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THE STRUCTURE OF ROBUST OBSERVERS

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16. ABSTRACT

In this paper conventional observers for linear time-invariant systems are shown to be structurally inadequate from a sensitivity standpoint. It is proved that if a linear dynamic system is to provide observer action despite arbitrary small perturbations in a specified subset of its parameters, it must: (1) be a closed loop system, i.e., be driven by the observer error; (2) possess redundancy, i.e., the observer must be generating, implicitly or explicitly, at least one linear combination of states that is already contained in the measuremen and (3) contain a perturbation-free model of the portion of the system observable from the external inpu. to the observer. The procedure for design of "robust" observers possessing the above structural features is established and discussed.

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1. INTRODUCTION

The accepted procedure (see Reference 1 and the references cited therein) for designing an observer for the system

$$\dot{x}(t) = A_0 x(t) \tag{1a}$$

$$y(t) = C_0 x(t) \quad , \tag{1b}$$

$$t \geqslant 0 \quad \text{and} \quad x(0) = x_0 \quad ,$$

is to specify the structure

$$\dot{z}(t) = N_0 z(t) + L_0 y(t) \quad , \tag{2}$$

$$t \geqslant 0 \quad \text{and} \quad z(0) = z_0 \quad ,$$

and choose (N_0, L_0) so that, for suitable V, $z(t) - Vx(t)$ converges to zero much "faster" than the dynamics of (1). Since (2) is an open loop device (i.e., it is not driven by the error $z - Vx$), one might suspect that it would fail to regulate the error in a satisfactory manner if the observer parameters deviated from (N_0, L_0); that this is indeed the case was proved in Reference 2 for the case of identity observers, where it was shown that the observer must be driven by the observation error if it is to tolerate small parameter perturbations. The purpose of the present paper is to treat the general case.

In Section 2 we show that every observer of the form of (2) fails to provide observer action for almost any small variation of observer parameters from the nominal. To dramatize this fact, we show that if (1) is completely unstable and (2) is a minimal order state observer, then the observer error diverges for each and every (infinitesimal) perturbation of the observer parameters. In Section 3 we consider the problem of designing an observer capable of tolerating perturbations and prove that if a linear dynamic system is to provide observer action in the face of arbitrary small perturbations in a specified subset of its parameters, it must be driven by the observer error and must contain a perturbation-free copy of the portion of the system observable from the external input to the observer. Such a device, if it is of high enough order, provides observer action despite small changes in the specified parameters; hence, we label such a device a robust observer. Some properties of robust observers are pointed out in Section 3. Among these is the fact that every robust observer possesses redundancy in the sense that at least one linear functional of the state provided by the observer is directly calculable from the measured outputs. In Section 4 we present some results on minimal order robust observers, formulate the general problem of designing a minimal order robust observer for evaluating a given set of linear functionals of the state, and present a (theoretical) procedure for determining such an observer. Some examples are presented in Section 5, and Section 6 is a concluding discussion.

Notation. In (1) and (2), $A_0 : X \to X$, $C_0 : X \to Y$, $N_0 : Z \to Z$, and $L_0 : Y \to Z$ are linear time-invariant maps with dimension $X = n$, dimension $Y = m$, and dimension $Z = q$. According to context A_0, C_0, etc., will also denote matrices representing the corresponding maps. The kernel and image of a map, C_0 (for example), will be denoted respectively by $\text{Ker } C_0$ and $\text{Im } C_0$. The subscript 0 in A_0, N_0, etc., denotes the fact that these are the nominal values of the corresponding parameters; deviations from a nominal value will be handled by writing, for instance, $N = N_0 + \delta N$, $L = L_0 + \delta L$, etc., and $\delta N(\delta L)$ will be called a perturbation of $N_0(L_0)$. To consider classes of perturbations we let, for $\epsilon > 0$, the symbol $\Omega_L(\epsilon)$ (for instance) denote the class of matrices $\{\delta L\}$, where δL has the same size as L_0 and the absolute value of each element of δL is strictly less than ϵ. When the precise value of ϵ is unimportant, we refer to $\Omega_L(\epsilon)$, $\Omega_N(\epsilon)$, etc., as a class of arbitrary small perturbations of L_0, N_0, etc. When the elements of a set of matrices, say (L_0, N_0) (of size $m \times q$, $q \times q$) are to be regarded as a point in parameter space, we denote this by writing $(\ell_0, n_0) \in \mathbb{R}^{mq+q^2}$, where ℓ_0 and n_0 represent lists of the elements of L_0 and N_0 respectively. $\mathbb{C}$ will denote the complex plane and $\mathbb{C}^+(\mathbb{C}^-)$ the closed right half (open left half) of $\mathbb{C}$. $\sigma(A_0)$ will denote the spectrum of A_0; A_0 is completely unstable if $\sigma(A_0) \subset \mathbb{C}^+$.

2. CONVENTIONAL OBSERVERS

To render the analysis more cogent, we first formulate a suitable formal definition of satisfactory observer action. Let the set of functionals to be evaluated be denoted by

$$w(t) = Hx(t) \quad , \tag{3}$$

where $H : \mathcal{X} \rightarrow \mathcal{W}$ is given. Now, z(t) <u>identifies</u> Vx(t) if

$$\lim_{t \to \infty} z(t) - Vx(t) = 0 \qquad \forall x_0 \in \mathcal{X} \ , \ z_0 \in \mathcal{Z} \quad . \tag{4}$$

Clearly, if z(t) identifies Vx(t), then w(t) can be evaluated from y(t) and z(t) if and only if there exists (E, D) for which

$$EC_0 + DV = H \quad . \tag{5}$$

The latter is equivalent to

$$\text{Ker } C_0 \cap \text{Ker } V \subset \text{Ker } H \quad . \tag{6}$$

We rule out the possibility $\text{Ker } C_0 \subset \text{Ker } H$ since no dynamic observer is required in this case.

