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ABSTRACT

Substantial advances have been made over the past decade in the prediction of turbulent flows. There has been extensive work in the development of turbulence models, particularly for use in boundary layer calculations. This review covers, in a common notation, the basic aspects of several important methods based on partial differential equations for the mean velocity field and turbulence quantities, including the relationship between the methods and suggestions for future development. New work on three-dimensional time-dependent large eddy simulations is discussed. The emphasis is on the hydrodynamics of incompressible flows, but sources for consideration of heat transfer and compressibility are mentioned.

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1. INTRODUCTION

The computation of turbulent flows has been a problem of major concern since the time of Osborne Reynolds. Until the advent of the high-speed computers, the range of turbulent flow problems that could be handled was very limited. The advances during this period were made primarily in the laboratory, where basic insights into the general nature of turbulent flows were developed, and where the behaviors of selected families of turbulent flows were studied systematically. For the engineer there were only a limited number of useful tools such as boundary layer prediction methods based on the momentum integral equation with a high empirical content. Features such as sudden changes in boundary conditions, separation, or recirculation could not be predicted by these early methods with any degree of reliability. Very specific empirical work remained an essential ingredient of any engineer's analysis.

Midway through this century computers began to have a major impact. First it became possible to handle more difficult boundary layers by complex integral analyses involving several first order ordinary differential equations. By the mid 1960's there were several workers actively developing turbulent flow computation schemes based on the governing partial differential equations, (pde's). The first such methods used only the equations for the mean motions but second generation methods began to incorporate turbulence pde's.

In 1968 Stanford hosted a specialists conference designed to assess the accuracy of the then current turbulent boundary layer prediction methods (Kline, et. al 1968). The main impact of this conference was to legitimize pde methods, which proved to be more accurate and more general than the best integral methods.

Vigorous development of more complex and supposedly more general pde turbulence models followed. Methods were first developed in which a pde for the turbulence energy was solved in conjunction with the pde's for the mean motion. Then, in an effort to reduce the empiricism required, models incorporating a pde relating to the turbulence length scales were studied.

More recently there has been intense development of models involving pde's for all of the nonzero components of the turbulent stress tensor.

The ability of these more complex models to produce predictions for the detailed features of turbulent flows has outstripped the available storehouse of data against which these predictions can be compared; moreover the output of these programs now includes quantities that are difficult, if not impossible to measure. At the same time these rapid developments were being made in computation, some totally new approaches to turbulence experiments were introduced (Laufer 1975). These centered on the observation that turbulent shear flows possess a remarkable degree of organization of their large-scale motions. New "selective sampling" techniques were introduced to study these structures, and a great deal has been learned. As yet the pde models have not made any use of the new experimental data, perhaps because large scale transport is not really consistent with the "local" ideas used in pde models.

One new approach that appears promising, and is just beginning to be carefully explored, is the idea of using a very fast, very large computer to solve three-dimensional time-dependent pde models for the large scale turbulence. These would incorporate a simple model of the small scale turbulence in some semi-empirical way. At present these methods are in their infancy, but already they have begun to shed some light on the simpler pde models, in some cases producing numerical values for constants used in the "simpler" two-dimensional steady pde models. As experience with this approach grows, and as machines improve, it seems quite likely that this type of calculation will eventually be useful at the engineering level.

This review will outline the essential ingredients and effectiveness of several levels of turbulent flow pde models:

1. zero-equation models - models using only the pde for the mean velocity field, and no turbulence pde's.
2. one-equation models - models involving an additional pde relating to the turbulence velocity scale.
3. two-equation models - models incorporating an additional pde related to a turbulence length scale.
4. stress-equation models - models involving pde's for all components of the turbulent stress tensor.

5. large-eddy simulations - computations of the three-dimensional time-dependent large eddy structure and a low-level model for the small-scale turbulence.

Zero-equation models are common practice in the more sophisticated engineering industries, and one-equation models find use there on occasion. Two-equation models, currently popular among academics, have not been used extensively for engineering applications, probably because one can do as well if not better in most problems with simpler methods. Stress-equation modeling is now under intensive development; it is essential for handling the more difficult flows, and will probably become standard practice in industry in ten years time. Large eddy simulations are just in their infancy, and are serving mainly to help assess the lower level models. However, in the long term, large-eddy simulation may be the only way to accurately deal with the difficult flows that stress-equation models are presently trying to handle.

Four other reviews have appeared recently covering selected aspects of the subject. Reynolds (1974), in a publication long delayed in press, outlined the state of affairs in 1970. Mellor and Herring (1973) provided an overview of one-equation, two-equation, and stress-equation modeling as of mid 1972. Cebeci and Smith (1974) have an entire book on the subject, concentrating primarily on their own zero-equation approach. Bradshaw (1972) wrote an incisive and delightful review of the interplay between model development and experimentation that should be mandatory reading for all students of the field.

The present review will concentrate on the hydrodynamic modeling of incompressible flows, but sources of insight for extension to compressibility and heat transfer will be mentioned.

2. ZERO-EQUATION MODELS

The equations describing the mean velocity field in incompressible turbulent flow are well known (Tennekes and Lumley 1972); they follow from the Navier-Stokes equation by the usual decomposition of the velocity field into mean and fluctuating components, $u_i = U_i + u_i'$, and may be written as

$$\dot{U}_i + U_j U_{i,j} = -\frac{1}{\rho} p_{,i} + (2\nu S_{ij} - R_{ij})_{,j} \qquad (2.1a)$$

$$U_{i,i} = 0 \qquad (2.1b)$$

Here we use the cartesian tensor summation convention, in which repeated indices are to be summed over all three coordinates. Subscripts after commas denote partial differentiation, e.g. $U_{i,j} = \partial U_i / \partial x_j$, and the overdot denotes a partial derivative with respect to time. $R_{ij} = \overline{u_i u_j}$ ($-\rho R_{ij}$ is the Reynolds stress tensor), and $S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ is the strain-rate tensor; ν is the kinematic viscosity, p is the pressure, and ρ the mass density. Note that $S_{ii} = 0$ by (2.1b).

To close eqs. (2.1), additional equations must be provided for R_{ij}. In the simplest models R_{ij} is described by a Newtonian constitutive equation of the form

$$R_{ij} = \frac{1}{3} q^2 \delta_{ij} - 2\nu_T S_{ij} \tag{2.2}$$

where $q^2 = R_{ii}$, and ν_T is a turbulent or eddy viscosity which must be prescribed in some suitable manner. The q^2 term can be absorbed in with p, and so need not be calculated explicitly.

In a zero-equation model ν_T is related directly to the mean velocity field U_i. For free shear flows (jets and wakes) one makes the usual boundary layer assumptions to simplify (2.1). Remarkable success is obtained with simple assumptions of the form

$$\nu_T = K \Delta U b \tag{2.3}$$

where ΔU is some appropriate velocity difference associated with the flow (e.g., the difference between jet centerline velocity and the velocity of the external flow), and b is a length scale characterizing the width of the jet. The constant K may vary from flow to flow, but is typically of the order 0.05-0.1. In this model the turbulent viscosity is constant across the shear layer at any given downstream station (see Schlichting 1968.) A similar sort of assumption also works very well in the outer (wake) region of turbulent boundary layers.

In the wall region of a turbulent boundary layer is is essential to consider the cross-stream variation of the turbulent viscosity. Outside of the viscous region a commonly used form is

$$\nu_T = \kappa u_* y \tag{2.4}$$

Here κ is the "Karman constant," (approximately 0.4), u_* is the "shear velocity," $u_* = \sqrt{\tau_w/\rho}$ where τ_w is the wall shear stress and $y = x_2$ is the distance from the wall. Very close to the wall, where viscous effects are important, success has been had with simple modifications of (2.4) that reflect the effect of the wall in suppressing turbulent transport, for example

$$\nu_T = \kappa u_* y(1 - e^{-y^+/A^+})^2 \tag{2.5}$$

where $y^+ = yu_*/\nu$, and A^+ is an empirical constant.

Alternatively, many have used the "mixing length model," which can be generalized by

$$\nu_T = \ell^2\sqrt{2S_{nm}S_{nm}} \tag{2.6}$$

where ℓ is the "mixing length." In the wall region of a turbulent boundary layer, but outside of the viscous region, the velocity field is known to behave as

$$\frac{\partial U}{\partial y} = \frac{u_*}{\kappa y} \tag{2.7}$$

where $U = U_1$ is the flow velocity parallel to the wall. This is the only important element of $U_{i,j}$. With $\ell = \kappa y$ in the wall region, (2.4) and (2.7) are equivalent.

Patankar and Spalding (1970) were among the first to document boundary layer computation methods of this type, and now make programs available on a commercial basis. More recently Cebeci and Smith (1974) devoted an entire book to the subject, emphasizing their own particular computational models and processes of this general type. A Stanford group under W. M. Kays and R. J. Moffat has been working with these methods for several years, with the distinct advantage of doing this in parallel with their comprehensive experimental program on turbulent boundary layers with wall suction, blowing, pressure gradient, and heat transfer. Their own particular model is certainly one of the most advanced of this type, and I have chosen to delve

into it in more detail to illustrate the empiricism and capabilities of such methods. Their present program is called STAN-5, and is available upon request for reproduction costs (Crawford and Kays, 1975).

The boundary layer simplifications of (2.1) produce

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial p^*}{\partial x} + \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial U}{\partial y}\right] \tag{2.8}$$

where we have used $U_i = (U,V,0)$, $x_i = (x,y,z)$, and $p^* = p/\rho + q^2/3$. In a boundary layer calculation, $p^*(x)$ is derived from the pressure distribution applied by the external flow. STAN-5 uses (2.6) specialized to boundary layer flows,

$$\nu_T = \ell^2 \left|\frac{\partial U}{\partial y}\right| \tag{2.9}$$

In the outer region it uses $\ell = \lambda\delta_{.99}$, where $\delta_{.99}$ is the thickness of the boundary layer to the point where U is 99% of the free-stream velocity U_∞. The factor λ is provided with a dependence on the momentum thickness Reynolds number $R_\theta = \theta U_\infty/\nu$ in order to better predict low Reynolds number flows,

$$\lambda = \min \left\{ \begin{matrix} 0.085 \\ \\ 0.25R_\theta^{-0.25}\ (1 - 67.5\ F) \end{matrix} \right\} \tag{2.10}$$

Here F is a wall layer blowing parameter, V_0/U_∞, where V_0 is the velocity of injection into the flow through the wall.

