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# PSYCHOPHYSICAL MODELS FOR SIGNAL DETECTION

## WITH TIME VARYING UNCERTAINTY

by

Eliezer Gai

B.S., Technion Israel Institute of Technology, 1965

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"CERTAINTY IS FOR THE ANGELS;

FOR MEN, THERE ARE ONLY PROBABILITIES"

Pierre Simon De Laplace
(1749 - 1827)

# PSYCHOPHYSICAL MODELS FOR SIGNAL DETECTION

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Submitted to the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, on January 15, 1975, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

## ABSTRACT

Psychophysical models for the behavior of the human operator in detection tasks which include change in detectability, correlation between observations, and deferred decisions are developed. Classical Signal Detection Theory (SDT) is discussed and its emphasis on the sensory processes is contrasted to decision strategies which are the subject of analysis in this thesis. The analysis of decision strategies utilizes detection tasks with time varying signal strength. The classical theory is modified to include such tasks and several optimal decision strategies are explored. Two methods of classifying strategies are suggested. The first method is similar to the analysis of ROC curves, while the second is based on the relation between the criterion level (CL) and the detectability.

Experiments to verify the analysis of tasks with changes of signal strength are designed. The results show that subjects are aware of changes in detectability and tend to use strategies that involve changes in the CL's.

The effect on the decision strategy of correlation between successive observations is studied. It is found that the present decision of the subject is dependent on his previous decision with a strong tendency to repeat the last decision even if it is wrong. The bias effects of correlation are described with the use of Markov process theory and the relation to classical SDT is also shown.

The case of deferred decisions applies to tasks in which the information rate is so high that the subject cannot make a decision after each observation. Thus, he is allowed to make more than one observation, but is asked to minimize the detection time. Such detection tasks are usually related to problems of

failure detection. The model that is suggested consists of two stages: linear estimation and a sequential decision mechanism whose decision function is the integral of the observation error. This model is found to be effective in predicting subjects performance in experiments that include "well behaved" processes. The model is also applied to the task of monitoring automatic landings for instrument failures. Although the processes that are involved are obtained by a non-linear high order time varying system and although the task is multidimensional, the predictions of the model fit the experimental data well.

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# CHAPTER I

## INTRODUCTION

### 1.1  Background, Motivation and Problem Statement

Psychophysics is that part of experimental psychology which deals with the quantitative relationship between stimuli and response of living mechanisms. The general framework can be divided into three different fields:  detection, recognition, and scaling. Detection deals with the question of the smallest amount of a stimulus that is needed to elicit a response. Recognition  deals with the question of resolution or the minimum difference between two stimuli that can be resolved.  The problem of relation between the strength of the stimuli and the amount of response is referred to as scaling. These three problems were the subject of extensive research in the last century when they were first posed in a methodical way by G.T. Fechner in 1860 (Swets, 1966).  Fechner also seems to have been the first one to notice the probabilistic nature of the problems, although this approach had already been implied by Laplace in his quote which was used as an epigram to this thesis.  The probabilistic approach was needed because of the the large variability in the sensitivity to the stimuli due to individual differences as well as internal and external conditions  of the subject.  Therefore, Fechner suggested the use of the method of replication, namely, to get repetitive yes/no responses of a subject to different stimuli

and to plot the proportion of positive responses as a function of stimulus magnitude. This method seems to be the backbone of any psychophysical research. The next step forward was taken by Thurstone in 1927 (Thurstone, 1927). He suggested that the stimulus can be represented as a random variable with some density function, and the recognition problem is that of separating two random variables on the psychological continuum. He also suggested the use of this method for cases in which the stimulus was not susceptible to physical measurements. A further step was made by Blackwell (1952) who related the psychophysical problems to the statistical theory of hypothesis testing.

The mathematical approach to hypothesis testing was formulated by Neyman and Pearson (1933) and was generalized later by Wald (1950). The application of this theory was first employed in communication theory for detection of electromagnetic signals in the presence of noise. It was further advanced because of its importance to the design of radar receivers during World War II, and it was then that the formulas and terminology of "false alarm", "hit" and "miss" were introduced.

The first rigorous presentation of what is now referred to as Signal Detection Theory (SDT) was given by Peterson et al (1954). It was followed by the work of Tanner and Swets (1954) which suggested the use of the theory in psychophysical

experiments. Later Swets et al (1961) embodied the theory in a psychophysical model for detection of visual signals. Most of the classical theory as well as the basic experiments were collected and summarized in books by Green and Swets (1966), Swets (1964) and Luce (1963). A good summary of the historical development of SDT and its applications was recently published by Swets (1973).

The principle appeal for utilizing SDT in psychophysical research was its ability to separate the detection process into two components, namely, the sensory process and the decision strategy. For the psychologists who were interrested mainly in the threshold mechanism, the sensory process seemed the more important of the two, and the separation characteristic was used only to eliminate the subjective bias of the subject that was reflected through his decision strategy (Trieshman and Watts, 1966). This approach motivated the use of a fixed signal strength within each experimental session and the evaluation of the results by Relative Operating (Receiver Operator) Characteristics (ROC) curves which are the heart of classical SDT. This approach was used in a wide field of applications which ultimately manifested the validity of the theory. The applications included cases in which a well defined signal was to be detected when the background noise had a known density function . Those experiments tested several sensory systems including vision (Tanner et al,

1953), auditory (Green, 1960) and tactile (Gaussin Hupet, 1972). However, it was also utilized in cases where the noise was the internal uncertainty of the decision maker due to the limited resolution of his senses. Such experiments were carried out for the detection of motion (Kinchala, 1969), visual monitoring (Gai and Curry, 1973), and manual control (Cohen and Ferrel, 1969).

Much less attention was paid to the second component of the detection process, that is, the decision strategy, although it seems that there are several areas in which this component is the dominant one. One such area is a more complicated visual monitoring task in which the signal strength is changing from one decision interval to the next. Such detection processes occur, for example, when a pilot uses traffic situation displays to avoid collisions with intruders. Since the input to the display is updated with radar information only every four seconds, the signal strength is fixed within the decision intervals but varies between the intervals. This is therefore the discrete case of signal detection with time varying signal strength. The main interest in such tasks lies in the decision strategy or more exactly in the subject's changing of his decision criterion when the signal strength is changed. These questions provided the motivation for the work presented in the first part of this thesis.

Little research work could be found in the literature concerning this approach to detection problems. Some work on

the problem of signal detection with varying signal strength

was done by Kinchala and·Smyzer (1967), Glorioso et al (1968)

and Thurmond et al (1970). However none of this work addressed

the question of decision strategy. Other work by Swets et al

(1967) and Birdsall and Roberts (1965, 1966) analyzed the

change of criterion between decision intervals but with fixed

signal strength. Decision strategies that were not based

on SDT were suggested by Parks (1966) and Thomas and Legge

(1970). Also, the problem of sequential effects between

decision intervals was analyzed by Kinchala (1965), Speeth

and Mathews (1961), and Tanner et al (1970, 1967).

This thesis suggests a unified theoretical analysis

of the problem as well as experimental analysis to support

the theory. It is shown that classical SDT can be modified

to analyze these problems, if the updating rate is slow

enough so that the signal strength is constant within each

decision interval. The difference between independent and

correlated input stimuli is also dealt with.

If, however, the information flow is fast or even

continuous, the problem is that of testing stochastic pro-

cesses rather than random variables. An example of such

detection tasks is a pilot monitoring the displayed outputs

of an automatic landing system based on ILS information

(Decelles et al, 1970). This problem is related to the

design of Failure Detection and Isolation (FDI) algorithms

for fully automatic systems. The question was first analyzed by control engineers·using linear filtering (Jazwenskii, 1970) and optimal control (Bryson and Ho, 1969) techniques to design optimal systems (Athans, 1971). Later the same ideas were used by man-machine researchers to model the human as a controller (Kleinman et al, 1970). This model was also used by Levison (1971) and Levison and Tanner (1971) to model the human monitoring performance. The problem of the human operator as an FDI system was investigated by Neimala and Krendel (1974) and Phatak et al (1969, 1972).

The second part of this thesis suggests still another approach to modelling the human operator as an FDI system which is based on sequential analysis techniques (Wald, 1974). This approach is similar to the method used by Chien (1972) in the design of FDI algorithms for strapdown inertial systems. Experiments were run to support this approach and the question of closed and open decision intervals is dealt with. The theory is also modified to multi-decision tasks where a share of attention was needed. This compound model is then applied to the case above, that is, a pilot monitoring an automatic landing system where his task is only to detect failures but not to identify and compensate them.

## 1.2 Thesis Organization

Chapter Two includes a detailed description of Signal Detection Theory (SDT) which is the foundation for the work presented in this thesis. In the discussion of SDT, we tried to combine the approaches of the psychophysicist and the communications engineer, as well as to emphasize the points that are important to our use of SDT in time-varying signal detection problems.

Chapter Three generalizes classical SDT to detection tasks with time varying detectability. Several decision strategies are discussed and the concept of Decision Rule (DR) curves is introduced for use in the analysis of these strategies. An alternative method for analyzing decision strategies when the underlying distributions are known is also described.

Chapter Four provides the description and the results of a visual discrimination experiment in which the signal strength is changed in a random order to avoid correlations between successive decisions. A model is suggested which describes the subjects behaviour and leads to the use of SDT. The results are used to verify the strategies that are suggested in Chapter Three.

In Chapter Five, correlation effects on the decision strategies are discussed. An experiment similar to the one in Chapter Four is described. In this experiment, the order

of presentation is changed to sequential in order to introduce correlations. The bias effect of the correlation is described as a repitition or alternate strategy. The analysis is based on the theory of Markov processes, and the relation to class-ical SDT is also discussed.

Chapter Six deals with those detection tasks in which the information rate is high and a decision is not required after each observation but can be delayed. The suggested model for the detection process consists of two parts: a linear estimation mechanism and a decision mechanism. There-fore, the chapter includes a short summary on linear estimation and sequential analysis. Results of a set of experiments that support the model are described for both open and closed decision intervals.

Chapter Seven presents an implementation of the model that is suggested in Chapter Six for the specific problem of modelling the behaviour of a pilot in monitoring an automatic landing system for failure detection. Detailed discussion of the problem, its simplifications, and the use of the previous model for multidimensional tasks are described.

Finally, in Chapter Eight, we summarize the results and conclusions, and suggest some ideas for future research.

# CHAPTER II

## CLASSICAL SDT AND PSYCHOPHYSICS

### 2.1  General Discussion

Classical Signal Detection Theory is the foundation
of the work done in this thesis.  It is, therefore, important
to repeat in some detail the basic concepts of the theory and
its application.  An historical background of the development
of the theory was given in Chapter I.  This chapter is a sum-
mary of the basic concepts of SDT and is primarily based on
two references representing two points of view.  One is the
communication engineer's approach (Van Trees, 1968) and the
other is the psychologist's approach (Green and Swets, 1966).
In addition, some of the results are presented in still another
form in order to clarify the generalization of the classical
theory to include the case of time varying signal strength
which is the topic of this thesis.

### 2.2  General Concepts of SDT

Signal detection is a theoretical approach to the problem
of discriminating between several hypotheses or states of the
world.  It is assumed that there exist M  well-defined states
of the world, each of them affecting in some way an entity
which is available to the decision mechanism and is referred

to as the observation. Based on these observations, the mechanism must decide which of the possible states of the world is true.

The simplest case of signal detection arises when there are only two possible states of the world, sometimes referred to as a simple binary hypothesis test, simple in the sense that the statistical characteristics of the signals are completely known. Analysis of this simplifed problem allows a reduction in the algebraic work required in the derivation of the equations without any loss of generality. The generalization to the composite M hypothesis case is straightforward and can be found in the literature (Van Trees, 1968). Therefore, in this work we will concentrate only on the simple binary case. We will also assume that a decision must be made after each observation, and that the observation is a scalar quantity. The generalization to the vector case with a fixed number of observations is given in Van Trees (1968). The case of a free number of observations is dealt with in section 6.3 of this work.

Let us assume that there are only two hypotheses $H_0$ and $H_1$. Under hypothesis $H_0$ the state of the world is $S_0$ and under hypothesis $H_1$ the state of the world is $S_1$. It is further assumed that the *a priori* probability of the appearance of $S_0$ and $S_1$, $P(S_0)$ and $P(S_1)$ are known and that

$$P(S_0) + P(S_1) = 1 \qquad (2.1)$$

The observer receives a sequence of N successive observations, and he must make a decision after each of these observations. Each response or answer A may take one of the two possible values

$$A_0 - \text{State } S_0 \text{ has happened}$$
$$A_1 - \text{State } S_1 \text{ has happened}$$

It is assumed that all observations are statistically independent.

The results of such procedures can be categorized into four groups:

$n(A_0/S_0)$ = number of decisions in which the answer was $A_0$ and the state of the world was $S_0$

$n(A_1/S_0)$ = number of decisions in which the answer was $A_1$ and the state of the world was $S_0$

$n(A_0/S_1)$ = number of decisions in which the answer was $A_0$ and the state of the world was $S_1$

$n(A_1/S_1)$ – number of decisions in which the answer was $A_1$ and the state of the world was $S_1$

Clearly $n(A_0/S_0)$ and $n(A_1/S_1)$ represent the number of correct decisions while $n(A_0/S_1)$ and $n(A_1/S_0)$ are the number of errors. These four numbers can be normalized and transformed into conditional probabilities as follows:

Let $\quad n_{S_0} = P(S_0) \cdot N$ = number of presentations of $S_0$ in N trials

$$n_{S_1} = P(S_1) \cdot N = \text{number of presentations of } S_1$$
$$\text{, in } N \text{ trials}$$

Then

$$P(A_0/S_0) = n(A_0/S_0)/n_{S_0} = \text{probability of a hit}$$

$$P(A_1/S_0) = n(A_1/S_0)/n_{S_0} = \text{probability of a miss}$$

$$P(A_0/S_1) = n(A_0/S_1)/n_{S_1} = \text{probability of a false alarm}$$

$$P(A_1/S_1) = n(A_1/S_1)/n_{S_1} = \text{probability of a correct rejection}$$

The above names are invoked out of tradition from the communication engineers who first used them in radar applications where state $S_0$ was the appearance of a signal and state $S_1$ was the appearance of noise without a signal. These four conditional probabilities are related as follows:

$$P(A_0/S_0) + P(A_1/S_0) = 1$$
$$P(A_0/S_1) + P(A_1/S_1) = 1$$

(2.2)

Equation (2.2) shows that only two of these conditional probabilities are needed to completely specify the expected results of the experiment. The two that are usually chosen are the probability of a hit $P(A_0/S_0)$ and the probability of a false alarm $P(A_0/S_1)$. The two quantities are a measure of the performance of the decision process.

The original goal in communications theory was to find a method of receiver design that optimized some performance measure. It was suggested that if the observation is represented as a random variable whose density function under both

states of the world is completely known, then a sufficient statistic would be the likelihood ratio (Van Trees, 1968) given by

$$\ell(x) = \frac{P(x/S_0)}{P(x/S_1)} \qquad (2.3)$$

where $P(x/S_0)$ is the conditional probability of x given $S_0$ and $P(x/S_1)$ is the conditional probability of x given $S_1$. It has been shown (Green and Swets, 1966) that if the set on which x is defined is divided into two subsets, one of which includes all those x which satisfy

$$\ell(x) \geq \beta \qquad (2.4)$$

while the other contains all x which satisfy

$$\ell(x) < \beta \qquad (2.5)$$

then the decision strategy (choosing a state of the world based on the value of $\ell(x)$ with respect to β) will be optimal for the following performance measures:

1. Maximizing the weighted combination

$$P(A_0/S_0) - \beta P(A_0/S_1)$$

2. Maximizing the expected value of the cost

3. Maximizing the percentage of correct responses

4. Minimizing the expected penalty for errors

5. Satisfying the Neyman-Pearson objective, namely, maximizing $P(A_0/S_0)$ for a fixed value of $P(A_0/S_1)$

The likelihood ratio criterion level (LRCL) β will take different values depending on the specific performance measure.

The discussion above suggests the likelihood ratio $\ell(x)$ as a decision function. From a psychological point of view, the question is whether such a model can characterize human behavior. It is, therefore, to be assumed that the sensory experience of the operator is somehow transformed onto a psychological continuum which is equivalent to the likelihood ratio. It is still not clear how such a transformation is accomplished. However, it has been found that after a period of training, the performance of the subjects is similar to the performance which is predicted by the likelihood ratio model. Evidence for such performance is found in experiments in a wide range of applications of detection tasks (Swets, 1973).

The major appeal of the SDT model to psychologists is its potential ability to separate the two processes which are involved in the detection task. One of the processes is the sensory process which is characterized by some distance measure between the two states of the world $S_0$ and $S_1$. It is usually referred to as the detectability of the signals and is written as d'. The value of d' is a function of the parameters of the ensemble density function of the stimulus. The other process is the decision strategy or the way in which the LRCL β is chosen. These two processes, represented by d' and β, determine the performance of the subject. Therefore the performance may be written as:

$$P(A_0/S_0) = g_1(d', \beta)$$
$$P(A_0/S_1) = g_2(d', \beta)$$

(2.6)

It should be noted that both $d'$ and $\beta$ might themselves be functions of several variables; however, those variables that affect $\beta$ do not affect $d'$.

If two inverse functions can be found such that equation (2.6) can be written in the following form

$$d' = g_3(P(A_0/S_0), P(A_0/S_1))$$
$$\beta = g_4(P(A_0/A_0), P(A_0/A_1))$$

(2.7)

then the two processes are completely separable. A necessary condition for this separation is that the density functions $f(x/S_0)$ and $f(x/S_1)$ are completely known. In many psychophysical applications this condition is difficult to satisfy.

In most of the past studies in psychophysics, the concentration was on the sensory process alone and the separation property was used only to eliminate the subjective bias of the subjects. In these cases there is a simple way to avoid the difficulties mentioned above. This is done by fixing the value of $d'$ for the whole experimental session, while changing $\beta$ in the range $[-\infty, +\infty]$. The performance is then a function of $\beta$ alone for some fixed value of $d'$. Using equation (2.6) the results can be plotted in the $P(A_0/S_0)-P(A_0/S_1)$ plane, yielding the Receiver (Relative) Operating Characteristic (ROC) curve.

By repeating the experiment with different values of d', a family of curves representing the sensory process is generated. A typical ROC curve based on experimental data is shown in Figure 2.1.

In the ROC approach, it is necessary to make the subject change his LRCL $\beta$. This LRCL is a function of three factors:

1.  The *a priori* possibilities of the state of the world $P(S_0)$ and $P(S_1)$

2.  The rewards given for the correct decision and the penalties for errors

3.  The detectability d'.

Since the detectability d' is fixed, only the first two can be used. If n different values of $\beta$ are used, the experiment will be n times longer. In order to shorten the total experiment time, it is possible to obtain different values for $\beta$ by requesting the subject to give rated answers rather than only two. For example, the following rated answers might be used: Sure $S_0$, Think $S_0$, Indifferent, Think $S_1$, Sure $S_1$. For n rated answers the subject must choose n-1 LRCL's and therefore produce n-1 values for $\beta$. Let these rated decisions be $R_i$ (i = 1,...,n). Then the performance is a function of 2n variables because for each decision $R_i$ the state of the world might be $S_0$ or $S_1$. However only 2(n-1) of these are independent. We will use the following notation
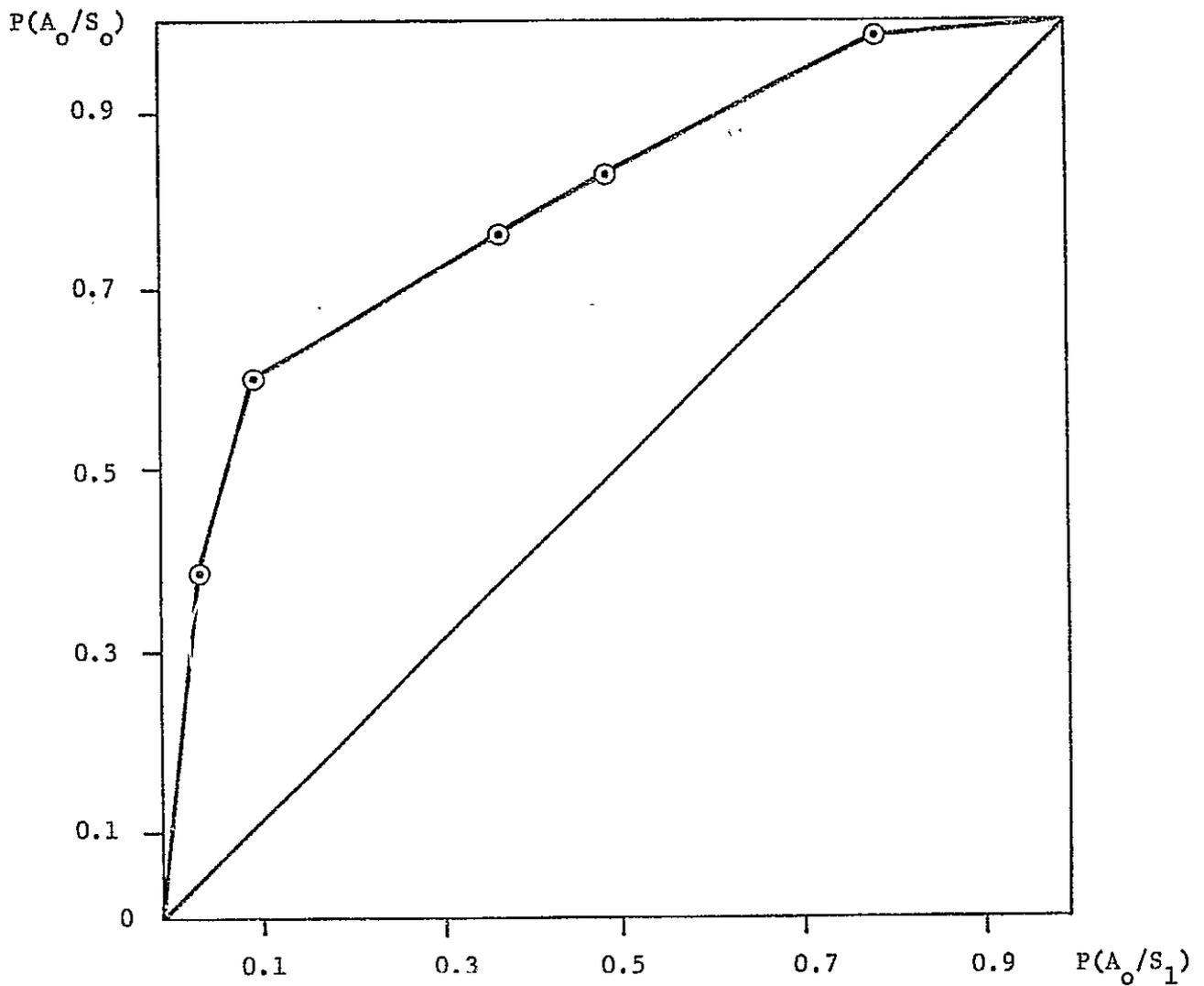
FIGURE 2.1  ROC DATA OBTAINED IN A VISUAL DETECTION TASK

(FROM GREEN AND SWEETS 1966)

$$P(R_i/S_0) = g_{2i-1}(d', \beta_i, \beta_{i-1})$$
$$P(R_i/S_1) = g_{2i}(d', \beta_i, \beta_{i-1}) \qquad i - 1, \ldots, n-1 \quad (2.8)$$

where

$$\beta_{i+1} \geq \beta_i$$

$$\beta_0 = -\infty \qquad \beta_n = +\infty$$

It might be noted that if the results of the experiment can be written in the form of equation (2.7), namely as a function of d' and $\beta_1, \ldots, \beta_n-1$, only n parameters are needed so that the number of parameters can be reduced by n-2.

Another approach to define the "detectability" without knowledge of the underlying density function is to use a non-parametric measure (Hammerton and Altman, 1972). The measure is based on the outcomes of a confidence rating experiment with n possible answers. A random variable y is defined on the set of all possible responses by assigning the value i to the response $R_i$. The probability of y = i is therefore:

$$P(y=i) = P(S_0) \cdot P(R_i/S_0) + P(S_1) \cdot P(R_i/S_1) \qquad (2.9)$$

Also two conditional expectations can be defined as follows:

$$\bar{y}_0 = E(y/S_0) = \sum_{i=1}^{n} iP(R_i/S_0)$$

$$\qquad (2.10)$$

$$\bar{y}_1 = E(y/S_1) = \sum_{i=1}^{n} iP(R_i/S_1)$$

The nonparametric measure of detectability is then defined by:

$$C = \frac{\overline{Y}_0 - \overline{Y}_1}{n - 1} \qquad (2.11)$$

It should be noted that although this measure does give information on the detectability of the signals, it is not equivalent to d'. It can be shown (Morgan, 1973) that C is a function not only of the parameters of the density functions of the signals but also a function of the LRCL's. This means that by using C the separation property of SDT is lost, and therefore the approach is not often used.

Another important property of the ROC analysis is that the area under the ROC curve in the $P(A_0/S_0)-P(A_1/S_1)$ plane is equal to the expected percent of correct answers in a four alternative forced choice experiment (Green and Swets, 1966).

The above methods enable the analysis of data from psychophysical experiments without the knowledge of the underlying distribution. However, if this information is available, much more powerful results can be obtained.

## 2.3 Mathematical formulation of SDT with known distributions

From the discussion in section 2.2 it is clear that if the underlying distributions are known, the values of d' and the LRCL's might be found so that the performance can be expressed in the form of equation (2.7) and the separation of the two processes is complete. Also an analytical expression can be found for the ROC curves in the form of

$$P_{A_0/S_0}(d',\beta) = \{P_{A_0/S_1}(d', \beta)\} \tag{2.12}$$

where

$$d' = \text{constant} \qquad -\infty \leq \beta \leq +\infty \tag{2.13}$$

These theoretical curves can be drawn and the subjects performance can be compared to the predicted performance.

In those experiments where both the signal and noise are included in the stimulus and the internal noise of the subject is considered negligible in comparison to the external noise, the statistical characteristics are virtually known (Lee, 1963). Then, the theoretical ROC curves can be plotted analytically before the experiment starts. However, for many other cases in which the internal noise is the main source of uncertainty, the experimenter has to assume the functional form of the density function, and then, on the basis of the outcome of the experiment, find its parameters. The general problem is, therefore, a parameter optimization problem and many algorithms have been suggested for the solution. For the particular

problem of fitting distributions to confidence rating experiments,

algorithms were suggested by Ogilvie and Creelman (1968),

Dorfman and Alf (1968), Abramson and Levitt (1969), and Grey

and Morgan (1972). A further discussion of this problem and a

suggestion for another algorithm are given in section 4.4. and

Appendix A.

## 2.3.1 The Gaussian assumption

By far the most commonly used assumption is that the

underlying probability density is Gaussian. In those cases

where the simulated noise is the dominant factor, this dis-

tribution is chosen because of the ease with which it can be

created. Moreover it has some appealing characteristics:

1. The distribution is completely defined by two
   parameters, the mean $m_x$ and the standard
   deviation $\sigma_x$.

2. Gaussian random variables remain Gaussian under
   linear operations.

3. Two jointly-Gaussian random variables which are
   uncorrelated are also independent.

In the cases where the internal noise is the dominant

noise source, the Gaussian assumption is supported by the

central limit theorem. This theorem states that the distri-

bution of the sum of a large number of independent random

variables with equal distributions and with finite first and

second moments is approximately Gaussian regardless of their individual distributions. • Since the sensory events are often considered to be composed of many similar but simple events, this theorem might apply. However, when the assumption about the distribution is based on this argument, the results should go through a careful significance test.

Using the Gaussian assumption, the conditional densities are given by:

$$f(x'/S_0) = \frac{1}{\sqrt{2\pi}\sigma_0}\exp\{- \frac{(x' - m_0)^2}{2\sigma_0^2} \}$$

$$f(x'/S_1) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\{- \frac{(x' - m_1)^2}{2\sigma_1^2} \}$$

(2.14)

and the likelihood ratio is given by

$$\ell(x') = \frac{\sigma_1}{\sigma_0} \exp\{\frac{(x' - m_1)^2}{2\sigma_1^2} - \frac{(x' - m_0)^2}{2\sigma_0^2} \}$$
(2.15)

The likelihood ratio is, therefore, a function of four parameters $m_1$, $\sigma_1$, $m_0$, $\sigma_0$. Since the decision is made by commparing the likelihood ratio to the LRCL, the performance would be invariant under a linear transformation. Therefore, let

$$x = \frac{x' - m_1}{\sigma_1}$$
(2.16)

so that equation (2.15) becomes

$$\ell(x) = \frac{1}{\sigma} \exp \frac{x^2}{2} - \frac{(x - m)^2}{2\sigma^2}$$
(2.17)

where

$$m = (m_0 - m_1)/\sigma_1 \qquad (2.18)$$

$$\sigma = \sigma_0/\sigma_1 \qquad (2.19)$$

so that the likelihood ratio is now a function of only two parameters. The decision is, as before, made as follows:

$$\text{if } \ell(x) \geq \beta \qquad \text{choose } S_0 \qquad (2.20)$$
$$\text{if } (x) < \beta \qquad \text{choose } S_1$$

Let K be that value of x which satisfies

$$\ell(x) = \beta \qquad (2.21)$$

Then the performance of the decision maker is given by:

$$P(A_0/S_0) = \text{Prob}(\ell(x) \geq \beta/S_0)$$
$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{K-m}{\sigma}}^{\infty} \exp\{-\zeta^2/2\} d\zeta \qquad (2.22)$$

$$P(A_0/S_1) = \text{Prob}(\ell(x) \geq \beta/S_1)$$
$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{K-m}{\sigma}}^{\infty} \exp\{-\zeta^2/2\} d\zeta \qquad (2.23)$$

Defining

$$\Phi\left(\frac{\eta-m}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\frac{\eta-m}{\sigma}} \exp\{-\zeta^2/2\} d\zeta \qquad (2.24)$$

where $\Phi$ is the Gaussian distribution function, equations (2.22) and (2.23) are written

$$P(A_0/S_0) = 1 - \Phi\left[\frac{K-m}{\sigma}\right].$$

$$P(A_0/S_1) = 1 - \Phi[K] \tag{2.25}$$

In order to express the performance in the form of equation (2.7), K in equation (2.25) is replaced by a function of $\beta$, m, and $\sigma$. From equations (2.17) and (2.20):

$$\ln\beta = \frac{K^2}{2} - \frac{(K-m)^2}{2\sigma^2} - \ln\sigma \tag{2.26}$$

The value of K is found to be the solution of the following quadratic equation:

$$(\sigma^2 - 1)K^2 + 2mK - m^2 - 2\sigma^2(\ln\sigma + \ln\beta) = 0$$

and this equation has a real solution only if

$$m^2 + 2(\sigma^2 - 1)(\ln\sigma + \ln\beta) \geq 0$$

When such a solution does exist then:

$$P(A_0/S_0) = 1 - \Phi\left[\frac{-m\sigma \pm \sqrt{m^2 + 2(\sigma^2-1)\ln\beta\sigma}}{\sigma^2-1}\right]$$

$$P(A_0/S_1) = 1 - \Phi\left[\frac{-m \pm \sigma\sqrt{m^2 + 2(\sigma^2-1)\ln\beta\sigma}}{\sigma^2-1}\right] \tag{2.27}$$

Thus the detectability d' is a function of two variables m and $\sigma$; however, the analytic expression for this relationship can not be found.

From equation (2.25) it is possible to get the equation

for the ROC curve in the $P(A_0/S_0)$-$P(A_0/S_1)$ plane. If d' is constant, $P(A_0/S_0)$ and $P(A_0/S_1)$ are functions of K only. By eliminating K from equation (2.25), the ROC equation is

$$\Phi^{-1}[1 - P(A_0/S_1)] = \sigma[\Phi^{-1}\{1-P(A_0/S_0)\}] + m \qquad (2.28)$$

where $\Phi^{-1}(\alpha)$ is the inverse of $\Phi(\alpha)$ and can be found in mathematical tables. For each curve in the family, m and $\sigma$ are constant, i.e. d' is constant. Such a family of curves for different values of m and $\sigma$ is shown in figure 2.2. Because these curves are usually used to validate experimental results, it would be helpful if the curvature could be eliminated. This can be done, by using a unit deviate scale rather than a linear one. Let:

$$z_1 = \Phi^{-1}[1 - P(A_0/S_0)]$$
$$\qquad (2.29)$$
$$z_2 = \Phi^{-1}[1 - P(A_0/S_1)]$$

Then the ROC is given by a straight line

$$z_2 = \sigma z_1 + m \qquad (2.30)$$

Therefore, if the Gaussian assumption holds, the experimental data points for fixed $\sigma$ and m should be on a straight line. However, even these curves do not alleviate the problem of unique measurement of detectability since it is a function of both the slope and the zero crossing point.
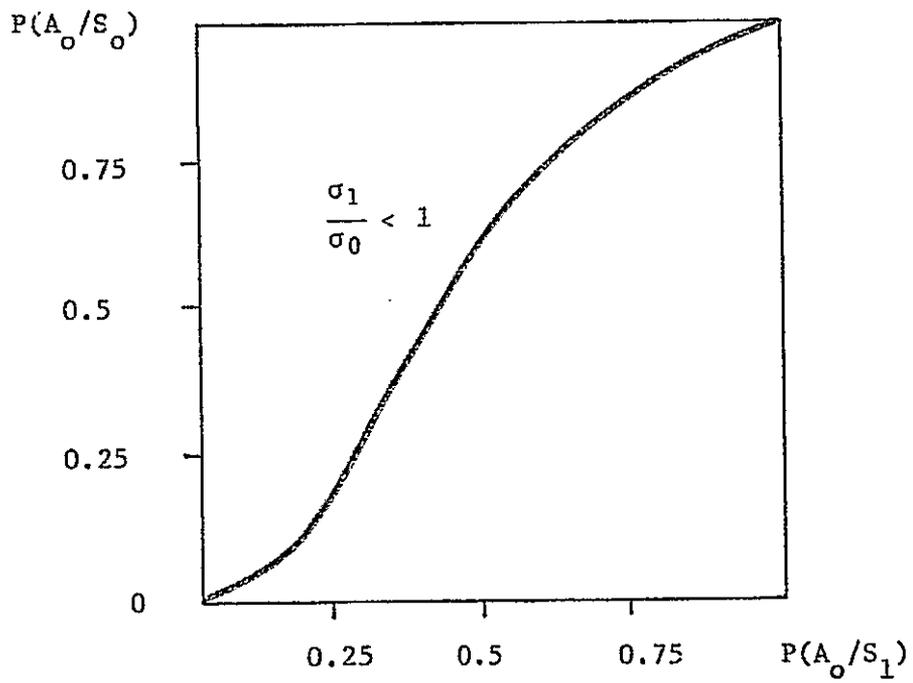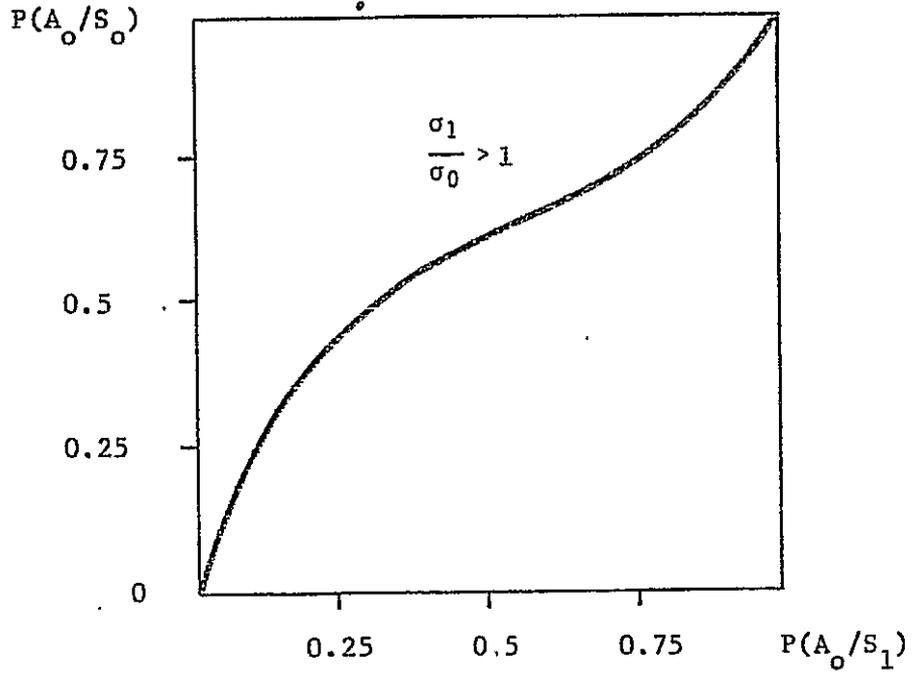
FIGURE 2.2  ROC CURVES IN THE GENERAL GAUSSIAN CASE

## 2.3.2  Special Cases:

The fact that the detectability in the general Gaussian case is a function of two variables and that it is difficult to express the performance through d' and $\beta$, leads to further assumptions in which these problems do not arise.

One assumption is that the distributions under both states of the world have the same variance, or:

$$\sigma_0 = \sigma_1 \longrightarrow \sigma = 1 \tag{2.31}$$

In those cases where the simulated noise is dominant, it is easy to satisfy the assumption. In the cases where the internal noise is dominant, the justification of the assumption is that, as the signals are deterministic, the internal noise source is the same for both signals, hence the variance of the stimuli is the same. Under this assumption, equation (2.26) which relates $\ln\beta$ to K is:

$$\ln\beta = K^2/2 \quad - (K - m)^2/2$$

Therefore, $\ln\beta$ is linearly related to K:

$$\ln\beta = Km - m^2/2 \tag{2.32}$$

The performance of the decision maker is given by

$$P(A_0/S_0) = 1 - \Phi(\ln\beta/m \quad - m^2/2)$$
$$P(A_0/S_1) = 1 - \Phi(\ln\beta/m \quad + m^2/2) \tag{2.33}$$

Therefore, the detectability of the signals in equation (2.33) is a function of only one quantity, so we may define

$$d' = m = \frac{m_0 - m_1}{\sigma} \qquad (2.34)$$

It is clear also that in this case the performance can be represented by $d'$ and $\beta$.

The ROC curve is given by the following equation

$$\Phi^{-1}\{1 - P(A_0/S_1)\} = \Phi^{-1}\{1 - P(A_0/S_0)\} + d' \qquad (2.35)$$

or if we use the unit deviate scale

$$Z_2 = Z_1 + d'$$

This is a family of parallel straight lines with unit slope, and with intercept $d'$. The detectability is, therefore, the distance of the line from the origin multiplied by $\sqrt{2}$.

Figure 2.3 shows ROC curves for equal variance Gaussian distributions in the $P(A_0/S_0)-P(A_0/S_1)$ plane. Figure 2.4 shows the same ROC curves in the $Z_2-Z_1$ plane.

Another possible assumption is that the means of both signals are the same and the variances differ, namely,

$$\sigma_0 = \sigma_1, \ m_0 = m_1 \rightarrow \ m = 0 \qquad (2.36)$$

In most of these cases both means are zero. The decision is therefore made on the basis of the difference between the variances of the signals. Equation (2.26) is reduced to

$$\ln\beta = \frac{K^2}{2} - \frac{K^2}{2\sigma^2} - \ln\sigma$$

Therefore, from (2.27), K is given by:
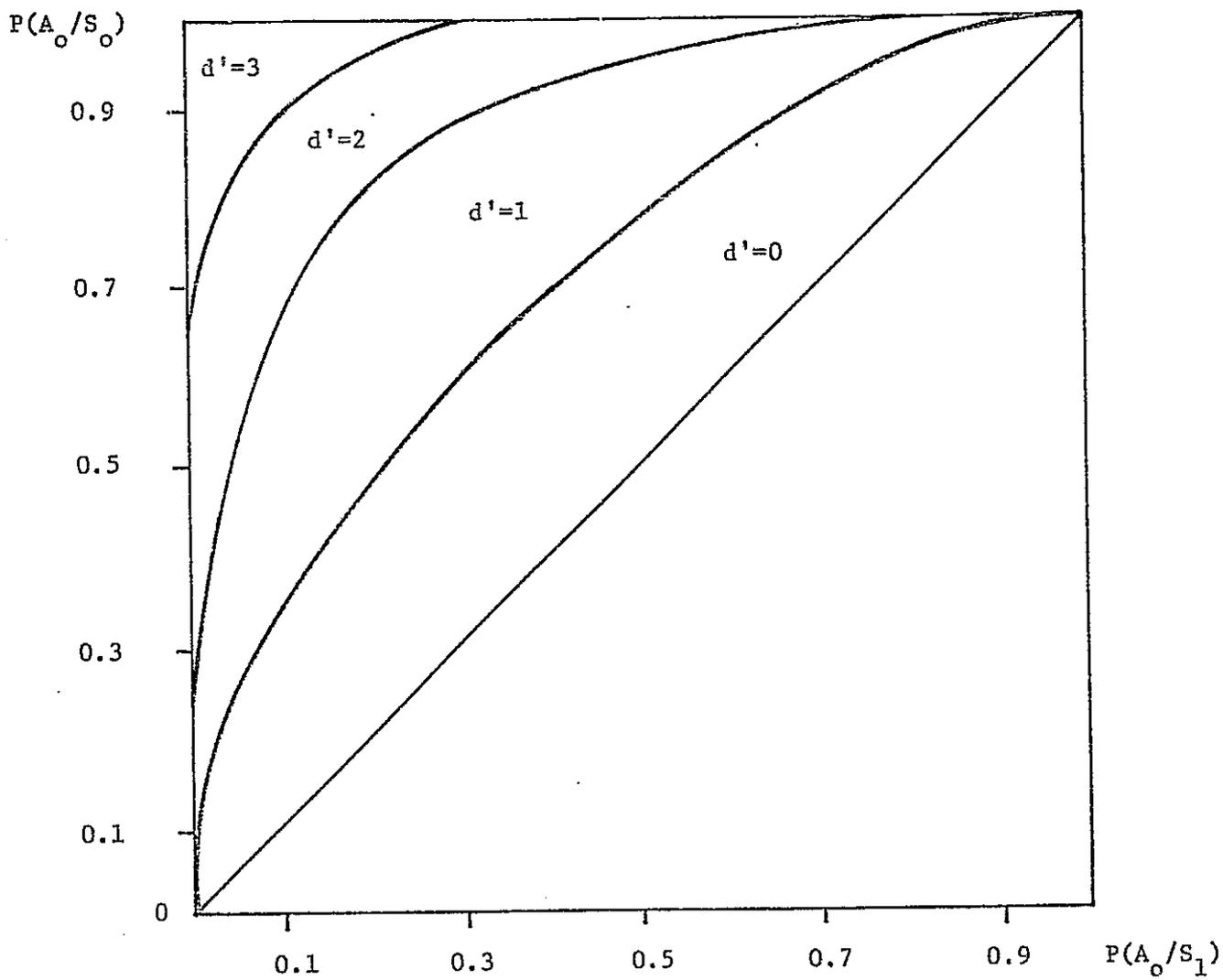
FIGURE 2.3  ROC CURVES FOR EQUAL VARIANCE GAUSSIAN
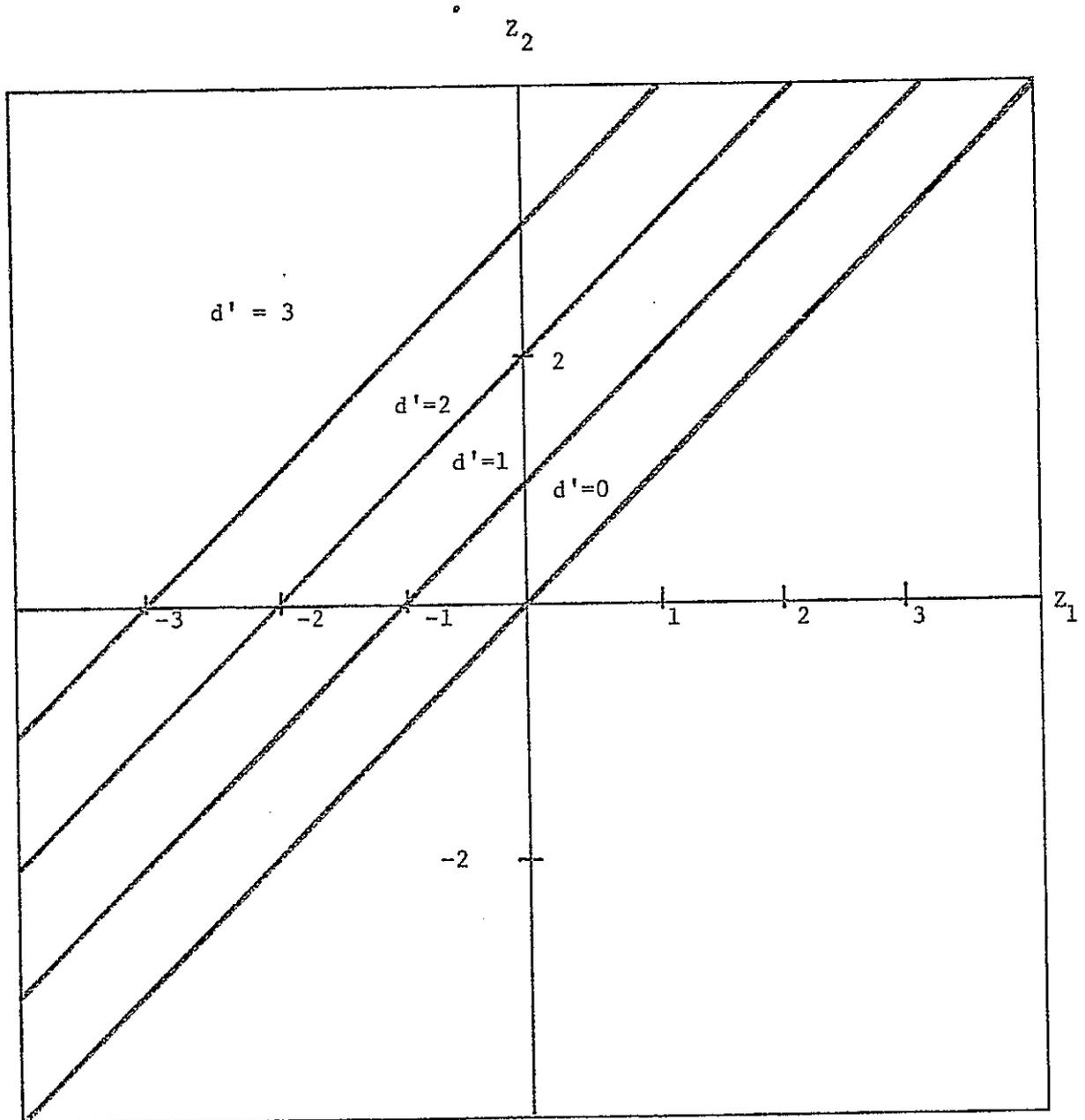
DISTRIBUTIONS

FIGURE 2-4   ROC CURVES OF EQUAL VARIANCE GAUSSIAN

DISTRIBUTION USING UNIT DEVIATE SCALE

$$K = \pm\sigma\sqrt{\frac{2(\ln\beta + \ln\sigma)}{\sigma^2 - 1}} \quad . \tag{2.37}$$

In contrast to the former case, the linear relationship between $\ln\beta$ and $d'$ does not hold, and the relation does not yield a unique answer. By substituting K from (2.37) into equation (2.25) (with $m = 0$), the performance of the decision maker is given by:

$$P(A_0/S_0) = 1 - \Phi(\pm\sqrt{\frac{2(\ln\beta\sigma)}{\sigma^2-1}})$$

$$\tag{2.38}$$

$$P(A_0/S_1) = 1 - \Phi(\pm\sigma\sqrt{\frac{2(\ln\beta\sigma)}{\sigma^2-1}})$$

Again the detectability of the signals affects the expected performance of the subjects through one quantity. Therefore, the detectability can be defined as:

$$d' = \sigma = \sigma_0/\sigma_1 \tag{2.39}$$

The ROC curves for this special case are defined by the equation

$$\Phi^{-1}\{1 - P(A_0/S_1)\} = \sigma\Phi^{-1}\{1 - P(A_0/S_0)\} \tag{2.40}$$

or in the $Z_2$-$Z_1$ plane

$$Z_2 = \sigma Z_1 \tag{2.41}$$

This is a family of straight lines with different positive

slopes, all passing through the origin. The detectability is given by the slope of the line. Note, however, that the detectability is a minimum when $\sigma = 1$, and it grows as a function of $|\sigma - 1|$.

### 2.3.3 The Logistic distribution assumption

Although in most applications in psychophysics the Gaussian assumption is used, several other density functions have been tried (Ogilvie et al, 1966). In particular, the class of density functions $f(x)$ that satisfy the condition (Thomas and Myers, 1972)

$$- \frac{\partial^2 \ln[f(x)]}{\partial x^2} \geq 0 \qquad \text{for all } x \qquad (2.42)$$

was found applicable in ROC analysis. This class includes the Gaussian, Logistic, Gamma, and Exponential distributions.

The Logistic distribution is sometimes preferred to the Gaussian distribution because the cumulative distribution function can be expressed in a closed form. This property is also useful in simplifying any parameter optimization algorithm that is used to fit the distribution. Furthermore, it has been found that the form of the ROC curves in the $P(A_0/S_0) - P(A_0/S_1)$ plane is very similar to the form under the Gaussian assumption (Luce, 1963).

The two conditional densities under this assumption are:

$$f(x'/S_0) = \frac{\exp\{-\frac{x'-m_0}{\sigma_0}\}}{[1 + \exp\{-\frac{x-m_0}{\sigma_0}\}]^2}.$$

$$(2.43)$$

$$f(x'/S_1) = \frac{\exp\{-\frac{x'-m_1}{\sigma_1}\}}{[1 + \exp\{-\frac{x-m_1}{\sigma_1}\}]^2}$$

where $m_0$ and $m_1$ are the means, and $\sigma_0$ and $\sigma_1$ are the variances of the distribution. The likelihood ratio is given by:

$$\ell(x') = \frac{\exp\{\frac{x'-m_1}{\sigma_1} - \frac{x'-m_0}{\sigma_0}\} [1 + \exp\{-\frac{x'-m_1}{\sigma_1}\}]^2}{[1 + \exp\{-\frac{x'-m_0}{\sigma_0}\}]^2}$$

$$(2.44)$$

The likelihood ratio is again a function of four variables; however, the same likelihood transformation as in equation (2.16) can be used so that:

$$\ell(x) = \frac{\exp\{x - \frac{x-m}{\sigma}\} [1 + \exp\{-x\}]^2}{[1 + \exp\{\frac{x-m}{\sigma}\}]^2}$$

$$(2.45)$$

where

$$\sigma = \sigma_0/\sigma_1 \qquad m = (m_0 - m_1)/\sigma_1$$

The performance of the subject is given by:

$$P(A_0/S_0) = [1 + \exp\{- \frac{K-m}{\sigma}\}]^{-1}$$

$$\tag{2.46}$$

$$P(A_0/S_1) = [1 + \exp\{-K\}]^{-1}$$

Where K satisfies

$$\ell(K) = \beta$$

The equation for the ROC curves is

$$\ln[P(A_0/S_1). -1] = \sigma\ln[P(A_0/S_0) - 1] - m \tag{2.47}$$

Defining new coordinates

$$Z_1 = \ln[P(A_0/S_0) - 1]$$

$$\tag{2.48}$$

$$Z_2 = \ln[P(A_0/S_1) - 1]$$

The ROC is again a straight line characterized by two parameters $\sigma$, m:

$$Z_2 = \sigma Z_1 -, m \tag{2.49}$$

In order to express the performance as a function of d' and $\beta$, the value of K has to be found by solving the equation:

$$\ln\beta = (1 - 1/\sigma)K + m/\sigma + 2(\ln[1 + \exp\{-K\}])$$
$$- 2(\ln[1 + \exp\{-(K-m)/\sigma\}]) \tag{2.50}$$

A closed form solution for this equation is not feasible, so a further assumption has to be made. One such assumption is

similar to (2.30), that is,

$$\sigma_0 = \sigma_1 \longrightarrow \sigma = 1$$

In this case

$$K = \ln\left\{\frac{\sqrt{\beta e^m} - 1}{1 - \sqrt{\beta/e^m}}\right\} \tag{2.51}$$

and the performance is given by:

$$P(A_0/S_0) = \left[1 + \frac{1 - \sqrt{\beta/e^m}}{\sqrt{\beta e^m} - 1}\right]^{-1}$$

$$P(A_0/S_1) = \left[1 + \frac{1 - \sqrt{\beta/e^m}}{\sqrt{\beta e^m} - 1}e^m\right]^{-1} \tag{2.52}$$

The performance is a function of only one parameter with respect to detectability; so we can define

$$d' = m = \frac{m_0 - m_1}{\sigma_1} \tag{2.53}$$

which is equivalent to the definition of detectability in the Gaussian case. The ROC curves will be straight lines with a slope of unity in the $Z_2$-$Z_1$ plane.

It can be shown that if the assumption of equal means is made, the detectability is defined by:

$$d' = \sigma = \sigma_0/\sigma_1 \tag{2.54}$$

and again the ROC curves and performance are similar to the Gaussian case.

# CHAPTER III

## ANALYSIS OF SIGNAL DETECTION WITH VARYING SIGNAL STRENGTH

### 3.1 General Discussion

As has already been shown in the last chapter the performance of any decision mechanism in a binary detection task can be characterized by the quantities $P(A_0/S_0)$ and $P(A_0/S_1)$. Each of these quantities is itself a function of two other parameters, the detectability d' and the likilihood ratio criterion level (LRCL) $\beta$, which are controlled by either the experimenter or the subject. Therefore, the most general question to be posed is how does the performance change when both d' and $\beta$ are changed within their full range? Since both $P(A_0/S_0)$ and $P(A_0/S_1)$ are functions of the same parameters, any change in either $\beta$ or d' will change both of them. Therefore, in general, the following relationship is sought:

$$P_{A_0/S_0}(\beta,d') = f\{P_{A_0/S_1}(\beta,d')\} \qquad (3.1)$$

where

$$-\infty \leq \beta \leq +\infty \qquad 0 \leq d' \leq \infty \qquad (3.2)$$

The fact that d' is a measure of the performance of the sensory process alone, while $\beta$ is a measure of the decision strategy, enables us to reduce the general case to some special interesting cases.

In Chapter 2 we dealt with the analysis of the sensory process. This is a special case of equation (3.1) in which d' is kept constant. In this chapter we investigate the manner in which the LRCL β, or decision strategy, is changed. A simple approach to this problem might be a dual approach to the ROC method, namely, to look at another special case of equation (3.1) in which β is kept constant. However, this approach cannot be justified as easily as in the sensory analysis case. There, the fixed detectability assumption could be based on the following arguments: If both signal and noise are simulated and the internal noise is negligible, then it is possible to *a priori* fix the detectability and by so doing to satisfy the assumption. If the internal noise is dominant, it can still be argued that the sensory process is prior to the decision mechanism so that the value of d' does not depend on β. Therefore, if the stimuli are kept constant, the internal noise will be stationary and d' is constant.

Those arguments cannot be used in the analysis of the decision strategy. The main reason is that the LRCL β, being based on the output of the sensory system, may depend on the value of d'. If we force the subject to fix his LRCL, we are actually dictating his strategy. Therefore, if the decision strategy is to be determined on the basis of the performance of the subject, the correct question to ask is how does the

decision strategy change with the change of d'.

It should be noted that d' is the distance measure of the two states of the world and is referred to by several names. Communications engineers use the names detectability or signal-to-noise ratio (SNR), while psychophysicists use the name signal strength. We will use these names interchangeably, according to the particular use of d'. Since we are interested in the analysis of the decision strategy, we have to deal with detection problems in which d' is changing in order to find how the LRCL is changed with d'.

## 3.2  Decision Rules

Since in many practical cases in psychophysics the
underlying density functions are not known, it is desirable
to work with the experimental raw data, namely, the various
values of $P(A_0/S_0)$ and $P(A_0/S_1)$. This leads to the analysis
of the decision strategy in the $P(A_0/S_0)-P(A_0/S_1)$ plane, the
same plane which is used in the ROC analysis. In this plane
we are looking for curves given by equations (3.1) and (3.2),
under the assumption that the changes in $\beta$ are the result of
the decision strategy. Therefore, the bias factors that control
$\beta$, namely, the *a priori* probabilities $P(S_0)$ and $P(S_1)$, as well
as the rewards for correct decisions and penalties for errors
must be kept constant. The resultant curves in the $P(A_0/S_0)-$
$P(A_0/S_1)$ plane will be referred to as the Decision Rule (DR)
curves. The shape of these curves is determined by the deci-
sion strategy of the subject.

The first strategy is the one already mentioned, namely,
a fixed LkCL decision rule. This means that the subject ig-
nores changes in the detectability so that $\beta$ is kept constant,
and this constant value is predetermined on the basis of the
bias factors. Therefore, in this strategy, the performance
will be a function of d' alone. As in the ROC analysis, an
analytic expression for the DR curve in this strategy is not
available unless the underlying distributions are known. It

should be noted that this strategy is optimal with respect to all the criteria that were discussed in section 2.2 except for the N.P. objective.

A second strategy is to try to satisfy the Neyman Pearson (N.P.) objective. In this strategy, the subject fixes his probability of false alarm $P(A_0/S_1)$ while maximizing his probability of hit. This strategy can be defined by

$$P_{A_0/S_1}(\beta, d') = \text{const.} \qquad P_{A_0/S_0}(\beta, d') \rightarrow \text{max} \qquad (3.3)$$

Two properties of this strategy are:

1.    For a fixed false alarm rate, increasing d' will make the task easier so the probability of hit should increase.

2.    By decreasing the false alarm rate, the subject also decreases his probability of hit for the same value of d'.

Since this strategy is governed directly by $P(A_0/S_0)$ and $P(A_0/S_1)$ rather than through d' and $\beta$, the DR curve is independent of the distribution function. The shape of the DR curve is a vertical straight line. A family of such DR lines, where the magnitude of $P(A_0/S_1)$ is the parameter, is shown in figure 3.1.

Vertical lines are not the only possible shape of the DR curves under the Neyman Pearson strategy. In a dual strategy to the one defined by equation (3.3) the subject might decide
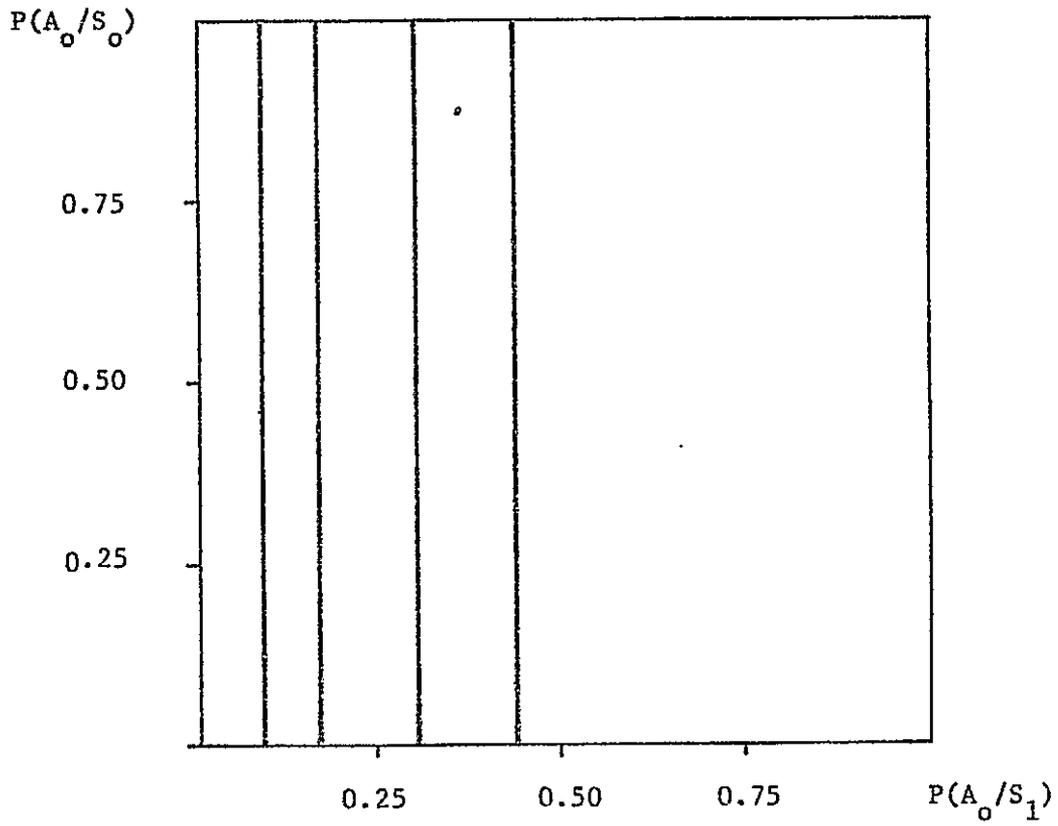
FIGURE 3.1  N.P DECISION RULE WITH $P(A_o/S_1)$ = CONSTANT
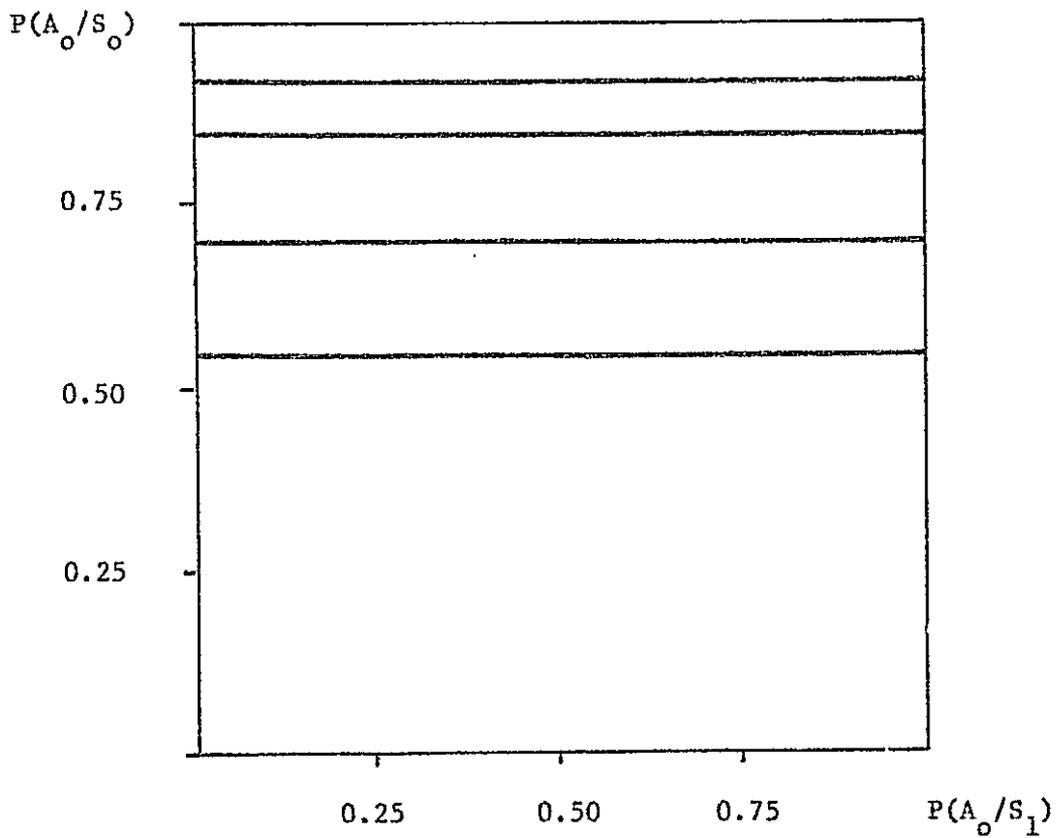


FIGURE 3.2  N.P. DECISION RULE WITH $P(A_1/S_o)$ = CONSTANT

to fix his probability of miss $P(A_1/S_0)$ and maximize the probability of correct rejections. This is formulized as

$$P(A_1/S_0) = 1 - P(A_0/S_0) = const.$$

$$P(A_1/S_1) = [1 - P(A_0/S_1)] \longrightarrow max$$

(3.4)

The use of equation (3.4) rather than (3.3) might happen when the magnitudes of $P(A_1/S_0)$ and $P(A_1/S_1)$ are more important to the decision maker than the magnitudes of $P(A_0/S_0)$ and $P(A_0/S_1)$. If the $P(A_0/S_0) - P(A_0/S_1)$ plane is used to draw the DR curves, their shape in this strategy would be that of horizontal straight lines. A family of such DR lines, where the magnitude of $P(A_1/S_0)$ is the parameter, is shown in figure 3.2.

In some cases the subject might use a strategy that involves both equation (3.3) and (3.4). This might happen in those cases where there is more than one LRCL to be determined, as in a confidence rating experiment. If the two states of the world are two signals rather than a signal and noise alone, the subject might use equation (3.3) for the LRCL for the sure state $A_0$, while using equation (3.4) for the LRCL determining the sure state $A_1$. The resultant DR lines under this strategy in a case of four LRCL's are shown in figure 3.3.