Now, to specify that the observer error is to converge faster than the modes of (1), we partition the complex plane $\mathbb{C}$ into symmetric (about the real axis) disjoint subsets $\mathbb{C}_g$, $\mathbb{C}_b$ such that

$$\mathbb{C}_g \text{ is open} \quad , \quad \mathbb{C}_g \subset \mathbb{C}^- \tag{7}$$

$$\sigma(A_0) \subset \mathbb{C}_b \tag{8}$$

and choose $\mathbb{C}_g$ sufficiently to the left of $\sigma(A_0)$.

<u>Definition</u>. A linear dynamic system with state z(t) is an <u>observer</u> (provides <u>observer action</u>) with respect to given $(A_0, C_0, H, \mathbb{C}_g)$ as in (1), (3), (7), and (8) if and only if: (a) there exists V satisfying (4) and (6), and (b) the convergence in (4) takes place with all exponents in $\mathbb{C}_g$.

Based on this definition we have the following preliminary fact.

Lemma 1. The system (2) is an observer if and only if

$$\sigma(N_0) \subset \mathbb{C}_g \tag{9}$$

and for some V satisfying (6),

$$N_0 V - V A_0 + L_0 C_0 = 0 \quad . \tag{10}$$

Proof. The proof follows from the differential equation for $e(t) \equiv z(t) - Vx(t)$. ■

Now introduce perturbations in the observer parameters by letting

$$N \equiv N_0 + \delta N$$
$$L \equiv L_0 + \delta L \quad .$$

If (2) is to continue to provide observer action with the new parameters (N, L), we require, from Lemma 1,

$$\sigma(N_0 + \delta N) \subset \mathbb{C}_g \tag{11}$$

and

$$(N_0 + \delta N)V - VA_0 + (L_0 + \delta L)C_0 = 0 \quad . \tag{12}$$

From (10) and (12), we have

$$\delta NV + \delta LC_0 = 0 \quad . \tag{13}$$

Condition (13) fails for almost every $(\delta N, \delta L)$, since those points $(\delta N, \delta \ell) \in \mathbb{R}^{q^2+qm}$ for which (13) holds is a proper variety determined by the orthogonal complement of $\mathrm{Im}\begin{bmatrix} C_0 \\ V \end{bmatrix}$. Therefore, we have proved the following.

Theorem 1. The system (2), acting as an observer for the nominal parameters (N_0, L_0), fails to provide observer action for almost every perturbation $(\delta N, \delta L)$ in these parameters.

To dramatize the content of this theorem, we specialize it to the case of minimal order state observers, with A_0 completely unstable. We define an observer to be non-redundant if

$$\text{rank}\begin{bmatrix} C_0 \\ V \end{bmatrix} = \text{rank}\, C_0 + \text{rank}\, V \quad . \tag{14}$$

If (14) fails, the observer has redundancy. This definition is motivated by the fact that (14) is equivalent to ruling out the possibility that some linear functional identified by the dynamic observer is already contained in the measurement y.

Corollary 1.1. If A_0 is completely unstable and (2) is a minimal order state observer, then the observer error fails to converge for each and every perturbation $(\delta N, \delta L)$ in (N_0, L_0).

Proof. Every minimal order state observer is nonredundant; also, Ker H = 0 for state observers, so from (6) $\text{Ker}\, C_0 \cap \text{Ker}\, V = 0$. It follows then that the variety in which $(\delta n, \delta \ell)$ may lie, subject to (13) and the above version of (6), shrinks to the origin of $\mathcal{R}^{q^2+qm}$. ∎

Remark 1. Theorem 1 and its corollary establishes that the structure (2) may be drastically inadequate in providing observer action if the plant contains unstable or underdamped modes. This however does not rule out the effectiveness of a device such as (2) acting as a compensator in a closed loop system about the plant. For example, consider the plant observer pair

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + B_0 u(t) \\ y(t) &= C_0 x(t) \end{aligned} \tag{15}$$

$$\dot{z}(t) = N_0 z(t) + L_0 y(t) + G_0 u(t) \quad , \tag{16}$$

where A_0 is unstable and the observer is required for implementing the control law $u = H_0 x$ such that

$$A_0 + B_0 H_0 \text{ is stable} \quad . \tag{17}$$

This is accomplished by choosing V so that for some (E_0, D_0)

$$E_0 C_0 + D_0 V = H_0 \quad , \tag{18}$$

(9) and (10) are satisfied, and

$$G_0 = V B_0 \tag{19}$$

and by setting

$$u = E_0 y + D_0 z \quad . \tag{20}$$

The following proposition may now be easily verified. The proof, which is omitted, depends on the fact that if $T_0 : X_1 \to X_1$ is such that $\sigma(T_0) \subset \mathbb{C}_1$, an open subset of $\mathbb{C}$, then there exists $\epsilon > 0$ such that for every $\delta T \in \Omega_T(\epsilon)$, $\sigma(T_0 + \delta T) \subset \mathbb{C}_1$.

Proposition. For the system (15) with A_0 unstable and the observer (16) determined by (17) through (20), the closed loop system consisting of plant and observer is stable for the nominal parameters and remains stable for a class of arbitrary small perturbations $(\delta N, \delta L, \delta G, \delta E, \delta D, \delta A, \delta C, \delta B)$ about the nominal.

