DECENTRALIZED REGULATION OF DYNAMIC SYSTEMS

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INTRODUCTION

In classical control and decision-making problems, the system is handled in a centralized fashion, namely, there is a single supervisor who handles all the information processing and decision making for the entire system. The decisions of control policies and their implementation are all made according to the preference of this central supervisor.

By contrast, when a large-scale system is considered, such as those arising from the studies of socio-economic problems and electrical power systems, information processing and control policy decision are delegated to a set of agents. Generally, these agents have different information, different permissible control notices, and different preference orderings. These agents may act in complete independence; some may coordinate or supervise the actions of others and hence they form a certain hierarchical decision structure in the system. The behavior of the entire system will result from the interaction of all the decisions by these agents. Such an environment is called here decentralization decision making.

This paper deals with a special class of decentralized control problem in which the objectives of the agents are to steer the state of the system to certain desirable levels. Such a problem is called the decentralized regulation problem. Each agent is concerned about certain aspects of the state of the entire system. The following defines notation:

$$\text{State:} \quad x^T = (x_1{}^T, x_2{}^T, \ldots, x_n{}^T) \qquad x_i \in R^{n_i}$$

$$\text{Control:} \quad u^T = (u_1{}^T, u_2{}^T, \ldots, u_n{}^T) \qquad u_i \in R^{m_i}.$$

The state variables are affected by the individual controls by the following differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x,u) \\ \dot{x}_2 &= f_2(x,u) \\ &\vdots \\ \dot{x}_n &= f_n(x,u) \; . \end{aligned} \tag{1}$$

The objective of agent i is to use control u_i to affect the system so that state x_i can approach a certain desirable level as time $t \to \infty$. Without losing generality, we can assume that such desirable levels are zero for all agents.

The information available to agent i at any time is assumed to be a mapping of the present state x to his data space, that is, for the ith agent information:

$$y_i = h_i(x), \qquad h_i : \underset{j}{\pi} R^{n_j} \to R^{p_i}. \tag{2}$$

Control strategy is assumed to be of a feedback form from information y_i to u_i, that is,

$$\text{Control: } u_i = \gamma_i[h_i(x)], \qquad \gamma_i : R^{p_i} \to R^{m_i}. \tag{3}$$

The mapping γ: above is chosen from the permissible set of functions Γ_i. Two sets of questions arise from the above function:

(1) Stabilization: Given the structure (f_i, h_i, Γ_i), is it feasible to find $\{\gamma_i\}$ such that $x_i \to 0$ as $t \to \infty$ for all i? If it is not feasible, what kind of structure modification will enable us to make it feasible?

(2) Optimization: When it is feasible to regulate all the states, $x_i \to 0$ as $t \to \infty$, what will be the optimal $\{\gamma_i\}$ to achieve such goals with respect to certain performance criteria? What is the impact of various information structures to the performance of regulation?

To be more specific, we shall limit ourselves to linear time-invariant systems:

$$\dot{x} = Ax + \sum_{i=1}^{n} B_i u_i \tag{1$'$}$$

where

$$x^T = (x_1{}^T, \ldots, x_n{}^T)$$
$$y_i = H_i x \tag{2$'$}$$

and

$$u_i = F_i y_i \tag{3$'$}$$

where F_i is a constant real matrix of appropriate dimension.

STABILITY AND COORDINATION

Agent i applies control $u_i = F_i z_i$ to regulate the state x_i so that its desirable level $x_i = 0$ will be achieved and maintained. Because the actions by all agents are coupled, the ability of agent i to regulate state x_i depends on the actions of the other agent. On the other hand, a control u_i of agent i supposedly to decrease the deviation of x_i from zero may affect the other system state variables x_i. This paper discusses the interactions of these individual regulation actions.

Individual Stability

System state variable x_i is said to be *individually stable* with feedback control gain F_i if agent i applies the control $u_i = F_i x_i$ and all other agents take no actions ($F_j = 0 \ \forall j \neq i$) and $t \to \infty$ implies $x_i \to 0$. The collection of all such F_i is denoted by S_i.