If the *a priori* probabilities of $P(S_0)$ and $P(S_1)$ are not known to the subject, he might use still another strategy. However all the costs must be known to him:
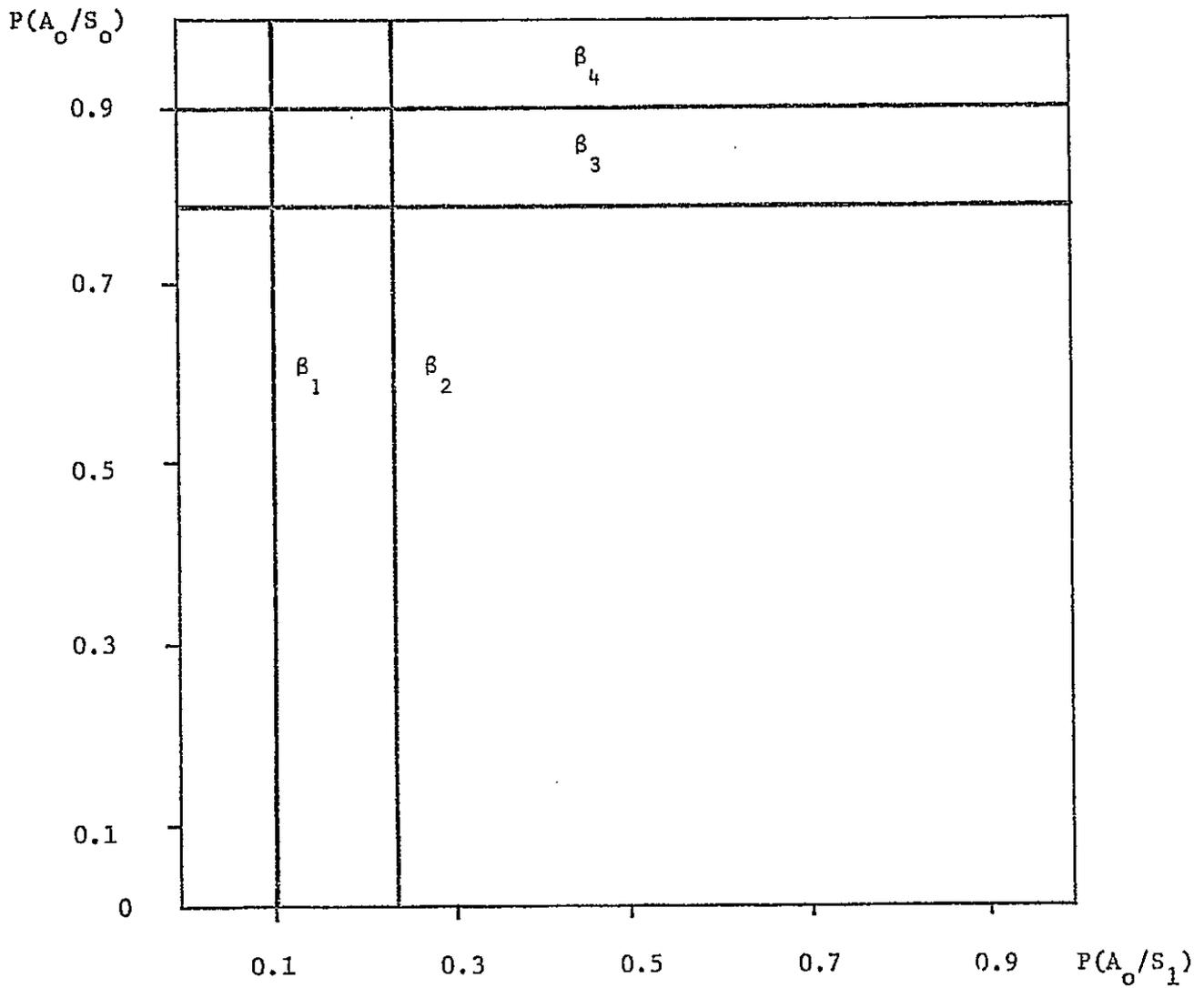
FIGURE 3.3  N.P. DR CURVES IN CONFIDENCE RATING EXPERIMENT

WITH FOUR DECISION CATEGORIES

$$C_{11} = \text{cost for deciding } A_1 \text{ when } S_1 \text{ is true}$$

$$C_{00} = \text{cost for deciding } A_0 \text{ when } S_0 \text{ is true}$$

$$C_{01} = \text{cost for deciding } A_0 \text{ when } S_1 \text{ is true}$$

$$C_{10} = \text{cost for deciding } A_1 \text{ when } S_0 \text{ is true}$$

Then the expected value of the total cost of making the decision is a function of $P(S_0)$ and there will be some value of $P(S_0)$ for which the total cost is maximized. The strategy then is to minimize this maximum total cost, referred to as a minimax strategy. The DR curves for the minimax strategy are given by (Van Trees, 1968):

$$P(A_0/S_0) = \frac{C_{11} - C_{01}}{C_{01} - C_{00}} P(A_0/S_1) + \frac{C_{10} - C_{11}}{C_{10} - C_{00}} \qquad (3.5)$$

Equation (3.5) represents straight lines with negative slopes. A family of such curves is shown in figure 3.4. Again these DR lines are independent of the underlying distributions.

Once the experimental $P(A_0/S_0)$ and $P(A_0/S_1)$ are found, the DR curves can be drawn. If these curves are vertical or horizontal straight lines, then it can be said that the Neyman Pearson rule was used. If the straight lines have a negative slope, than the strategy is equivalent to the minimax rule. If, however, the DR curves are not straight lines, little can be said unless an assumption about the distributions is made.

In those cases where the distributions are known, the
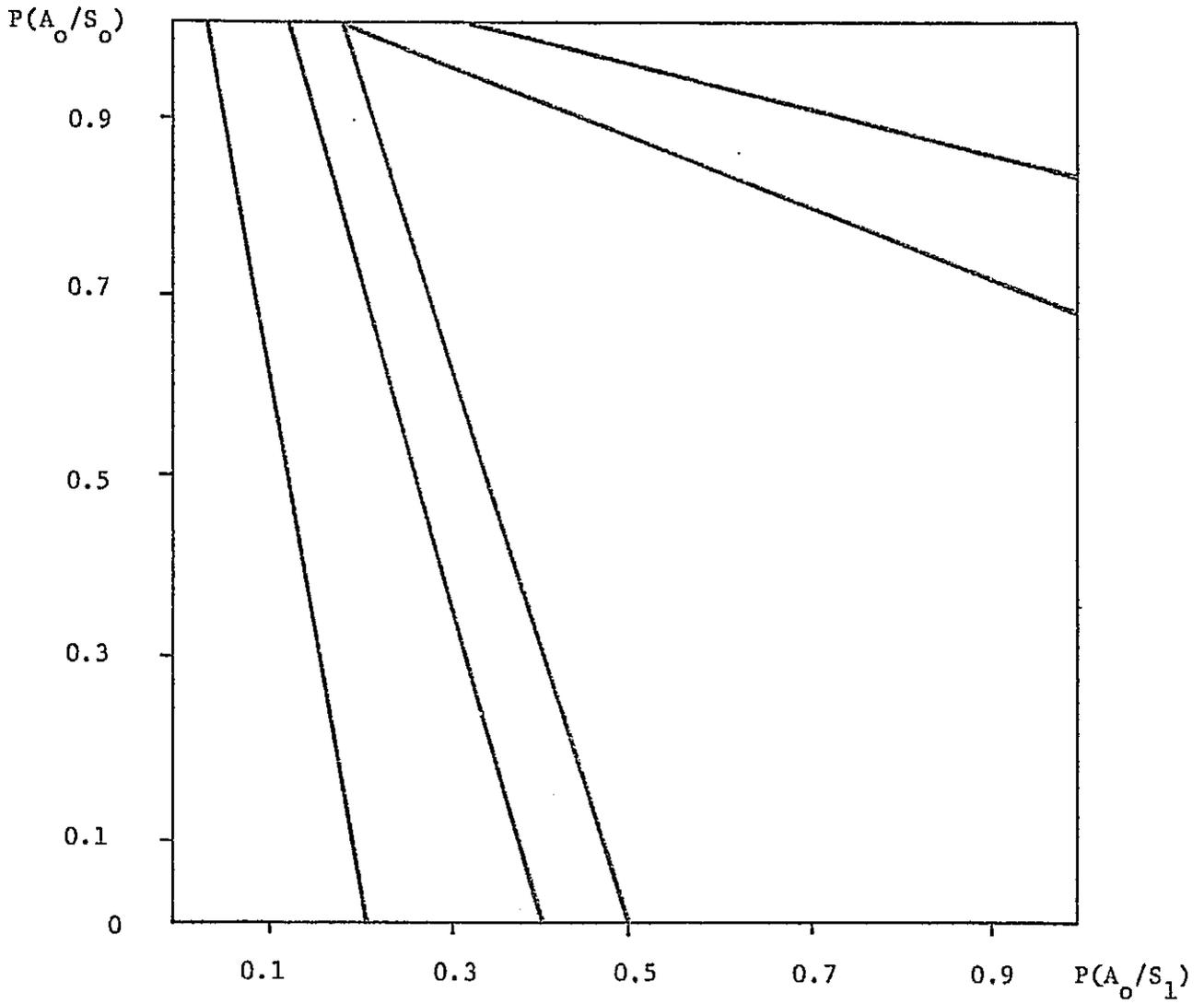
FIGURE 3.4   DR CURVES FOR MINIMAX DECISION STRATEGY

$(C_{01}, C_{10}, C_{11}, C_{00}$ PARAMETERS)

decision rules can be expressed in somewhat different form.
The performance of the subject can be described by $\beta$ and $d'$
directly, rather than through $P(A_0/S_0)$ and $P(A_0/S_1)$. Since
$d'$ is independent of the decision strategy and is the only
variable, the decision rule is defined by the relation

$$\beta = f(d') \tag{3.6}$$

In equation (3.6) the decision rule can be viewed as a relation
between a stimulus ($d'$) and a response ($\beta$). Possible decision
rules are:

      1. $\beta$ is independent of $d'$.

      2. $\beta$ is a monotonic function of $d'$.

      3. $\beta$ is a nonmonotonic function of $d'$.

Clearly, any decision rule that is given by equation
(3.6) can also be expressed as a DR in the $P(A_0/S_0)-P(A_0/S_1)$
plane. For example, case 1 above is equivalent to the fixed
LRCL DR. The analysis can, therefore, be done either way
depending on the case in hand. However, whenever possible
both methods should be used for a complete analysis.

## 3.3  Decision Rules for the Gaussian Case

As explained in section 2.3, the assumption that is usually made and the one that agrees with experimental results is that the underlying distributions are Gaussian. Equation (2.27), which relates the performance of the decision maker to $\beta$ and d', provides a basis for starting the DR analysis. However, for the general Gaussian case the detectability cannot be expressed by one parameter, d', but with two parameters, m and $\sigma$;  the easiest way to express the relationships describing the DR curve equations is the parametric form given by equation (2.27) with $\beta$, m, and $\sigma$ as parameters.

The DR curve equation for constant LRCL strategy is derived by eliminating both m and $\sigma$ from equation (2.27). However, since there are two equations, only one parameter can be eliminated so that a closed form expression for the DR curve is not feasible, and the parametric form (2.27) is used.

In order to satisfy the Neyman Pearson criteria in the general Gaussian case, one of the following relations should hold

$$P(A_0/S_1) = C_1 \qquad P(A_1/S_0) = C_2 \qquad (3.7)$$

substituting from equation (2.27) into (3.7)

$$\frac{-m \pm \sigma\sqrt{m^2 - 2(\sigma^2 - 1)\ln\beta\sigma}}{\sigma^2 - 1} = C_3 \qquad (3.8)$$

or

$$\frac{- m\sigma \pm \sqrt{m^2 - 2(\sigma^2 - 1)\ln\beta\sigma}}{\sigma^2 - 1} = C_4 \qquad (3.9)$$

where $C_1$, $C_2$, $C_3$, and $C_4$ are constants.

Therefore the Neyman Perarson strategy in the form of equation (3.6) is

$$\ln\beta = \frac{m^2}{2(\sigma^2 - 1)} - \frac{[C_3(\sigma^2 - 1) + m]^2}{2\sigma^2(\sigma^2 - 1)} - \ln\sigma \qquad (3.10)$$

or

$$\ln\beta = \frac{m^2}{2(\sigma^2 - 1)} - \frac{[C_4(\sigma^2 - 1) + m\sigma]}{2(\sigma^2 - 1)} - \ln\sigma \qquad (3.11)$$

The relations obtained for the general Gaussian case are rather complicated and further simplifications will be made. The first assumption is that of equal variances, that is, $\sigma = 1$. Now the detectability is defined by one quantity $d' = m$, and the performance can be specified directly by $\beta$ and $d'$. For the strategy of fixed LRCL or $\beta =$ constant, the DR equation is obtained by eliminating m from equation (2.33). Rewriting (2.33):

$$\phi^{-1}[1-P(A_0/S_0)] = \ln\beta/m - m/2$$

$$\phi^{-1}[1-P(A_0/S_1)] = \ln\beta/m + m/2$$

Squaring both sides of the above equations and then substituting the second into the first, we get:

$$\{\Phi^{-1}[1-P(A_0/S_0)]\}^2 = \{\Phi^{-1}[1-P(A_0/S_1)]\}^2 - 2(\ln\beta) \qquad (3.12)$$

Equation (3.12) is the analytic expression for the DR curve in the $P(A_0/S_0)-P(A_0/S_1)$ plane for a fixed threshold strategy. A family of such curves with $\beta$ as a parameter is shown in figure 3.5. For the special case where $\beta = 1$, equation (3.12) reduces to:

$$\Phi^{-1}[1-P(A_0/S_0)] = \pm\Phi^{-1}[1-P(A_0/S_1)] \qquad (3.13)$$

which are straight lines on the two diagonals of the $P(A_0/S_0)-P(A_0/S_1)$ plane as shown in figure 3.6. It should be mentioned that for $\beta = 1$, these DR lines are also the DR lines for the general Gaussian case.

Using unit deviates, the equation of the DR curve is

$$z_1^2 - z_2^2 = -2(\ln\beta) \qquad (3.14)$$

This is a family of hyperbolas that can be seen in figure 3.7. The asymptotes of the hyperbolas are obtained for $\beta = 1$. As in the ROC analysis, it is helpful to transform the DR curves into straight lines. Then the following transformation is made:

$$\tilde{z}_2 = z_2^2 \qquad\qquad \tilde{z}_1 = z_1^2$$

so that the DR curves are:

$$\tilde{z}_2 - \tilde{z}_1 = -2(\ln\beta) \qquad (3.15)$$
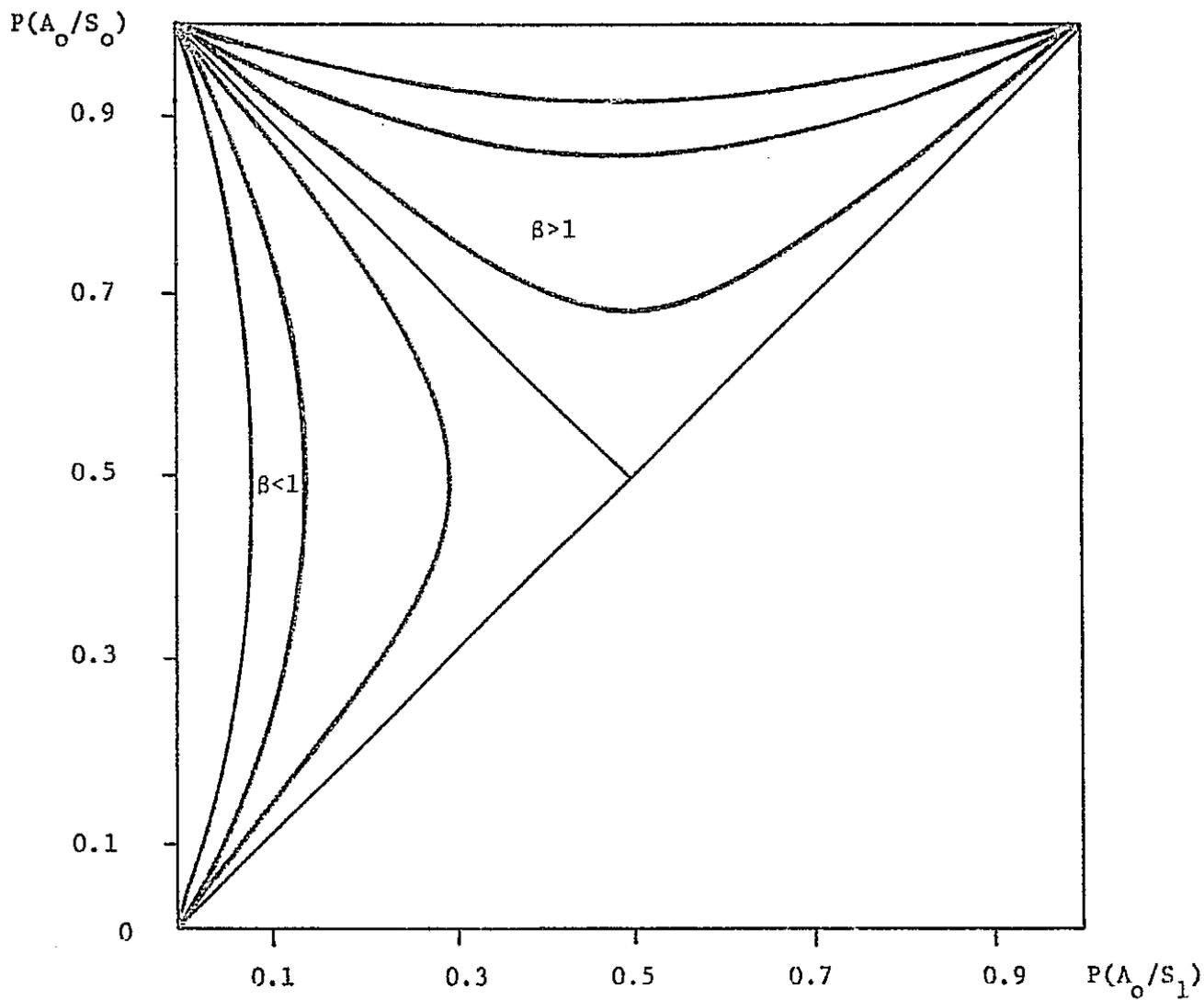
FIGURE 3.5   DR CURVES FOR A FIXED LRCL DECISION

STRATEGY, UNDER EQUAL VARIANCE

GAUSSIAN DISTRIBUTIONS
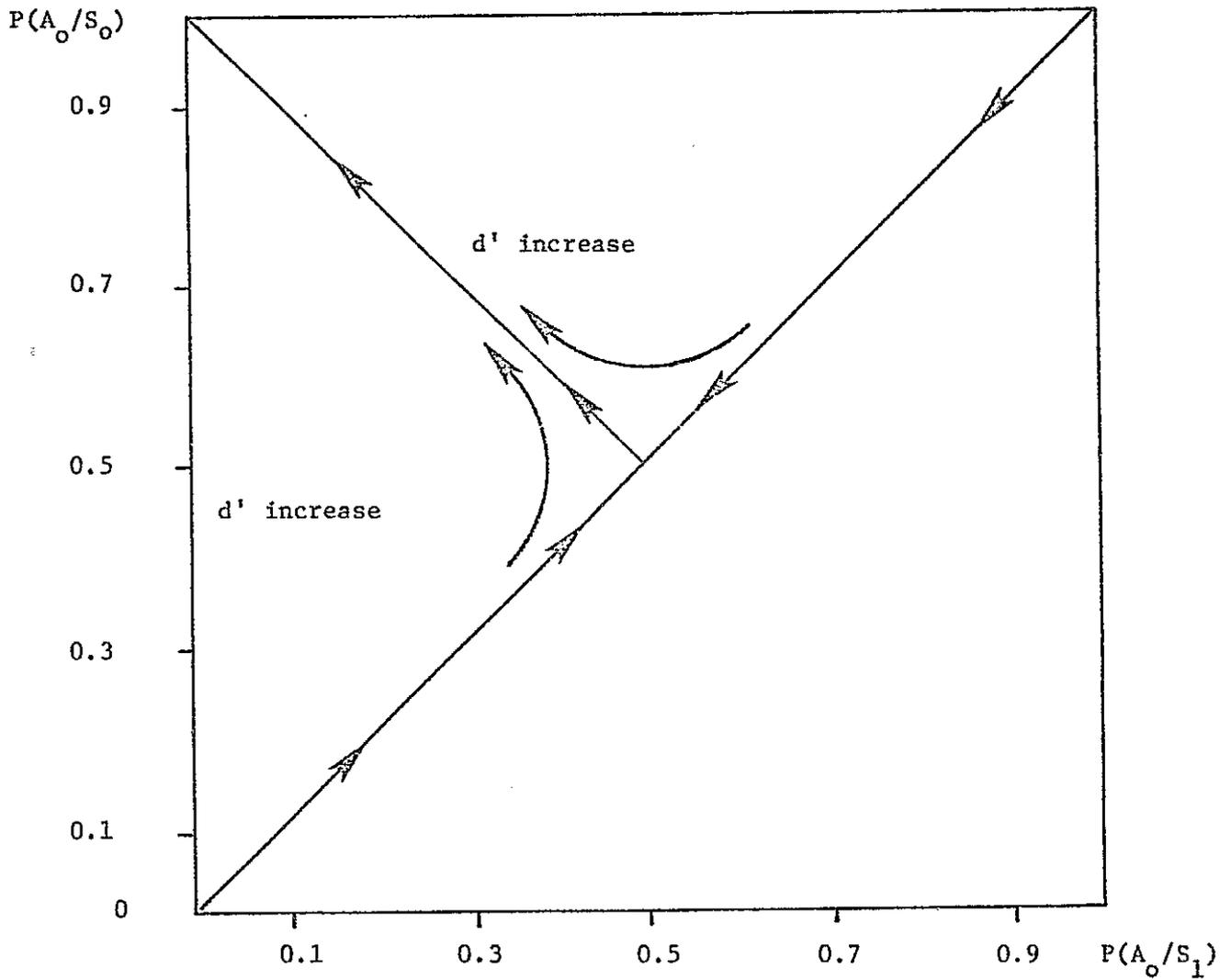
FIGURE 3.6   DR CURVE FOR FIXED LRCL DECISION STRATEGY, GENERAL

GAUSSIAN DISTRIBUTION, AND $\beta = 1$
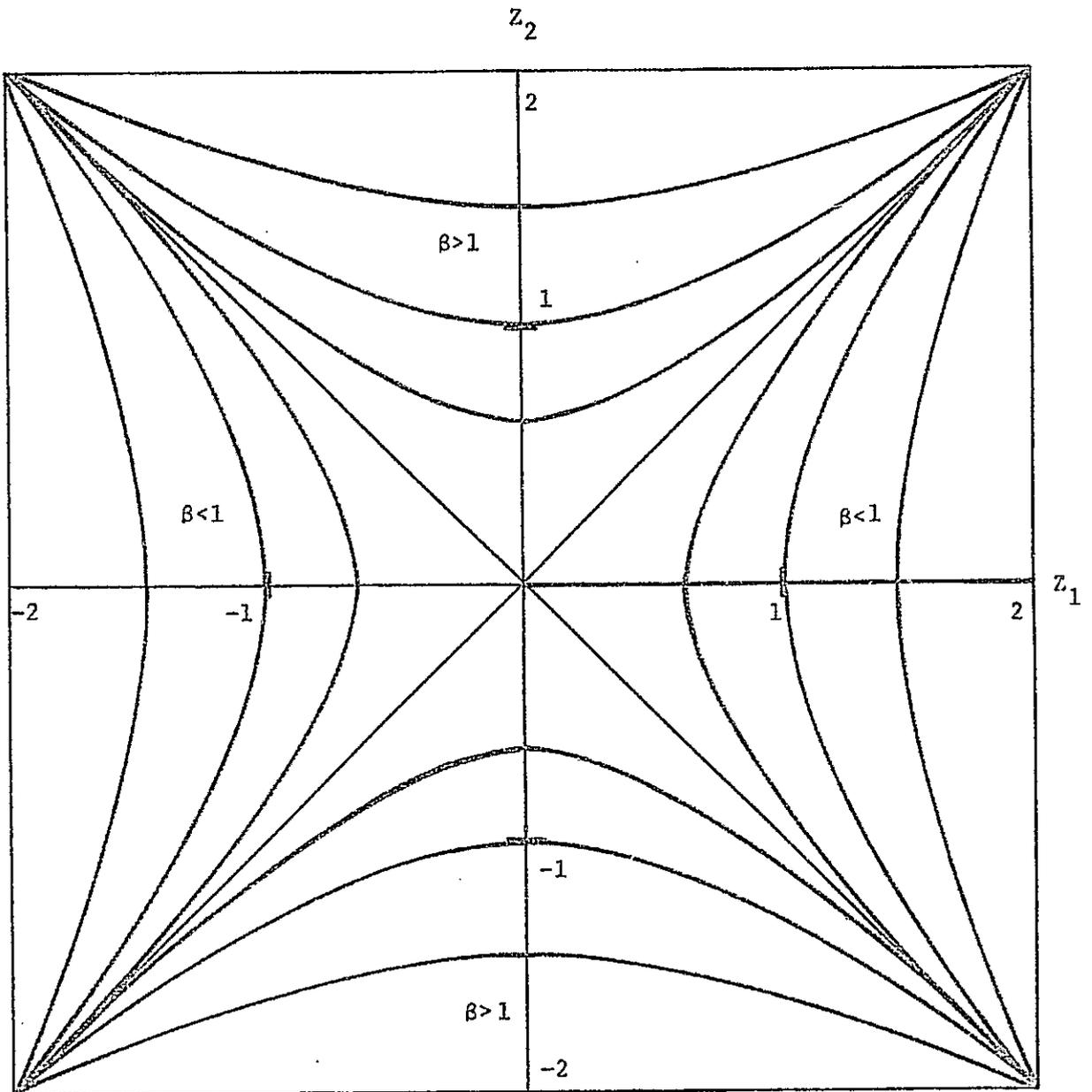
FIGURE 3.7  DR CURVES FOR FIXED THRESHOLD DECISION

STRATEGY ON UNIT DEVIATE SCALE

A family of such curves on a unit deviate squared scale is shown in figure 3.8.

To satisfy the Neyman Pearson objective, equation (3.7) should hold. In the equal variance Gaussian case, this means:

$$1 - \Phi(\ln\beta/d' - d'/2) = c_1$$

or
$$1 - \Phi(\ln\beta/d' + d'/2) = c_2$$

(3.16)

Therefore this decision strategy in the form of equation (3.7) is:

$$\ln\beta = c_3 d' + d'^2/2 \tag{3.17}$$

$$\ln\beta = c_4 d' - d'^2/2 \tag{3.18}$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are constants. The Neyman Pearson criterion implies a quadratic relation between the log LRCL and the detectability. It is important to note that in this strategy $\beta$ might not be monotonic with $d'$.

Since we have already seen a decision rule in which $\beta$ is constant, and the Neyman Pearson decision rule implies a quadratic relation between $\ln\beta$ and $d'$, it seems reasonable to suggest another decision rule in which $\ln\beta$ is linear with $d'$. Such a decision rule might be:

$$\ln\beta = c_5 + c_6 d' \tag{3.19}$$

where $c_5$ and $c_6$ are constants. Substituting equation (3.19)
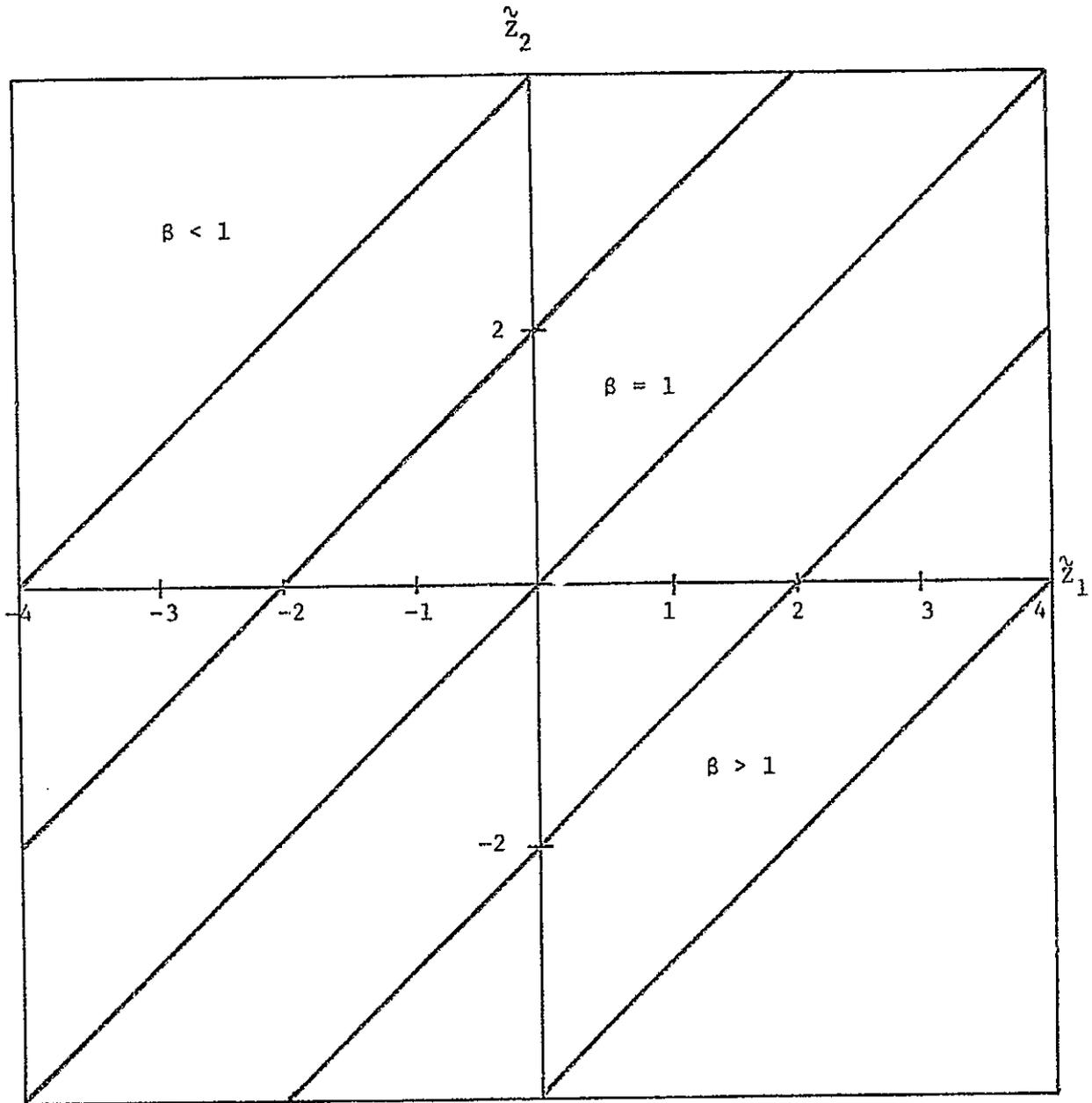
FIGURE 3.8   DR CURVES FOR FIXED LRCL DECISION STRATEGY

ON UNIT DEVIATE SQUARE SCALE

into equation (3.12), we get the performance of the decision maker as a function of d'. When d' is changed in the range $[0,\infty]$ points on the DR curve are obtained. A family of such curves in the $P(A_0/S_0)-P(A_0/S_1)$ plane, with $C_5$ and $C_6$ as parameters, is shown in figure 3.9.

Another interesting special case of the Gaussian assumption is the case of two Gaussian distributions with equal means and different variances, where m = 0. In this case, the detectability is defined as $\sigma$, and the performance is a function of only $\beta$ and $\sigma$. In order to get an equation for the constant threshold decision strategy $\sigma$ must be eliminated from equation (2.38). Rewriting equation (2.38)

$$\{\Phi^{-1}[1-P(A_0/S_0)]\}^2 = 2(\ln\beta\sigma)/\sigma^2-1$$

$$\{\Phi^{-1}[1-P(A_0/S_1)]\}^2 = 2\sigma^2\ln\beta\sigma/\sigma^2-1$$

(3.20)

subtracting the two:

$$\{\Phi^{-1}[1-P(A_0/S_1)]\}^2 - \{\Phi^{-1}[1-P(A_0/S_0)]\}^2 = 2(\ln\beta\sigma)$$

but from equation (2.40)

$$\sigma = \Phi^{-1}\{1-P(A_0/S_1)\}/\Phi^{-1}\{1-P(A_0/S_0)\}$$

Therefore the equation for the DR is given by

$$\{\Phi^{-1}[1-P(A_0/S_1)]\}^2 - \{\Phi^{-1}[1-P(A_0/S_0)]\}^2 = 2(\ln\beta\,(\frac{\Phi^{-1}[1-P(A_0/S_1)]}{\Phi^{-1}[1-P(A_0/S_0)]}))$$

(3.21)
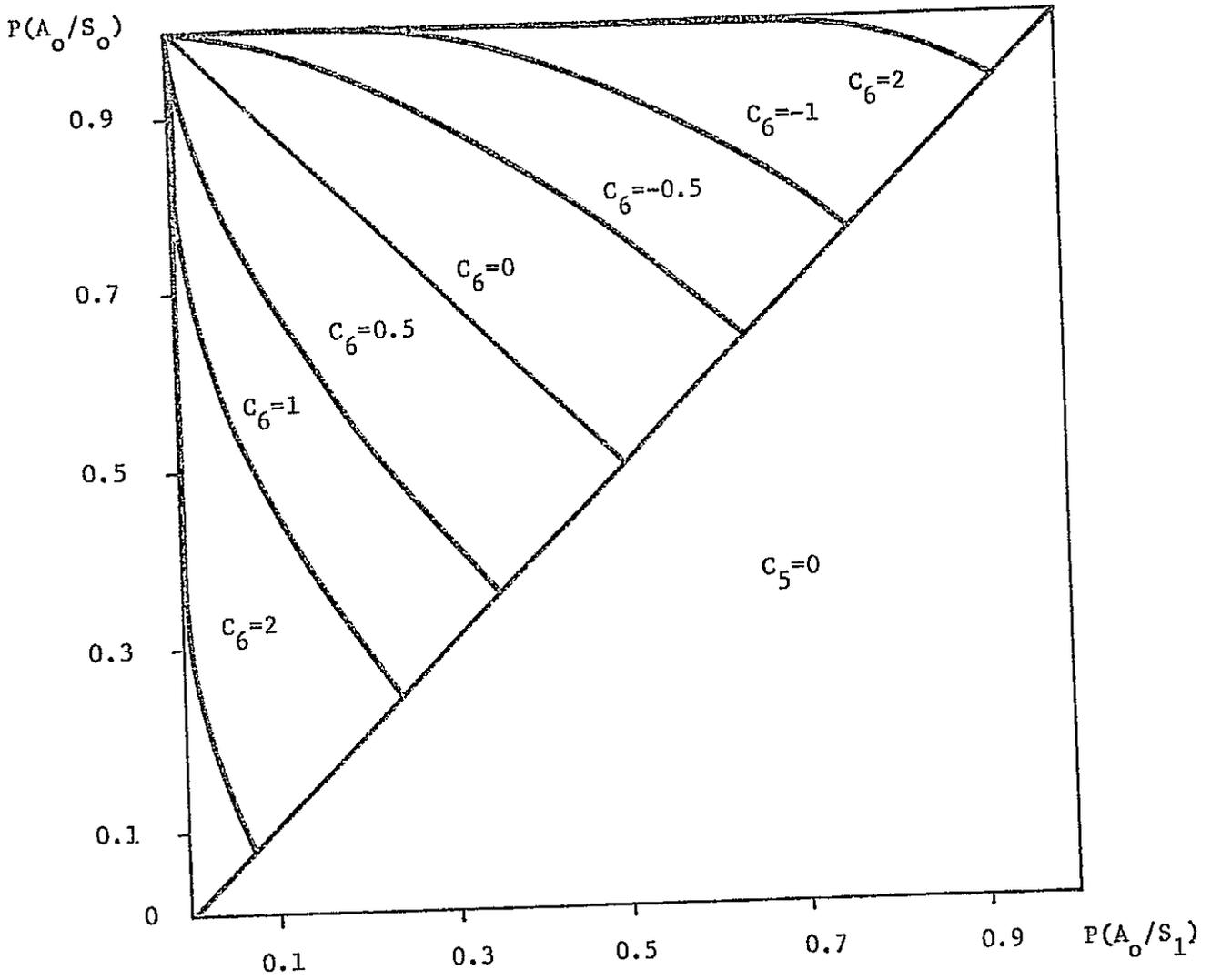
FIGURE 3.9  DR CURVES FOR LINEAR DECISION RULE

When the Neyman Pearson decision strategy is used, the following equation holds (from eg. (2.37))

$$1 - \Phi\left(\sqrt{\frac{2(\ln\beta + \ln\sigma)}{\sigma^2 - 1}}\right) = C_1$$

or

$$1 - \Phi\left(\sigma\sqrt{\frac{2(\ln\beta + \ln\sigma)}{\sigma^2 - 1}}\right) = C_2$$

(3.22)

Therefore

$$\sqrt{\frac{2(\ln\beta + \ln\sigma)}{\sigma^2 - 1}} = C_3$$

or

$$\sqrt{\frac{2(\ln\beta + \ln\sigma)}{\sigma^2 - 1}} = C_4/\sigma$$

(3.23)

This decision strategy in the form of equation (3.7) is

$$\ln\beta = (C_3\sigma^2 - C_3 - 2(\ln\sigma))/2$$

(3.24)

or

$$\ln\beta = (C_4 - C_4/\sigma^2 - 2(\ln\sigma))/2$$

(3.25)

since

$$\sigma^2 \gg \ln\sigma \qquad \text{for} \quad \sigma > 1$$

The decision rule given by (3.24) and (3.25) is, therefore, approximately quadratic, as was true in the equal variance case. Again a third decision rule might be suggested in which $\ln\beta$ is linear with $\sigma$.

It should be noted that similar expressions for the

possible decision rules can be derived for other distributions satisfying equation (2.41). The derivations are similar to the ones for the Gaussian distribution and therefore will not be repeated.

CHAPTER IV

DETECTION OF SIGNALS WITH UNCORRELATED SIGNAL STRENGTH

## 4.1   General Discussion

Chapter 3 dealt with a theoretical approach to the analysis of decision strategies in signal detection tasks. An important conclusion of that discussion was that any study of decision strategy  should include experiments in which the signal strength (the detectability) is time varying.  Among the few experiments of this type that have been reported are Kinchala and Smyzer (1967), Donaldson and Murdock (1968), Glorioso et al (1968), Thurmond et al (1970) and Healy and Jones (1973).  To support the theoretical analysis of earlier chapters, we collected our own data from experiments designed to explore the decision strategy of the subjects.  A description of this experiment and the results are given in this chapter.

Since our major concern was the general concept of decision strategy, the question of which sensory system to use was of minor importance.  We chose to test the decision strategy in a visual discrimination task because a computer with a graphics terminal was available so that the simulation of visual stimuli was relatively easy.

A major preliminary question in any experiment with varying signal strength concerns the time structure of these

changes. In classical signal detection, it is assumed that although repetitive decisions are made in each experimental session, these decisions are statistically independent. This implies that in cases with time varying signal strength, the change in signal strength should be designed in such a way as to prevent correlation between successive decisions. One possible way to avoid correlation is to change the signal strength in a random manner. It might be argued that in real life situations random changes rarely occur, so that such an approach is impractical. However, since we intend to relate our results to classical SDT, we chose to start the analysis with experiments in which the decisions would be uncorrelated. Once the decision strategy in this basic case is evaluated, it will be easier to analyze the more complicated experiments which involve correlations.

In order to simplify the experiment as much as possible without affecting the generality of the results, the following three constraints were adopted:

1. The input signals are deterministic so that the uncertainty is due only to the internal uncertainty of the subject.

2. The change in signal strength is instantaneous; namely, there are no dynamics and, therefore, no transient effects. Thus within each presentation the signal strength is fixed, and the transient

process of changing levels between presentations
is not shown to the subject.

3.  No feedback is given to the subject after his
    decision is made.  Also, his decisions do not
    affect in any way the format of the data present-
    ations.

The principal aim of the experiment is to determine the
decision strategy of the subject in the form of DR curves as
well as the functional relationship between the change in
Likelihood Ratio Criterion Level (LRCL) and the detectability.
Since the input signals are deterministic and the associated
noise is attributed to internal noise, two more questions should
be answered:

1.  Can the assumption of Gaussian distributions with
    equal variances be used for this particular visual
    task?  The answer to this question is important
    because only if the answer is positive can the
    major results of Chapter 3 be employed.

2.  How does the ensemble discrimination of the subject
    relate to the signal strength in the presentation?
    This relation is important because it might be
    used to describe the error sources in the subject's
    behavior.

## 4.2 Experimental Method

### 4.2.1 Motivation

In the previous section, we discussed the general importance of signal detection experiments with time varying signal strength. However, the specific motivation for the experiment that is discussed in this chapter was the study of the decision task of a pilot who uses a traffic situation display to avoid collisions. Such a display shows the pilot the relative position of the intruder and updates this position every four seconds. The pilot's task is to decide whether the intruder will pass to his left or right. Since the decision becomes easier when the intruder is closer, the pilot faces a signal detection task with time varying uncertainty. A simplified version of this problem is discussed in this chapter, and the correlation effects are studied in Chapter 5.

### 4.2.2 Apparatus

The ADAGE Model 30 graphics computer with a 17 inch CRT was used to simulate and display the input data. The function switch box which contains 12 push buttons was used to sort the decisions of the subject which were stored for data analysis.

A horizontal line in the center of the screen along with a small vertical cursor would appear on the CRT during all the

experimental sessions. In addition, a pair of quarter inch circles appeared at the beginning of each decision interval and disappeared at the end of the interval. The presentation on the display during an arbitrary decision interval is shown in figure 4.1.

The location of the circles relative to the cursor is changed from one decision interval to the next and can take ten different values. Let $(x_1, y_1)$ define the center of the upper circle, $(x_2, y_2)$ define the center of the lower circle, and the height of the CRT be 2L. Then the ten locations of the pairs are given by

$$x_1 = \frac{\pm 14(i-1)}{1500} L \qquad y_1 = \frac{12-2i}{10} L \qquad i = 1,..,5 \quad (4.1)$$

$$x_2 = \frac{\pm 14i}{1500} L \qquad y_2 = \frac{11-2i}{10} L \qquad i = 1,..,5 \quad (4.2)$$

Note that five of the pairs are on a straight line with a slope of 7/150, while the remaining five pairs are on a straight line with a slope of -7/150, as can be seen in figure 4.2.

These pairs represent the visual stimuli. The subject's task was to indicate whether the pair that was presented was to the left or the right of the vertical cursor. Since the location of each pair was fixed within each decision interval and was always either to the right or to the left of the cursor, the stimuli can be considered as deterministic (no noise). Any
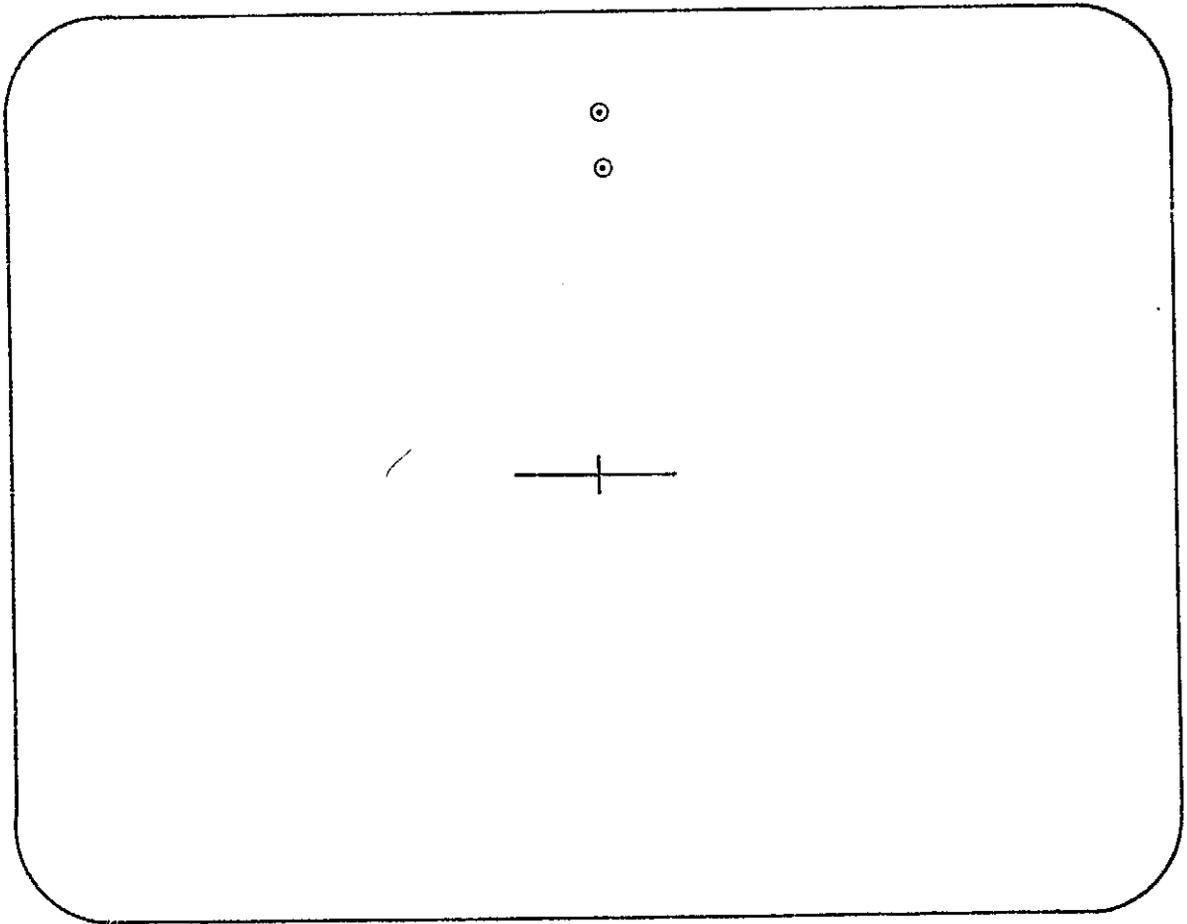
FIGURE 4.1  DISPLAY PRESENTATION DURING AN ARBITRARY

DECISION INTERVAL
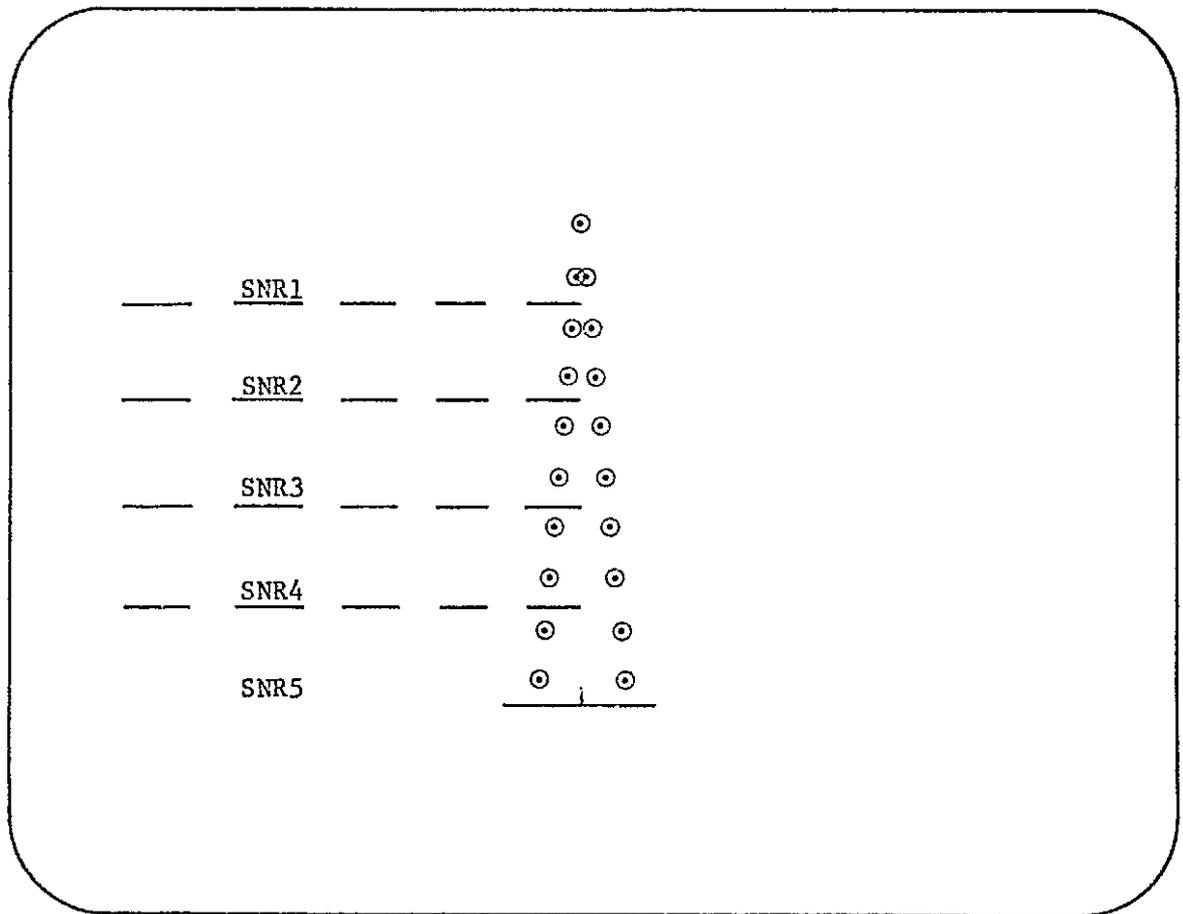
FIGURE 4.2   THE FIVE LEVELS OF SIGNAL STRENGTH

uncertainty involved was due to the internal uncertainty of the subject because the position of the dots was close to the limit of his discrimination ability. It can also be seen from figure 4.2 that each pair to the left has a symmetric pair to the right so that the stimuli are symmetric with respect to the vertical and any asymmetry in the results should be attributed to an internal bias of the subject.

Equations (4.1) and (4.2) and figure 4-2 show that any two symmetric pairs lie at the same distance from the horizonal line; and therefore, they constitute fiv different distances from that line. Since the discrimination between left and right should be easier for a pair that is closer to the horizontal line, each two systematic pairs represent a different level of signal strength. Therefore there are five levels of SNR which are shown in figure 4-2 and referred to as SNR1 – SNR5. SNR1 represents the smallest SNR and SNR5 the largest.

The reason for limiting the number of SNR levels to five is to limit the length of the experiment. Green and Swets (1966) suggested that for a classical signal detection experiment with fixed detectability a subject has to make a large number of successive decisions before his ensemble performance can be related to his internal uncertainty. If there are n SNR levels, both the total number of decisions and the total length of the experiment are multiplied by n. If every decision interval takes six seconds

the experiment would take around four hours for five SNR levels. This was found to be the limit for our subjects' patience. Furthermore, five points seemed to be sufficient to draw DR curves.

### 4.2.3 Subjects

Six subjects participated as observers in the experiments. All six subjects were graduate students in the Man Vehicle Laboroatory of M.I.T. Each one of them had a basic knowledge of probability theory and hypothesis testing, and they all were familiar with the terminology used in psychophysical experiments.

The subjects' participation was on a voluntary basis. However, in order to motivate them to perform their best, they were told that they were competing against each other. The competition was based on the total score of each subject, and after all the subjects finished their task, the table of the individual scores was posted. An informal preliminary test showed that the competition factor improved subject's performance considerably.

### 4.2.4 Procedure

The subject sat in front of the CRT while holding the function switch box in his hand. He could adjust his distance from the display so that he could get the best view of the

stimuli. The pair of circles would appear on the display and remain for 4 seconds; the pair would then disappear for 2 seconds and reappear at a different location for 4 more seconds and so on.

The two second blanking interval was used in an attempt to clear the subject's memory and thereby prevent him from making judgements on the basis of the previous stimuli.

The position of the pair at each four second decision interval was determined by a random number generator. This random number generator picked one of the ten possible locations (see Figure 4.2) during the blanking interval in such a way that the probability of each location appearing was equal. Since the subject was told *a priori* about the random order, he knew that the present position was statistically independent from any previous presentation. Therefore, it was expected that the successive decisions of the subject would also be statistically independent as was demanded by the experimental design.

During each decision interval which included 4 seconds of stimulus presentation and 2 seconds blanking period (6 seconds total), the subject was to indicate whether the pair of circles was to the right or the left of the cursor. The response was given by pushing one of the three buttons on the function switch box which corresponded to

THINK LEFT

DON'T KNOW                                              (4.3)

THINK RIGHT

The subject was told that his decision could be given at any
time during the 6 second interval and the scoring method
accounted only for his correctness but not for the time of
response. Only a single decision could be given, and the
subject was not allowed to change his decision within the 6
seconds.

In classical signal detection, rated decision procedures
with n possible decisions provide (n-1) points for drawing
the ROC curve. In view of this aim, the use of only three
possible decisions may seem insufficient. However, it should
be remembered that in dealing with the decision strategy, the
objective is to obtain the DR curves rather than the ROC curves,
and the number of points on a single DR curve is related to the
number of SNR levels and not to the number of response categories.
It should be noted that an increase in the number of response
categories also increases the overall number of decision inter-
vals to be used in one session (to get sufficient data) and the
length of the experiment is increased. We decided to choose the
minimal number of response categories required for a confidence
rating experiment, and therefore, chose three categories.
The odd number of categories is also helpful to avoid a central
DR curve which leads to the choice of $\beta$ close to unity, a case
in which the decision of which strategy was used is more compli-
cated. (For example, for $\beta = 1$ the DR for fixed thresholds and

for linear strategy might be the same, as can be seen by comparing figure 3.6 to figure 3.9

A standard set of instructions was read to the subjects describing the experimental set-up. They were told that the probability of a left or a right presentation was 0.5. They were also introduced to the scoring method which was as follows:

+3 points for a correct deci

-3 points for an incorrect decision

0 points for "Don't know" decision

The ten possible locations of the pairs on the CRT were shown to them, and the fact that all pairs lie on one of the two straight lines (figure 4.2) was explained.

There was no feedback after each trial, and the subjects were not advised as to what level of confidence they should choose in making a positive decision.

Each of the subjects had a ten minute practice session during which he could interrupt in the event that he had any questions or problems. After practice, the first half hour session of 300 decisions was started. Later the subject participated in three more half hour sessions, each on a different day. Each subject had made a total of 1200 decisions or 200 decisions per SNR level. The data analysis was based on the 1200 pooled decisions, rather than on the results from each session. One of the subjects (L.L.) participated in only two sessions.

## 4.3   A Model for the Change in Subject Uncertainty

In Chapter 3 we dealt with the analysis of decision strategies in a general SDT experiment.   In the previous section we described in detail the experimental design.   In this section we suggest a model that describes the particular decision process that might apply to our experiment.

Since it was assumed that the decision made by the subject in each decision interval is independent of decisions in other intervals and since at each interval the task is the same, it is reasonable to further assume that the method of decision used by the subject in each interval is also the same.   Therefore, we shall limit our discussion to the subject's performance within one arbitrary decision interval.   The information given to t' subject within this interval is time invariant and is shown in figure 4.1.

Figure 4.3 shows the displayed information but includes additional notation that is needed for the analysis.   The horizontal line on the screen is the line SS'.   The vertical cursor crosses this line at point C.   The lines OA and OE are the lines on which the pairs may be located as was shown in figure 4.2.   The lengths of the two intervals $\overline{CE}$ and $\overline{CA}$ are the same and are given by

$$\overline{AC} = -d_0 \qquad \overline{CE} = d_0 \qquad\qquad (4.4)$$
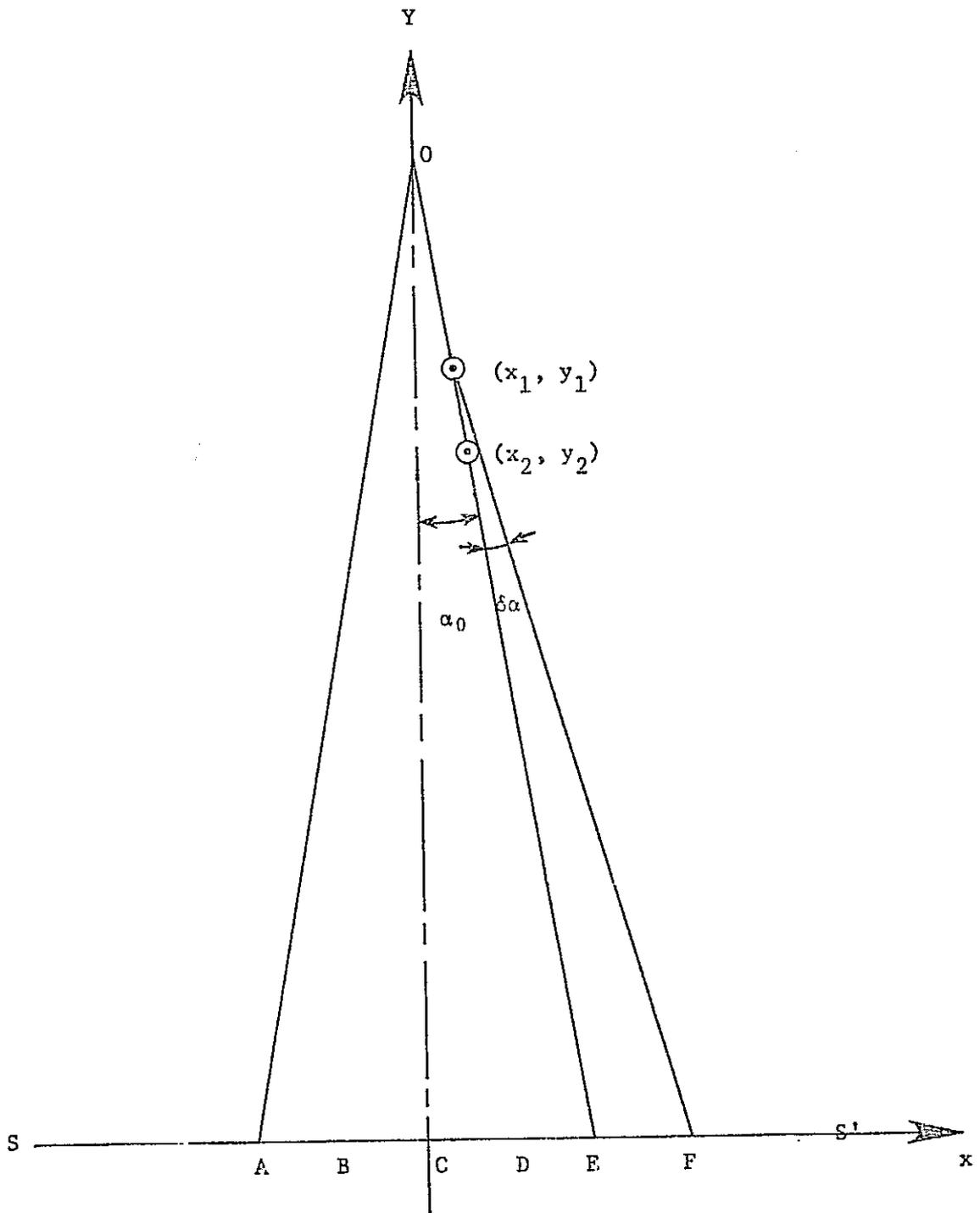
FIGURE 4.3   DISPLAYED INFORMATION WITH THE DETAILED

NOTATIONS THAT IS USED FOR THE MODEL

In a coordinate system with origin at C, the y axis in the CO direction, and the x axis in the CE direction, the location of the centers of the pair of circles are at points $(x_1, y_1)$ and $(x_2, y_2)$. The length of the line $\overline{CO}$ is L, and the angle between OC and OE is $\alpha_0$.

The task of the decision maker is to decide whether the two circles are to the left or to the right of the cursor. A possible method to do the discrimination (particularly in view of the subjects' knowledge that the pairs lie on a straight line) is to extrapolate the straight line that passes through $(x_1, y_1)$ and $(x_2, y_2)$ and to find the intersection point with SS'. For a perfect sensory system those intersection points would be either A or E according to the state of the world at this decision interval. However, because of his internal noise, the subject might reach point F rather than E when the state of the world is R (Right). The length of the line $\overline{CF}$ is

$$\overline{CF} = d_1(i) \tag{4.5}$$

where i is the SNR index and changes from 1 to 5. The distance $d_1$ is a function of the SNR index since the discrimination is easier if the circles are closer to SS' and therefore the difference $(d_1 - d_0)$ decreases as i increases. In the limit, when the lower circle almost touches SS':

$$\lim_{i \to 5} d_1(i) = d_0 \tag{4.6}$$

It should be noted that the value of $d_1(i)$ is a random variable.

Once the location of point F is found the decision is straightforward. If the subject were forced to a right/left decision, he would choose point C as a CL (note that this CL is not the LRCL, but the value of K that is defined in equation (2.21)) and decide

Right - if F is to the right of C

Left - if F is to the left of C

Since he is given the option of saying "do not know", he will choose two CL's at points B and D and decide

Right - if F is to the right of D

Left - if F is to the left of B

Don't know - if F is between D and B

In any of these cases, this model leads to the use of classical SDT for the analysis of the experimental data.

The second question that was posed was how to relate the internal uncertainty of the subject $u(i)$ to the SNR. The uncertainty of the subject is defined as the reciprocal of the detectability of the signals, or the reciprocal of the normalized difference between the values of $d_1(i)$ (see equation (2.34)). Let us use the notation $d_R(i)$ for $d_1(i)$ when the state of the world is right, and $d_L(i)$ when the state of the world is left. Therefore

$$d_R(i) = d_0 + \Delta d(i)$$
$$d_L(i) = -d_0 + \Delta d(i) \tag{4.6}$$

where $\Delta d(i)$ is the error of the subject in locating the point E or A and is a random variable. Becuase of the symmetry of the deterministic stimuli, the same d(i) which is still a function of i can be used for both states of the world. From equation (4.2)

$$y = (12 - 2i)L/10 \qquad\qquad i = 1,\dots,5$$

Since $\alpha_0$ is a small angle (7/150 radians), a good approximation for $\Delta d(i)$ is (see figure 4.3):

$$\Delta d(i) = (12 - 2i)L\Delta\alpha(i)/10 \tag{4.7}$$

where $\Delta\alpha(i)$ is the angular error that the subject makes when he tries to extrapolate the line OE. Also from figure 4.3, $d_0$ can be approximated by

$$d_0 = L\alpha_0 \tag{4.8}$$

The angular error $\Delta\alpha(i)$ is a random variable and a function of i. However, since the angle $\alpha_0$ is the same for every i, we might assume that the statistics of $\Delta\alpha(i)$ are stationary with respect to i, namely

$$\overline{\Delta\alpha(i)} = \overline{\Delta\alpha} \qquad \overline{(\Delta\alpha(i) - \overline{\Delta\alpha})^2} = \sigma_\alpha^2 \qquad (4.9)$$

Therefore substituting (4.9) into equations (4.7) and (4.6)

$$d_R(i) = d_0 + [(12-2i)/10]L\overline{\Delta\alpha} \qquad (4.10)$$

$$d_L(i) = -d_0 + [(12-2i)/10]L\overline{\Delta\alpha}$$

and the standard deviation of $d_R(i)$ and $d_L(i)$ are:

$$\sigma_R(i) = [(12-2i)/10]L\sigma_\alpha = C_1 - C_2 i \qquad (4.11)$$

$$\sigma_L(i) = [(12-2i)/10]L\sigma_\alpha = C_1 - C_2 i$$

where $C_1$ and $C_2$ are constants.

$\overline{\Delta\alpha}$ is a measure of the internal bias (to the left or to the right) of the subject in estimating the angle $\alpha_0$. For an unbiased observer $\overline{\Delta\alpha} = 0$. On the other hand, $\sigma_L$ and $\sigma_R$ are possible measures of the uncertainty of the subject in locating the points A and E. Since the stimuli are symmetric with respect to the y axis, $\sigma_R$ and $\sigma_L$ are equal as shown in equation (4.11). Equation (4.11) also shows that $\sigma_L$ and $\sigma_R$ are linearly decreasing with increases of the SNR index i.

It should be noted that the values $\overline{d_R}(i)$, $\overline{d_L}(i)$, $\sigma_R$, and $\sigma_L$ are not included implicitly in the data collected in the experiment. In order to evaluate them it is necessary to make an assumption about the underlying distribution function of $d_R(i)$ and $d_L(i)$. Using the arguments of section 2.3, we might assume the distribution functions to be Gaussian. Also from

equation (4.11) we deal with Gaussian distributions with equal variances. For these assumptions, the detectibility d' is defined by

$$d'(i) = [\overline{d_R(i)} - \overline{d_L(i)}]/\sigma_R$$

and therefore the uncertainty is

$$u(i) = 1/d'(i) = \sigma_R/[\overline{d_R(i)} - \overline{d_L(i)}] = \sigma_R/2d_0 \qquad (4.12)$$

so the uncertainty is linearly related to $\sigma_R$. Substituting (4.11) into (4.12):

$$u = [(12-2i)/20d_0]L\sigma_\alpha = C_3 - C_4 i \qquad (4.13)$$

where $C_3$ and $C_4$ are constants. Note that the uncertainty as defined by equation (4.13) is independent of the bias $\overline{\Delta\alpha}$.

The uncertainty u and the bias $\overline{\Delta\alpha}$ can be found from the raw data by fitting Gaussian distributions to the experimental results as will be shown in the next section. However, the values of $\sigma_R$, $\overline{d_R(i)}$, and $\overline{d_L(i)}$ that are found as a solution to the optimization problem and are used to evaluate u and $\overline{\Delta\alpha}$, are not unique under linear transformations (see Appendix A). Nevertheless, by defining u as in equation (4.12) it can be shown (appendix A) that this expression for the uncertainty is invariant to the transformation mentioned above.

## 4.4 Data Processing

The raw data collected in a rated decision signal detection experiment with n possible outcomes consists of 2n numbers that represent conditional probabilities as defined by equation (2.2). Without any additional information, the analysis is limited to the use of the $P(A_0/S_0)$-$P(A_0/S_1)$ plane, namely, drawing the ROC curves in classical SDT and drawing DR curves for SDT with changing signal strength. Since only finite (and usually small) numbers of points are used to draw the curves, an efficient method of curve fitting is needed to compare the raw data points to the analytic results. Tanner and Swets (1954) suggested the use of a visual fit for data. This might seem reasonable if the aim is to test whether the underlying distribution is Gaussian since the Gaussian assumption implies that the ROC curves are straight lines on a Gaussian unit deviate scale. However, if the aim is to obtain estimates for all the parameters that define the underlying distribution as well as the n-1 thresholds, a more rigorous method should be used.

As a start, an assumption has to be made about the functional form of the underlying distributions. These distributions are usually assumed to be continuous and unimodal and also satisfy equation (2.42). Once the functional form of the distribution is chosen, the unknowns in the problem are the parameters that define the distribution and the n-1

LRCL's. The problem now is how to choose values for these unknowns so that some criterion will be optimized. Since all the parameters are completely unknown, the maximum likelihood is an appropriate criterion, (Ogilvie and Creelman, 1968; Dorfman and Alf, 1968; Abramson and Levitt, 1969; Grey and Morgan, 1972). Under this criterion, we are seeking those values of the unknown parameters which are most likely to produce values that are equivalent to the experimental results.

Since decisions are taken repetitively and since the 2n possible outcomes define mutually exlusive and exhaustive events, the distribution which is associated with these events is multinomial (for n=1, the special case of the binomial distribution is obtained). A simple example explaining the occurence of such a distribution is when a die is thrown. The probability of getting any one of the numbers 1 to 6 in a single toss is assumed to be known and is referred to as $p_i$ (for an unbiased die, $p_i = 1/6$ for $i = 1, \ldots, 6$). When the die is thrown N times, it would show the number i $n_i$ times where

$$\sum_{i=1}^{6} n_i = N$$

The probability that a set of given $n_i = m_i$ will occur is

$$P(n_i = m_i) = N! \prod_{i=1}^{6} p_i^{m_i}/m_i! \qquad (4.14)$$

Equation (4.14) defines the multinomial distribution. In our optimization problem, the values of the $m_i$'s are given by the raw data and, therefore, are known. However, the probabilities

$p_i$ are not known explicitly, but are functions of the unknown parameters. Our optimization problem is to find the parameters in such a way that they will maximize the probability that the $m_i$ did appear, where this probability has the form of equation (4.14).

Several parameter optimization techniques have been suggested for the solution of such problems, such as the gradient method, the conjugate gradient method, and the Davidon method (Vander Velde, 1972). These methods were used to solve the specific problem of fitting an underlying distribution to data from SDT experiments by the authors mentioned above. In this work we applied still another method that was suggested recently by Jacobson and Oksman (1971) which seems to converge more rapidly than other methods.

The algorithm fits in its general case, two Gaussian distributions with different means and different variances; however many special cases can be implemented. Since we deal with experiments in which d' changes, new values for the unknown parameters must be computed for each level of signal strength. The algorithm has been programmed to repeat itself automatically as many times as required, so that all necessary information is available in one run. When a set of new values are found, a special subroutine checks the goodness of fit by the use of a chi square test. A detailed description of the algorithm is given in Appendix A. Since the expressions involved in the computation are all rather complicated, the

probability of programming errors is high. In the same Appendix, we suggest a method to test the algorithm with simulated data; a method that proved to be very helpful in our work.

## 4.5 Experimental Results

As has been discussed before, the DR curves can be drawn directly from the raw data in the $P(A_0/S_0)-P(A_0/S_1)$ plane. Since in our experiment there were three possible decisions for each subject, we will get two DR curves. These two DR curves for each subject are shown in figures 4-4 to 4-9 and are referred to by the LRCL's $\beta_1$ and $\beta_2$. Without any further data processing the only way to analyze these results is by visual inspection, that is, comparing these curves to the theoretical curves that were drawn in the $P(A_0/S_0)-P(A_0/S_1)$ plane in Chapter 3. In many cases, the decision is quite complicated so the question has to be resolved on the basis of the processed data. Table 4.1 summarizes the conclusions of the visual inspection method.

