0}{\partial X} + \left[\frac{\partial S_0}{\partial X}\frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X}\frac{\partial \phi_0}{\partial X}\right] e^{i\Omega T}\right. \\
&\left. + \frac{\partial \tilde{S}}{\partial X}\frac{\partial \tilde{\phi}}{\partial X}\, e^{i2\Omega T}\right] = 0
\end{aligned} \tag{4.9}$$

Assume that the surface is given in the form

$$S = +\left[Z - Z_u(X,Y) - \tilde{Z}_u(X,Y)e^{i\Omega T}\right] = 0 \qquad \text{(Upper surface)}$$

$$S = -\left[Z - Z_\ell(X,Y) - \tilde{Z}_\ell(X,Y)e^{i\Omega T}\right] = 0 \qquad \text{(Lower surface)}$$

(4.10)

with

$$Z_{u,\ell}(X,Y) = O(\varepsilon) \tag{4.11}$$

$$\tilde{Z}_{u,\ell}(X,Y) = O(\varepsilon^2) \tag{4.12}$$

or, in general by Eq. (E.7) with

$$S_0 = \pm\left[Z - Z_{u,\ell}(X,Y)\right] = O(1) \tag{4.13}$$

$$\frac{\partial S_0}{\partial X} = \mp \frac{\partial Z_{u,\ell}}{\partial X} = O(\varepsilon) \tag{4.14}$$

$$\nabla S_0 = O(1) \tag{4.15}$$

and

$$\tilde{S} = \pm \tilde{Z}_{u,\ell}(X,Y) = O(\varepsilon^2) \tag{4.16}$$

$$\frac{\partial \tilde{S}}{\partial X} = O(\varepsilon^2) \tag{4.17}$$

Assume also

$$\Omega = O(1) \tag{4.18}$$

This implies (see Eqs. E.25 and E.26) that

$$\phi_0 = O(\varepsilon) \tag{4.19}$$

$$\tilde{\phi} = O(\tilde{\varepsilon}) \tag{4.20}$$

Neglecting the terms which contain $e^{i2\Omega T}$ (which are of order ε^4) and separating the steady from the oscillatory terms, one obtains

$$-\nabla_{XYZ} S_0 \odot \nabla_{XYZ} \phi_0 + \frac{1}{B}\frac{\partial S_0}{\partial X} + \frac{M^2}{B^2}\frac{\partial S_0}{\partial X}\frac{\partial \phi_0}{\partial X} = 0 \tag{4.21}$$

$$-\nabla_{XYZ} S_0 \odot \nabla_{XYZ} \tilde{\phi} - \nabla_{XYZ}\tilde{S} \odot \nabla_{XYZ}\phi_0 + \frac{B}{M} i\Omega\tilde{S} + \frac{1}{B}\frac{\partial \tilde{S}}{\partial X} + \frac{M^2}{B^2}\left(\frac{\partial S_0}{\partial X}\frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X}\frac{\partial \phi_0}{\partial X}\right) = 0 \tag{4.22}$$

Introducing $\hat{\phi}$ such that

$$\tilde{\phi} = \hat{\phi}\, e^{-i\Omega M X} \tag{4.23}$$

Equation (4.22) reduces to

$$-\nabla_{XYZ} S_0 \odot \nabla_{XYZ} \hat{\phi}\, e^{-i\Omega MX} + i\Omega M \frac{\partial S_0}{\partial X}\hat{\phi}\, e^{-i\Omega MX} - \nabla_{XYZ}\tilde{S} \odot \nabla_{XYZ}\phi_0 + \frac{B}{M} i\Omega\tilde{S} + \frac{1}{B}\frac{\partial \tilde{S}}{\partial X} + \frac{M^2}{B^2}\left[\frac{\partial S_0}{\partial X}\left(\frac{\partial \hat{\phi}}{\partial X} - i\Omega M\hat{\phi}\right)e^{-i\Omega MX} + \frac{\partial \tilde{S}}{\partial X}\frac{\partial \phi_0}{\partial X}\right] = 0 \tag{4.24}$$

Finally, neglecting terms of order ε^2 in Eq. (E.21) and terms of order ε^3 in Eq. (E.24), one obtains

$$-\nabla_{xyz} S_o \odot \nabla_{xyz} \phi_o = -\frac{1}{B}\frac{\partial S_o}{\partial X} \tag{4.25}$$

$$-\nabla_{xyz} S_o \odot \nabla_{xyz} \hat{\phi} = -\left(i\frac{B\Omega}{M}\tilde{S} + \frac{1}{B}\frac{\partial \tilde{S}}{\partial X}\right) e^{i\Omega MX} \tag{4.26}$$

In particular, for

$$S = \pm\frac{1}{\ell}\left[z - z_{u,\ell}(x,y) - \tilde{z}_{u,\ell}(x,y)\, e^{i\omega t}\right] \tag{4.27}$$

(where the upper[lower] sign holds on the upper[lower] surface), one obtains

$$S_o = \pm\frac{1}{\ell}\left[z - z_{u,\ell}(x,y)\right] \tag{4.28}$$

$$\tilde{S} = \mp\frac{1}{\ell}\,\tilde{z}_{u,\ell}(x,y) \tag{4.29}$$

$$\frac{1}{|\nabla_{xyz} S_o|} = |N_z| = \pm N_z \tag{4.30}$$

and

$$\frac{\partial \hat{\phi}}{\partial N^c} = \frac{\nabla_{xyz} S_o \odot \nabla_{xyz}\hat{\phi}}{|\nabla_{xyz} S_o|} = -N_z\left(iK\frac{\tilde{z}}{\ell} + \frac{\partial \tilde{z}}{\partial x}\right) e^{i\Omega MX} \tag{4.31}$$

where

$$K = \frac{\beta\Omega}{M} = \frac{\omega \ell}{U_\infty} \tag{4.32}$$

4.3 Pressure Coefficient

The pressure coefficient can be evaluated by using the linearized Bernoulli theorem, as

$$c_p = -\frac{2}{U_\infty^2}\left(\frac{\partial \phi}{\partial t} + U_\infty \frac{\partial \phi}{\partial x}\right)$$

$$= -2\left(\frac{B}{M}\frac{\partial \phi}{\partial T} + \frac{1}{B}\frac{\partial \phi}{\partial X}\right) \tag{4.33}$$

For oscillatory flow, setting

$$\phi = \tilde{\phi}\, e^{i\Omega T} = \hat{\phi}\, e^{i\Omega(T - MX)} \tag{4.34}$$

$$c_p = \tilde{c}_p\, e^{i\Omega T} \tag{4.35}$$

one obtains

$$\tilde{c}_p = -2\left(\frac{B}{M}\, i\Omega\tilde{\phi} + \frac{1}{B}\frac{\partial \tilde{\phi}}{\partial X}\right)$$

$$= -2\left[i\Omega\left(\frac{B}{M} - \frac{M}{B}\right)\hat{\phi} + \frac{1}{B}\frac{\partial \hat{\phi}}{\partial X}\right] e^{-i\Omega MX}$$

$$= -\frac{2}{B}\left[-\frac{i\Omega}{M}\hat{\phi} + \frac{\partial \hat{\phi}}{\partial X}\right] e^{i\Omega MX}$$

$$= -\frac{2}{B}\left[e^{i\Omega X/M}\frac{\partial}{\partial X}\left(\hat{\phi}\, e^{-i\Omega X/M}\right)\right] e^{-i\Omega MX}$$

$$= -\frac{2}{B}\, e^{-i\Omega B^2 X/M}\frac{\partial}{\partial X}\left(\hat{\phi}\, e^{-i\Omega X/M}\right)$$

$$= -\frac{2}{B}\, e^{-ikBX}\frac{\partial}{\partial X}\left(\hat{\phi}\, e^{-ikX/B}\right) \tag{4.36}$$

SECTION 5

NUMERICAL RESULTS

5.1 Introduction

The formulation presented in the previous sections was imbedded into a computer program, called SOSSA ACTS (Steady and Oscillatory, Subscnic and Supersonic Aerodynamic for Aerospace Complex Transportation Systems). Typical results of this computer program are presented in this section.

5.2 Rectangular Wing in Both Steady and Oscillatory Flow

The results in Figs. 2 to 4 are relative to a rectangular wing with aspect ratio AR = 3 and with a biconvex circular arc section, 5% thickness, with sharp leading and trailing edges. Fig. 2 shows the distribution of the pressure coefficient C_p on the lower and upper surfaces of the wing with $\alpha = 0°$ and M = 1.3. Fig 3a shows the distribution of the lift coefficient on the wing with $\alpha = 5°$ and M = 1.3, while Fig. 3b shows the distribution of C_p on the lower and upper surfaces of the wing with $\alpha = 5°$ and M = 1.3; these results are obtained with NX = NY = 7. Fig. 4 shows the distributions of the absolute values and phase angles of the lift coefficient C_ℓ, of the same wing oscillating in bending mode

$$Z = 0.18043\,|2y/b| + 1.70255\,|2y/b|^2 - 1.13688\,|2y/b|^3 + .25387\,|2y/b|^4$$

with $K = \omega c/2U_\infty$ = .1, M = 1.3 and NX = NY = 10. All the above results are compared with the ones obtained by Lessing, Troutman and Menees (Ref. 4). The results obtained for Fig. 2 are also compared with the analytical two dimensional solution which can be easily evaluated. For, this problem can be treated as a two dimensional problem in the central region of the wing. For this case the pressure coefficient is given by (Ref. 4)

$$c_p = \frac{2}{\beta}\frac{\partial Z}{\partial X} \tag{5.1}$$

For biconvex circular-arc wing, the equation of wing section is approximately given as

$$Z = -2\tau\,(X^2 - 0.25) \qquad , \quad (-0.5 \le X \le 0.5) \tag{5.2}$$

Noting that $\tau = 0.05$ and $\beta = 0.83$, one obtains

$$c_p = \frac{2}{\beta}\frac{\partial Z}{\partial X} = -0.48\ X \tag{5.3}$$

i.e. c_p varies from -0.24 to 0.24 linearly. The results shown in Fig. 2 are in excellent agreement with Eq. 5.3.

5.3 Convergence Analysis

The convergence analysis of the problem considered in Figs. 2 and 4, is presented here. The distribution of the velocity potential along $Y/b = 0$ for the problem for Fig. 2 (for different numbers of elements) is shown in Fig. 5. The curves are obtained with NX = NY = 5,6 and 7. These curves, indicate that the convergence is very fast and that 144 elements on the whole wing, or NX = NY = 6 are sufficient for an accurate analysis. For oscillatory flows, the distributions of the real and imaginary parts of the velocity potential along $2Y/b = 0.5$ for the problem for Fig. 4 are shown in Fig. 6. The curves are obtained with NX = NY = 5, 6, and 7. From these curves, it is safe to say that 144 elements (i.e. NX = NY = 6) are sufficient for an accurate analysis.

5.4 Delta Wing with Supersonic Leading Edge

Fig. 7 shows the distribution of lift coefficient per unit angle of attack for a delta wing with supersonic leading edge and

$$m = B/\tan\Lambda = 1.2$$

where Λ is the sweep angle of the leading edge. The results obtained with NX = 8, NY = 12 and M = 1.2 are compared with the exact solution which is given by (Ref. 5).

$$\Delta C_p = \frac{4\alpha}{\pi B}\frac{m}{\sqrt{m^2-1}} R_e\left[\cos^{-1}\frac{1-m\gamma}{m-\gamma} + \cos^{-1}\frac{1+m\gamma}{m+\gamma}\right]$$

where $\gamma = By/x$. The numerical results obtained are remarkably accurate.