Collective Stability

System state variable x is said to be *collectively stable* with feedback control gains $(F_i, \ldots, F_n)$ if, for all i, agent i applies control $u_i = F_i y_i$ and $t \to \infty$ implies $x_i \to 0$ for all i. The collection of all such $(F_i x \ldots, x F_n)$ is denoted by S_c.

The following two observations can be made:

(1) $F_i \in S_i \ \forall i$ does not imply $(F_i x \ldots x F_n) \in S_c$

(2) $F_i \notin S_i \ \forall i$ does not imply $(F_i x \ldots x F_n) \notin S_c$.

These two facts are easily demonstrated by the following scalar example:

$$\dot{x}_i = -x_1 + ax_2 + u_1 + u_2$$

$$\dot{x}_2 = -x_2 + ax_1 + u_1 + u_2.$$

Both u_1 and u_2 influence x_1 and x_2. Agent 1 has access only to information x_1 and agent 2 has access only to information x_2, that is, $u_1 = f_1 x_1$ and $u_2 = f_2 x_2$. For collective stability, the example requires that $f_1 + f_2 < 1 - a$. For individual stability, it requires that $f_1 < 1 - a$ and $f_2 < 1 - a$. Set S_1, S_2, and S_c are illustrated in figure 1(a) and (b). When $-1 < a < 1, S_1 \times S_2 \not\subset S_c$ and when $1 < a, S_1 \times S_2 \subset S_c$. For $-1 < a < 1$, when both agents use their $F_i \subset S_i$ purposely to regulate their state, it is possible that none of them will achieve that goal. For $1 < a$, any individual action $F_i \subset S$ will guarantee the results $x_i \to 0$ as $t \to \infty$ for $i = 1$ and 2. For $1 < a$, it may also be interesting to note that it is possible to have $F_i \not\subset S_i$ for both $i = 1$ and 2 while $x_{1,2} \to 0$ as $t \to \infty$.

If $F_i \subset S_i$ for all i implies $(F_i x \ldots x F_n) \subset S_c$, the system is called coordinated. In a coordinated system, each agent need only to assess his stability regime S_i. If each of them pick an $F_i \subset S_i$, everyone will achieve $x_i \to 0$.

A noncoordinated system can be made coordinated by imposing constraints on the permissible control policies each agent can use. Such constraints should be imposed by a certain *central coordinator* (planner). For instance, in the example given above, when $-1 < a < 1$, if both agents are first forbidden to pick a feedback gain $\geqslant (1 - a)/2$, the system becomes coordinated.

The concept of coordination in a system is important. In a coordinated system, agents have greater degree of autonomy. Both the decision of his control policy and implementation of such control can be done by total decentralization. In a noncoordinated system, the solution of all control policies for each agent may require the help of a central coordinator, which may involve very difficult computation. Substituting $u_i = F_i y_i$ and $y_i = H_i x$ into $\dot{x} = Ax = \Sigma B_i u_i$, it becomes

$$\dot{x} = (A + \Sigma B_i F_i H_i)x \tag{4}$$

Obviously, the condition for collective stability is that all eigenvalues of $A + \Sigma B_i F_i H_i$ have negative real parts.

A more basic question to ask is that, for given structure matrices (A_i, B_i, H_i), is it ever possible to find matrices $(F_1, \ldots, F_n)$ that stabilize the system? Fisher and Fuller (ref. 1), McFadden (refs. 2, 3), and Aoki (ref. 4) have all tried to answer this *stabilizability* problem in certain aspects. The general stabilizability problem for system equation (4) has been solved recently by Wang and Davison (ref. 5). The problem of finding conditions on (A_i, B_i, H_i, F_i) such that the system is coordinated, controlled by agents and can fully be decentralized is still open.