This proposition shows that although (2) fails to act under parameter perturbations as an observer, it does function robustly as a stabilizing compensator.

Remark 2. Lemma 1 can be used to determine the class of perturbations (or modelling inaccuracies) in the plant equations (1) that may be tolerated by an observer. Setting

$$A = A_0 + \delta A \qquad \text{and} \qquad C = C_0 + \delta C$$

and using (10), we have, for observer action with (A, C) in place of (A_0, C_0),

$$-V\delta A + L_0 \delta C = 0 \quad . \tag{21}$$

Since a particular choice (N_0, L_0) subject to (8), (9), and (10) uniquely determines V, it is clear that (21) characterizes the admissible $(\delta A, \delta C)$ for a given observer; the latter class corresponds to a proper variety in $\mathbb{R}^{n^2+nm}$ determined by (21). Clearly, arbitrary plant parameter variations are not tolerable.

3. ROBUST OBSERVERS

The inadequacy of the observer (2) pointed out by Theorem 1 is due to its inherent open loop nature. To provide for the possibility of closed loop performance, we introduce the following feedback structure:

$$\dot{z}(t) = M_0 z(t) + K_0 p(t) \tag{22a}$$

$$p(t) = R_0 y(t) - T_0 z(t) \quad , \tag{22b}$$

in which we call p the driving signal to distinguish it from the external input $R_0 y$. Note that it is customary to say, when $T_0 \neq 0$ that (22) has feedback; however, the mere presence of feedback does not make the above system closed loop. In fact we shall say that (22), acting as an observer, is a closed loop system if and only if for some $Q \neq 0$,

$$p(t) = Qe(t) \qquad t \geqslant 0 \quad , \tag{23}$$

where $e(t)$ is the error that the observer regulates. We emphasize that for given (M_0, K_0, R_0, T_0), (22) can certainly be written in the form of equation (2) by identifying

$$N_0 = M_0 - K_0 T_0 \tag{24}$$

and

$$L_0 = K_0 R_0 \quad . \tag{25}$$

However, the realization of (22) as written involves, in general, hardware and signals that are entirely different from those involved in realizing (2); in particular, as will be shown in the following, the sensitivity properties of the two realizations can be totally different.

If (22) is to be an observer and identifies $Vx(t)$, it follows from Lemma 1 and (24) and (25) that

$$\sigma(M_0 - K_0 T_0) \subset \mathbb{C}_g \tag{26}$$

and

$$(M_0 - K_0 T_0)V - VA_0 + K_0 R_0 C_0 = 0 \quad . \tag{27}$$

We establish at the outset that any perturbation in (M_0, T_0, R_0) is almost always inadmissible. Write

$$M = M_0 + \delta M$$
$$T = T_0 + \delta T$$
$$R = R_0 + \delta R \quad .$$

Then, if (22) is to continue to provide observer action with the new parameters (M, K_0, R, T), it is required that

$$(M_0 + \delta M - K_0T_0 - K_0\delta T)V - VA_0 + K_0(R + \delta R)C_0 = 0 \quad . \tag{28}$$

From (27) and (28),

$$(\delta M - K_0\delta T)V + K_0\delta RC_0 = 0 \quad , \tag{29}$$

which clearly restricts $(\delta m, \delta t, \delta r)$ to a proper variety. Thus any perturbation $(\delta M, \delta T, \delta R)$ almost certainly causes observer action to fail. On the other hand, examples show that it is possible for (22) to act as an observer despite arbitrary small variations in K_0. Based on these observations, we formulate the following problem.

Robust Observer Design Problem

Given $(A_0, C_0, \mathbb{C}_g, H)$ as in (1), (3), (7), and (8), determine conditions under which there exists a nominal set of parameters (M_0, K_0, R_0, T_0) and $\epsilon > 0$ so that (22) with the parameters $(M_0, K_0 + \delta K, R_0, T_0)$ is an observer for every $\delta K \epsilon\, \Omega_K(\epsilon)$.

We proceed to solve this problem by first deriving necessary conditions for robustness. Writing $K = K_0 + \delta K$ and once again applying Lemma 1, we require, for observer action with the parameters (M_0, K, T_0, R_0) in (22), that

$$\sigma(M_0 - K_0T_0 - \delta KT_0) \subset \mathbb{C}_g \tag{30}$$

and for some V satisfying (6),

$$[M_0 - (K_0 + \delta K)T_0]V - VA_0 + (K_0 + \delta K)R_0C_0 = 0 \quad . \tag{31}$$

Now, (27) and (31) yield

$$\delta K(T_0V - R_0C_0) = 0 \quad . \tag{32}$$

If (32) is to hold for every $\delta K \epsilon\, \Omega_K(\epsilon)$ for some $\epsilon > 0$, it is necessary that

$$T_0V = R_0C_0 \quad . \tag{33}$$

From (33) and the fact that the observer error is $e(t) = z(t) - Vx(t)$, we obtain

$$p(t) = -T_0 e(t) \quad . \tag{34}$$

Equations (33) and (34) form the basis of the following structure theorem.

Theorem 2. If (22) is a robust observer it must: (1) be a closed loop system and (2) possess redundancy.