The inner regions are handled by assuming that

$$\ell = \kappa y(1 - e^{-y^+/A^+}) \tag{2.11}$$

with $\kappa = 0.41$. The parameter A^+ is given as a complicated function of both the pressure gradient and blowing rate, shown in Fig. 1. There $v_0 = V_0/u_*$, and $p^+ = (dp/dx)(\nu/\rho u_*^3)$. An empirical fit to Fig 1 is used in STAN-5. The parameter A^+ determines the thickness of the viscous region; this will not change suddenly if p^+ or v_0^+ changes suddenly; to accomodate this delay, STAN-5 uses a "lag" equation,

$$\frac{dA^+}{dx^+} = \frac{A^+ - A_e^+}{4000} \qquad (2.12)$$

where A_e^+ is determined from Fig. 1, and $x^+ = xu_*/\nu$. In handling the heat transfer problem, similar models and empiricism are required; for details see Crawford and Kays (1975).

For a particular flow of interest, $U_\infty(x)$ and $p(x)$ are known, and a "starting" profile $U(x_0,y)$ must be prescribed. The numerics are actually executed in STAN-5 using the stream function as a dependent variable and the mean vorticity as independent variable, as in Patankar and Spalding (1970). The mesh points are closely spaced in the wall region, and then expand out away from the wall.

The resulting velocity distributions, temperature distributions, skin friction, and heat transfer are typically in excellent agreement with experiments, except for layers very close to separation. Figure 2 shows one of the greater triumphs of the STAN-5 model, the heat transfer predictions for a turbulent boundary layer subjected at first to strong blowing, which is removed midway through a section of very strong acceleration, which in turn is terminated downstream. The rapid changes in heat transfer coefficient that accompany the cessation of blowing and acceleration are extremely difficult to predict; every element of the empiricism reflected above is essential to the success of this calculation.

Two other groups are experienced in the use of zero-equation methods for a wide variety of problems. The first is that of T. Cebeci and A. M. O. Smith at the Douglas Aircraft Corporation. They have extended their calculations to compressible flows, flows over axisymmetric bodies and bodies with longitudinal curvature, and have done extensive calculations on aircraft wing and body systems. Their particular model, as well as their numerical technique, is outlined in detail in their book (1974), which is highly recommended to potential user of zero-equation methods. Cebeci (1975) has extended their procedures to three-dimensional turbulent boundary layers. A second group is that at Imperial College, under D. B. Spalding. Patankar and Spalding's book (1970) describes their zero-equation approach for turbulent boundary layers, and another book by Gosman et. al (1969) describes their

modeling of recirculating flows. The most finely tuned zero-equation model for boundary layers is probably the STAN-5 program developed at Stanford as an extension of the Patankar-Spalding approach (Crawford and Kays 1975).

Zero-equation models like STAN-5 are extremely useful in engineering analysis. However, they fail to handle some important effects, such as strong surface curvature and free-stream turbulence, all important on turbine blades. Nor are they accurate near separation points, or in boundary layers subjected to extremely strong accelerations. The more advanced models, which incorporate a pde for the turbulence kinetic energy, were originally introduced in the hope of providing additional generality and at the same time to reduce the extensive empiricism that is essential to success in a zero-equation model.

3. ONE-EQUATION MODELS

An equation describing the dynamics of the turbulence kinetic energy can be derived from the Navier-Stokes equations by simple manipulations (Tennekes and Lumley 1972),

$$\dot{q}^2 + U_j q^2{}_{,j} = 2(\mathcal{P}-\varepsilon) - J_{j,j} \qquad (3.1)$$

Here $\mathcal{P} = -R_{ij}U_{i,j}$ is the rate of production of turbulence energy, $\varepsilon = 2\nu\overline{s'_{ij}s'_{ij}}$ is the rate of energy dissipation, and $J_j = \overline{(u'_i u'_i u'_j} + \frac{1}{\rho}\overline{p'u'_j} - 2\nu\overline{u'_i s'_{ij}})$ is the twice diffusive flux of turbulent kinetic energy, all per unit of mass. We use $s'_{ij} = \frac{1}{2}(u'_{i,j} + u'_{j,i})$.

Alternatively, (3.1) can be written with ε replaced by the "isotropic dissipation" $\mathcal{D} = \nu\overline{u'_{i,j}u'_{i,j}}$ and J_j replaced by $J_j^* = \overline{u'_i u'_i u'_j} + \frac{1}{\rho}\overline{p'u'_j} - \nu q^2{}_{,j}$. This second form is appealing because of the direct appearance of the gradient diffusion of q^2 by ν in J_j^*. Some authors have incorrectly termed $\mathcal{D}$ the dissipation. At high Reynolds numbers the isotropy of the small scale turbulence renders $\mathcal{D} = \varepsilon$, but this is not true at low Reynolds numbers, or near a wall.

In one-equation turbulence models, (3.1) forms the basis for a model equation for the turbulence velocity scale q. Typically (2.2) is used as a constitutive equation, and the turbulent viscosity is modeled by

$$\nu_T = c_2 q \ell \tag{3.2}$$

The length scale ℓ is prescribed, much as in the zero-equation approach typified by STAN-5. The dissipation and transport are modeled in terms of the scales q and ℓ. It is well known that, at high Reynolds numbers, the rate of energy dissipation is controlled by inviscid mechanisms (non-linear interactions that cascade energy to smaller scales) and that the small scale motions adjust in size to accomodate the imposed energy dissipation. Hence, by dimensional analysis

$$\mathcal{D} = c \tau^3/\ell \tag{3.3}$$

The diffusive flux is usually treated by a gradient diffusion model,

$$J_j^* = -(c_4 \nu_T + \nu)\, q^2{}_{,j} \tag{3.4}$$

STAN-5 has the capability of incorporating this one-equation model for boundary layer analysis. The zero-equation approach described above is used for $y^+ \leq 2A^+$; for $y^+ > 2A^+$ (3.1)-(3.3) are employed, using (2.10) and (2.11) to prescribe ℓ. Guidance in selection of the constants is obtained using the well-known fact that, immediately outside of the viscous layer, (2.7) holds, and the production and dissipation terms are essentially in balance. Using (2.7), (2.9), and (3.2), in this region $c_2 = u_*/q$. Setting $\mathcal{P}-\mathcal{D} = 0$ in this region, one obtains $c_3 = (u_*/q)^3 = c_2^3$. STAN-5 uses $c_2 = 0.38$, $c_3 = 0.055$, suggested by experiments which show $q^2/u_*^2 \approx 7$ in this region, and $c_4 = 0.59$, which was determined by comparing calculations with the one-equation model with those of the zero-equation model. As a "boundary" condition on the q^2 calculation, which is carried out only for $y^+ \geq 2A^+$, STAN-5 requires that q^2 be such that ν_T at $y^+ = 2A^+$ matches ν_T generated by the mixing length model (2.9) at this point. Kays and his co-workers have used this model to explore the effects of free-stream

turbulence on boundary layer heat transfer (Kearney et. al 1970) and presently are using the model to study the effects of rapid changes in free-stream conditions ("non-equilibrium" boundary layer behavior).

Norris and Reynolds (1975) proposed a one-equation model that shows promise as an alternative to the highly empirical A^+ correlation and empirical lag equation needed if one is to get good results in the viscous region. Their intent was to develop a one-equation model that is valid right down to the wall. Noting that at low Reynolds numbers the dissipation should scale as $\nu q^2/\ell^2$, they use

$$\mathcal{D} = c_3 \frac{q^3}{\ell} \left[1 + \frac{c_5}{q\ell/\nu} \right] \tag{3.5}$$

They argue that the length scale should do nothing special in the viscous region, but should behave like $\ell = \kappa y$ right down to the wall. Near the wall, $q \sim y$, and so (3.5) near the wall becomes $\mathcal{D} = c_3 c_5 \nu q^2/\ell^2$ and $\mathcal{D}$ approaches a constant as $y \to 0$. This is indeed the proper physical behavior of the dissipation. Finally, they use (3.4), but assume that the turbulent transport is suppressed by the presence of the wall, and hence

$$\nu_T = c_2 q \ell (1 - e^{-c_6 q y/\nu}) \tag{3.6}$$

Note that this produces $\nu_T \sim y^4$ as $y \to 0$. At the wall (3.1) becomes $-2\mathcal{D} + \nu \partial q^2/\partial y^2 = 0$, which requires $c_3 c_5/\kappa^2 = 1$ if $q \sim y$ near $y = 0$. Having established c_3, this determines c_4. Finally, a value for c_6 can be estimated from the known behavior for a flat-plate boundary layer, and they used $c_6 = 0.014$.

Norris and Reynolds applied this model to channel flow with blowing from one wall and equal suction on the other. For ℓ they used a smooth fit between $\ell = 0.4y$ near the wall and $\ell = 0.13\delta$ in the center, where δ is the channel half-width. The mean velocity profiles calculated in the wall region, and the change in skin friction over the no-blowing case, are in excellent agreement with the corresponding data for flat-plate boundary layers. Since the main effect on A^+ is that of v_o^+, and the

Norris-Reynolds model seems to handle that quite well, it does seem likely that it will handle the pressure gradient system as well. A boundary layer version of this model is now being prepared to study this conjecture.

A similar approach was adopted by the Imperial College group, reported by Wolfshtein (1969). However, Wolfshtein allowed the length scale to depart from κy in the viscous region, but kept the same behavior (3.3) for the dissipation. When placed in comparable forms, the constants used by Wolfshtein and by Norris and Reynolds are quite similar.