OPTIMIZATION OF DECENTRALIZED CONTROL

When the system can be stabilized collectively, we would like to consider how to optimize the choice of $(F_1, \ldots, F_n)$. Costs are associated with the deviation of the states from their desirable values and the magnitudes of the control "forces." The objective of the central coordinator (planner) is to choose $(F_1, \ldots, F_n)$ to minimize these costs. Suppose that the costs are described by the following quadratic loss function:

$$J = \frac{1}{2} \int_o^x (x^T Q x + \Sigma u_i^T R_i u_i) dt \tag{5}$$

where $Q \geqslant 0$ and $R_i > 0\ V\ i$. Since

$$\dot{x} = Dx$$

where

$$D = A + \Sigma B_i F_i H_i \tag{6}$$

we have

$$x = e^{Dt}\, x_0 \tag{7}$$

where x_0 is the state deviation at $t = 0$. Then

$$J = \frac{1}{2}\int_{o}^{\infty} x_o{}^T e^{D^T t}\left(Q + \Sigma H_i{}^T F_i{}^T R_i F_i H_i\right) e^{Dt} x_o \, dt$$

$$= \frac{1}{2} \operatorname{trace}\left[\int_{o}^{\infty} e^{D^T t}\left(Q + \Sigma H_i{}^T F_i{}^T R_i F_i H_i\right) e^{Dt} \, dt \, x_o x_o{}^T\right]. \tag{8}$$

It can be shown that, if D is stable, J can be expressed as (ref. 6)

$$J = \frac{1}{2} \operatorname{trace} \left(K x_o x_o{}^T\right) \tag{9}$$

where

$$KD + D^T K + Q + \Sigma H_i{}^T F_i{}^T R_i F_i H_i = 0 . \tag{10}$$

Since D is linear in $\{F_i\}$, K solved by equation (10) is a rational function of $\{F_i\}$. The determination of optimal $\{F_i\}$ falls in the framework of classical nonlinear programming.

It can be shown (ref. 7) that the gradient at the optimal choice of $\{F_i\}$ must satisfy

$$0 = \frac{\partial T}{\partial F_i} = R_i F_i H_i L H_i{}^T + B_i K L H_i{}^T \, ; \quad i = 1, \ldots, n \tag{11}$$

where

$$LD^T + DL + x_o x_o{}^T = 0 . \tag{12}$$

The necessary condition of optimality (eqs. (10)–(12)) is given here as the generalization of the single-agent output optimization problems studied by Levine and Athans (ref. 8). Note that sometimes the initial value x_o is not known exactly. The product $x_o x_o{}^T$ in equations (9) and (12) should be replaced by its expected value. Furthermore, if we wish to find the optimal $\{F_i\}$ independent of the initial states, we could use an approach by assuming that the initial state is random and distributed uniformly in a sphere, namely, the expected value of $x_o x_o{}^T$ is expressed as an identity matrix.

What is expressed in equations (10), (11), and (12) is the necessary condition of optimal $\{F_i\}$; it is entirely possible to have more than one solution to such conditions. In any case, the determination of optimal $\{F_i\}$ involves many structure parameters and we cannot possibly expect elegant solutions unless the system is of very low dimension or of high dimension with nice structures (e.g., in symmetry or repetition).

Since in practice only a high-dimension system justifies decentralized control, we should examine in the sequel some examples of high-dimension decentralized problems where the system is nicely structured. With such a system, we will be able to derive some results and to understand the relations between structure and control system behavior in more explicit terms.

Sequential Systems (ref. 9)

The system can be decomposed into a sequence of subsystems, all identical in structure (see fig. 2). The interactions among subsystems depends only on the distance, that is, the difference in the subsystem indices.

Subsystem i: state x_i and control u_i

$$x^T = (\ldots, x_{i-1}^T, x_i^T, x_{i+1}^T, \ldots) \tag{13}$$

$$\dot{x}_i = \sum_j A_{i-j} x_j + \sum_j B_{i-j} u_j \tag{14}$$

$$y_i = \sum_j H_{i-j} x_j \,. \tag{15}$$

For simplicity, the number of subsystems in the string is assumed to be infinite so that there are no ends in the string and the roles of all subsystems are identical. If a system comprises a large but finite number of subsystems, it can be treated as if the number were infinite by assuming fictitious subsystems at both ends of the string.