5.5 Wing-body Configuration

The present method is general enough to extend to any arbitrary configuration. Following is an example of this application. A wing-body combination in supersonic flow with M = 1.48 is considered in Figs. (8a) and (8b). The combination is composed of a wing with chord C = 3, span S = 9, thickness τ = 0.05, a forebody of length LA = 6.0 and radius varying from 0.0 to 0.75 linearly and a midsection of length L_m = 3.0 and radius r = 0.75. Wake and aftbody are not considered. The angle of attack of the wing is α = 1.92, while the angle of attack of the body is $\alpha_B = 0$. To obtain the results, 580 elements on the whole configuration (NX = NY = 10 on the wing, NX = 5, NY = 3 on the body, NX = 10, NY = 3 on the middle section) are used. In Fig. (8a), the distributions of the lift coefficient per unit wing angle of attack along chordwise direction are presented. The curves are plotted at different values of y/r, and are compared with the experimental, as well as analytical, ones obtained by Nielsen (Ref. 6) and Woodware, Tincoco and Larsen (Ref. 7). In Fig. 8b, the distributions of the same quantity along fuselage at different meridian angles are shown.

5.6 Computer Time

All the above results are obtained on the IBM 370/145 available at the Boston University Computer Center. The computer time for the problem for Figs. 5 and 6 are given in Tables 1 and 2 respectively, where N_T is the total number of the elements on the whole wing. N_{eq} is the number of equations to be solved (by using IBM Subroutine GELG), and N_d is the number of diaphragm element.

TABLE 1 Steady Case

N_T	NX, NY	Neg.	N_d	Computing Time
64	4	28	12	11 sec.
100	5	40	15	22.8sec.
144	6	54	18	42.5sec.
196	7	70	21	140.5sec.

TABLE 2 Oscillatory Case

N_T	NX, NY	Neg.	N_d	Computing Time
64	4	25	9	21.4 sec.
100	5	34	9	38.0 sec.
144	6	45	9	65.4 sec.
196	7	58	9	118.9 sec.

It should be mentioned that the advantage of symmetry with respect to z and y was taken. Therefore, the number of unknowns, or equations, to be solved is only one-fourth of the number of the elements on the whole wing plus the number of diaphragm elements.

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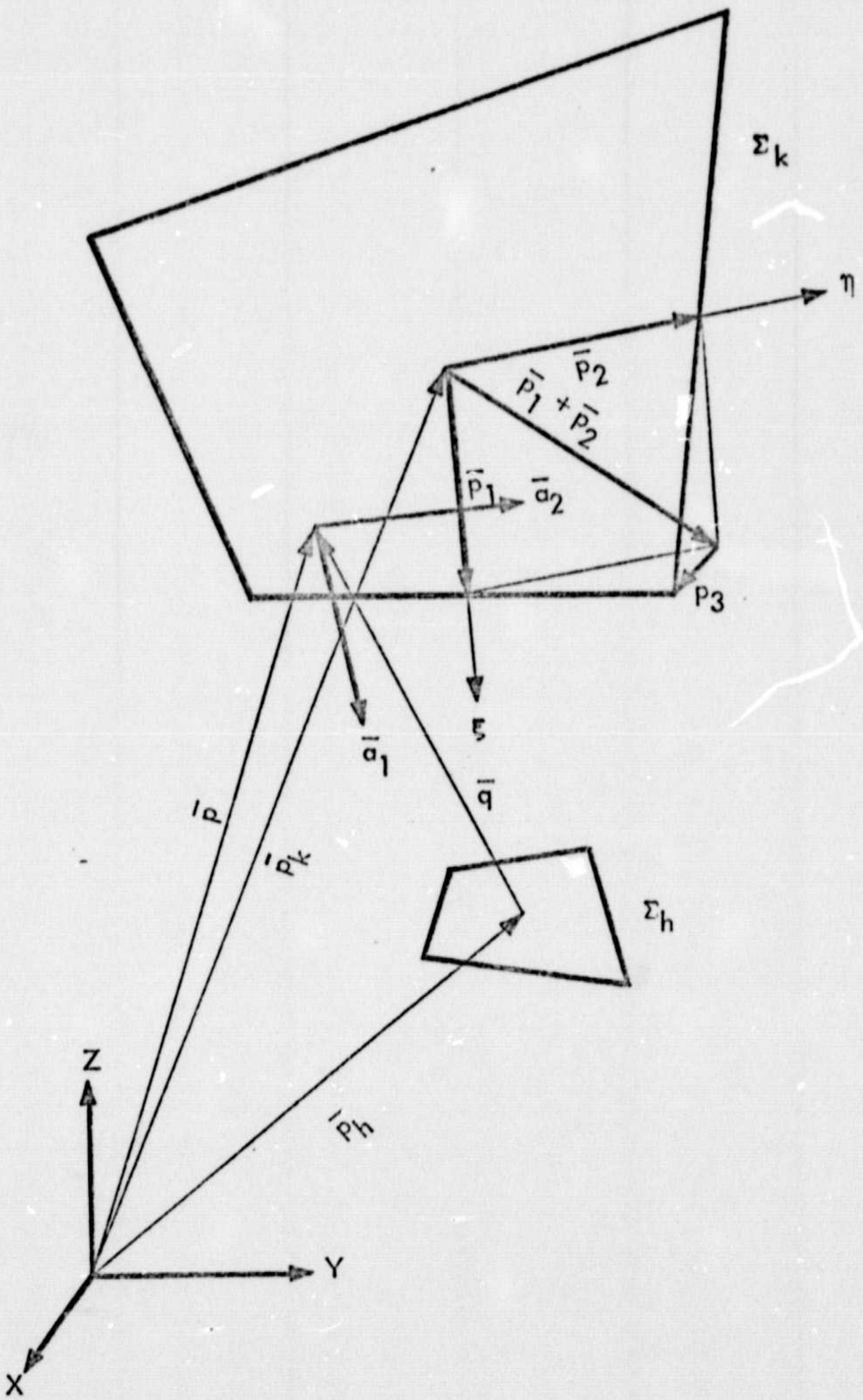

Figure 1. Hyperboloidal Element

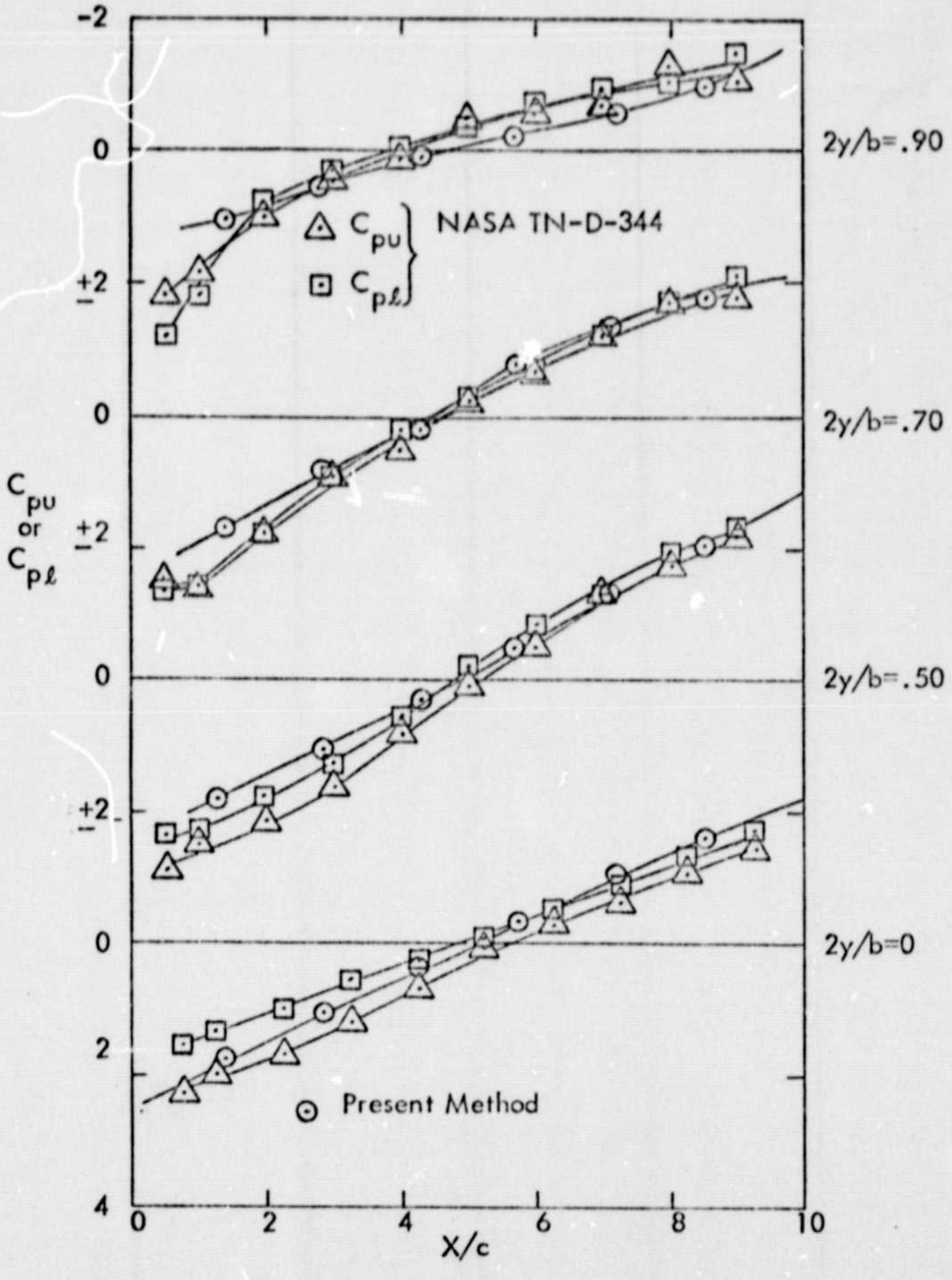

Figure 2 . The pressure distribution on a symmetric rectangular wing with AR = 3, τ = 5%, α = 0°, M = 1.3 and NX = NY = 7 for the comparison with results of Ref. 4.