Proof. It will result that item (2) implies item (1); therefore, we shall prove item (2) first. For this it is necessary (and sufficient) to show that (14) fails. In view of (33) this is accomplished if it is shown that $T_0 V \neq 0$. Suppose $T_0 V = 0$; then, from (33), $R_0 C_0 = 0$, so that $p(t) = -T_0 z(t)$. Therefore,

$$z(t) = e^{(M_0 - K_0 T_0 - \delta K T_0)t} z_0 \quad , \tag{35}$$

and from (30) the exponents in $z(t)$ lie in $\mathbb{C}_g$; thus, the exponents in $e(t) = z(t) - Vx(t)$ lie (for $x_0 \neq 0$) in $\mathbb{C}_b$ unless $V = 0$, a case that is ruled out [see the remark following (6)] by assumption since no observer is then required. Therefore, if $T_0 V = 0$, (22) ceases to provide observer action. This proves item (2). To prove item (1) it is only necessary, from (23) and (34), to show that $T_0 \neq 0$; this is implicitly proved in establishing item (2). ■

Corollary 2.1. A lower bound on the order of a robust observer (if it exists) is $\text{rank}\begin{bmatrix} C_0 \\ H \end{bmatrix} - \text{rank}\, C_0 + 1$. The lower bound on the order of a full state observer is $n - m + 1$, $m = \text{rank}\, C_0$.

Proof. The proof follows from item (2) of Theorem 2, the fact that $q \geq \text{rank}\, V$, and condition (6) which may be rewritten

$$\text{rank}\begin{bmatrix} C_0 \\ V \end{bmatrix} = \text{rank}\begin{bmatrix} C_0 \\ V \\ H \end{bmatrix} \quad .$$ ■

To proceed, substitute (33) into (31); this yields

$$M_0 V = V A_0 \quad , \tag{36}$$

an equation which enables stating the next three Lemmas.

Lemma 2. If (22) is a robust observer and identifies $Vx(t)$, then: (1) Ker V is A_0 invariant, (2) Im V is M_0 invariant, and (3) the restriction $\tilde{M}_0$ of M_0 to Im V is similar to $\bar{A}_0$, the map induced by A_0 in $X/\text{Ker } V$.

Proof. The lemma is a mere restatement of (36); items (1) and (2) are obvious and (3) is best illustrated in a particular coordinate system. Let rank $V = q_1$; then, there exists S^{-1}, J^{-1} such that

$$J^{-1} V S^{-1} = \begin{bmatrix} 0 & I_{q_1} \\ 0 & 0 \end{bmatrix} \equiv \tilde{V} \quad . \tag{37}$$

Substituting (37) into (36),

$$J^{-1} M_0 J \tilde{V} = \tilde{V} S A_0 S^{-1} \quad . \tag{38}$$

Clearly, $J^{-1}(S^{-1})$ induces a coordinate transformation in $Z(X)$. Writing $J^{-1}M_0J$, SA_0S^{-1} partitioned conformably with $\tilde{V}$ as

$$J^{-1} M_0 J = \begin{bmatrix} M_0{}^1 & M_0{}^3 \\ M_0{}^4 & M_0{}^2 \end{bmatrix} \qquad S A_0 S^{-1} = \begin{bmatrix} A_0{}^1 & A_0{}^3 \\ A_0{}^4 & A_0{}^2 \end{bmatrix} \quad ,$$

it follows from (37) and (38) that

$$M_0{}^4 = 0 \qquad A_0{}^4 = 0 \qquad M_0{}^1 = A_0{}^2 \quad .$$

The first two of the above equalities correspond to items (2) and (1), respectively, of Lemma 2; item (3) of Lemma 2 corresponds to the third equality since $M_0{}^1(A_0{}^2)$ is a representation of $\tilde{M}_0(\bar{A}_0)$ in the coordinate system specified by (37). ■

The above lemma shows that if (22) is a robust observer, $\sigma(\bar{A}_0) \subset \sigma(M_0)$. Since observer action requires that $\sigma(M_0 - K_0 T_0) \subset \mathbb{C}_g$ and since by definition $\sigma(\bar{A}_0) \subset \sigma(A_0) \subset \mathbb{C}_b$, it is clear that the (M_0, T_0) pair must possess "sufficient observability" to accomplish the desired shift of poles. The next result is addressed to this problem. Let $\tilde{T}_0 : \text{Im } V \to P$ denote the unique map that agrees with T_0 on Im V (P is the codomain of T_0).

Lemma 3. Subject to (8) and (36), there exists (M_0, K_0, T_0) such that

$$\sigma(M_0 - K_0T_0) \subset \mathbb{C}_g$$

if and only if the pair $(\tilde{M}_0, \tilde{T}_0)$ is observable.

Proof. By assumption $V \neq 0$, so that $\sigma(\bar{A}_0) \neq \phi$. Suppose now that $\sigma(\bar{A}_0)$ has only real elements. Then, if $(\tilde{M}_0, \tilde{T}_0)$ is not observable, there exists, by Lemma 2 and (36) and (8), $\lambda \in \sigma(\bar{A}_0)$ and $z \in \text{Im } V$ (i.e., $0 \neq z = Vx$ for some x) such that

$$\tilde{M}_0 z = \lambda z$$
$$\tilde{T}_0 z = 0 \quad .$$

Then,

$$(M_0 - K_0T_0)z = (\tilde{M}_0 - K_0\tilde{T}_0)z = \lambda z \quad ,$$

so that $\sigma(M_0 - K_0T_0)$ contains $\lambda \in \mathbb{C}_b$ for every K_0. The proof for the case that $\sigma(\bar{A}_0)$ has complex elements is similar and is omitted. This proves that observability of $(\tilde{M}_0, \tilde{T}_0)$ is necessary.