Since all subsystems are identical, the feedback gains used in each control can also be assumed identical, that is,

$$u_i = F y_i \quad \text{for all } i \tag{16}$$

Bilateral transformations can be used to convert the sequential subsystems into a lumped system in the z domain:

$$\{x_i\} \xrightarrow{z} X(z) = \sum_{1=-\infty} x_i z^{-i} \tag{17}$$

$$\{u_i\} \xrightarrow{z} U(z) = \sum_{1=-\infty} u_i z^{-i} \tag{18}$$

.
.
.
etc.

Optimal conditions of F can be obtained by converting equations (9), (10), (11), and (12) into appropriate counterparts involving a z transformation (ref. 9).

This configuration is illustrated in figure 3. It is similar to the sequential system in the previous section, except that the subsystems are distributed in two-dimensional arrays. If the structures and interactions of all subsystems are all identical subject only to index translation, double bilateral z-transformation techniques can be used to solve the optimization problem. Such models could represent situations of street traffic control, huge electric network control, etc.

Regulation of Vehicular Strings (ref. 10)– A string of high-speed, densely packed vehicles are moving along a certain guideway. It is desirable to keep the spacings and velocities of all vehicles in the string as close as possible to certain predetermined values (see fig. 4). The position deviation of the kth vehicle from its predetermined reference is denoted by x_k. The dynamics of the kth vehicle can be described by the second-order, normalized differential equation:

$$\ddot{x}_k + \alpha \dot{x}_k = u_k \; ; \qquad -\infty < k < \infty . \tag{19}$$

The total information data for the kth vehicle is denoted by a vector y_k given by

$$y_k = \sum_j H_{k-j} \begin{bmatrix} x_j \\ \dot{x}_j \end{bmatrix} \; ; \qquad -\infty < k < \infty . \tag{20}$$

The structure of equation (20) includes the following special cases:

(i) $y_k^T = (x_{k-1}, x_k, x_{k+1}, \dot{x}_{k-1}, \dot{x}_k, \dot{x}_{k+1})$

(ii) $y_k^T = (x_k, \dot{x}_k)$

(iii) $y_k^T = (x_{k-1}, x_k, x_{k+1})$

(iv) $y_k^T = (x_k)$

(v) $y_k^T = (x_j, \dot{x}_j | -\infty < j < \infty)$.

The linear feedback control used by each vehicle is of the same form:

$$u_k = F\, y_k \; ; \qquad -\infty < k < \infty . \tag{21}$$

The regulation cost is assumed to be of the following form where the magnitude of the control forces and relative vehicle position errors are penalized.

$$J = \frac{1}{2} \int_0^\infty \sum_k \left[q(x_{k-1} - x_k)^2 + u_k^2 \right] dt . \tag{22}$$

Results— If $\alpha = 1$, $q = 10$, and the information is given as

$$z_k = (x_{k-1}, x_k, x_{k+1}, \dot{x}_{k-1}, \dot{x}_k, \dot{x}_{k+1})^T . \tag{23}$$

The optimal control for each vehicle is

$$u_k = f_0 x_k + f_1 (x_{k-1} + x_{k+1}) + g_0 \dot{x}_k + g_1 (\dot{x}_{k-1} + \dot{x}_{k+1})$$

where

$$f_0 = -4.06 , \quad f_1 = 1.43 , \quad g_0 = 1.94 , \quad g_1 = 0.52 . \tag{24}$$

The associated minimal cost is

$$J^* = 7.59 . \tag{25}$$

Figure 5 represents the stability region as well as the sensitivity of J to various choices of feedback gains. Figure 6 shows the behavior of a string of 10 vehicles when the optimal control scheme is applied. The position of each vehicle is plotted relative to a coordinate system moving with the desired velocity. At $t = 0$, all vehicles are subject to random perturbations in position and velocity. Also, vehicle 5 is constantly driven with a sinusoidal disturbance force. For comparison, figure 7 shows the response of the 10 vehicles when the controllers chosen are too sensitive to the motion of the neighboring vehicles – the result is a chain collision.

INFORMATION STRUCTURE DESIGN

Structure Design versus Stabilizability

The problem of stabilizability for a given structure (A, B_i, H_i) was discussed under Stability and Coordination. A more basic problem is how the information structures H_i influence the stabilizability. In other words, if a system cannot be stabilized initially by the imposed decentralized scheme, what kind of change in information structure can induce stabilizability?