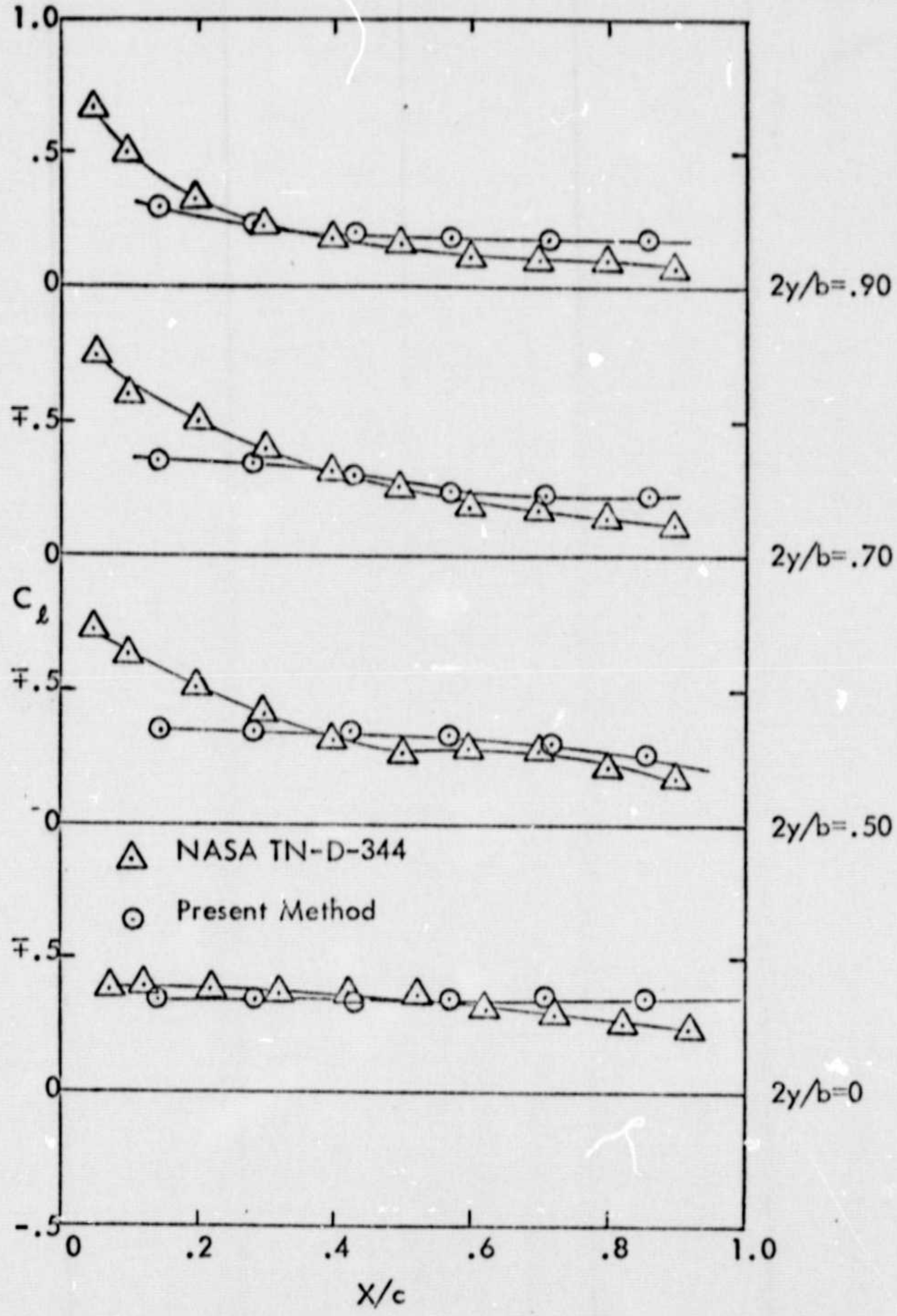

Figure 3a . The lift distribution on symmetric rectangular wing with AR = 3, τ = 5%, α = 5°, M = 1.3 and NX = NY = 7 for the comparison with results of Ref. 4.

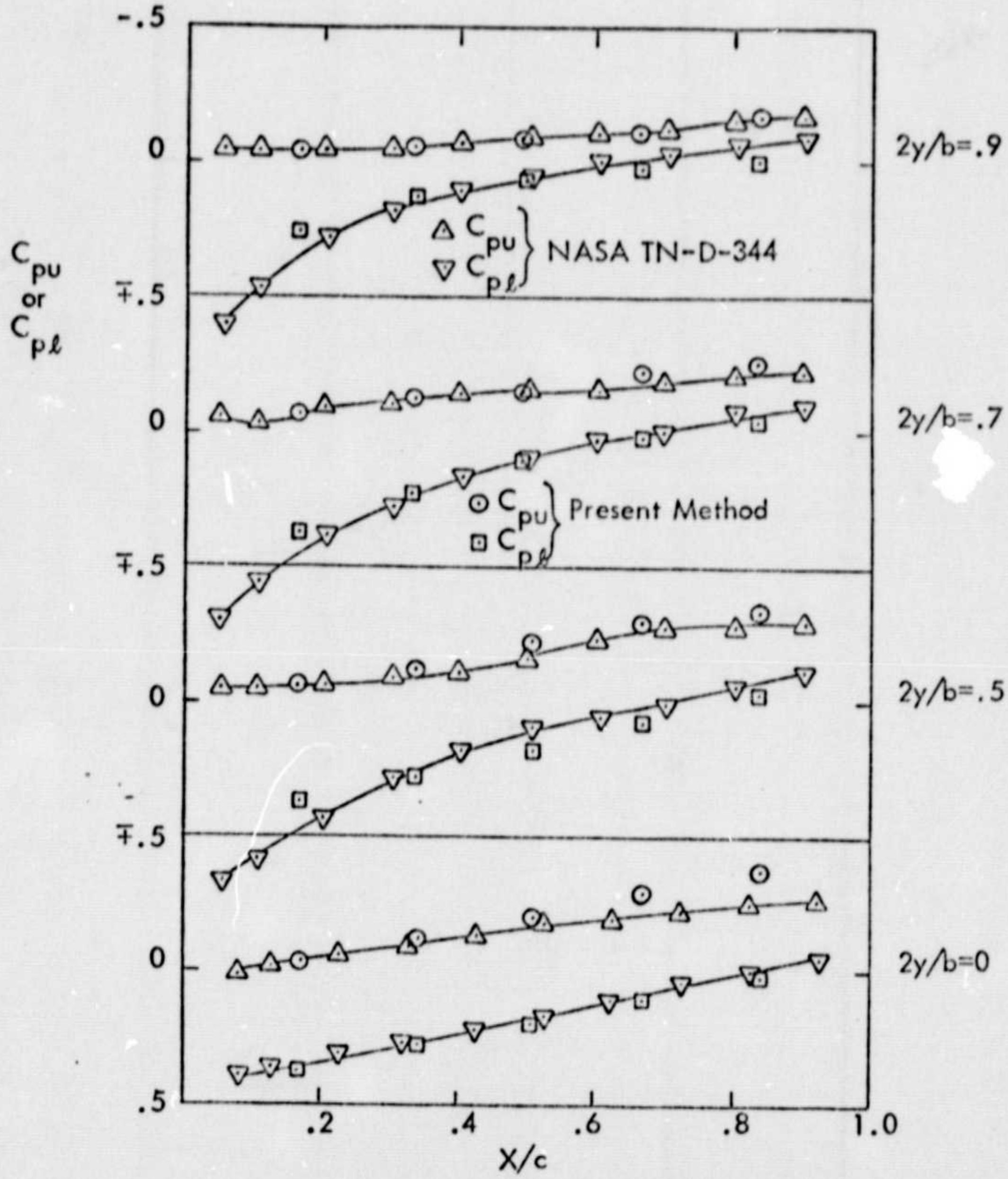

Figure 3b. The distribution of C_p on the upper and lower surfaces of a symmetric rectangular wing with AR = 3, τ = 5%, $\alpha = 5^o$, M = 1.3 and NX = NY = 6 for comparison with results of Ref. 4.

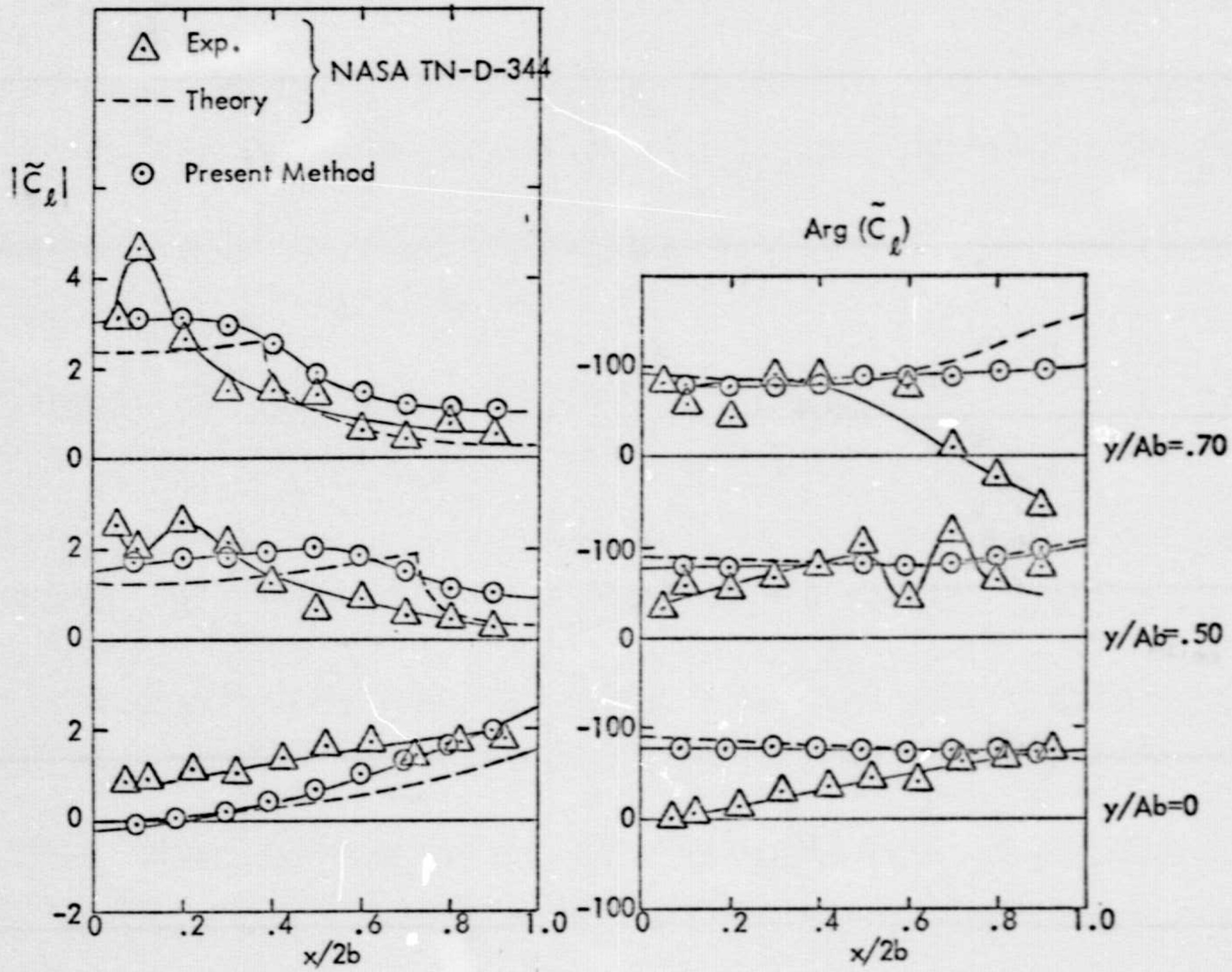

Figure 4 . Lift coefficient, $\tilde{C}_l$, for a rectangular wing oscillating in bending mode with $k = \omega c/2U_\infty = 0.1$, $M = 1.3$, $AR = 3$ and $NX = NY = 10$.