Now, assume $(\tilde{M}_0, \tilde{T}_0)$ is observable. Then, there exists $\tilde{K}_0 : P \to \text{Im } V$ such that $\sigma(\tilde{M}_0 - \tilde{K}_0\tilde{T}_0) \subset \mathbb{C}_g$. Let $\bar{M}_0$, the map induced by M_0 in $Z/\text{Im } V$, be chosen so that $\sigma(\bar{M}_0) \subset \mathbb{C}_g$ and extend $\tilde{K}_0$ to $K_0 : P \to Z$ in the natural way. Then, $\sigma(M_0 - K_0T_0) \subset \mathbb{C}_g$. ■

Lemma 4. Subject to (33) and (36), the pair $(\tilde{M}_0, \tilde{T}_0)$ is observable if and only if

$$\text{Ker } V = \bigcap_{i=0}^{n-1} \text{Ker } R_0C_0A_0{}^i \quad . \tag{39}$$

Proof. From (33) and (36),

$$T_0 M_0^i V = R_0 C_0 A_0^i \quad , \quad i = 0, 1, 2, \ldots \tag{40}$$

so that

$$\text{Ker } V \subset \bigcap_{i=0}^{n-1} \text{Ker } R_0 C_0 A_0^i \quad .$$

Now, assume that

$$\text{Im } V = Z \quad ; \tag{41}$$

then, $\tilde{M}_0 = M_0$ and $\tilde{T}_0 = T_0$. Let (M_0, T_0) be observable and write

$$\Theta \equiv \bigcap_{i=0}^{n-1} \text{Ker } R_0 C_0 A_0^i \quad .$$

Now, (40) implies that if $x \in \Theta$, then $Vx \in \bigcap_{i=0}^{q-1} \text{Ker } T_0 M_0^i = 0$ so that $\Theta \subset \text{Ker } V$. If (M_0, T_0) is not observable, there exists, subject to (41), $x \in X$ such that $Vx \neq 0$ and $T_0 M_0^i Vx = 0, i = 0, 1, 2, \ldots$. This implies by (40) that $x \in \Theta$, i.e.. $x \in \Theta$ and $x \notin \text{Ker } V$, so (39) fails. This proves the lemma subject to assumption (41). If (41) does not hold, write $\text{Im } V = \tilde{Z}$ and note that (33) and (36) imply, respectively,

$$\tilde{T}_0 \tilde{V} = R_0 C_0 \tag{42}$$

and

$$\tilde{M}_0 \tilde{V} = \tilde{V} A_0 \quad , \tag{43}$$

where $\tilde{V} : X \rightarrow \text{Im } V$ is the map that agrees with V on X. Therefore,

$$\text{Im } \tilde{V} = \tilde{Z}$$

and the preceding arguments establish that observability of $(\tilde{M}_0, \tilde{T}_0)$ is equivalent to

$$\text{Ker } \tilde{V} = \Theta \quad .$$

Since $\text{Ker } \tilde{V} = \text{Ker } V$, this completes the proof. ■

The content of the preceding three lemmas has established the following important result.

Theorem 3. If (22) is a robust observer, it contains explicitly a perturbation-free replica of that part of the system (1) that is observable from the external input to the observer $y_0 \equiv R_0 y$.

The proof is a direct consequence of Lemmas 2 through 4. To clarify the meaning of the theorem, we choose a basis for X so that (A_0, R_0C_0) have the representations

$$A_0 = \begin{bmatrix} \tilde{A}_0 & \hat{A}_0 \\ 0 & \bar{A}_0 \end{bmatrix} \qquad R_0C_0 = [0 \quad \bar{C}_0] \quad , \tag{44}$$

with $(\bar{A}_0, \bar{C}_0)$ observable. Then, the theorem states that there is a coordination of Z in which (M_0, T_0) have the representations

$$M_0 = \begin{bmatrix} \bar{A}_0 & \hat{M}_0 \\ 0 & \bar{M}_0 \end{bmatrix} \qquad T_0 = [\bar{C}_0, \bar{T}_0] \quad . \tag{45}$$

In these coordinates (1a) becomes, with $x = \begin{bmatrix} \tilde{x} \\ \bar{x} \end{bmatrix}$,

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}_0\tilde{x} + \hat{A}_0\bar{x} \\ \dot{\bar{x}} &= \bar{A}_0\bar{x} \quad , \end{aligned} \tag{46}$$

and, with $z \equiv \begin{bmatrix} \tilde{z} \\ \bar{z} \end{bmatrix}$,

$$\begin{aligned} y_0 &\equiv R_0 y = \bar{C}_0 x \\ p &= y_0 - C_0\tilde{z} - \bar{T}_0 \bar{z} \end{aligned} \tag{47}$$

so that the observer contains a model of the portion of the system observable from y_0 (i.e., $\bar{A}_0, \bar{C}_0$).

The preceding conditions have stated only necessary conditions for solvability of the robust observer design problem. The following result now completes the solution.

Theorem 4. There exists a solution to the robust observer design problem if and only if for some R_0,

$$\bigcap_{i=0}^{n-1} \text{Ker } R_0 C_0 A_0{}^i \cap \text{Ker } C_0 \subset \text{Ker } H \tag{48}$$

For a fixed R_0 satisfying (48), the lowest order observer has the order $\left(n - \text{dimension} \bigcap_{i=0}^{n-1} \text{Ker } R_0 C_0 A_0{}^i\right)$, and such an observer is obtained by setting (M_0, T_0, K_0) equal to a matrix representation of $(\bar{A}_0, \bar{C}_0, \bar{K}_0)$, where $\bar{A}_0$ is the map induced by A_0 in $X/\Theta\left(\Theta \equiv \bigcap_{i=0}^{n-1} \text{Ker } R_0 C_0 A_0{}^i\right)$, $\bar{C}_0$ is the unique map satisfying $\bar{C}_0 P = R_0 C_0$ where $P : X \rightarrow X/\Theta$ is the canonical projection, and $\bar{K}_0$ is chosen so that $\sigma(\bar{A}_0 - \bar{K}_0 \bar{C}_0)$ has the desired spectrum in $\mathbb{C}_g$.