Example— In the vehicular string regulation problem presented earlier, it can be shown that, if the information of vehicle k's state x_k is available to him, the system can be made stable by appropriate choice of the feedback gains. If the information available to the controllers is only velocity data with no position data, the entire system can never be stabilized by any choice of feedback gains.

Example – Assignment Problem (ref. 3)– In a special decentralized regulation problem,

$$\dot{x} = \Sigma B_i u_i . \tag{26}$$

If control agent i knows only about the ith component of x, the feedback will be of the following form:

$$u_i = F_i x_i \tag{27}$$

hence

$$x = [B_1, B_2, \ldots, B_n] \begin{bmatrix} F_1 & & 0 \\ & F_2 & \\ 0 & & \ddots F_n \end{bmatrix} x \,. \tag{28}$$

A nonstabilizable system could possibly be stabilized by permiting the information available to each agent. It has been shown that if the matrix $[B_1, B_2, \ldots, B_n]$ is nonsingular, it is always possible to stabilize the system of equation (28) by an appropriate permutation of the information structure, that is,

$$\begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = Px$$

where P is a permutation matrix (refs. 3, 11).

Structure versus Regulation Performance

As expressed in equation (5), the regulation performance of the entire system is given by a certain cost function J. When the regulation is feasible (stabilizable), it is interesting to ask how each piece of information contributes to the optimization of J. A useful concept of the *value of information* can be defined. Simply speaking, the value of information can be visualized as the difference between the optimal J^* with the given information and the optimal J^* obtained without that particular information.

Reconsider the vehicular string regulation problem. A list of certain information structures with the optimal J^* of each is given below (see also fig. 8).

Information available to the kth vehicle		Optimal J^*
Case A:	$x_{k-1}, x_k, x_{k+1}, \dot{x}_{k-1}, \dot{x}_k, \dot{x}_{k+1}$	7.59
Case B:	x_{k-1}, x_k, x_{k+1}	11.30
Case C:	$x_k, \dot{x}_k$	8.13
Case D:	x_k	11.81
Case E:	$\{\phi\}$ (nothing)	∞
Case F:	$\{x_j, \dot{x}_j \mid -\infty < j < \infty\}$	7.54

If each vehicle is provided with its own state information x_k, $J^*_{(D)}$ is 11.81. The following are values of certain added pieces of information for the kth vehicle:

(i) $\dot{x}_k$	3.68	$J^*(D) - J^*(C)$
(ii) x_{k-1} and x_{k+1}	.50	$J^*(D) - J^*(B)$
(iii) $x_{k-1}, x_{k+1}, \dot{x}_{k-1}, \dot{x}_{k+1}$	.54	$J^*(C) - J^*(A)$
(iv) $\dot{x}_{k-1}$ and $\dot{x}_{k+1}$	.03	$J^*(\text{iii}) - J^*(\text{ii})$
(v) $\{x_j, \dot{x}_j \mid \text{for all } \lvert j-k \rvert \geqslant 2\}$	.05	$J^*(A) - J^*(F)$.

For regulating this vehicular system, the controller's own velocity data $\dot{x}_k$ is much more important than the data x_{k-1} and x_{k+1}. Moreover, the remote input data $\{x_j, \dot{x}_j \text{ for } \lvert j-k \rvert \geqslant 2\}$ do not significantly add to the optimization of the regulation performance. If the structure in case A is adopted, despite the fact that much less information is required than for case F, the performance index value is very close to the ultimate minimum.

In real implementation of various control schemes, it is important to consider the cost of installing various measurement and control mechanisms and the feasibility of establishing various links for data and controls. The concept of information value discussed here provides a quantitative method for measuring and comparing the relative merits of different information structures and hence the usefulness of the different information provided.