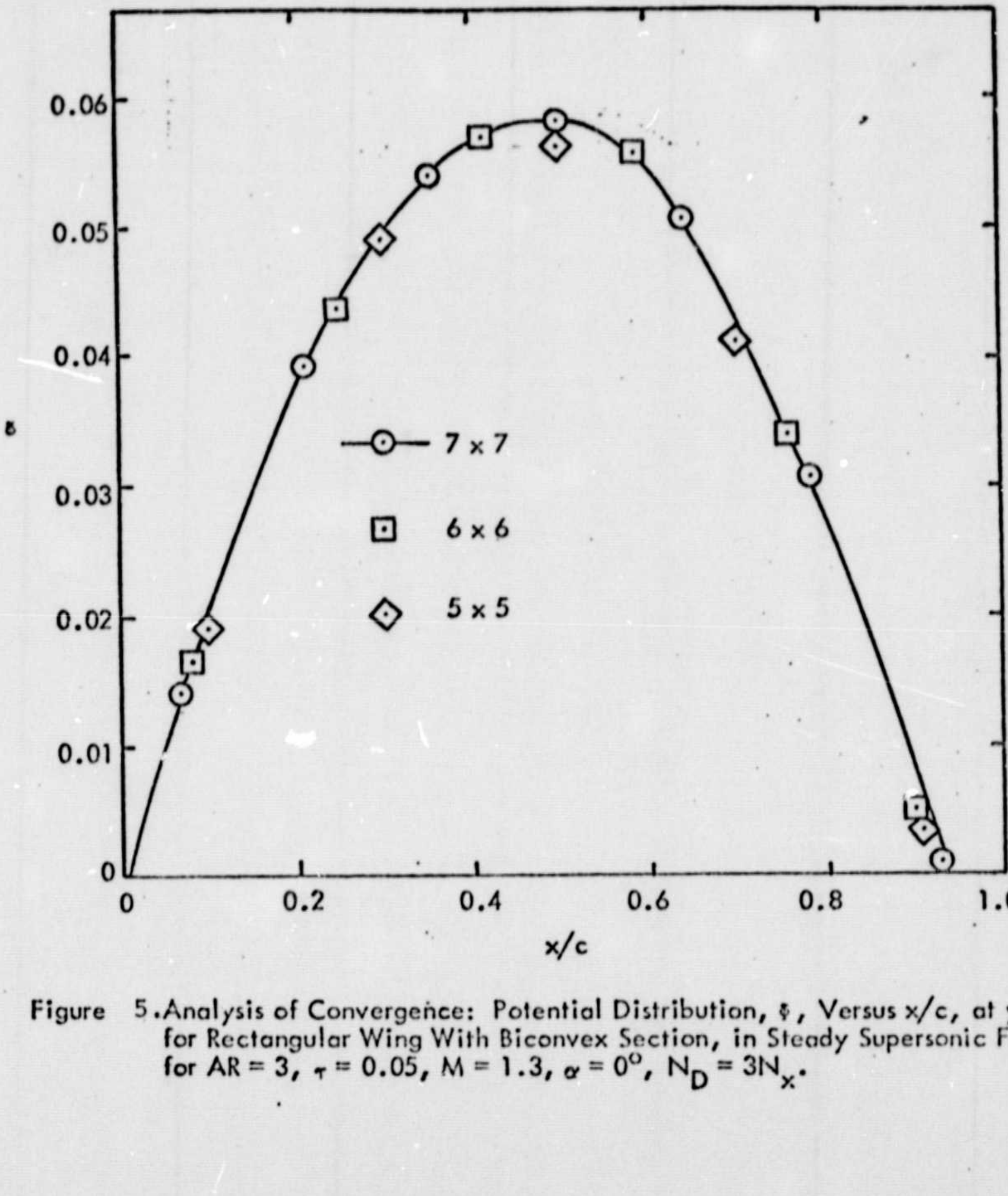

Figure 5. Analysis of Convergence: Potential Distribution, ϕ, Versus x/c, at y = 0, for Rectangular Wing With Biconvex Section, in Steady Supersonic Flow, for AR = 3, $\tau = 0.05$, M = 1.3, $\alpha = 0^{\circ}$, $N_D = 3N_x$.

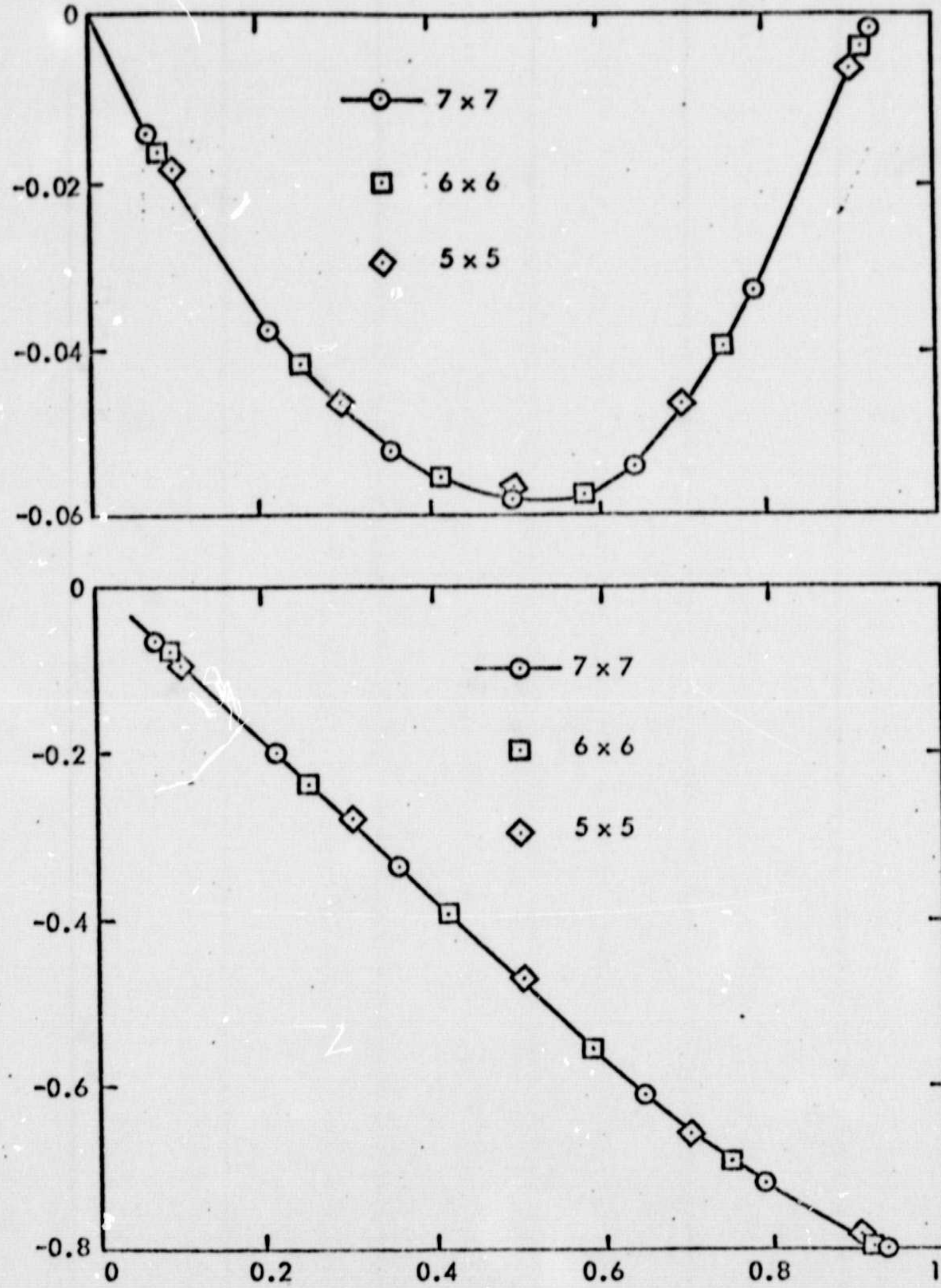

Figure 6. Analysis of Convergence: Distribution of $\hat{\phi} = \tilde{\phi} e^{i\Omega MX}$ Versus x/c, at 2y/b = 0.5, for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Supersonic Flow, for AR = 3, τ = 0.01, M = 1.3, K = 0.1, $\alpha = 0^o$, $N_D = 3N_x$.

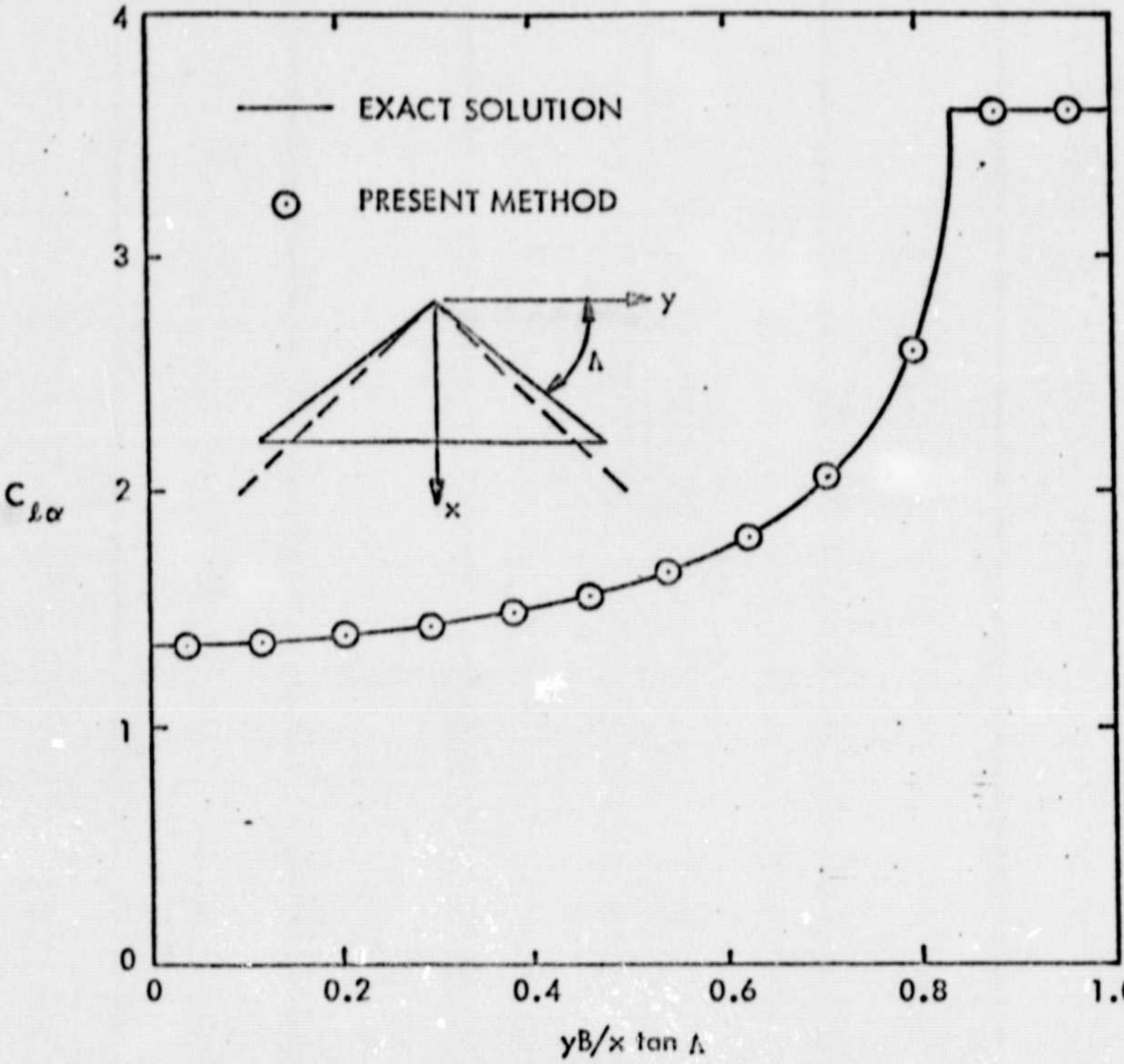

Figure 7. Lift Distribution Coefficient, $C_{\ell\alpha}$, for Delta Wing With Supersonic Trailing Edge, in Steady Supersonic Flow, With $B/\tan \Lambda = 1.2$, $\tau = 0$, $N_x = 8$, $N_y = 12$. Comparison With Exact, Conical-Flow Solution, Reference 5.