Norris and Reynolds discovered an interesting aspect of the behavior of their model. They solved the channel flow equations by guessing a wall dissipation, integrating outwards from the wall, and then adjusting the wall dissipation until the proper conditions were satisfied at the channel centerline. The calculation proved enormously sensitive to the wall dissipation, and a double precision integrating scheme had to be used. The guessed dissipation had to be within one part in 10^8 of the proper value before the calculation could even continue to the centerline (if the value was further off, q^2 either blew up quickly or went negative). This very narrow window meant that a wide variety of centerline conditions could be satisfied with almost identical distributions of mean velocity and kinetic energy in the viscous regions; computationally the model confirmed the concept of the law of the wall!

Most workers have abandoned one-equation models in favor of two-equation or even stress equation models. However, it may be that one can do better with this sort of one-equation model in most flows of interest, for it may be easier to specify the length scale distribution than to compute it with a pde. This would be particularly true if the length scale really should be governed by the global features of the flow through an integro-differential equation. Hence, further study of extended one-equations models is encouraged.

Mellor and Herring (1973) discuss some of the earlier work on one equation models, citing numerous references of particular calculations. The serious student of this subject will find their review particularly useful as a resource for computational examples.

4. TWO-EQUATION MODELS

In attempts to eliminate the need for specifying the turbulence length scale ℓ as a function of position throughout the flow, several workers have explored the use of a second turbulence pde which in effect gives ℓ. The groups at Imperial College and at Stanford both experimented with ad-hoc transport equations for ℓ, with no real success. However, success has been had by both groups and others using a model equation based on the exact equation for the isotropic dissipation $\mathcal{D}$; this equation can be developed from the Navier-Stokes equations by appropriate differentiation, multiplication, and averaging, and is

$$\dot{\mathcal{D}} + U_j \mathcal{D}_{,j} = -W - H_{j,j} \tag{4.1a}$$

Here

$$W = 2\nu\overline{u'_{i,j}u'_{j,k}u'_{k,i}} + 2\nu^2\overline{u'_{i,jj}u'_{i,kk}}$$

$$+ 2\nu\left(\overline{u'_{i,j}u'_{i,k}}U_{j,k} + \overline{u'_{i,k}u'_{j,k}}U_{i,j}\right) + 2\nu\overline{u'_j u'_{i,k}}U_{i,jk} \tag{4.1b}$$

$$H_j = \nu\overline{u'_{i,k}u'_{i,k}u'_j} + 2\nu\overline{u'_{j,k}p'_{,k}} - \nu\mathcal{D}_{,j} \tag{4.1c}$$

H_j represents the diffusive flux of $\mathcal{D}$ in the j direction.

The systematic workers have insisted that their two-equation models first describe properly the decay of isotropic turbulence, and then have worried about the behavior of their models in homogeneous flows where the transport terms vanish. For the isotropic decay problem, (3.1) and (4.1) reduce to

$$\dot{q}^2 = -2\mathcal{D} \qquad \dot{\mathcal{D}} = -W \tag{4.2a,b}$$

W is a scalar for which a closure assumption is needed. In this problem W must be a function of the only other variables around, q^2 and $\mathcal{D}$, and from dimensional arguments must be (at high Reynolds number)

$$W = c_7\mathcal{D}^2/q^2 \tag{4.3}$$

The exact solution for the decay is

$$q^2 = q_0^2 (1+t/a)^{-n} \qquad \varepsilon = \varepsilon_0 (1+t/a)^{-(n+1)} \qquad (4.4a,b)$$

$$a = nq_0^2/(2\varepsilon_0) \qquad n = 2/(c_7-2) \qquad (4.4c,d)$$

Here q_0^2 and ε_0 are the initial values. Early experiments suggested $n = 1$, which gives $c_7 = 4$. Comte-Bellot and Corrsin (1966, hereafter denoted by C-BC) took special care to obtain better isotropy, and their data reveal n values in the range 1.1-1.3. Lumley and Khajeh-Nouri (1974b, hereafter denoted by LK-N2) suggested that slight anisotropies are responsible for these differences, and proposed a higher-order model to take this into account. But this theory does not explain the different values observed in truly isotropic decay, as revealed in Table 3 of C-BC. It seems more reasonable that the structure of the low wavenumber portion of the spectrum is responsible for these differences.

The influence of the low wavenumber spectrum on n can be shown using the spectrum of Fig. 3, following a similar analysis of C-BC. The low wavenumber part of the spectrum is assumed to be permanent, and the high wavenumber portion moves down as ε becomes smaller. The peak, which corresponds to the energy-containing scale, occurs at wavenumber k_L. To the left of the peak we take $E = Ak^m$; it is known that $E \sim k^4$ for $k \to 0$, but this might not include the energy containing range and so we allow a less gradual growth in this range. For $k < k_0$ the k^4 behavior would obtain, but we shall not need to deal with this region. To the right of k_L we use the Kolmogoroff inertial sub-range spectrum $E \sim k^{-5/3}$. The constant α is universal for this spectrum, and has a value of about 1.5. In the inertial sub-range energy is transported up the wavenumber scale by non-linear interactions, and the spectrum is controlled solely by the rate at which energy is being processed upscale (i.e., by the dissipation ε). At high wavenumbers viscosity is important, but this range does not contain significant energy and need not be considered here in detail. It is a simple matter to calculate the energy contained in this model spectrum from $q^2/2 = \int E(k)dk$, assuming $k_0 \ll k_L \ll k_d$. One finds

$$q^2 = \alpha\left(\frac{1}{m+1} + \frac{3}{2}\right) k_L^{-2/3} \varepsilon^{2/3} \qquad (4.5)$$

It is interesting that the form of the large-eddy spectrum enters through m, but its strength (A) does not. Eqn. (4.5) shows that the length scale of the energy-containing eddies is $q^3/\mathcal{D}$ (compare 3.3), and hence the time scale is $q^2/\mathcal{D}$.

Matching the two portions of the spectrum gives

$$k_L = \left[a\mathcal{D}^{2/3}/A\right]^{3/(3m+5)}$$

Then, using (4.5)

$$\mathcal{D} = C[q^2]^{(3m+5)/(2m+2)} \qquad (4.6)$$

where C is a constant. Substituting in (4.2a), and solving for q^2, one obtains (4.4a) with $n = (2m+2)/(m+3)$. So, $m = 4$ gives $n = 10/7$, $m = 2$ gives $n = 6/5$, and $m = 1$ gives $n = 1$.

It is clear that the details of the low wavenumber portion of the spectrum are instrumental in determining n; since these details are in no way represented by the scales q^2 and $\mathcal{D}$, there is no way that this model can exactly predict the decay of laboratory grid turbulence. However, it is possible to make a fairly rational choice of c_7. We really should expect the model to work only when the large-scale structure is devoid of any scales, i.e. when the large-scale energy is uniformly distributed over all wave vectors. This occurs only when $\phi_{ij}(\underline{k})$ is the same at all $\underline{k}$ low wavenumbers. The three-dimensional energy spectrum function used above is $E(k) = 2\pi k^2 \phi_{ii}(\underline{k})$, and represents the energy associated with a shell of wave vector space. Hence, in "equipartioned" large-scale turbulence,* $E(k) \sim k^2$. On this basis we recommend $n = 6/5$, which gives $c_7 = 11/3$. This is close to the value used by LK-N2 and the Imperial College workers.

When strain is applied to the flow, there is every reason to expect an alteration in W; something must provide a "source" of $\mathcal{D}$, and this must depend in some way on the mean flow. Lumley has argued that this can not come from the terms in W explicitly containing the mean velocity, but must come from the first two terms in W (see 4.1b), which are very large but of opposite sign. Lumley feels that the alteration of W by strain should be modeled in terms of the anisotropy of the Reynolds stress tensor. If we

* There is no real reason to require $E(k) \sim k^4$, as required by analyticity in $\underline{k}$ as $k \to 0$ (see Hinze 1959). The box-like grid certainly could create a directionally-dependent $dE/d\underline{k}$ for $\underline{k} \to 0$.

follow this approach, and represent the anisotropy through

$$b_{ij} = (R_{ij} - q^2\delta_{ij}/3)/q^2 \tag{4.7}$$

then the first scalar that can be formed from the anisotropy measure is $b^2 = b_{ij}b_{ij}$. Lumley therefore proposes

$$W = (c_7 - c_8 b^2)\mathcal{D}^2/q^2 \tag{4.8}$$

LK-N2 use $c_7 = 3.73$ and $c_8 = 30$.

In a two-equation model b^2 must be produced from the constitutive equation (2.2), with ν_T given by

$$\nu_T = c_9 q^4/\mathcal{D} \tag{4.9}$$

To match (3.2) and (3.3), $c_9 = c_2c_3$. Then, $b_{ij} = 2c_9q^2S_{ij}/\mathcal{D}$, $b^2 = 4c_9^2q^4S^2/\mathcal{D}$, where $S^2 = S_{ij}S_{ij}$. The turbulence production is $\mathcal{P} = 2c_9q^4S^2/\mathcal{D}$, and hence $b^2 = 2c_9\mathcal{P}/\mathcal{D}$. Hence, in this model (4.8) may be written as

$$W = (c_7 - c_{10}\mathcal{P}/\mathcal{D})\mathcal{D}^2/q^2 \tag{4.10}$$

where $c_{10} = 2c_8c_9$.

Using Lumley's value of $c_8 = 30$ and the other constants given earlier, $c_{10} = 1.25$. The group under B. E. Launder at Imperial College have explored two equation models extensively, using forms equivalent to (4.10) with $c_3 = 3.1$.

It seems most desirable to determine c_{10} by reference to experiments in nearly homogeneous flow, where the transport would not confuse the issue. There are two types of such flows, those involving pure strain and those involving pure shear. Tucker and Reynolds (1968, hereafter denoted by TR) and Marechal (1972) studied the pure strain case; Champagne, Harris, and Corrsin (1970, hereafter refered to by CHC) and Rose (1966) studied homogeneous shearing flows. In 1970 (4.10) was proposed as a generalization of models used by Launder and others, and the constants were evaluated by reference to the TR and CHC flow (see Reynolds 1974). For that evaluation $c_7 = 4$ was

used. Recently L. H. Norris and I repeated the evaluation for the preferred value of $c_7 = 11/3$. We carefully evaluated the production term from the data in these two flows, and used this as input to (4.10) . The q^2 history was carefully differentiated to get an initial value for $\mathcal{D}$, the $\mathcal{D}$ and q^2 equations were solved simultaneously by an accurate forward-difference integration, and the q^2 histories were compared with the experimental data. We found that $c_{10} = 2$ gives excellent agreement in both flows, as found in the earlier work. Hence, if one elects to use (4.10) in any model, the choices $c_7 = 11/3$, $c_{10} = 2$ are recommended.