Proof:Necessity. Suppose (M_0, K_0, T_0, R_0) is a solution. Then, there exists V satisfying (6), (33), and (36) and, by Lemma 4, Ker $V = \Theta$. Therefore (48) is implied by (6).

Sufficiency. With R_0 chosen to satisfy (48), let $P : X \rightarrow X/\Theta$ denote the canonical projection, $\bar{A}_0 : X/\Theta \rightarrow X/\Theta$ the map induced by A_0, and $\bar{C}_0$ the unique map for which

$$\bar{C}_0 P = R_0 C_0 \quad . \tag{49}$$

Then,

$$\bar{A}_0 P = P A_0 \tag{50}$$

and applying P to both sides of (1a) and writing $\bar{x} = Px$, we have

$$\dot{\bar{x}}(t) = \bar{A}_0 \bar{x}(t) \tag{51a}$$

and

$$y_0(t) \equiv R_0 y(t) = \bar{C}_0 \bar{x}(t) \quad . \tag{51b}$$

The pair $(\bar{A}_0, \bar{C}_0)$ is observable, and there exists $\bar{K}_0$ such that

$$\sigma(\bar{A}_0 - \bar{K}_0 \bar{C}_0) \subset \mathbb{C}_g \quad . \tag{52}$$

The robust observer may now be designed simply as a closed loop identity observer [2] for (51) by setting dimension $\mathcal{Z}$ = dimension X/Θ = n - dimension Θ and letting

$$\dot{z}(t) = M_0 z(t) + K_0 [y_0(t) - T_0 z(t)] \quad , \tag{53}$$

with

$$M_0 = \bar{A}_0 \quad , \tag{54a}$$

$$T_0 = \bar{C}_0 \quad , \tag{54b}$$

and

$$K_0 = \bar{K}_0 \quad . \tag{54c}$$

The observer (53) identifies $\bar{x}(t) = Px(t)$. With

$$V = P \quad , \tag{55}$$

it follows from the definition of P and (49), (50), and (52) that (6), (26), (27), (33), and (36) are satisfied. Then (31) clearly holds for all δK, and from (52) and the fact that $\mathbb{C}_g$ is open, it follows that there exists $\epsilon > 0$ such that (30) holds for all $\delta K \in \Omega_K(\epsilon)$. Therefore, from Lemma 1, (22) with the parameters $(M_0, K_0 + \delta K, T_0, R_0)$ provides observer action for all $\delta K \in \Omega_K(\epsilon)$ and hence is a robust observer. That this is the lowest order observer for a fixed choice of R_0 is clear from Theorem 3. This completes the proof. ■

For the sake of completeness we state the following.

Corollary 4.1. If (A_0, C_0) is observable, for arbitrary H there exists a robust observer of order n.

Proof. Set $R_0 = Im$; then, since (A_0, C_0) is observable, $\Theta = 0$ and an nth order robust observer is given by $M_0 = A_0$, $T_0 = C_0$, and $K_0 = \bar{K}_0$ where $\sigma(A_0 - \bar{K}_0C_0) \subset \mathbb{C}_g$. ■

Remark. For a given robust observer it is clear that almost any perturbation in (A_0, C_0) will cause (33) and (36), and hence observer action, to fail. It is clear that as long as the perturbations satisfy

$$\text{Im}\, \delta A \subset \bigcap_{i=0}^{n-1} \text{Ker}\, R_0 C_0 A_0^{\,i}$$

$$\text{Im}\, \delta C \subset \text{Ker}\, R_0 \quad ,$$

observer action will be preserved. Further, if the robust observer has lowest order for the chosen R_0, any perturbation in (M_0, T_0) will cause failure of observer action; this is easily seen from Theorem 3. In general, as may be verified from (33) and (36), arbitrary small perturbations $(\delta M, \delta T, \delta R)$ will not disrupt observer action as long as

$$\text{Im}\, V \subset (\text{Ker}\, \delta M) \cap (\text{Ker}\, \delta T)$$

$$\text{Im}\, C_0 \subset \text{Ker}\, \delta R \quad .$$

4. MINIMAL ORDER ROBUST OBSERVERS

We assume throughout this section that (A_0, C_0) is observable. Under this assumption existence of a robust observer of order n is guaranteed, independent of the functionals to be evaluated. Since the external input to the robust observer is $R_0 y$, we shall say that the observer uses the entire output information dynamically if $\text{rank}\, R_0 = m$. The following interesting result may now be stated.

Theorem 5. The minimal order of a robust observer that uses the entire output information dynamically is n, regardless of the particular functionals to be evaluated.

Proof. If $\text{rank}\, R_0 = m$, the unobservable subspace Θ of (A_0, R_0C_0) is 0 and from Theorem 4 and its corollary, the minimal order is n. ■

Corollary 5.1. The minimal order of every robust observer for a single output system $(m = 1)$ is n (or 0), regardless of the particular functionals to be evaluated.