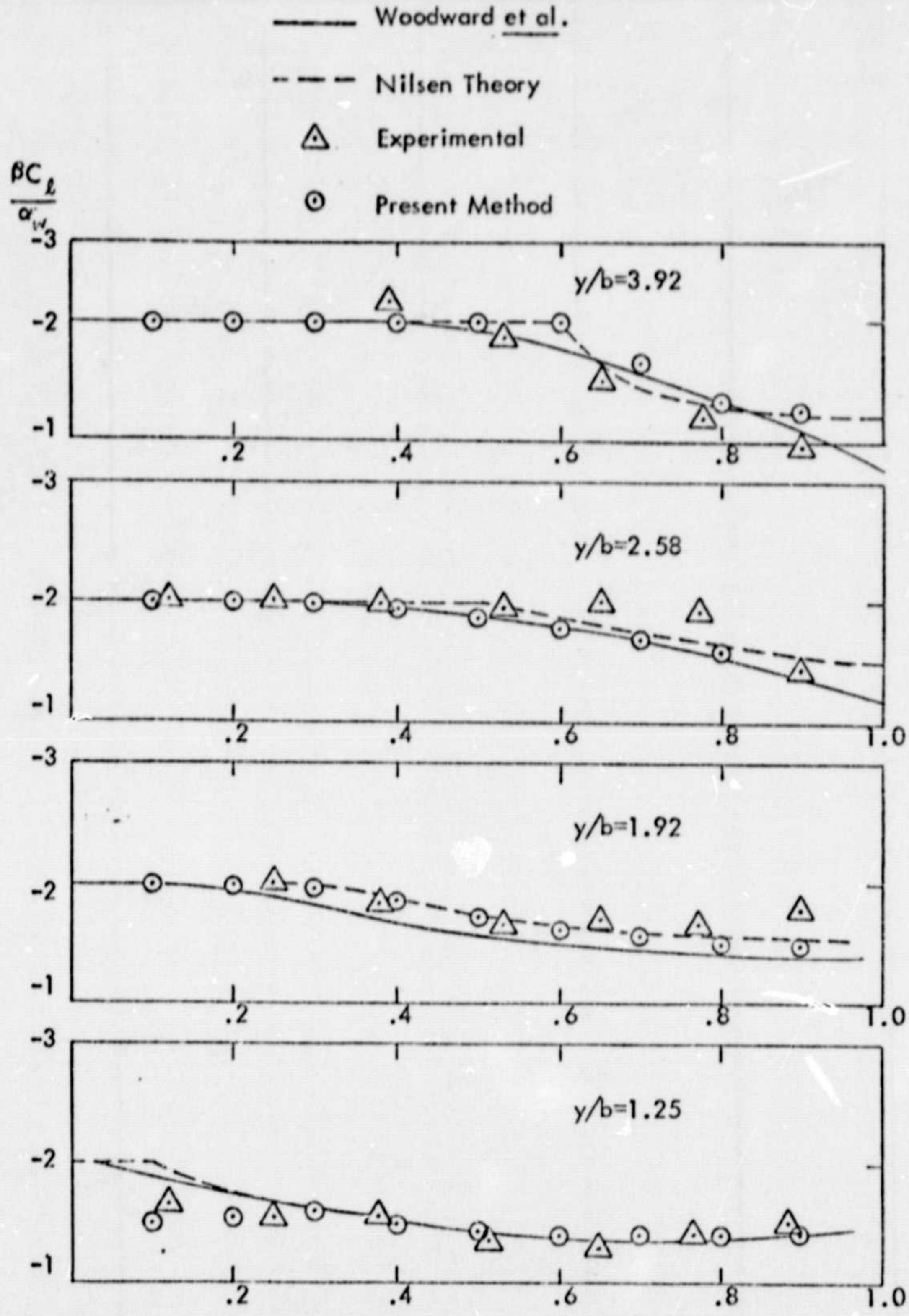

Figure 8a. The distribution of $\beta C_\ell/\alpha w$ on the wing section for a wing-body configuration (shown in Figure 27c) with M = 1.48, α_w = 1.92° and α_B = 0°, for the comparison with results of Ref. 21.

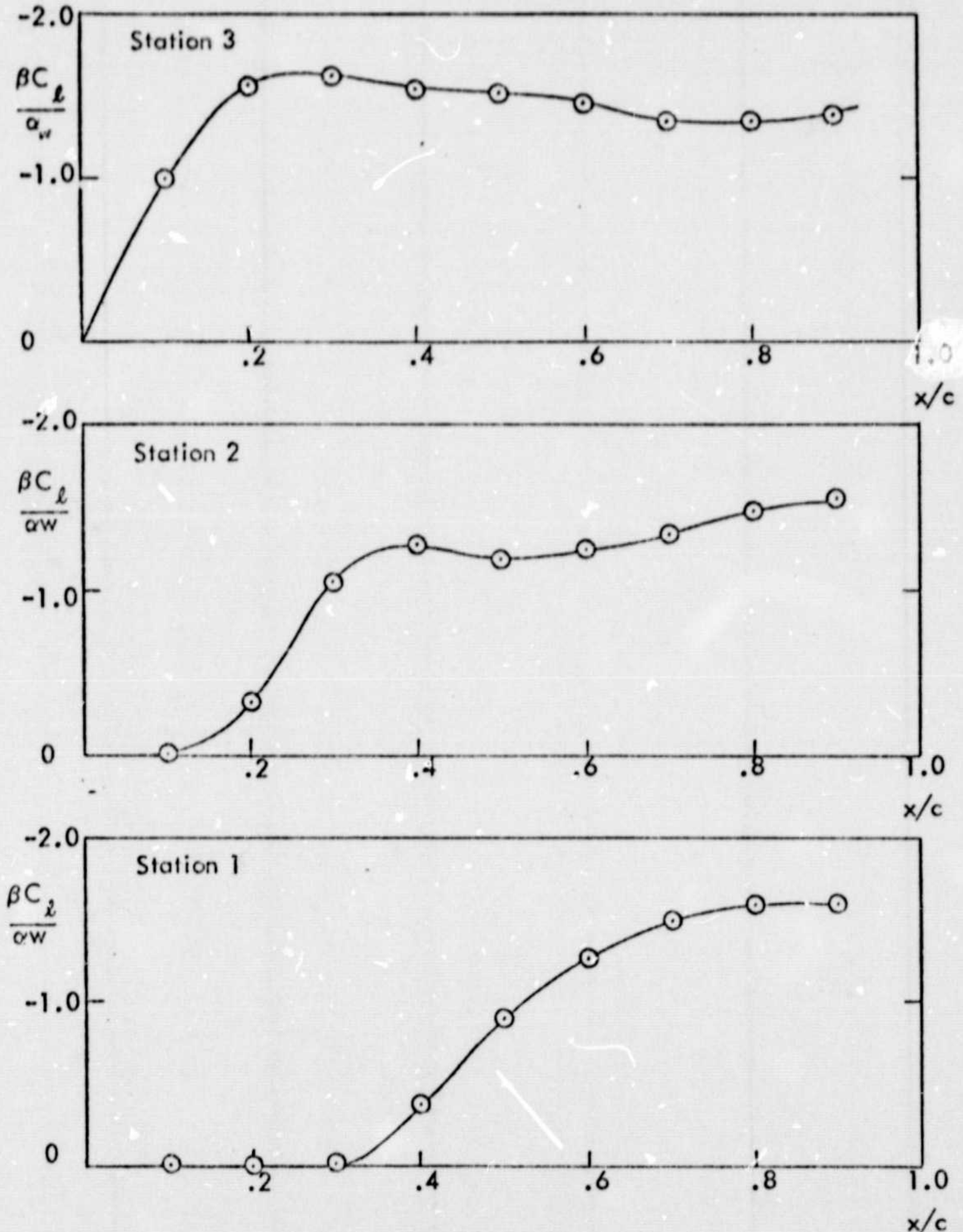

Figure 8b. The distribution of $\beta C_\ell/\alpha_w$ on the fuselage at three circumferential stations for the same problem of Figure 27a.

APPENDIX A

SUPERALGEBRA

A.1 Super-product

As mentioned in Section 1, in order to simplify the algebraic manipulation for the supersonic flow theory, it is convenient to introduce a special algebra, called supersonic vector algebra or super-algebra. In addition to the rules of the ordinary vector algebra, the super-algebra includes a supersonic dot product or super-product

$$\bar{a} \odot \bar{b} = a_x b_x - a_y b_y - a_z b_z \qquad . \tag{A.1}$$

The additive and distributive rules are obviously valid for the super-product. Note that $\bar{a} \odot \bar{a}$ is

$$\bar{a} \odot \bar{a} \gtreqless 0 \qquad \text{for} \quad a_x \gtreqless \sqrt{a_y^2 + a_z^2} \tag{A.2}$$

that is for $\bar{a}$ pointed, respectively, inside, on, outside the Mach cone (Fig. 1). Hence, in addition to the ordinary norm of a vector (or dot-norm)

$$|a| = \sqrt{\bar{a} \cdot \bar{a}} \tag{A.3}$$

it is convenient to introduce the supersonic norm (or super-norm)

$$\| a \| = \sqrt{|\bar{a} \odot \bar{a}|} \tag{A.4}$$

Finally, it is convenient to introduce the concept of covector

$$\bar{a}^c = \begin{Bmatrix} a_x \\ -a_y \\ -a_z \end{Bmatrix} \tag{A.5}$$

With these notations, it is immediately verified that

$$\bar{a} \odot \bar{b} = \bar{a}^c \cdot \bar{b} = \bar{a} \cdot \bar{b}^c \tag{A.6}$$

It may be worth noting that

$$\bar{a} \cdot \bar{b} \times \bar{c} = \bar{a}^c \odot \bar{b} \times \bar{c} = \bar{a} \odot (\bar{b} \times \bar{c})^c = \bar{a} \odot b^c \times c^c \tag{A.6a}$$

A.2 First Super-rule

Throughout the subsonic finite-element formulation (Ref. 6) the following rule is used

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c}) \tag{A.7}$$

The corresponding supersonic rule, called for convenience, first super-rule, is also valid

$$(\bar{a} \times \bar{b}) \odot (\bar{c} \times \bar{d}) = (\bar{a} \odot \bar{c})(\bar{b} \odot \bar{d}) - (\bar{a} \odot \bar{d})(\bar{b} \odot \bar{c}) \tag{A.8}$$