At this point we have a two-equation model that can be tested against the homogeneous TR and CHC flows. In a prediction the R_{ij} and hence $\mathcal{P}$ must be derived using the constitutive equation (2.2) with (4.9) . Remarkably good results are obtained for the TR flow with $c_9 = 0.025$. As noted below (4.9) , gives $c_9 = 0.020$ using STAN-5 constants. With this value the two-equation model underpredicts $\mathcal{P}$ in the TR flow, and does not produce enough anisotropy in the Reynolds stresses. When applied to the CHC flow, the two-equation model fails miserably in prediction of both shearing and normal stresses.

A weakness of (2.2) is that it forces the principal axes of R_{ij} and S_{ij} to be alighned. This is true in pure strain (the TR flow), but not true in any flow with mean vorticity (e.g., CHC) . One is tempted to try a modified constitutive equation (see Saffman 1974)

$$R_{ij} = \frac{q^2}{3}\delta_{ij} - 2\nu_T S_{ij} - c_{11}\ell^2(S_{ik}\Omega_{kj} + S_{jk}\Omega_{ki}) \qquad (4.11)$$

where $\Omega_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i})$ is the rotation tensor. In a two-equation model ℓ could be expressed in terms of q^2 and $\mathcal{D}$. Eqn. (4.11) does produce the right sort of normal stress anisotropy in shear flows, but the new terms don't alter the shear stress, and hence (4.11) works no better than (2.2) for the CHC flow. Two-equation models also fail to predict the return to isotropy after the removal of strain, or the isotropizing of grid-generated turbulence (C-BC). This failure arises because of the need for a constitutive equation for the R_{ij} . Thus, one should not really expect two-equation models to be very general, although they might be made to work well with specific constants in specific cases, such as boundary layers.

In spite of these difficulties with models based on constitutive equations, their simplicity makes them attractive. Two-equation models have been studied by a number of groups, and it is significant that these workers inevitably turn to stress equation models because of the difficulties outlined above. Stress equation models have their own problems, and so there probably is still considerable room for development of two-equation models. Of particular interest is turbulent boundary layer separation, where anisotropy of the normal stresses is known to be important. Since (2.2) won't give this properly in a shear layer, but (4.11) can, the use of (4.11) in conjunction with two-equation models should be explored further.

To use the two-equation model outlined above in an inhomogeneous flow, one needs to assess (or neglect) the effects of inhomogeneity on W, and also to model the transport term H_j. Jones and Launder (1972, 1973, hereafter referred to by JL1 and JL2) assume that W is not modified by inhomogeneity and use a gradient diffusion model for H_j,

$$H_j = -(\nu + c_{12}\nu_T)\mathcal{D}_{,j} \tag{4.12}$$

with $c_{12} = 0.77$. Lumley (see Lumley and Khajeh-Nouri 1974a, hereafter denoted by LK-N1) argues on formal grounds that the diffusive flux of dissipation should depend as well on the gradients in turbulence energy, and vice versa, in the manner of coupled flows such as thermoelectricity and thermodiffusion studied by the methods of irreversible thermodynamics. If this is true, one really should use models of the form

$$J_j = -A_{11}q^2_{,j} - A_{12}\mathcal{D}_{,j} \tag{4.13a}$$

$$H_j = -A_{21}q^2_{,j} - A_{22}\mathcal{D}_{,j} \tag{4.13b}$$

Lumley and his coworkers have done this in their stress-equation modeling, but as yet no users of two-equation models have adopted this approach. Eqn. (4.13) would allow for up-gradient diffusion of turbulence energy, a real phenomena in the central region of a wake, while the simpler uncoupled models do not. This is an area worthy of further experimentation within the structure of two-equation models.

The ε equation model described above works fairly well at high Reynolds numbers, but fails near a wall where viscous effects are important. JL proposed ad-hoc low Reynolds number modifications that seem to work reasonably well in the wall region, and Hanjalic and Launder (1974, hereafter denoted by HL) proposed further modifications of the ε equation for use with their stress-equation model. Clearly the W term has to be modified, for in the "final period" of decay of isotropic turbulence $q^2 \sim t^{-5/2}$ instead of $t^{-6/5}$. If the turbulence Reynolds number $R_T = q^4/(\varepsilon\nu)$ is small, then the inertial terms in the dynamical equation are unimportant, and in isotropic turbulence W is dominated by the second term in (4.1b). At low R_T, $\varepsilon \sim \nu q^2/\ell^2$, so $\ell \sim (\nu q^2/\varepsilon)^{1/2}$, and $W \sim \nu^2 q^2/\ell^4 \sim \varepsilon^2/q^2$. Hence, at low R_T, $W = c_7^* \varepsilon^2/q^2$, which is of the same form as the high R_T behavior (see 4.3). Setting $n = 5/2$ in (4.4), $c_7^* = 14/5$, which is consistent with the models of HL and JL. A smooth transition between c_7 and c_7^* is needed; a form similar to that used by JL and HL but consistent with $c_7 = 11/3$ and $c_7^* = 14/5$, is

$$W = \frac{11}{3} f_1(R_T) \varepsilon^2/q^2 \qquad (4.14a)$$

where

$$f_1 = 1 - \frac{33}{55} \exp\left[-(R_T/12)^2\right] \qquad (4.14b)$$

Remember that this is just for the part of W that is non-zero in homogeneous isotropic turbulence.

Eqn. (4.14) presents problems near a wall, where $\varepsilon \to$ const. and $q^2 \to 0$. Launder and his coworkers get around this by ad-hoc modifications of their model equations. HL replace ε^2 by $\varepsilon\hat{\varepsilon}$ where $\hat{\varepsilon} = \tilde{\varepsilon} - \nu(\partial q/\partial x_i)^2$. Unfortunately they refer to $\tilde{\varepsilon}$ as the isotropic dissipation, for some reason confusing it with ε. In spite of this semantic problem, their assumption does seem to work in boundary layers. However, this reviewer would prefer an approach in which $\varepsilon \to$ const. as $y \to 0$, which physically is correct.

An alternative approach to handling this part of W near a wall is

$$W = \frac{11}{3} f_1(R_T)(1 - e^{-c_{13}qy/\nu})\mathcal{D}^2/q^2 \qquad (4.15)$$

This gives $W \to$ const. as $y \to 0$. The use of (4.15) should be explored.

The third and fourth terms on the right in (4.1b) vanish at high R_T because of the small scale isotropy. HL planned to include these at low R_T by lumping them with the first two terms in (4.1b) by further modification of f_1. However, they found that this was not necessary. The last term in (4.1b) was neglected by JL. It was modeled by HL in a complex way involving products of two second derivatives of the mean velocity and the Reynolds stresses.

JL2 used the two-equation model to study a limited number of boundary layers, including the "difficult" flow shown in Fig. 2. The predictions of their model are seen to be noticeably less accurate than those of the STAN-5 one-equation model shown.

One difficulty with using the $\mathcal{D}$ equation as the basis for a second model equation has escaped the model developers. This arises from the second term in (4.1c), the pressure gradient-velocity gradient term in the transport H_j. Since the pressure field depends explicitly upon the mean velocity field (see §5), mean velocity gradients can explicitly give rise to $\mathcal{D}$ transport. This could be an extremely important effect, especially near a wall. The omission of this consideration would seem to be a serious deficiency in all $\mathcal{D}$ equation models that have been studied to date.

Other two-equation models have been heuristically conceived. Of these the most well developed is the Saffman-Wilcox (1974, hereafter denoted by SW) model. Instead of a $\mathcal{D}$ equation they use an equation for a "pseudovorticity" $\underline{\Omega}$,

$$\underline{\Omega}^{\cdot 2} + U_j \underline{\Omega}^2_{,j} = \left[\alpha\sqrt{U_{i,j}U_{i,j}} - \beta\underline{\Omega}\right]\underline{\Omega}^2 + \left[(\nu + \sigma\nu_T)\underline{\Omega}^2_{,j}\right]_{,j} \qquad (4.16)$$

In conjunction with this they use the q^2 equation (3.1) with

$$\mathcal{P} = \alpha^*\sqrt{2S^2}\, q^2/2 \qquad \mathcal{D} = \beta^* q^2 \underline{\Omega}/2 \qquad (4.17a,b)$$

They use (3.4) for the q^2 transport, setting $c_4 = \sigma^*$, and for ν_T they set

$$\nu_T = q^2/(2\underline{\Omega}) \qquad (4.18)$$

The constitutive equation (2.2) is used to provide R_{ij} for the mean momentum equations. Their recommended constants are $\alpha = 0.1638$, $\alpha^* = 0.3$, $\beta = 0.15$, $\beta^* = 0.09$, $\sigma = 0.5$, $\sigma^* = 0.5$.

The production term $\mathcal{P}$ as given by (4.17a) is inconsistent with the R_{ij} constitutive model; this seems to be an internal inconsistency in the model, but it may in fact be a strength. The $\mathcal{P}$ model is based on the experimental fact that the structure of the turbulence in the wall region of a boundary layer is essentially independent of the strain rate, and hence $\mathcal{P}$ should be proportional to q^2. Hence, the SW model is a curious blend of the "Newtonian" and "structural" alternatives (Reynolds 1974).

For isotropic turbulence decay the SW model equations may be solved exactly. The high-R_T behavior, $q^2 \sim t^{-6/5}$, is obtained if $\beta^*/\beta = 3/5$, as suggested by SW. I recently tested the SW model against the TR and CHC flows, using "starting" values for $\underline{\Omega}$ carefully calculated from the initial q^2 decay rate. In neither case were the results at all impressive. Moreover, the SW model does not display the proper decay of isotropic turbulence at low R_T. Therefore, it does not appear that the SW model is or can be any more general than any other two-equation model. Indeed, both Saffman and Wilcox are independently exploring stress equation models (Saffman 1974, Wilcox 1975), neither version of which presently works very well in the TR and CHC flows.