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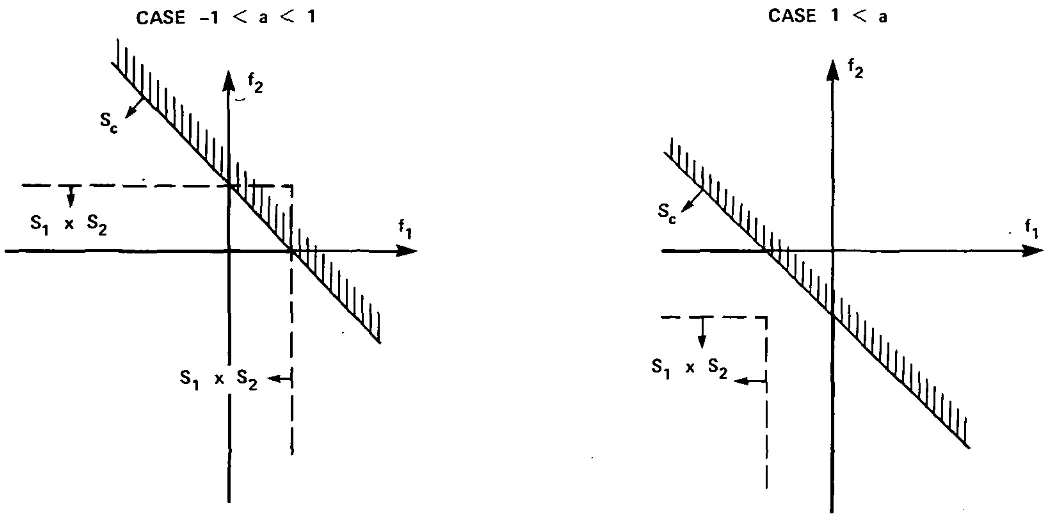


(a) Not coordinated. (b) Coordinated.

Figure 1.– Example relations between individually and collectively stable systems.

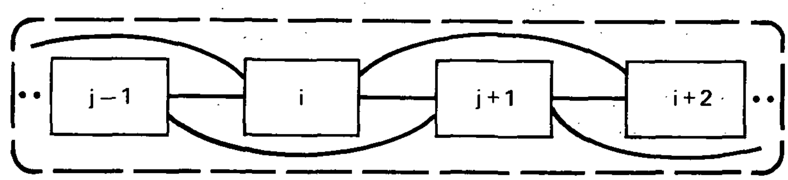


Figure 2.– A section of a sequential system and some of its interactions.

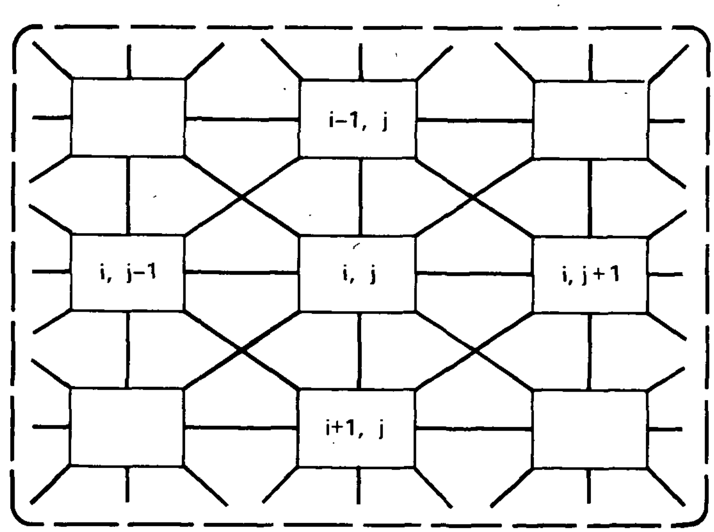


Figure 3.– A section of a planar array of identical subsystems and some of its interactions.