For

$$
\begin{aligned}
&(\bar{a} \times \bar{b}) \odot (\bar{c} \times \bar{d}) \\
&= (a_y b_z - a_z b_y)(c_y d_z - c_z d_y) \\
&\quad - (a_z b_x - a_x b_z)(c_z d_x - c_x d_z) \\
&\quad - (a_x b_y - a_y b_x)(c_x d_y - c_y d_x) \\
&= a_y b_z c_y d_z + a_z b_y c_z d_y - a_y b_z c_z d_y - a_z b_y c_y d_z \\
&\quad - a_z b_x c_z d_x - a_x b_z c_x d_z + a_z b_x c_x d_z + a_x b_z c_z d_x \\
&\quad - a_x b_y c_x d_y - a_y b_x c_y d_x + a_x b_y c_y d_x + a_y b_x c_x d_y
\end{aligned} \tag{A.9}
$$

while

$$(\bar{a}\circ\bar{c})(\bar{b}\circ\bar{d})-(\bar{a}\circ\bar{d})(\bar{b}\circ\bar{c})$$

$$= (a_x c_x - a_y c_y - a_z c_z)(b_x d_x - b_y d_y - b_z d_z)$$

$$- (a_x d_x - a_y d_y - a_z d_z)(b_x c_x - b_y c_y - b_z c_z)$$

$$= a_x c_x (\cancel{b_x d_x} - b_y d_y - b_z d_z)$$

$$- a_y c_y (b_x d_x - \cancel{b_y d_y} - b_z d_z)$$

$$- a_z c_z (b_x d_x - b_y d_y - \cancel{b_z d_z})$$

$$- a_x d_x (\cancel{b_x c_x} - b_y c_y - b_z c_z)$$

$$+ a_y d_y (b_x c_x - \cancel{b_y c_y} - b_z c_z)$$

$$+ a_z d_z (b_x c_x - b_y c_y - \cancel{b_z c_z})$$

$$= a_y b_z c_y d_z + a_z b_y c_z d_y - a_y b_z c_z d_y - a_z b_y c_y d_z$$

$$- a_z b_x c_z d_x - a_x b_z c_z d_z + a_z b_x c_x d_z + a_x b_z c_z d_x$$

$$- a_x b_y c_y d_y - a_y b_x c_y d_x + a_x b_y c_y d_x + a_y b_x c_x d_y \qquad \text{(A.10)}$$

A.3 Second Super-rule

A second rule of the super-algebra is

$$(\bar{a} \circ \bar{a}')(\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b} \times \bar{c})(\bar{a}' \cdot \bar{b} \times \bar{c})$$

$$= \bar{a} \circ \bar{c}\, (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{a}') + \bar{a} \circ \bar{b}\, (\bar{c} \times \bar{b}) \circ (\bar{c} \times \bar{a}')$$

$$= \bar{a}' \circ \bar{c}\, (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{a}) + \bar{a}' \circ \bar{b}\, (\bar{c} \times \bar{b}) \circ (\bar{c} \times \bar{a}) \tag{A.11}$$

Note that the dot product appears in the triple product. In order to prove Eq. (A.11), consider the regular vector algebra rule

$$\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}\, (\bar{a} \cdot \bar{c}) - \bar{c}\, (\bar{a} \cdot \bar{b})$$

This yields, for the covector $\bar{a}^c$, (see Eq. A.6)

$$\bar{a}^c \times (\bar{b} \times \bar{c}) = \bar{b}\, (\bar{a}^c \cdot \bar{c}) - \bar{c}\, (\bar{a}^c \cdot \bar{b})$$

$$= \bar{b}\, (\bar{a} \circ \bar{c}) - \bar{c}\, (\bar{a} \circ \bar{b}) \tag{A.12}$$

On the other hand, according to Eqs. (A.6) and (A.8)

$$\bar{a}^c \times (\bar{b} \times \bar{c}) \circ \bar{a}'^c \times (\bar{b} \times \bar{c})$$

$$= \bar{a}^c \circ \bar{a}'^c\, (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - [\bar{a}^c \circ (\bar{b} \times \bar{c})][\bar{a}'^c \circ (\bar{b} \times \bar{c})]$$

$$= \bar{a} \circ \bar{a}'\, (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b} \times \bar{c})(\bar{a}' \cdot \bar{b} \times \bar{c}) \tag{A.13}$$

while, according to Eqs. (A.6) and (A.8)

$$[\bar{b}\, (\bar{a} \circ \bar{c}) - \bar{c}\, (\bar{a} \circ \bar{b})] \circ [\bar{b}\, (\bar{a}' \circ \bar{c}) - \bar{c}\, (\bar{a}' \circ \bar{b})]$$

$$= \bar{b} \circ \bar{b}\ \bar{a} \circ \bar{c}\ \bar{a}' \circ \bar{c} - \bar{b} \circ \bar{c}\ \bar{a} \circ \bar{c}\ \bar{a}' \circ \bar{b}$$

$$= - \bar{c} \circ \bar{b}\ \bar{a} \circ \bar{b}\ \bar{a}' \circ \bar{c} + \bar{c} \circ \bar{c}\ \bar{a} \circ \bar{b}\ \bar{a}' \circ \bar{b}$$

$$= \bar{a} \circ \bar{c}\, (\bar{b} \times \bar{c} \circ \bar{b} \times \bar{a}') + \bar{a} \circ \bar{b}\, (\bar{c} \times \bar{b} \circ \bar{c} \times \bar{a}')$$

$$= \bar{a}' \circ \bar{c}\ \bar{b} \times \bar{c} \circ \bar{b} \times \bar{a} + \bar{a}' \circ \bar{b}\, (\bar{c} \times \bar{b} \circ \bar{c} \times \bar{a}) \tag{A.14}$$

Combining Eqs. (A.12), (A.13) and (A.14) yields

$$\bar{a}\circ\bar{a}\,(\bar{b}\times\bar{c})\circ(\bar{b}\times\bar{c}) - (\bar{a}\cdot\bar{b}\times\bar{c})(\bar{d}\cdot\bar{b}\times\bar{c})$$

$$= [\bar{a}^c\times(\bar{b}\times\bar{c})]\circ[\bar{d}^c\times(\bar{b}\times\bar{c})]$$

$$= [\bar{b}(\bar{a}\circ\bar{c}) - \bar{c}(\bar{a}\circ\bar{b})]\odot[\bar{b}(\bar{d}\circ\bar{c}) - \bar{c}(\bar{d}\circ\bar{b})]$$

$$= \bar{a}\circ\bar{c}\,(\bar{b}\times\bar{c})\odot(\bar{b}\times\bar{d}) + \bar{a}\circ\bar{b}\,(\bar{c}\times\bar{b})\odot(\bar{c}\times\bar{d})$$

$$= \bar{d}\odot c\,(\bar{b}\times c)\odot(\bar{b}\times\bar{a}) + \bar{d}\odot\bar{b}\,(\bar{c}\times\bar{b})\odot(\bar{c}\times\bar{a}) \qquad \text{(A.15)}$$

that is the second super-rule, Eq. (A.11). In particular, for $\bar{a}\equiv\bar{d}\equiv\bar{q}$, $\bar{b}\equiv\bar{a}_1$, $\bar{c}\equiv\bar{a}_2$, one obtains

$$\bar{q}\circ\bar{q}\;\bar{a}_1\times\bar{a}_2\circ\bar{a}_1\times\bar{a}_2 - |\bar{q}\cdot\bar{a}_1\times\bar{a}_2|^2$$

$$= \bar{q}\circ\bar{a}_2\;\bar{a}_1\times\bar{a}_2\circ\bar{a}_1\times\bar{q} + \bar{q}\circ\bar{a}_1\,(\bar{a}_2\times\bar{a}_1\circ\bar{a}_2\times\bar{q}) \qquad \text{(A.16)}$$

A.4 Third Supersonic Rule

A third useful formula, called the third supersonic rule, is

$$\bar{a}\times\bar{d}\circ\bar{b}\times\bar{c}\;\;\bar{a}\cdot\bar{f}\times\bar{g} + \bar{a}\times\bar{f}\circ\bar{b}\times\bar{c}\;\;\bar{a}\cdot\bar{g}\times\bar{d}$$
$$+\,\bar{a}\times\bar{g}\circ\bar{b}\times\bar{c}\;\;\bar{a}\times\bar{d}\cdot\bar{f} = 0 \qquad \text{(A.17)}$$

The proof of this rule follows

$$(\bar{a}\times\bar{d}\circ\bar{b}\times\bar{c})(\bar{a}\cdot\bar{f}\times\bar{g}) - (\bar{a}\times\bar{g}\circ\bar{b}\times\bar{c})(\bar{a}\cdot\bar{f}\times\bar{d})$$

$$= (\bar{a}\circ\bar{b}\;\bar{c}\circ\bar{d} - \bar{a}\circ\bar{c}\;\bar{b}\circ\bar{d})\;\bar{a}\times\bar{f}\cdot\bar{g}$$

$$-\,(\bar{a}\circ\bar{b}\;\bar{c}\circ\bar{g} - \bar{a}\circ\bar{c}\;\bar{b}\circ\bar{g})\;\bar{a}\times\bar{f}\cdot\bar{d}$$

$$= \bar{a}\circ\bar{b}\,(\bar{c}\circ\bar{d}\;\bar{a}\times\bar{f}\cdot\bar{g} - \bar{c}\circ\bar{g}\;\bar{a}\times\bar{f}\cdot\bar{d})$$

$$-\,\bar{a}\circ\bar{c}\,(\bar{b}\circ\bar{d}\;\bar{a}\times\bar{f}\cdot\bar{g} - \bar{b}\circ\bar{g}\;\bar{a}\times\bar{f}\cdot\bar{d})$$

$$= \bar{a}\cdot\bar{b}^c\,(\bar{c}^c\cdot\bar{d}\ \bar{a}\times\bar{f}\cdot\bar{g} - \bar{c}^c\cdot\bar{g}\ \bar{a}\times\bar{f}\cdot\bar{d})$$
$$- \bar{a}\cdot\bar{c}^c\,(\bar{b}^c\cdot\bar{d}\ \bar{a}\times\bar{f}\cdot\bar{g} - \bar{b}^c\cdot\bar{g}\ \bar{a}\times\bar{f}\cdot\bar{d})$$
$$= \bar{a}\cdot\bar{b}^c[\bar{c}^c\times(\bar{a}\times\bar{f})]\cdot\bar{d}\times\bar{g} - \bar{a}\cdot\bar{c}^c[\bar{b}^c\times(\bar{a}\times\bar{f})]\cdot\bar{d}\times\bar{g}$$
$$= \bar{a}\cdot\bar{b}^c[(\bar{c}^c\cdot\bar{f})(\bar{a}\cdot\bar{d}\times\bar{g}) - (\bar{c}^c\cdot\bar{a})(\bar{f}\cdot\bar{d}\times\bar{g})]$$
$$- \bar{a}\cdot\bar{c}^c[(\bar{b}^c\cdot\bar{f})(\bar{a}\cdot\bar{d}\times\bar{g}) - (\bar{b}^c\cdot\bar{a})(\bar{f}\cdot\bar{d}\times\bar{g})]$$
$$= (\bar{a}\cdot\bar{b}^c\ \bar{c}^c\cdot\bar{f} - \bar{a}\cdot\bar{c}^c\ \bar{b}^c\cdot\bar{b})\,\bar{a}\cdot\bar{d}\times\bar{g}$$
$$= (\bar{a}\circ\bar{b}\ \bar{c}\circ\bar{f} - \bar{a}\circ\bar{c}\ \bar{b}\circ\bar{f}) = (\bar{a}\times\bar{f}\circ\bar{b}\times\bar{c})\,\bar{a}\cdot\bar{d}\times\bar{g}$$