From table 4.1 it is clear that all possible decision rules were used, and there does not exist one dominant strategy. Furthermore, for most subjects the visual inspection shows a mixed strategy, i.e., the subject used different strategies in obtaining $\beta_1$ and $\beta_2$. The decision strategies that are shown in brackets in table 4.1 indicate the strategy that would best fit both DR curves. A clear understanding of the decision strategy is expected on the basis of data processing results; however, it should be noted that statistical analysis for testing the decision rule that was used on the basis of raw data alone is possible and was done by Curry (1974).
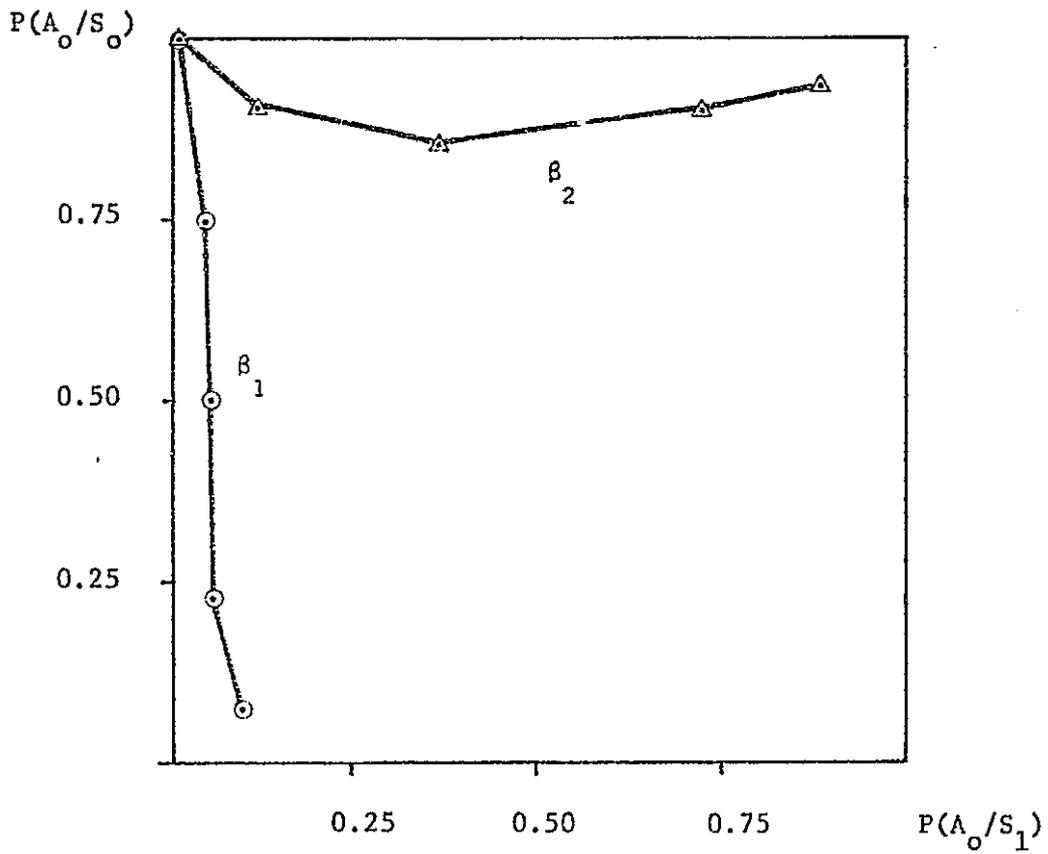
FIGURE 4.4   DR CURVES FOR SUBJECT J.TA
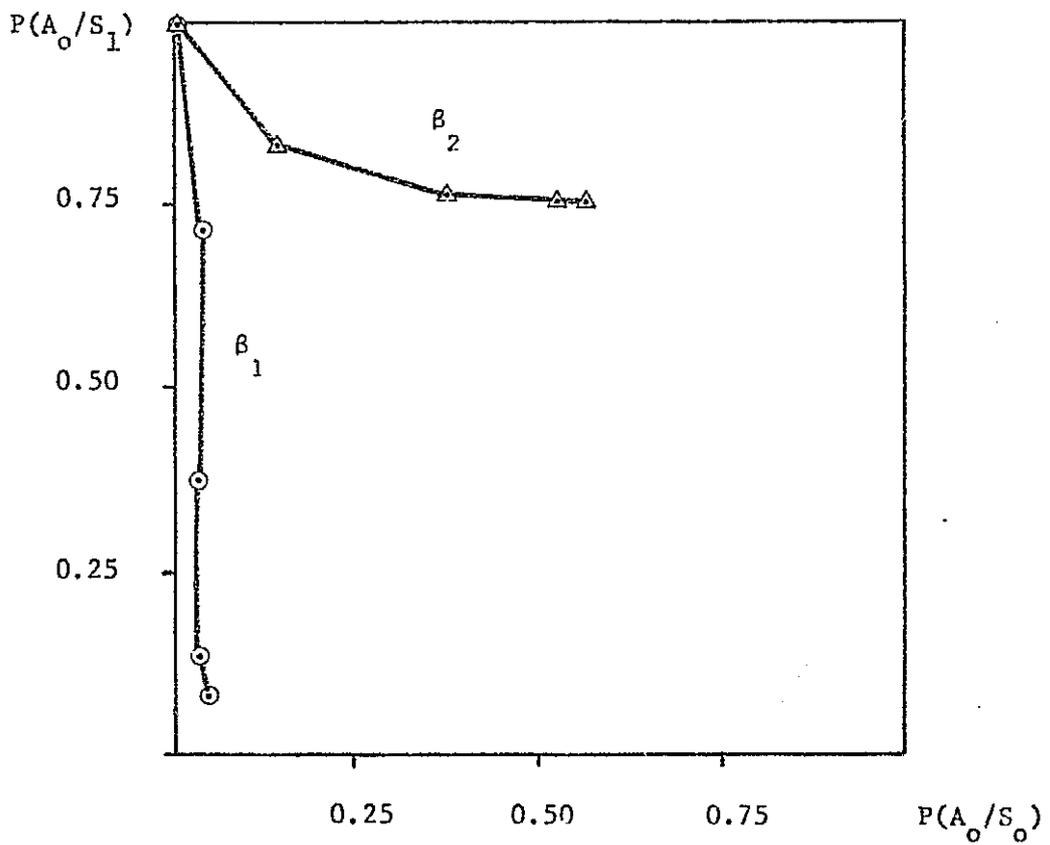


FIGURE 4.5   DR CURVES FOR SUBJECT L.L
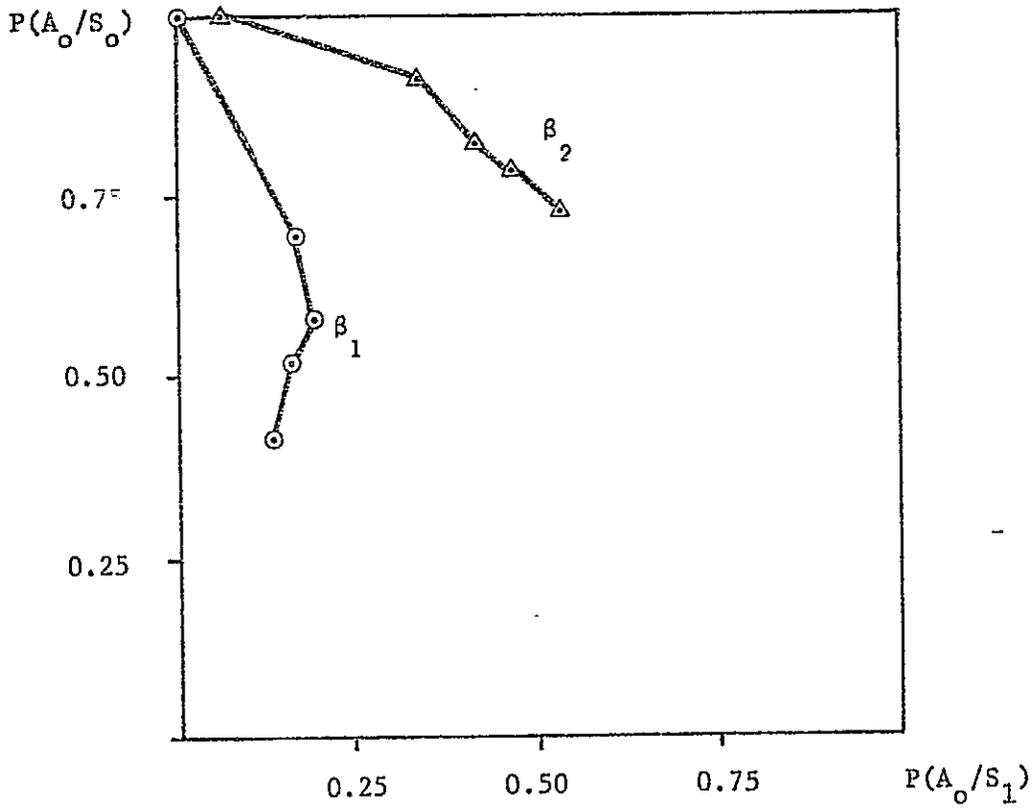
FIGURE 4.6  DR CURVES FOR SUBJECT J.TO
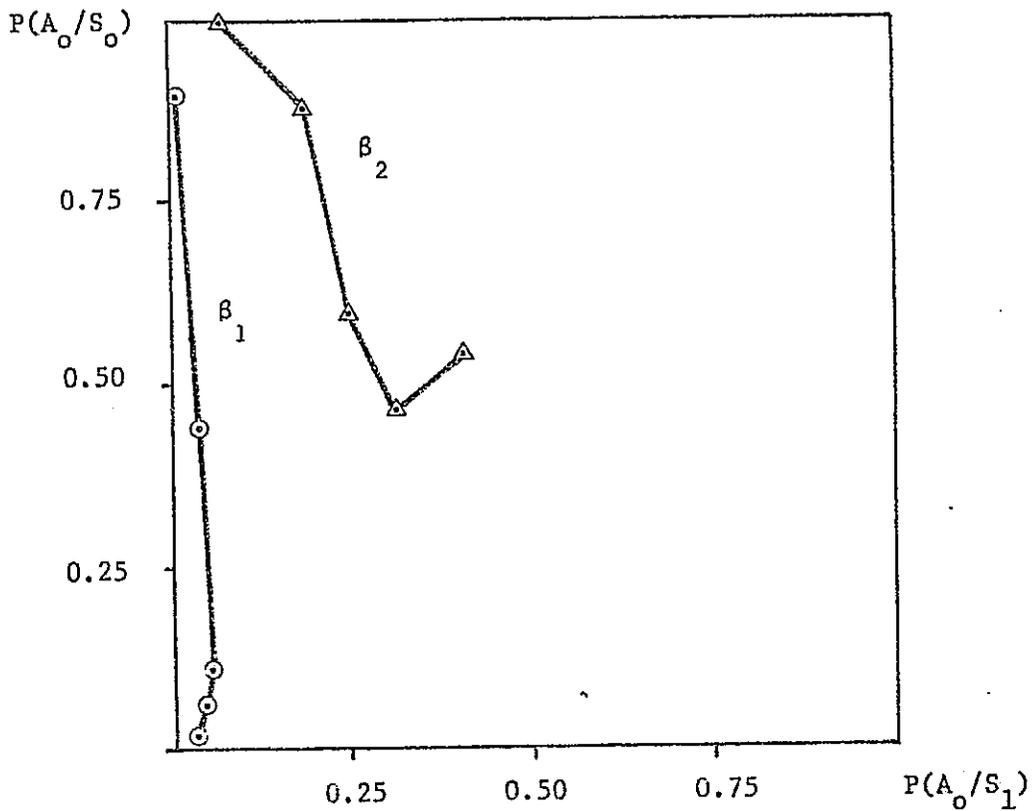


FIGURE 4.7  DR CURVES FOR SUBJECT A.E

C - 2
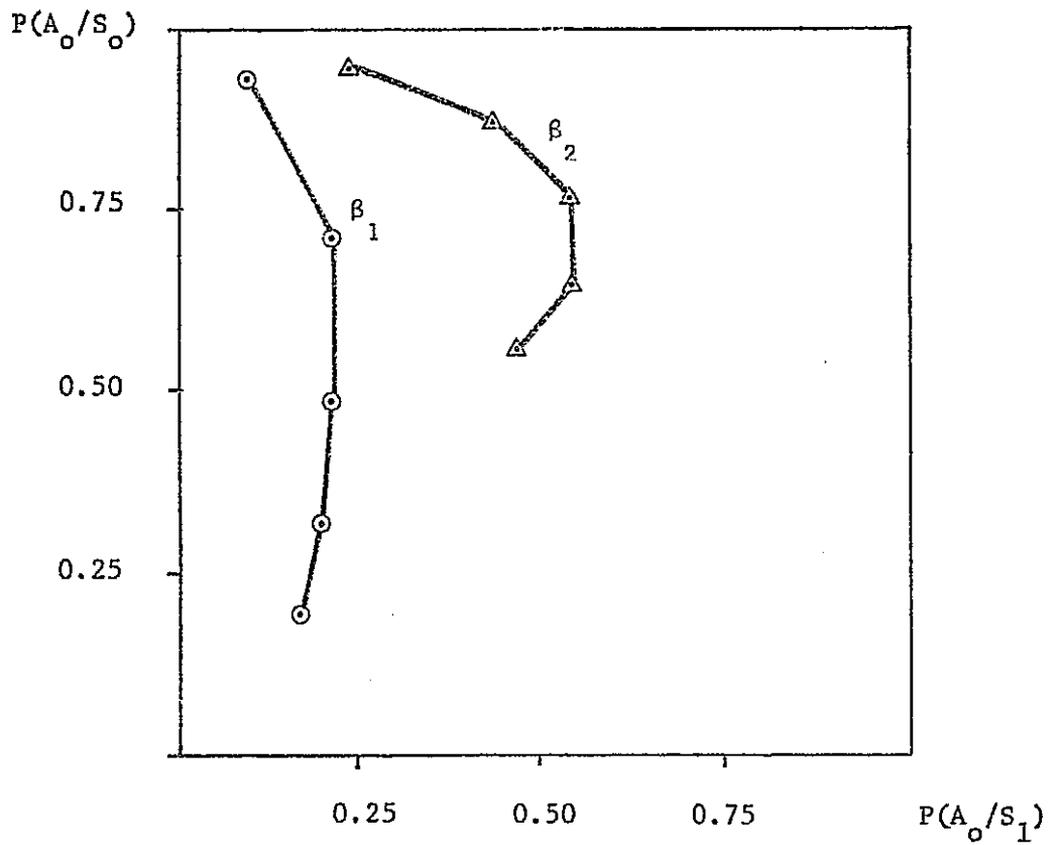
FIGURE 4.8   DR CURVES FOR SUBJECT C.B
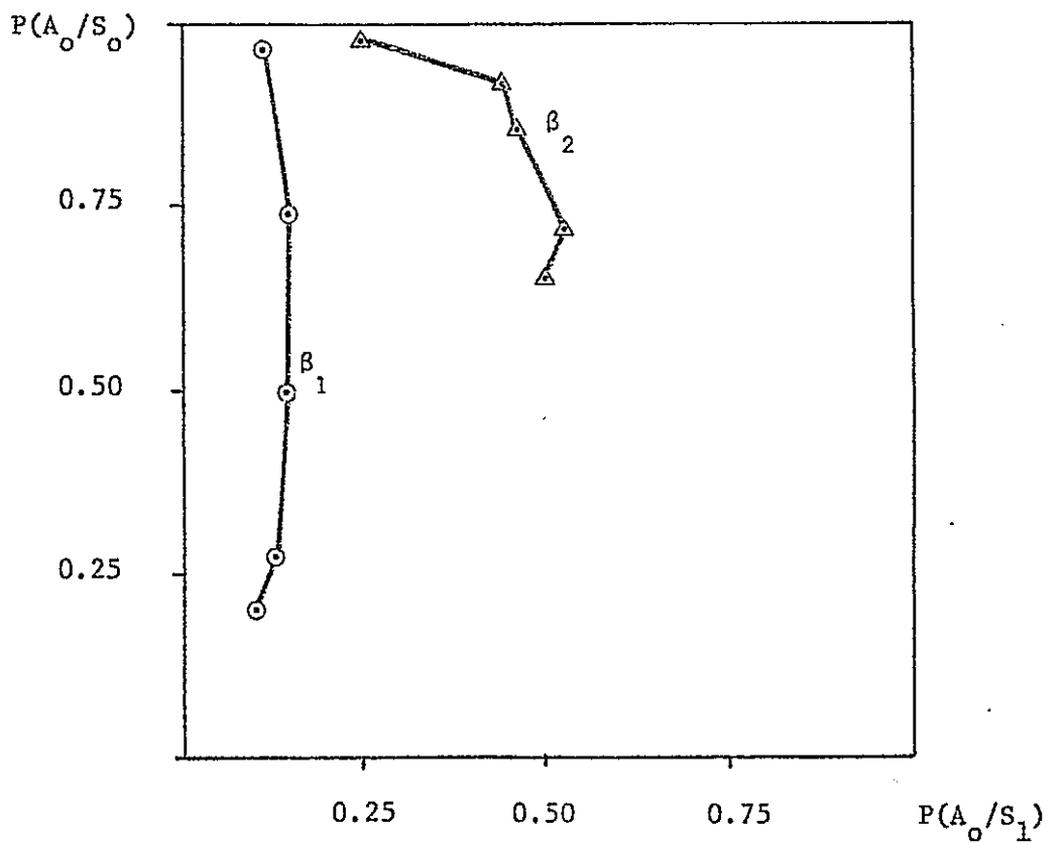


FIGURE 4.9   DR CURVES FOR SUBJECT L.M.L

| Subject | Fig No | D.R. for $\beta_1$ | D.R. for $\beta_2$ |
|---------|--------|--------------------|--------------------|
| J.TA.   | 4.4    | constant $\beta$   | N.P. (constant $\beta$) |
| L.L.    | 4.5    | N.P. (constant $\beta$) | constant $\beta$ |
| J.TO.   | 4.6    | linear             | linear             |
| A.E.    | 4.7    | N.P.               | linear (N.P.)      |
| C.B.    | 4.8    | constant $\beta$ (linear) | linear      |
| L.M.L.  | 4.9    | N.P. (linear)      | linear             |

Table 4.1  Decision Rules based on DR Curves in the $P(A_0/S_0)-P(A_0/S_1)$ Plane (N.P. — Neyman Pearson)

In order to better understand the detection process, it
was assumed that the underlying distributions were Gaussian,
and the parameter estimation algorithm described in Appendix A
was used. Since the left and right signals had the same
magnitude (but different signs), it was further assumed that
both distributions have the same variance. However, it was
expected that each subject might have some bias to the right
or left, therefore we chose $\overline{\Delta\alpha} \neq 0$ or

$$\overline{d_R(i)} \neq \overline{d_L(i)} \qquad\qquad i = 1,\ldots,5 \qquad (4.15)$$

To test for the significance of these assumptions, the chi square
test was used. The results of this test for each subject and
for each signal level are shown in table 4-2.

The values that are presented in table 4-2 are derived
from a chi square distribution with one degree of freedom (see
Appendix A). Each of these numbers represents the probability
that the observed chi square values (or smaller values) will
be obtained if the experiment were repeated a large number of
times with the unknown parameters taking the values that were
found as a solution to the optimization algorithm (Hoel, 1966).
The probabilities that were obtained indicate that the hypothesis
that the underlying distribution is Gaussian cannot.be rejected.

The data processing algorithm provides the values of d'
(and u = 1/d') as a function of i. The values u(i) for each
subject are summarized in table 4-3 (as a function of i). They
are also drawn as a function of the SNR index i in figures
4.10 to 4.15.

| SNR | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| SUBJECT | Percentage Points of the $\chi^2$ Test | | | | |
| J.TA. | 0.50 | 0.75 | 0.90 | 0.90 | 0.60 |
| L.L. | 0.90 | 0.70 | 0.40 | 0.55 | 0.90 |
| J.TO. | 0.25 | 0.70 | 0.92 | 0.88 | 0.92 |
| A.E. | 0.50 | 0.75 | 0.75 | 0.85 | 0.25 |
| C.B. | 0.70 | 0.85 | 0.85 | 0.70 | 0.55 |
| L.M.L. | 0.85 | 0.80 | 0.90 | 0.90 | 0.85 |

Table 4.2   Results of $\chi^2$ Significance Tests

| SNR | 1 | 2 | 3 | 4 | 5 | TESTS | |
|---|---|---|---|---|---|---|---|
| SUBJECT | UNCERTAINTY | | | | | $\rho_{ui}$ | t |
| J.TA. | 5.50 | 1.32 | 0.66 | 0.43 | 0.12 | -0.56 | 1.90 |
| L.L. | 2.30 | 1.66 | 0.88 | 0.50 | 0.12 | -0.99 | 6.10 |
| A.E. | 3.80 | 2.90 | 1.10 | 0.63 | 0.27 | -0.86 | 3.80 |
| J.TO. | 1.80 | 1.10 | 0.95 | 0.60 | 0.14 | -0.99 | 3.70 |
| C.B. | 7.60 | 3.55 | 1.58 | 0.88 | 0.38 | -0.93 | 2.96 |
| L.M.L. | 2.66 | 1.95 | 0.96 | .0.70 | 0.34 | -0.96 | 9.60 |

Table 4.3   Uncertainty u as a Function of SNR i
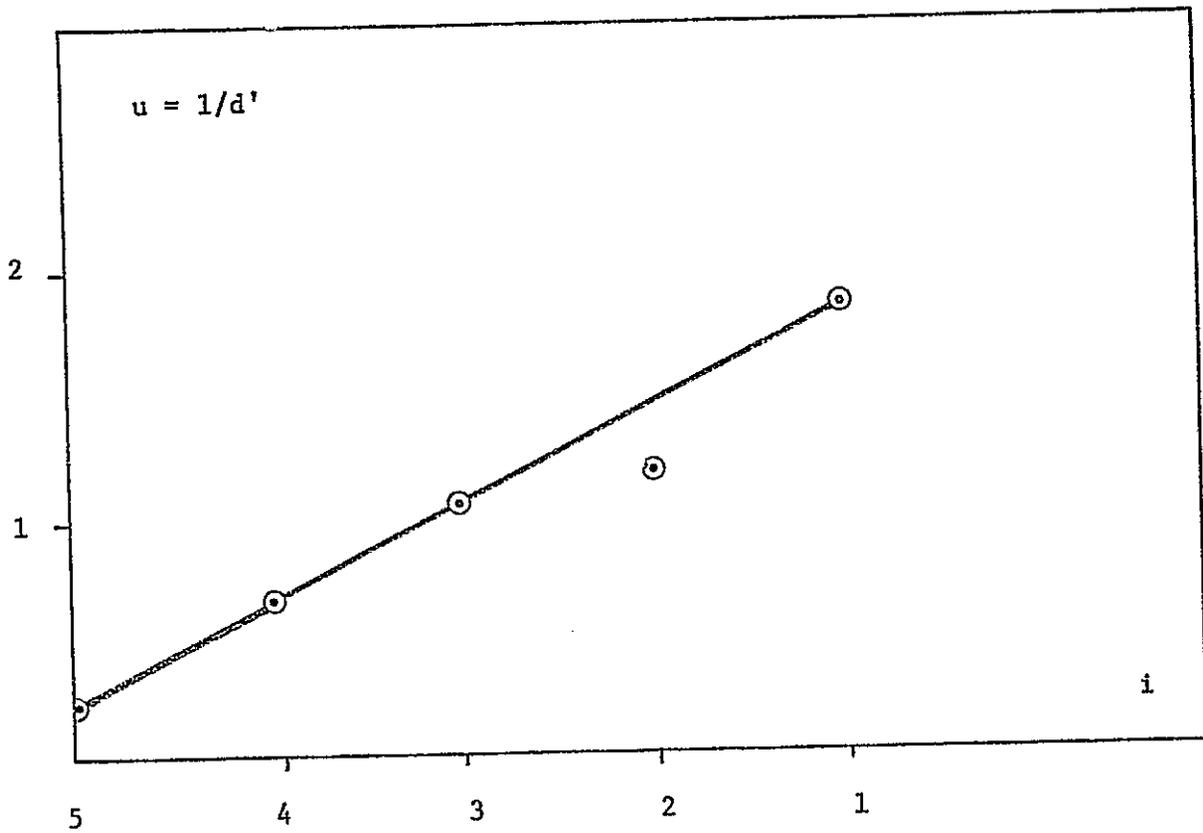
FIGURE 4.10   UNCERTAINTY VERSUS SNR INDEX (SUBJECT J.TO)
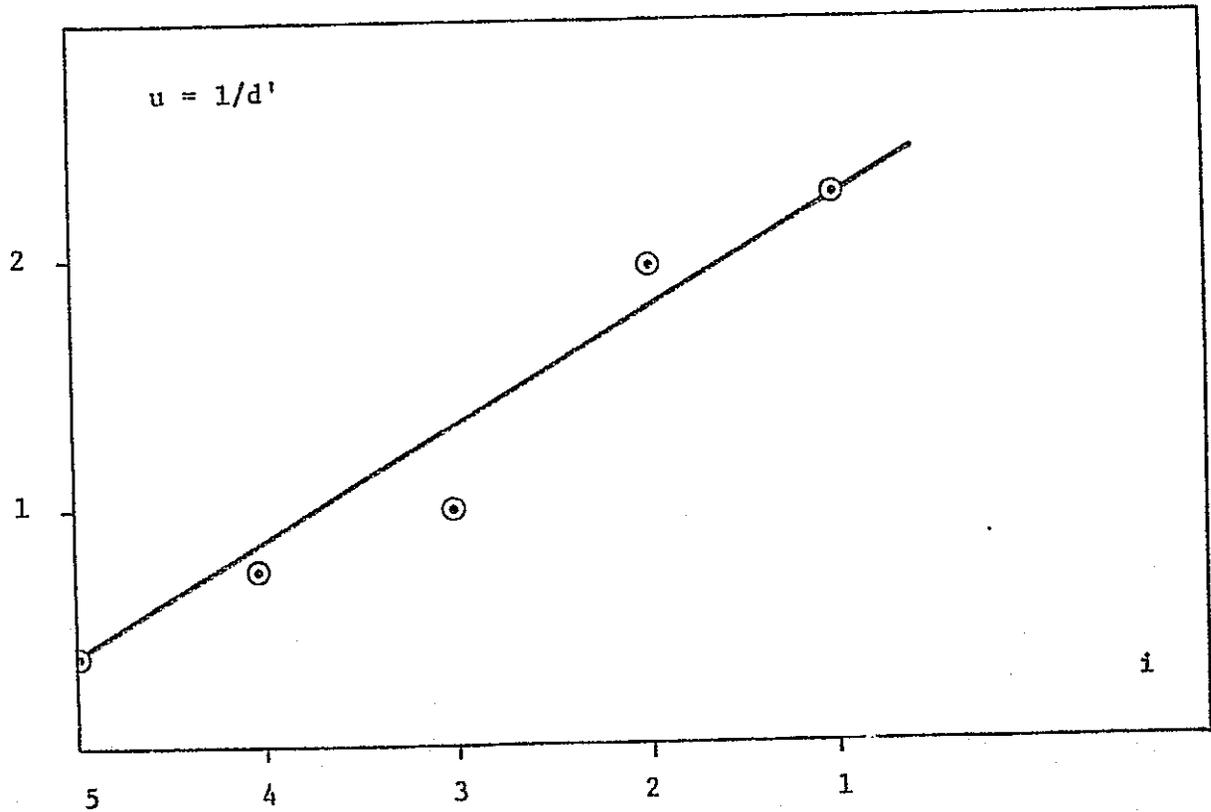


FIGURE 4.11   UNCERTAINTY VERSUS SNR INDEX (SUBJECT L.M.L.)
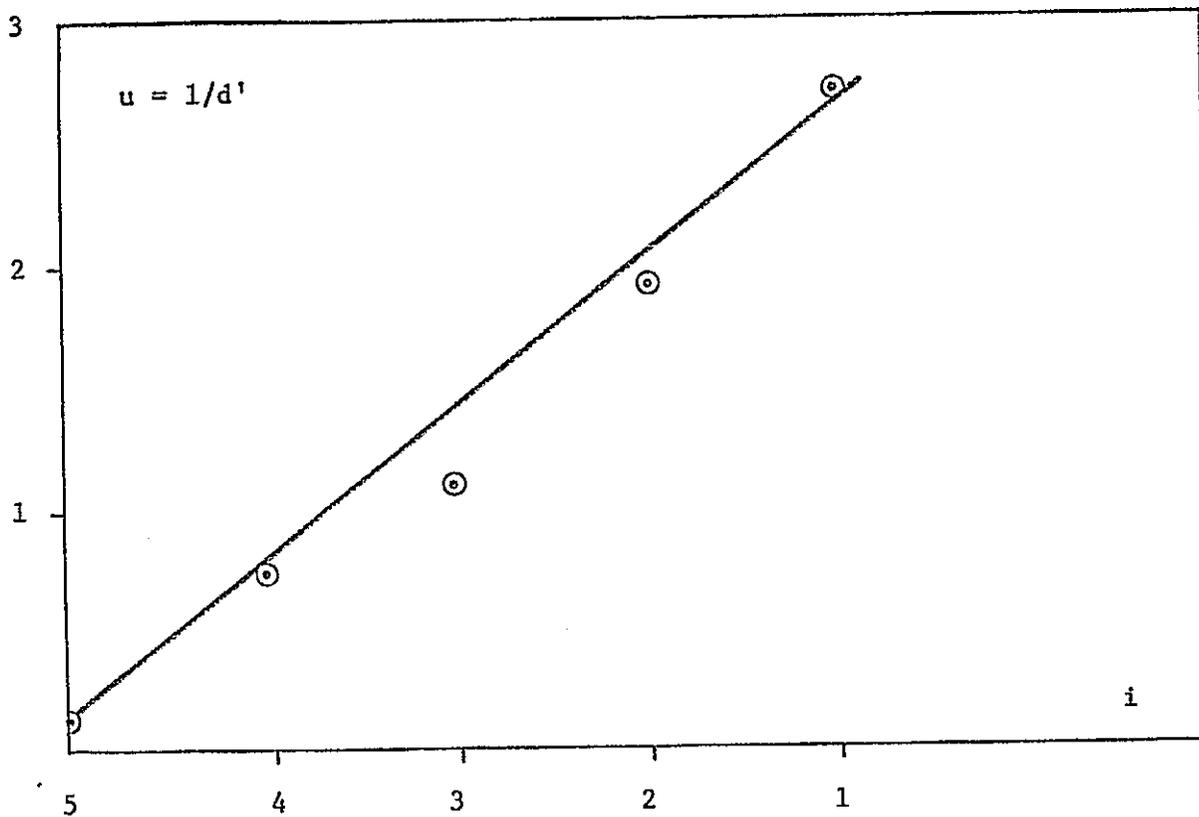
FIGURE 4.12   UNCERTAINTY VERSUS SNR INDEX (SUBJECT L.L.)
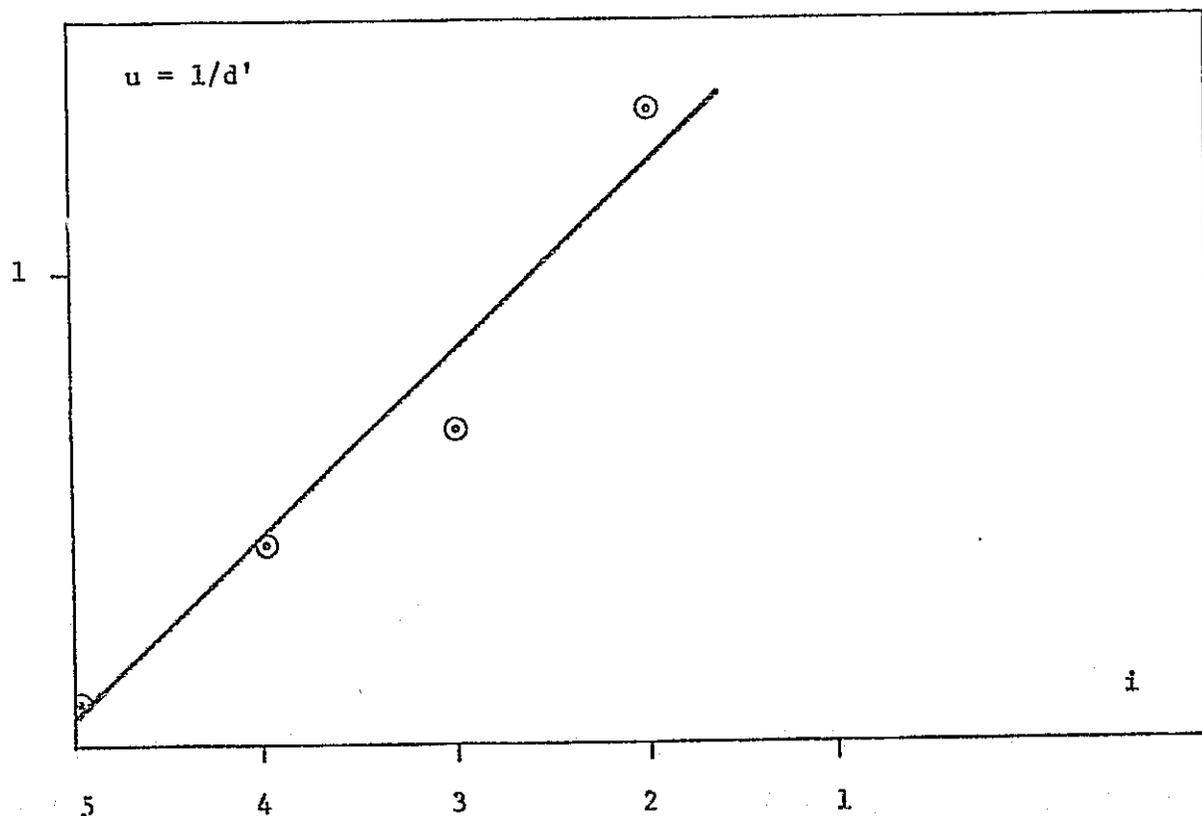


FIGURE 4.13   UNCERTAINTY VERSUS SNR INDEX (SUBJECT J.TA.)
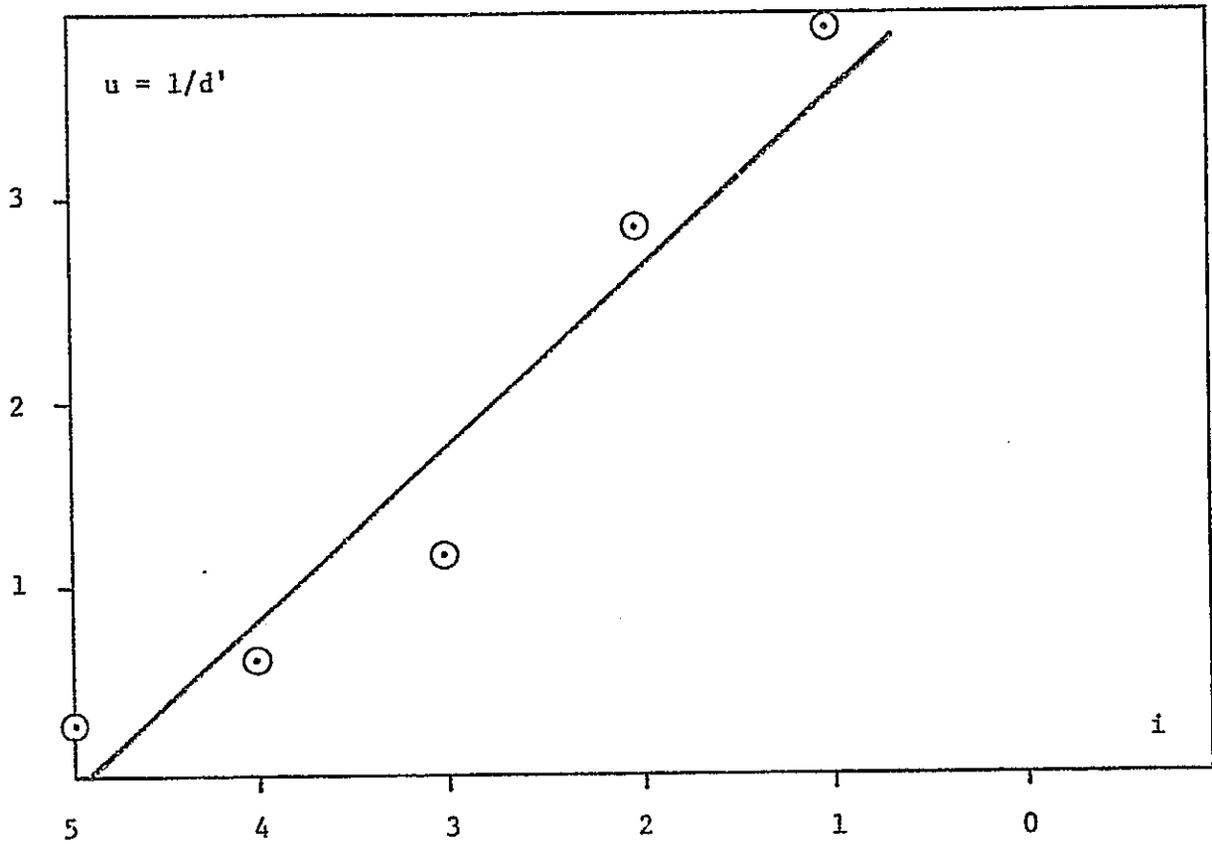
FIGURE 4.14   UNCERTAINTY VERSUS SNR INDEX (SUBJECT A.E)
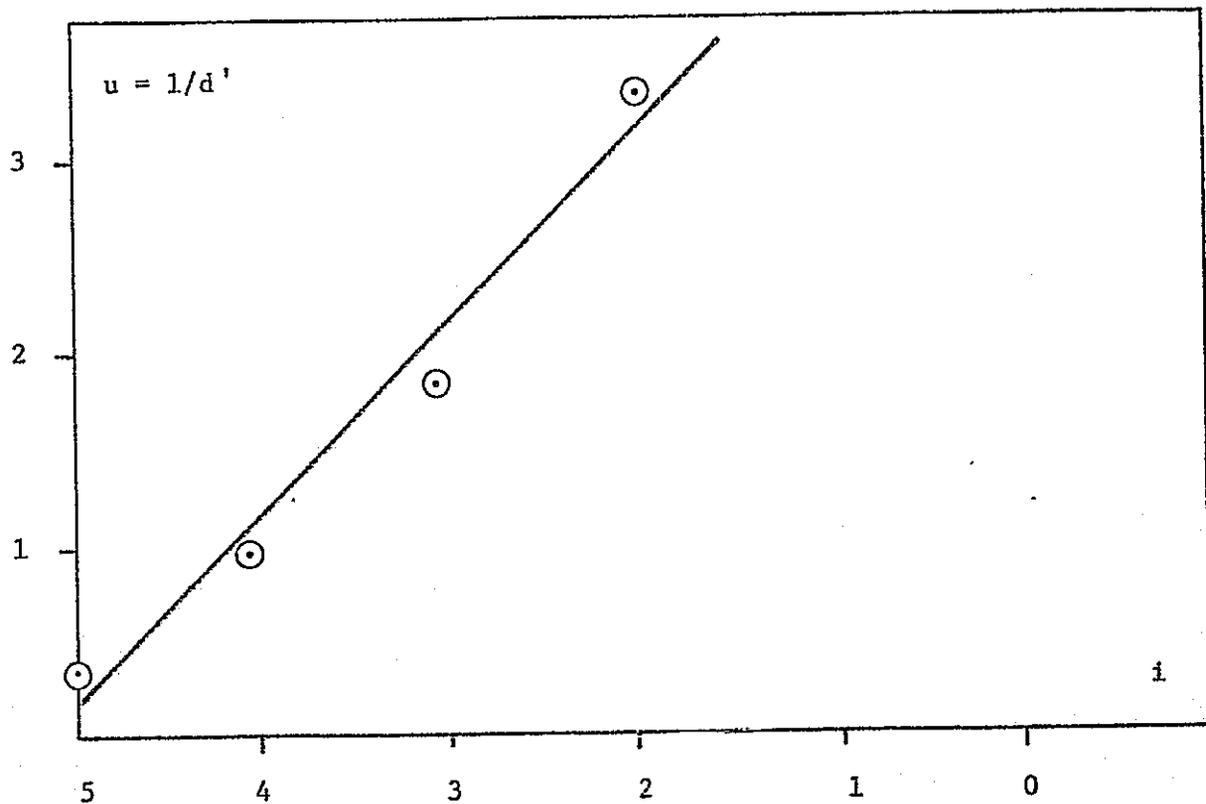


FIGURE 4.15   UNCERTAINTY VERSUS SNR INDEX (SUBJECT C.B)

The model that was suggested in section 4.3 implies a linear relationship between the uncertainty u and the SNR index i in such a way that u is decreasing with i increasing. Therefore, linear regression was used to fit a straight line to the data points. The correlation coefficient $\rho_{ui}$ was computed as a figure of merit to the linearity assumption. The computed values for $\rho_{ui}$ are shown in the sixth column of table 4-3. For five of the subjects, these coefficients were close to one which indicates a strong tendency to linearity. For one subject (J.TA) the value was relatively low; however, this was due for the most part to one data point. When this point was eliminated, $\rho_{ui}$ jumped to 0.97. It is also desirable to test the hypothesis that u and i are not linearly related. Since the number of the data points is small, either the t or the F test should be used. It can be shown (Draper and Smith, 1966) that for inferences concerning linear regression, these tests are equivalent. The t value for the test is evaluated as follows:

$$t = \frac{b}{S} \left( \sum_{i=1}^{5} (i - \bar{i}) \right)^{1/2} \tag{4.16}$$

where b is the slope of the regression line and S is given by

$$S = \sqrt{\frac{1}{n-2} \sum_{i=1}^{5} (u(i) - \overline{u(i)})^2} \tag{4.17}$$

The t scores for each of the subjects are shown in the last column of table 4-3. Using tables of the t distribution, it can be seen that the hypothesis of nonlinearity can be rejected

for five of the six subjects. The probability for an error of type I, namely, rejecting the nonlinearity hypothesis when it is actually true, is 0.01 for the subjects L.L. and L.M.L. and 0.05 for the subjects C.B., A.E., and J.TO. For the sixth subject (J.TA.) the nonlinearity hypothesis can be rejected with a probability of error of 0.2. Therefore, we might conclude that the experimental results agree with the data that was predicted by the model suggested in section 4.3.

As shown in Appendix A, the results of the optimization algorithm include, in addition to the parameters of the distributions, the two values $K_1$ and $K_2$ which satisfy

$$\ell(K_j) = \beta_j \qquad j = 1,2 \qquad (4.18)$$

Since we assumed that the distributions have equal variances and since the value of m is known, equation (2.3 ) can be used in calculating the values of $\ln\beta_1$ and $\ln\beta_2$. These values are summarized in table 4-4 as a function of the SNR index for each subject. However, we are interested in the relation

$$\ln\beta_j = f(d') \qquad j = 1,2 \qquad (4.19)$$

in order to classify the decision strategy. Under our assumption (Gaussian distributions, equal variances) the following strategies might be considered:

1.    $\ln\beta_j$ = constant (fixed LRCL decision strategy)

2.    $\ln\beta_j = C_5 + C_6 d'$ .   (linear decision strategy)

3.    $\ln\beta_j = C_3 d' \pm d'^2/2$ (N.P. decision strategy)

| SUBJECT | SNR | 2 | 3 | 4 | 5 |
|---------|-----|---|---|---|---|
| | | LRCL's | | | |
| J.TA. | $\ln\beta_1$ | -0.27 | -0.92 | -2.05 | -1.3 |
| | $\ln\beta_2$ | 0.20 | 0.63 | 0.53 | 0.04 |
| L.L. | $\ln\beta_1$ | -1.10 | -1.10 | -0.95 | -0.75 |
| | $\ln\beta_2$ | 0 | 0.018 | 0.018 | -0.015 |
| A.E. | $\ln\beta_1$ | -0.58 | -0.57 | -1.10 | -1.70 |
| | $\ln\beta_2$ | -0.03 | -0.08 | -0.18 | -0.08 |
| J.TO. | $\ln\beta_1$ | -0.30 | -0.35 | -0.22 | 0 |
| | $\ln\beta_2$ | -0.18 | 0.31 | 0.37 | 0.38 |
| C.B. | $\ln\beta_1$ | -0.13 | -0.18 | -0.24 | -0.09 |
| | $\ln\beta_2$ | 0 | 0.06 | 0.24 | 0.57 |
| L.M.L. | $\ln\beta_1$ | -0.37 | -0.41 | -0.54 | -0.33 |
| | $\ln\beta_2$ | 0.07 | 0.16 | 0.50 | 0.86 |

Table 4.4  $\ln\beta_i$ as a Function of SNR Index i

Since both $\beta_j$ and d' are known for each level of SNR, $\ln\beta_j$ can be found as a direct function of d'. Figures 4-16 to 4-21 show $\ln\beta_1$ and $\ln\beta_2$ as a function of d' for each of the subjects.

Table 4-5 indicates for each subject which one of the three strategies was used (on the basis of visual inspection). The strongest conclusion drawn from the figures is that in most cases the threshold was changed with the SNR level. This finding cannot be predicted on the basis of classical SDT (Donaldson and Murdock, 1968); however, similar data was obtained by Kinchala and Smyzer (1967) and Healy and Jones (1973).

Although the use of the data processing results helped in showing the change in the LRCL, still for four of the six subjects mixed strategy gave the best fit. An attempt to settle this question was made by Curry et al (1974). Their argument was that the subject is actually using only one decision rule, and this decision rule is one of those that were discussed above. However, instead of using the objective probabilities to form the likelihood ratio, the subjects used subjective probabilities which are linearly related to the true values. The fit of DR curves on this strategy, for the results in our experiment are given in the above reference.
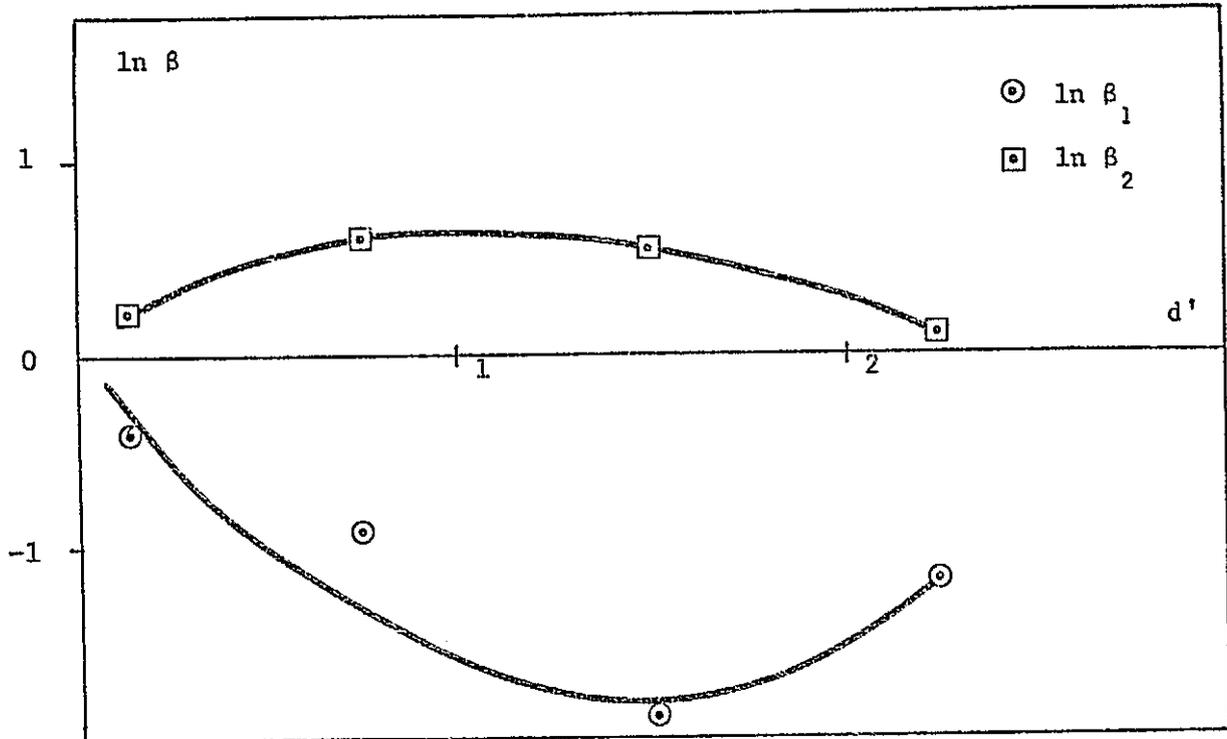
FIGURE 4.16 ln β AS A FUNCTION OF d' (SUBJECT T.A)
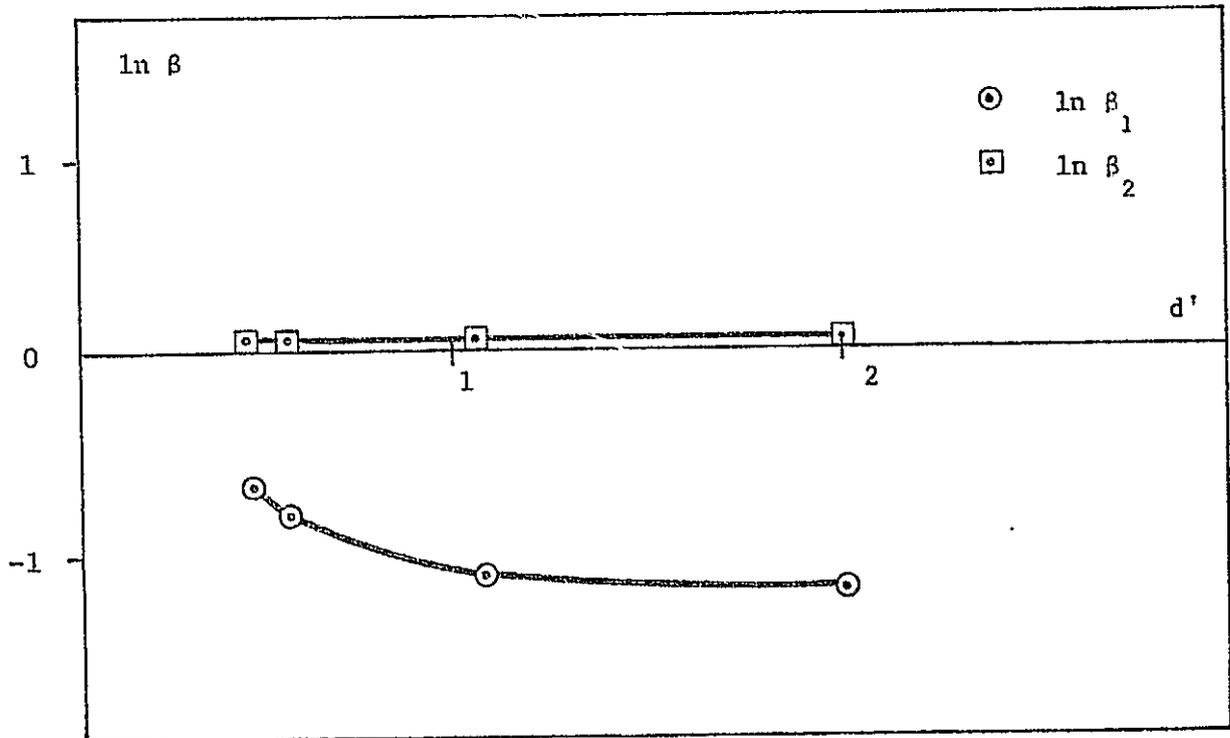


FIGURE 4.17 ln β AS A FUNCTION OF d' (SUBJECT L.L)
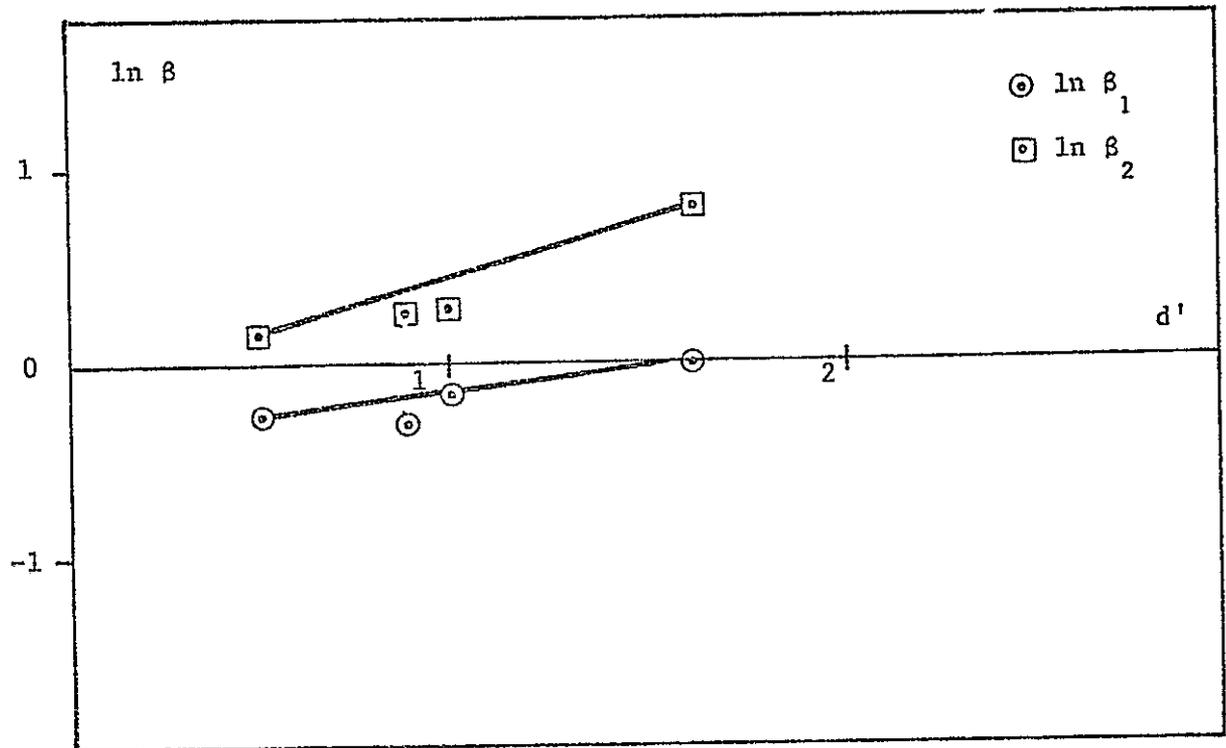
FIGURE 4.18   ln β AS A FUNCTION OF d' (SUBJECT J.TO)
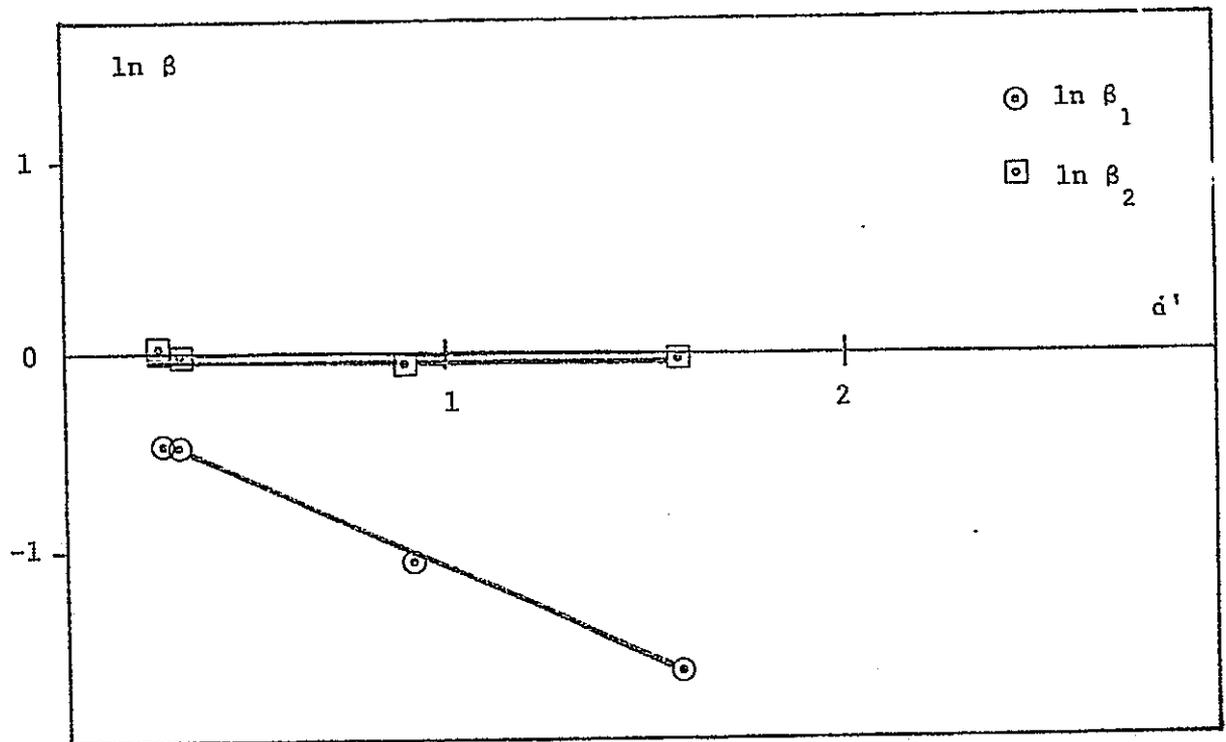


FIGURE 4.19   ln β AS A FUNCTION OF d' (SUBJECT A.E)
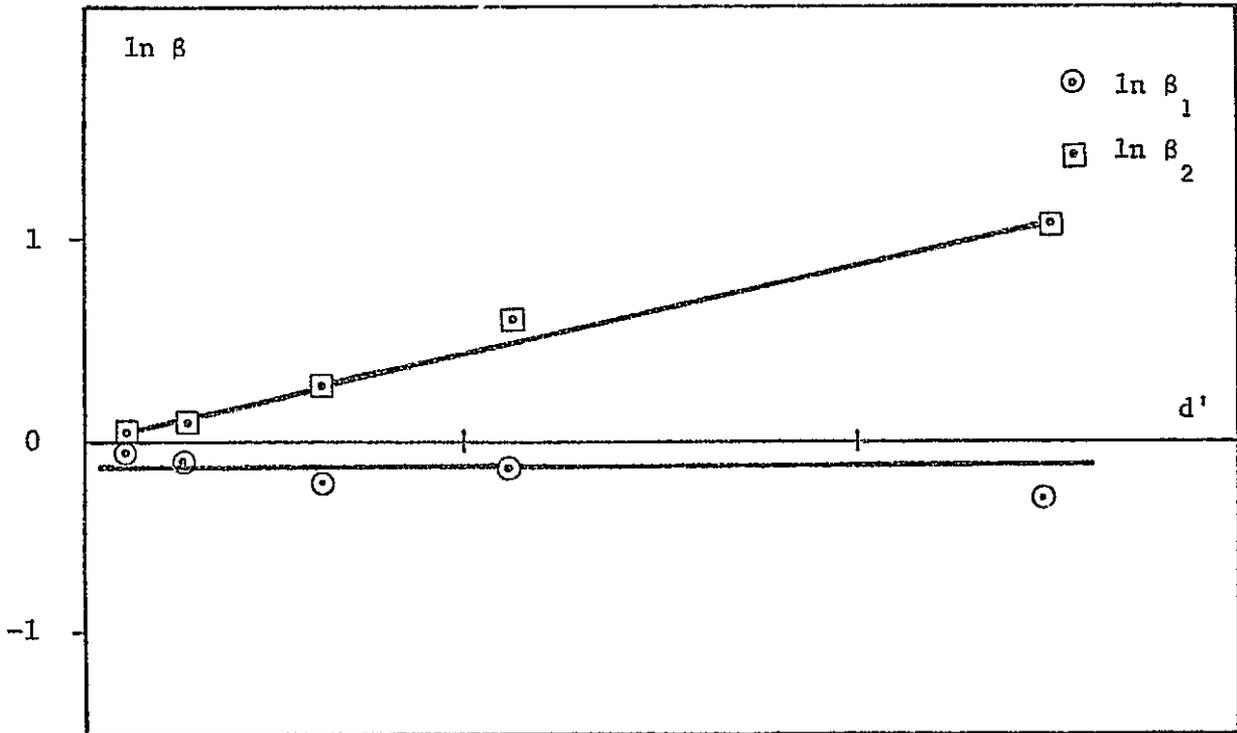
FIGURE 4.20   ln β AS A FUNCTION OF d' (SUBJECT C.B)
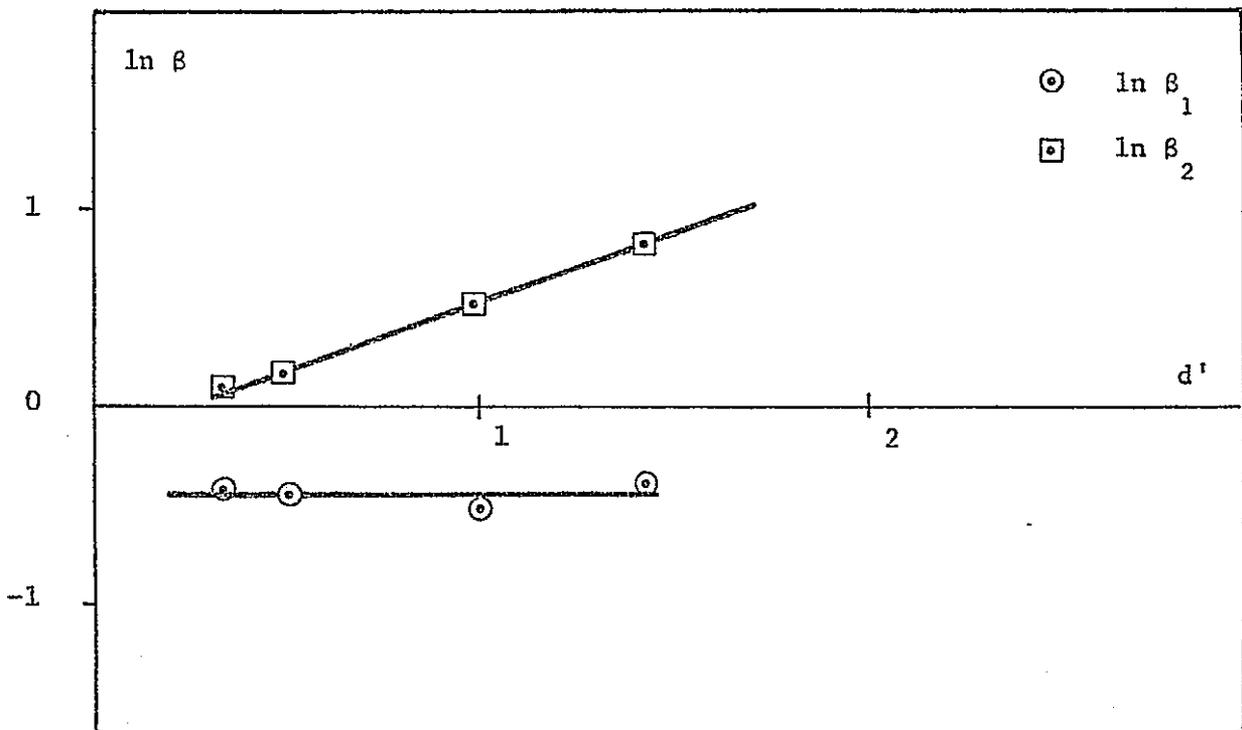


FIGURE 4.21   ln β AS A FUNCTION OF d' (SUBJECT L.M.L)

| Subject | Fig | DR for $\beta_1$ | DR for $\beta_2$ |
|---------|------|-----------------|------------------|
| J.TA. | 4.16 | N.P. | N.P. |
| L.L. | 4.17 | N.P. | constant $\beta$ |
| J.TO. | 4.18 | linear | linear |
| A.E. | 4.19 | linear | constant $\beta$ |
| C.B. | 4.20 | constant $\beta$ | linear |
| L.M.L. | 4.21 | constant $\beta$ | linear |

Table 4-5  Decision strategy evaluation on the basis of the processed data

# CHAPTER V

DETECTION OF SIGNALS WITH SEQUENTIAL CHANGE

OF SIGNAL STRENGTH

## 5.1 General Discussion

In the previous chapter we limited our discussion of decision strategies to those cases in which the effect of correlation between successive decisions could be neglected. The lack of correlation was due to the following properties of the input data:

1.  The order of changes in the signal strength were chosen with the use of a uniformly distributed random number generator. Therefore, the subject was unable to predict on the basis of the past information and had to consider each stimulus independently.

2.  A blanking period of two seconds was introduced between successive decision intervals to help the subject forget the location of the circles in the previous presentation.

However, both of these properties are somewhat artificial and were chosen to satisfy classical SDT assumptions of independent decisions. In particular, property 1 implies that the input signal has no time structure, a property which is usually associated wtih noise rather than with signals. For most real

life detection tasks in which the signal is changing, the change is governed by some specific rule that determines the mean time structure. In such cases, this rule might be used by the decision maker to base his decision not only on the current information but also on the past information. Therefore, his decisions will be correlated in some yet unspecified manner.

Since we have already studied decision strategies under independent decisions, we are now in a position to analyze by comparison the effect of the temporal correlation of the signal on the overall performance of the subject. It should be noted that our main interest is the effect of the correlation, so that the functional form of the time structure is of secondary importance as long as it introduces correlation effects into the subject's strategies.

The time structure that was chosen is referred to as a "sequential" change of SNR, and is related to some practical decision tasks that are of interest. The definition of sequential change of SNR is as follows:

1. The true state of the world is the same for all decision intervals within a sequence. With the use of a uniform random number generator, this state of the world is determined before the presentation of the sequence in such a way that all states of the world are equiprobable.

2.    The SNR level is fixed within each decision
      interval, but increases with the index i of
      the interval within the sequence.

3.    There is no blanking period between the
      decision intervals so that the current
      decision interval starts when the previous
      interval ends.

The signal strength in each sequence is a deterministic
process. However, randomness does exist and is due to the
random choice of the state of the world, which is determined
at the beginning of the sequence. Once the state of the world
is determined, the sequence is deterministic. Therefore, there
are two possible sequences for which the magnitude of the
signal is the same but the sign is different depending on the
true state of the world ($S_0$ or $S_1$), and the appearance of each
of these is equiprobable.

The real life decision task that we had in mind, while
using the sequential presentation was that of a pilot who is
using a traffic situation display (TSD) to avoid mid air
collisions. Let us again consider such a case and assume that
the TSD is updated by radar information with a change in infor-
mation every four seconds to show the present and previous
(four seconds before) position of all intruders. This infor-
mation is always translated in such a way that the plane of

of the TSD user stays in the center of the display. We further assume that all airplanes are flying in a linear motion with constant velocity; an assumption that usually holds near airfields. The pilot must decide whether a specific intruder is going to pass to the right or to the left. If we consider each 4 second interval between radar updating as a decision interval, these decision intervals, starting when the intruder is at the far end of the display and ending when it is closest to center, constitute a sequence with the aforementioned properties.

Clearly the true state of the world (namely, the intruder passing to the left or right) does not change within this sequence (property 1). Also the discrimination becomes easier as the intruder approaches the center, so that the SNR is increasing with time (property 2). Finally, there are no blanking periods between these intervals (property 3). Therefore, the pilot's task is a decision task with sequential change of SNR which might lead to correlation between successive decisions.