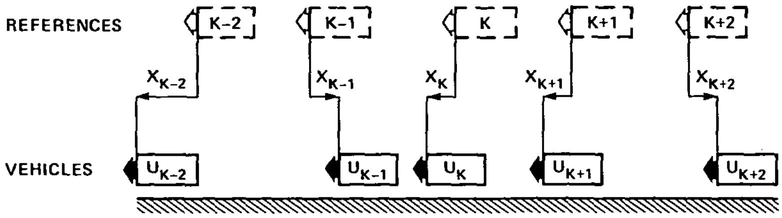
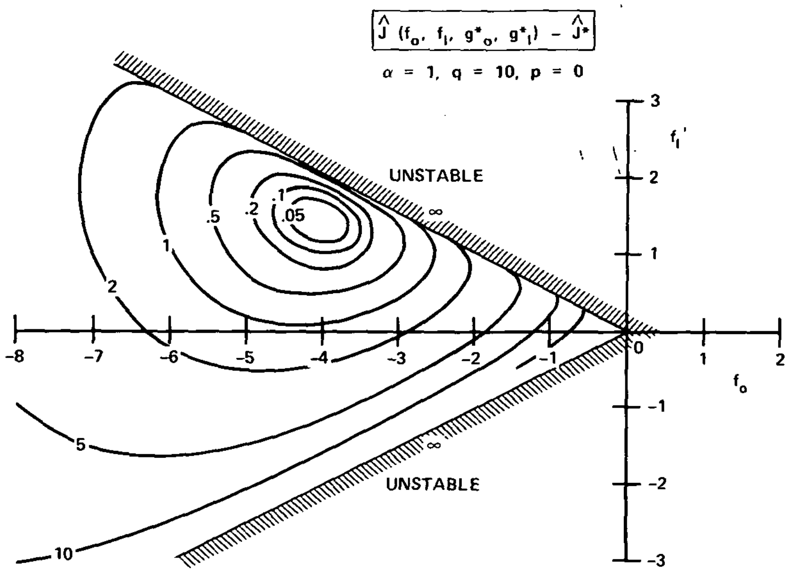
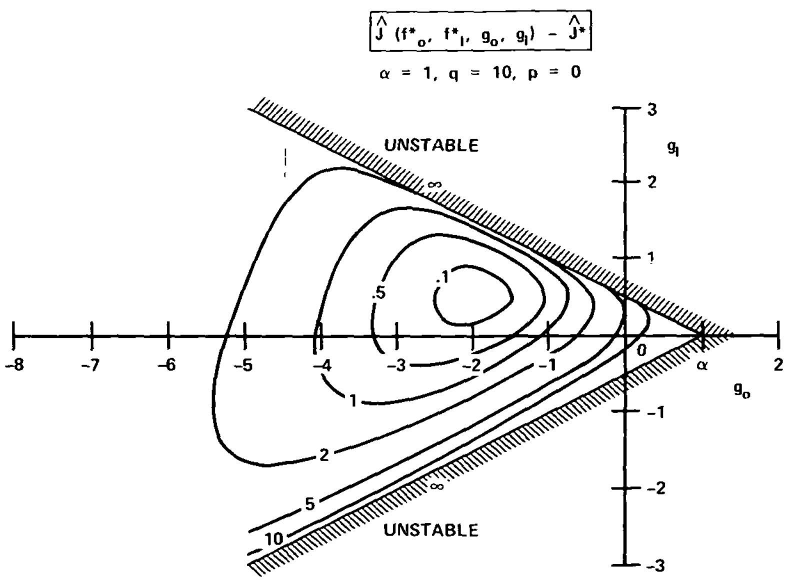


Figure 4.– Regulated vehicular strings: a sequential system.



(a) Not coordinated.



(b) Coordinated.

Figure 5.– Stability regions for a regulated vehicular string.