(A.18)

In particular, for $\bar{a} = \bar{b} = \bar{q}$, $\bar{c} = \bar{f} = \bar{a}_2$, $\bar{d} = \bar{p}_3$, $\bar{g} = \bar{a}_1$, the third superrule reduces to

$$(\bar{q}\times\bar{p}_3\circ\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_2\times\bar{a}_2) - (\bar{q}\times\bar{a}_1\circ\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_2\times\bar{p}_3)$$
$$= (\bar{q}\times\bar{a}_2\circ\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{p}_3\times\bar{a}_1) \qquad \text{(A.19)}$$

A.5 Fourth Supersonic Rule

In subsonic theory, it is easy to show that

$$\bar{q}\cdot\bar{q}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2 + (\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2)^2 \equiv |\bar{q}\times\bar{a}_1|^2\,|\bar{q}\times\bar{a}_2|^2 \qquad \text{(A.20)}$$

The corresponding equation for supersonic theory is called fourth supersonic rule

$$\bar{q}\circ\bar{q}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2 + (\bar{q}\times\bar{a}_1\circ\bar{q}\times\bar{a}_2)^2 \equiv \|\bar{q}\times\bar{a}_1\|^2\,\|\bar{q}\times\bar{a}_2\|^2$$

(A.21)

The proof is shown as following, according to the second super-rule.

$$
\begin{aligned}
&\bar q \odot \bar q\,(\bar q \cdot \bar a_1 \times \bar a_2)^2 + (\bar q \times \bar a_1 \odot \bar q \times \bar a_2)^2 \\
&= \bar q \odot \bar q\,[\,\bar q \odot \bar q\,(\bar a_1 \times \bar a_2 \odot \bar a_1 \times \bar a_2 - \bar q \odot \bar a_2\;\bar a_1 \times \bar a_2 \odot \bar a_1 \times \bar q \\
&\quad - \bar q \odot \bar a_1\;\bar a_2 \times \bar a_1 \odot \bar a_2 \times \bar q)] + (\bar q \odot \bar q\;\bar a_1 \odot \bar a_2 - \bar q \odot \bar a_1\;\bar q \odot \bar a_2)^2 \\
&= (\bar q \odot \bar q)[\bar a_1 \odot \bar a_1\;\bar a_2 \odot \bar a_2 - (\bar a_1 \odot \bar a_2)^2] - (\bar q \odot \bar q)(\bar q \odot \bar a_2) \times \\
&\quad [\bar a_1 \odot \bar a_1\;\bar a_2 \odot \bar q - \bar a_1 \odot \bar q\;\bar a_2 \odot \bar a_1] - \bar q \odot \bar q\;\bar q \odot \bar a_1\,(\bar a_2 \odot \bar a_2\;\bar a_1 \odot \bar q - \bar a_2 \odot \bar q\;\bar a_1 \odot \bar a_2) \\
&\quad + (\bar q \odot \bar q)^2(\bar a_1 \odot \bar a_2)^2 - 2(\bar q \odot \bar q)(\bar a_1 \odot \bar a_2)(\bar q \odot \bar a_1)(\bar q \odot \bar a_2) \\
&\quad + (\bar q \odot \bar a_1\;\bar q \odot \bar a_2)^2 \\
&= [\bar q \odot \bar q\;\bar a_1 \odot \bar a_1 - (\bar q \odot \bar a_1)^2][\bar q \odot \bar q\;\bar a_2 \odot \bar a_2 - (\bar q \odot \bar a_2)^2] \\
&= (\bar q \times \bar a_1 \odot \bar q \times \bar a_1)(\bar q \times \bar a_2 \odot \bar q \times \bar a_2) \\
&= \|\bar q \times \bar a_1\|^2\,\|\bar q \times \bar a_2\|^2 \qquad \text{(A.22)}
\end{aligned}
$$

APPENDIX B

$\frac{\partial F_1}{\partial \xi}$ AND $\frac{\partial F_2}{\partial \eta}$

In order to prove Eq. (2.1) is valid for quadrilateral planar element, it is necessary to prove the following two equations

$$\frac{\partial F_1}{\partial \xi} = \frac{1}{\|\bar{q}\|} \tag{B.1}$$

$$\frac{\partial F_2}{\partial \eta} = \frac{1}{\|\bar{q}\|} \tag{B.2}$$

Proof of Eq. (B.2) is shown below. Proof of Eq. (B.1) is similar except for the fact that $\bar{a}_2$ is replaced by $\bar{a}_1$. There are three different cases in F_2. Consider the first case $\bar{a}_2 \odot \bar{a}_1 > 0$
Note that

$$\frac{\partial}{\partial \eta} \|\bar{a}_2\| = 0 \tag{B.3}$$

since $\bar{a}_2 = \frac{\partial}{\partial \eta} \bar{q} = \bar{p}_2 \times \xi \bar{p}_3$

and

$$\begin{aligned} &\frac{\partial}{\partial \eta} (\bar{q} \times \bar{a}_2 \odot \bar{q} \times \bar{a}_2) \\ &= 2 \left(\frac{\partial \bar{q}}{\partial \eta} \times \bar{a}_2 \right) \odot (\bar{q} \times \bar{a}_2) \\ &= 2 \, (\bar{a}_2 \times \bar{a}_2) \odot (\bar{q} \times \bar{a}_2) = 0 \end{aligned} \tag{B.4}$$

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta} \left[\frac{1}{\|\bar{a}_2\|} \ln \left| \frac{\|\bar{q}\| \|\bar{a}_2\| + \bar{q} \odot \bar{a}_2}{\|\bar{q} \times \bar{a}_2\|} \right| \right] =$$

$$= \frac{1}{\|\bar{a}_2\|}\left[\frac{\partial}{\partial\eta}\ln\left|\|\bar{q}\|\,\|\bar{a}_2\| + \bar{q}\circ\bar{a}_2\right| - \frac{\partial}{\partial\eta}\ln\|\bar{q}\times\bar{a}_2\|\right]$$

$$= \frac{1}{\|\bar{a}_2\|}\;\frac{1}{\|\bar{q}\|\,\|\bar{a}_2\| + \bar{q}\circ\bar{a}_2}\left(\frac{\dot{\bar{q}}\circ\bar{a}_2}{\sqrt{\bar{q}\circ\bar{q}}}\sqrt{\bar{a}_2\circ\bar{a}_2} + \bar{a}_2\circ\bar{a}_2\right)$$

$$= \frac{1}{\|\bar{a}_2\|}\;\frac{1}{\|\bar{q}\|\,\|\bar{a}_2\| + \bar{q}\circ\bar{a}_2}\left(\bar{q}\circ\bar{a}_2\;\|\bar{a}_2\| + \|\bar{q}\|\,\|\bar{a}_2\|^2\right)\frac{1}{\|\bar{q}\|}$$

$$= \frac{1}{\|\bar{q}\|} \qquad \text{(B.5)}$$

Consider the second case, $\bar{a}_2 \circ \bar{a}_2 = 0$

$$\frac{\partial F_2}{\partial\eta} = \frac{\partial}{\partial\eta}\left(\frac{\|\bar{q}\|}{\bar{q}\circ\bar{a}_2}\right)$$

$$= \frac{1}{\bar{q}\circ\bar{a}_2}\;\frac{\partial}{\partial\eta}\|\bar{q}\|$$

$$= \frac{1}{\bar{q}\circ\bar{a}_2}\;\frac{\bar{q}\circ\bar{a}_2}{\sqrt{\bar{q}\circ\bar{q}}} = \frac{1}{\|\bar{q}\|} \qquad \text{(B.6)}$$

since

$$\frac{\partial}{\partial\eta}\left(\bar{q}\circ\bar{a}_2\right) = \left(\frac{\partial}{\partial\eta}\bar{q}\right)\circ\bar{a}_2 = \bar{a}_2\circ\bar{a}_2 = 0$$

Consider the third case, $\bar{a}_2 \circ \bar{a}_2 < 0$

$$\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta}\left[-\frac{1}{\|\bar{a}_2\|}\sin^{-1}\left(\frac{\bar{q}\circ\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right)\right]$$

$$= -\frac{1}{\|\bar{a}_2\|}\frac{\partial}{\partial \eta}\sin^{-1}\left(\frac{\bar{q}\circ\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right)$$

$$= -\frac{1}{\|\bar{a}_2\|}\frac{1}{\sqrt{1-\left(\frac{\bar{q}\circ\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}\right)^2}}\frac{\bar{a}_2\circ\bar{a}_2}{\|\bar{q}\times\bar{a}_2\|}$$

$$= \frac{1}{\|\bar{a}_2\|}\frac{\|\bar{q}\times\bar{a}_2\|}{\sqrt{-\bar{q}\times\bar{a}_2\circ\bar{q}\times\bar{a}_2-(\bar{q}\circ\bar{a}_2)^2}}\frac{1}{\|\bar{q}\times\bar{a}_2\|}\|\bar{a}_2\|^2$$

$$= \frac{\|\bar{a}_2\|}{\sqrt{(\bar{q}\circ\bar{q})(-\bar{a}_2\circ\bar{a}_2)}}$$

$$= \frac{1}{\|\bar{q}\|} \tag{B.7}$$

Combining (B.5), (B.6), and (B.7), yields

$$\frac{\partial F_2}{\partial \eta} = \frac{1}{\|\bar{q}\|} \qquad \bar{a}_2\circ\bar{a}_2 \gtreqless 0 \tag{B.8}$$

Similarly, yields

$$\frac{\partial F_2}{\partial \xi} = \frac{1}{\|\bar{q}\|} \qquad \bar{a}_1\circ\bar{a}_1 \gtreqless 0 \tag{B.9}$$

APPENDIX C

DERIVATIVES OF I_D AND I_{S3}

C.1 Introduction

In this Appendix, it will be shown that the second mixed derivatives of

$$I_D = \frac{\bar{q}\cdot\bar{n}}{|\bar{q}\cdot\bar{n}|}\tan^{-1}\frac{-\bar{q}\times\bar{a}_1 \odot \bar{q}\times\bar{a}_2}{\|\bar{q}\|\,|\bar{q}\cdot\bar{a}_1\times\bar{a}_2|} \tag{C.1}$$

$$I_{S3} = |\bar{q}\cdot\bar{n}|\tan^{-1}\frac{-\bar{q}\times\bar{a}_1 \odot \bar{q}\times\bar{a}_2}{\|\bar{q}\|\,|\bar{q}\cdot\bar{a}_1\times\bar{a}_2|} \tag{C.2}$$

are given by

$$\frac{\partial^2 I_D}{\partial\xi\,\partial\eta} = \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\|\bar{q}\|^3} \tag{C.3}$$

and

$$\frac{\partial^2 I_{S3}}{\partial\xi\,\partial\eta} = \bar{q}\cdot\bar{n}\;\frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\|\bar{q}\|^3} \tag{C.4}$$

where I_{S3} is the third term of I_S. Note that

$$\frac{\partial\bar{a}_1}{\partial\xi} = \frac{\partial\bar{a}_2}{\partial\eta} = 0 \tag{C.5}$$

$$\frac{\partial\bar{a}_1}{\partial\eta} = \frac{\partial\bar{a}_2}{\partial\xi} = \bar{\rho}_3 \tag{C.6}$$

and

$$\frac{\partial \bar{q}}{\partial \xi} = \bar{a}_1 \tag{C.7}$$

$$\frac{\partial \bar{q}}{\partial \eta} = \bar{a}_2 \tag{C.8}$$

C.2 Derivative of I_D

Consider Eq. (C.1) or

$$I_D = S_n \tan^{-1}\left(\frac{-\bar{q}\times\bar{a}_1 \circ \bar{q}\times\bar{a}_2}{\|\bar{q}\|\, S_n\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}\right) \tag{C.9}$$

where

$$S_n = \frac{\bar{q}\cdot\bar{n}}{|\bar{q}\cdot\bar{n}|} = \mathrm{sign}\,(\bar{q}\cdot\bar{n}) \tag{C.10}$$