To analyze the results of a signal detection experiment with correlated signal presentations, the classical methods (chapter 2) have to be modified. In particular, the outcome is not based on the current decision alone, but must be further sorted on the basis of past decisions. This might not be feasible if the capacity of the memory involved were large enough to store all the information from the start of the

sequence. However, it seems reasonable to assume that the decision maker has a finite memory and that his present decision is correlated only to the previous one while all further past information is ignored. This assumption reduces the sorting problem considerably and leads to the use of the well established theory of Markov processes. Since Markov models will be used in the analysis of the data, some notation and terminology of this theory will be presented in the next section.

## 5.2 Discrete Markov Processes

Let us start with a sequence of discrete random variables

$$\{X(i)\} \qquad i = 1,\ldots,N \qquad (5.1)$$

where each $X(i)$ can take its value from a finite set of real numbers

$$\{S_p\} \qquad p = 1,\ldots,M \qquad (5.2)$$

The set $\{S_p\}$ defines the state of the system and the equation

$$X(i) = S_p$$

defines the state of the system to be $S_p$ at interval $i$ of the sequence. In our particular application $X(i)$ is the decision of the subject at the decision interval $i$ and the states are

$$S_1 = A_0 \qquad\qquad S_2 = A_1$$

and therefore $M = 2$.

If the random variables $X(i)$ are independent (as was the case in Chapter 4), the state of the system in each interval does not depend on its states in the past. Therefore, all that was needed to describe the system were the probabilities

$$P\{X(i) = S_p\} \qquad i = 1,\ldots,N; \; P = 1,\ldots,M \qquad (5.3)$$

However, if the random variables are dependent, then the probabilities that define the system are the joint probabilities

$$\text{Prob}\{X(1) = S_p, \; X(2) = S_q,\ldots,X(N-1) = S_r, \; X(N) = S_\ell\} \quad (5.4)$$

or using Bayes rule

$$\text{Prob}\{X(N) = S_\ell/X(1) = S_p,\ldots,X(N-1) = S_r\}\text{Prob}\{X(1)$$
$$= S_p,\ldots X(N-1) = S_r\} \qquad (5.5)$$

Equation (5.5) implies that the joint probability at time N is a function of the conditional probability that depends on all past history. In some cases though, this dependence can be relieved by limiting it only to the previous decision interval. Then the following relation holds:

$$\text{Prob}\{X(N) = S_\ell / X(1) = S_p, \ldots, X(N-1) = S_r\} =$$

$$\text{Prob }\{X(N) = S_\ell / X(N-1) = S_r\} \tag{5.6}$$

Equation (5.6) is usually referred to as the Markov assumption and a sequence $\{X(i)\}$ that satisfies it is called a discrete Markov process. Therefore, a Markov process is completely defined if:

1. $\text{Prob }\{X(1) = S_p\}$

2. $\text{Prob }\{X(i) = S_\ell / X(i-1) = S_n\}$   $i = 2, \ldots N$

are known.

To simplify our notation, the conditional probabilities defined in (5.6) above, will be written as follows:

$$\text{Prob }(X(i) = S_k / X(i-1) = S_\ell) = P_{k,\ell}(i, i-1) \tag{5.7}$$

where:

$$i = i, \ldots, N; \quad k = 1, \ldots, M; \quad \ell = 1, \ldots, M$$

They are referred to as the one step transition probabilities. If the number of states of the process is M, equation (5.7)

defines $M^2$ such probabilities, which can be written in matrix notation as

$$P(i,i-1) = \begin{bmatrix} P_{1,1}(i,i-1) & \cdots\cdots\cdots\cdots & P_{1,M}(i,i-1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ P_{M,1}(i,i-1) & & P_{M,M}(i,i-1) \end{bmatrix} \tag{5.8}$$

The matrix P is called the one step probability transition matrix. Since at each interval the system must be in one of the states

$$\sum_{\ell=1}^{M} P_{k,\ell}(i,i-1) = 1 \tag{5.9}$$

i.e. the elements in each row of $P(i,i-1)$ sum to unity. A stationary Markov process is a Markov process for which the one step probability transition matrix satisfies

$$P(i,i-1) = P(i-(i-1)) = P(1) \qquad i = 1,\ldots,N \tag{5.10}$$

In many cases we are interested in transitions which include more than one step. We therefore define

$$\Phi_{k,\ell}(i,j) = \text{Prob}\{X(i) = S_\ell/X(j) = S_k)\}$$

and the probability transition matrix is defined as

$$\Phi(i,j) = \begin{bmatrix} \phi_{1,1}(i,j) & \cdots \cdots \cdots & \phi_{1,M}(i,j) \\ \vdots & & \vdots \\ \phi_{M,1}(i,j) & & \phi_{M,M}(i,j) \end{bmatrix} \tag{5.11}$$

by definition

$$\Phi(i,i) = 0 \qquad \Phi(i,i-1) = P(i,i-1) \tag{5.12}$$

Also it can be shown (Howard, 1971) that

$$\Phi(i,j) = \prod_{r=0}^{r=i-j-1} P(i-r,i-r-1) \tag{5.13}$$

If the process is stationary, equation (5.13) is reduced to

$$\Phi(i-j) = P^{i-j} \tag{5.14}$$

Also of interest is the probability of being at a state p in the interval i, regardless of the state in the interval (i-1). We therefore define this probability as

$$\Pi_p(i) = \text{Prob}\{X(i) = S_p\} \tag{5.15}$$

If the number of states is M, there are M probabilities $\Pi_p(i)$ so that an M dimensional vector can be defined

$$\underline{\Pi}^T(i) = \{\Pi_1(i), \Pi_2(i), \ldots \Pi_M(i)\} \tag{5.16}$$

It can be shown (Howard, 1971) that the following relation holds for stationary processes

$$\underline{\Pi}(i) = \Phi(i)\,\underline{\Pi}(0) \tag{5.17}$$

## 5.3 Experimental Method and Preliminary Results

In order to study the effect of the correlated signals on the decision strategy, an experiment was designed in which the subject participated in two tasks (each in a different session) with equivalent stimuli, but with different order of presentation. In the first task, the signal strength was changed in a random order (as in the experiment in Chapter 4) and, therefore, independent decisions were expected, while in the second task the data was presented in a "sequential" order to induce correlation.

The detailed description of the data presentation in the experiment with random change of signal strength was given in section 4.2. Since the SNR level in that experiment is time varying, it was easy to modify the presentation to "sequential". Each sequence included all 5 levels of SNR ($N = 5$) in increasing order to satisfy property 2 in section 5.1. For each SNR level the pair of circles could be either on the right or on the left (Figure 4.2). In order to satisfy property 3, the two seconds blanking period between decision intervals was eliminated. Before displaying the first pair of circles of a new sequence, a random number generator was used to choose whether the state of the world during this sequence was $S_0$ or $S_1$ with the following probabilities:

$$P(S_0) = P(S_1) = 0.5$$

The subject had the same decision task as in the previous experiment, i.e. to discriminate right from left, except that now he has four response categories (n = 4).

SURE LEFT

THINK LEFT

THINK RIGHT

SURE RIGHT

The reason for the change from three to four response categories was mainly due to the use of Markov models for the analysis. Four response categories imply a Markov process with four states, however, by combining the sure and think states together the number of states can be reduced to two. Such a reduction simplifies the computations as well as the analysis of the decision strategy. Clearly such a reduction is not feasible for an odd number of response categories.

Two new subjects (who did not take part in the previous experiment) participated in this experiment. Both were graduate students in the Man Vehicle Laboratory at M.I.T. and were participating on a voluntary basis. Each subject took part in 4 experimental sessions. In two half hour sessions the presentation was in a sequential order and each subject made a total of 900 decisions. In the other two sessions of 45 minutes each, the presentation was in a random order and again, the total number of decisions was 900. One of the subjects started with the sequential presentation, while the other started with the random presentation in order to balance

learning effects.  Instructions and the training session were the same as in the previous experiment.

For a preliminary comparison of the performance of the subjects in the two tasks, the DR curves in the $P_H-P_{FA}$ plane were used.  Two of the DR curves (those between sure and think) are shown in Figures 5.1 and 5.2.  Those figures show that a difference in performance does exist; and, in particular, there is an increase in the probability of a false alarm for the task with the sequential presentation.  When the data processing program (section 4.4) was used, the performance could be expressed in terms of $\beta$ and $d'$.  Figures 5.3 and 5.4 show the same two LRCL's as a function of $d'$, and again there is a difference in performance between the two tasks.  It should be noted that a difference in performance due to correlated decisions in auditory signal detection experiments was reported by Speeth and Matthews (1961), and by McGill (1954).

The next step is to describe these differences and to provide a modification to the theory of uncorrelated decisions that would describe the subject's behavior when correlations are present.  Furthermore, since analysis through SDT can separate the sensory and decision processes, it might be possible to find whether the change is due to only one of these processes or both.
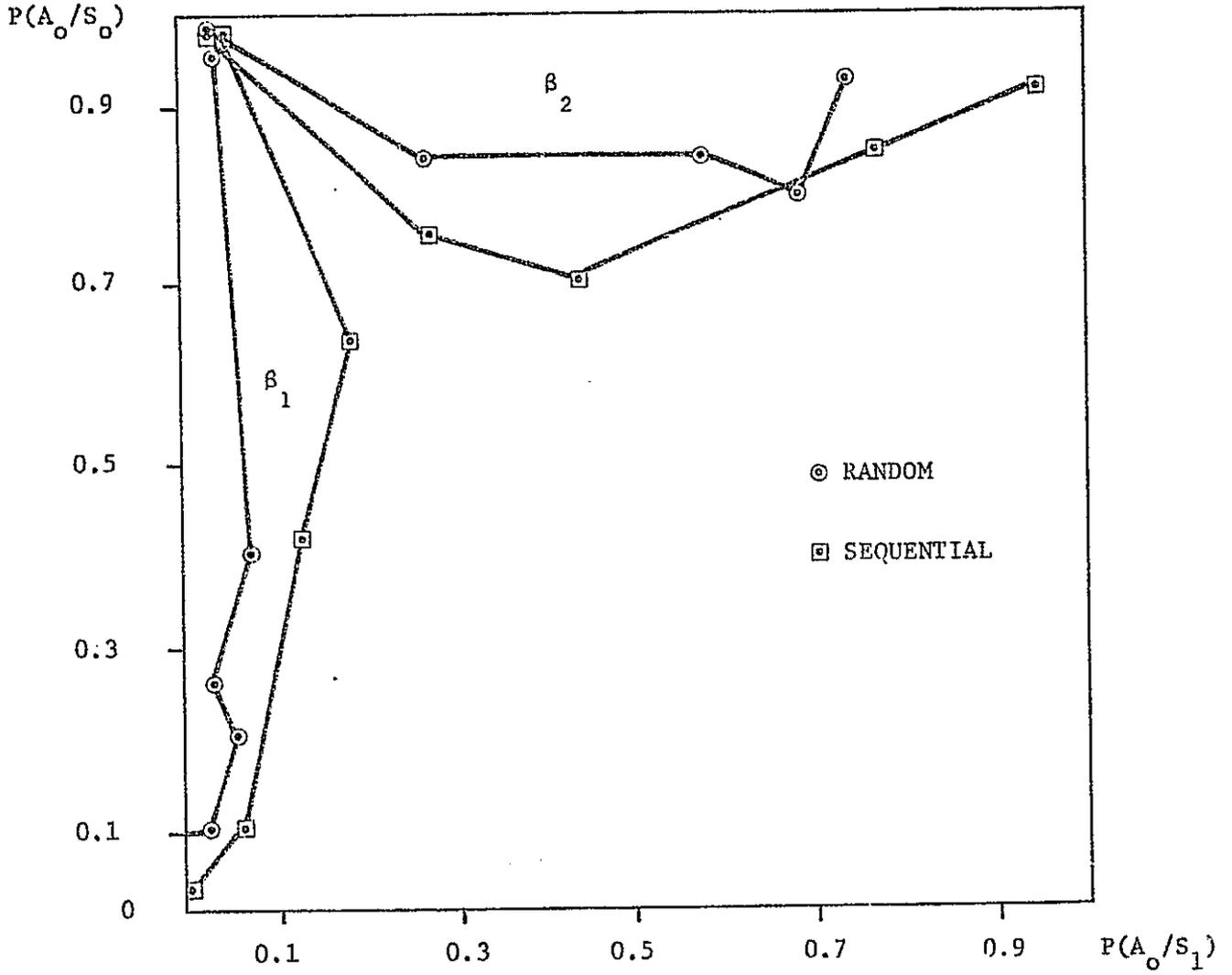
FIGURE 5.1   DR CURVES FOR RANDOM AND SEQUENTIAL

PRESENTATIONS (SUBJECT A.C.)
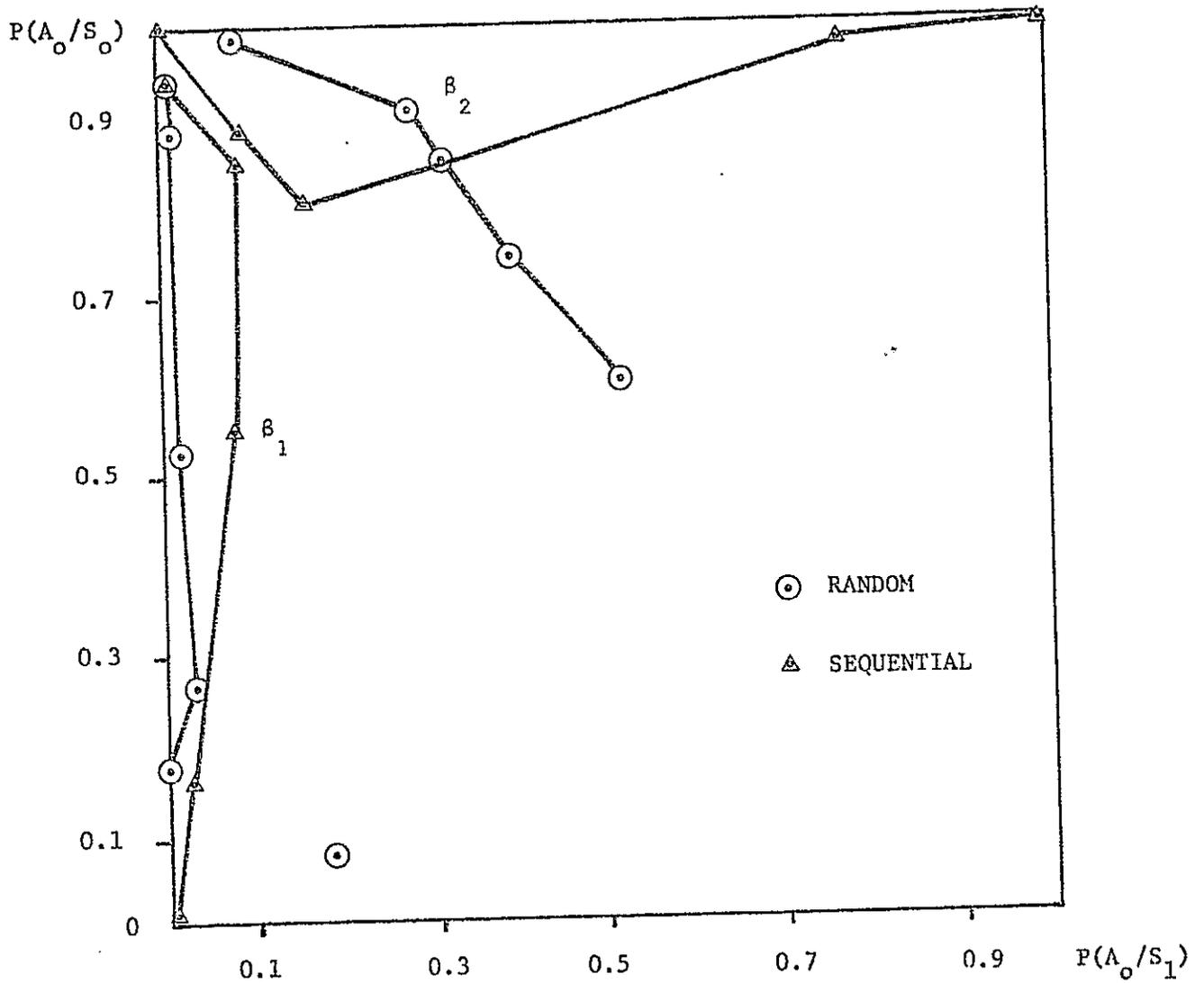
FIGURE 5.2   DR CURVES FOR RANDOM AND SEQUENTIAL·
PRESENTATIONS (SUBJECT A.T)
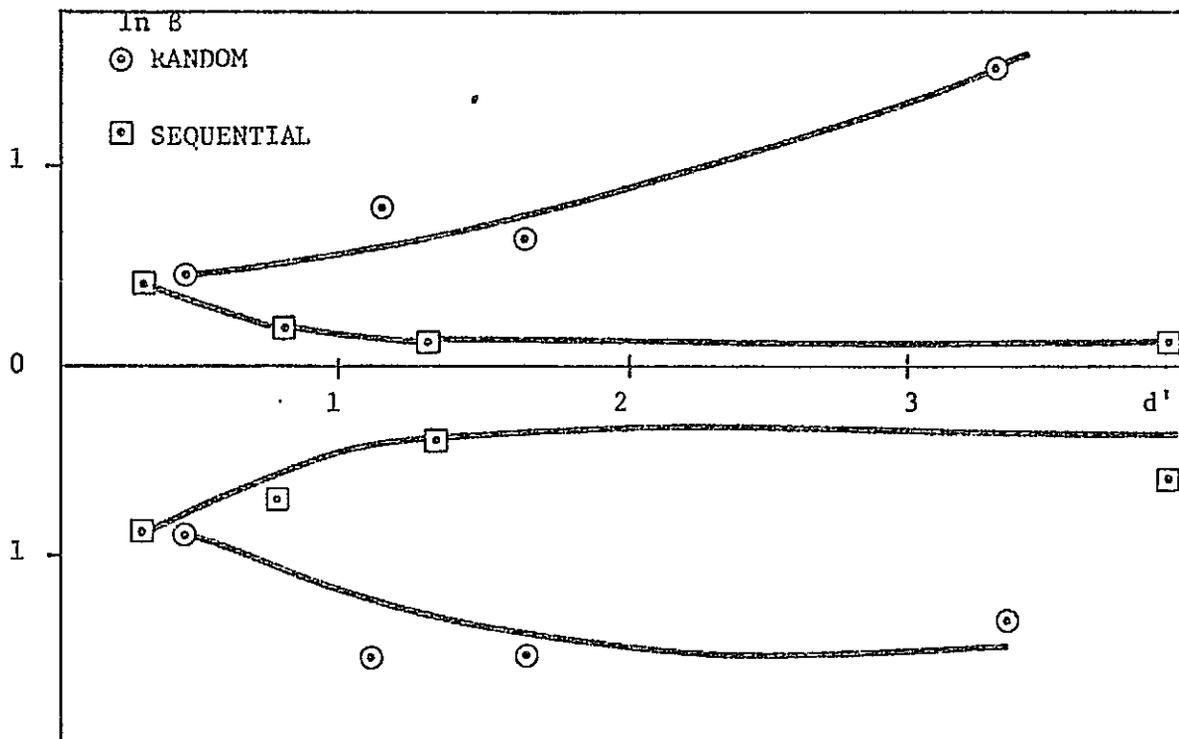
FIGURE 5-3   DIFFERENCES IN LRCL BETWEEN RANDOM AND SEQUENTIAL PRESENTATIONS
(SUBJECT A.C)
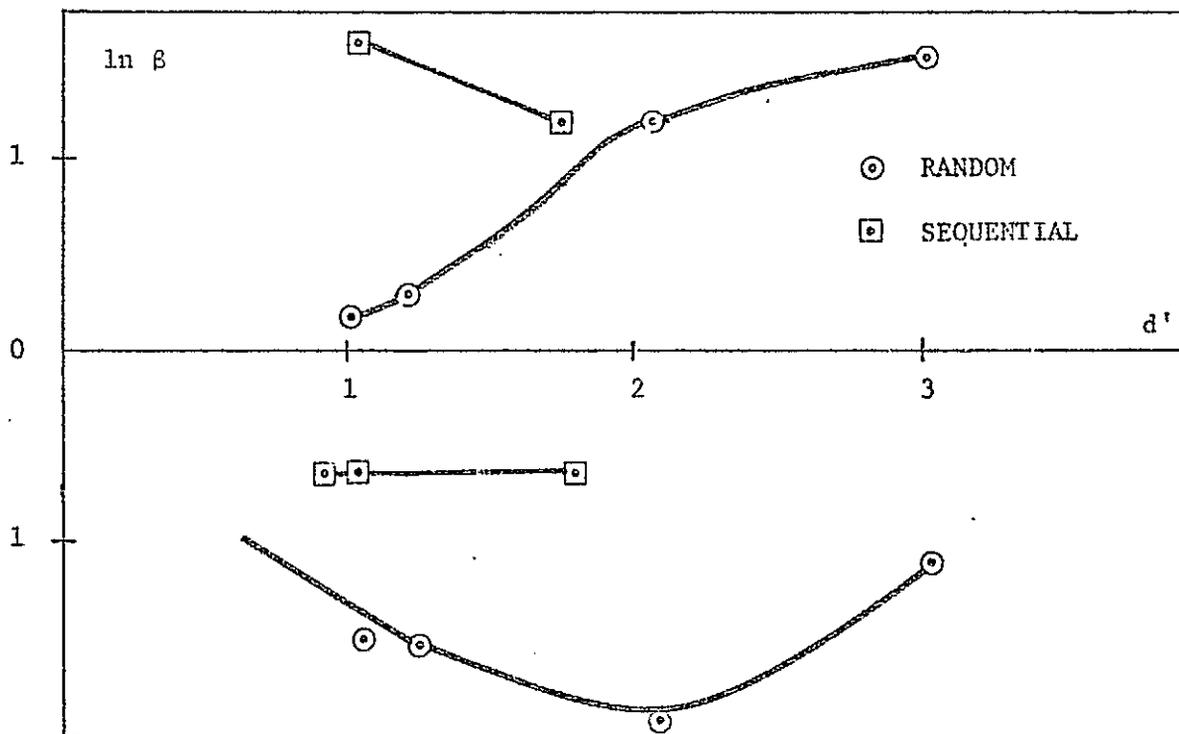


FIGURE 5-4   DIFFERENCES IN LRCL BETWEEN RANDOM AND SEQUENTIAL PRESENTATIONS
(SUBJECT A.T)

## 5.4  A Model for the Decision Strategy

As was suggested in section 5.1, a basic assumption that we will make is that the subject has a "limited" memory, so that a correlation exists between each successive decision interval, but is weak enough to be neglected between intervals that are more than one interval apart.  This assumption leads to the use of the Markov models that were discussed in Section 6.2.

Markov processes have previously been used in psycho-physics for modelling human behavior in auditory recognition tasks (Tanner et al, 1961).  In these experiments, the subject was asked to discriminate between two signals with the same tone but two different amplitudes which were presented in a random order.  Tanner suggested that the recognition was based on the difference between the present and previous amplitudes rather than on the current stimulus alone. They also assumed that the subjects used two LRCL's in such a way that a high amplitude was chosen if the difference was larger than the higher LRCL, and the low amplitude was chosen if it were smaller than the the lower LRCL.  If the difference was between the two LRCL's, the previous decision would be repeated.

The main argument to support this model, which is based on the difference between successive stimuli, is that within each recognition interval the subject does not have any objective reference on which he can base his judgement.  In our detection

experiments, the situation differs because the reference is presented in each interval so that no external references are needed.      Since,  the  results  show that  a  change does exist, we would assume that the decisions are based on the current information only.  If there is no accumulation of information from one interval to the next, the information is the same for both random and sequential presentations, and we might conclude that the difference in performance is due to the effects of past decisions, rather than past information. The suject's motivation for dependency on past decisions can be explained by his knowledge that the state of the world is the same during an entire sequence, and by changing decisions he admits previous errors.

The hypothesis that the change in performance is due to a previous decision and not to past information can be tested by the use of the separation property of SDT.  If this hypothesis is true, then the detectibility d', which is sensitive only to changes in input information, should be the same for both presentations, and the linear relationship between d' and the SNR index i that was found for the random presentation should hold for the sequential presentation.  If that is so, the change in performance can be attributed to the decision strategy alone, and the model which will be suggested will apply to the decision process in the detection task.

For simplicity, we will start our discussion of the decision strategy with the assumption that the subject can make only two decisions, $A_0$ and $A_1$ (n=2). Therefore his performance in each decision interval i is given by the pair:

$$P_H(i) = P(A_0(i)/S_0(i)) \quad P_{FA}(i) = P(A_0(i)/S_1(i)) \qquad (5.18)$$

Since the true state of the world is the same for all intervals in the sequence, equation (5.18) can be written as:

$$P_H(i) = P(A_0(i)/S_0) \qquad P_{FA}(i) = P(A_0(i)/S_1) \qquad (5.19)$$

Now if we assume a Markov model in which the current decision is based on the previous one, then:

$$P_H(i) = P(A_0(i)/S_0, A_0(i-1))P(A_0(i-1)) +$$
$$P(A_0(i)/S_0, A_1(i-1))P(A_1(i-1))$$
$$\qquad (5.20)$$
$$P_{FA}(i) = P(A_0(i)/S_1, A_0(i-1))P(A_0(i-1)) +$$
$$P(A_0(i)/S_1, A_1(i-1))P(A_1(i-1))$$

Since there are two response categories, the number of states in the process is two. However, for our discussion, it will be convenient to define two (rather than one) processes: one in which the subject was correct in his previous decision and the other in which he was wrong.

Using the notation of section 5.2, the one step transition probabilities for the "correct" process are:

$$P^C_{00}(i,i-1) = P(A_0(i)/S_0,A_0(i-1))$$

$$P^C_{01}(i,i-1) = P(A_0(i)/S_1,A_1(i-1))$$

$$P^C_{10}(i,i-1) = P(A_1(i)/S_0,A_0(i-1))$$

$$P^C_{11}(i,i-1) = P(A_1(i)/S_1,A_1(i-1))$$

(5.21)

and the same probabilities for the "incorrect" processes are:

$$P^{NC}_{00}(i,i-1) = P(A_0(i)/S_1,A_0(i-1))$$

$$P^{NC}_{01}(i,i-1) = P(A_0(i)/S_0,A_1(i-1))$$

$$P^{NC}_{10}(i,i-1) = P(A_1(i)/S_1,A_0(i-1))$$

$$P^{NC}_{11}(i,i-1) = P(A_1(i)/S_0,A_1(i-1))$$

(5.22)

or in matrix notation

$$\underline{P}^C(i,i-1) = \begin{bmatrix} P^C_{00}(i,i-1) & P^C_{01}(i,i-1) \\ \\ P^C_{10}(i,i-1) & P^C_{11}(i,i-1) \end{bmatrix}$$

(5.23)

and

$$\underline{P}^{NC}(i,i-1) = \begin{bmatrix} P^{NC}_{00}(i,i-1) & P^{NC}_{01}(i,i-1) \\ \\ P^{NC}_{10}(i,i-1) & P^{NC}_{11}(i,i-1) \end{bmatrix}$$

(5.24)

Note that the elements in each row of the above matrices sum to one.

If the detection is perfect for all intervals in the sequence:

$$p^C(i,i-1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad p^{NC}(i,i-1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (5.25)$$

and for pure guessing:

$$p^{NC}(i,i-1) = p^C(i,i-1) \quad = \quad \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \qquad (5.26)$$

These performances are the two extremes which are not expected in well designed experiments.

The values which are found for these matrices can show whether or not the subject was biased by his previous decisions. An unbiased subject is expected (since the SNR is increasing) to stick to his previous decision if he was correct and to change his decision if he was wrong.

The strategy of an unbiased decision maker is, therefore, defined by the following inequalities:

$$p^C_{00}(i,i-1) > 0.5 \qquad p^C_{11}(i,i-1) > 0.5 \qquad (5.27)$$

$$p^{NC}_{10}(i,i-1) > 0.5 \qquad p^{NC}_{01}(i,i-1) > 0.5 \qquad (5.28)$$

The values for these probabilities should increase with the

value of i and in the final interval ($i = 5$) should approach unity. Such an unbiased decision strategy would lead to performance that is similar to the performance of the subject in an experiment with independent decision intervals.

The biased subjects can be divided into two types, those who repeat and those who alternate. A repeat strategy (RS) subject would prefer to repeat his previous decisions whether he was right or wrong. For such a decision maker, the inequalities in (5.28) are reversed so that:

$$P^{NC}_{10} (i,i-1) < 0.5 \qquad P^{NC}_{01}(i,i-1) < 0.5 \qquad (5.29)$$

On the other hand, an alternate strategy (ALS) subject will tend to change his decisions even if he was correct before. For this type of decision maker, the inequalities in (5.27) are reversed so that

$$P^{C}_{00}(i,i-1) < 0.5 \qquad P^{C}_{11}(i,i-1) < 0.5 \qquad (5.30)$$

Since the simplest method to define a decision strategy is through the DR curves, it is important to analyze the effect of the three strategies that were suggested above through these curves. It has already been noted that the performance of an unbiased decision maker is similar to the performance in a random data experiments; therefore, the DR curves will be similar to those found in section 4.5.

For the analysis of the strategy of the biased subject, let us rewrite equations (5,20).

$$P_H(i) = P^C_{00}(i,i-1)P(A_0(i-1)) + P^{NC}_{01}(i,i-1)P(A_1(i-1))$$

$$\tag{5.31}$$

$$P_{FA}(i) = P^{NC}_{00}(i,i-1)P(A_0(i-1)) + P^C_{01}(i,i-1)P(A_1(i-1))$$

Since the probabilities $P(A_0(i-1))$ and $P(A_1(i-1))$ are not known, it would be helpful to separate the performance of the subject into two categories. His performance conditioned on a previous decision of $A_0$, is given by

$$P^0_H(i) = P^C_{00}(i,i-1) \qquad P^0_{FA}(i) = P^{NC}_{00}(i,i-1) \tag{5.32}$$

and his performance conditioned on a previous decision $A_1$ is given by:

$$P^1_H(i) = P^{NC}_{01}(i,i-1) \qquad P^1_{FA}(i) = P^C_{01}(i,i-1) \tag{5.33}$$

Each of the equations above, (5.32) and (5.33), defines a different DR curve for values of i from 1 to 5. So, for every LRCL there are two DR curves, and if there were n response categories, the number of DR's would be 2(n-1). It should be noted, though, that the family of DR curves defined by equations (5.32) and (5.33) cannot be plotted in the same $P_H$-$P_{FA}$ plane because the functional relationship between the above probabilities is not known. Using the superscript terminology defined above, equation (5.29) can be rewritten as:

$$P_H^1(i) < 0.5 \qquad P_{FA}^0(i) = 1 - P_{10}^{NC}(i,i-1) > 0.5 \qquad (5.34)$$

These inequalities characterize the behavior of the RS subjects, as they describe the tendency of this type of biased subject to repeat his decisions when he is wrong. Furthermore, it is reasonable to assume that he will repeat his decisions when he is correct, therefore

$$P_H^0(i) > 0.5 \qquad P_{FA}^1(i) < 0.5 \qquad (5.35)$$

Substituting these inequalities into equations (5.32) and (5.33) defines the two DR curves of an RS subject which are shown in Figures 5.5 and 5.6. The straight lines in these drawings are the DR curves based on the N.P. strategy, and the curved lines are based on a fixed LRCL. In a similar way, the inequalities which describe the behavior of the ALS subject are (from 5.30):

$$P_H^0(i) < 0.5 \qquad P_{FA}^1(i) > 0.5 \qquad (5.36)$$

and again we might add that this type of decision maker will also tend to change his mind when he is wrong, therefore:

$$P_H^1(i) > 0.5 \qquad P_{FA}^0(i) < 0.5 \qquad (5.37)$$

The DR curves for an ALS subject are reversed as compared to those of an RS subject and are shown in Figures 5.7 and 5.8.
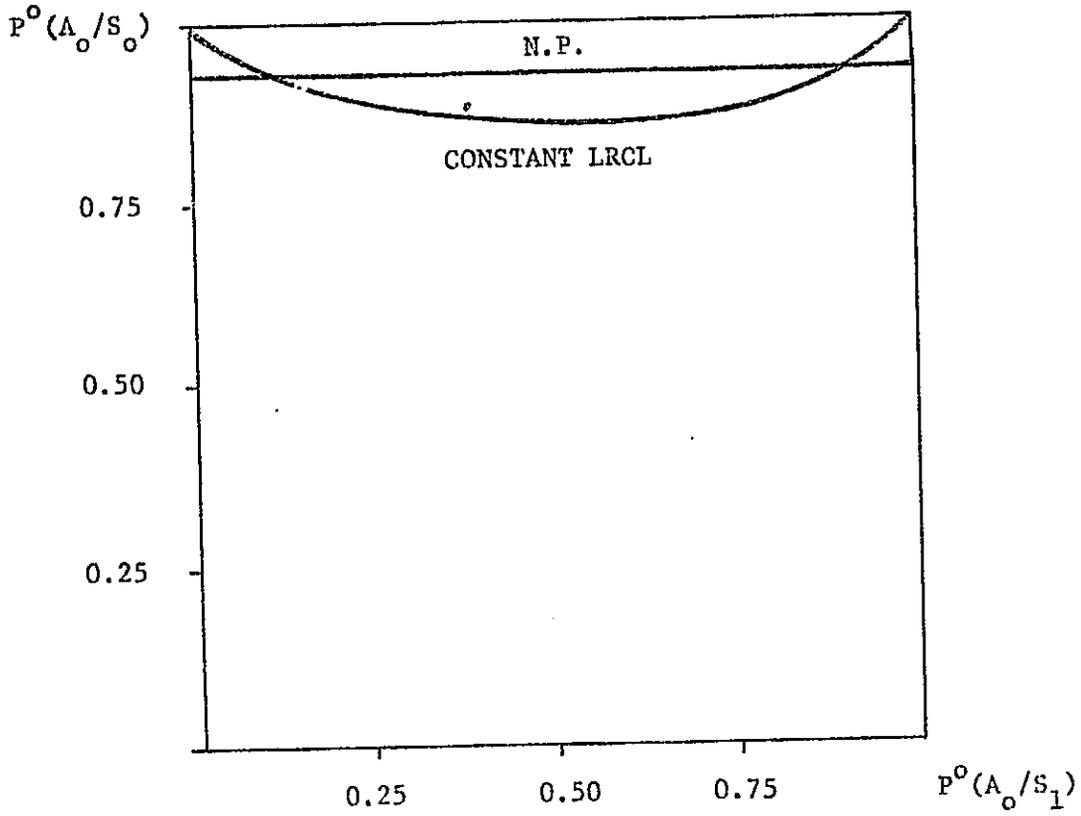
FIGURE 5.5   DR CURVES GIVEN $A_o$ FOR "REPEAT" STRATEGY
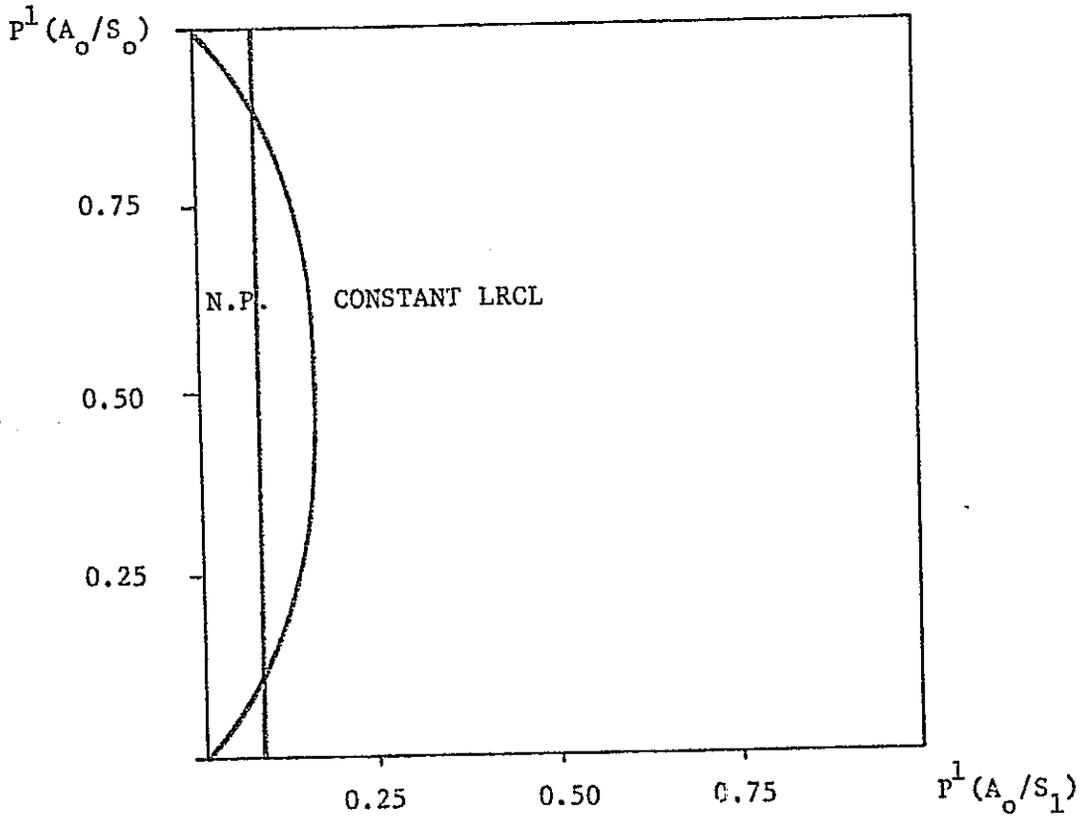


FIGURE 5.6   DR CURVES GIVEN $A_1$ FOR "REPEAT" STRATEGY
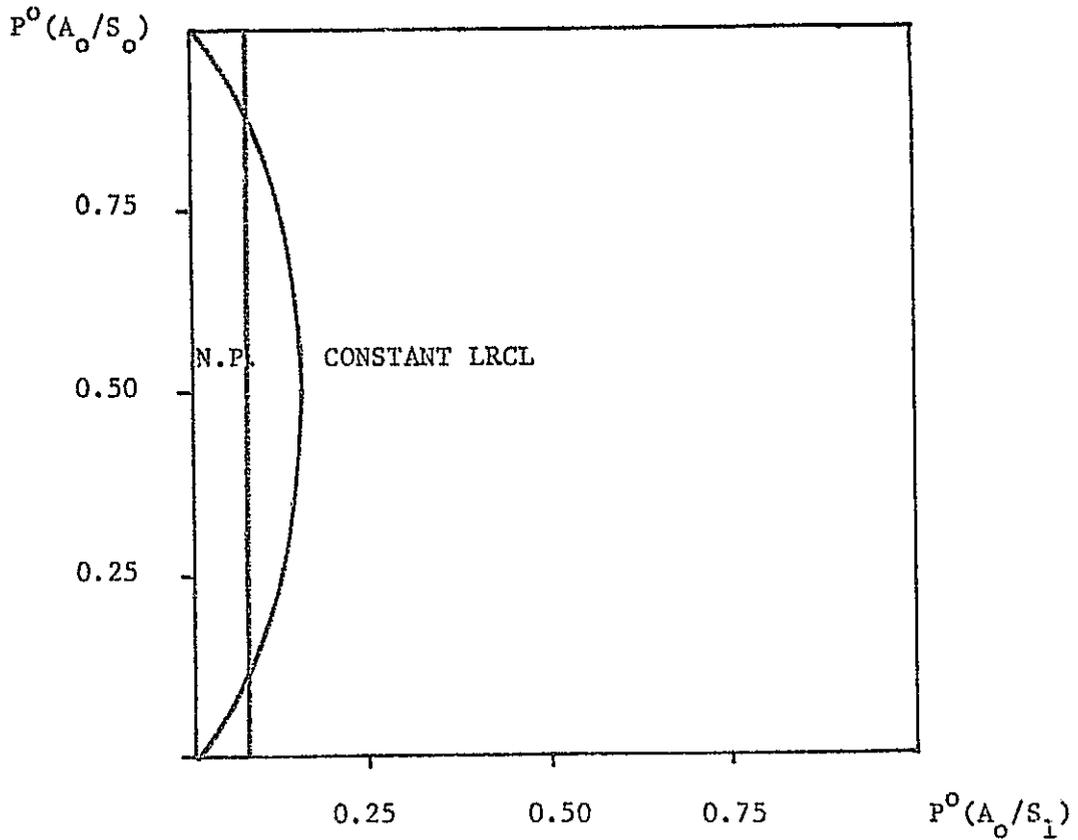
FIGURE 5.7   DR CURVES GIVEN $A_o$ FOR "ALTERNATE" STRATEGY
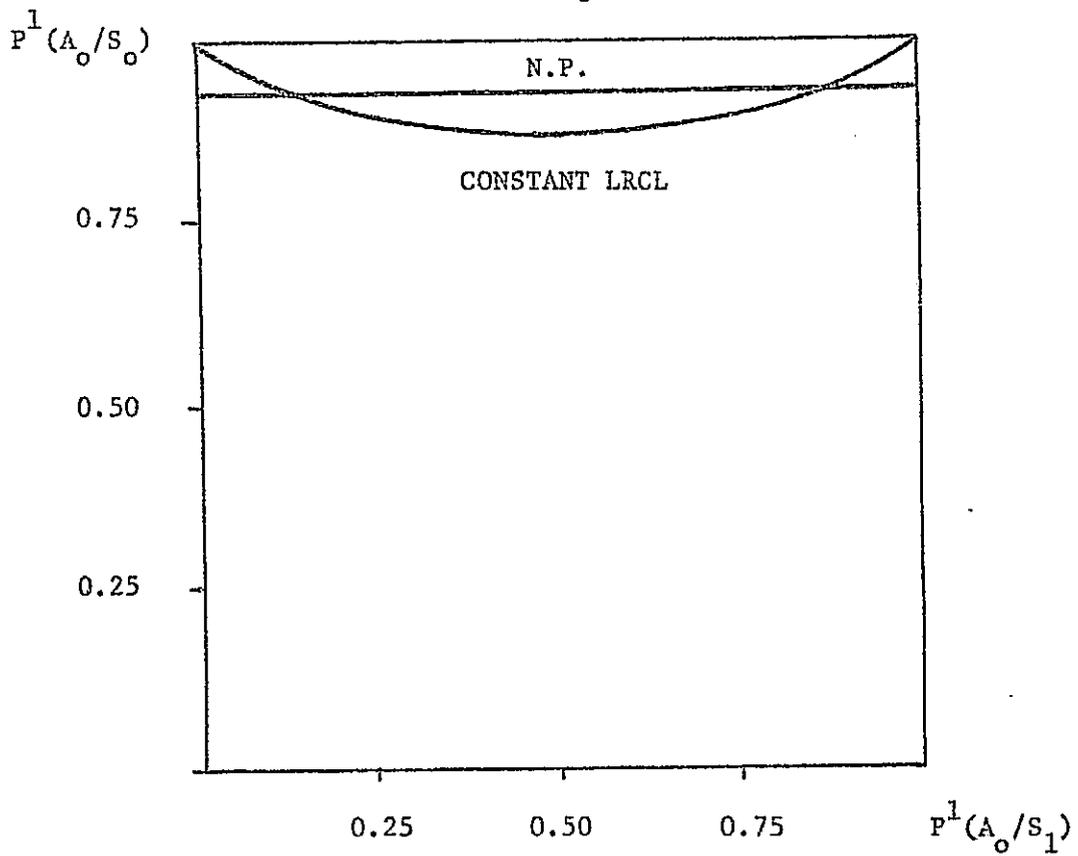


FIGURE 5.8   DR CURVES GIVEN $A_1$ FOR "ALTERNATE" STRATEGY

The difference between the two DR curves (one given that the decision before was $A_0$ and the other given that it was $A_1$), can easily be explained on the basis of classical SDT. Figure 5.9 shows the conditional probability density of the observation under the two states of the world. Let us now assume that $P(S_0) = P(S_1) = 0.5$ and also that the regret ratio is equal to 1. Under these conditions, an unbiased decision maker will choose $\beta_0$ as his LRCL. However, if the subject is RS he will tend to repeat his previous decision. Therefore, if his previous decision was $A_0$, he would move his threshold to $\beta_{A_0}$ so that

$$\beta_{A_0} < \beta_0$$

This will increase his probability of hit, but will also increase his probability of false alarm, which is in agreement with equations (5.34) and (5.35). If his previous decision was $A_1$, then he will move his threshold to $\beta_{A_1}$ such that

$$\beta_{A_1} > \beta_0$$

Therefore, both $P^1_H(i)$ and $P^1_{FA}(i)$ will decrease, which is again in agreement with equations (5.34) and (5.35). An ALS subject will behave in the opposite way (see Figure 5.10). If his previous decision was $A_0$, he would choose the LRCL $\beta_{A_0}$ rather than $\beta_0$ such that:
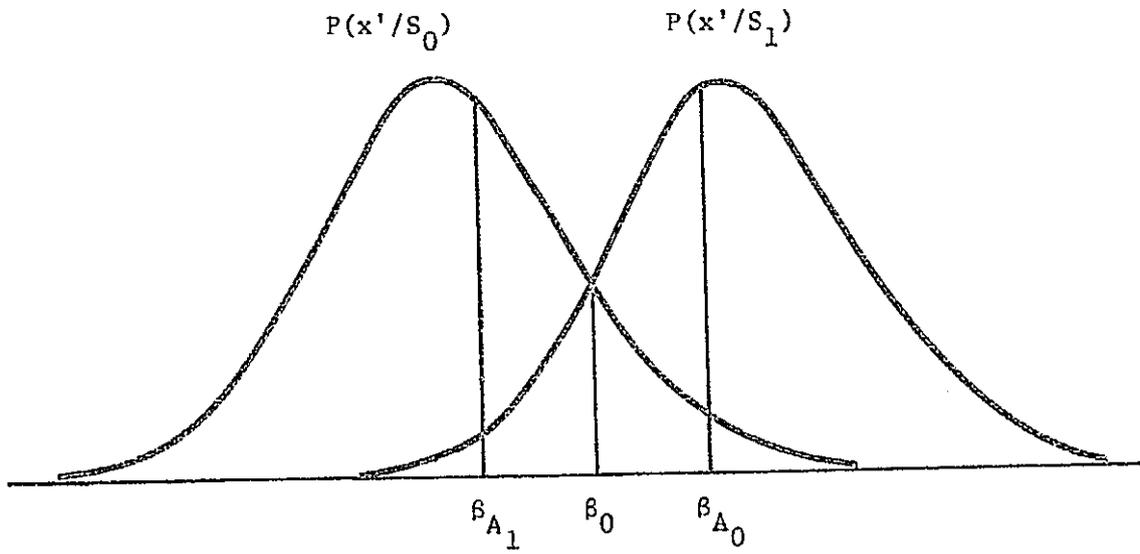
FIGURE 5.9   LRCL CHANGE FOR "REPEAT" BIAS
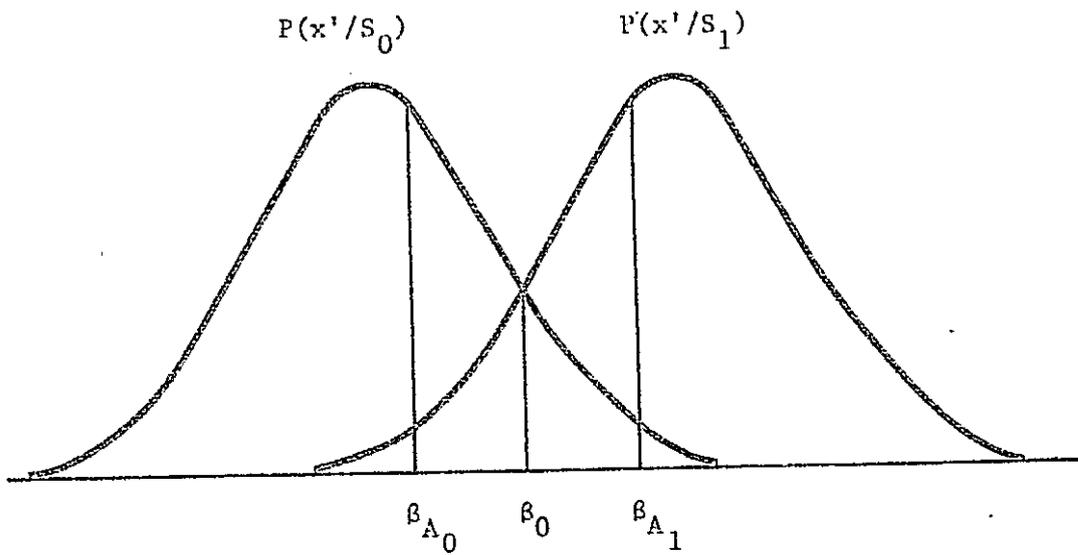


FIGURE 5.10   LRCL CHANGE FOR "ALTERNATE" BIAS

$$\beta_{A_0} > \beta_0$$

Therefore, $P^0_H(i)$ and $P^0_{FA}(i)$ will both decrease which is in agreement with equations (5.36) and (5.37). If his previous decision were $A_1$, the LRCL $\beta_{A_1}$ will satisfy

$$\beta_{A_1} < \beta_0$$

and $P^1_H(i)$ and $P^1_{FA}(i)$ will increase.

Finally, we would like to discuss the method by which the subject might change his LRCL for the various decision rules that are used. If the decision rule is to satisfy the N.P. objective, then the LRCL is based on the probability of false alarm and the subject is working with two values of $P_{FA}$ rather than one. For an RS subject, the value for $P^0_{FA}$ will be much larger than the value for $P^1_{FA}$, and for an ALS subject, the opposite will happen. If the decision rule is to keep the LRCL $\beta$ constant, the value of $\beta_0$ in classical SDT is determined by (Van Trees, 1968)

$$\beta_0 = \frac{P(S_0)}{P(S_1)} [ \frac{C_{00} - C_{10}}{C_{11} - C_{01}} ] \tag{5.38}$$

Where $C_{00}$, $C_{01}$, $C_{10}$ and $C_{11}$ were defined in Section 3.2 and represent the costs that are associated with the four outcomes of the experiment. From a mathematical point of view, in order to change $\beta_0$, the subject can either change his current values of the *apriori* probabilities $P(S_0)$ and $P(S_1)$, or the regret

ratio (the expression in square brackets). From psychophysical aspects, it is more reasonable to assume that the apriori probabilities were changed. For example, an RS subject will tend to increase $P(S_0)$ if his last decision was $A_0$ or to increase $P(S_1)$ if his last decision was $A_1$ (recall that $P(S_0) + P(S_1) = 1$). Therefore, for biased subjects equation (5.38) has to be modified to:

$$\beta_j = K_j \frac{P(S_0)}{P(S_1)} [\frac{C_{00} - C_{10}}{C_{11} - C_{01}}] \qquad j = 0,1 \qquad (5.39)$$

For an RS subject:

$$K_0 > 1 \qquad K_1 < 1 \qquad (5.40)$$

and for an ALS subject:

$$K_0 < 1 \qquad K_1 > 1 \qquad (5.41)$$

It should be noted that the values of $K_j$ in equation (5.39) are functions of the decision interval index i.

The same arguments that were used in developing the above model can be generalized to confidence rating SD experiments. If there are n response categories, a Markov model with n states will be used. A biased subject will alter his (n-1) LRCL's in a way that is similar to the case of a single threshold as described above. For a particular type of decision maker, all LRCL's will be moved in the same direction for a given previous decision. However, the magnitude of this movement may differ for different LRCL's.

## 5.5 Experimental Results

As mentioned in section 5.3, the preliminary results of the experiment showed that a difference in performance between sequential and random presentation did exist for both subjects. The first question to be raised concerning the validity of the model is whether our assumption that the detectability d' was almost the same for both present- ations is supported by the experimental results. Figures 5.11 and 5.12 show the values of d' as a function of the decision interval index i for both subjects. As in the previous experiment, the linear relation seems to hold as follows:

$$u = 1/d' = a + bi \qquad (5.42)$$

The least square estimates for the parameters a and b (using linear regression) are shown in Table 5-1. This table also includes the value for the correlation coefficient $\rho$ which indicates the "goodness" of the linear relationship:

| SUBJECT | DATA | a | b | $\rho$ |
|---------|------|------|------|------|
| A.C. | Seq. | 0.44 | 0.29 | 0.98 |
| | Ran. | 0.36 | 0.35 | 0.91 |
| A.T. | Seq. | 0.24 | 0.13 | 0.98 |
| | Ran. | 0.22 | 0.32 | 0.94 |

Table 5-1  Summary of Linear Regression Results
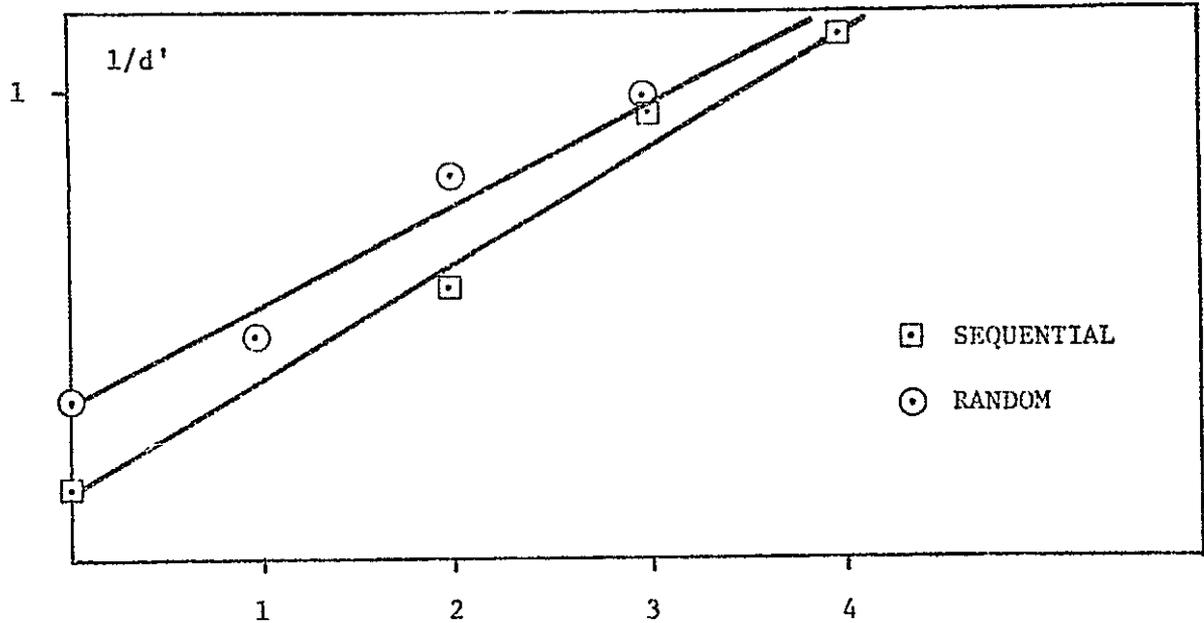
FIGURE 5.11    UNCERTAINTY AS A FUNCTION OF INTERVAL INDEX FOR
RANDOM AND SEQUENTIAL PRESENTATIONS (SUBJECT A.T)
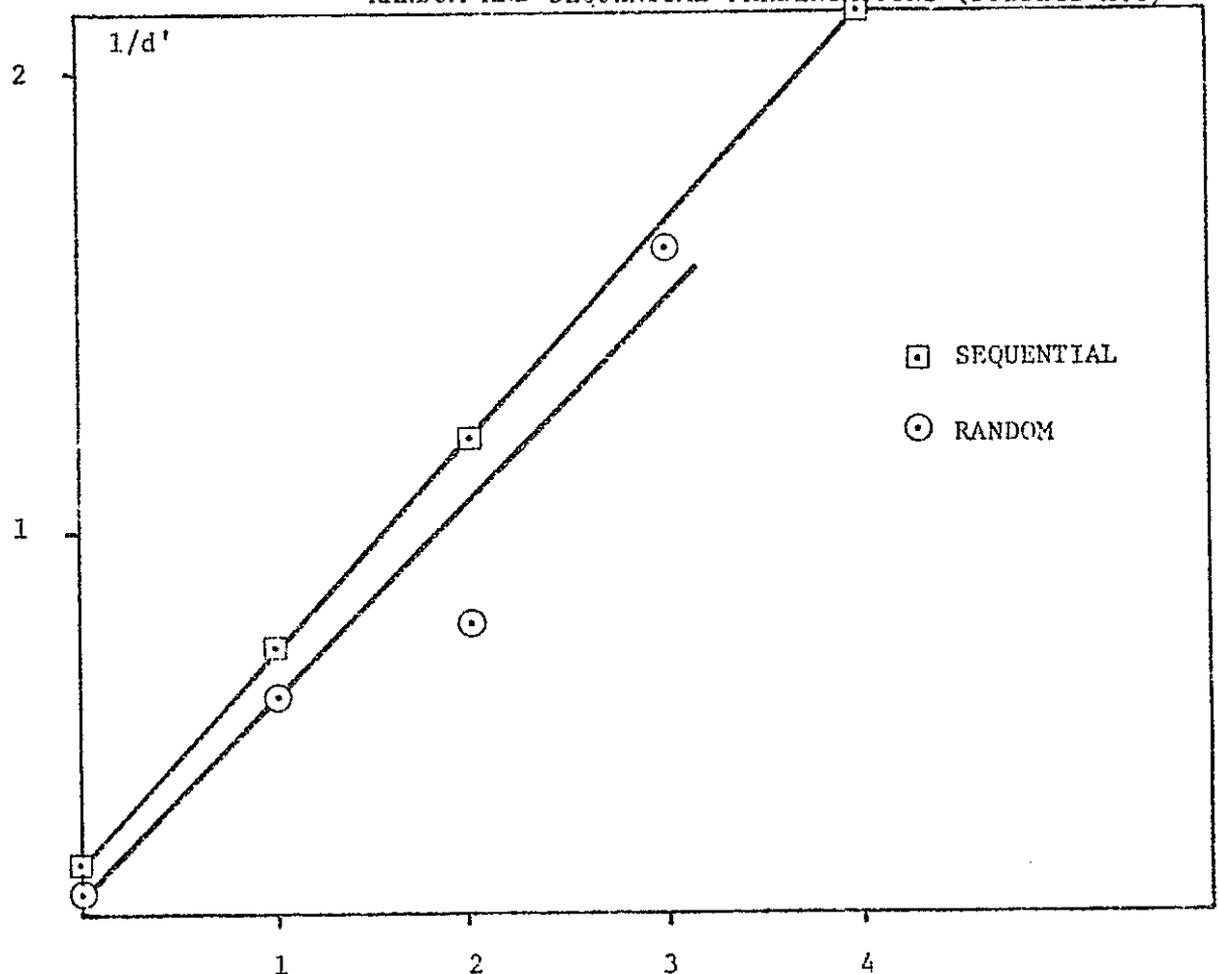
FIGURE 5.12    UNCERTAINTY AS A FUNCTION OF INTERVAL INDEX FOR
RANDOM AND SEQUENTIAL PRESENTATIONS (SUBJECT A.C)

Although the regression lines for random and sequential presentations indicate that d' was almost the same for both presentations, a statistical test should be used. As mentioned in section 4.4 the use of our data processing program does not provide information about the variance of the computed parameters d' and $\beta$. Therefore, we will base the analysis on a method that was suggested by Gourevitch and Galanter (1967) which provides good approximations (Marasculo, 1970). For this analysis the data should be regrouped to a form with one LRCL. This can easily be done by pooling the "think" and "sure" decisions for both A and B. Moreover, since the analysis of the decision strategy is easier for a two state Markov model, this regrouping will be useful for later discussions. After the regrouping, the experimental results are defined by only two parameters:

$$P_H = P(A_0/S_0) \qquad\qquad P_{FA} = P(A_0/S_1)$$

An approximation for the mean value of d' is given by (Gourevitch et al, 1967)

$$\overline{d}' = \Phi^{-1}[1 - P_H] - \Phi^{-1}[1 - P_{FA}] \tag{5.43}$$

where $\Phi$ is defined by equation (2.24). It is further assumed that d' is a Gaussian random variable with the above mean (5.43) and variance:

$$\sigma^2_{d'} = \frac{P_H(1-P_H)}{N_0[ORD(1-P_H)]^2} + \frac{P_{FA}(1-P_{FA})}{N_1[ORD(1-P_{FA})]^2} \tag{5.44}$$

where $N_0$ and $N_1$ are the total number of presentations of $S_0$ and $S_1$ respectively, and

$$ORD(\alpha) = \frac{1}{\sqrt{2\pi}} \exp[-\frac{1}{2}\alpha^2] \tag{5.45}$$

Let $d'_R$ denote the detectability for random presentations and $d'_S$ the detectability for sequential presentation. We wish to test the null hypothesis

$$H_0: \quad d'_R = d'_S$$

Therefore, let us define:

$$Z = \frac{\overline{d}'_R - \overline{d}'_S}{\sigma^2_{d'_S} + \sigma^2_{d'_R}} \tag{5.46}$$

and the null hypothesis can be rejected with a confidence level of 95% if

$$|Z| > 1.96$$

Table 5-2 presents the values of $d'_R$, $d'_S$, $\sigma^2_{d'_R}$ and $\sigma^2_{d'_S}$ as a function of the decision interval index for both subjects (for $N_0 = N_1 = 60$). As can be seen from the table, only two out of the ten Z values shown are greater than 1.96, while all the others are considerably less. Therefore, the hypothesis that $d'_R = d'_S$ cannot be rejected even if a larger confidence level is used.

| SUBJECT | SNR LEVEL | 1 | 2 | 3 | 4 | 5 |
|---------|-----------|-----|-----|-----|-----|-----|
| A.C. | $\hat{d}'_R$ | 0.890 | 0.840 | 1.310 | 1.870 | 5.960 |
| | $\hat{d}'_S$ | 0.660 | 0.650 | 0.910 | 1.680 | 3.960 |
| | $SE^2_R$ | 0.060 | 0.056 | 0.063 | 0.075 | $10^3$ |
| | $SE^2_S$ | 0.035 | 0.035 | 0.036 | 0.055 | $10^3$ |
| | z | 0.746 | 0.631 | 1.260 | 0.520 | $\simeq 0$ |
| A.T. | $\hat{d}'_R$ | 0.060 | 1.230 | 1.140 | 2.560 | 3.800 |
| | $\hat{d}'_S$ | 0.900 | 1.030 | 1.840 | 2.750 | 7.800 |
| | $SE^2_R$ | 0.055 | 0.071 | 0.059 | 0.092 | $10^3$ |
| | $SE^2_S$ | 0.042 | 0.040 | 0.046 | 0.068 | $10^3$ |
| | z | 2.700 | 0.598 | 2.140 | 0.475 | $\simeq 0$ |

TABLE 5.2   Values found for Significance Test of

$$d'_R = d'_S$$

Now that we have shown that the change in performance is essentially not due to a change in the detectability, we can test our model for the decision strategy. As stated before, the two state Markov model will be used. The one step probability transition matrices for $i = 2,...,5$ for both subjects are shown in Table 5-3. In this table both $\underline{p}^C(i,i-1)$ and $\underline{p}^{NC}(i,i-1)$ are shown. These matrices clearly show the tendency of the two subjects to repeat their previous decisions even though they were wrong. Both subjects show a biased strategy which we refered to as RS; however, subject A.C. was more biased than subject A.T. It can also be seen that the bias effect decreases when the SNR increases, so the Markov process is non-stationary. This dependency of $P^C_{00}$, $P^{NC}_{00}$, $P^C_{11}$ and $P^{NC}_{11}$ on the decision interval index $i$ is shown in Figure 5.13. It should be noted that these four probabilities completely define the RS bias of the subject.