The SW model has been tested against only a limited body of boundary layer flows. The model works surprisingly well in the viscous region, but has the troublesome point that $\underline{\Omega}$ must be infinity at a perfectly smooth wall. SW use a "large" value of $\underline{\Omega}$ at the wall to produce mean velocity curves that are in excellent agreement with experiments for smooth walls. In effect, SW match their solution to experimental data by judicious choice of the value of the wall $\underline{\Omega}$. In SW they considered only zero pressure gradients with no transpiration. More recently Wilcox (1975) examined a few cases of pressure gradient and transpiration, and made a useful comparison of the SW

model with other two-equation models, including JL. By judicious selection of the wall value of $\underline{\Omega}$ he could match some of the Stanford transpired boundary layer; his calculations indicated a strong effect of blowing on the wall $\underline{\Omega}$. Thus, the SW method will require a graph of the wall $\underline{\Omega}$ as a function of pressure gradient and blowing parameter, similar to Fig. 1. Wilcox also found it essential to use accurate values for the free-stream value of $\underline{\Omega}$, which he also had to carefully deduce from experimental data. It appears that the sensitivity of the SW model to free-stream conditions may be significantly greater than that of the JL model, and certainly is much greater than that of one-equation models.

One is led to conclude that the SW model should not be used as an engineering tool until such time as it has been developed much further. Regarding $\underline{\Omega}$ as a reciprocal time scale may be useful in guiding these developments.

5. STRESS-EQUATION MODELS

In turbulent shear flows, the energy is usually first produced in one component and then transferred to the others by turbulent processes. Exact equations for R_{ij} can be derived from the Navier-Stokes equations (Tennekes and Lumley, 1972); for an incompressible fluid,

$$\dot{R}_{ij} + U_k R_{ij,k} = P_{ij} + T_{ij} - D_{ij} - J_{ijk,k} \qquad (5.1a)$$

Here P_{ij} is the "production tensor,"

$$P_{ij} = -R_{ik}U_{j,k} - R_{jk}U_{i,k}$$

$$= -(R_{ik}S_{kj} + R_{jk}S_{ki}) + (R_{ik}\Omega_{kj} + R_{jk}\Omega_{ki}) \qquad (5.1b)$$

Note that $P_{ii} = 2\mathcal{P}$. T_{ij} is the "transfer tensor",

$$T_{ij} = \frac{1}{\rho}\overline{p'(u'_{i,j} + u'_{j,i})} \qquad (5.1c)$$

This "pressure-strain" term is responsible for energy exchange between components. Note that $T_{ii} = 0$ by continuity. D_{ij} is the "isotropic dissipation tensor,"

$$D_{ij} = 2\nu \overline{u'_{i,k} u'_{j,k}} \tag{5.1d}$$

and $D_{ii} = 2\varepsilon$. J_{ijk} is the diffusive flux of R_{ij},

$$J_{ijk} = \frac{1}{\rho}(\overline{p'u'_i}\delta_{jk} + \overline{p'u'_j}\delta_{ik}) + \overline{u'_i u'_j u'_k} - \nu R_{ij,k} \tag{5.1e}$$

Note that $J_{iik} = J^*_k$.

P_{ij} is explicit, but models are needed for T_{ij}, D_{ij}, and J_{ijk}. In addition, one must either specify ℓ or use a ε equation. We will first discuss the high-Reynolds number modeling of (5.1), particularly as applied to homogeneous flows, and then discuss the problems and status of extending this model to inhomogeneous regions, particularly near walls where R_T is small.

The one fact that seems very clear from experiments is that, at high R_T the small-scale dissipative structures are isotropic. Hence all workers now use

$$D_{ij} = \frac{2}{3}\varepsilon\delta_{ij} \tag{5.2}$$

The transfer term T_{ij} has been the subject of most controversy and experimentation. In a flow without any mean strain, this term is responsible for the return to isotropy. However, in deforming flows the situation is much more complicated. Guidance is provided by the exact equation for the fluctuation pressure, derivable from the Navier-Stokes equation (see Tennekes and Lumley 1972),

$$p'_{,ii} = -2u'_{i,j}U_{j,i} - u'_{i,j}u'_{j,i} = g_1 + g_2 \tag{5.3}$$

The source term in this Poisson equation contains two parts, each of which will be responsible for a part of the pressure field. The part determined by g_1, which involves the mean deformation explicitly, we denote by p'_1,

and the remainder by p_2' . Following LK-N2, the explicit dependence of the p_1' contribution to T_{ij} can be obtained for homogeneous fields in terms of the Fourier transform of the velocity field. Let

$$p_1' = \int \hat{p}(\underline{k}) e^{i\underline{k}\cdot\underline{x}} d\underline{k} \tag{5.4}$$

In homogeneous flows the mean gradients are constants, so (5.3) gives

$$\hat{p} = 2\hat{u}_i \frac{k_j}{k^2} U_{j,i} \tag{5.5}$$

Then, the part of the pressure-strain term associated with p_1' is (we adopt the subscript choice of LK-N2 for convenience in comparison)

$$T_{1pq} = \overline{p_1'(u_{p,q}+u_{q,p})} = \iint \overline{\hat{p}(k)\left[\hat{u}_p^*(k')k_q' + \hat{u}_q^*(k')k_p'\right]} dk'dk \tag{5.6}$$

Using (5.5) and the statistics of random transforms, (5.6) becomes

$$T_{1pq} = 2U_{j,i}G_{ijpq} \tag{5.7a}$$

where

$$G_{ijpq} = \int \left(\frac{k_j k_q}{k^2} \Phi_{ip}(\underline{k}) + \frac{k_j k_p}{k^2} \Phi_{iq}(\underline{k}) \right) d\underline{k} \tag{5.7b}$$

Eqn (5.7a) is identical to an expression developed by Rotta (1951) from slightly different arguments.

Models for G_{ijpq} have been proposed by Launder and Lumley and their coworkers. There are various constraints that G_{ijpq} must satisfy. From continuity, $G_{ijpp} = 0$, $G_{iipq} = 0$. Also, $G_{ijjq} = R_{iq}$. For isotropic turbulence these suffice to define G_{ijpq} . Hanjalic and Launder (1972) first used a model of G_{ijpq} that involved linear and quadratic terms in the R_{ij} . Later Launder, Reece, and Rodi (1973, hereafter denoted by LRR) dropped the quadratic terms. LK-N2 also used non-linear terms, but later Lumley (1975) argued that the model <u>must</u> be be linear in the Reynolds stresses because for a field that is the sum of two uncorrelated fields T_{1ij} should be the sum of their individual T_{1ij}'s . Lumley (1975) sought to resolve certain inconsistencies between the calculations and experiments by allowing

G_{ijpq} to depend in a complicated way on scalars developed from combinations of the mean deformation and b_{ij} (see 4.7). But this violates the condition that the G_{ijpq} should not be changed by a sudden change in the mean strain-rate. If this condition is imposed, and we insist on linearity in the Reynolds stresses, then the G_{ijpq} model (in a homogeneous field) must be of the form

$$\begin{aligned} G_{ijpq} = & \left\{ -\frac{1}{15}\delta_{ij}\delta_{pq} + \frac{1}{10}(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \right\} q^2 \\ & + \frac{1}{5}\left\{ \frac{5}{3}(b_{ip}\delta_{jq} + b_{iq}\delta_{jp}) - \frac{5}{3}(b_{jp}\delta_{iq} + b_{jq}\delta_{ip}) \right. \\ & + A_1[b_{ij}\delta_{pq} - \frac{1}{6}(b_{ip}\delta_{jq} + b_{iq}\delta_{jp}) \\ & \left. - \frac{4}{3}(b_{jp}\delta_{iq} + b_{jq}\delta_{ip}) + b_{pq}\delta_{ij}] \right\} q^2 \end{aligned} \tag{5.8}$$

Using this in (5.7a), the part of T_{ij} explicitly related to the mean field must be

$$\begin{aligned} T_{1ij} = & \frac{2}{5}(1+A_1)\, S_{ij}q^2 - \frac{3}{5}A_1\left[R_{ik}S_{kj} + R_{jk}S_{ki} + \frac{2}{3}\mathcal{P}\delta_{ij}\right] \\ & - \frac{4}{5}\left(\frac{5}{3} + \frac{7}{12}A_1\right)\left[R_{ik}\Omega_{kj} + R_{jk}\Omega_{ki}\right] \end{aligned} \tag{5.9}$$

This is precisely the form used by HL.

The part of T_{ij} associated with g_2, which we denote by T_{2ij}, should not change instantly when the mean deformation is changed, and hence should not depend explicitly on the mean deformation. LK-N% ignored this requirement, and allowed T_{2ij} to depend on the rotation tensor. Lumley (1975) has now abandoned this position. Launder and his coworkers, and others, have followed Rotta in assuming

$$T_{2ij} = -A_o\mathcal{D}b_{ij} \tag{5.10}$$

The constant A_o determines the rate of return to isotropy. Its value has been the subject of much uncertainty. The TR flow implies a value $A_o = 6$, while the C-BC data suggest a much lower value is appropriate.

HL and LRR use $A_o = 3.0$, LK-N2 use $A_o = 3.21$. LRR point to the advantages that would be obtained if a lower value of approximately 0.6 could be used, in which case the behavior near a wall would be much more accurately modeled. Lumley and his coworkers add additional non-linear terms in the b_{ij}, feeling that the rate of return to isotropy should depend upon the degree of anisotropy. It does not seem that the data justify the inclusion of higher-order terms, and so (5.10) is recommended, at least for homogeneous flows away from boundaries at high R_T.