Proof. In this case rank R_0 is either 1 or 0. If rank $R_0 = 0$, the unobservable subspace of (A_0, R_0C_0) is $\Theta = X$, so that (48) reduces to $\text{Ker } C_0 \subset \text{Ker } H$, which in turn means that no dynamic observer is required (or the minimal order is 0). When rank $R_0 = 1$, the result follows from the theorem. ∎

To consider the general case we recast the solvability condition (48) in matrix terms. Define the matrices,

$$W_0(R_0) \equiv \begin{bmatrix} R_0C_0 \\ R_0C_0A \\ \vdots \\ R_0C_0A^{n-1} \end{bmatrix} , \tag{56a}$$

$$W_1(R_0) \equiv \begin{bmatrix} W_0(R_0) \\ C_0 \end{bmatrix} , \tag{56b}$$

and

$$W_2(R_0) \equiv \begin{bmatrix} W_1(R_0) \\ H \end{bmatrix} . \tag{56c}$$

Then (48) is equivalent to

$$\text{rank } W_1(R_0) = \text{rank } W_2(R_0) ,$$

and the lowest order observer for a given matrix R_0 is rank $W_0(R_0)$. The problem of determining the minimal order of a robust observer then reduces to the following. Determine a real matrix $R_0{}^*$ so that

$$\text{rank } W_1(R_0{}^*) = \text{rank } W_2(R_0{}^*) , \tag{57}$$

and for every real matrix R_0 with m columns,

$$\text{rank } W_0(R_0{}^*) \leq \text{rank } W_0(R_0) . \tag{58}$$

In satisfying (57) and (58) there is clearly no loss of generality in restricting R_0 to be $m \times m$. Now, let $r = (r_1, \ldots r_N)$ denote the elements of R_0 regarded as a point in $\mathcal{R}^N$, $N = m^2$ and let $\psi_i^j(r)$, $i = 0, 1, 2$ denote the polynominal obtained by summing the squares of the jth order minors, $j = 1, 2, \ldots n$ of $W_i(R_0)$; let $\Gamma_i^j \subset \mathcal{R}^N$ denote the locus of $\psi_i^j(r)$, i.e.,

$$\Gamma_i^j = \left\{ r | r \in \mathcal{R}^N \quad , \quad \psi_i^j(r) = 0 \right\}$$

and define

$$\Lambda_i^j \equiv \Gamma_i^{j+1} - \Gamma_i^j \equiv \left\{ r | r \in \mathcal{R}^N \quad , \quad r \in \Gamma_i^{j+1} \quad , \quad r \notin \Gamma_i^j \right\} \quad ,$$

$$i = 0, 1, 2 \qquad j = 1, 2, \ldots n-1$$

$$\Lambda_i^n \equiv \mathcal{R}^N - \Gamma_i^n \quad , \quad i = 0, 1, 2 \quad .$$

Now, it is easily seen that rank $W_i(R_0) = j$ if and only if $r \in \Lambda_i^j$, $i = 0, 1, 2$ $j = 1, 2, \ldots n$. Noting also that for every R_0

$$\text{rank } W_0(R_0) \leq \text{rank } W_1(R_0) \leq \text{rank } W_2(R_0) \quad ,$$

we have the following result; the proof is obvious.

<u>Theorem 6.</u> There exists a robust observer of order q if and only if for some integer $t \in \left\{0, 1, 2, \ldots n-q\right\}$

$$\Lambda_0^q \cap \Lambda_1^{q+t} \cap \Lambda_2^{q+t} \neq \phi \quad . \tag{59}$$

The minimal order of a robust observer is the smallest integer q^* such that (59) holds with $q = q^*$.

<u>Remark.</u> For the case of full state observers, the equality (57) must hold with both ranks equal to n, and (59) accordingly becomes

$$\Lambda_0^q \cap \Lambda_1^n \cap \Lambda_2^n \neq \phi \quad . \tag{60}$$

Determination of q^* using (59), assuming that the sets Λ_i^j are available, would be an iterative process, beginning, for instance, with the lower bound given by Corollary 2.1, verifying (59), and increasing the order by one at each step that (59) fails.

5. EXAMPLES

Example 1a

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$C_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$H = (1 \quad 0 \quad 0 \quad 0)$$

$$\mathbb{C}_g = \mathbb{C}^- , \; \mathbb{C}_b = \mathbb{C}^+ \quad .$$

(A_0, C_0) is observable so a robust fourth order observer can always be designed. If a lower order observer is to be designed, rank $R_0 = 1$; therefore, let $R_0 = (r_1, r_2)$. Then,

$$W_0(R_0) = \begin{pmatrix} r_1+r_2 & r_1 & r_2 & r_2 \\ 0 & r_1 & 2r_2 & 3r_2 \\ 0 & r_1 & 4r_2 & 9r_2 \\ 0 & r_1 & 8r_2 & 27r_2 \end{pmatrix} \quad .$$

By elementary operations,

$$\text{rank } W_0(R_0) = \text{rank} \begin{pmatrix} r_1+r_2 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_2 \end{pmatrix} \quad ,$$

$$\operatorname{rank} W_1(R_0) = \operatorname{rank}\begin{pmatrix} r_1+r_2 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_2 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} ,$$

and

$$\operatorname{rank} W_2(R_0) = \operatorname{rank}\begin{pmatrix} r_1+r_2 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_2 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

From Corollary 2.1 the order of every robust observer is $\geqslant 2$; by inspection

$$\Lambda_0{}^2 = \left\{(r_1, r_2) | r_2 = 0 \quad , \quad r_1 \neq 0\right\}$$

and

$$\Lambda_1{}^3 = \Lambda_2{}^3 = \Lambda_0{}^2 \qquad ;$$

therefore the minimal order is 2. A second order observer is obtained by setting $r_1 = 1$ and $r_2 = 0$, i.e.,

$$R_0 = (1, 0) \qquad ,$$

so that

$$R_0C_0 = (1, 1, 0, 0)$$

and the unobservable subspace Θ of (A_0, R_0C_0) is

$$\left\{\begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}\right\} .$$

The matrix representations of $P : X \to X/\Theta$, $\bar{A}_0 : X/\Theta \to X/\Theta$, and $\bar{C}_0$ satisfying $\bar{C}_0 P = R_0 C_0$ are