The DR curve in the sequential presentation task in the $P_H-P_{FA}$ plane are shown for both subjects in Figure 5.14 and 5.15. Since there are only two states, there is only one pair of DR curves for each subject. Although each one of the DR curves should be shown in a separate plane ($P^0_H-P^0_{FA}$ and $P^1_H-P^1_{FA}$) the same drawing was used with one axis used for $P^0_H$ and $P^1_H$ and the other for $P^0_{FA}$ and $P^1_{FA}$. These two DR curves are referred to in the drawing as 0 (for a previous decision of $A_0$) and 1 (for a previous decision $A_1$). The curves agree well with the theoretical curves predicted by the model for an RS subject. In order to see the effect of the correlation, the same DR curves for the random presentation task

| Subject | A.T | | A.C | |
|---|---|---|---|---|
| n | $\underline{P}^C(n,n-1)$ | $P^{nc}(n,n-1)$ | $\underline{P}^{\,C}(n,n-1)$ | $\underline{P}^{nc}(n,n-1)$ |
| 2 | $\begin{bmatrix} 1 & 0 \\ 0.03 & 0.97 \end{bmatrix}$ | $\begin{bmatrix} 0.94 & 0.06 \\ 0.11 & 0.89 \end{bmatrix}$ | $\begin{bmatrix} 0.93 & 0.03 \\ 0.08 & 0.92 \end{bmatrix}$ | $\begin{bmatrix} 0.86 & 0.14 \\ 0.14 & 0.86 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 0.97 & 0.03 \\ 0.03 & 0.97 \end{bmatrix}$ | $\begin{bmatrix} 0.67 & 0.33 \\ 0.52 & 0.48 \end{bmatrix}$ | $\begin{bmatrix} 0.97 & 0.03 \\ 0.08 & 0.92 \end{bmatrix}$ | $\begin{bmatrix} 0.84 & 0.16 \\ 0.17 & 0.83 \end{bmatrix}$ |
| 4 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0.72 & 0.29 \\ 0.17 & 0.83 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0.74 & 0.26 \\ 0.11 & 0.89 \end{bmatrix}$ |
| 5 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0.04 & 0.96 \\ 0.92 & 0.08 \end{bmatrix}$ |

TABLE 5-3    Two Dimensional Probability Transition Matrices
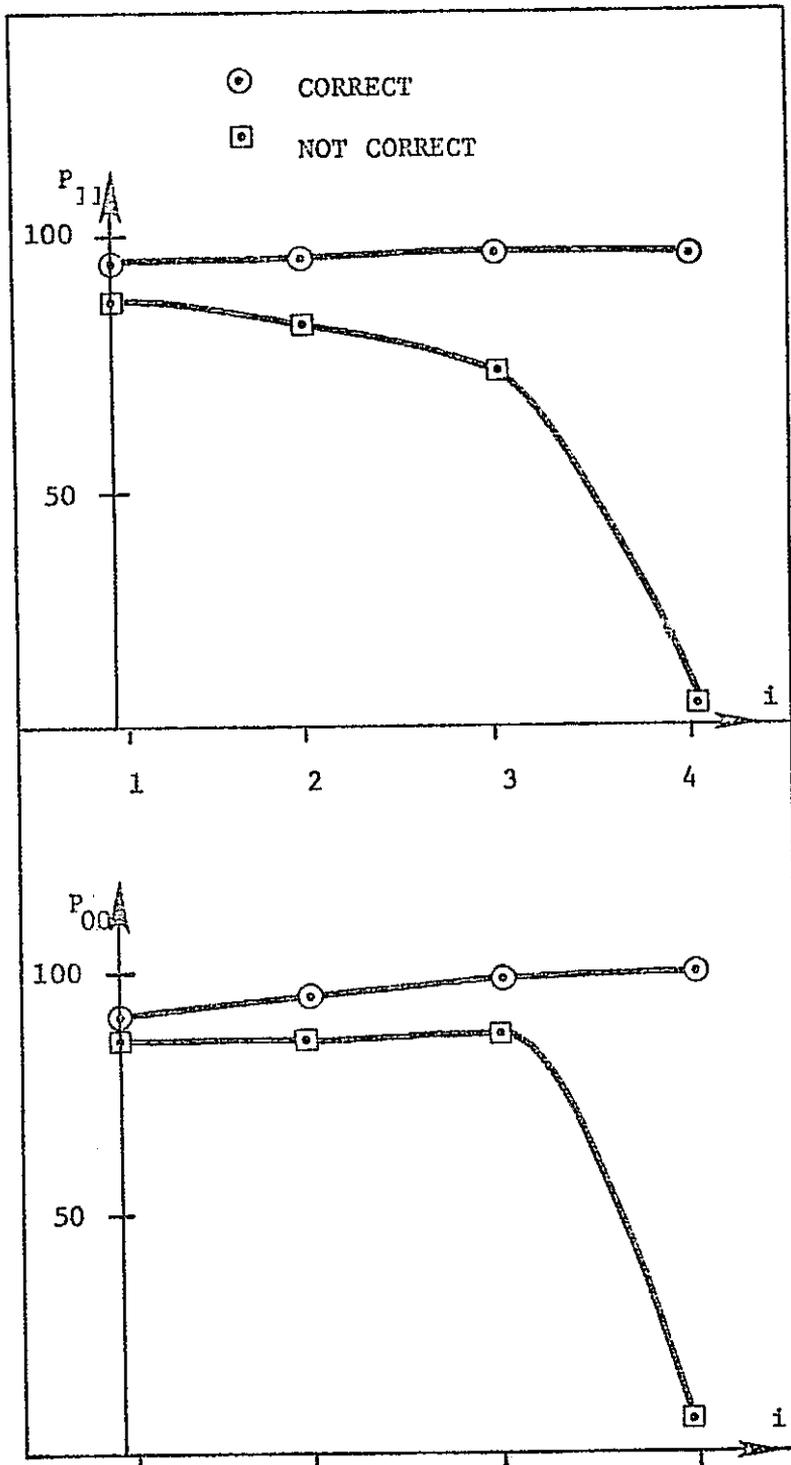
FIGURE 5.13   TRANSITION PROBABILITIES OF THE MARKOV

PROCESS AS A FUNCTION OF THE INTERVAL

INDEX (SUBJECT A.C)
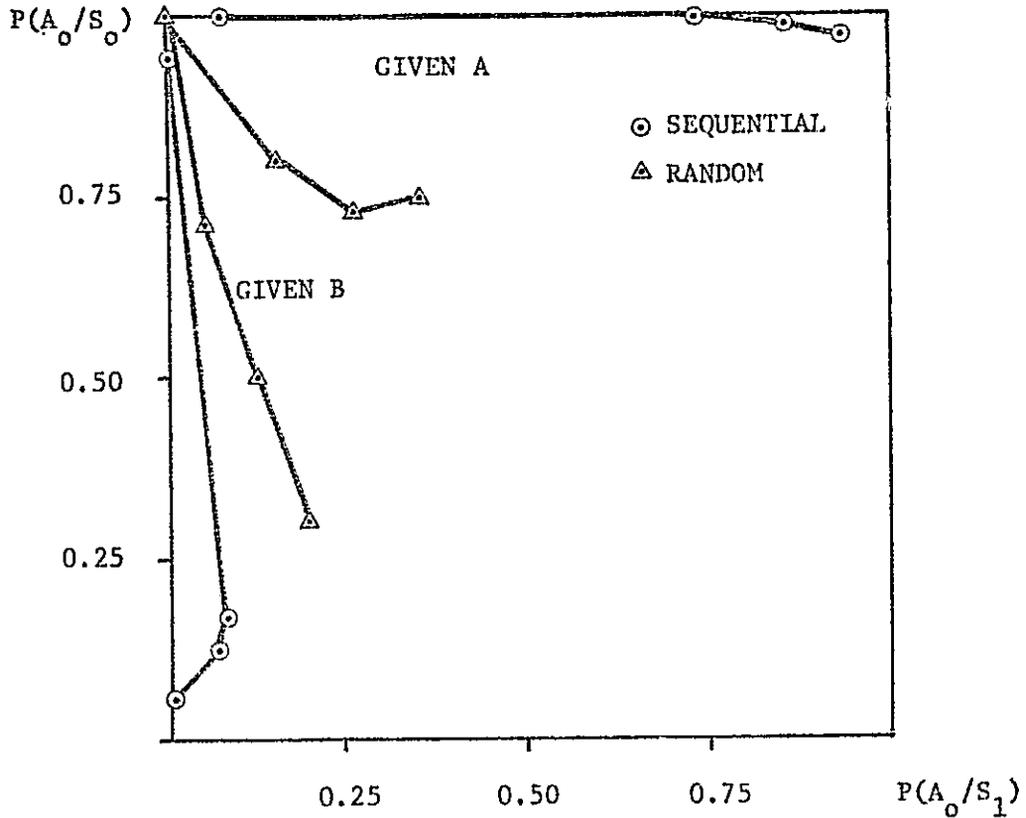
FIGURE 5.14    CONDITIONAL DR CURVES FOR SEQUENTIAL AND
                RANDOM PRESENTATIONS, SUBJECT A.C
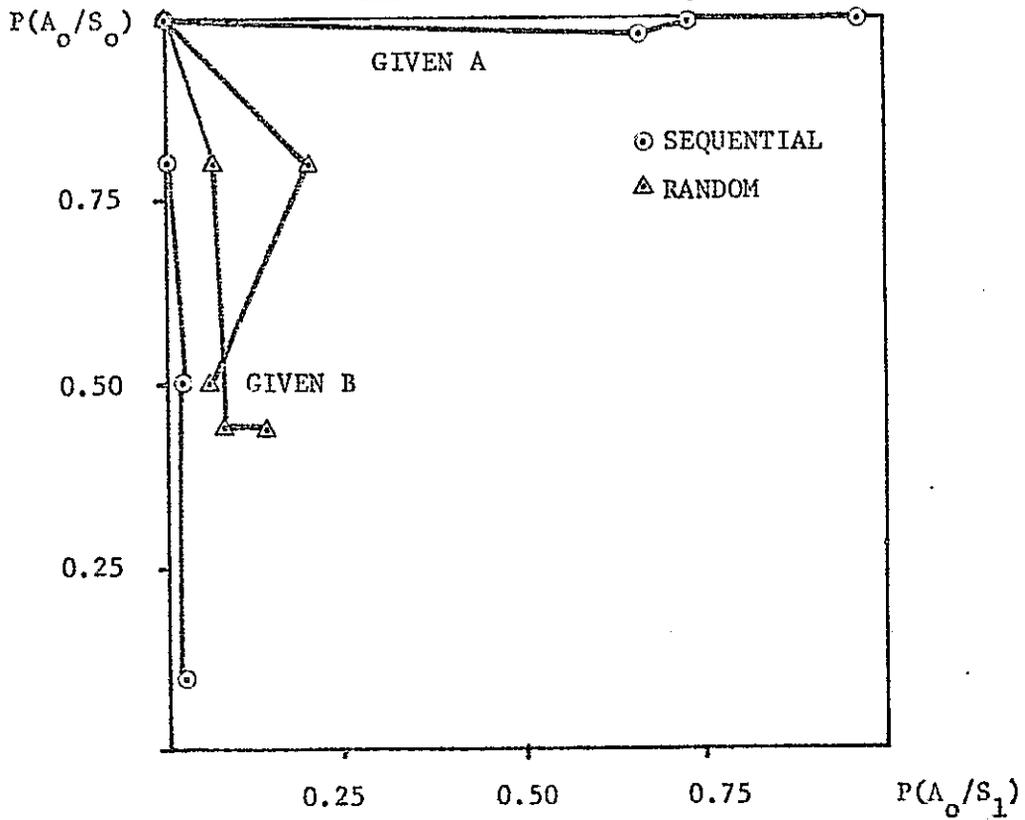


FIGURE 5.15    CONDITIONAL DR CURVES FOR SEQUENTIAL AND
                RANDOM PRESENTATIONS, SUBJECT A.T

are also shown in figures 5.14 and 5.15. These curves show that the strategy of the same subjects, when the correlation was eliminated, was unbiased.

Since in our experiment the subject had four response categories, a four state Markov model can also be applied. However the larger number of states has some disadvantages:

1.  The number of decisions that were collected per state will decrease by a factor of two. Since we used a small number of decisions, the data collected might not be sufficient for statistical analysis.

2.  Since there are two states of the world and four Markov states, there are three different LRCLs. One LRCL is actually separating the states of the world, while the other two represent different confidence levels for each of the two states of the world.

In spite of these disadvantages, the four state analysis was carried out using the following notation:

$R_1$    The decision is Sure $S_0$

$R_2$    The decision is Think $S_0$

$R_3$    The decision is Think $S_1$

$R_4$    The decision is Sure $S_1$

$\beta_1$      Threshold between $R_1$ and $R_2$

$\beta_2$      Threshold between $R_2$ and $R_3$

$\beta_3$      Threshold between $R_3$ and $R_4$

Tables 5-4 and 5-5 show the one step probability transition matrices for both subjects for $i = 2,\ldots,5$.

These tables show again the RS bias of the two subjects when they were in one of the two sure states $R_1$ or $R_4$. For the two think states $R_2$ and $R_3$, the tendency to repeat the previous decision was weaker. When they did not repeat their think decision, they changed it to the sure decision for the same states of the world even if this state of the world was not correct and even though the SNR had increased. This can be explained as a result of the subject's knowledge that a think decision should be followed by a sure decision, otherwise they are admitting an error.

Since this is a four state model, there are three LRCLs and six DR curves for each subject (two DR curves for each LRCL). Figures 5.16 and 5.17 show the two DR curves that are related to $\beta_2$ for each of the subjects. In these curves, the same RS bias that was implied by the two state model is exhibited. The DR curves given $R_1$ and $R_2$ are closer to the line $P^0_H = 1$, while the DR curves given $R_3$ or $R_4$ are close to the line $P^1_{FA} = 0$.

In Figures 5.18 and 5.19 the DR curves that are related to the LRCL $\beta_1$ are shown. The DR curves given $R_1$, $R_4$ and $R_3$

| n | $\underline{P}^C(n,n-1)$ | | | | $\underline{P}^{NC}(n,n-1)$ | | | |
|---|---|---|---|---|---|---|---|---|
| n=2 | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.175 & 0.8 & 0.025 & 0 \\ 0 & 0.083 & 0.652 & 0.265 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | | | | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.223 & 0.629 & 0.111 & 0.037 \\ 0 & 0.166 & 0.727 & 0.112 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | | | |
| n=3 | $\begin{bmatrix} 0.875 & 0.125 & 0 & 0 \\ 0.725 & 0.25 & 0.025 & 0 \\ 0 & 0.1 & 0.12 & 0.78 \\ 0 & 0.04 & 0 & 0.96 \end{bmatrix}$ | | | | $\begin{bmatrix} 0.875 & 0 & 0.143 & 0 \\ 0.434 & 0.392 & 0.174 & 0 \\ 0 & 0.23 & 0.192 & 0.577 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | | | |
| n=4 | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.882 & 0.118 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | | | | $\begin{bmatrix} 0.75 & 0 & 0.125 & 0.125 \\ 0.601 & 0.133 & 0.133 & 0.133 \\ 0.17 & 0.17 & 0.33 & 0.33 \\ 0 & 0.044 & 0.174 & 0.782 \end{bmatrix}$ | | | |
| n=5 | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | | | | $\begin{bmatrix} 0 & 0.046 & 0.136 & 0.818 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.88 & 0 & 0 & 0.12 \end{bmatrix}$ | | | |

TABLE 5-4   Probability Transition Matrices for Subject A.C

| n | $\underline{P}^C(n,n-1)$ | | | | $P^{NC}(n,n-1)$ | | | |
|---|---|---|---|---|---|---|---|---|
| n=2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.364 | 0.636 | 0 | 0 | 0.059 | 0.882 | 0.059 | 0 |
| | 0.026 | 0 | 0.781 | 0.253 | 0.02 | 0.078 | 0.902 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| n=3 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| | 0.8 | 0.16 | 0 | 0.04 | 0.357 | 0.215 | 0 | 0.428 |
| | 0.017 | 0.017 | 0.078 | 0.896 | 0.288 | 0.244 | 0.156 | 0.342 |
| | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| n=4 | 1 | 0 | 0 | 0 | 0.9 | 0 | 0 | 0.1 |
| | 0.615 | 0.385 | 0 | 0 | 0.25 | 0 | 0 | 0.75 |
| | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 1 | 0.5 | 0.25 | 0 | 0.25 |
| n=5 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| | 0.4 | 0.6 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 1 | 0.5 | 0.5 | 0 | 0 |

TABLE 5-5  Probability Transition Matrices for Subject A.T
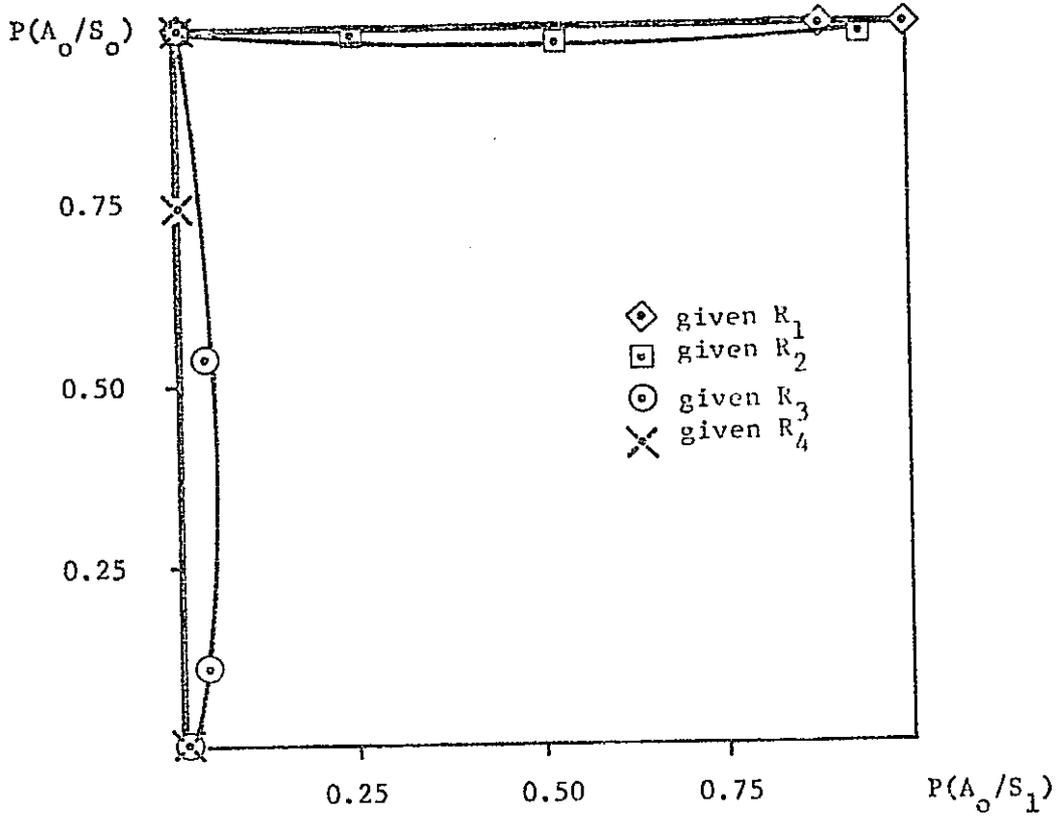
FIGURE 5.16    DR CURVES RELATED TO LRCL $\beta_2$    (SUBJECT A.T)
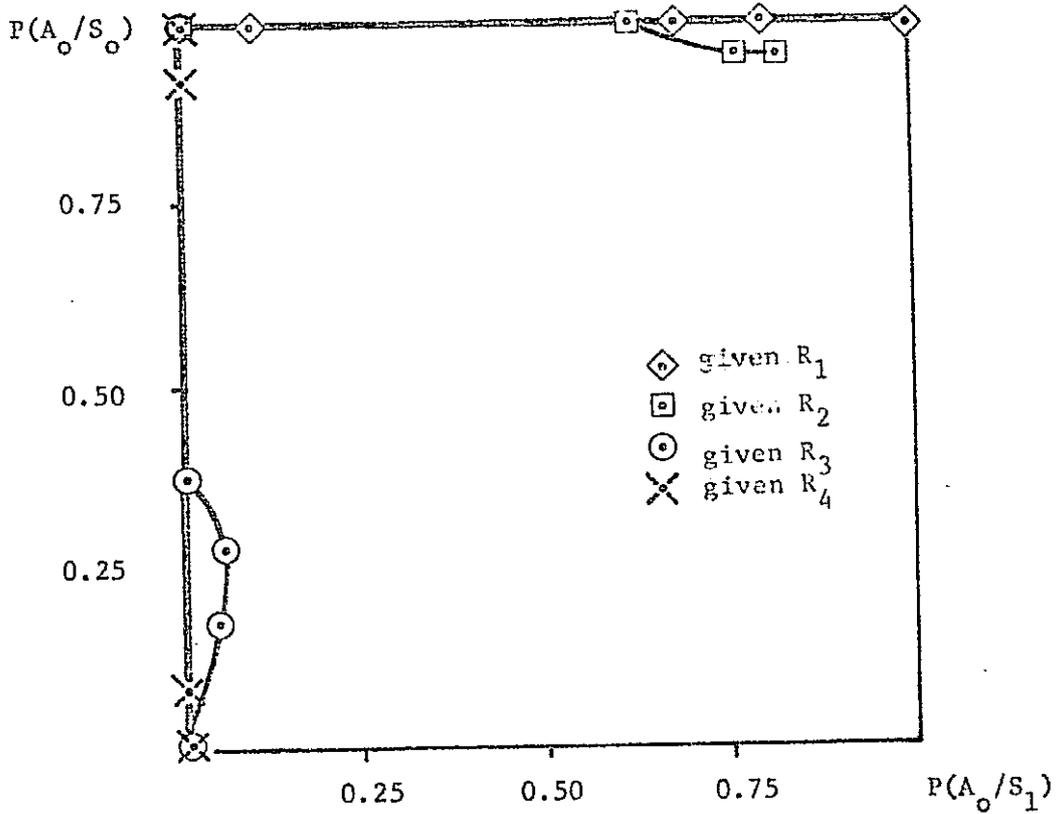


FIGURE 5.17    DR CURVES RELATED TO LRCL $\beta_2$    (SUBJECT A.C)
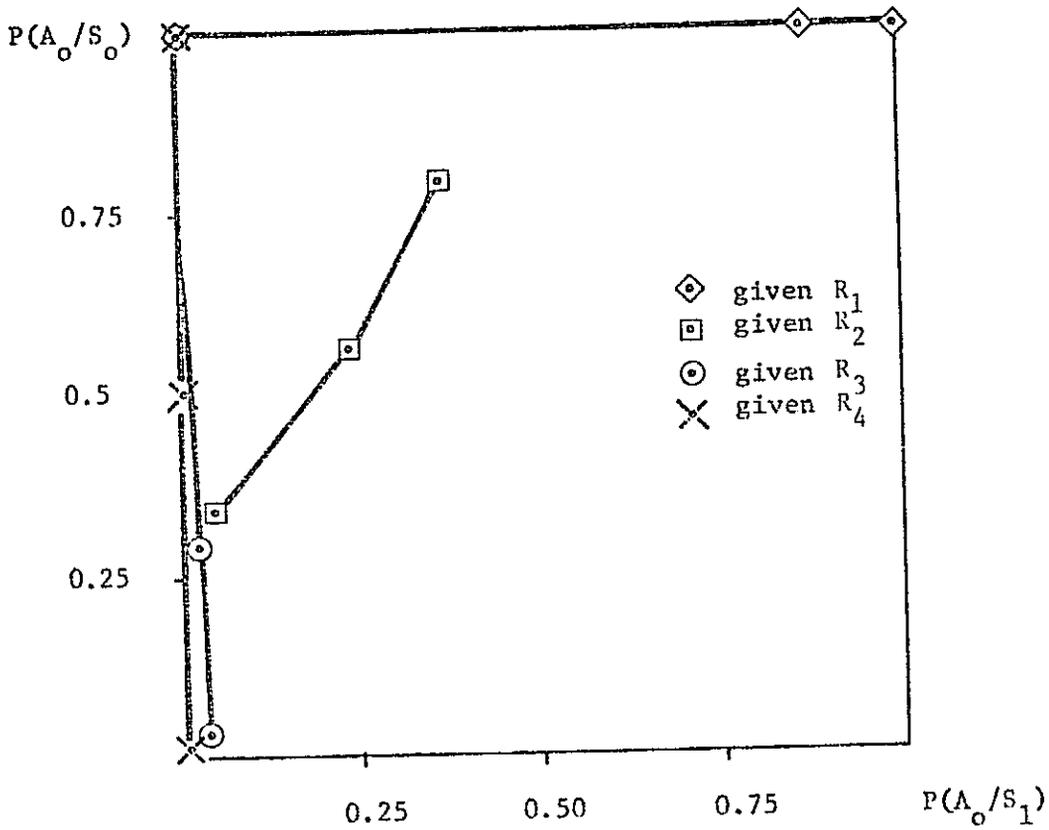
FIGURE 5.18   DR CURVES RELATED TO LRCL $\beta_1$  (SUBJECT A.T)
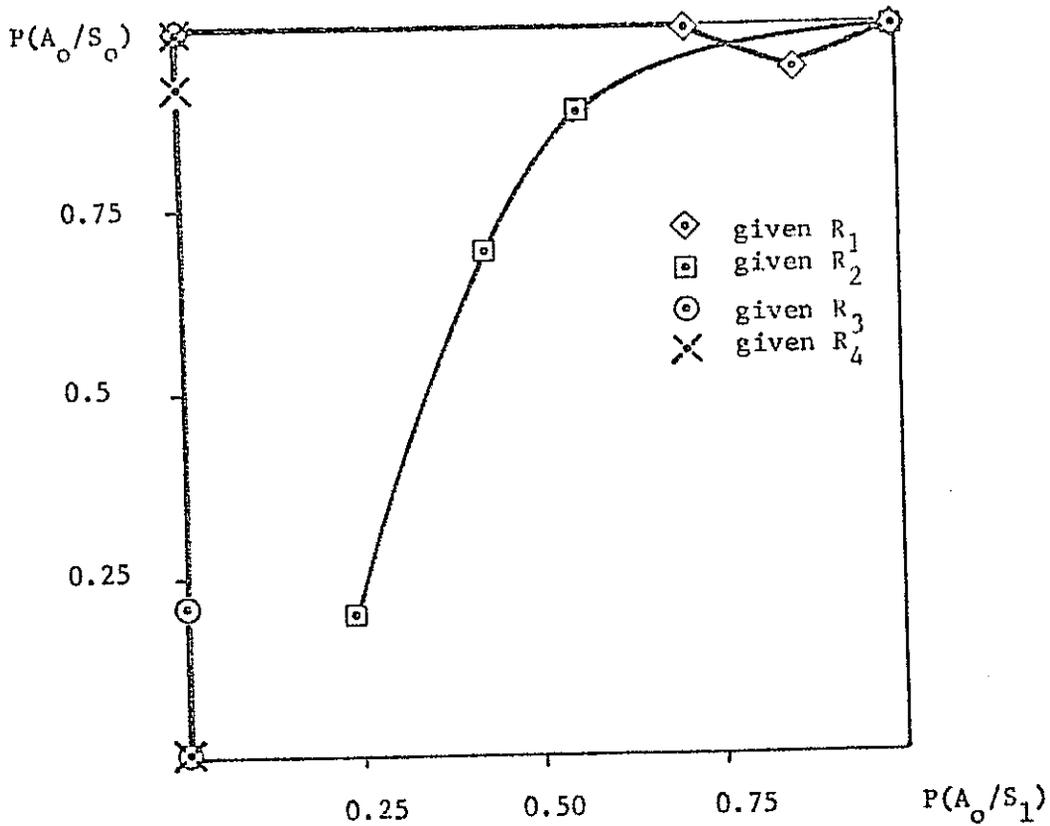


FIGURE 5.19   DR CURVES RELATED TO LRCL $\beta_1$  (SUBJECT A.C)

are similar to the ones for $\beta_2$. However, the DR curve given $R_2$ shows different bias. This bias can be explained by recalling that $\beta_1$ represents a threshold between two confidence levels of the subjects and not between the two states of the world, so the bias is expressed by moving $\beta_2$ rather than $\beta_1$. A symmetric phenomenon exists for DR curves related to $\beta_3$ which are s..own in Figures 5.20 and 5.21. The DR curves given $R_1$, $R_2$, and $R_4$ are similar to the ones for $\beta_1$, while the DR curve given $R_3$ shows the same bias as in $R_2$ for $\beta_1$.

The location of these six DR's show that all of them are shifted according to the bias of the decision maker, but the magnitude of the shift is different for different LRCLs.
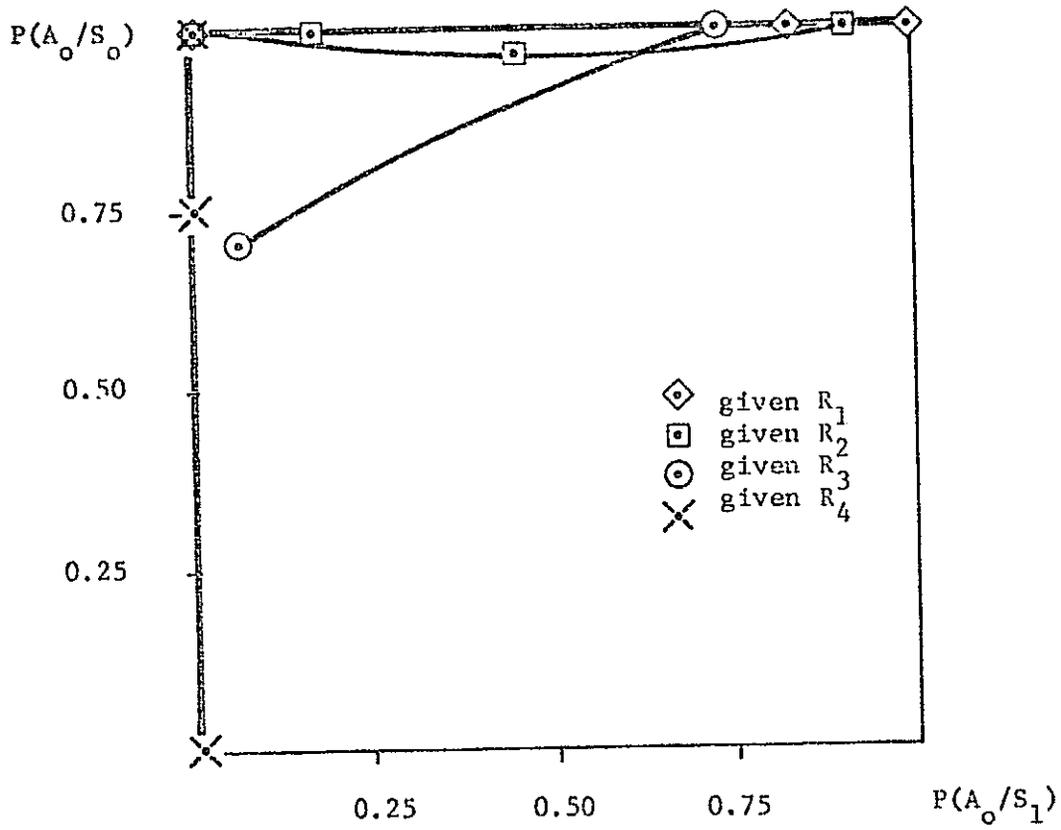
FIGURE 5.20   DR CURVES RELATED TO LRCL $\beta_3$   (SUBJECT A.T)
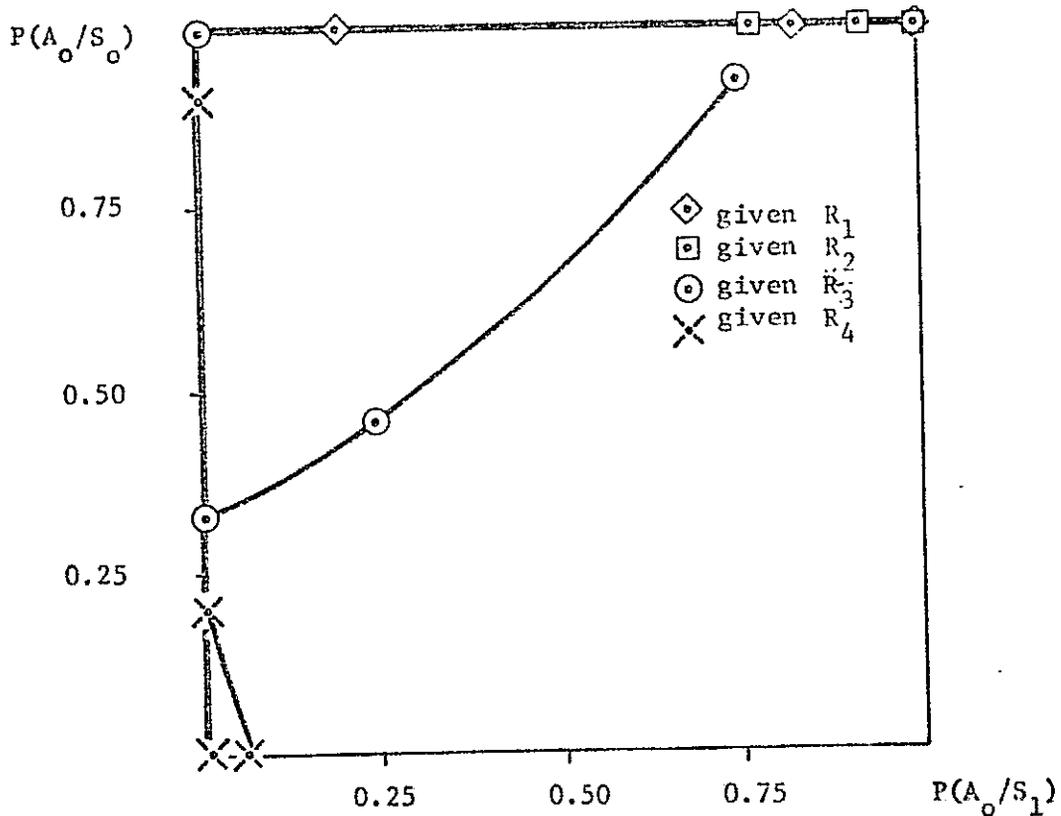


FIGURE 5.21   DR CURVES RELATED TO LRCL $\beta_3$   (SUBJECT A.C)

CHAPTER VI

DETECTION OF A CHANGE IN RANDOM PROCESSES

6.1  General Discussion

In the previous chapters  we dealt with detection tasks

which required a decision after each observation.  However,

in some tasks the observation rate is too high (the observation

might be continuous at the extreme) so that the decision maker

is allowed to delay his decision and take more observations

until he collects enough information to make a decision.  In

binary decision tasks of this form, the decision maker is

told to use two CLs as was the case in the experiments that

were described in Chapter 4;  however, instead of giving

"I do not know" as a decision, he takes another observation.

A typical case of a deferred decision situation is the

task of failure detection.  In such cases the observation

gives the subject information about the state of some oper-

ating system.  The decision maker must decide on the basis

of this observation whether the system is operating in its

normal mode $H_0$ or in the failure mode $H_1$.  The subject is

free to take more than one observation before making a

decision, but he is asked to minimize the time between

the occurance of the failure and its detection.

Since the observation under both modes of operation

is a stochastic process, it is assumed that the detection

process consists of two steps.  In the first step the

subject tries to estimate the statistical parameters of the observation, and then on the basis of his estimates, he makes the decision.

The next two sections in this chapter include a short discussion of linear estimation theory and sequential analysis which provide the theoretical basis for our model. The model itself is described in sections 6.4 and 6.5. The last two sections deal with the experiments that were run to verify the validity of the model.

## 6.2  Linear Estimation

Estimation theory deals with the problem of obtaining the best estimate, in some sense, of a process which cannot be exactly measured because of the associated measurement noise (Lee, 1964).  If the statistics of all noise sources are completely known, the problem is sometimes referred to as Bayesian estimation (Schweppe, 1973).  Linear estimation is a special case in which the estimates are constrained to be linear functions of the measurements.  The most common criterion for optimality in the linear Bayesian problem (LBP) is the minimization of the mean square error.

There have been two approaches to the LBP which lead to the same solution, and the corresponding numerical effort was basically the same (Kailath, 1974).  The first approach is the so called Wiener filtering theory (Wiener, 1949) in which the information about the signal to be estimated is given by its covariance matrix.  The second and more recent approach is that of Kalman filtering theory (Kalman, 1960) in which the signal is represented as the output of a dynamical system which is driven by a white process.  Because of the  identical results of the two methods and the equivalence of their numerical difficulty, the choice between the two is usually based on the way in which the problem at hand is posed.  Since in our case the dynamical model or the "shaping filter" of the signal is known, it would be practical to use

the second method, namely the Kalman filter (Kalman, Bucy, 1961).

Therefore let us assume that the signal to be estimated $\underline{z}(t)$ is given by

$$\underline{z}(t) = H(t)\underline{x}(t) \tag{6.1}$$

where $\underline{x}(t)$ is the state vector of the shaping filter given by

$$\dot{\underline{x}}(t) = F(t)\underline{x}(t) + G(t)\underline{u}(t) \tag{6.2}$$

Here $\underline{u}(t)$ is a zero mean white process with covariance matrix

$$E[\underline{u}(t)\underline{u}^T(s)] = Q(t)\delta(t-s) \tag{6.3}$$

and $H(t)$, $F(t)$, $G(t)$, and $Q(t)$ are known matrices. Also the first and second order statistics of $\underline{x}_0$ are given by

$$E[\underline{x}_0] = 0 \qquad\qquad E[\underline{x}_0\underline{x}_0^T] = \Pi_0$$

and $\underline{x}_0$ and $\underline{u}(t)$ are uncorrelated:

$$E[\underline{x}_0\underline{u}(t)] = 0$$

The solution to (6.2) is given by (Deyst, 1972):

$$\underline{x}(t) = \phi(t,t_0)\underline{x}_0 + \int_{t_0}^{t} \phi(t,\tau)G(\tau)\underline{u}(\tau)d\tau \tag{6.4}$$

where $\phi(t,t_0)$ is the state transition matrix which satisfies

$$\dot{\phi}(t,t_0) = F(t)\phi(t,t_0) \qquad \phi(t,t_0) = I$$

The observation vector $\underline{y}(t)$ is expressed as

$$\underline{y}(t) = H(t)\underline{x}(t) + \underline{v}(t) \tag{6.5}$$

where $\underline{v}(t)$ is the observation noise which is assumed to be a zero mean white process with covariance matrix

$$E[\underline{v}(t)\underline{v}^T(s)] = R(t)\delta(t-s) \tag{6.6}$$

and it is also uncorrelated to the process noise $\underline{u}(t)$ and to $\underline{x}_0$

$$E[\underline{v}(t)\underline{u}^T(s)] = 0 \qquad E[\underline{v}(t)\underline{x}_0^T] = 0$$

The optimal estimate of the observation $\hat{\underline{z}}(t)$ is then (Deyst, 1972)

$$\hat{\underline{z}}(t) = H(t)\hat{\underline{x}}(t) \tag{6.7}$$

where $\hat{x}(t)$ is the state estimate given by the differential equation

$$\dot{\hat{\underline{x}}}(t) = F(t)\hat{\underline{x}}(t) + K(t)[\underline{y}(t) - H(t)\hat{\underline{x}}(t)] \tag{6.8}$$

The term in the square brackets is referred to as the measurement residual, and $K(t)$ is the Kalman gain:

$$K(t) = P(t)H(t)R^{-1}(t) \tag{6.9}$$

$P(t)$ is the error covariance matrix of the state, namely,

$$P(t) \triangleq E[(\hat{\underline{x}}(t) - \underline{x}(t))(\hat{\underline{x}}(t) - \underline{x}(t))^T] \tag{6.10}$$

which is the solution to the Riccati equation:

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t)$$
$$- P(t)H^T(t)R^{-1}(t)H(t)P(t) \tag{6.11}$$

with $\quad P(t_0) = \Pi_0$

Equations (6.7) through (6.11) are the equations of the
continuous Kalman filter and their recursive characteristics
makes them easily adaptable to solution on a digital
computer.

In most practical cases the observation are taken
in a discrete manner rather than continuously. Also, because
of the use of a digital computer, even the state equation
must be transformed into its discrete form. Therefore the
optimal filter is the solution to a set of difference
equations rather than differential equations. The trans-
formation must be carried out carefully because of the
stochastic nature of the problem. More details about the
practical implementation of the filter are given in
Appendix B.

A special case of equations (6.8) through (6.11) is
the time invarient case, for which the matrices H,F,G,Q,
and R are time invariant. The filter will still be time
varying because of the variation of the gain K(t) through
the covariance P(t). However, if the system given by
equation (6.1) and (6.2) is completely controllable and
observable and if Q > 0 and R > 0, the covariance matrix
will reach a steady state value $P_{SS}$ (Schweppe, 1973).
This value can be evaluated as the positive definite
solution to (6.11) under the stationary assumption:

$$\dot{P}(t) = 0 \qquad\qquad (6.12)$$

The filter then becomes a time invariant system. It is interesting to note that the steady state requirements do not include the stability of the original system (6.2).

The quantity in the square brackets in equation (6.8)

$$\underline{\varepsilon}(t) = \underline{y}(t) - H(t)\hat{\underline{x}}(t) \tag{6.13}$$

has an important role in filtering theory. We referred to it as the residual, but it is also called the innovation, the new information, or the measurement error. It can be shown (Deyst, 1972) that this residual is orthogonal to all the past measurements. This means that the filter gleans all the new information out of each measurement. Also, it has been shown that the residual is a zero mean white process (Kailath, 1970) with covariance

$$E\{\underline{\varepsilon}(t)\underline{\varepsilon}^T(t)\} = H(t)P(t)H^T(t) + R(t) \tag{6.14}$$

This property of the residual is used for evaluation of the system model or the implemented filter algorithm.

Up to this point nothing has been said about the distribution functions of the stochastic processes involved; only the first and second order moments were used. This is a result of the mean square error criterion and the linearity constraint on the filter. Under these conditions only first and second order statistics are needed to obtain the best linear filter (Vander Velde, 1972). If, however, all the white processes are assumed to be Gaussian, then, because Gaussian random variables are invariant under linear trans-

formations, $\hat{\underline{x}}(t)$, $\hat{\underline{z}}(t)$, and $\underline{\varepsilon}(t)$ will also be Gaussian.

Furthermore, it is possible to show that $\hat{x}(t)$ given by (6.8)

is also the conditional expectation estimate (Schweppe, 1973).

Therefore, for Gaussian processes the Kalman filter is not

only the best linear filter but also the best possible filter.

Since in our application the Gaussian assumption is used,

we will refer from here only to the Gaussian case.

As stated before, if the system model is correct, the

residual $\underline{\varepsilon}(t)$ will be a zero mean white Gaussian process. If

the system is time invariant and controllable and observable,

the filter will achieve steady state and $\underline{\varepsilon}(t)$ becomes a

stationary Gaussian process with covariance:

$$E\{\underline{\varepsilon}(t)\underline{\varepsilon}^T(t)\} = HP_{SS}H^T + R \tag{6.15}$$

Let us assume now that a failure has occurred in the system

so that the measurement $\underline{y}(t)$ is now

$$\underline{y}(t) = H\underline{x}(t) + \underline{v}(t) + \underline{m} \tag{6.16}$$

rather than the value given by (6.5). Furthermore, let us

assume that the additional signal $\underline{m}$ in (6.16) is a deterministic

constant. Since $\underline{m}$ is deterministic it will not affect the

covariance of either $\hat{\underline{x}}(t)$ or $\underline{\varepsilon}(t)$ but will certainly alter

their means. Because the filter is linear, the superposition

property can be used to find the change in the means of $\hat{\underline{x}}(t)$

and $\underline{\varepsilon}(t)$ by computing the response of the filter to a step

function with magnitude $\underline{m}$. As the filter has already reached its steady state, the Laplace transform can be used. Transforming equation (6.8) we get

$$S\hat{\underline{X}}(S) - \hat{\underline{X}}_{m_0} = [F - KH]\hat{\underline{X}}_m(S) + K\underline{m}(S)$$

where $\hat{X}_m$ is the Laplace transform of the state estimate due to $\underline{m}$ alone. Now let

$$\underline{X}_{m_0} = 0 \qquad \underline{m}(S) = \frac{1}{S}\underline{m}$$

then

$$\hat{\underline{X}}_m(S) = [SI - F + KH]^{-1}K(1/S)\underline{m} \tag{6.17}$$

and the residual due to $\underline{m}$ alone is

$$\underline{\varepsilon}_m(S) = (1/S)\underline{m} - H\hat{\underline{X}}(S) = \{I - H[SI - F + KH]^{-1}K\}(1/S)\underline{m} \tag{6.18}$$

In particular, the steady state value of the $_m$ is given by

$$\underline{\varepsilon}_{m_{SS}} = \lim_{S \to 0} \{I - H[SI - F + KH]^{-1}K\}\underline{m} \tag{6.19}$$

From equation (6.18) it is clear that, after the filter has reached steady state with respect to the failure, the new residual will be a white Gaussian process with mean $\varepsilon_{m_{SS}}$ and covariance given by (6.15)

The detection of the failure can be accomplished on the basis of the change in the residual mean. The problem is

that of discriminating between two Gaussian processes with equal variances but unequal means. A possible method to perform such discrimination is described in the next section.

## 6.3 Sequential Analysis

Sequential analysis deals with those cases of hypothesis testing for which the sample size is not fixed, namely, the decision maker is free to take as many observations as he wants before making a decision of some prescribed confidence. The mathematical theory of the optimal strategy in such situations is usually associated with the name of Abraham Wald (1949). The use of this method in the analysis of signal detection experiments was suggested by Birdsall et al (1965), and later by Phatak et al (1972) and Sheridan and Ferrell (1974).

For simplicity, but without loss of generality, let us assume that the decision task is to test between two hypotheses $H_0$ and $H_1$. Two further assumptions will be made:

1.  The hypotheses to be tested are simple hypotheses. This means that under either hypothesis the density function of the observed random variable is completely known.

2.  The observations that are made are independent.

Under those assumptions the problem is formulated as follows. Let x be a random variable whose density function is given by

$$\text{Under hypothesis } H_0 \sim f(x, \theta_0) \tag{6.20}$$
$$\text{Under hypothesis } H_1 \sim f(x, \theta_1)$$

where $\theta_1$ and $\theta_0$ are two values for the distribution parameter $\theta$. Now assume that m observations have been made with the random variable taking the values $x_i$, $i = 1,2,\ldots,m$. Then the likelihood of hypohtesis $H_0$ given m observations is defined by

$$P_{0m} = \prod_{i=1}^{m} f(x_i, \theta_0) \tag{6.21}$$

and the likelihood of hypothesis $H_1$ is

$$P_{1m} = \prod_{i=1}^{m} f(x_i, \theta_1) \tag{6.22}$$

Since the test of a simple hypothesis $(H_1)$ against another simple hypothesis $(H_0)$, the Neyman Pearson Lemma (Hoel, 1971) suggests the use of the likelihood ratio

$$P_{1m}/P_{0m} \tag{6.23}$$

to decide between $H_1$ and $H_0$. The idea of using the likelihood ratio as the decision function is similar to its use in classical SDT. There the decision is done by choosing one LRCL $\beta$ and deciding

$$H_0 \text{ if } P_{1m}/P_{0m} \leq \beta$$
$$H_1 \text{ if } P_{1m}/P_{0m} > \beta$$

In sequential analysis two LRCL's A and B are set so that the decision has three possible outcomes:

1.  $H_1$ if $P_{1m}/P_{0m} \geq A$

2.  $H_0$ if $P_{1m}/P_{0m} \leq B$  (6.24)

3.  Continue the observation if

$$B < P_{1m}/P_{0m} < A$$

The next problem is to choose the LRCL in some optimal way. Intuitively it would seem desirable to choose the two LRCLs A and B in such a way that would relate them to some prescribed values of the two types of error defined as:

$P_{FA}$ - probability of rejecting $H_0$ when $H_0$ is true

$P_{miss}$ - probability of accepting $H_0$ when $H_1$ is true

The values of these two errors are predetermined by the decision maker. Unfortunately the exact functions

$$B = g(P_{FA}, P_{miss}), \quad A = g(P_{miss}, P_{FA})$$

are not available. However, very good approximations were found by Wald (1947). These approximations are

$$A = (1 - P_{miss})/P_{FA} \qquad B = P_{miss}/(1 - P_{FA}) \qquad (6.25)$$

The use of equations (6.24) and (6.25) is referred to as the sequential probability ratio test.

Some advantages of this test are:

1.  There is no need to derive the density function of a statistic such as t or F to carry out the test.

2.      The desired size of the two types of error can be chosen apriori to the test.

3.      Although the number of samples needed to terminate the process is a random variable, the mean of this random variable can be computed.

It is our interest to use this sequential ratio test for detection of failures in linear systems driven by white Gaussian process. It has already been shown in the previous section that if the detection is based on the first and second order statistics of the residuals of the optimal filter, the problem is that of testing between two stationary Gaussian processes with equal variances and different means. Next we shall find the LRCLs for this special case.

For the above problem the density functions under the two hypotheses are

$$\text{under } H_0 - f(x_i, \theta) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} (x_i - \theta_0)^2\}$$

$$\text{under } H_1 - f(x_i, \theta) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} (x_i - \theta_1)^2\}$$

(6.26)

Substituting in equation (6.23) using equations (6.21) and (6.22)

$$P_{1m}/P_{0m} = \prod_{i=1}^{m} \exp\{-\frac{1}{2} (x_i - \theta_1)^2\} / \prod_{i=1}^{m} \exp\{-\frac{1}{2} (x_i - \theta_0)^2\} .$$

using this expression in (6.24)

$$B < \exp\{-\frac{1}{2} \sum_{i=1}^{m} (x_i - \theta_1)^2 + \frac{1}{2} \sum_{i=1}^{m} (x_i - \theta_0)^2\} < A$$

or

$$B \lessgtr \exp\{(\theta_1 - \theta_0)\sum_{i=1}^{m} x_i + (\theta_0{}^2 - \theta_1{}^2)m/2\} < A$$

Taking logarithms and substituting for the values for A and B

$$\ln\frac{P_{miss}}{1-P_{FA}} \leq \{(\theta_1 - \theta_0)\sum_{i=1}^{m} x_i + (\theta_0{}^2 - \theta_1{}^2)m/2\} \leq \ln\frac{1-P_{miss}}{P_{FA}} \quad (6.27)$$

Therefore if $\theta_1 > \theta_0$ the decision would be

choose $H_0$ if $\sum_{i=1}^{m} x_i \leq \frac{1}{\theta_1-\theta_0}\ln\frac{P_{miss}}{1-P_{FA}} + (\theta_1+\theta_0)m/2$

$$(6.28)$$

Choose $H_1$ if $\sum_{i=1}^{m} x_i \geq \frac{1}{\theta_1-\theta_0}\ln\frac{1-P_{miss}}{P_{FA}} + (\theta_1+\theta_0)m/2$

and continue if (6.28) is not satisfied. If $\theta_1 < \theta_0$, the decision would be

choose $H_0$ if $\sum_{i=1}^{m} x_i \geq -\{\frac{1}{\theta_1-\theta_0}\ln\frac{P_{miss}}{1-P_{FA}} + (\theta_1+\theta_0)m/2\}$

$$(6.29)$$

choose $H_1$ if $\sum_{i=1}^{m} x_i \leq -\{\frac{1}{\theta_1-\theta_0}\ln\frac{1-P_{miss}}{P_{FA}} + (\theta_1+\theta_0)m/2\}$

and continue if (6.29) is not satisfied. These decision

regions for both cases are shown in Figures 6.1 and 6.2.

The above basic theory has to be modified if it is to

be applied to modelling failure detection mechanisms. Since

the theory is limited to the testing of simple hypotheses,

the values of $\theta_0$ and $\theta_1$ should be completely known apriori

to the test. In failure detection, the value of $\theta_0$ (the

normal mode) is known, however the value of $\theta_1$ (the failure)
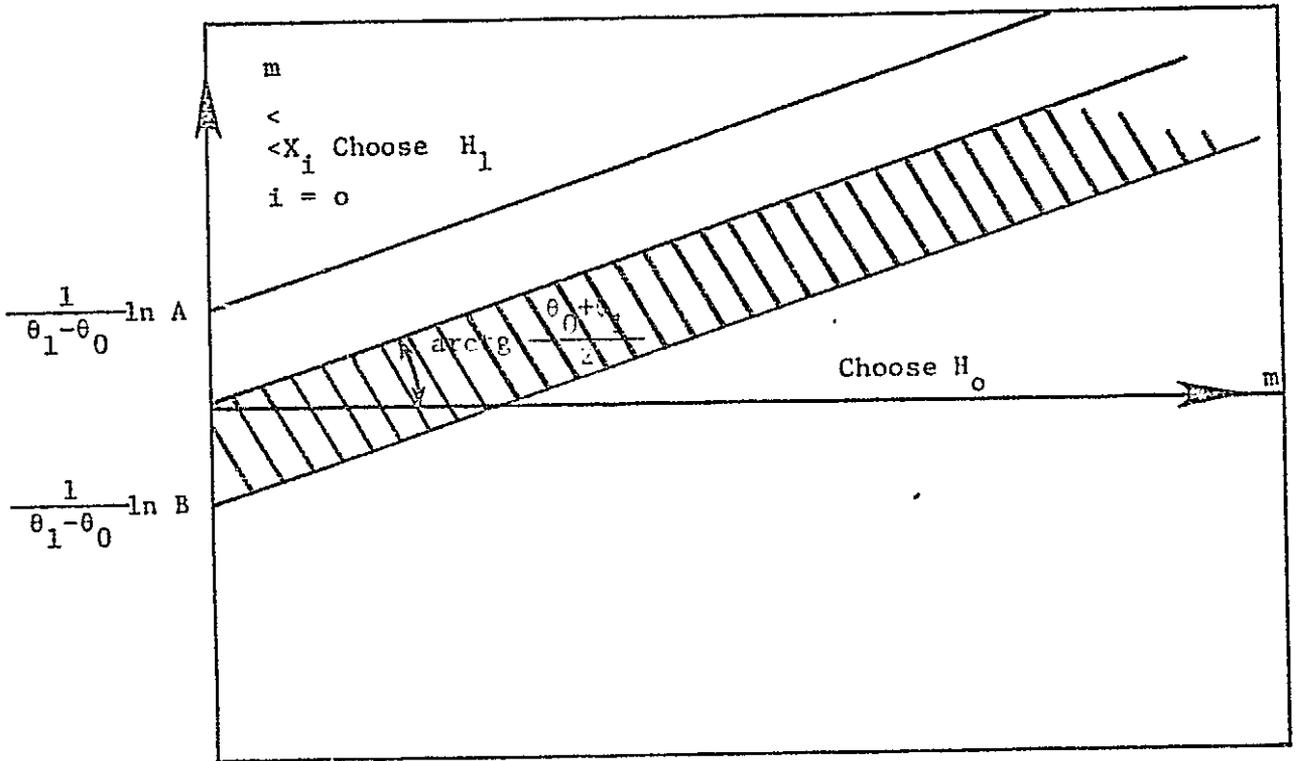
FIGURE 6.1  DECISION REGIONS FOR $\theta_1 > \theta_0$
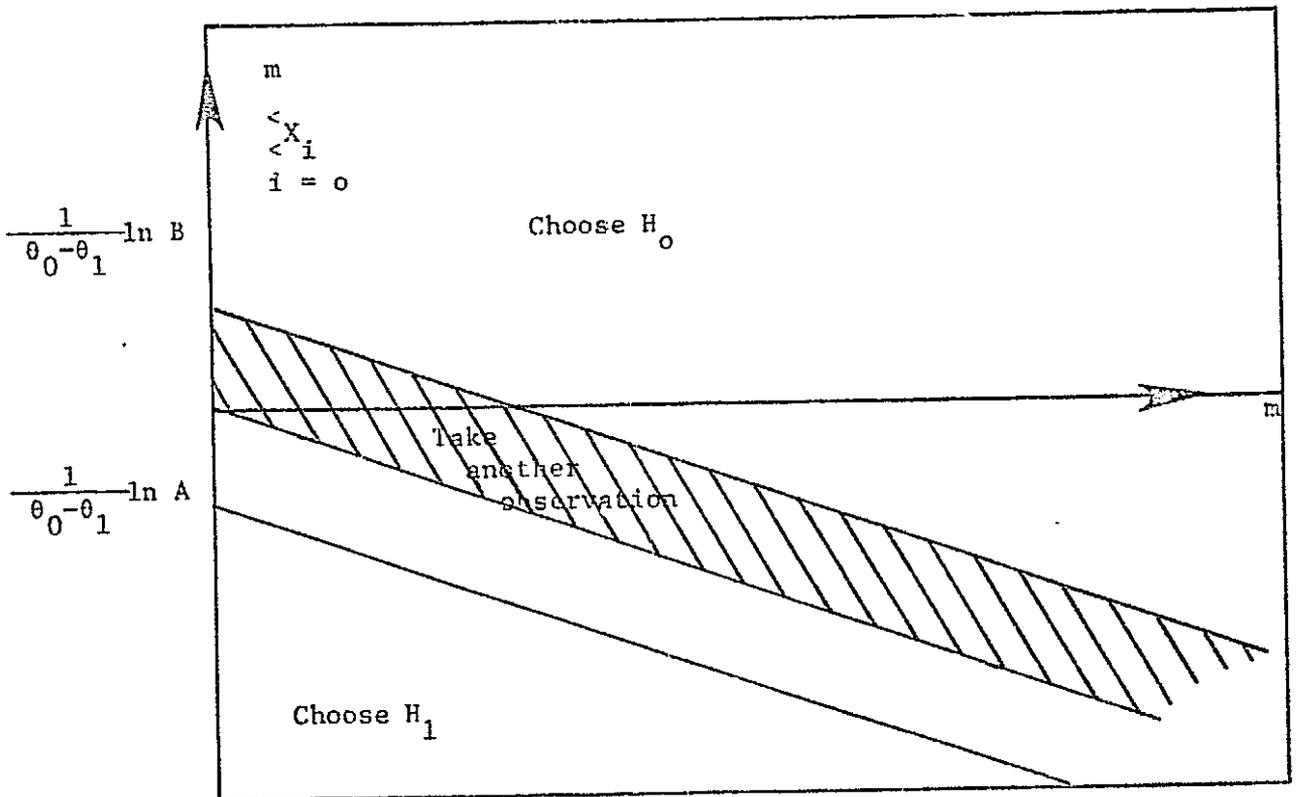


FIGURE 6.2  DECISION REGIONS FOR $\theta_1 < \theta_0$

is not known.  A solution to this problem was suggested by Wald (1949).  His suggestion was to choose on the basis of the physical properties at hand, an artificial parameter $\theta_1$ that would replace $\theta_1$ in equation (6.27).

A more severe difficulty is that in the basic sequential test  no transition of modes is assumed to occur during the whole observation process, while the failure detection task is characterized by such a transition.  A method to overcome this difficulty was suggested by Chien (1972).  His idea is based on the fact that in failure detection tasks, a decision in favor of the normal mode leads the subject to take more observations since he is not asked to report when the system is in its normal mode.  Therefore, a suboptimal strategy would be to reset the decision function to its initial value whenever the current value is in the region indicating that the normal mode is more likely (the shaded area in figures 6.1 and 6.2).  In this way, when a failure does occur, the number of observations required to drive the decision function into the failure region is less than if there had been no resetting (Chien, 1972).  Therefore, this resetting helps to reduce  the time between the onset of a failure and its detection and thereby eliminates the effect of the unknown transition time.

From a system engineer's point of view, the resetting procedure is equivalent to the addition of a feedback loop to the decision mechanism. The decision function is defined (equations (6.28) and (6.29)) as:

$$\overset{\curvearrowright}{\lambda}(m) = \sum_{i=1}^{m} x_i \tag{6.30}$$

or in a recursive form:

$$\overset{\curvearrowright}{\lambda}(m) = \overset{\curvearrowright}{\lambda}(m-1) + x_m \qquad\qquad \overset{\curvearrowright}{\lambda}(1) = x_1$$

By employment of the resetting, the modified decision function is

$$\lambda(m) = \lambda(m-1) + x_m + \xi_m \qquad\qquad \lambda(1) = \overset{\curvearrowright}{\lambda}(1) \tag{6.31}$$

where $\xi_m$ is the feedback. Let $\xi_b$ be defined as

$$\xi_b = \frac{(\theta_0 + \theta_1)m}{2}$$

then from equations (6.28) and (6.29) $\xi_b$ is the border between the normal and failure modes (see also figure 6.1). Therefore the value of $\xi_m$ in equation (6.31) for $\theta_1 > \theta_0$ is

$$\begin{aligned} \xi_m &= 0 & &\text{if } \lambda(m-1) + x_m > \xi_b \\ \xi_m &= \xi_b - \lambda(m-1) - x_m & &\text{if } \lambda(m-1) + x_m < \xi_b \end{aligned} \tag{6.32}$$

and for $\theta_1 < \theta_0$ is

$$\xi_m = -\xi_b - \lambda(m-1) - x_m \qquad \lambda(m-1) + x_m > -\xi_b$$
$$\xi_m = 0 \qquad\qquad\qquad\qquad \lambda(m-1) + x_m < -\xi_b \qquad (6.33)$$

The effect of the feedback on the decision function is shown in figure 6.3. When the modified decision function $\lambda(m)$ is used, only one CL is needed since the CL for the normal mode will never be met due to the feedback. However if the same CL that was suggested by Wald (equation 6.25) is used, more false alarms should be expected due to the feedback. In order to keep the same mean time between two false alarms, as in the original sequential test, the CL A in equation (6.25) should be modified to $A_1$, where $A_1$ is given by the solution to the following equation (Chien, 1972):

$$A_1 - \ln A_1 - 1 = -[\ln A + \frac{A - 1}{1 - B}\ln B] \qquad (6.34)$$
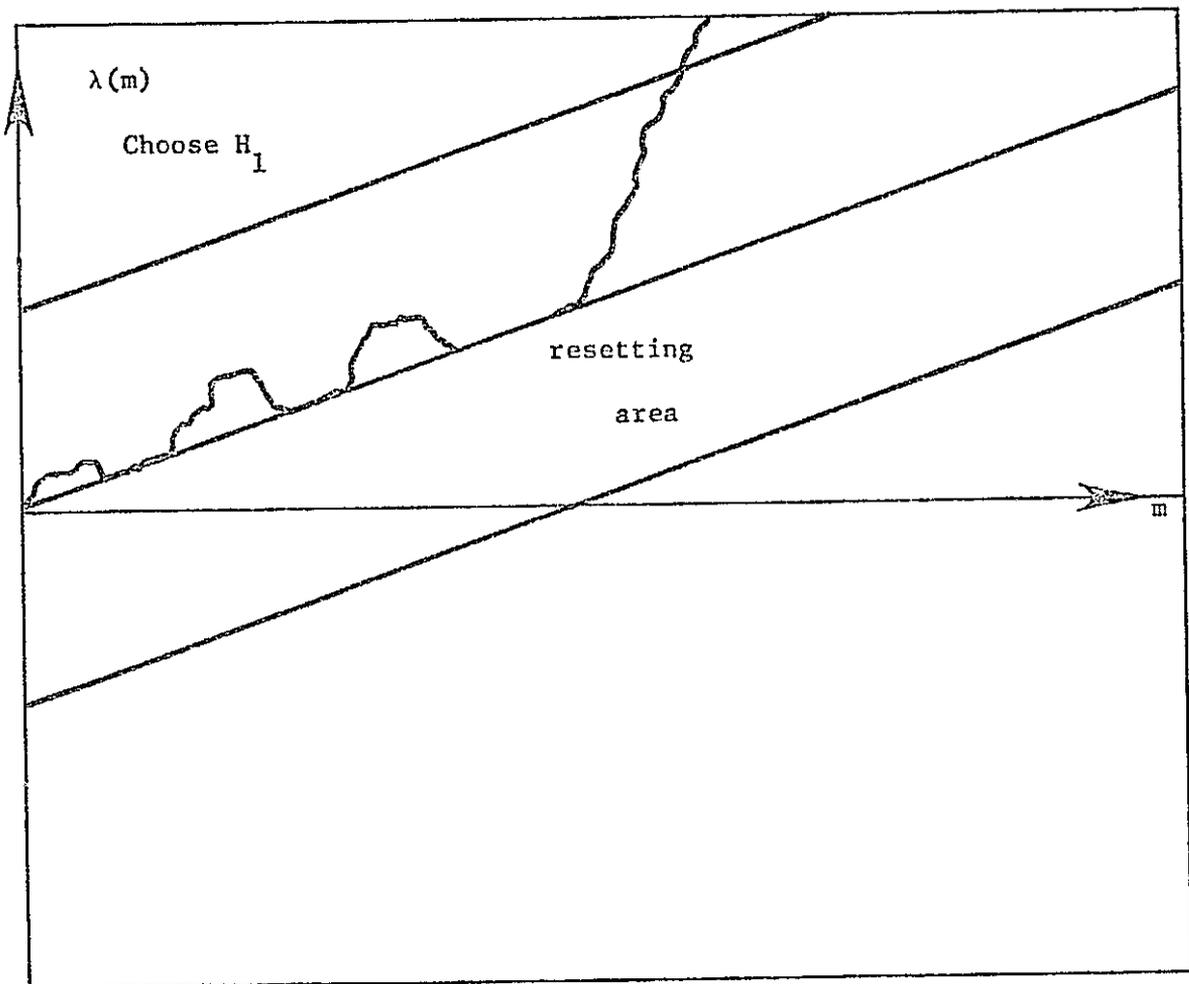
FIGURE 6.3  EFFECTS OF FEEDBACK ON THE DECISION FUNCTION

## 6.4  A Model for Decision Strategy

The discussion of estimation theory and sequential analysis in the last two sections provides the basis for our model of the human operator in failure detection tasks.

Let us assume that the process which is displayed to the subject is unidimensional, namely $\underline{z}(t)$ in equation (6.1) is a scalar.  As the subject observes $z(t)$, his observations are corrupted by additive noise $v(t)$ which is modelled as a zero mean white-Gaussian process (Levison et al, 1969).  Thus the input to the failure detection system $y(t)$ can be described by equation (6.5).  Since this input is a stochastic process, the detection system is assumed to consist of two stages: linear estimation and decision mechanism (Levison, 1971; Phatak et al, 1972).  The functional block diagram of the detection system is shown in figure 6.4.
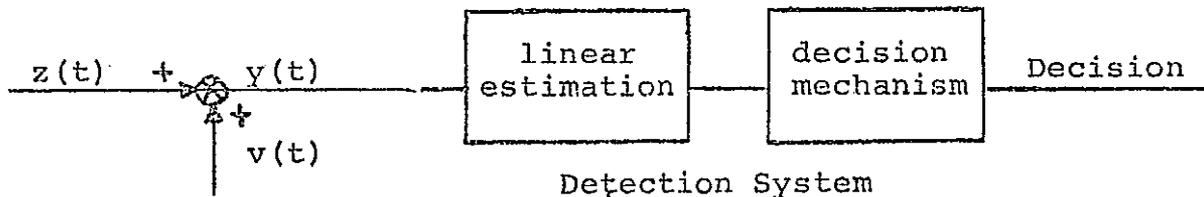


Figure 6.4  Functional Block Diagram of Decision Mechanism

We will now assume that the matrices $F(t)$, $G(t)$, and $H(t)$ in equations (6.1) and (6.2) are known, so that the input

y(t) is given by the state space description of the "shaping filter". Therefore, the overall optimal filter (recall that z(t) is Gaussian), is given by equations (6.7) through (6.10), and its block diagram is shown in figure 6.5. As seen from the block diagram the Kalman estimator is a linear system of the same order as the shaping filter. If the shaping filter is of high order, it is reasonable to assume that a low (second or third) order approximation will suffice for the human operator. As the estimator will also be of this order, it could be implemented easily. If the conditions that were specified in section 6.2 hold, the estimator will also be a.time invariant system. This means that the data processing done by the subject prior to the decision mechanism is equivalent to low pass filtering. The linear estimation approach provides us with an elegant way to define th parameters of this low pass filter.

Since both the shaping filter and the estimator are linear and the input is zero mean Gaussian process, both the state estimates and the observation estimate are zero mean Gaussian processes, and both can be used as inputs to the decision mechanism. It seems more reasonable to use the observation estimates in the model for the following reasons:

1.    The states are abstract non unique variables that can be defined in different ways while the observation is unique and well defined for the subject.

2.      The dimension of the state is usually larger
        than the dimension of the observation so that
        using the observation estimates simplifies the
        decision algorithm.

It is further assumed that the input to the decision mechanism
is the observation error (residual) rather than the observation
itself.   The reasons for this assumption are:

1.      The error is more sensitive to the effect of
        failure than the observation estimate (Schweppe,
        1973).

2.      The observation error is a white Gaussian
        process, so successive observations are in-
        dependent.

Once the observation residual is used as the input to the
decision mechanism, the question of the dimension of the
observation arises.   Although only a scalar observation
(position) is directly presented to the subject, there is
some evidence to claim that independent direct measurements
of the rate are also taken.   This claim is supported by the
fact that in some animals there are cells that are sensitive
only to the rate of the input.   Also, in the model of the
human operator as a controller (Kleinman and Baron, 1970),
the addition of the rate component improved the fitting of
the model to the experimental data.   In our model the addi-
tion of direct rate measurements did not improve the results,

but only complicated the decision algorithm. Therefore, we decided to use a scalar (position) measurement.

Since the covariance of the residual is also known (equation (6.15)), the actual input to the decision mechanism is the normalized residual. Therefore, for the normal mode, the residual is a zero mean white Gaussian process with unit variance; and, for the failure mode the residual is also a white Gaussian process with unit variance but with a specified mean.

Our first approach was to base the decision on the instantaneous values of the residual. However, checking the value of the residual at the particular time which the subject pressed the button (minus his reaction time) showed that this value did not have any special property that would explain why the detection occurred there. Therefore, we assumed that the decision was based on the accumulated information and decided to use the sequential analysis.

Let $\varepsilon_i$ be the value of the residual at the observation interval i, and let us assume that the failure is positive in sign, i.e.

$$m(t) > 0 \qquad t > t_f \qquad (6.35)$$

then by adding the bias term in equation (6.28) to the decision function in equation (6.30)

$$\tilde{\lambda}(m) = \sum_{i=1}^{m} (\varepsilon_i - \xi_b) \qquad (6.36)$$

with the resetting feedback the decision function takes the
form

$$\lambda(m) = \overset{\sim}{\lambda}(m) \qquad\qquad \text{if} \quad \overset{\sim}{\lambda}(m) > 0$$

$$\lambda(m) = 0 \qquad\qquad\quad \text{if} \quad \overset{\sim}{\lambda}(m) < 0$$

(6.37)

where $\overset{\sim}{\lambda}(m)$ is given by the recursive equation

$$\overset{\sim}{\lambda}(m) = \lambda(m-1) + (\epsilon_m - \xi_b) \qquad \overset{\sim}{\lambda}(1) = \epsilon_1 - \xi_b$$

A block diagram of the decision mechanism is shown in figure
6.6 where the CL $A_1$ is defined by equation (6.34).