L. H. Norris and I recently studied this problem using the exact solution of the model equations for the return to isotropy in homogeneous turbulence without strain. Using (5.2) and (5.10) in (5.1), for this case

$$\dot{b}_{ij} = -(A_o - 2)\frac{\varepsilon}{q^2} b_{ij} \tag{5.11}$$

Eqns. (4.2) again describe q^2 and ε. The exact solution for the decay is (see 4.4)

$$b_{ij} = b_{ijo}(1 + t/a)^{-(A_2-2)n/2} \tag{5.12}$$

where b_{ijo} are the initial values. Note that A_o must be at least 2 if isotropy is to be restored. Norris and I used the data of C-BC's Table 1, and first simply solved (5.11) for (A_o-2). Subsequently we compared the solution (5.12) to the data, using $n = 6/5$. There is a great deal of scatter, because the anisotropies are rather small. There was absolutely no systematic dependence of A_o on either anisotropy or R_T. Based on this work, we recommend $A_o = 5/2$.

Kwak and Reynolds (1975) studied the TR flow in a numerical simulation, and found a much slower return to isotropy than indicated in the TR experiments. However, different components return at decidedly different rates. Shaanan and Ferziger (1975) carried out a similar calculation for a shear flow similar to that studied by CHC. In a computation the shearing can be removed, which can not be done experimentally. These calculations also showed a marked difference in the return rate for different components, probably because of great difference in the length scales in the three directions. We conclude that current stress equation models will not do a very good job in handling the return to isotropy; however, the models may work well in flows dominated by other effects.

The constant A_1 should be evaluated by reference to homogeneous flows, such as the TR and CHC flow. LK-N2 used -2.456, which was obtained by a comparison with a rapid distortion analysis of homogeneous strain. Later Lumley (1975) argued against this approach, and settled on -1.23 (in a more involved model). LRR use a value of -1.45 (their $c_2 = -(\frac{2}{3} + \frac{33}{45} A_1)$), which they base on homogeneous experiments. I recently found that -1.5 is a reasonable compromise between -1 which works better for the TR flow and -2 which works better for CHC, and now recommend $A_1 = -\frac{3}{2}$

Inhomogeneities greatly complicate the T_{ij} modeling, especially T_{1ij}. LRR add a complicated term inversely proportional to the distance from the wall. Recently M. Acharya and I extended LK-N2's analysis for T_{1ij} to a flow near a wall. We took Fourier transforms in only the x_1 and x_3 directions, and solved the ordinary differential equation for the transform amplitude $\hat{p}(y)$. This leads one to a messy integral expression in which T_{1ij} depends upon the mean velocity gradients <u>at all points in the flow</u>. In a wall region one might well expect T_{1ij} to be determined by a region at least as wide as the distance to the wall, and hence a complex integral model is really needed for such flows. This is a very unsatisfactory aspect of present stress-equation modeling, and an area that should receive considerable attention in the future.

In addition to modifications in T_{ij}, inhomogeneities require modeling of J_{ijk}. The gradient diffusion model is usually employed; HL and LRR set

$$J_{ijk} = -A_2 \frac{q^2}{\mathcal{D}} \left(R_{in} R_{jk,n} + R_{jn} R_{ik,n} + R_{kn} R_{ij,n} \right) \qquad (5.13)$$

Hanjalic and Launder (1972) gave some justification for this form by consideration of the dynamical equation for $\overline{u_i' u_j' u_k'}$. Lumley (1975) used somewhat more extensive arguments to in effect provide further justification for this form. Noting that J_{ijk} contains one pressure-velocity term, and since p' will have a part (p_1') that depends explicitly on the mean velocity gradients, it does seem that J_{ijk} also should be explicitly linear in the mean gradients, though this need has escaped notice.

Other modifications necessary near a wall have been suggested by LRR. In particular, they propose to allow anisotropy in D_{ij} at low R_T, and have concocted a smooth transition between (5.2) and

$$D_{ij} = 2R_{ij}\mathcal{D}/q^2 \qquad (5.14)$$

which they incorrectly imply is exact as $R_T \to 0$.

Two approaches have been used in stress-equation modeling. The earlier work (Donaldson 1972) involved specification of the length scale and use of (3.3) to determine $\mathcal{D}$. HL used the $\mathcal{D}$ equation model outlined above in conjunction with the R_{ij} equations. At this writing this work is in a state of rapid development, and undoubtedly improvements will be made by the time this article is released. Interested persons should follow most carefully the work of Launder and Lumley. It will be some time before these models are sufficiently well developed to be better than simpler models for use in engineering analysis.

An interesting use of stress equation models is suggested by a contraction of (5.13),

$$J_{iik} = -A_2\frac{q^2}{\mathcal{D}}\left(R_{kn}q^2_{,n} + 2R_{in}R_{ik,n}\right) \qquad (5.15)$$

If this is compared with (3.4), its counterpart in the one- or two-equation models, an important difference is seen; Eqn. (5.14) does allow for a flux of q^2 to be driven by gradients of other than q^2. Moreover, if the constitutive equation (2.2) is used with (3.2), the q^2 flux will be driven by mean velocity gradients These effects are not incorporated in (3.4); an approach to improving the simpler one- and two-equation models might be to use the more complex stress equation model as a guide to the nature of new terms that should be included.

There is a basic difficulty in this general approach to turbulence models. One would like to model only terms that respond on time scales short compared to that of the computed quantities. It is well known that the small scales respond to change much faster than the large scales, and hence it is reasonable to express a quantity dominated by small scales, such as D_{ij}, as a function of quantities dominated by large scales, such as R_{ij}.

However, terms like J_{ijk} have time scales comparable with that of R_{ij}, and thus one really should not expect an equilibrium constitutive relationship to exist between J_{ijk} and R_{ij}. In general, it seems that higher order statistical quantities take longer to reach steady state than lower order statistics; for example, in a channel flow the "entrance length" for the mean velocity is rather short, while the entrance length for the R_{ij} is known to be quite long. Any model obtained by truncation at some statistical order would suffer from this difficulty. What one really needs to do is truncate at some level of scale, and thereby take advantage of the fact that the smaller scales do adjust faster to local conditions. Then, by truncating at smaller and smaller scales, one has at least some hope of convergence, a hope that is at best dim when one truncates at higher and higher orders of statistical quantities that have comparable time scales. The large-eddy simulation described in the next section provides one avenue to a scale-truncation approach.

An interesting identity that might be useful in a different approach to turbulence modeling is

$$R_{ij,j} = -\varepsilon_{ijk}\,\overline{u_j'\omega_k'} + \tfrac{1}{2}q^2{}_{,i} \qquad (5.16)$$

where $\omega_i' = \varepsilon_{ijk}u_{k,j}'$ is the fluctuation vorticity. When this is used in (2.1a), the Reynolds "stresses" disappear (except for a "Reynolds pressure" $q^2/2$), and are replaced by "Reynolds body forces" $F_i = \varepsilon_{ijk}\overline{u_j'\omega_k'}$. Stress equation models try to model R_{ij}, and then take their gradients. It might be easier to model the body forces F_i directly. For a physical discussion of the F_i, see Tennekes and Lumley (1972).

6. LARGE EDDY SIMULATIONS

This line of approach is just beginning to bear fruit. The idea is to do a three-dimensional time-dependent numerical computation of the large scale turbulence. It is impossible to compute the smallest scales in any real flow at high R_T (and will be forever), so they must be modeled. Care must be taken to define what it is that is being computed and the early work was not done with sufficient care.

In 1973 we began a systematic program of development and exploration of this method, in close cooperation with NASA-Ames Laboratory. The first contribution was made by Leonard (1973), who clarified the need for spatial filtering. We now define the large-scale variables by (see Kwak and Reynolds 1975)

$$\overline{f}(\underline{x}) = \int G(\underline{x}-\underline{x}')\, f(\underline{x}')d\underline{x}' \tag{6.1a}$$

where the filter function is

$$G(\underline{x}-x') = \left(\sqrt{\frac{6}{\pi}}\,\frac{1}{\Delta_a}\right)^3 \exp\left[-6(\underline{x}-\underline{x}')^2/\Delta_a^2\right] \tag{6.1b}$$

Here Δ_a is the averaging scale, which need not and should not be the same as the grid mesh width. We use this particular filter because of its advantages in Fourier transformation. When this operation is applied to the Navier-Stokes equation, and an expansion is carried out, one finds (neglecting molecular viscosity)

$$\dot{\overline{U}}_i + \overline{U}_j\overline{U}_{i,j} = -\frac{1}{\rho}\overline{P}_{,i} + \left[\frac{\Delta_a^2}{24}\left(\overline{U}_i\overline{U}_j\right)_{,kk} - R_{ij}\right]_{,j} + O(\Delta_a^4) \tag{6.2}$$

where $-\rho R_{ij}$ are the "sub-grid scale Reynolds stresses." The unusual term appearing before R_{ij} is an additional stress-like term resulting from the filtering of the non-linear terms; we now call these the "Leonard terms," and view $\rho\Delta_a^2\left(\overline{U}_i\overline{U}_j\right)_{,kk}/24$ as the "Leonard stresses."

We have explored two models for the R_{ij}. Both are based on (2.2); the first is Smagorinsky's (1963) model,

$$\nu_T = B_1\Delta_a^2\sqrt{S_{nm}S_{nm}} \tag{6.3a}$$

The second uses the rotation in place of the strain rate,

$$\nu_T = B_2\Delta_a^2\sqrt{\Omega_{nm}\Omega_{nm}} \tag{6.3b}$$

In these expressions S_{ij} and Ω_{ij} are the strain rate and rotation of the calculated local time-dependent large scale field. The q^2 term in

(2.2) is again absorbed with the pressure. Note that the sub-grid terms R_{ij} are $O(\Delta_a^2)$, and hence if they are important the Leonard stresses are also likely to be important. Moreover, a difference scheme must be used that is accurate to $O(\Delta_a^2)$; this important requirement was overlooked by many of the early workers.

Kwak and Reynolds (1974) solved the isotropic decay problem, adjusting B_1 or B_2 to obtain the proper rate of energy decay. The calculations were started using an isotropic field with zero skewness, but the proper skewness develops in only a few time steps. The predicted results for the large scale field are compared with the experimental results of Comte-Bellot and Corrsin (1971) filtered with (6.1). We find that the averaging scale Δ_a must be twice the computational mesh scale Δ for a satisfactory calculation of the spectral evolution. We find that calculation in a mesh containing as few as 16^3 points gives remarkably good spectral predictions; better results are obtained with 32^3 points, and it is reassuring that the same constants B_1 or B_2 fit both sizes. The skewness, which is dominated by smaller scales, is predicted much more accurately in the 32^3 calculation. Good results are obtained with both (6.3a) and (6.3b). Fig. 4 shows the results for the 16^3 calculation. On the basis of this work, we now use $B_1 = 0.06$ or $B_2 = 0.09$. It is surprising that $B_2 \neq B_1$, because, as Tennekes and Lumley (1972 - Eqn. 3.3.44) show, $S^2 \approx \Omega^2$ for large R_T. This paradox remains to be understood.