The derivative of I_D with respect to η is given by

$$\frac{\partial I_D}{\partial \eta} = \frac{\partial}{\partial \eta} S_n \tan^{-1}\left(\frac{-\bar{q}\times\bar{a}_1 \circ \bar{q}\times\bar{a}_2}{\sqrt{\bar{q}\circ\bar{q}}\; S_n\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}\right)$$

$$= -S_n^2 \frac{1}{1+\left(\dfrac{\bar{q}\times\bar{a}_1 \circ \bar{q}\times\bar{a}_2}{\sqrt{\bar{q}\circ\bar{q}}\;\; \bar{q}\cdot\bar{a}_1\times\bar{a}_2}\right)^2} \times$$

$$\times \left\{ \left[\bar{a}_2\times\bar{a}_1 \circ \bar{q}\times\bar{a}_2 + \bar{q}\times\bar{\beta}_3 \circ \bar{q}\times\bar{a}_2 + \bar{q}\times\bar{a}_1 \circ \bar{a}_2\times\bar{a}_2\right] \frac{1}{\|\bar{q}\|\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2} \right.$$

$$\left. - \frac{\bar{q}\times\bar{a}_1 \circ \bar{q}\times\bar{a}_2}{\bar{q}\circ\bar{q}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2}\left[\frac{\bar{a}_2\circ\bar{q}}{\sqrt{\bar{q}\circ\bar{q}}}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 + \sqrt{\bar{q}\circ\bar{q}}\,\left(\bar{a}_2\cdot\bar{a}_1\times\bar{a}_2 + \bar{q}\cdot\bar{\beta}_3\times\bar{a}_2\right)\right]\right\}$$

$$= - \frac{\bar{q} \circ \bar{q} \, (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2}{(\bar{q} \circ \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2)^2} \, \frac{1}{\|\bar{q}\|^3 (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2} \times$$

$$\times \Big\{ (\bar{a}_2 \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2 + \bar{q} \times \bar{p}_3 \circ \bar{q} \times \bar{a}_2) \, \bar{q} \circ \bar{q} \;\; \bar{q} \cdot \bar{a}_1 \times \bar{a}_2$$

$$- \bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2 \, [\, \bar{a}_2 \circ \bar{q} \;\; \bar{q} \cdot \bar{a}_1 \times \bar{a}_2 + \bar{q} \circ \bar{q} \;\; \bar{q} \cdot \bar{p}_3 \times \bar{a}_2 \,] \Big\}$$

$$= \frac{-1}{(\bar{q} \circ \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2)^2} \, \frac{1}{\sqrt{\bar{q} \circ \bar{q}}} \times$$

$$\times \Big\{ [(\bar{a}_2 \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2) \, \bar{q} \circ \bar{q} - (\bar{q} \times \bar{a}_2 \circ \bar{q} \times a_2) \, \bar{a}_2 \circ \bar{q}\,] \, \bar{q} \cdot \bar{a}_1 \times \bar{a}_2$$

$$+ [(\bar{q} \times \bar{p}_3 \circ \bar{q} \times \bar{a}_2) \, \bar{q} \cdot \bar{a}_1 \times \bar{a}_2 - (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2) \, \bar{q} \cdot \bar{p}_3 \times \bar{a}_2 \,] \, \bar{q} \circ \bar{q} \Big\} \tag{C.11}$$

Next note, as shown in Appendix A, Eq. (A.20),

$$\bar{q} \cdot \bar{q} \, (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2) = \|\bar{q} \times \bar{a}_1\|^2 \, \|\bar{q} \times \bar{a}_2\|^2 \tag{C.12}$$

Moreover, note that (see Eq. A.19)

$$[(\bar a_2 \times \bar a_1 \circ \bar q \times \bar a_2)\bar q \circ \bar q - (\bar q \times \bar a_1 \circ \bar q \times \bar a_2)\bar a_2 \circ \bar q]\, \bar q \cdot \bar a_1 \times \bar a_2$$
$$+ [(\bar q \times \bar p_3 \circ \bar q \times \bar a_2)(\bar q \cdot \bar a_1 \times \bar a_2) - (\bar q \times \bar a_1 \circ \bar q \times \bar a_2)(\bar q \cdot \bar p_3 \times \bar a_2)]\, \bar q \circ \bar q$$
$$\equiv \{(\bar a_2 \circ \bar q \;\, \bar a_1 \times \bar a_2 - \bar a_2 \circ \bar a_2 \;\, \bar a_1 \circ \bar q)\, \bar q \circ \bar q$$
$$-(\bar q \circ \bar q \;\, \bar a_1 \times \bar a_2 - \bar q \circ \bar a_2 \;\, \bar q \circ \bar a_1)\, \bar a_2 \circ \bar q\}\, \bar q \cdot \bar a_1 \times \bar a_2$$
$$+\{(\bar q \circ \bar q \;\, \bar p_3 \circ \bar a_2 - \bar q \circ \bar a_2 \;\, \bar q \circ \bar p_3)\, \bar q \cdot \bar a_1 \times \bar a_2$$
$$-(\bar q \circ \bar q \;\, \bar a_1 \circ \bar a_2 - \bar q \circ \bar a_2 \;\, \bar q \circ \bar a_1)\, \bar q \cdot \bar p_3 \times \bar a_2\}\, \bar q \circ \bar q$$
$$= -\{\bar q \circ \bar q \;\, \bar a_2 \circ \bar a_2 - (\bar q \circ \bar a_2)^2\}\, \bar q \circ \bar a_1 \;\, \bar q \cdot \bar a_1 \times \bar a_2$$
$$+\{\bar q \circ \bar q\, (\bar a_2 \circ \bar p_3 \;\, \bar q \cdot \bar a_1 \times \bar a_2 - \bar a_1 \circ \bar a_2 \;\, \bar q \cdot \bar p_3 \times \bar a_2)$$
$$\bar q \circ \bar a_2\, (\bar q \circ \bar p_3 \;\, \bar q \cdot \bar a_1 \times \bar a_2 - \bar q \circ \bar a_1 \;\, \bar q \cdot \bar p_3 \times \bar a_2)\}\, \bar q \circ \bar q$$
$$= -\{\bar q \circ \bar q \;\, \bar a_2 \circ \bar a_2 - (\bar q \circ \bar a_2)^2\}\, \bar q \circ \bar a_1 \;\, \bar q \cdot \bar a_1 \times \bar a_2$$
$$+ (\bar q \times \bar a_2 \circ \bar q \times \bar a_2)(\bar q \cdot \bar a_1 \times \bar p_3)(\bar q \circ \bar q)$$
$$= \|\bar q \times \bar a_2\|^2 (\bar q \circ \bar a_1 \;\, \bar q \cdot \bar a_1 \times \bar a_2 - \bar q \circ \bar q \;\, \bar q \cdot \bar a_1 \times \bar p_3) \tag{C.13}$$

since

$$\bar q \circ \bar q \;\, \bar a_2 \circ \bar a_2 - (\bar q \circ \bar a_2)^2 \equiv \bar q \times \bar a_2 \circ \bar q \times \bar a_2 = -\|\bar q \times \bar a_2\|^2 \tag{C.14}$$

Finally, combining Eqs. (C.11), (C.12), and (C.13) yields

$$\frac{\partial I_v}{\partial \eta} = \frac{\|\bar q \times \bar a_2\|^2 (\bar q \circ \bar a_1 \;\, \bar q \cdot \bar a_1 \times \bar a_2 - \bar q \circ \bar q \;\, \bar q \cdot \bar a_1 \times \bar p_3)}{\|\bar q \times \bar a_1\|^2 \, \|\bar q \times \bar a_2\|^2 \, \|\bar q\|}$$

$$= \frac{1}{\|\bar{q}\times\bar{a}_1\|\,\|\bar{q}\|}\left(\bar{q}\circ\bar{a}_1\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 - \bar{q}\circ\bar{q}\;\bar{q}\cdot\bar{a}_1\times\bar{p}_3\right) \tag{C.15}$$

Next, consider the second mixed derivative, noting that

$$\frac{\partial}{\partial\xi}(\bar{q}\times\bar{a}_1) = \frac{\partial}{\partial\xi}\left[(\bar{p}_0+\eta\bar{p}_2)\times(\bar{p}_1+\eta\bar{p}_3)\right] \equiv 0$$

one obtains

$$\frac{\partial^2 I_p}{\partial\xi\,\partial\eta} = \frac{1}{\|\bar{q}\times\bar{a}_1\|^2}\frac{\partial}{\partial\xi}\left(\frac{\bar{q}\circ\bar{q}\;\bar{q}\cdot\bar{a}_1\times\bar{p}_3 - \bar{q}\circ\bar{a}_1\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\sqrt{\bar{q}\circ\bar{q}}}\right)$$

$$= \frac{1}{\|\bar{q}\times\bar{a}_1\|^2}\Bigg\{-\frac{\bar{q}\circ\bar{a}_1}{(\bar{q}\circ\bar{q})^{3/2}}\left(\cancel{\bar{q}\circ\bar{q}\;\bar{q}\cdot\bar{a}_1\times\bar{p}_3} - \bar{q}\cdot\bar{a}_1\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2\right)$$

$$+\frac{1}{\sqrt{\bar{q}\circ\bar{q}}}\Big(\cancel{2\,\bar{q}\circ\bar{a}_1\;\bar{q}\cdot\bar{a}_1\times\bar{p}_3} + \cancel{\bar{q}\circ\bar{q}\;\bar{a}_1\cdot\bar{a}_1\times\bar{p}_3} - \bar{a}_1\circ\bar{a}_1\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2$$

$$- \cancel{\bar{q}\circ\bar{a}_1\;\bar{a}_1\cdot\bar{a}_1\times\bar{a}_2} - \cancel{\bar{q}\circ\bar{a}_1\;\bar{q}\cdot\bar{a}_1\times\bar{p}_3}\Big)\Bigg\}$$

$$= \frac{1}{\|\bar{q}\times\bar{a}_1\|^2}\,\frac{1}{\|\bar{q}\|^3}\left[(\bar{q}\circ\bar{a}_1)^2 - \bar{q}\circ\bar{q}\;\bar{a}_1\circ\bar{a}_1\right]\bar{q}\cdot\bar{a}_1\times\bar{a}_2$$

$$= \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{\|\bar{q}\|^3}$$