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ,$$

$$\bar{A}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ,$$

and

$$\bar{C}_0 = (1, 1) .$$

Choose $\bar{K}_0 = \begin{pmatrix} \bar{K}_1 \\ \bar{K}^2 \end{pmatrix}$ so that the characteristic polynomial of $\bar{A}_0 - \bar{K}_0\bar{C}_0$ has roots in $\mathbb{C}^-$, i.e., if $s^2 + \alpha s + \beta$ is the desired stable polynomial, set

$$\bar{K}_1 = -\beta$$

and

$$\bar{K}_2 = \alpha + \beta + 1 .$$

Now, the robust observer is given by

$$M_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \bar{K}_0 = \begin{pmatrix} \bar{K}_1 \\ \bar{K}_2 \end{pmatrix}$$

$$R_0 = (1, 0) \qquad T_0 = (1, 1) .$$

The driving signal p is given by

$$p = y_1 - (z_1 + z_2)$$

and w is "estimated" by

$$\hat{w} \equiv y_1 - z_2 \quad .$$

Example 1b

If a full state observer is required for the system of Example 1a, it is readily seen from Corollary 2.1 that the order of every robust observer is ≥ 3. Again, by inspection

$$\Lambda_0{}^3 = \left\{ (r_1, r_2) | r_1 = 0 \quad , \quad r_2 \neq 0 \right\}$$

and

$$\Lambda_1{}^4 = \Lambda_2{}^4 = \Lambda_0{}^3 \quad ,$$

so that the minimal order is 3. Since a third order observer must have y_2 as an external input, M_0 must have eigenvalues (0, 2, 3). In fact with $R_0 = (0, 1)$, the parameters of a robust third order state observer are

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and

$$T_0 = (1, 1, 1)$$

and K_0 is chosen to stabilize $M_0 - K_0 T_0$. Also,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the observer "estimates" x_1, x_2, x_3, and x_4 as

$$\hat{x}_1 = z_1 \quad ,$$

$$\hat{x}_3 = z_2 \quad ,$$

$$\hat{x}_4 = z_3 \quad ,$$

and

$$\hat{x}_2 = y_1 - z_1 \quad .$$

The observer possesses redundancy since it is (implicitly) "estimating" $x_1 + x_3 + x_4$, a quantity directly measurable as y_2.

Example 2

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H = I_6 \quad .$$

The system is observable so that a sixth order robust observer exists. If a lower order observer is to be designed, rank $R_0 \leq 2$. Let

$$R_0 = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \end{pmatrix} \quad ;$$

then,

$$W_0(R_0) = \begin{pmatrix} r_1 & 0 & r_2 & 0 & 0 & r_3 \\ 0 & r_3 & 0 & r_1 & r_2 & 0 \\ 0 & 0 & r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_3 & 0 \\ r_4 & 0 & r_5 & 0 & 0 & r_6 \\ 0 & r_6 & 0 & r_4 & r_5 & 0 \\ 0 & 0 & r_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_6 & 0 \end{pmatrix}$$

and

$$W_1(R_0) = \begin{pmatrix} W_0(R_0) \\ C_0 \end{pmatrix} .$$

By elementary operations on $W_1(R_0)$, we have

$$\text{rank } W_1(R_0) = \text{rank} \begin{pmatrix} 0 & r_3 & 0 & r_1 & r_2 & 0 \\ 0 & 0 & 0 & 0 & r_3 & 0 \\ 0 & r_6 & 0 & r_4 & r_5 & 0 \\ 0 & 0 & 0 & 0 & r_6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

It is clear that rank $W_1(R_0) = 6$ implies that rank $W_0(R_0) = 6$ for every R_0 ; therefore, the minimal order is 6 and a sixth order robust observer results by choosing $R_0 = I_3$, $M_0 = A_0$, $T_0 = C_0$, and K_0 so that $M_0 - K_0 T_0$ is sufficiently stable. The minimal order of a conventional observer would be 3 in the present case.

6. CONCLUDING REMARKS

There has been a heavy emphasis in the past few years on the use of observers as devices for generating missing state information. This paper brings to attention, initially, that the conventional observers invariably used in these applications suffer from serious sensitivity problems and, secondly, that even if a robust observer is used, its order is likely to be high and extremely precise plant models and hardware for simulating these models may be required to implement such observers. The precise tolerances permissible would, of course, depend on the particular application. The results of the paper merely specify qualitatively the nature of these requirements.

Some aspects of the design of robust observers are inadequately treated here. The most serious of these is that Theorem 6, which provides a procedure for determining the minimal order, may be totally unsuited for computation since it involves the determination of the loci of polynomials in many variables. The problem of designing minimal order robust observers for use in a closed-loop system implementing an "observed" state feedback control law is not treated here. The structural results of Section 3 may be generalized to some extent by choosing a configuration more general than (22). These and other related problems are to be treated in future articles.

The results of this paper extend the ideas introduced in Reference 2. Some of the structural results on robust observers are similar to those reported recently in Reference 3 in the context of the servomechanism problem [4]. This is because the observer is, of course, a servomechanism since it must drive the observer error to zero; however, in the servomechanism problem the variables to be tracked are specified a priori, whereas the variables to be tracked by an observer are determined in the process of design.

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APPROVAL

THE STRUCTURE OF ROBUST OBSERVERS

By S. P. Bhattacharyya

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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