In real life detection tasks, assumption (6.37) cannot
usually be made because the sign of the failure is not known
apriori. Even in predesigned experiments it is preferred
that the sign of the failure not be known to the subject
apriori. The reason is that this uncertainty prevents the
subject from guessing if he has to identify the sign of the
failure in addition to detecting it. Therefore, we assumed
that the decision maker is actually involved with the following
two simultaneous hypothesis tests. The first is:

$$H_1^+ : \overline{\epsilon}(t) = m(t) > 0$$

$$H_0 : \overline{\epsilon}(t) = 0$$

and the other is

$$H_1^- : \overline{\epsilon}(t) = -m(t)$$

$$H_0 : \overline{\epsilon}(t) = 0$$

For the first test the decision function is defined by
equation (6.37) while for the second test the decision function
is
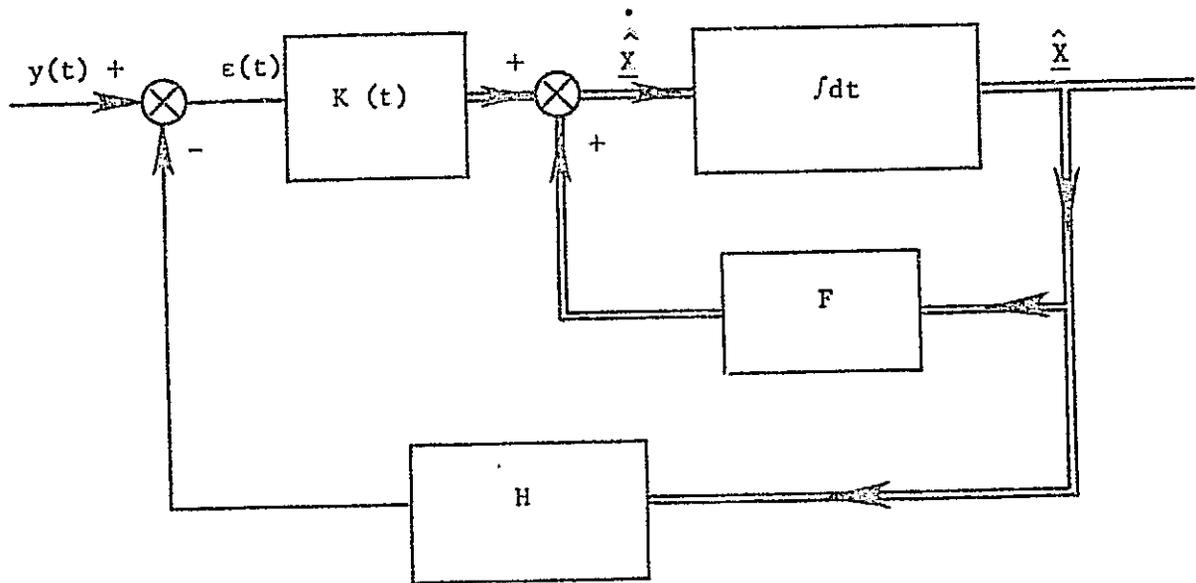
FIGURE 6.5 LINEAR ESTIMATOR (KALMAN FILTER)
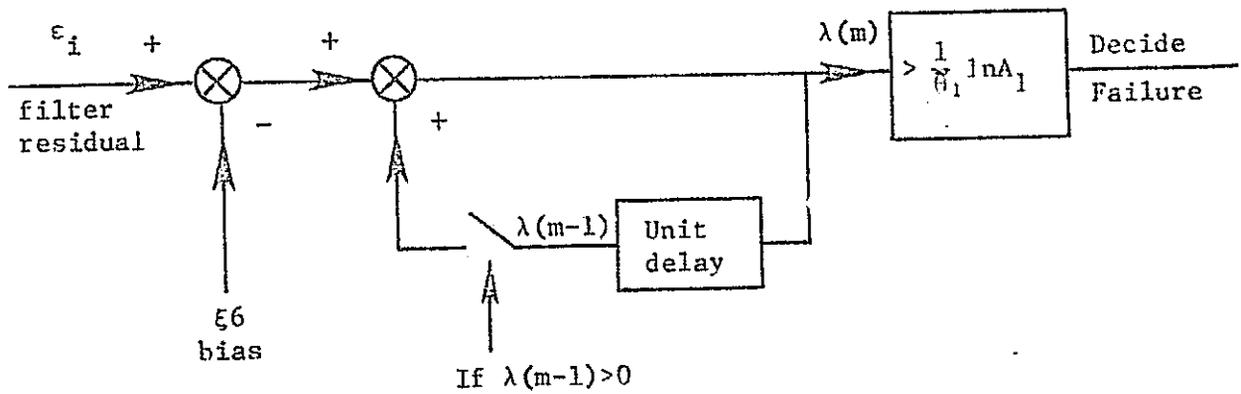


FIGURE 6.6 DECISION MECHANISM

$$\lambda^-(m) = 0 \qquad\qquad \text{if } \overset{\sim}{\lambda}{}^-(m) > 0$$

$$\lambda^-(m) = \overset{\sim}{\lambda}(m) \qquad\qquad \text{if } \overset{\sim}{\lambda}{}^-(m) < 0 \tag{6.38}$$

where $\overset{\sim}{\lambda}{}^-(m)$ is given by the recursive equation

$$\overset{\sim}{\lambda}{}^-(m) = \overset{\sim}{\lambda}{}^-(m-1) + (\epsilon_m + \xi_b) \qquad \overset{\sim}{\lambda}{}^-(1) = \epsilon_1 + \xi_b \tag{6.39}$$

A block diagram of the complete model for the decision mechanism is shown in figure 6.7.

A simulation of the model for a process that is the output of a second order shaping filter with

$$\xi = 0.7 \qquad\qquad w_0 = 4.24$$

was implemented. The performance of the model for four levels of step failures is shown in figure 6.8. Since the detection time $t_d$ is a random variable, both its mean and variance are shown (computed on the basis of 40 samples).

The sensitivity of the mean detection time to several parameters of the model was also studied. Figure 6.9 shows the sensitivity to the value of the two types of errors $P_{miss}$ and $P_{FA}$. The curves show a strong decrease in the mean detection time when the value of these errors is increased. Figure 6.10 shows the sensitivity to the parameter that defines the failure $\overset{\sim}{\theta}_1$. An increase in $\overset{\sim}{\theta}_1$ decreases the mean detection time. Finally, figure 6.11 shows the sensitivity to the ratio between the variance of the observation process $\sigma_z^2$ and the variance of the measurement noise. The effect of the measurement noise is minimized due to the good performance of the filter.
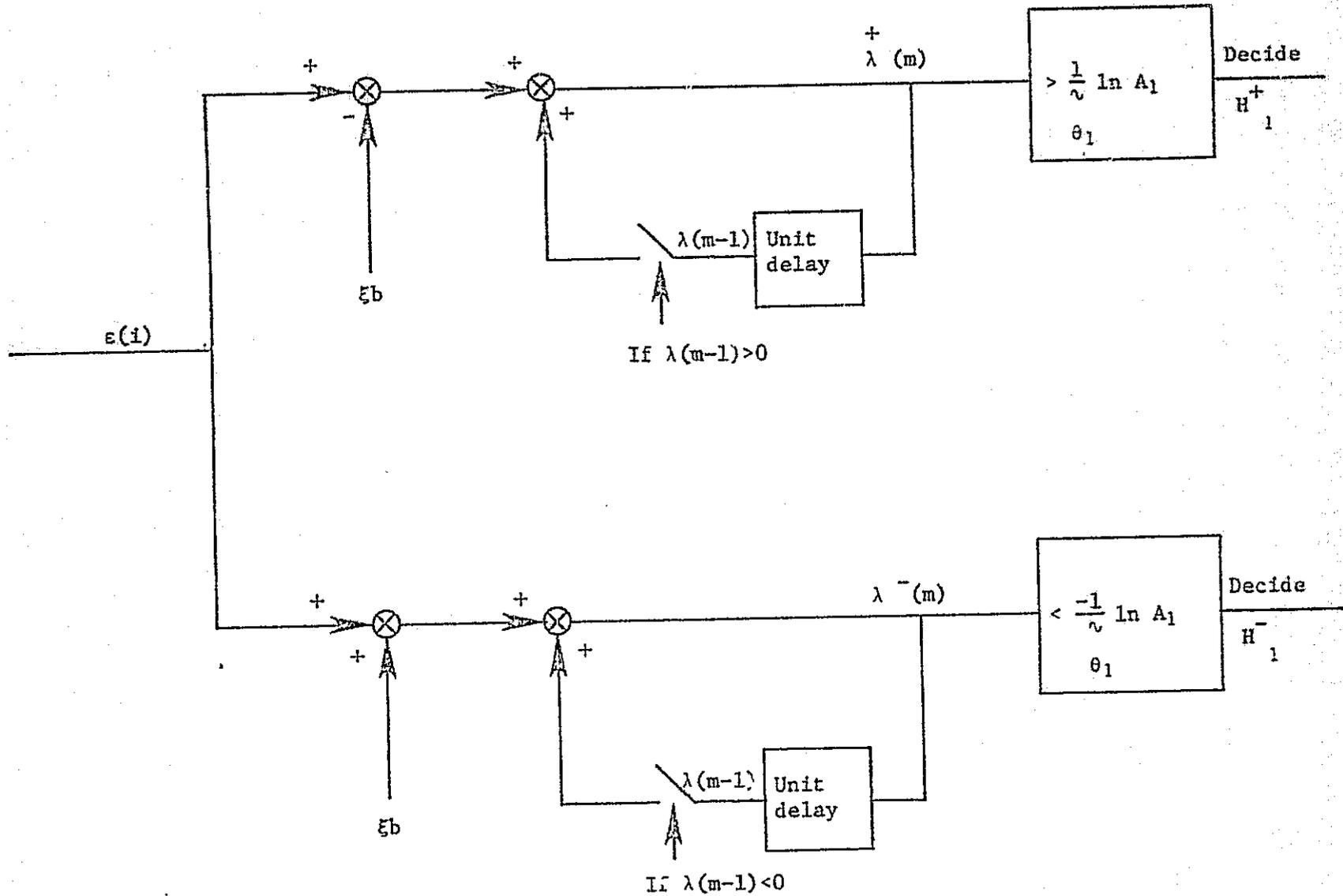
FIGURE 6.7   COMPLETE BLOCK DIAGRAM OF THE DECISION MECHANISM
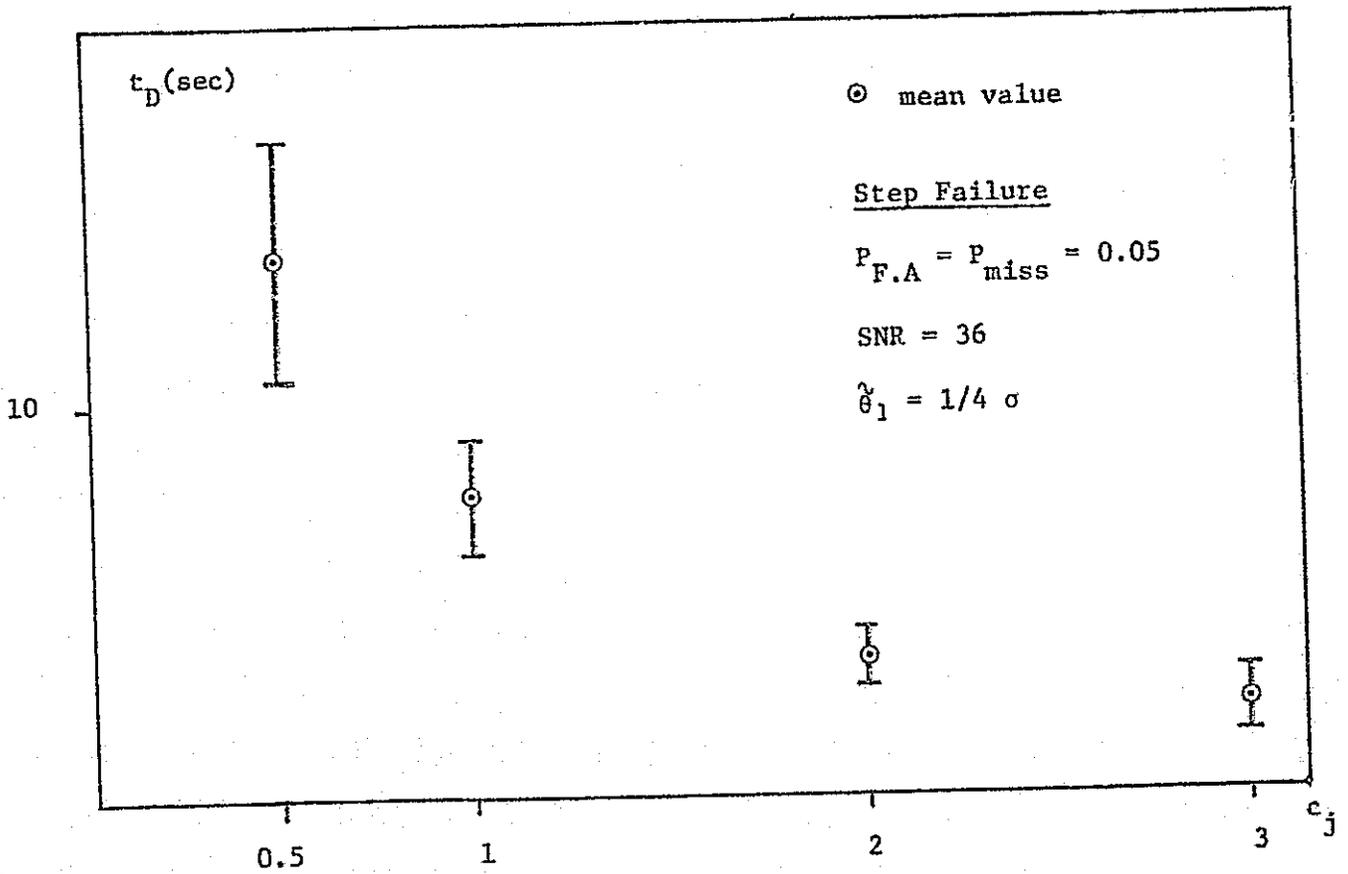
FIGURE 6.8  DETECTION TIME PREDICTED BY THE MODEL

Step Failure

SNR = 36

$\tilde{\theta}_1 = 1/4\ \sigma$

$P_{F.A} = P_{miss}$

$E(t_D)$

$P_{F.A} = 0.01$

$P_{F.A} = 0.05$

$P_{F.A} = 0.1$

$c_j$
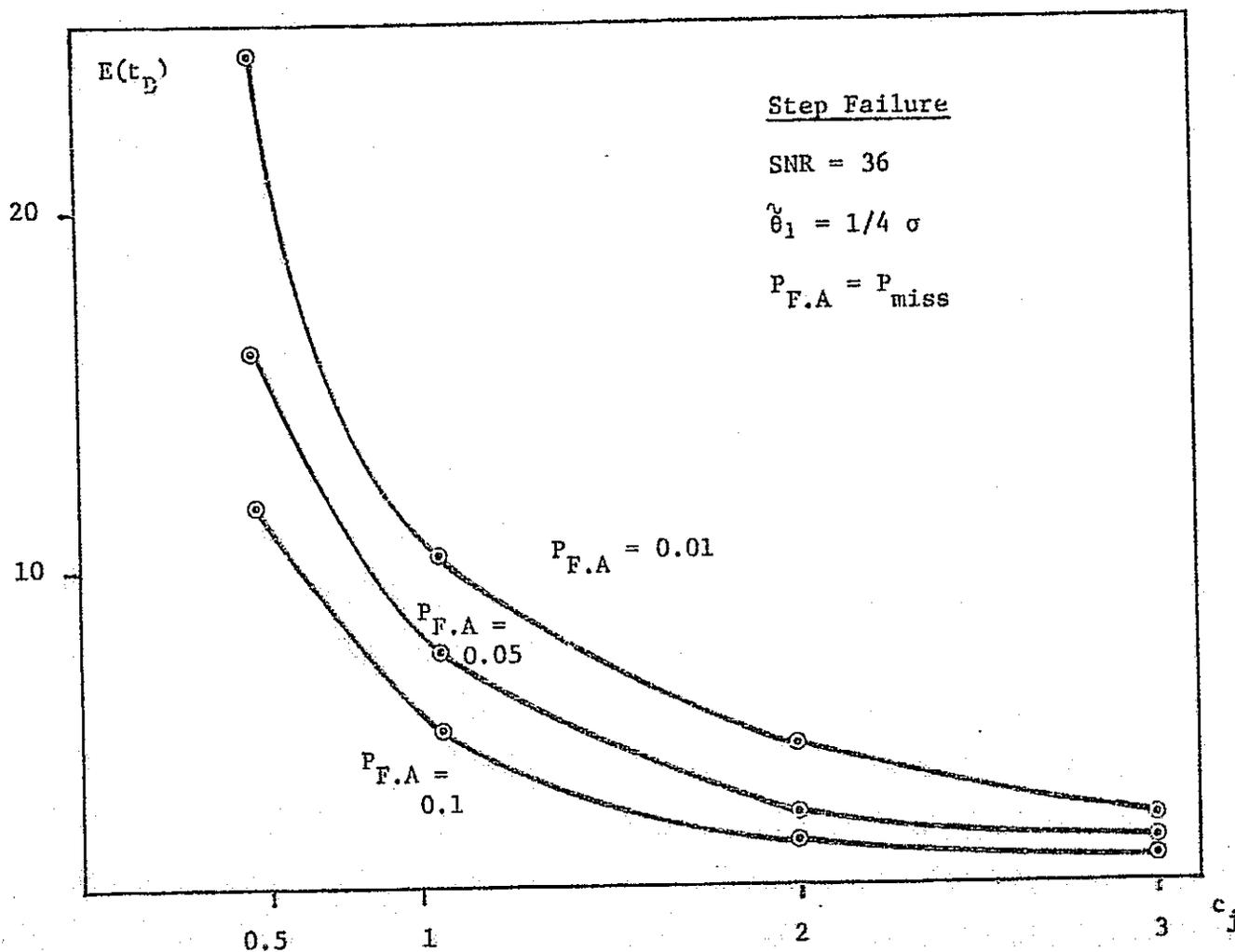
FIGURE 6.9   SENSITIVITY OF $E(t_D)$ TO DETECTION ERRORS

$E(t_D)$

Step Failure

$P_{F.A} = P_{miss} = 0.05$

SNR = 36

$\tilde{\theta}_1 = 1/8 \; \sigma$

$\tilde{\theta}_1 = 1/4 \; \sigma$

$\tilde{\theta}_1 = 1/2 \; \sigma$

20

10

0.5    1    2    3    $c_j$
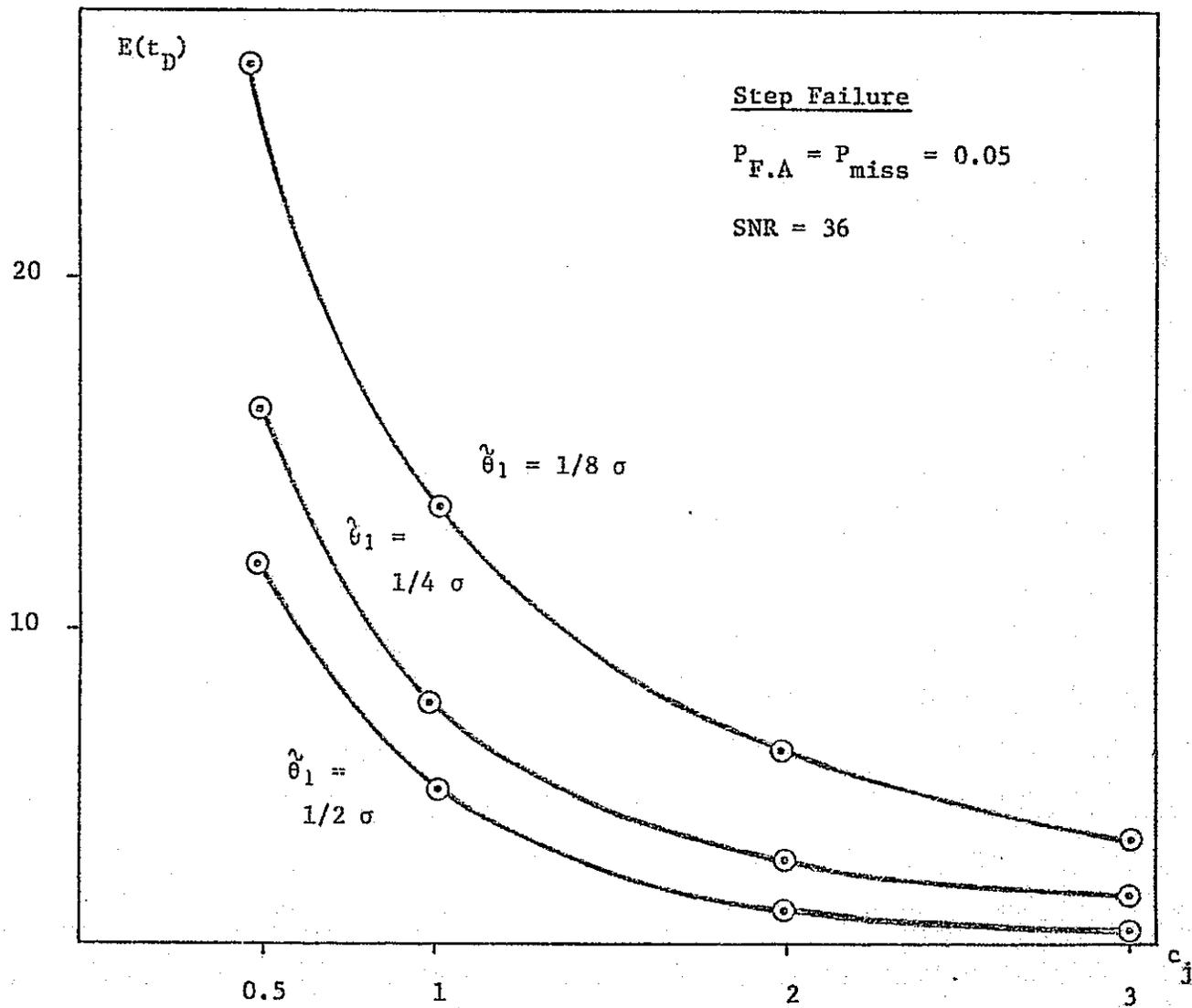
FIGURE 6.10   SENSITIVITY OF $E(t_D)$ TO THE THRESHOLD $\tilde{\theta}_1$
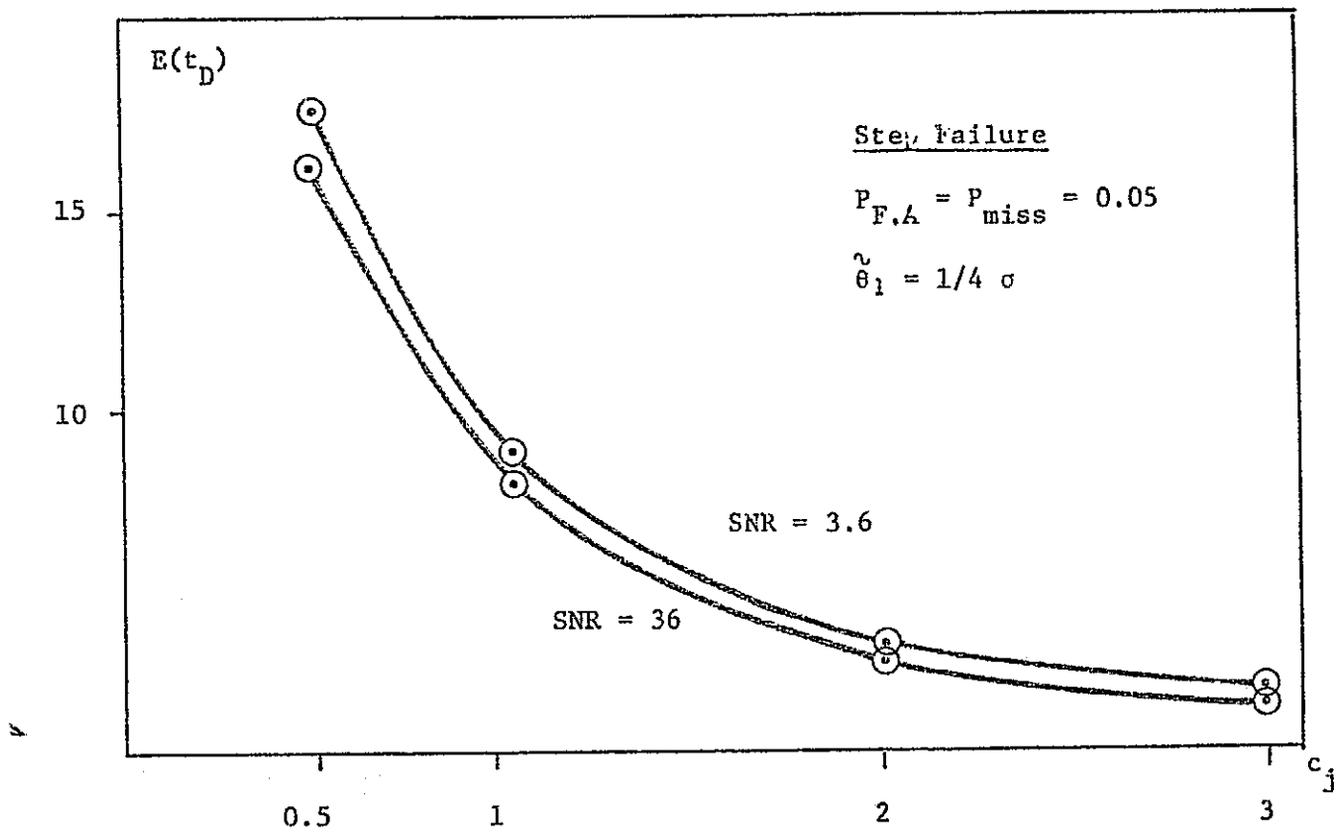
FIGURE 6.11   SENSITIVITY OF $E(t_D)$ TO SNR

## 6.5 Closed and Open Decision Intervals

In the model that was discussed in the previous section, a basic assumption was that the decision maker is free to take as many observations as he needed so that the length of the decision interval depel ed only on his performance. We will refer to such decision intervals as open decision intervals.

However, in many real life situations, the observation interval is limited because the observed process has a pre-determined finite duration. For example, consider the human operator whose task is to monitor the airplane instruments during the final phase of an automatic landing. We will refer to these types of observation intervals as closed decision intervals.

It is obvious from out discussion of sequential analysis (section 6.3) that the classical theory does not apply to such closed decision interval tasks, and some modifications must be made. In particular, in the classical sequential analysis, it is assumed that the value of the probability of the two types or error, $P_{miss}$ and $P_{FA}$, are kept constant during the whole observation interval. However, when the observation interval is limited, the subject might consider changing these probabilities (Birdsall et al, 1965). In the experiment that is described in the next section, the subject was told apriori that a change must occur within each interval. Therefore, it seems reasonable to assume that as time goes by the subject's willingness to accept the hypothesis $H_1$ will increase. This

means that the subject is increasing $P_{FA}$ with time. This time dependency of $P_{FA}$ can take several functional forms. We obtained the best fit to the subject's data when the following relationship was used:

$$P_{FA}(i) = P_{FA}(T)(1 + \tanh(\frac{i}{12} - 5)) \qquad i = 1,60 \qquad (6.40)$$

where i is the observation index. This time dependency is shown in figure 6.12.
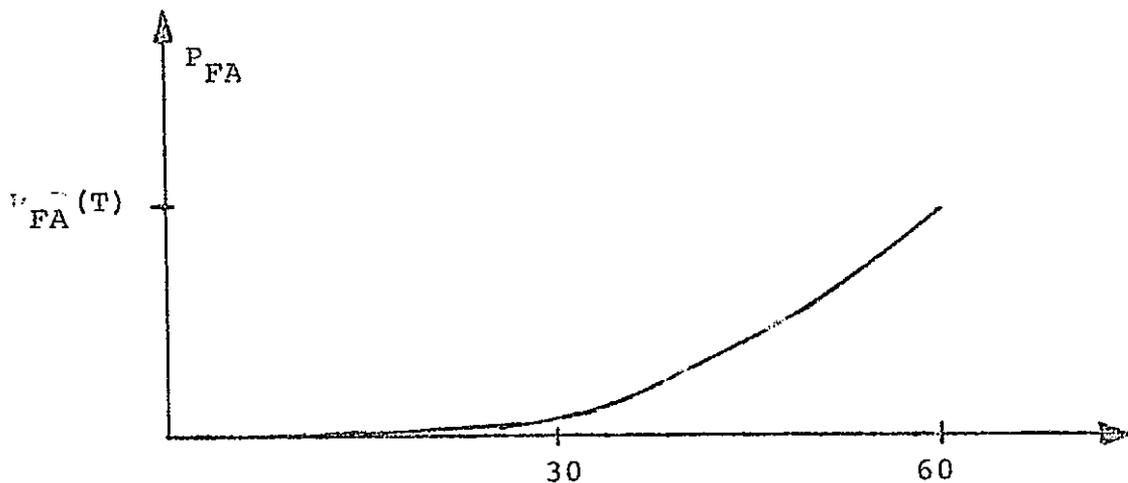


Figure 6.12 TIME DEPENDENCE OF $P_{FA}$

Figure 6.13 shows the performance of the model that is described in the previous section with the modification of equation (6.40). The failures are four levels of step failures equivalent to the ones that were used to produce the data for figure 6.8. When compared to the open interval results, these results show a decrease in the mean detection time; however, this decrease is at the expense of an increase in $P_{FA}$.
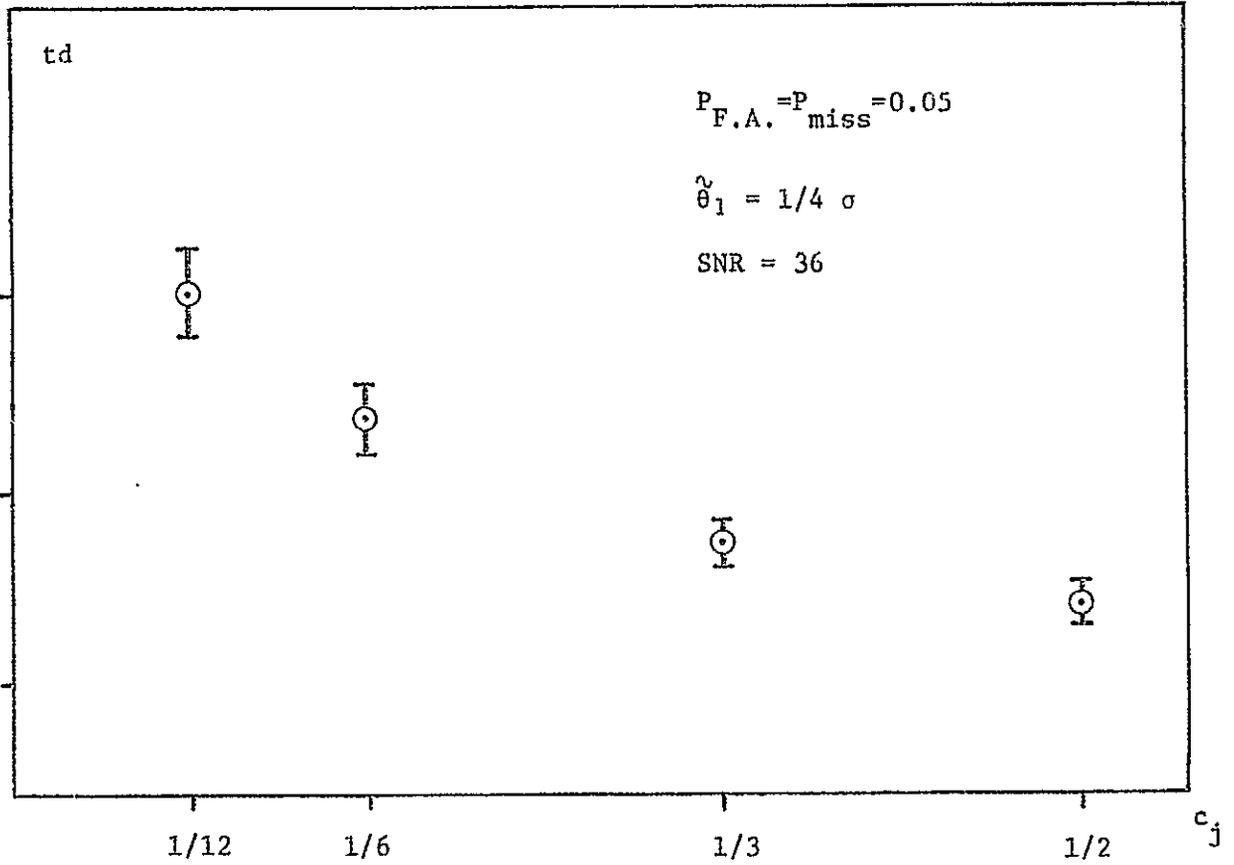
FIGURE 6.13   DETECTION TIME PREDICTED BY THE MODEL FOR CLOSED

OBSERVATION INTERVALS

## 6.6 Experimental Method

### 6.6.1 Apparatus

Again, the Adage Model 30 graphics computer was used for the simulation and display of the observed variables. The computer function switch box was used as a control by the subject to make his decisions.

The displayed information included two fixed cursors that indicated the horizontal (x) axis. Also, a horizontal bar represented the displacement of the process to be monitored from the x axis (see figure 6.14). This displacement $z(t)$ was a zero mean Gaussian process which was generated by driving a time invariant second order system with a white Gaussian sequence. The transfer function for the second order system was

$$G(s) = \frac{1}{s^2 + 2\xi w_0 s + w_0^2} \tag{6.41}$$

where

$$\xi = 0.7 \qquad w_0 = 4.24$$

The covariance of the white sequence was chosen in such a way that the steady state standard deviation of the observed variable was 1/16 of the display height. The continuous process was approximated by its discrete equivalent at a time interval of 0.2 seconds (see Appendix B).

The failure in the process was defined by a change in the mean of $z(t)$, and this change was added directly to the

$$Z(t)$$
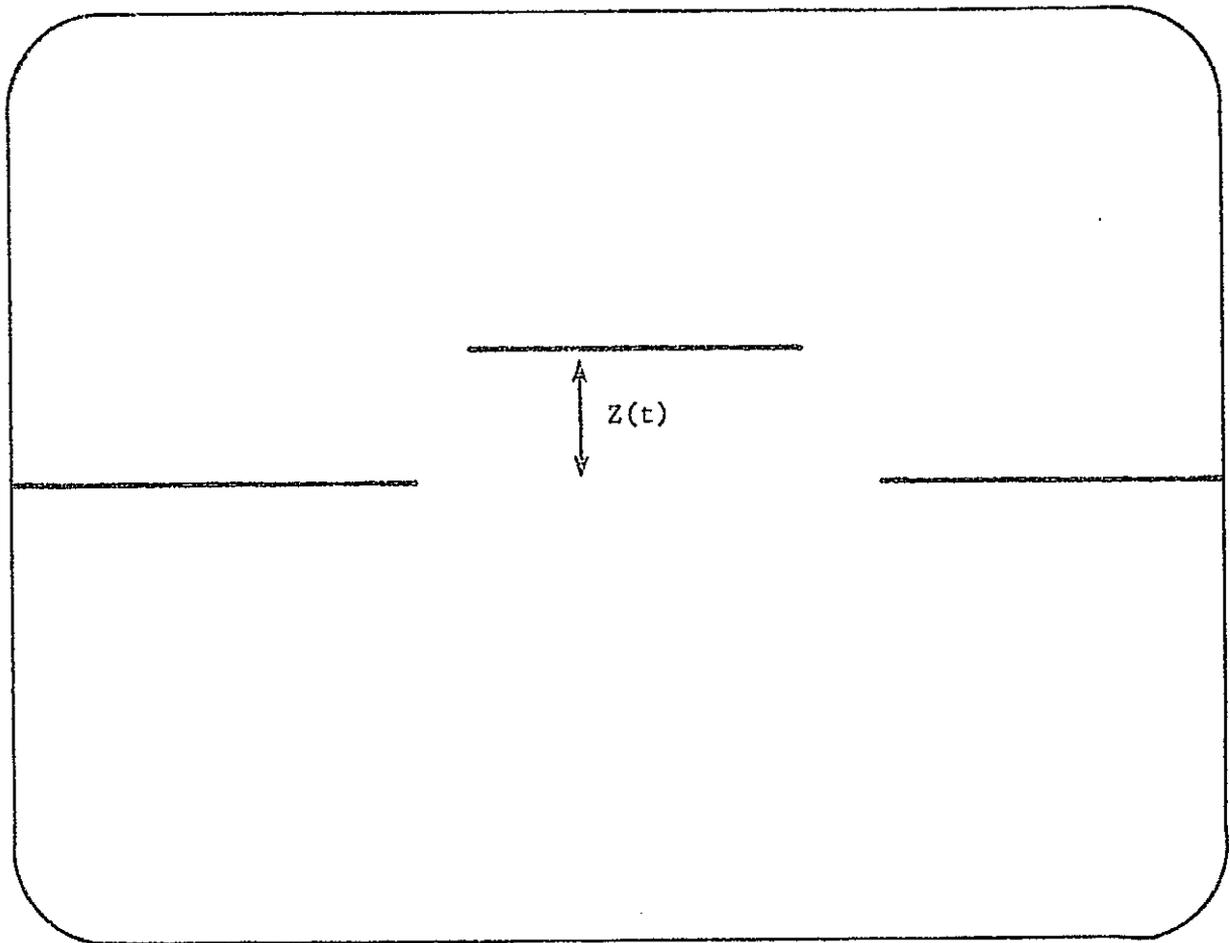
FIGURE 6.14  DISPLAY PRESENTATION FOR THE EXPERIMENT

output of the system so that the states of the dynamic system remained unchanged. Therefore in the normal mode $H_0$, the output was a Gaussian process with

$$H_0 : \overline{z}(t) = 0 \qquad \sigma_z(t) = L/4$$

where L is the display height. In the failure mode $z(t)$ was a Gaussian process with

$$H_1 : \overline{z}(t) = m \qquad \sigma_z(t) = L/4$$

## 6.6.2 Subjects

Two subjects participated as observers in the experiment. Both were graduate students in the Man-Vehicle Laboratory and were familiar with decision analysis terminology. Their participation was on a voluntary basis, and no rewards were given on the basis of performance.

## 6.6.3 Procedures

All of the experimental sessions consisted of 160 observation intervals. In each interval the subject made a single decision. The subject sat in front of the display while holding the function switch box in his hand. Every observation interval started with the process in its normal mode ($H_0$). Failures (changes to $H_1$ mode) occurred in each interval, and the time of occurance was determined by a random number generator. The generator picked with equal probabilities one of the following four values for $t_f$ (seconds)

$$3.50 \qquad 3.75 \qquad 4.25 \qquad 4.50$$

The subject's task was to indicate as soon as possible whether he had perceived a change.

Closed and open intervals were used in different experimental sessions. In the closed interval sessions, termination of the observation occurred after exactly ten seconds. In the open interval sessions, the termination occurred immediately after the subject made his decision. For both sessions, each observation interval was followed by a two second blanking period, after which a new observation was started.

To minimize subject guessing, he was asked to use two push buttons: to press one when the change in the mean was positive and the other when the change in the mean was negative. Positive and negative changes in the mean had the same magnitudes, but the opposite sign, and each happened with equal probability.

In each interval, one and only one change occurred, and the subject was made aware of this fact. He was also told that he had only one chance to make a decision, and he would not be allowed to change his mind after he pressed one of the buttons. The level of the change (i.e., the magnitude of $m(t)$) had four different values, so that four levels of difficulty or SNR were presented. The appearance of each level was equiprobable and was determined by a random number generator.

Each of the subjects participated in three different sessions. In the first session, the length of the observation interval was open, i.e., the observation was terminated only after the subject pressed a decision button. The change in the process was a step function so that

$$m(t) = C_j \qquad j = 1,\ldots,4 \qquad (6.42)$$

where

$$|c_1| = \tfrac{1}{2}\,\sigma_z \qquad |c_2| = \sigma_z \qquad |c_3| = 2\sigma_z \qquad |c_4| = 3\sigma_z$$

As stated before each of the $C_j$ could be positive or negative with equal probability. In order to prevent the subject from making his decision on the basis of the instantaneous jump, this jump was replaced by a one second ramp that changed $z(t)$ from zero to $C_j$. It should be noted that this transient time was short compared to the average decision time.

In the second session, the same failure modes that are described above were reused. However, this time the length of the observation interval was fixed to 10 seconds. The observations were not terminated when detection occurred, and the system operated in the failure mode until the end of the 10 second period.

In the third session, the length of the observation interval was free again, however, the changes in the mean of $z(t)$ were ramp functions, so:

$$m(t) = \overset{\sim}{C}_j (t-t_f) \qquad j = 1,\ldots,4 \qquad (6.43)$$

where

$$\overset{\sim}{c}_1 = \frac{1}{12}(\sigma_z/\text{sec}) \quad \overset{\wedge}{c}_2 = \frac{1}{6}(\sigma_z/\text{sec}) \quad \overset{\sim}{c}_3 = \frac{1}{3}(\sigma_z/\text{sec}) \quad \overset{\sim}{c}_4 = \frac{1}{2}(\sigma_z/\text{sec})$$

Before the beginning of each session, a standard set of instructions was read to the subjects. They were told that they were allowed to make only one decision per interval and that a change definitely occurred in each interval. They were also told that there were four levels of failure and all levels, as well as their signs, are equiprobable. The subjects were not advised what value of $P_{miss}$ or $P_{FA}$ to use; however, they were told that the penalties and rewards were the same.

After the instructions, the normal mode was presented to the subjects until they declared that they were familiar with the process. Before the second session, the normal mode was shown in intervals of ten seconds to acquaint the subjects with the fixed interval length. Then some samples of the failure mode were shown. This was followed by still another observation interval in which there was no change from normal mode to further increase their familiarity with this mode before the detection intervals started.

## 6.7 Experimental Results

As stated in the previous section, the experiments were divided into three different sessions. In the first session, four levels of step failures with open observation intervals were included in the presentation. The mean $E(d_t)$ and the standard deviation $\sigma_{t_d}$ of the detection time for both subjects are shown in Table 6.1 (j - level of failure, see equation (6.42)).

| Subject | j | 1 | 2 | 3 | 4 |
|---------|---|---|---|---|---|
| A.C. | $E(t_d)$ Seconds | 20.90 | 11.50 | 5.15 | 4.17 |
|  | $\sigma_{t_d}$ Seconds | 10.00 | 4.50 | 1.50 | 1.30 |
|  | $\overline{K}_j$ | 0.62 | 0.69 | 0.62 | 0.75 |
|  | $\sigma^2_{K_j}$ | 0.09 | 0.07 | 0.03 | 0.05 |
| B.C. | $E(t_d)$ Seconds | 17.00 | 7.50 | 4.20 | 3.10 |
|  | $\sigma_{t_d}$ Seconds | 8.00 | 3.30 | 1.50 | 0.90 |
|  | $\overline{K}_j$ | 0.51 | 0.45 | 0.50 | 0.56 |
|  | $\sigma^2_{K_j}$ | 0.06 | 0.04 | 0.03 | 0.03 |

TABLE 6.1   Results from First Experimental Session

The mean detection times that were found justify our assumption that the transient in the failure (1 second) is negligible compared to the detection time.

The results that are presented in Table 6.1 are also

shown in Figures 6.15 and 6.16. Those figures include, in addition, the mean detection time that is predicted by our model using the following parameters:

$$SNR = 36 \qquad P_{miss} = P_{FA} = 0.05 \qquad \hat{\theta}_1 = 1/4 \qquad (6.44)$$

The number for $P_{FA}$ was determined on the basis of the actual number of false alarms for the subjects (8 out of 150). A false alarm was scored when the subject pressed the button before the occurence of the failure.

Equations (6.36) and (6.37) show that the value of the decision function in the period between occurence and detection of a failure is given by

$$\lambda(t) = \sum_{i_x}^{i_y} (\epsilon_i \pm \xi_b) \qquad (6.45)$$

where $i_x$ is the first observation after the failure had occurred and $i_y$ is the observation after which detection was made. If $\xi_b$ is small compared to $\sigma_{\epsilon_i}$ than equation (6.45) implies that the subject is integrating the residual and makes a decision when this integral is equal to some CL. Therefore, for all levels of failures the following relation holds:

$$\int_{t_f}^{t_d} \epsilon(t)dt = constant \qquad (6.46)$$

within the integration interval

$$\epsilon(t) = \epsilon_n(t) + \epsilon_m(t) \qquad t_f < t < t_d \qquad (6.47)$$
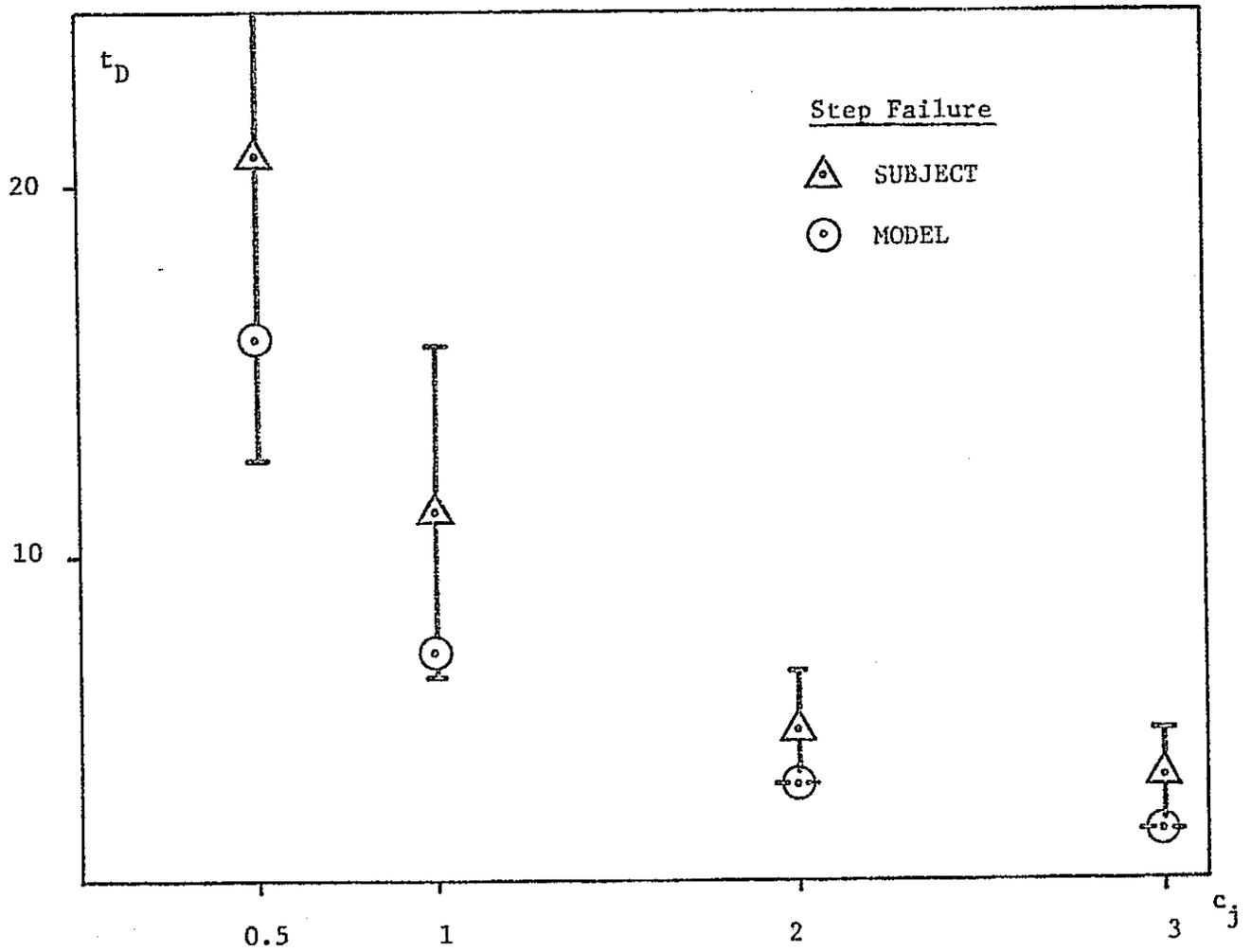
FIGURE 6.15   DETECTION TIME FOR STEP FAILURES (SUBJECT A.C)
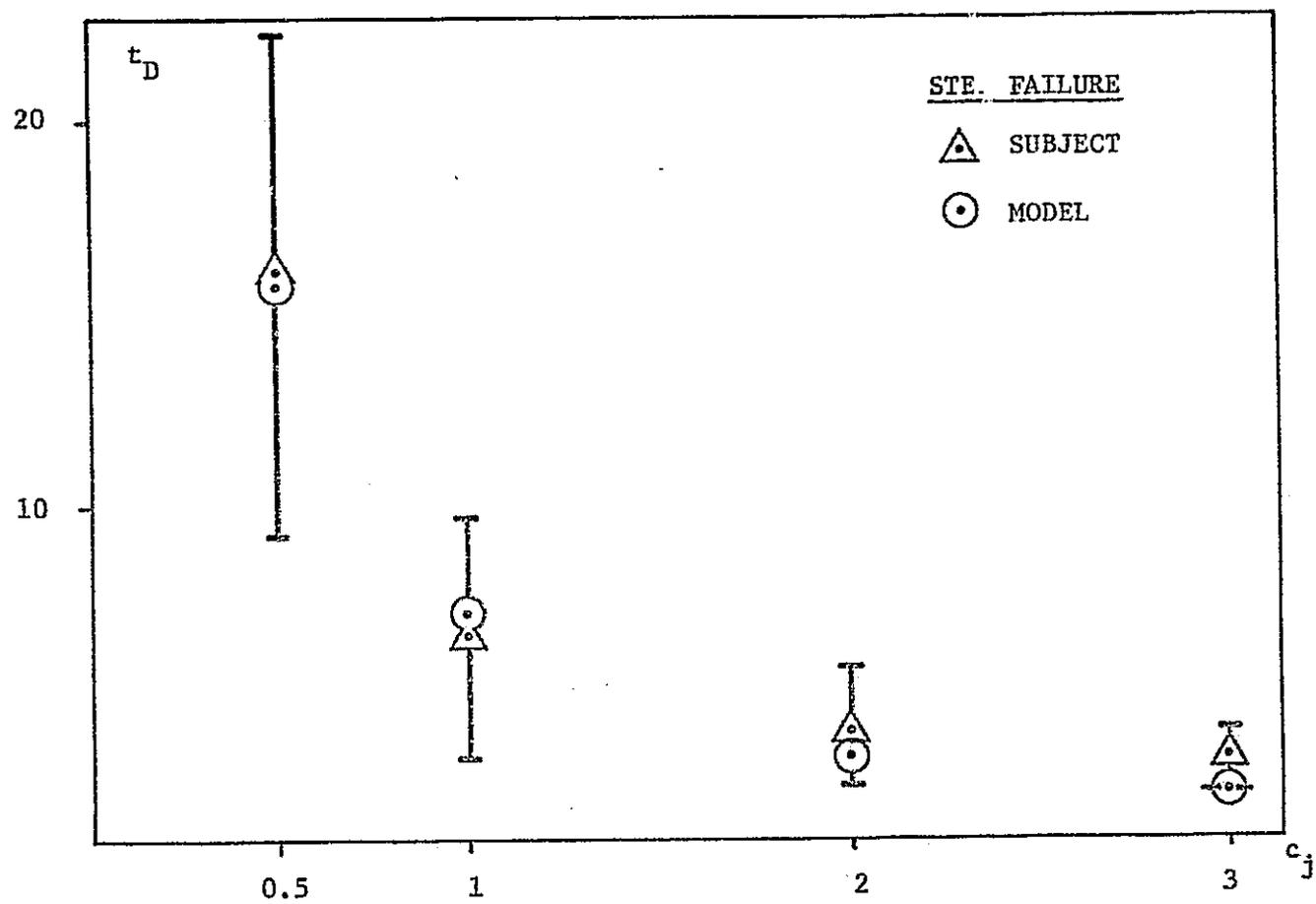
FIGURE 6.16   DETECTION TIME FOR STEP FAILURES (SUBJECT B.C)

where $\varepsilon_n(t)$ - value of the residual for the normal mode

$\varepsilon_m(t)$ - filter response to the deterministic failure

Therefore, equation (6.43) can be written:

$$\int_{t_f}^{t_d} \dot{\varepsilon}_n(t)\,dt + \int_{t_f}^{t_d} \varepsilon_m(t)\,dt = \text{constant} \qquad (6.48)$$

since

$$E\{\varepsilon_n(t)\} = 0$$

taking the expectation value of equation (6.48) the first integral vanishes and the result is

$$E\{\int_{t_f}^{t_d} \varepsilon_m(t)\,dt\} = \text{constant} \qquad (6.49)$$

For the first experimental session

$$\varepsilon_m(t) = \alpha C_j \qquad j = 1,\ldots,4$$

where $\alpha$ is the steady state attenuation of the filter. Substituting into equation (6.49) gives

$$C_j E\{(t_d - t_f)_j\} = K_j = \text{constant} \quad j = 1,\ldots,4 \qquad (6.50)$$

Equation (6.50) shows that for a step failure the product of the magnitude of the step and the mean time to detection is a constant value. The $\overline{K}_j$ values for both subjects are shown in Table 6.1 as well as $\sigma^2_{k_j}$ which is defined as

$$\sigma_{k_j}^2 = C_j{}^2 \sigma^2 t_d \tag{6.51}$$

Let us now test the null hypothesis $H_0$:

$$H_0 : \quad \overline{K}_1 = \overline{K}_2 = \overline{K}_3 = \overline{K}_4'$$

using analysis of variance. The ratio of variances F is defined as follows

$$F = n\sigma_b{}^2/\sigma_p{}^2 \tag{6.52}$$

where

$$\sigma_p^2 = \frac{1}{r} \sum_{j=1}^{r} \sigma^2{}_j$$

$$\sigma_b^2 = \frac{1}{r-1} \sum_{j=1}^{r} (K_j - \overline{K})$$

and

n - number of samples within each group

r - number of groups

$\overline{K}$ - the mean of $\overline{K}_j$     $j = 1,\ldots,r$

The results are summarized in Table 6.2.

| Subject | $\overline{K}$ | n | r | $\sigma^2{}_b$ | $\sigma^2{}_p$ | F | $F_{.05}$ |
|---------|------|----|---|--------|------|------|------|
| A.C. | 0.67 | 32 | 4 | 0.0039 | 0.06 | 2.08 | 2.68 |
| B.C. | 0.50 | 32 | 4 | 0.0020 | 0.04 | 1.60 | 2.68 |

TABLE 6.2   Results of Analysis of Variance for $\overline{K}$

The results of table 6.2 show that the hypothesis $H_0$ cannot be rejected.

In the second session, the same step failures as in the first session were included in the presentation but with

a closed observation interval of ten seconds. Table 6.3 gives the mean and variance of the detection time for both subjects.

| Subject | | 1 | 2 | 3 | 4 |
|---------|--------------|------|------|------|------|
| A.C. | $E(t_d)$ | 4.40 | 3.52 | 2.82 | 2.57 |
| | $\sigma_{t_d}$ | 1.25 | 1.15 | 0.90 | 0.57 |
| B.C. | $E(t_d)$ | 4.82 | 4.22 | 3.42 | 2.45 |
| | $\sigma_{t_d}$ | 1.65 | 1.27 | 0.80 | 0.55 |

TABLE 6.3   Results from Second Experimental
Session   (seconds)

These results are also shown in figures 6.17 and 6.18. The figures also include the prediction of our model with the same parameters as in (6.43) but with the modification for closed intervals (equation 6.40). The results show considerable decrease in detection times as expected. Also, the hyperbolic relation of equation (6.50) does not hold because the CL is time varying.

In the third session, the failures were ramp functions of time with open observation intervals. The main objective for including time dependent failures was to test the integration property that is suggested by equation (6.46). For ramp failures, the value of $\varepsilon_m(t)$ is given by

$$\varepsilon_m(t) = \beta \overset{\alpha}{C}_j (t - t_f) \qquad\qquad j = 1,\ldots,4$$
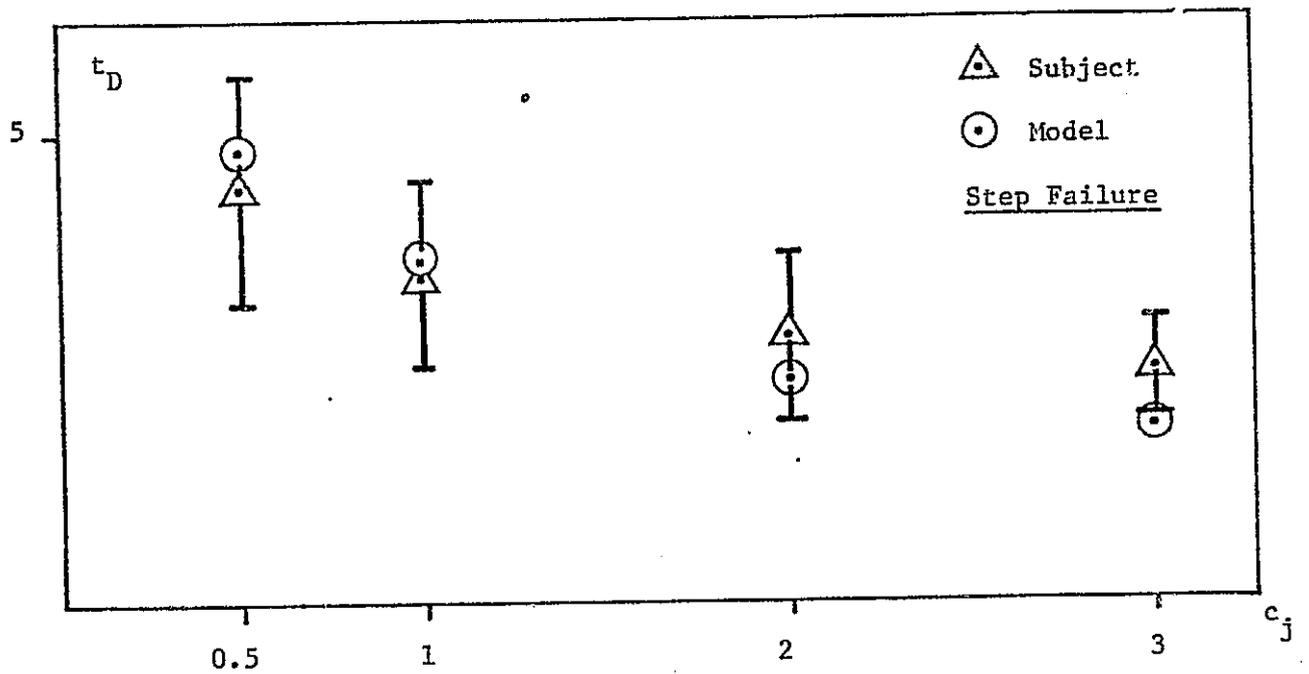
FIGURE 6.17   DETECTION TIME FOR CLOSED DECISION INTERVALS (SUBJECT A.C)
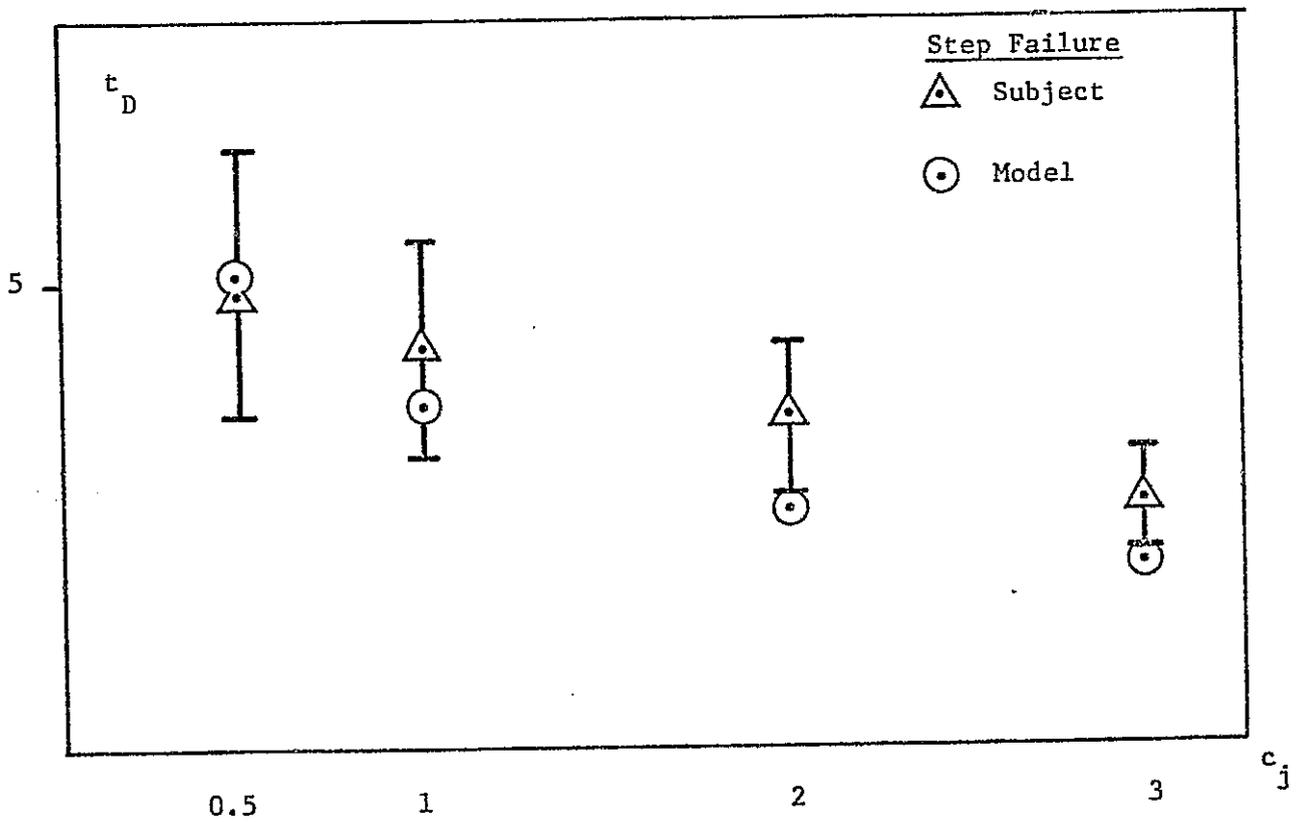


FIGURE 6.18   DETECTION TIME FOR CLOSED DECISION INTERVALS (SUBJECT B.C)

Substituting into equation (6.46):

$$(\hat{c}_j/2)E[(t_d - t_f)^2] = \hat{K}_j = \text{constant} \quad j = 1,\ldots,4 \quad (6.53)$$

Table 6.4 shows the mean and the variance of the detection time $(t_d - t_f)$ for both subjects (based on 128 samples, 32 for each level). The table also shows the values of $\overline{\hat{K}}$ and $\sigma^2\overline{\hat{K}}$. These results are also plotted in figures 6.19 and 6.20. The figures also include the prediction of our model with the following parameters:

$$\text{SNR} = 36 \quad P_{miss} = P_{FA} = 0.05 \quad \hat{\theta}_1 = 1/4$$

The predictions seem to fit the experimental results well. Next,,the relation that is suggested by equation (6.53) is tested to show that the decision function is the integral of the residuals. The hypothesis $H_0$ to be te ted by the analysis of variance is:

$$H_0: \quad \overline{\hat{K}}_1 = \overline{\hat{K}}_2 = \overline{\hat{K}}_3 = \overline{\hat{K}}_4$$

The results of the test are summarized in Table 6.5 and show that the hypothesis $H_0$ cannot be rejected.