Next we simulated the TR flow, first with an initial distribution that matched the anisotropy of the TR flow and later with an isotropic initial distribution. One has problems in setting anisotropic initial conditions that are free of shearing stresses, and so the isotropic starting is probably a better approach. It is remarkable that the salient features of the TR experiments are captured quite well in a computation using only 16^3 points! The results are shown in Fig. 5; the calculation was executed on a CDC 7600, using 120 time steps, in approximately 5 minutes.

Ferziger and Shaanan (1975) are experimenting with a staggered grid approach that is second-order accurate and does not require explicit inclusion of the Leonard stresses. They have validated the constants B_1 and B_2 with this method, and have also explored the CHC flows. There

are some difficulties in providing suitable initial conditions, so comparison with experiments is not easy. Nevertheless, the salient features of the CHC flow can be produced with 16^3 calculations!

We have started work on the two-stress mixing layer, in which we expect to find a sharp boundary between the turbulent field and irrotational external flow. The vorticity model (6.3b) will offer the advantage of yielding zero subgrid scale stresses in a vorticity-free region, and this is the reason that it is of interest. We plan to extend the computation to "infinity" using inviscid flow theory, and the matching of the computation with the inviscid analysis will require use of a difference scheme that does not produce vorticity improperly. It will be some time before we will feel prepared to handle a wall flow accurately. In the meantime, the simulations of Deardorff (1970) and Schumann (1973) provide some initial experience with channel flows.

One objective of this work is to test the turbulence models, particularly the stress equation model. We can compute the pressure strain terms directly (both T_{1ij} and T_{2ij}), and are doing this presently. We had hoped that the calculations would serve as a basis for evaluating constants in the stress-equation models; instead they seem to be highlighting the weaknesses of these models, as discussed in §5. However, the fact that a very coarse grid produces such remarkably good results leads us to believe that large eddy simulations might, after considerable development, eventually be useful for actual engineering analysis. Interested readers should also follow the work of Orzag and Israelli (1972), who are carrying out similar calculations using Fourier rather than grid methods.

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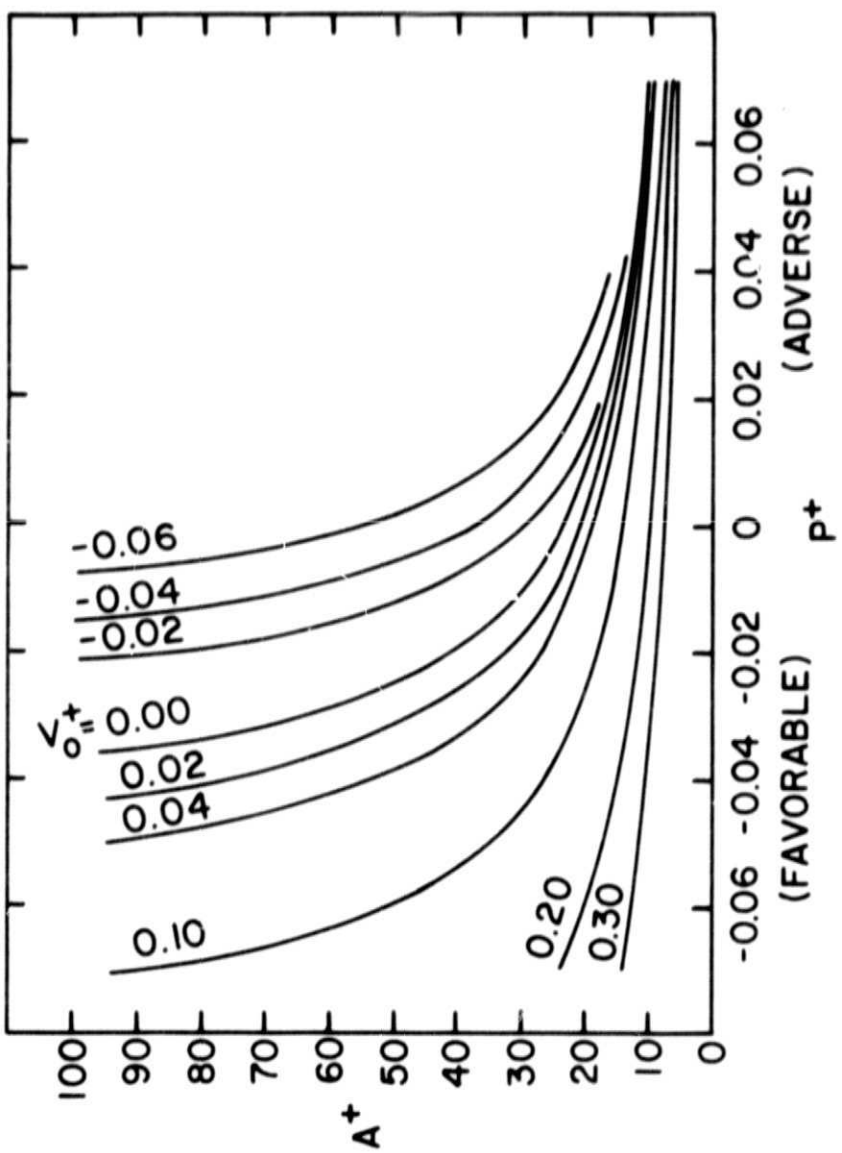

Figure 1. Sublayer thickness parameter.

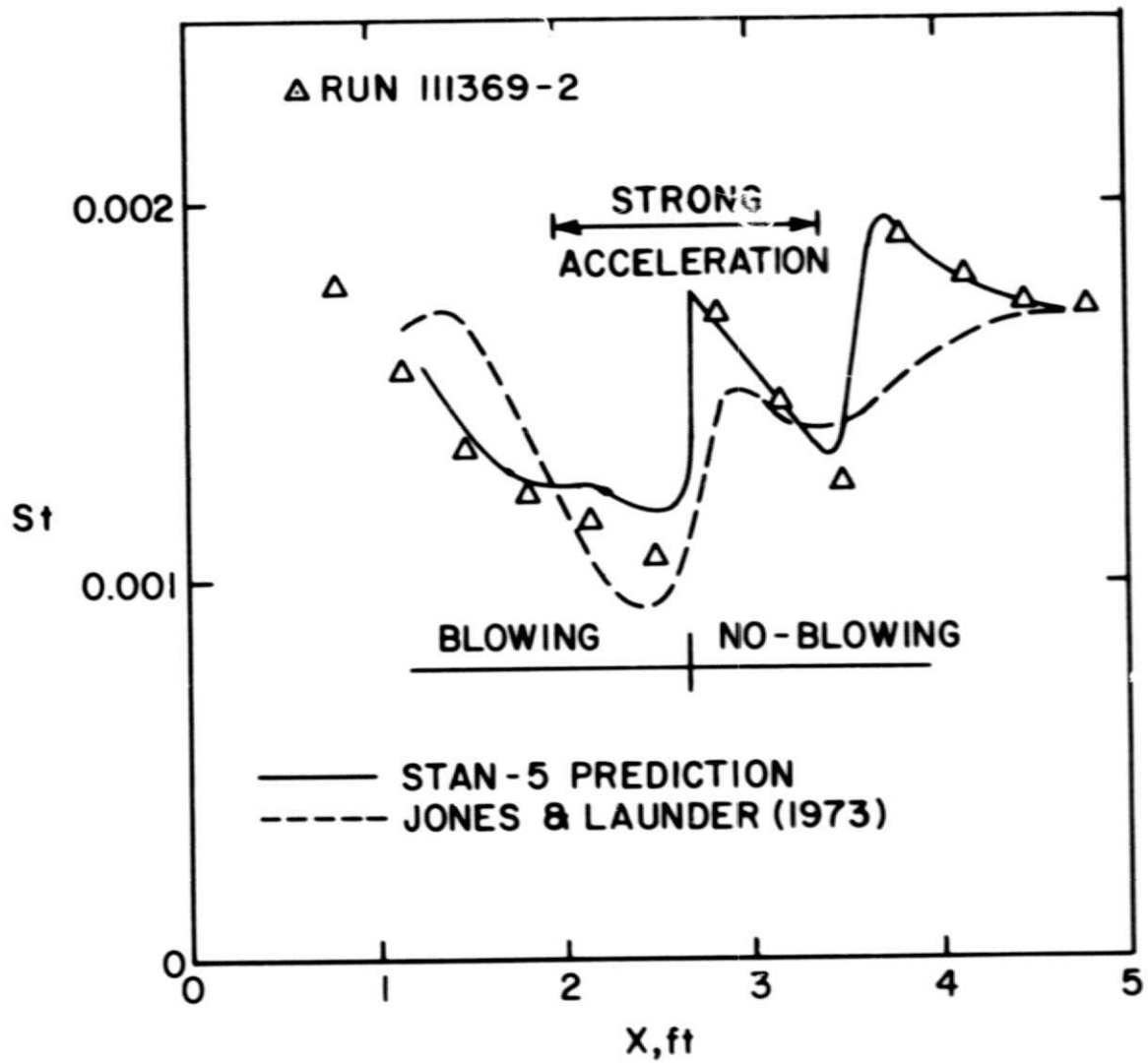

Figure 2. STAN-5 prediction.