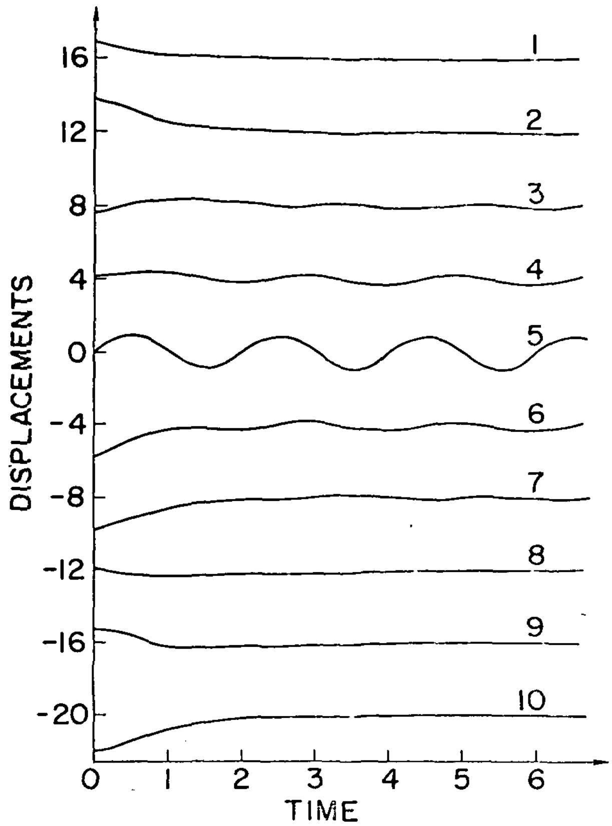


Figure 6.– Behavior of an optimally regulated string of 10 vehicles with the 5th vehicle oscillating.

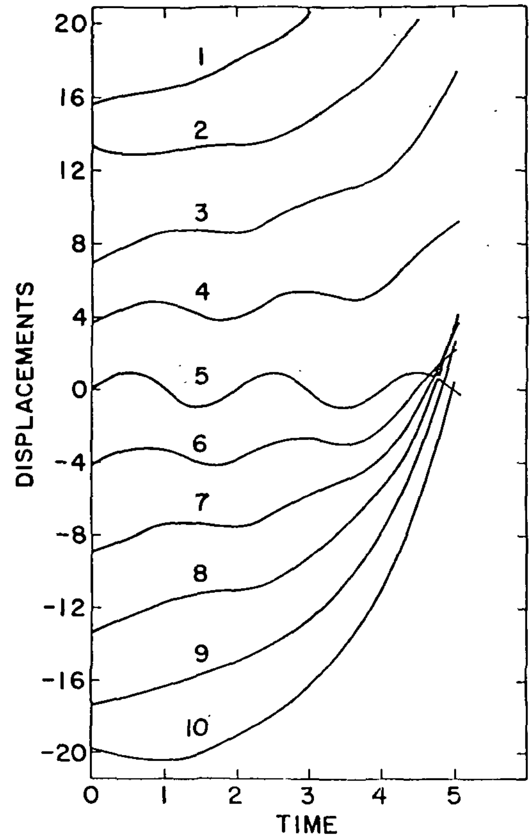


Figure 7.– Behavior of a vehicular string with excessive regulation.

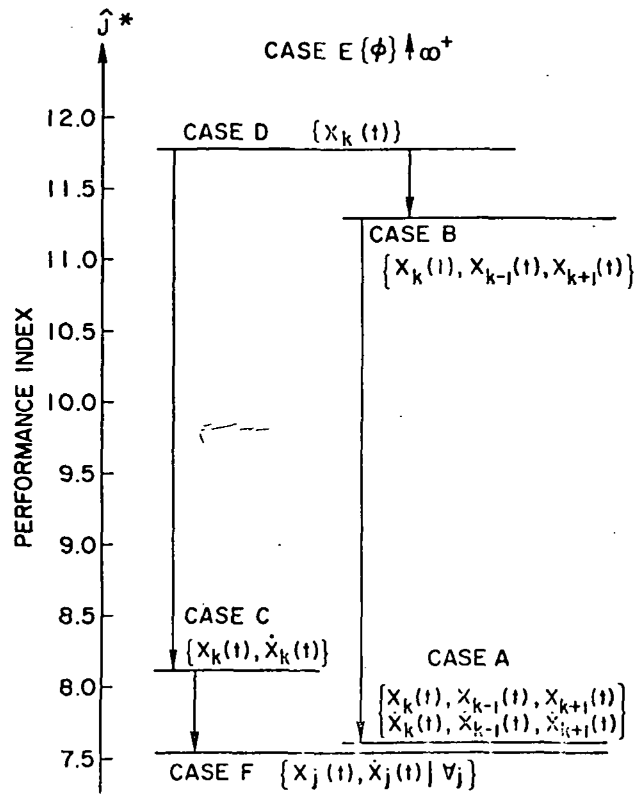


Figure 8.– Information value of various information structures in a regulated vehicular string.