The results that were presented in this section were based on the first and second order statistics of the data that was collected from the subjects and the simulation. In order to complete the analysis, the values of the decision function $\lambda(m)$ at each detection time that was found in the experiment were computed from the simulation. Figures 6.21 to 6.24

| Subject | | 1 | 2 | 3 | 4 |
|---------|---|---|---|---|---|
| A.C. | $E(t_d)$ | 12.30 | 9.22 | 6.77 | 5.32 |
| | $\sigma_{t_d}$ | 3.35 | 2.27 | 1.32 | 1.05 |
| | $\overline{\tilde{K}}$ | 0.41 | 0.45 | 0.47 | 0.44 |
| | $\sigma^2_{\tilde{K}}$ | 0.050 | 0.042 | 0.035 | 0.038 |
| B.C. | $E(t_d)$ | 13.20 | 9.85 | 7.07 | 5.70 |
| | $\sigma_{t_d}$ | 3.80 | 2.36 | 2.27 | 1.025 |
| | $\overline{\tilde{K}}$ | 0.47 | 0.51 | 0.54 | 0.50 |
| | $\sigma^2_{\tilde{K}}$ | 0.048 | 0.046 | 0.230 | 0.032 |

TABLE 6.4   Results from Third Experimental Session

($E(t_d)$   and $\sigma_{t_d}$   are in seconds)
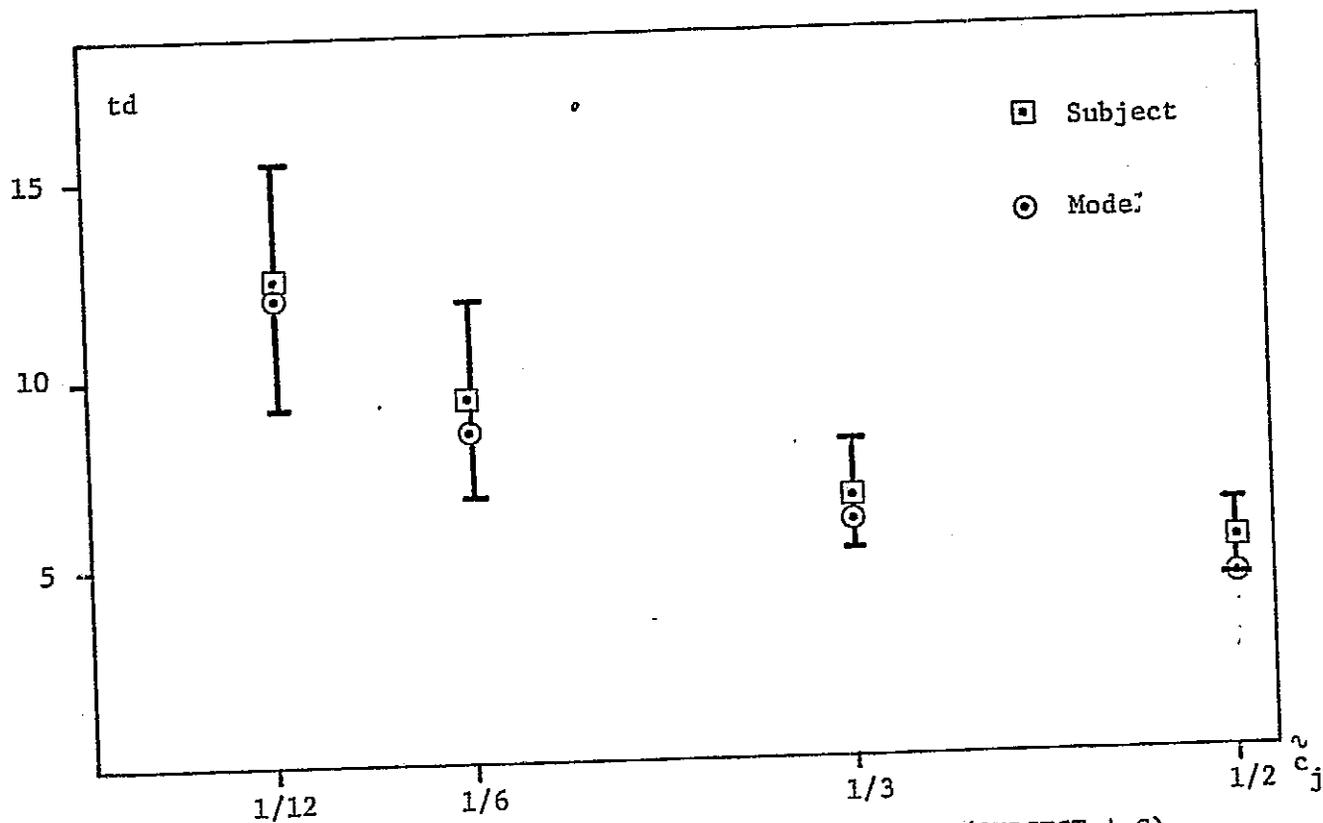
FIGURE 6.19   DETECTION TIME FOR RAMP FAILURES (SUBJECT A.C)
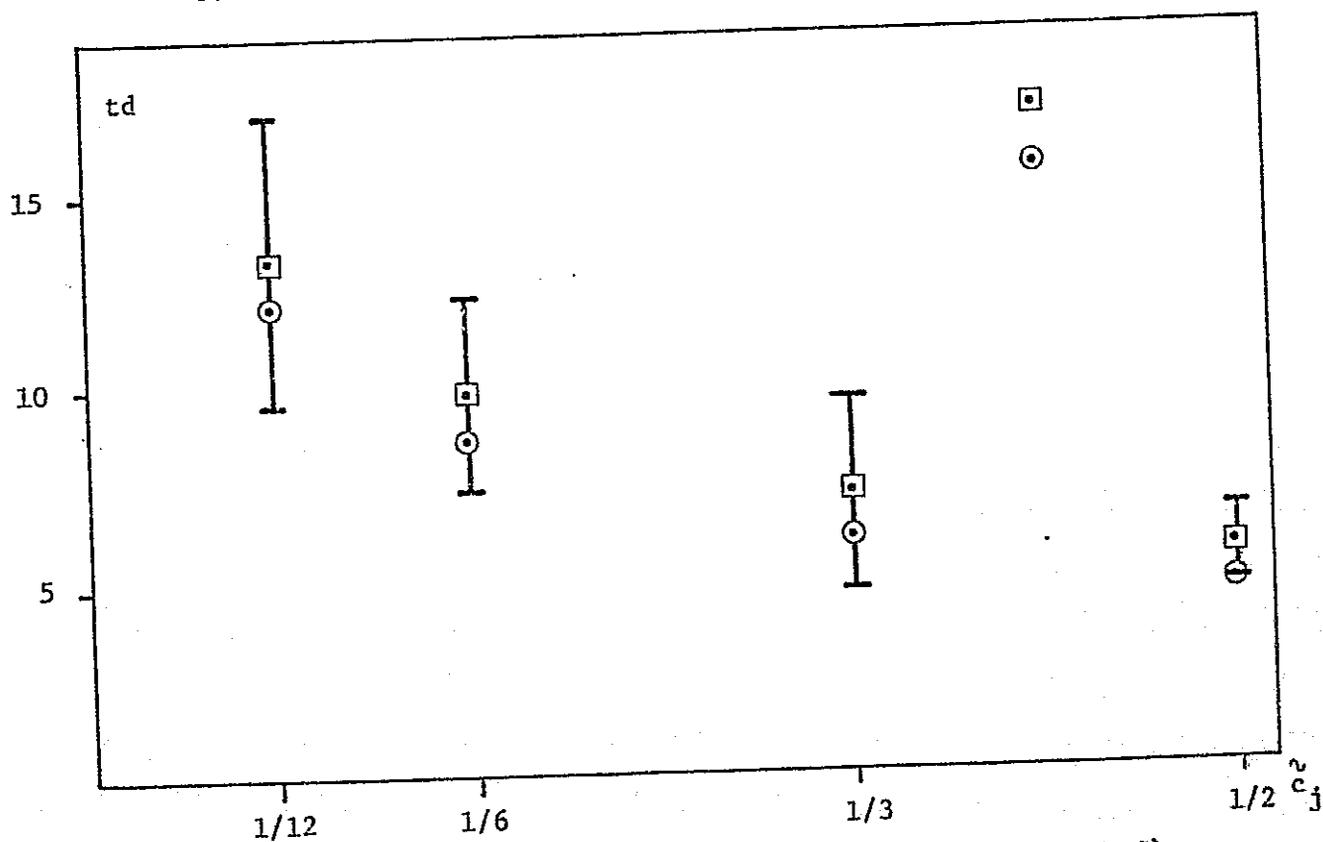


FIGURE 6.20   DETECTION TIME FOR RAMP FAILURES (SUBJECT B.C)

| Subject | $\bar{K}$ | n | r | $\sigma^2_b$ | $\sigma^2_p$ | F | $F_{.05}$ |
|---------|-----------|---|---|--------------|--------------|---|-----------|
| A.C. | 0.44 | 32 | 4 | 0.041 | 0.00063 | 0.49 | 2.68 |
| B.C. | 0.50 | 32 | 4 | 0.081 | 0.00086 | 0.34 | 2.68 |

TABLE 6.5  Results of Analysis of Variance for $\tilde{K}$
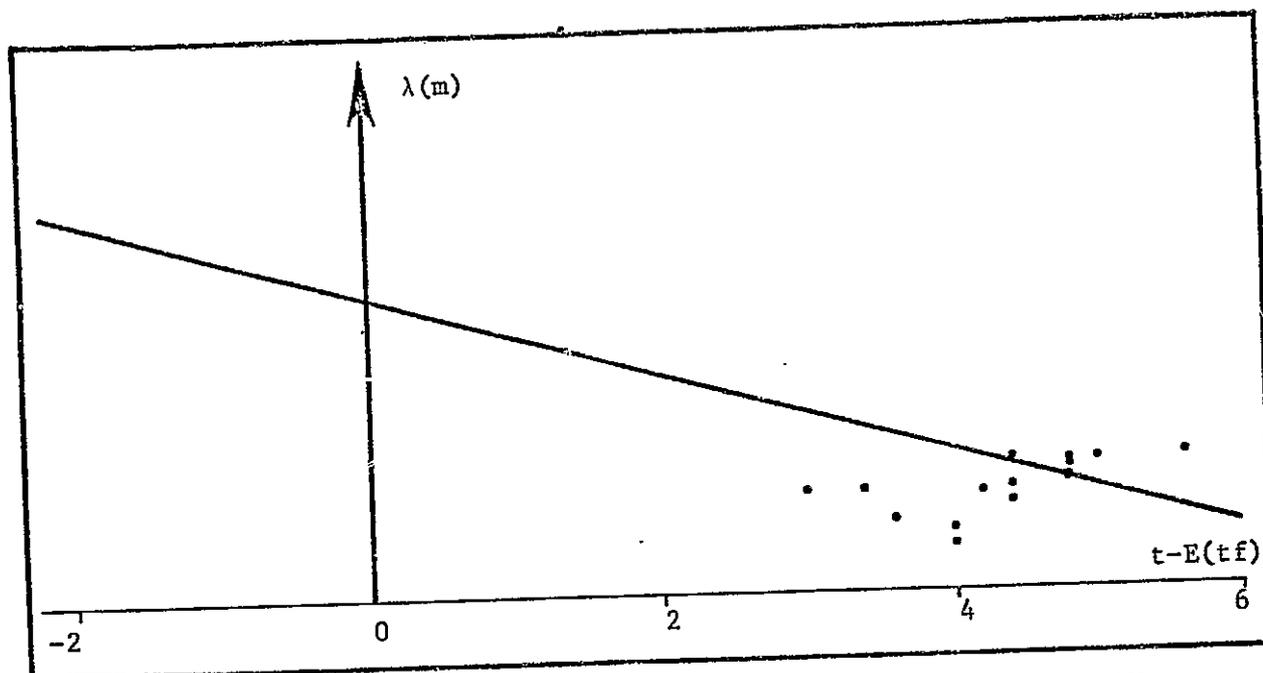
6.21 VALUES OF THE DECISION FUNCTION AT THE ACTUAL DETECTION
TIME OF THE SUBJECT (CLOSED INTERVALS, $c_j = 0.5$)
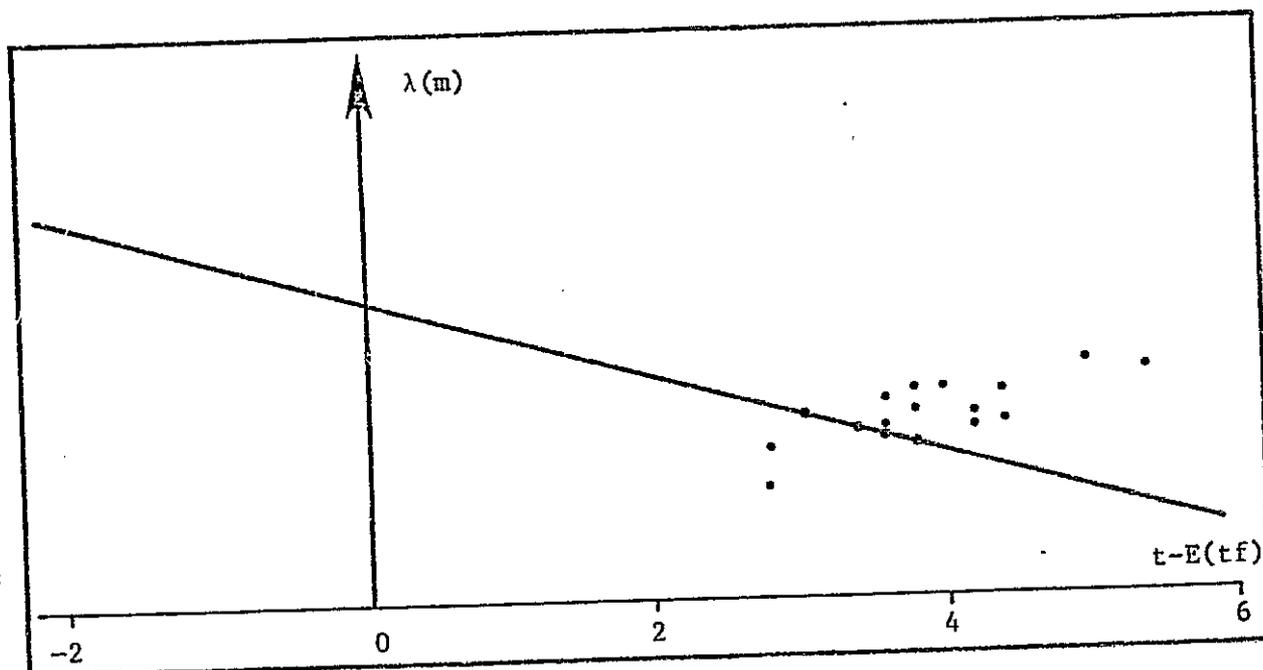


6.22 VALUES OF THE DECISION FUNCTION AT THE ACTUAL DETECTION
TIME OF THE SUBJECT (CLOSED INTERVALS, $c_j = 1$)
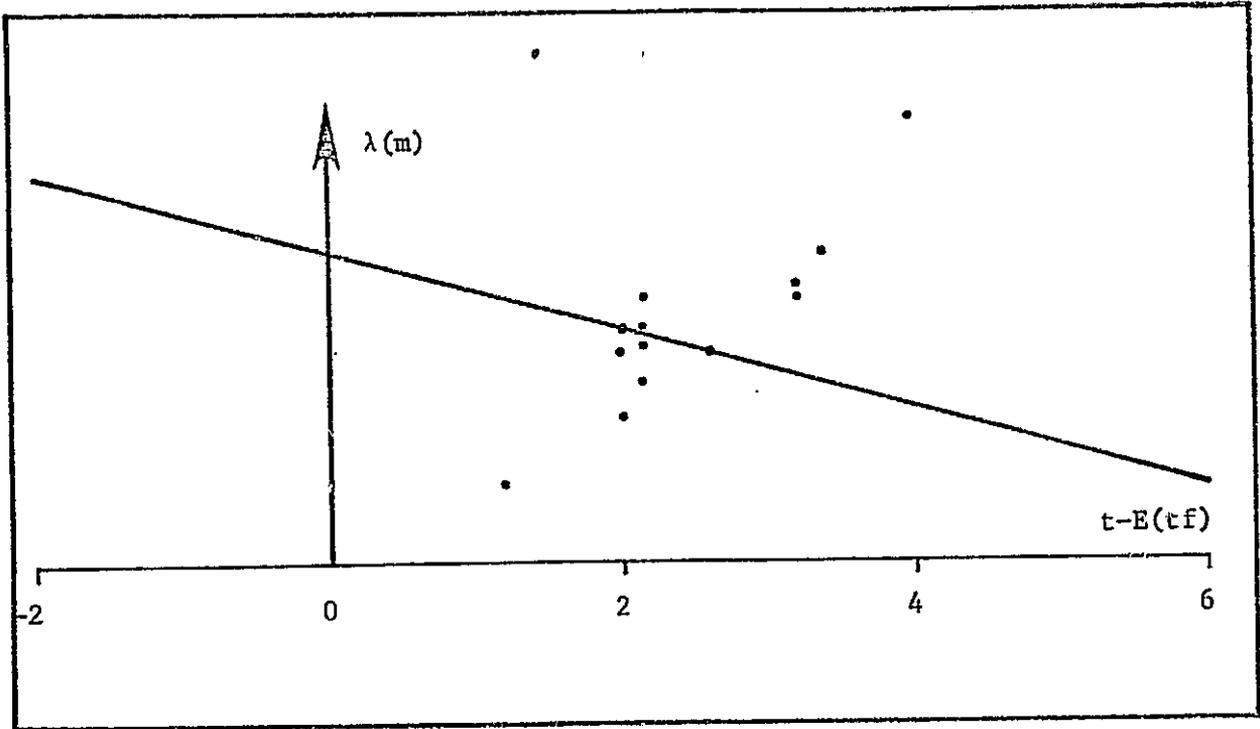
6.23 VALUES OF THE DECISION FUNCTION AT THE ACTUAL DETECTION
TIME OF THE SUBJECT (CLOSED INTERVALS, $c_j = 2$)
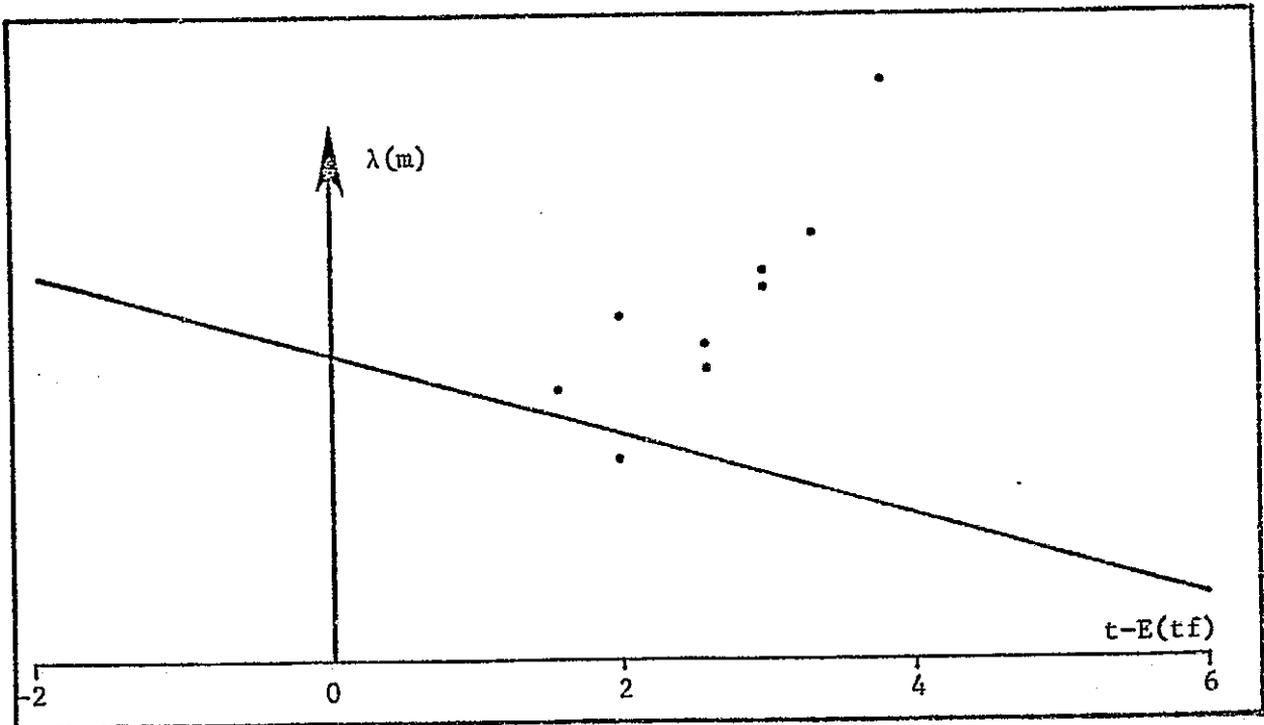


6.24 VALUES OF THE DECISION FUNCTION AT THE ACTUAL DETECTION
TIME OF THE SUBJECT (CLOSED INTERVALS, $c_j = 3$)

show the values of the decision function for the four failure

levels in the first experimental session (open intervals, step

failure). Figures 6.25 to 6.28 show these values for the

second experimental session (closed interval, step failure).

These figures also show the LRCL that was used in the simulation.

Those results that are due to the two stage operation of the

model give a unique opportunity to observe an internal quan-

tity which cannot be directly measured.
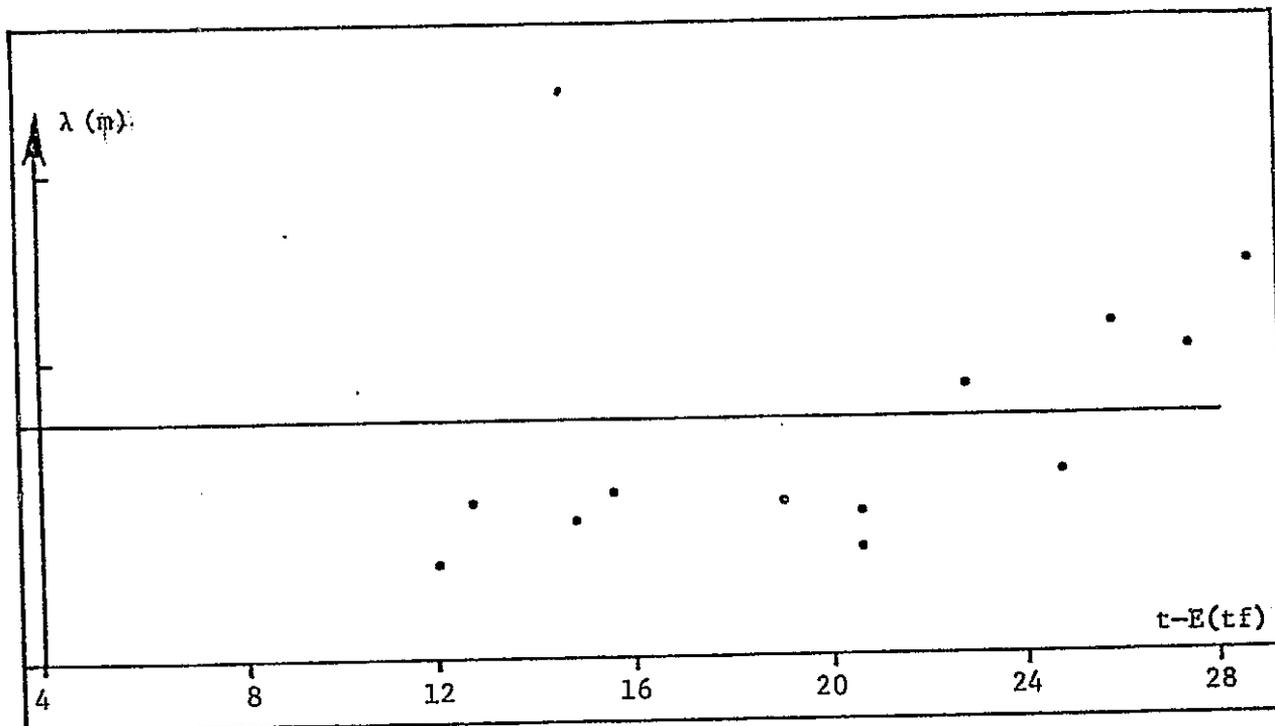
6.25  VALUES OF THE DECISION FUNCTION AT THE ACTUAL DETECTION
TIME OF THE SUBJECT (OPEN INTERVALS, $c_j = 0.5$)
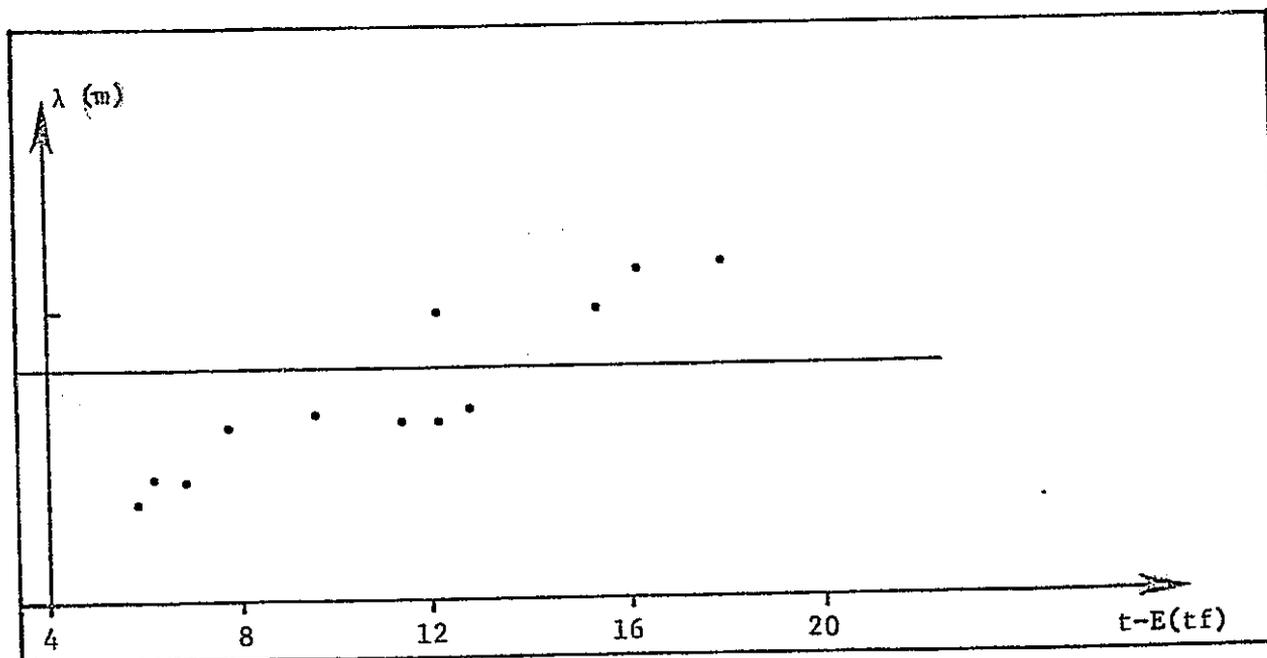


6.26  VALUES OF THE DECISION FUNCTION AT THE ACTUAL DETECTION
TIME OF THE SUBJECT (OPEN INTERVALS $c_j = 1$)
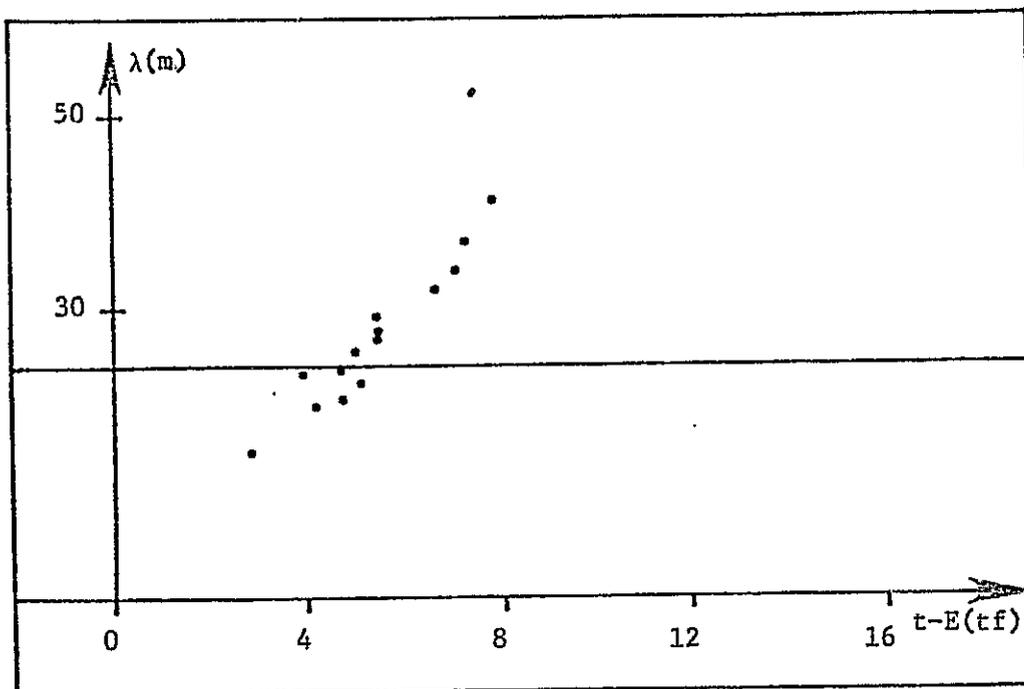
FIGURE 6.27 VALUES OF THE DECISION FUNCTION AT THE ACTUAL
DETECTION TIME OF THE SUBJECT (OPEN INTERVALS $c_j$ = 2)
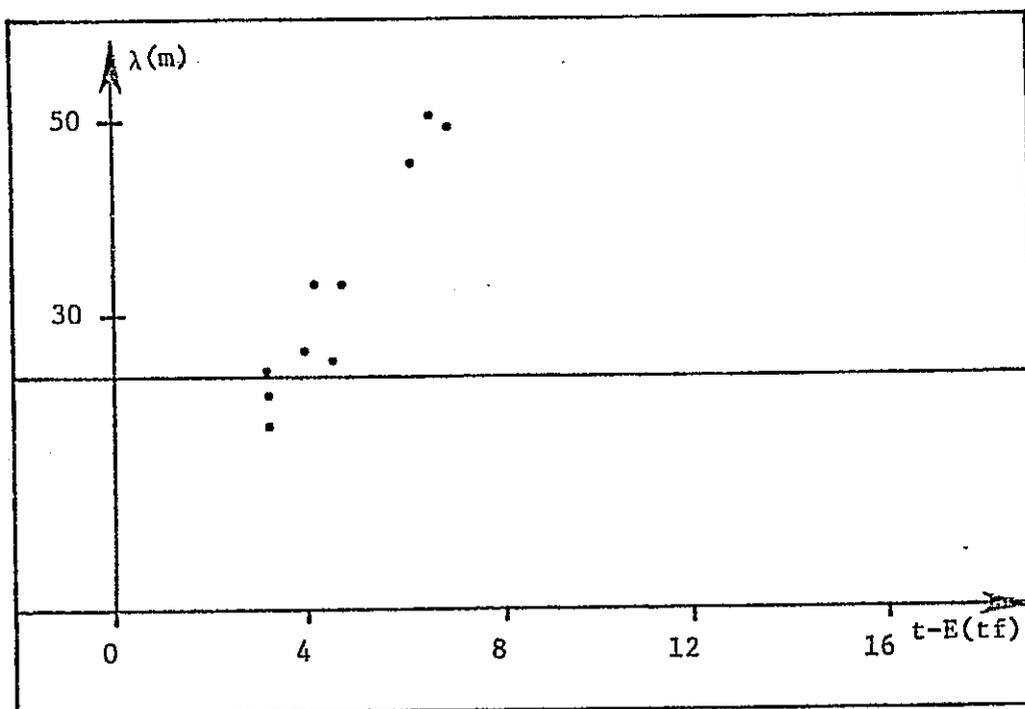


FIGURE 6.28 VALUES OF THE DECISION FUNCTION AT THE ACTUAL
DETECTION TIME OF THE SUBJECT (OPEN INTERVALS $c_j$ = 3)

# CHAPTER VII

## THE USE OF THE FAILURE DETECTION MODEL FOR MONITORING AUTOMATIC LANDINGS

### 7.1  General Discussion

In Chapter 6 we presented a model for the human observer in a failure detection task.  The experiments that were run for the evaluation of the validity of the model included a "well-behaved" process for the normal mode of operation.  In particular, the shaping filter was a stable linear second order time invariant system and the observation was a scalar. These characteristics simplified the implementation of the detection model so that its performance could be easily compared to the performance of the subjects.

In this chapter we would like to show that the suggested model can be applied in more complicated situations that arise in real life detection tasks.  Even if the processes involved do  not have any of the nice properties that characterized the former experiments, the model can still be used with some modifications.

The task of monitoring airplane instruments  during an automatic landing is an appropriate example.  The processes that are involved are characterized by a non linear, high order and time varying system.  In addition, there are several instruments to be monitored simultaneously, so that the observations are multidimensional.  Other reasons for this choice

are the current interest in the problem due to the introduction of "all weather" landing systems and the availability of the equipment to perform an accurate simulation of the task.

## 7.2 Simulation of a Jet Transport during Automatic Landing

This section deals with the description of the equations of motion for a jet transport during automatic landing. These equations are the basis for the simulation that was used in our experiments.

Let us define a coordinate system (X', Y', Z') with the origin at the touch down point, the X' axis in the direction of the north, Z' is perpendicular to the ground (positive upward) and the Y' axis completes the right orthogonal triad. If the initial position (at $t = t_0$) of the airplane (X'$_0$, Y'$_0$, Z'$_0$) is given, then its position at any future time ($t > t_0$) is completely defined by the following three variables

$v(t)$ - airplane velocity

$\psi(t)$ - course (rotation of the velocity vector with respect to the Z' axis)

$\gamma(t)$ - vertical inclination (rotation of the velocity vector with respect to the Y axis)

where the frame (X, Y, Z) is obtained by rotation of the frame (X', Y', Z') by $\psi(t)$ around the Z' axis.

For a complete knowledge of the airplane attitude, three additional variables are needed and are defined by:

$\alpha(t)$ - angle of attack

$\phi(t)$ - bank angle

$\beta(t)$ - side slip angle

These six time functions are the state variables that define the motion of a rigid body with six degrees of freedom when angular accelerations are neglected. The state equations are given by

$$\dot{v} = \frac{g}{w}(T\cos\alpha\cos\beta - D - L\sin\alpha - w\sin\gamma)$$

$$\dot{\psi} = \frac{g}{wv\cos\gamma}(L\sin\phi - T\cos\alpha\sin\beta\cos\phi)$$

$$\dot{\gamma} = \frac{g}{wv}[(L\cos\alpha + T\sin\alpha\cos\beta)\cos\phi - w\cos\gamma]$$

$$\dot{\alpha} = q\cos\beta - p\sin\beta - \dot{\gamma}\cos\beta - \dot{\psi}\cos\gamma\sin\phi$$

$$\dot{\beta} = \dot{\psi}(\cos\alpha\cos\gamma\cos\phi - \sin\alpha\sin\gamma) - \dot{\gamma}\cos\alpha\sin\phi - r$$

$$\dot{\phi} = p\cos\alpha\cos\beta + q\cos\alpha\sin\beta + r\sin\alpha + \dot{\psi}\sin\gamma$$

(7.1)

where

g — gravitational acceleration

w — airplane weight

T — thrust

D — drag

L — lift

The weight during landing and the coefficients of drag and lift for a DC8 (which is similar to a Boeing 707) were taken from Tepper (1969). The three variables p, q, and r in equation (7.1) are the control angular velocities of the

airplane around the roll, pitch and yaw axes respectively, and are given by:

$$p = \frac{1}{I_x}(K_1\phi_c + k_2\psi_c)v^2$$

$$q = \frac{1}{I_y}[k_3\theta_c v^2 + k_6(Lcos\alpha cos\phi - wcos\gamma)] \qquad (7.2)$$

$$\dot{r} = \frac{1}{I_z}[k_4(\psi_c + \beta - 12490r/v)v^2 - k_5\phi_c v^2]$$

where $\theta_c$, $\phi_c$, $\psi_c$ are the pitch, roll and yaw commands which are given by (Ephrath, 1975):

$$\theta_c = 0.5FD_p + 3F\dot{D}_p$$

$$\psi_c = FD_r + 3F\dot{D}_r \qquad (7.3)$$

$$\phi_c = -3\beta$$

$FD_p$ and $FD_r$ are pitch and roll commands of a linear flight director system and their Laplace transform is given by (Weir et al, 1970)

$$FD_p = -0.0003h_e - \frac{0.3s}{s + 0.34}\theta$$

$$\qquad (7.4)$$

$$FD_r = \frac{-0.62s(8.6\psi + 0.9\phi + 180\varepsilon)}{(s+1.06)(s+0.16)} - 0.27\phi - 1.56\varepsilon$$

where:

$h_e$ – vertical error between aircraft position and glideslope beam

$\varepsilon$ – horizontal angular error between aircraft position and localizer beam

An additional feedback loop that changes the thrust in such a way as to keep the velocity consant ($\dot{v} = 0$) is also included.

The above equations were used by Ephrath (1975) for a simulation of an automatic landing with the Adage model 30 graphics computer and a Boeing 707 fixed-base simulator. A detailed description of the derivation of the equations and the simulator is given in Ephrath (1970).

Since the landing of the airplane is fully automatic, the pilot task is monitoring and detecting failures. The instruments which are displayed in the cockpit and which the pilot can use for his monitoring tasks are (the variables that are displayed by each instrument are shown in brackets)

Glide slope indicator  $[h_e]$

Localizer indicator  $[\varepsilon]$

Attitude indicator  $[\theta, \phi]$

Horizontal situation indicator  $[\psi, \beta, \varepsilon]$

Air speed indicator  $[v]$

Altimeter  $[z]$

Vertical speed indicator  $[\dot{z}]$

where

$$h_e = \frac{z}{x} \tan(-3°) \qquad \text{($z$, $x$ are in ft)}$$

$$\varepsilon = \frac{y}{1.23-x} \qquad \text{($y$, $x$ are in nm)} \qquad (7.5)$$

$$\theta = \alpha + \gamma - 2 \qquad \text{(degrees)}$$

and the frame $(X, Y, Z)$ is obtained by rotation of the frame $(X', Y', Z')$ by 35° clockwise around the $Z'$ axis.

## 7.3 Simplification of the Airplane Dynamics

As mentioned in the previous section equations (7.1) and (7.2) were used to simulate the dynamics of a Boeing 707 during landing. Several professional 707 pilots landed the simulated airplane and were completely satisfied by the resemblance between the simulated dynamics and the performance of the real airplane. Furthermore, when the automatic landing system was applied, it proved to be capable of landing the airplane within the designed specifications. Also, the mean values of all the variables that were presented to the monitoring pilot had the specified nominal values, with perturbations due to outside disturbances. Therefore it is reasonable to assume that for the analysis of the performance of the monitoring pilot, it is possible to linearize equations (7.1) and (7.2) around the nominal values. Such an assumption is usually made for a preliminary design of the control loops (Blacklock, 1965).

Even when the system linearized, the dimension of the state vector is large (9), a fact that considerably complicates the computations. Therefore, for design purposes, another simplification is made by assuming that there is no coupling between the longitudinal and lateral dynamics. Instead of dealing with one nine state system, there are two independent four state subsystems and one scalar subsystem. This last scalar subsystem controls the airplane velocity, and is needed to guarantee proper behavior of the longitudinal control (Blacklock, 1965).

The basic block diagrams for the three control loops were taken from Blacklock (1965). The velocity control loop is shown in the following block diagram:
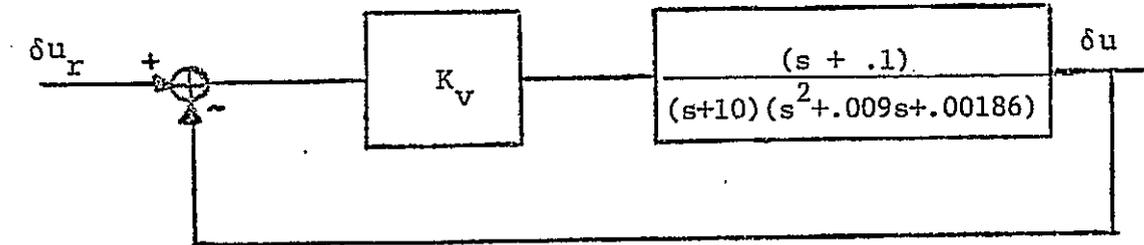


where $\delta u_r$ is the perturbation around the nominal velocity and is modelled as a zero mean white Gaussian process with variance $\sigma^2_{\dot{u}_r}$. The real pole represents the throttle servo, and the complex pair, the phugoid oscillations. The root locus of the velocity control system is shown in Figure 7.1. For $K_v$ equal to 10, the closed loop transfer function is given by:

$$\frac{\delta u(s)}{\delta u_r(s)} = \frac{10(s + 0.1)}{(s + 8.8)(s + 0.98)(s + 0.13)} \qquad (7.6)$$

The transfer function for the vertical inclination control is

$$\frac{\delta\gamma(s)}{\delta\theta(s)} = \frac{0.535}{s+0.585} \qquad (7.7)$$

where
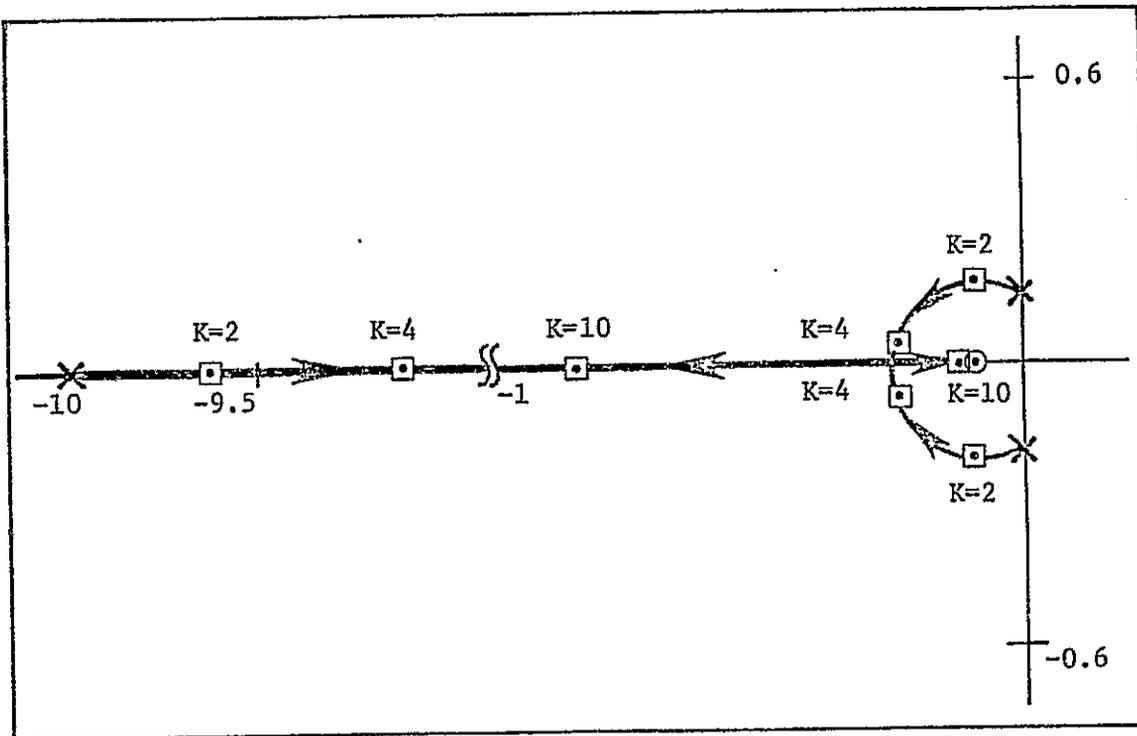
FIGURE 7.1   ROOT LOCUS FOR AIR SPEED CONTROL SYSTEM
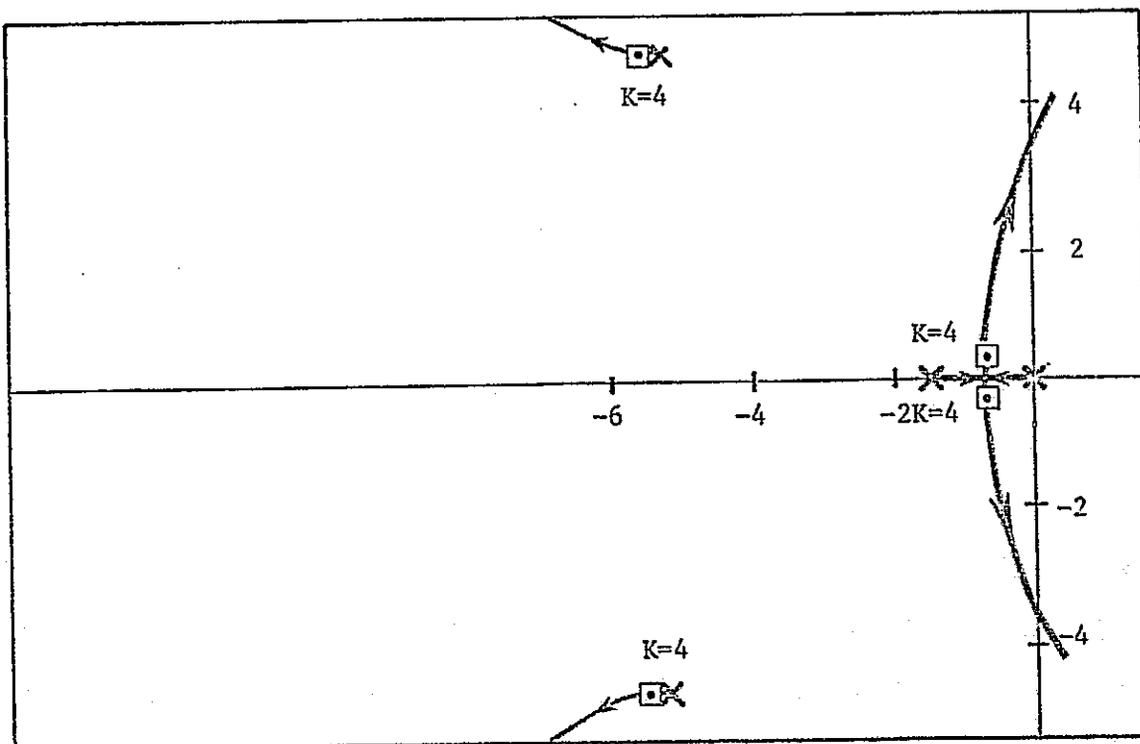


FIGURE 7.2   ROOT LOCUS FOR HEADING CONTROL SYSTEM

$$\frac{\delta\theta(s)}{\delta\theta_r(s)} = \frac{53.5(s + 0.585)}{(s + 0.5)(s + 5.5)(s^2 + 5.4s + 11.4)} \tag{7.8}$$

Again, $\delta\theta_r$ is the perturbation around the nominal pitch angle and is modelled as a zero mean white Gaussian process with variance $\sigma^2_{\theta_r}$. Sustituting equations (7.8) into 7.7):

$$\frac{\delta\gamma(s)}{\delta\theta_r(s)} = \frac{31.3}{(s + 0.5)(s + 5.5)(s^2 + 5.4s + 11.4)} \tag{7.9}$$

The third control loop is the heading control and is shown in the following block diagram
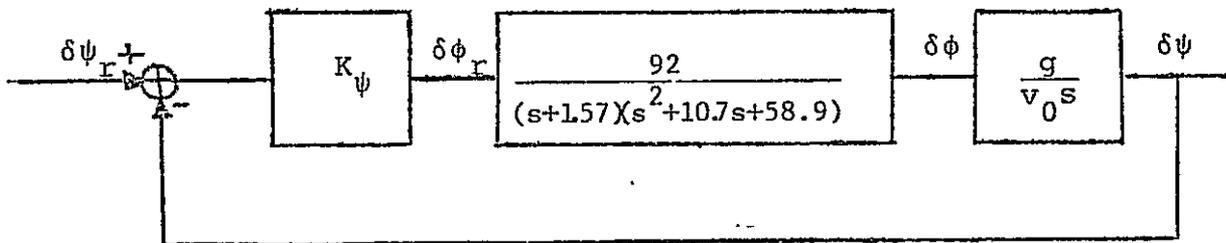


where $v_0$ is the nominal air speed (150 knots), and $\delta\psi_n$ is the heading perturbation modelled as zero mean white Gaussian process. The root locus for the heading control system is shown in Figure 7.2. For $K_\psi = 3.8$, the closed loop transfer function is

$$\frac{\delta\psi(s)}{\delta\psi_r(s)} = \frac{47}{(s^2 + 11s + 58)(s^2 + 1.5s + 0.81)} \tag{7.10}$$

The position of the airplane (x, y, z) is a function of the three controlled variables and is given by:

$$\dot{x} = v \cos\psi \cos\gamma$$
$$\dot{y} = v \sin\psi \cos\gamma \qquad (7.11)$$
$$\dot{z} = v \sin\gamma$$

Since $\gamma_0$ is small (3.0°) a small angle approximation can be used, so that

$$\dot{x} = v \cos\psi$$
$$\dot{y} = v \sin\psi \qquad (7.12)$$
$$\dot{z} = v\gamma$$

The perturbations in the airplane position $\delta x$, $\delta y$, $\delta z$ around the nominal values are therefore

$$\delta\dot{x} = \cos\psi_0 \delta v - v_0 \sin\psi_0 \delta\psi$$
$$\delta\dot{y} = \sin\psi_0 \delta v + v_0 \cos\psi_0 \delta\psi \qquad (7.13)$$
$$\delta\dot{z} = \gamma_0 \delta v + v_0 \delta\gamma$$

where the nominal values are:

$$v_0 = 150 \text{ knots}; \quad \gamma_0 = -3.0°; \quad \psi_0 = 35°$$

Equations (7.6),(7.9), (7.10) and (7.13) imply that the state space description of the whole system inovlves a total of 14 states. Although these states can be divided into three independent groups of four, five and five states respectively, it is assumed that the monitoring pilot bases his decision on a further simplified system in which he uses only the dominant poles. Therefore, the next step would be to simplify the three

basic subsystems given by equations (7.6), (7.9) and (7.10). For the velocity control system we will neglect the pole at s = -8.8, and also assume that the zero is cancelled by one of the other poles. Also the gain is adjusted to give an equivalent steady state gain. The simplified transfer function is therefore as follows:

$$\frac{\delta u(s)}{\delta u_r(s)} = 1/(s+1) \qquad (7.14)$$

for the vertical inclination control loop only two real poles would be used so the transfer function is:

$$\frac{\delta \gamma(s)}{\delta \theta_r(s)} = \frac{1.35}{s^2 + 3.2s + 1.35} = \frac{1.35}{(s + 0.5)(s + 2.7)} \qquad (7.15)$$

The new pole and the new gain were chosen in such a way that the steady state gain, as well as the steady state variance for a given stationary random input would be the same as for the original system (7.9). The time response of the original and simplified system to a step input are shown in Figure 7.3. The difference in the transient seems to be small enough to justify the approximation.

For the heading control system the far left half plane pair of complex poles was omitted and the gain was adjusted to give an equivalent steady state gain. The simplified transfer function is:

$$\frac{\delta \psi(s)}{\delta \psi_r(s)} = \frac{0.81}{s^2 + 1.5s + 0.81} \qquad (7.16)$$
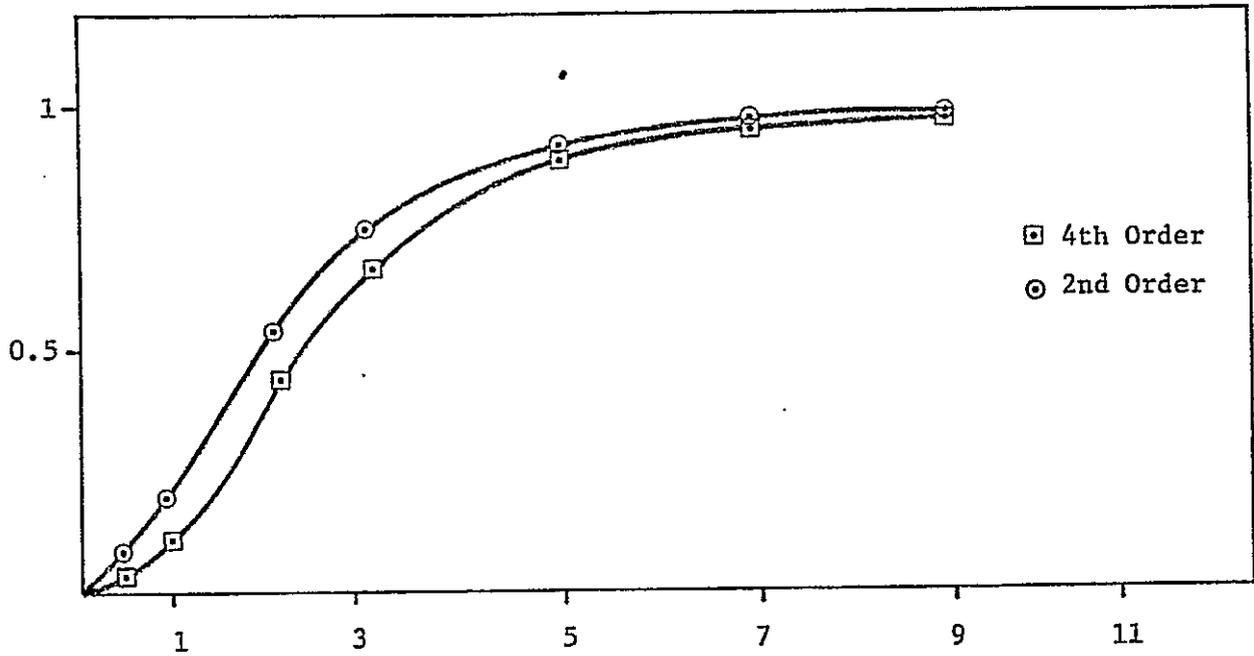
FIGURE 7.3   RESPONSE TO A UNIT STEP INPUT OF ORIGINAL AND
SIMPLIFIED VERTICAL INCLINATION CONTROL SYSTEM
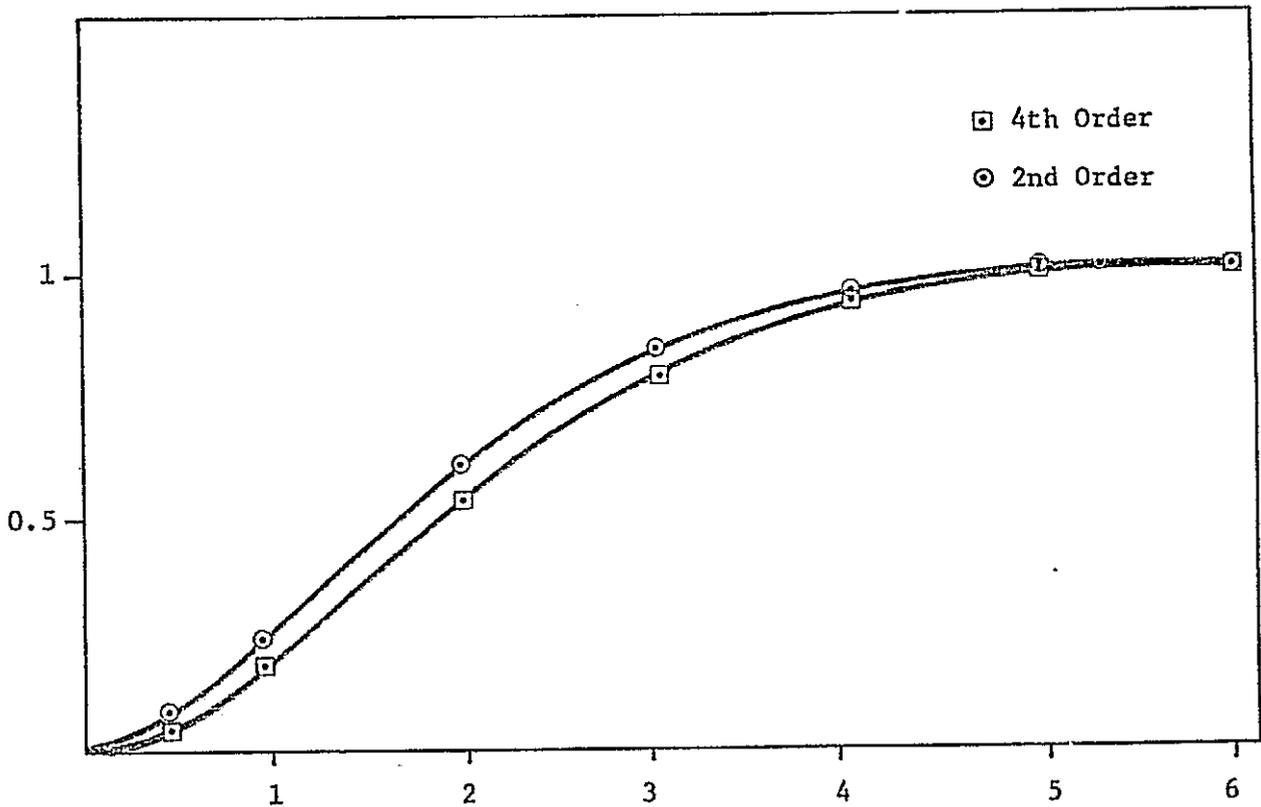


FIGURE 7.4   RESPONSE TO A UNIT STEP INPUT OF ORIGINAL AND
SIMPLIFIED HEADING CONTROL SYSTEM

The time response of the simplified and original systems to a step input are shown in Figure 7.4. Again the small difference justifies the simplification.

The simplified version is eight dimensional, and can be divided into three independent subsystems of order two, three and three respectively. Let $\underline{x}$ be the eight dimensional state vector

$$\underline{x}^T = (x_1, x_2, \ldots, x_8)$$

and let us define

$$x_2 = \delta u \qquad x_4 = \delta \gamma \qquad x_7 = \delta \psi \qquad (7.17)$$

also let

$$\dot{x}_1 = x_2, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = x_5, \quad \dot{x}_6 = x_7, \quad \dot{x}_7 = x_8 \qquad (7.18)$$

then the state equation for the three subsystems is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u_r \qquad (7.19)$$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1.35 & -3.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1.35 \end{bmatrix} \delta \theta_r \qquad (7.20)$$

and

$$\begin{bmatrix} \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.81 & -1.5 \end{bmatrix} \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.81 \end{bmatrix} \delta\psi_r \qquad (7.21)$$

Let $F_1$, $F_2$, and $F_3$ be the state matrices defined in equations (7.19), (7.20) and (7.21) respectively. Then the eight dimensional system is given by the vector differential equation:

$$\dot{\underline{x}} = F\underline{x} + G\underline{u} \qquad (7.22)$$

where

$$F = \begin{bmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix} \qquad G^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .81 \end{bmatrix} \qquad (7.23)$$

and

$$\underline{u}^T = (\delta u_r, \ \delta\theta_r, \ \delta\psi_r) \qquad (7.24)$$

Since equation (7.22) represents a linear time invariant system, the transition matrix can be found by the use of the Laplace transform. Furthermore, since the matrices in (7.23) can easily be reduced to three independent subsystems (7.19), (7.20), and (7.21), the same expression for the transition matrix that was described in appendix B can be used with slight modifications.

The perturbations of the outputs that were presented to the monitoring pilot in the simulation that is described in section 7.2 can now be expressed as linear functions of the

state variables.  Substituting the state variables from (7.17)
to (7.13)

$$\delta\dot{x} = (\cos\psi_0)x_2 - v_0(\sin\psi_0)x_7$$
$$\delta\dot{y} = (\sin\psi_0)x_2 + v_0(\cos\psi_0)x_7 \qquad (7.25)$$
$$\delta\dot{z} = \gamma_0 x_2 + v_0 x_4$$

and using equation (7.18)

$$\delta x = \cos\psi_0 x_1 - v_0\sin\psi_0 x_6$$
$$\delta y = \sin\psi_0 x_1 + v_0\cos\psi_0 x_6 \qquad (7.26)$$
$$\delta z = \gamma_0 x_1 + v_0 x_3$$

also, from equations (7.5)

$$\delta h_e = -1/x_n^2(\delta x) + 1/x_n(\delta z) \qquad (7.27)$$

$$\delta\varepsilon = 1/(1.23 - x_n)^2(\delta x) + 1/(1.23 - x_n)(\delta y) \qquad (7.28)$$

where $x_n$ is the nominal x value which is time varying.  There-
fore, using the state variables, the perturbations of the dis-
played variables $\underline{y}$ are as follows:

1.  Glide slope indicator

$$y_1 = (-\cos\psi_0/x_n^2 + \gamma_0/x_n)x_1 + v_0 x_3/x_n$$
$$+ v_0\sin\psi_0 x_6/x_n^2$$

2. Localizer:

$$Y_2 = [\cos\psi_0/(1.23 - x_n)^2 + \sin\psi_0/(1.23 - x_n)]x_1$$

$$+ [v_0\cos\psi_0/(1.23 - x_n) - v_0\sin\psi_0/(1.23 - x_n)^2]x_6$$

3. Attitude indicator:

$$y_3 = x_5/0.585 + x_4$$

$$y_4 = v_0 x_8/g$$

4. Horizontal situation display:

$$y_5 = x_7$$

5. Air speed indicator:

$$y_6 = x_2$$

6. Altimeter

$$y_7 = \gamma_0 x_1 + v_0 x_3$$

7. Vertical speed indicator

$$y_8 = \gamma_0 x_2 + v_0 x_4$$

It should be noted that $y_1$ and $y_2$ are time varying linear functions of the state while $y_3 - y_8$ are time invariant.

## 7.4 The Multidimensional Failure Detection Model

From the description of the equations of motion and the control loops that were discussed in section 7.2, it is clear that the processes with which the subject has to deal in this task differ considerably from the processes that were involved in the experiments described in Chapter 6. The main differences are:

1. The equations are highly nonlinear.
2. The order of the system is high.
3. The statistics of the observed variables are time varying.
4. The observation is multidimensional.

The nonlinearity difficulty can be relieved by linearization of equations (7.1) around the nominal values of the states. This is possible because the control loops are expected to keep the state variables at their nominal values so that only the perturbations are exposed to the subject. The linearization of the system is described in detail in section 7.3.

The second problem, that of the high dimensionality, can be solved by the decoupling of the system into several subsystems of lower order. If, after the decoupling, the subsystems are still of high order, we will assume that the human observer considers only the most important modes. In

general, it is felt that the human observer will not consider more than three dominant modes. The reduction of the dimension of the problem at hand is described in the previous section. It should be noted that although the state equations can be decoupled, an eight dimensional Kalman filter should be used due to the coupled form of the observation.

Once the processes are simplified to the level that is shown in section 7.3, our model can be applied. The time variability of the observation does not affect the performance of the linear estimator, and the only disadvantage is that the Kalman gain K(t) will not reach steady state. This means that the operator must update the gain with each observation.

The fourth point that is mentioned above is the multidimensionality of the observation. This means that the operator must share his attention among several instruments. It was found (Yntema, 1963; Senders et al, 1966) that in such cases the human observer will concentrate only on the most important instruments while using the others for verification purposes. In monitoring the automatic landing, it is expected that the pilot will spend 90% of his time monitoring the glideslope localizer and airspeed indicators.

When a linear estimator is used in the model, it is possible to account for this sharing of attention through the observation noise (Levison et al, 1971). If the subject is observing more than one instrument, then his internal observation noise for each of the observations is increased by a constant factor that is inversely proportional to the time that

the subject spends in monitoring that specific instrument.
Let $t_i$ be the total time that the subject is spending observing
instrument i during the whole observation interval. Define:

$$K_i = t_i / \sum_{i=1}^{N} t_i \qquad (7.29)$$

where N is the total number of instruments that are observed.
The observation noise that is associated with the $i^{th}$ instrument is then multiplied by a factor of $1/K_i$.

For example, let us consider the situation in which the
pilot spends 40% of his time monitoring the glideslope indicator,
40% monitoring the localizer and 10% monitoring the airspeed
indicator. The block diagram of our model for such an assumption
is shown in Figure 7.5. It should be noted that since all the
instrument variables are linear functions of the state, the
number of observations included in the model is not limited;
however, any increase will cause more complicated numerical
computations.
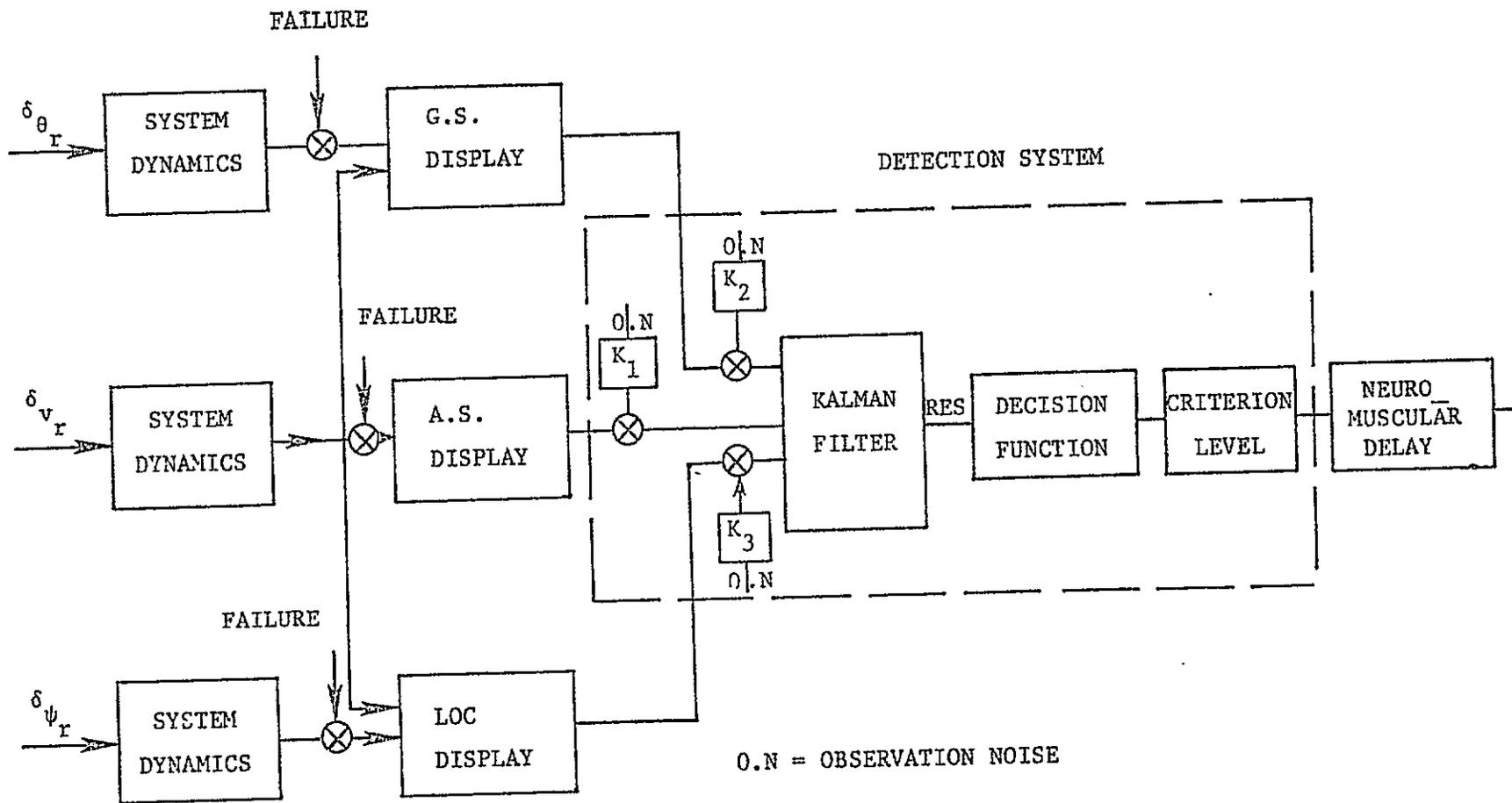
FIGURE 7.5   FUNCTIONAL DIAGRAM OF FAILURE DETECTION DURING AUTOMATIC LANDING

## 7.5  Experimental Method

### 7.5.1  Apparatus

The Adage Model 30 graphic computer was used to simulate the full equations of motion (7.1). All of the outputs that were defined in seciton 7.2 were fed into the instrument panel of a fixed base Boeing 707 simulator. This panel is shown in Figure 7.6. The simulation included only the last five minutes of flight prior to touch down, and the landing was fully automatic. The failures that were defined were instrument failures so that they affected only the output variables but were not fed back into the system. In order to minimize the dimensionality of the task but still have a multidimensional task, failures occured in two instruments. Those instruments were the glide slope indicator (GS) and the air speed indicator (AS). Four levels of failures were included for each of the two instruments. All failures were deterministic step changes that were fed to the instrument through a low pass filter with 0.1 second time constant. The magnitude of the failures for the AS indicator were

$$C_1 = 2\sigma_v \qquad C_2 = 3\sigma_v \qquad C_3 = 4\sigma_v \qquad C_4 = 5\sigma_v \qquad (7.30)$$

and for the glideslope indicator

$$C_1 = \sigma_{GS} \qquad C_2 = 1.5\sigma_{GS} \qquad C_3 = 2\sigma_{GS} \qquad C_4 = 2.5\sigma_{GS} \qquad (7.31)$$
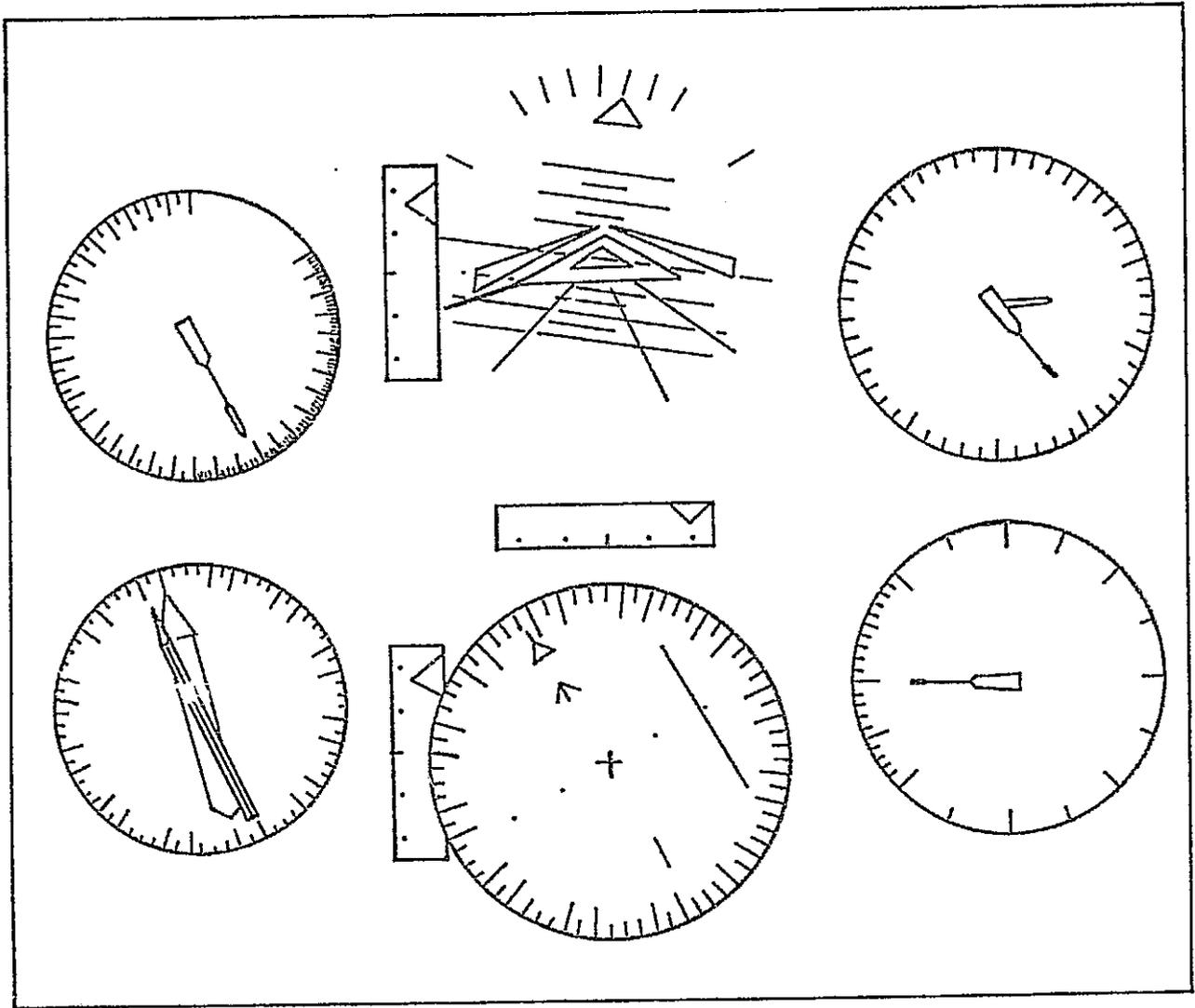
7.6   THE INSTRUMENT PANEL

Two random number generators were used to choose the failure in each run. One determined the instrument and the other the size of the failure. In addition, a third random number generator was used to determine the time of failure $t_f$. The value of $t_f$ had four discrete magnitudes with time difference of 15 seconds. The mean of these four values was the time at which the airplane passed the outer marker.