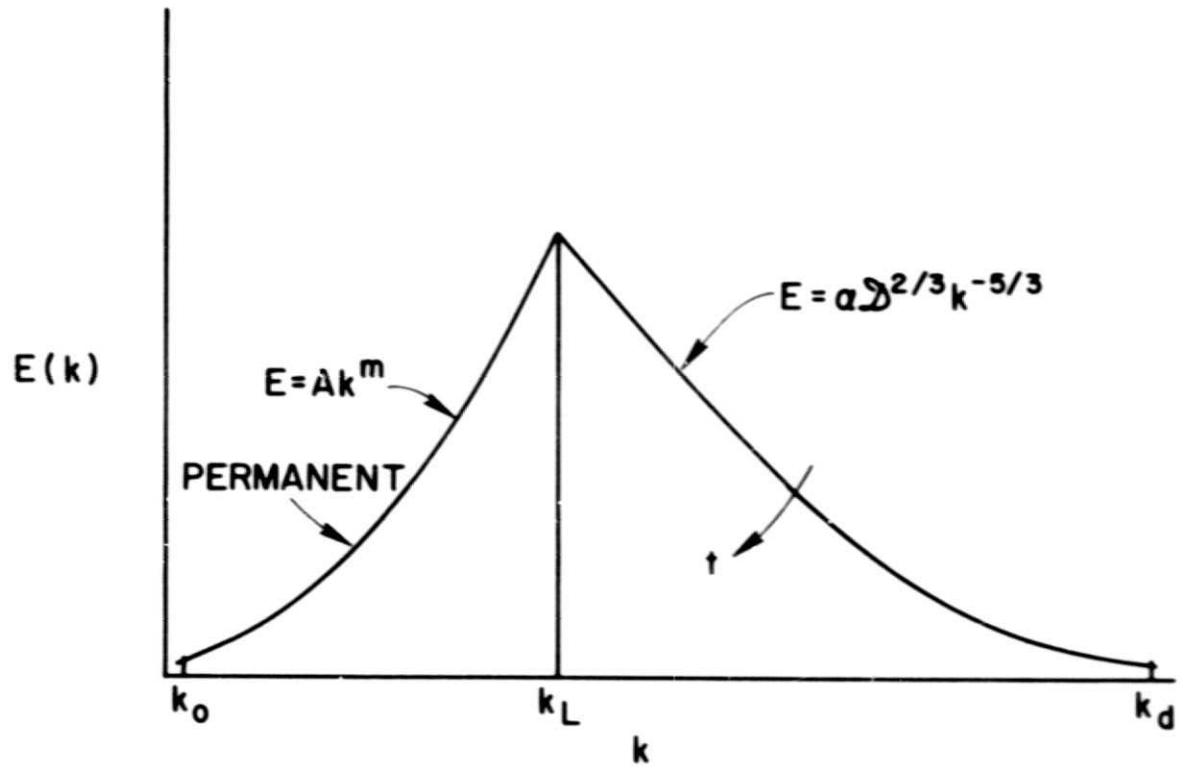

Figure 3. Model Spectrum.

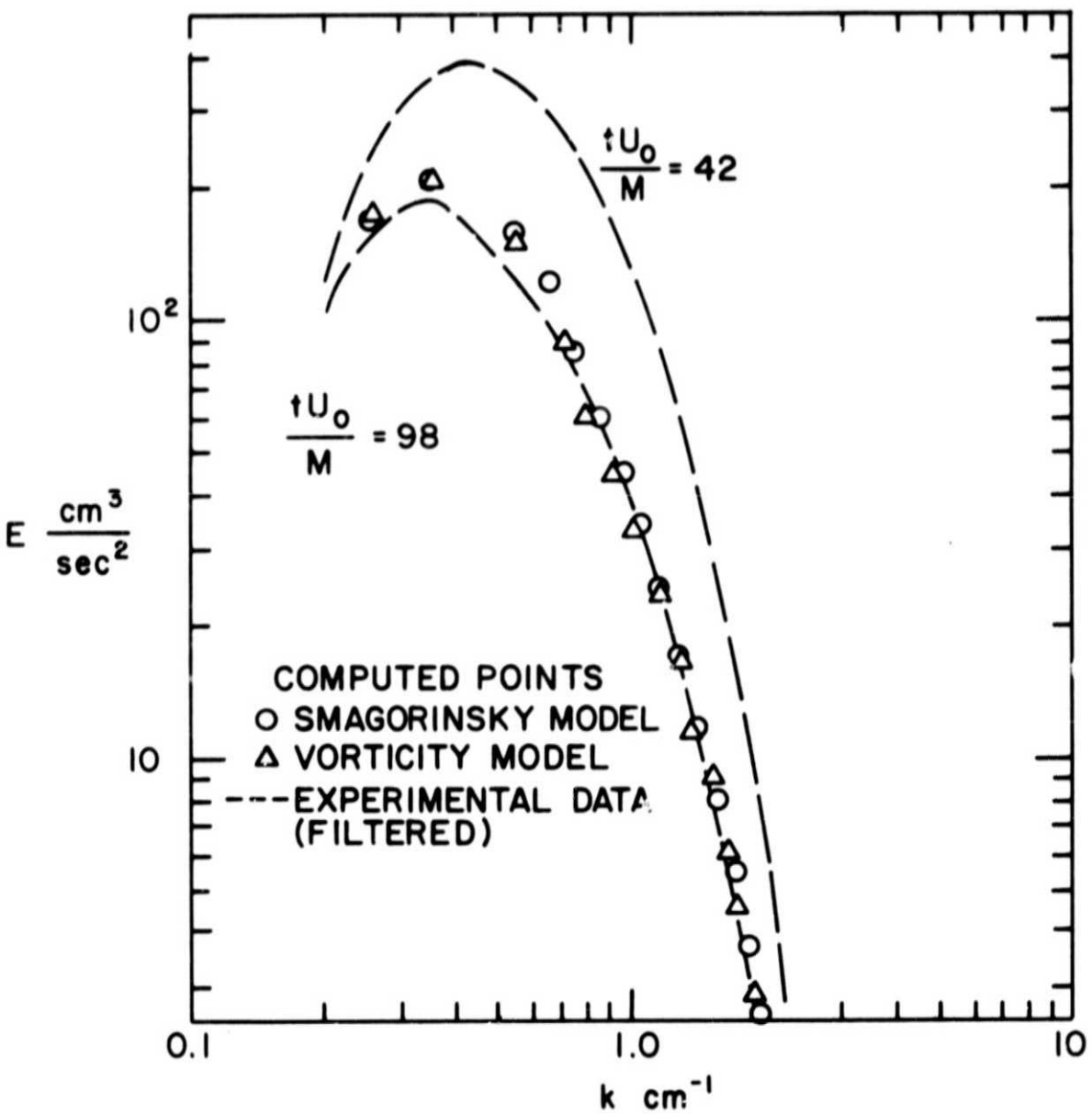

Figure 4. Decay of isotropic turbulence - 16^3 calculation

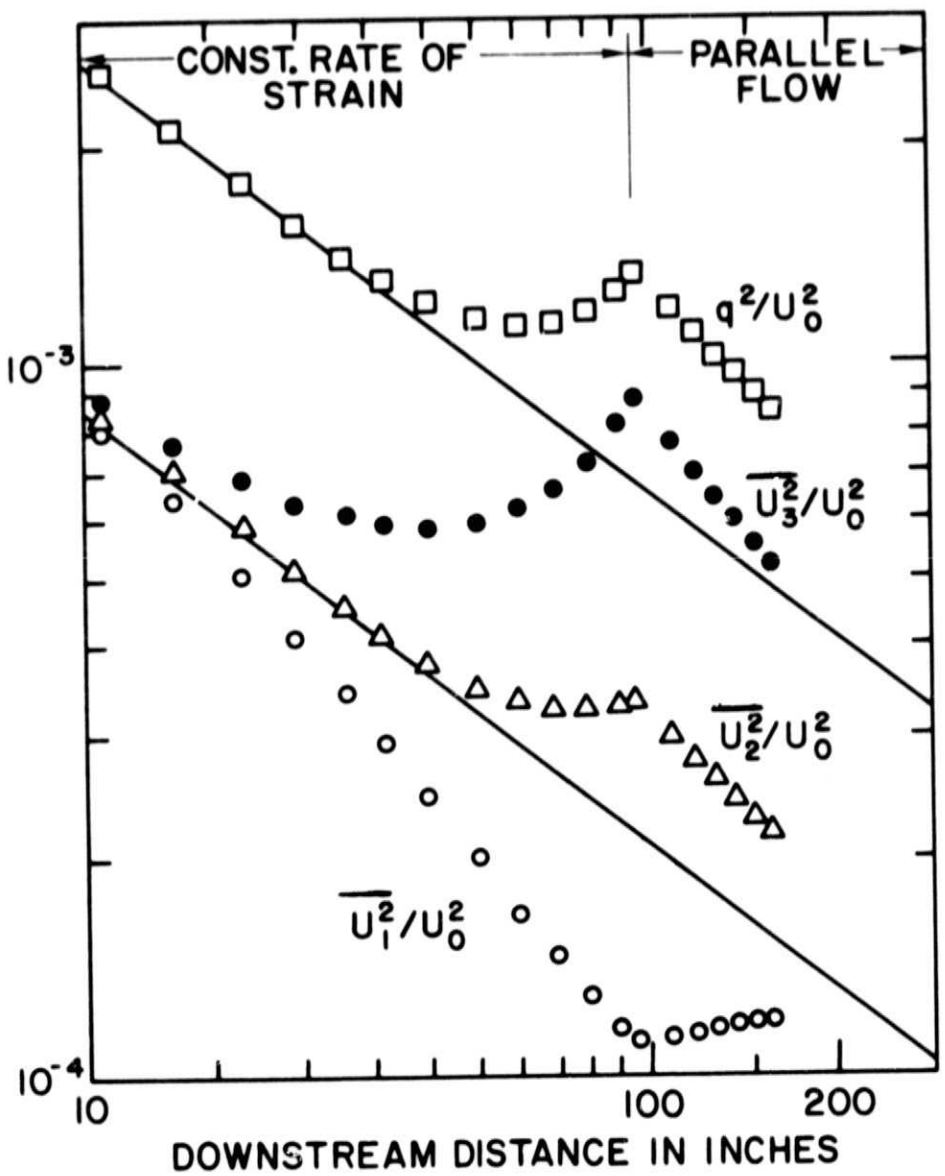

Figure 5. Large eddy simulation (16^3) of the Tucker-Reynolds flow.