There was a single failure in 90% of the runs. The high percentage of runs with failures provided enough data in a reasonable experimental time. There was no feedback to the pilot concerning his performance. It was felt that feedback would induce correlations between successive runs, and therefore, it was not used.

## 7.5.2 Subjects

Two subjects participated in the experiment. One did not have any practical flight experience; however, he did have a lot of experience flying the simulated airplane (he was using it for his own experiments). The other subject had experience as an Air Force pilot where he flew a T38 jet trainer.

In the first set of experiments, the participation was on a voluntary basis. At the end of this set it was evident that the enthusiasm of the subjects had faded due to the fact that their task was only monitoring. Therefore, it was decided that in the next set of experiments, the subjects would be paid $4 an hour in order to keep the same level of performance as in the first set.

## 7.5.3 Procedure

As has already been mentioned, there were two sets of experiments and the same subjects participated in both. The two sets were equivalent except for the differences in disturbance characteristics. In the second set the frequency of the disturbances $\delta u_r, \delta \theta_r, \delta \psi_r$ were reduced by a factor of ten (compared to the first set in which this frequency was $\pi/6$).

The first set consisted of three experimental sessions. Each session included 16 runs with a ten minute intermission after 8 runs. The second set consisted of only one session with 26 runs and two intermissions.

The subject was seated in the cockpit in the pilot's seat; however, the presentation was completely automatic and he could not affect its behavior. Each run started when the airplane was ten miles out from the touch down point and at an altitude of 2500 feet. The three random numbers that controlled the failures were typed in by the experimenter before the start of each run. When the pilot detected a failure he pressed a button and the run was terminated. Then the subject was asked to fill out a form in which he stated which instrument failed and how he detected the failure.

At the beginning of each session, a set of instructions was read to the subject. In particular he was told that failure would either be in the AS or GS indicator, but he could use other instruments for the detection. Then the subject was

shown two runs without failures, and then two runs with failures:
one in the AS and one in the GS.  The data runs followed these
familiarization trials.

## 7.6  Experimental Results .

The results from the first experimental set in which the frequency of all disturbances was $\pi/6$ radians per second are summarized in Table 7.1.  The table shows the mean and standard deviation of the detection time for the failures in the AS and GS indicators for the two subjects.

The results are also shown in Figures 7.7 through 7.10. These figures include the mean detection times that were predicted by the model using the 0.1 seconds filter for the failure.  The following parameters were used in the model

$$SNR = 36 \qquad P_{FA} = P_{MS} = 0.05 \qquad \ddot{\theta}_1 = 1/4 \qquad (7.32)$$

Again, the level of the $P_{FA}$ was determined on the basis of the actual false alarm rate that was found in the experimental data. For both subjects, the predicted results seem to fit the experimental data well.  It should be noted that a better fit for the data from subject C.C. can be obtained by changing the parameters in (7.32).

The results from the second experimental set, in which the frequency of all disturbances was reduced to 0.5 radians per second, and the time constant of failure appearance was raised to 20 seconds, are summarized in Table 7.2.

It should be noted that the values for $\overset{\sim}{c}_j$ $j = 1,\ldots,4$ for GS failures are only one half of the values in (7.31), namely

$$\overset{\sim}{c}_1 = 0.5\sigma_{GS} \qquad \overset{\sim}{c}_2 = 0.75\sigma_{GS} \qquad \overset{\sim}{c}_3 = \sigma_{GS} \qquad \overset{\sim}{c}_4 = 1.25\sigma_{GS}$$

| SUBJECT | INSTR. | . | C1 | C2 | C3 | C4 |
|---------|--------|---|------|------|------|------|
| B.M. | A S | $E(t_d)$ | 20.80 | 13.80 | 10.80 | 6.30 |
| | | $\sigma_{t_d}$ | 5.9 | 2.7 | 2.1 | 2.7 |
| | G S | $E(t_d)$ | 16.40 | 9.80 | 7.70 | 5.98 |
| | | $\sigma_{t_d}$ | 3.6 | 4.9 | 2.4 | 1.1 |
| C.C. | A S | $E(t_d)$ | 25.40 | 20.80 | 16.90 | 8.20 |
| | | $\sigma_{t_d}$ | 5.9 | 4.0 | 2.5 | 2.8 |
| | G S | $E(t_d)$ | 14.00 | 6.90 | 6.30 | 5.00 |
| | | $\sigma_{t_d}$ | 2.8 | 1.0 | 0.9 | 0.9 |

TABLE 7.1  Subjects Performance in First
Experimental Set  (seconds)

FIGURE 7.7  DETECTION TIMES FOR GS FAILURES (FIRST SET, SUBJECT B.M)



FIGURE 7.8  DETECTION TIMES FOR AS FAILURES (FIRST SET, SUBJECT B.M)

FIGURE 7.9   DETECTION TIMES FOR GS FAILURES (FIRST SET, SUBJECT C.C)



FIGURE 7.10   DETECTION TIMES FOR AS FAILURES (FIRST SET, SUBJECT C.C)

| SUBJECT | INSTR. | | C1 | C2 | C3 | C4 |
|---------|--------|------|------|------|------|------|
| B.M. | A S | $E(t_d)$ | 62.40 | 42.80 | 32.90 | 20.20 |
| | | $\sigma_{t_d}$ | 6.0 | 8.5 | 5.2 | 3.0 |
| | G S | $E(t_d)$ | 14.20 | 9.00 | 6.50 | 4.90 |
| | | $\sigma_{t_d}$ | 2.0 | 2.1 | 3.0 | 1.6 |
| C.C. | A S | $E(t_d)$ | ---- | 46.80 | 34.20 | 28.40 |
| | | $\sigma_{t_d}$ | ---- | 5.8 | 3.2 | 3.8 |
| | G S | $E(t_d)$ | 28.50 | 14.20 | 8.90 | 5.90 |
| | | $\sigma_{t_d}$ | 4.0 | 1.4 | 2.3 | 1.3 |

TABLE 7.2  Subjects Performance in Second Experiment
(seconds)

The increase in detection time is mainly due to the change in the failure time constant.' The results are plotted in Figures 7.11 through 7.14. The figures also include the predictions of the model with the parameters SNR, $P_{FA}$ and $P_{MS}$ as in (7.32). The values of the parameter $\tilde{\theta}_1$ were changed to obtain a good fit. The values of $\tilde{\theta}_1$ that were used are shown in the figures.
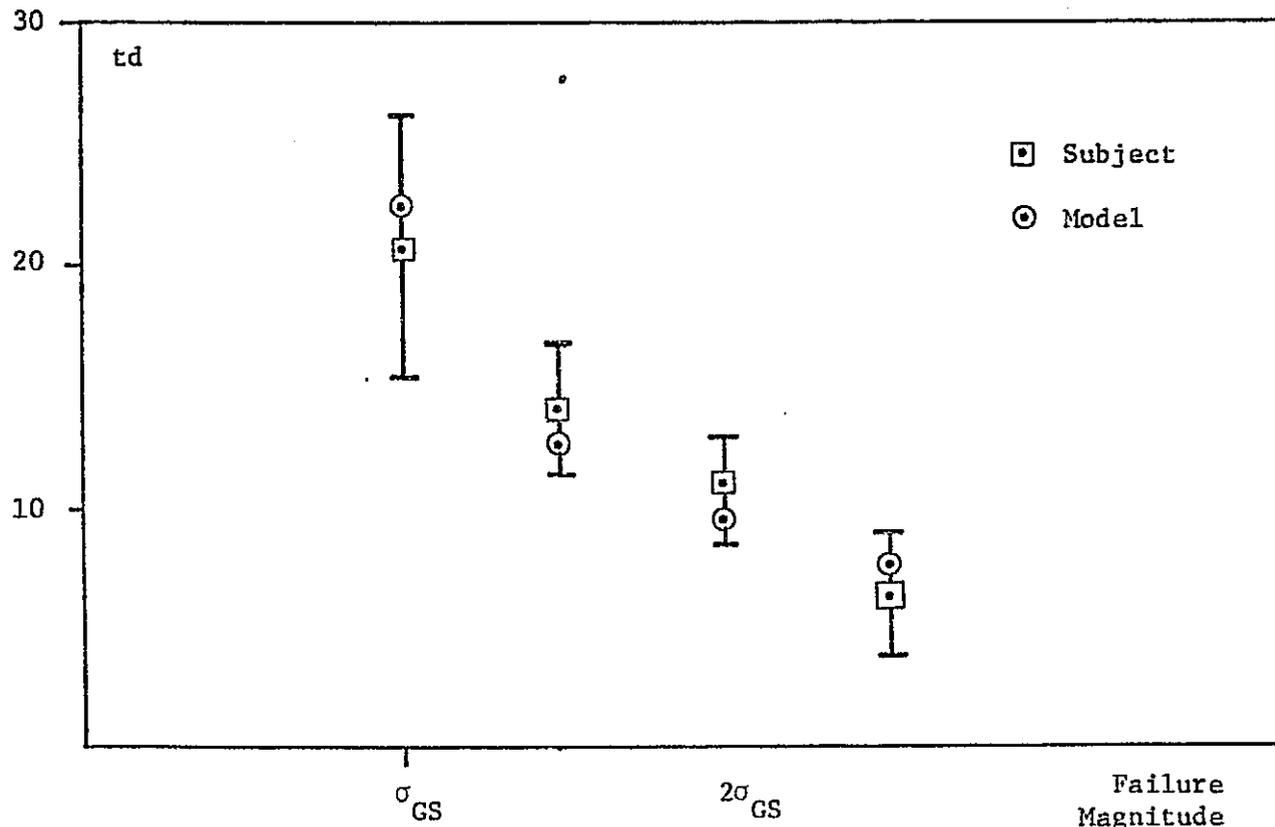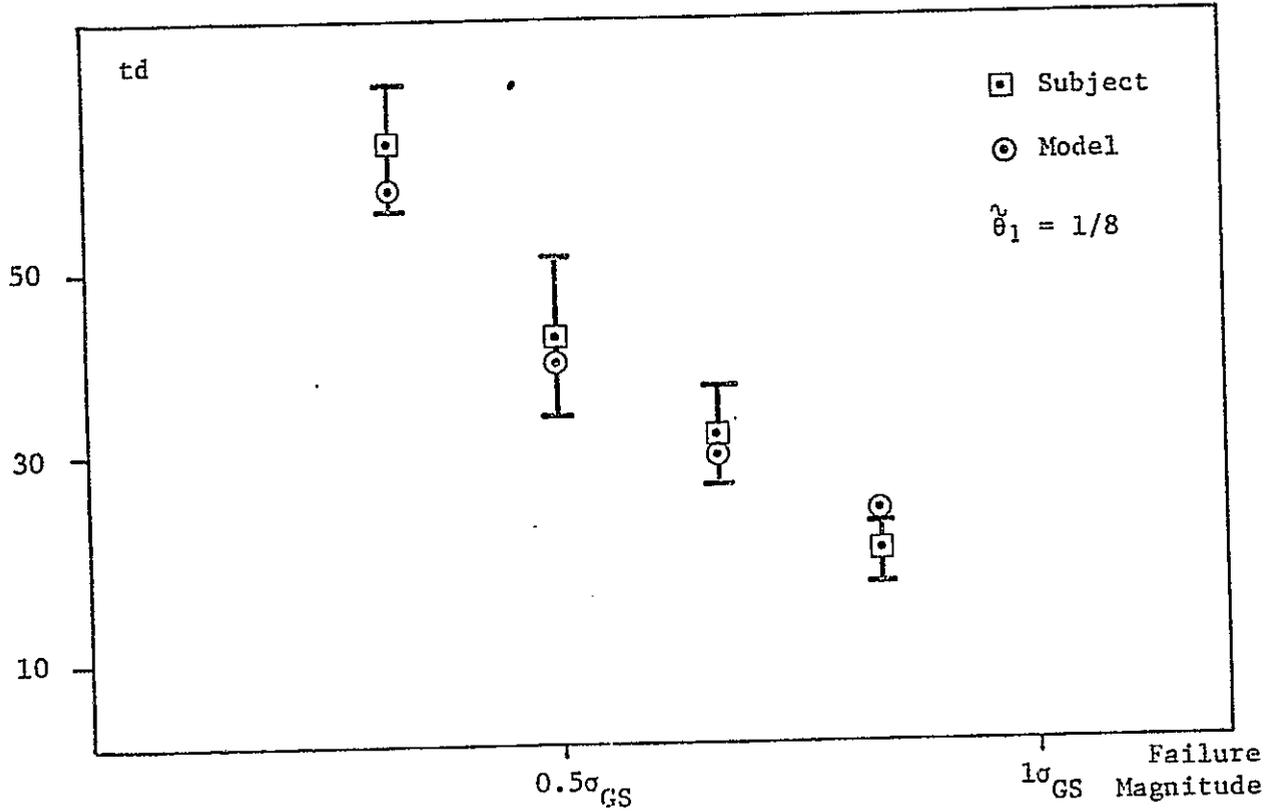
FIGURE 7.11   DETECTION TIMES FOR GS FAILURES (SECOND SET, SUBJECT B.M)
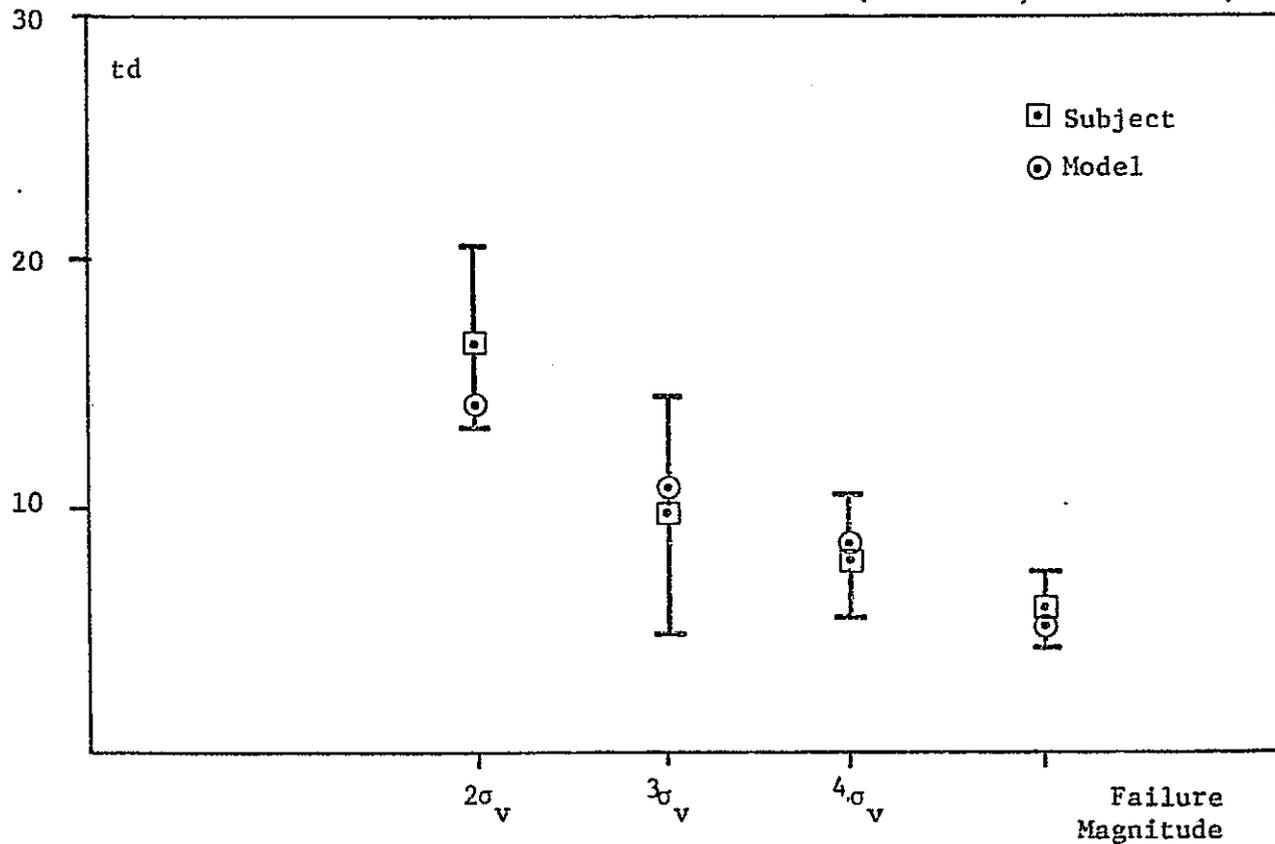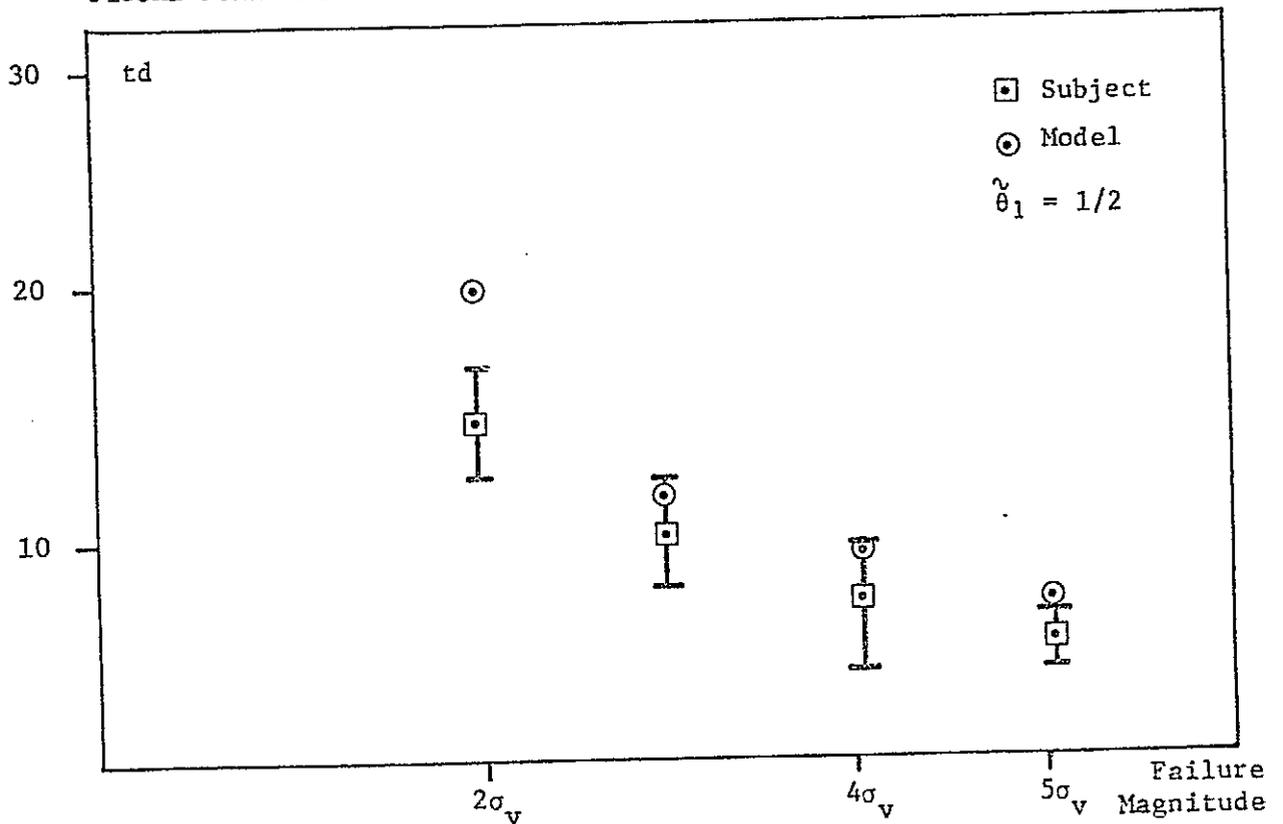


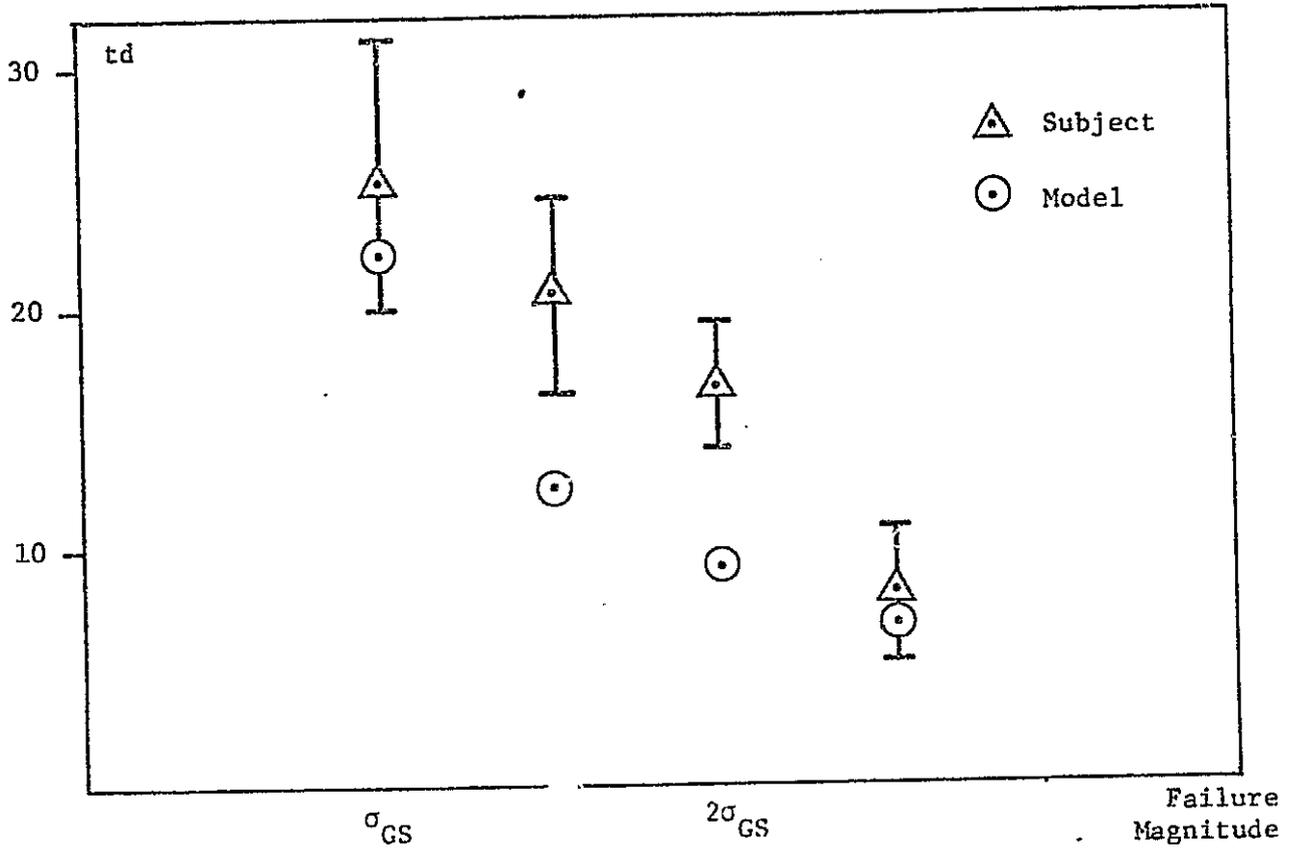FIGURE 7.12   DETECTION TIMES FOR AS FAILURES (SECOND SET, SUBJECT B.M)

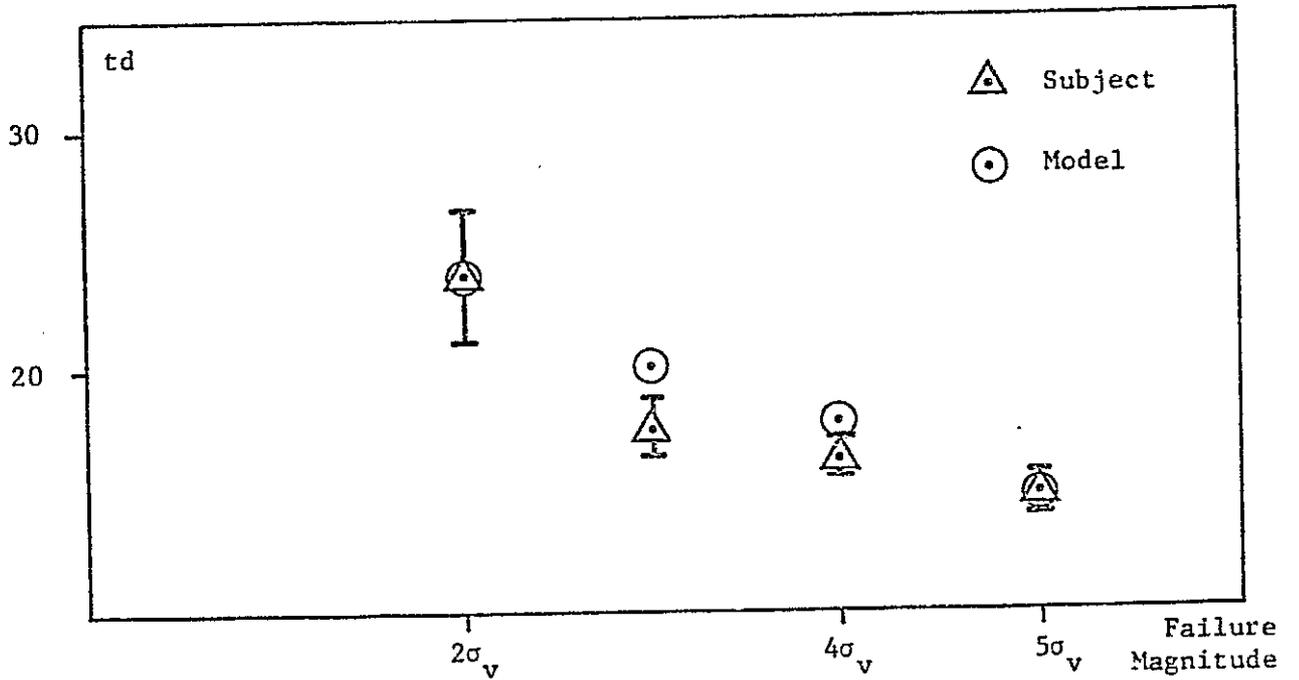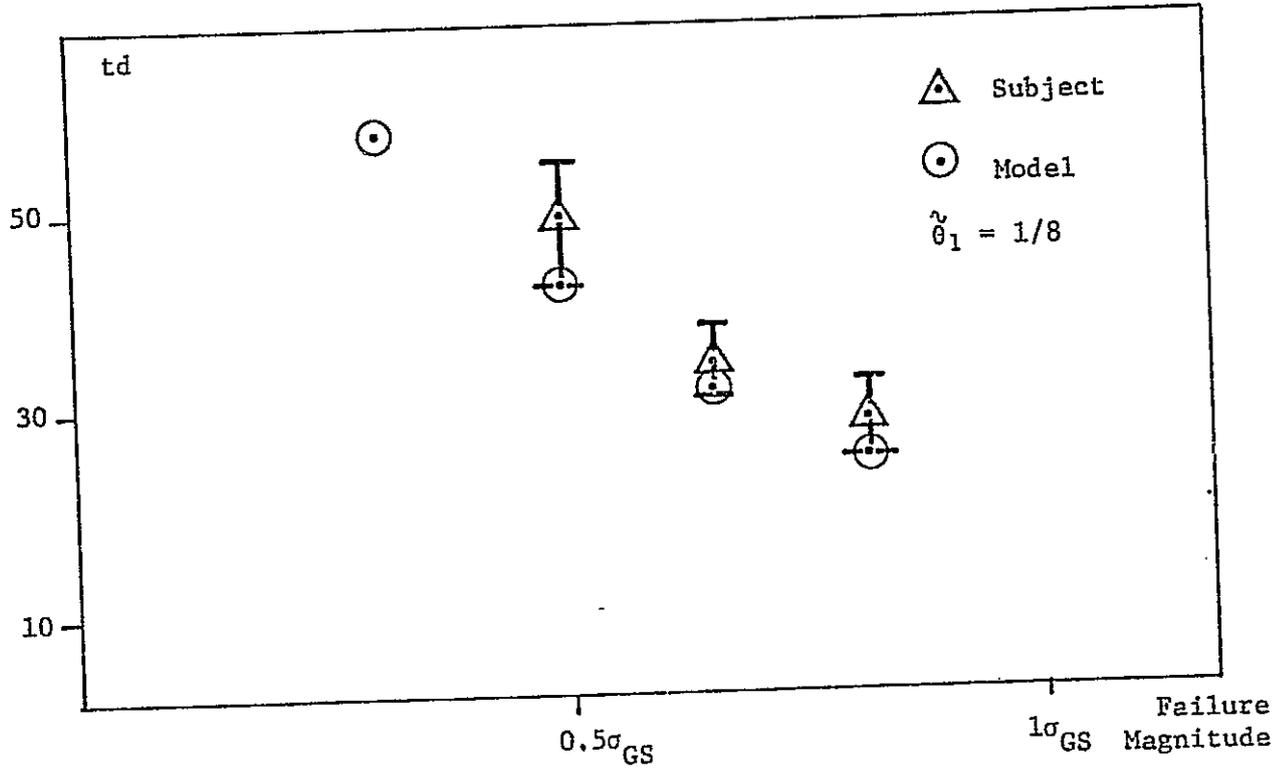FIGURE 7.13   DETECTION TIMES FOR GS FAILURES (SECOND SET, SUBJECT C.C)
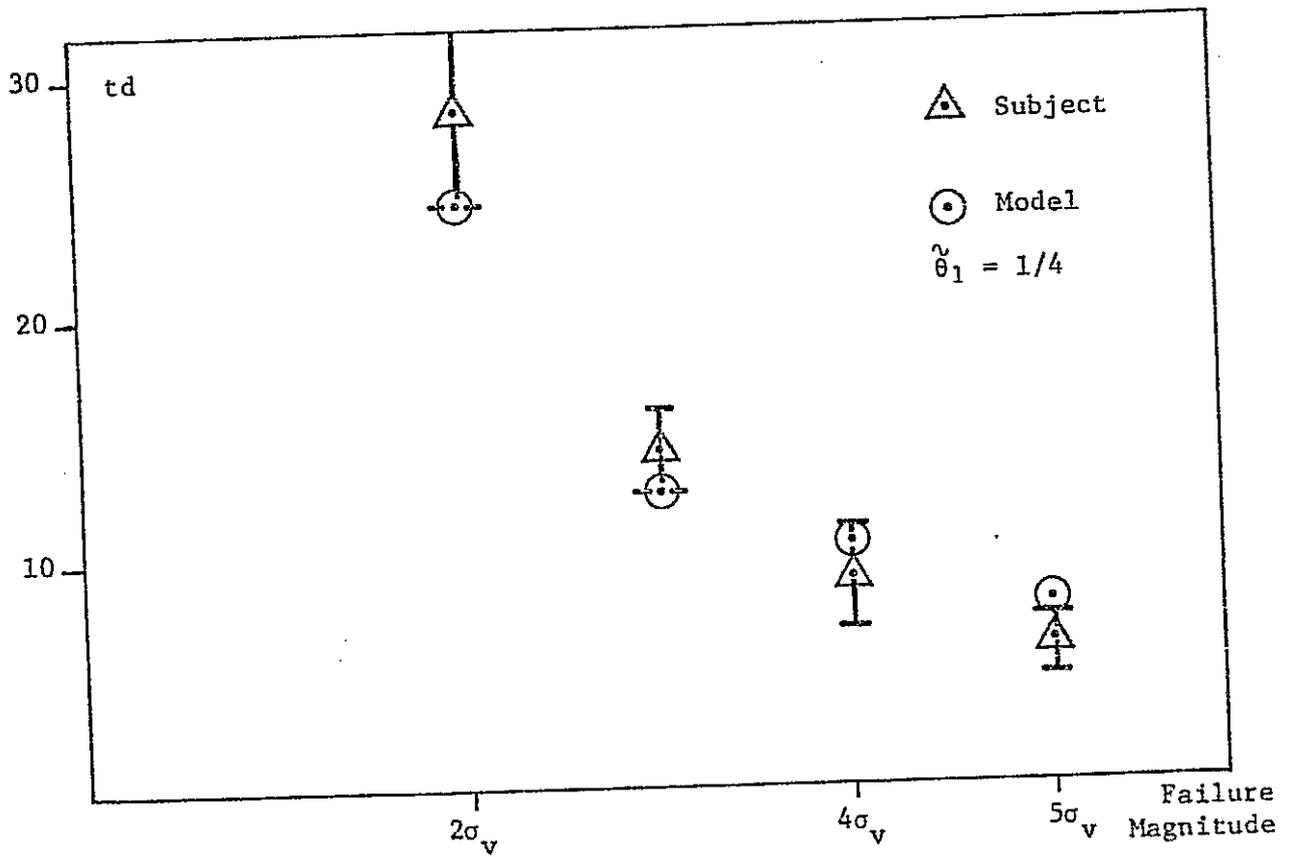


FIGURE 7.14   DETECTION TIMES FOR AS FAILURES (SECOND SET, SUBJECT C.C)

CHAPTER VIII

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

8.1  Conclusions

This thesis investigated some psychophysical aspects of the behavior of a human operator in several signal recognition tasks. The study was based on classical SDT, but the emphasis was not on the capability of the sensory system, but on the decision mechanism of the operator. Since almost all optimal decision strategies lead to the use of the likelihood ratio as the decision function, the decision strategy can only be analyzed through the detection tasks with time varying detectability. The analytical study included a discussion of several possible decision strategies as well as a suggestion of two methods for the classification of these strategies on the basis of experimental results. One of these methods is similar to the well known ROC analysis in classical SDT, and used decision rule (DR) curves in the $P_H$-$P_{FA}$ plane. The other method, which can only be used when the underlying distributions of the observations are known, relates the likelihood ratio criterion levels (LRCL) to the detectability.

Experiments in which the subject was to detect signals with discrete change of signal strength were described. The observations were designed to be independent and the subject had to make a decision after each observation. The main conclusion that can be drawn on the basis of the experimental results is

that the subject is aware of the change in the uncertainty, and changes his LRCL accordingly. However, no one strategy could be identified, and the subject's performance revealed that different subjects used different strategies.

The next step was to study the effect of correlated signal presentations on the performance of the subjects. It was found that the correlation did not affect the sensory process (the detectability) in our experiments, but caused the subjects to modify the parameters of the decision strategy (the LRCL's). The theory of Markov Processes was applied to the experimental data, and the probability transition matrices showed that the LRCL of the subject in the current decision interval was strongly dependent on the previous decision, regardless of it's correctness. In particular, when the signal strength was changed in a sequential order, the decisions along the sequence were influenced by the decision in the first interval, although the detectability in this interval was the lowest. These results can be explained on the basis of classical SDT when the LRCL's are modified by biasing the *a priori* probabilities.

In some detection tasks, the information rate is too high, so that the subject cannot respond after each observation. In such cases, the subject is allowed to take more than one observation, but he is asked to minimize the detection time. This type of detection task is often related to failure detection problems. It was found that the subject's behavior can be modelled as a two stage process. The first stage consists of a

linear estimator whose measurement residual is fed to the
second stage which is the decision mechanism. The main con-
clusion from our experimental results is that the decision
function is a pure integration of the observation error, and a
decision is made when some CL is reached. It was also found that
if the monitored process is of finite (short) length, and if
the decision maker knows that a change will occur, his proba-
bility of false alarm is time varying and causes changes in
the criterion level.

An application of the above model to predict the performance
of a pilot in a task of failure detection in automatic landings
showed that the model is applicable even when the processes
that are involved are complicated. In particular, the experi-
ment showed that a simplified linearized model gives good
prediction even if the system is nonlinear, the order of the
dynamical system is high, and the observations are time varying
and multidimensional.

## 8.2 Suggestions for Further Research

The theory and results that are presented in this thesis could be extended by further research in the following directions:

1. The analysis of experimental results of signal detection experiments with time varying uncertainty as well as the decision strategies that were discussed are not limited to visual discrimination tasks. The general results can, therefore, be verified by applications of the theory to other sensory processes such as auditory and tactile processes and other detection tasks that have been analyzed with the use of the classical theory.

2. In many detection tasks, the human operator is a part of the control loop so that the pilot is not only monitoring but can also influence the system before and after a failure. This additional control task might affect the performance of the subject as compared to his performance in monitoring tasks. It is therefore suggested to modify the model to include this additional task and design experiments that can show its validity in these cases.

3. Our experiments did not include feedback: however, it is felt that the addition of feedback may change the performance of the decision maker. Feedback can be given directly to the subject or only to the system

or to both. The study of the effects of feedback seems to be very valuable in increasing our knowledge of the decision strategies of the human operator, because feedback is used in many real life detection tasks.

4. This research considered only the psychophysical aspects of signal detection, and the question of how the processing is actually done was avoided. Lately, the use of EEG measurements has been found to be very valuable in arousal studies. The use of EEG techniques in experiments of signal detection with time varying uncertainty might give more insight into the reactions of subjects to the change in the difficulty of the task.

5. In our model of the human observer as a failure detection system there are three parameters that control the performance. The value for these parameters were chosen by "trial and error" method. Since the running of the simulation is relatively expensive a more efficient method to find the parameters that best fit the experimental data is needed. Systems identification techniques can be applied for this purpose.

6. In Chapter 7, we applied our model of the human observer to the task of monitoring automatic landings. This task should be reexamined to include failures on all instruments as well as failures of the system

itself (airplane or control). Also, the experimental
sessions should be spread over a longer period of
time so that the pilots can face realistic situations
of only a few landings per session (day), and a low
failure rate.

# APPENDIX A

## ALGORITHM FOR FITTING GAUSSIAN DISTRIBUTIONS TO DATA FROM SD EXPERIMENTS

### Problem Statement

Consider a signal detection (SD) experiment with two states of the world ($S_0$ and $S_1$) and n response categories. There are a total of N decision intervals, in $N_0$ of which the true state of the world is $S_0$. Therefore, the state of the world $S_1$ will appear in $N_1$ intervals, where:

$$N_1 = N - N_0 \qquad \text{(A.1)}$$

After the experiment is finished, the decisions of the subject can be sorted into 2n categories as follows. Let $D_j$ (j = 1,...,n) define the set of intervals in which the subject has decided on category j. This set can be divided into two subsets $D_{0j}$ and $D_{1j}$ in which the state of the world is $S_0$ or $S_1$ respectively. The $D_{0j}$ and $D_{1j}$ for j = 1,..., n constitute 2n exclusive and exhastive events. The number of decisions that correspond to the event $D_{0j}$ is $N_{0j}$ (j = 1,..., n) and the number of decisions that correspond to $D_{1j}$ is $N_{1j}$ (j = 1,..., n). Clearly the following relations hold

$$\sum_{j=1}^{n} N_{0j} = N_0 \qquad\qquad \sum_{j=1}^{n} N_{1j} = N_1 \qquad \text{(A.2)}$$

Now we assume that the raw data represented by $N_{0j}$ and $N_{1j}$ corresponds to the results of an optimal procedure to discriminate between two random variables with continuous distribution functions. The first step is to make a decision about

the functional form of the distributions to be used. For this discussion we will assume that the distribution functions for both variables are Gaussian. These variables have means $m_0$ and $m_1$ and variances of $\sigma_0^2$ and $\sigma_1^2$.

The decision procedure is assumed to be based on a choice of (n-1) criterion levels $K_j$, $j = 1,\ldots,n-1$, thereby dividing the observation space into n exclusive and exhaustive subspaces each corresponding to one of the n response categories. Figure A-1 shows this procedure for n = 3.

Although the functional form of the distributions is now established; the parameters of the distributions as well as the CL's are still unknown. Let us define an n+3 dimensional vector of all the unknown variables:

$$\underline{x}^T = (m_0, \ \sigma_0, \ m_1, \sigma_1, \ k_1, \ k_2,\ldots, \ k_{n-1}) \tag{A.3}$$

If the value of the vector $\underline{X}$ is known, then it is possible to find the probability that each one of the 2n events would happen. Those probabilities are the areas under the Gaussian distribution which are shown in Figure A-2 for n = 3. The analytic expressions for these probabilities are

$$P_{0j} = \frac{1}{\sqrt{2\pi}\,\sigma_0} \int_{k_{j-1}}^{k_j} \{-\frac{(\xi-m_0)^2}{2\sigma_0^2}\}d\xi \qquad\qquad j = 1,\ldots,n$$

$$P_{1j} = \frac{1}{\sqrt{\pi}\,\sigma_1} \int_{k_{j-1}}^{k_j} \{-\frac{(\xi-m_1)^2}{\sigma_1^2}\}d\xi \qquad\qquad j = 1,\ldots,n \tag{A.4}$$

where

$$k_0 = -\infty \qquad\qquad k_n = +\infty$$
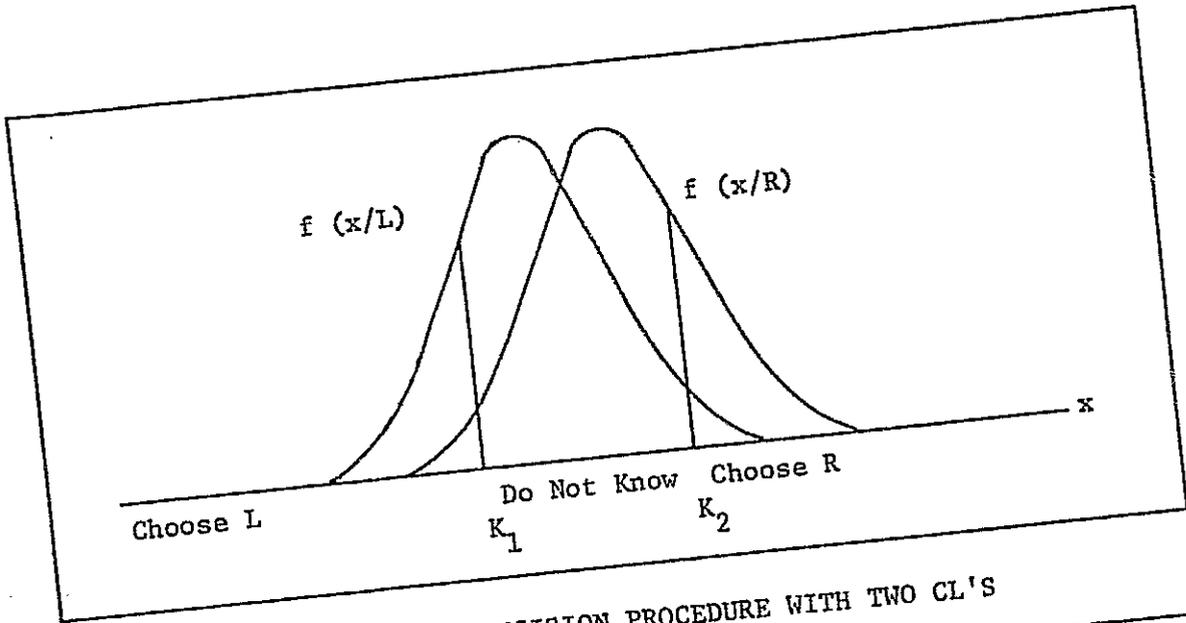
FIGURE A-1  DECISION PROCEDURE WITH TWO CL'S
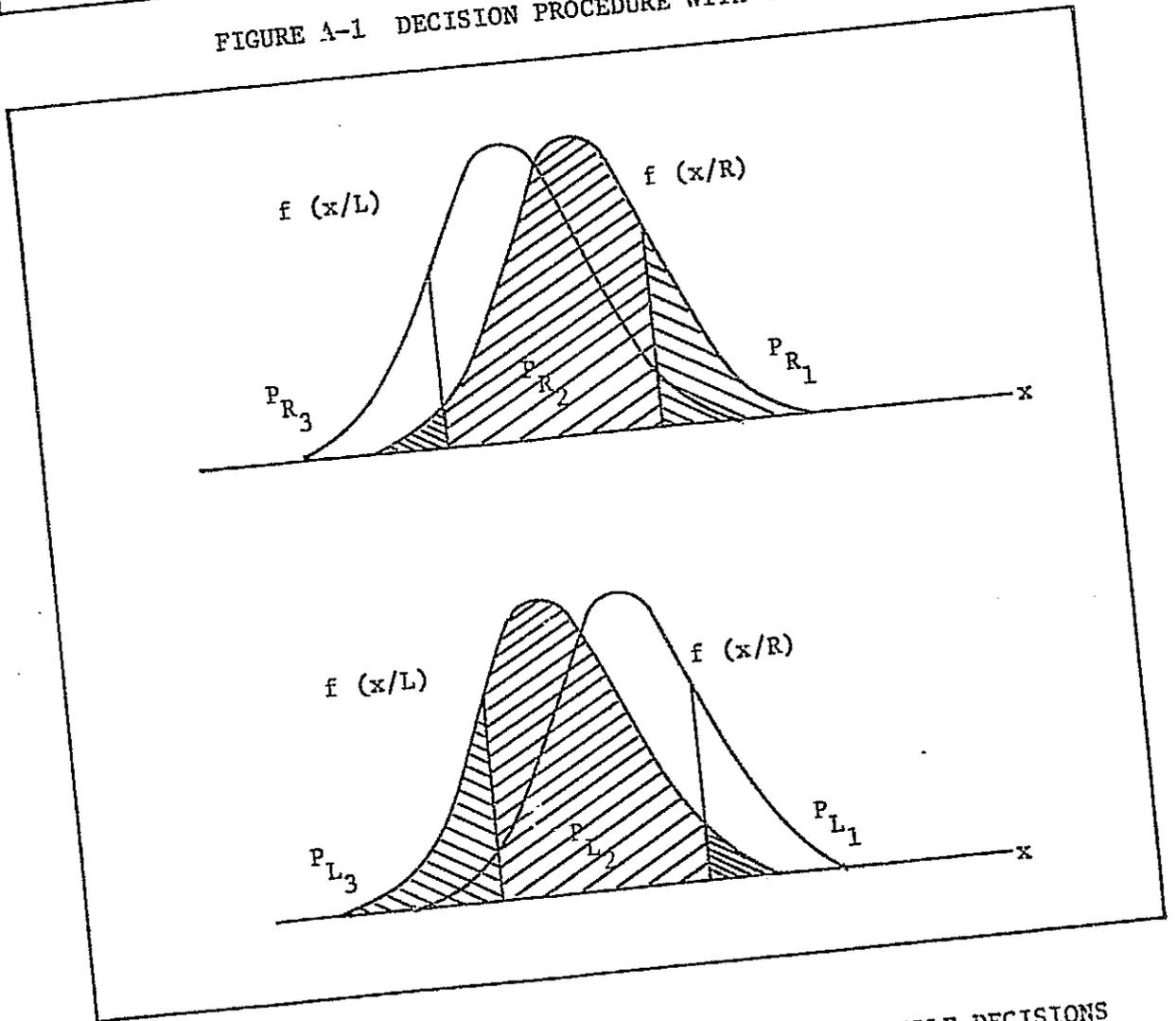


FIGURE A-2  PROBABILITIES CORRESPONDING TO POSSIBLE DECISIONS

Defining

$$\Phi_0(k) = \frac{1}{\sqrt{2\pi}\, \sigma_0} \int_{-\infty}^{k} \exp\left\{ \frac{-(\xi - m_0)}{2\sigma_0^2} \right\} \, d\xi$$

$$\Phi_1(k) = \frac{1}{\sqrt{2\pi}\, \sigma_1} \int_{-\infty}^{k} \exp\left\{ \frac{-(\xi - m_1)}{2\sigma_1^2} \right\} \, d\xi$$

(A.5)

Then

$$P_{0j} = \Phi_0(k_j) - \Phi_0(k_{j-1})$$

(A.6)

$$P_{1j} = \Phi_1(k_j) - \Phi_1(k_{j-1})$$

On the basis of these probabilities, the expected number of decisions for each event is

$$\hat{N}_{0j} = P_{0j} N_0 \qquad\qquad \hat{N}_{1j} = P_{1j} N_1 \qquad j = 1, \ldots, n \quad (A.7)$$

Our problem is to find these probabilities $P_{0j}$ and $P_{1j}$ that would give rise to $N_{0j}$ and $N_{1j}$ and that are as close as possible to the experimental results. These probabilities are functions of the vector $\underline{X}$; therefore, the problem is to find a value for this vector rather than the probabilities. Since the components of $\underline{X}$ are completely unknown (their distributions are not known) a feasible criterion to be used is to maximize the likelihood function. Under this criterion we try to maximize the conditional probability that the data values $N_{0j}$ and $N_{1j}$ occurred, given some value for $\underline{X}$. This conditional probability is referred to as the likelihood function L and is given for our case by

$$L = P(N_{01}, N_{02}, \ldots, N_{0n}, N_{11}, \ldots, N_{1n}) =$$

$$N! \prod_{j=1}^{n} P_{0j} P_{1j} / (N_{0j}! N_{1j}!) \tag{A.8}$$

Since only the value of $\underline{X}$ that maximizes L is of interest and since the ln function is monotonic with its argument, it is possible to maximize lnL rather than L where:

$$\ln L = \ln N! + \sum_{j=1}^{n} [N_{0j} \ln P_{0j} + N_{1j} \ln P_{1j} - \ln(N_{0j}! N_{1j}!)] \tag{A.9}$$

Since the $\ln(N_{0j}! N_{1j}!)$ and $\ln N!$ terms are not functions of $\underline{X}$ they can be dropped from the expression to be maximized. Substituting from (A.6) into (A.9) the final cost function is

$$C(\underline{X}) = \sum_{j=1}^{n} \{N_{0j} \ln[\Phi_0(k_j) - \Phi_0(k_{j-1})]$$

$$+ N_{1j} \ln[\Phi_1(k_j) - \Phi_1(k_{j-1})]\} \tag{A.10}$$

$C(\underline{X})$ is now a function of the parameters of the distribution and the CL's through $\Phi_0$ and $\Phi_1$. However, if the following transformation is applied to $\xi$

$$\eta = \frac{\xi - m}{\sigma} \tag{A.11}$$

Equation (A.5) can be written as:

$$\phi_0(k) = \frac{1}{\sqrt{2\pi} \ \sigma_0} \int_{-\infty}^{k-m_0/\sigma_0} \exp\{-\eta^2/2\} \ d\eta \tag{A.12}$$

$$\phi_1(k) = \frac{1}{\sqrt{2\pi} \ \sigma_1} \int_{-\infty}^{k-m_1/\sigma 1} \exp\{-\eta^2/2\} \ d\eta$$

Therefore the components of the unkown vector $\underline{X}$ affect the cost function through the expressions

$$\frac{k_j - m_0}{\sigma_0} \quad , \quad \frac{k_j - m_1}{\sigma_1} \quad , \qquad j = 1, \ldots, n \qquad \text{(A.13)}$$

Relation (A.13) implies that the solution for the minimization problem is invariant to a linear transformation. Specifically, if:

$$\underline{X}^T = (d_{0m}, \sigma_{0m}, d_{1m}, \sigma_{1m}, k_{jm}(j = 1, \ldots, n))$$

is a solution that maximized $C(\underline{X})$ then $\underline{\tilde{X}}_m$ given by

$$\underline{\tilde{X}}_m^T = (c_1 d_{0m} - c_2, \; c_1 \sigma_{0m}, \; c_1 d_{1m} - c_2, \; c_1 \sigma_{1m},$$

$$c_1 k_{jm} - c_2 (j = 1, \ldots, n))$$

is also a solution. $c_1$ and $c_2$ in the above expression are arbitrary constants.

The conclusion to be drawn from the above ambiguity is that if the algorithm is to be used repeatedly and the results are to be compared, the basis for comparison should be the invariant expression (A.13) rather than the actual parameters.

## Optimization Method

A necessary condition for an extremum point $\underline{X}_m$ of the cost function $C(\underline{X})$ given by (A.10) is

$$g(\underline{X}) = \frac{\partial C(\underline{X})}{\partial \underline{X}}\bigg|_{\underline{X}=\underline{X}_m} = 0 \qquad\qquad (A.14)$$

where $g(\underline{X})$ is the gradient of $C(\underline{X})$ with respect to $\underline{X}$. Equation (A.14) is a set of n+3 nonlinear algebraic equations, and their explicit form for the Gaussian case is given in Table A.1.

Several numerical methods were suggested for the solution of this parameter optimization problem. The most popular method is the Davidon algorithm (Fletcher and Powell, 1963), which is available as a subroutine in the IBM Scientific Subroutine Package. However, in this work we decided to use a more recent algorithm that was suggested by Jacobson and Oksman (1970). This algorithm seems to be superior to the Davidon algorithm in the following ways:

1. It converges to the minimum in fewer iterations for some classical test functions (Rosenbrook function, helical valley, etc.)

2. It does not require that a minimum be found along a line for each iteration.

3. It converges in (n+2) iterations for a homogeneous cost function, namely, cost functions that satisfy:

$$C(\underline{X}) = C(\underline{X}_m) + \frac{1}{d}g^T(\underline{X}_m)(\underline{X} - \underline{X}_m) \qquad\qquad (A.15)$$

where

$m$ — dimension of $\underline{X}$

$\underline{X}_m$ — the value of $\underline{X}$ that maximizes $C(\underline{X})$

$g(\underline{X})$ — the gradient of $C(\underline{X})$

$d$ — constant

$$g_1 = -\frac{1}{\sqrt{2\pi}\,\sigma_0} \sum_{j=2}^{n+1} \frac{N_{0,j-1}}{\Phi_0(k_j) - \Phi_0(k_{j-1})} \{\exp(-\frac{(k_j - m_0)^2}{2\sigma_0^2}) - \exp(-\frac{(k_{j-1} - m_0)^2}{2\sigma_0^2})\}$$

$$g_2 = -\frac{1}{\sqrt{\pi}\,\sigma_0} \sum_{j=2}^{n+1} \frac{N_{0,j-1}}{\Phi_0(k_j) - \Phi_0(k_{j-1})} \{\frac{k_j - m_0}{\sqrt{2}\,\sigma_0}\exp(-\frac{(k_j - m_0)^2}{2\sigma_0^2}) - \frac{k_{j-1} - m_0}{\sqrt{2}\,\sigma_0} \exp(-\frac{(k_{j-1} - m_0)^2}{2\sigma_0^2})\}$$

$$g_3 = -\frac{1}{\sqrt{2\pi}\,\sigma_1} \sum_{j=2}^{n+1} \frac{N_{1,j-1}}{\Phi_1(k_j) - \Phi_1(k_{j-1})} \{\exp(-\frac{(k_j - m_1)^2}{2\sigma_1^2}) - \exp(-\frac{(k_{j-1} - m_1)^2}{2\sigma_1^2})\}$$

$$g_4 = -\frac{1}{\sqrt{\pi}\,\sigma_1} \sum_{j=2}^{n+1} \frac{N_{1,j-1}}{\Phi_1(k_j) - \Phi_1(k_{j-1})} \{\frac{k_j - m_1}{\sqrt{2}\,\sigma_1} \exp(-\frac{(k_j - m_1)^2}{2\sigma_1^2}) - \frac{k_{j-1} - m_1}{\sqrt{2}\,\sigma_1} \exp(-\frac{(k_{j-1} - m_1)^2}{2\sigma_1^2})\}$$

$$g_j = -\frac{1}{\sqrt{2\pi}} \cdot \{\frac{1}{\sigma_0} \frac{N_{0,j-1}}{\Phi_0(k_j) - \Phi_0(k_{j-1})} - \frac{N_{0,j-1}}{\Phi_0(k_{j-1}) - \Phi_0(k_{j-2})}\}\exp(-\frac{(k_j - m_0)^2}{2\sigma_0})$$
$$+ \frac{1}{\sigma_1}\{\frac{N_{1,j-1}}{\Phi_1(k_j) - \Phi_1(k_{j-1})} - \frac{N_{1,j-1}}{\Phi_1(k_{j-1}) - \Phi_1(k_{j-1})}\}\exp(-\frac{(k_j - m_1)^2}{2\sigma_1^2})$$
$$j = 4,\ldots,n+3$$

Table A.1  Components of the gradient $\underline{g}(\underline{x})$

It should be noted that the class of functions that satisfies (A.15) is much larger than the class of quadratic functions for which the Davidon method converges in (n+1) iterations.

The method also has some disadvantages:

1. A strong step size control is needed to avoid divergence of the algorithm in the initial iterations

2. The algorithm does not provide an approximation to the matrix of second derivatives:

$$A = \frac{\partial^2 C(X)}{\partial \underline{X}^2}$$

and therefore *a posteriori* error analysis is not feasible.

A detailed description of the algorithm is given in the original report of Jacobson, so it will not be repeated. Listing of the algorithm in FORTRAN IV is given in Appendix B of a progress report by Curry (1973).

Notes on the program

Although the FORTRAN program was written for the general Gaussian case, some special cases can be applied by changing some code numbers (Curry, 1973). These special cases include:

1. A case without bias (symmetric means) with equal variances for which

$$m_0 = - m_1 \qquad \qquad \sigma_0 = \sigma_1$$

2. A case that includes bias with equal variances for which

$$m_0 \neq -m_1 \qquad \qquad \sigma_0 = \sigma_1$$

3.  A case without bias and with unequal variances for which

$$m_0 = -m_1 \qquad \sigma_0 \neq \sigma_1$$

In addition the number of CL's (n-1) can be varied from 1 to 25.

The stopping condition for the algorithm was based on the value of the norm of the gradient, namely,

$$||g(\underline{x})|| \leq 0.1$$

However, other values can be used as well, such as the use of

$$|C(\underline{x}_{M+1}) - C(\underline{x}_M)|$$

where M is the iteration number. The "goodness of the solution was tested by the use of the Chi Square test. The values for the Chi Square test were computed as follows:

$$\chi^2 = \sum_{j=1}^{n-1} \left\{ \frac{[N_{0j} - N_0(\Phi_0(k_{j+1}) - \Phi_0(k_j))]^2}{\Phi_0(k_{j+1}) - \Phi_0(k_j)} + \frac{[N_{1j} - N_1(\Phi_1(k_{j+1}) - \Phi_1(k_j))]^2}{\Phi_1(k_{j+1}) - \Phi_1(k_j)} \right\}$$

The number of degrees of freedom is:

$$2(n-1) - \text{number of estimated parameters.}$$

## Apriori test for the algorithm

Since a parameter optimization program is usually involved with complicated expressions and since in our particular use the equations for the gradient (Table A.1) are also complicated, programming errors are very likely to happen.  Therefore, it might be useful to use an *apriori* test to check the validity of the program  before it is used.

The method that is suggested here seems to be more general than the ones suggested by Grey and Morgan (1972).  The idea is to synthesize artificial data $N_{0j}$ and $N_{1j}$ for which the solution is known, and check whether this solution is actually obtained from the algorithm.

Let us choose some arbitrary values for the components of $\underline{X}$.  These values will be

$$\underline{X}_1^T = (m_{01}, \sigma_{01}, m_{11}, \sigma_{11}, k_{j1}(j = 1,\ldots,n-1)) \qquad (A.16)$$

On the basis of this vector we define the data as follows:

$$\begin{aligned} N_{0j} &= \Phi_0(k_{j+1}) - \Phi_0(k_j) \\ N_{1j} &= \Phi_1(k_{j+1}) - \Phi_1(k_j) \end{aligned} \qquad j = 1,\ldots,n \qquad (A.17)$$

where
$$k_0 = -\infty \qquad\qquad k_n = +\infty$$

For the data given by (A.17) it is possible to show that $\underline{X}_1$ as given by (A.16) satisfies the necessary conditions (A.14) and, therefore, constitutes a possible solution.  To prove this

statement (A.16) and (A.17) must be substituded into the components of the gradient in Table A.1 and the results should zero the gradient.

Now let

$$g^T(\underline{X}) = (g_1, g_2, \ldots, g_{n+3})$$

and consider each component by itself. From Table A.1

$$g_1 = \frac{1}{\sqrt{2\pi}\,\sigma_0} \sum_{j=1}^{n} \frac{N_{0,j-1}}{\Phi(k_j) - \Phi(k_{j-1})} \{\exp\{\frac{-(k_j - m_0)^2}{2\,\sigma_0^2}\}$$

$$- \exp\{\frac{-(k_{j-1} - m_0)^2}{2\,\sigma_0^2}\}\}$$

therefore

$$g_1 = \frac{1}{\sqrt{2\pi}\,\sigma_0} \{\exp[\frac{-(k_0 - m_0)^2}{2\sigma_0^2}] - \exp[\frac{(k_n - m_0)^2}{2\sigma_0^2}]\} \qquad (A.19)$$

and because of (A.18) this expression is equal to zero. The expression for $g_3$ is equivalent to (A.19) when the subscript 0 is changed to 1. Therefore the expression for $g_3$ is also equal to zero. Again from Table A.1:

$$g_2 = \frac{1}{\sqrt{2\pi}\,\sigma_0} \sum_{j=1}^{n} \frac{N_{0,j-1}}{\Phi_0(k_j) - \Phi_0(k_{j-1})} \{\frac{k_j - m_0}{\sqrt{2}\,\sigma_0} \exp[-\frac{(k_j - m_0)^2}{2\sigma_0^2}]$$

$$- \frac{k_{j-1} - m_0}{\sqrt{2}\sigma_0} \exp[-\frac{(k_{j-1} - m_0)^2}{2\sigma_0^2}]\}$$

Therefore, by carrying out the summation

$$g_2 = \frac{1}{\sqrt{\pi}\ \sigma_0} \left\{ \frac{k_n - m_0}{\sqrt{2}\ \sigma_0}\ \exp\left[-\frac{(k_n - m_0)^2}{2\sigma_0^2}\right] - \frac{k_0 - m_0}{\sqrt{2}\ \sigma_0}\ \exp\left[-\frac{(k_0 - m_0)^2}{2\sigma_0^2}\right]\right\}$$

(A.20)

Substituting from (A.18) and using the limit

$$\lim_{y \to \infty} y e^{-y^2} \to 0$$

$g_2$ is equal to zero. Again the expression for $g_4$ is similar to (A.20) if we replace the subscript 0 with 1. Therefore, $g_4$ is also equal to zero. $g_1$ to $g_4$ were the derivatives of $C(\underline{X})$ with respect to the parameters of the distribution, and $g_j$, $j = 5,\ldots,n+3$ are the derivatives of $C)\underline{X})$ with respect to the CL's. From Table A.1:

$$g_j = -\frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{\sigma_0}\left[\frac{N_{0,j-1}}{\Phi_0(k_j) - \Phi_0(k_{j-1})} - \frac{N_{0,j-2}}{\Phi_0(k_{j-1}) - \Phi_0(k_{j-2})}\right] \right.$$
$$\exp\left(-\frac{(k_j - m_0)^2}{2\sigma_0^2}\right) + \frac{1}{\sigma_1}\left[\frac{N_{1,j-1}}{\Phi_1(k_j) - \Phi_1(k_{j-1})}\right.$$
$$\left. - \frac{N_{1,j-2}}{\Phi_1(k_{j-1}) - \Phi_1(k_{j-2})}\right]\exp\left(-\frac{(k_j - m_1)^2}{2\sigma_1^2}\right) \right\}$$

$$j = 5,\ldots,n+3$$

Substituting (A.17) into the above, the expressions in the square brackets are equal to 0, and therefore, $g_j$, $j=5,\ldots,n+3$ are equal to 0. This completes the proof of our statement that $\underline{X}_1$ satisfies the necessary conditions if the data in (A.17) is used.

## APPENDIX B

## SIMULATION FOR THE EXPERIMENT OF CHAPTER 6

This appendix gives a detailed description of the analytical and numerical methods that were used to simulate the displayed process used in the experiments of Chapter 6. Reference is made to Figure B.1.

$$\underrightarrow{\begin{array}{c} W(t) \\ \text{input process} \end{array}} \boxed{\dfrac{1}{s^2 + 2w_0 C_1 s + w_0^2}} \underrightarrow{\begin{array}{c} y(t) \\ \text{observation} \end{array}}$$
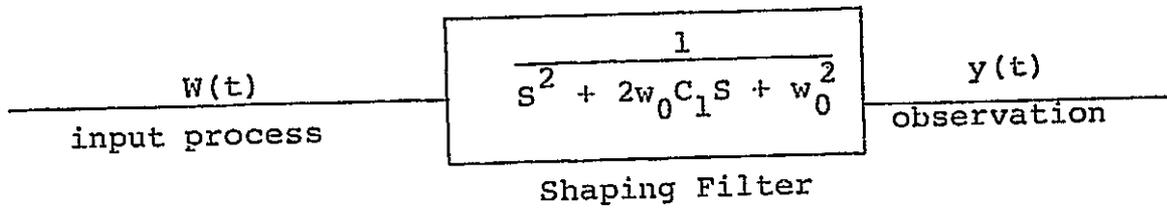
Shaping Filter

Figure B.1  Simulation of the Observed Process

### Input Process

The input process to the shaping filter is a scalar zero mean white Gaussian process. Since a digital computer is used, we want to form a white Gaussian sequence $w(t_n)$, where for every $t_n$ $w(t_n)$ is a Gaussian random variable with zero mean and unit variance.

The autocorrelation function of such a process is:

$$\Phi_{ww}(t) = (1 - |\tau|/\Delta t) \qquad |\tau| < \Delta t \qquad (B.1)$$
$$= 0 \qquad |\tau| > \Delta t$$

where $\Delta t = t_{n-1} - t_n = $ constant for all $n$

If $\Delta t$ is much smaller than the time constant of the shaping filter, $w(t_n)$ can be considered a white process over the bandwidth of the shaping filter.

A random number generator has to be formed, from which random numbers can be drawn at each $t_n$. To generate random numbers with uniform distribution, the linear congruential method is used (Knut, 1969) in the following way:

$$\eta_{n+1} = (a\eta_n + c) \text{ modulo } m \tag{B.2}$$

where:

$\eta_{n+1}$ = new random number

$\eta_n$ = last random number

If:

1. c is relatively prime to m

2. (a-1) is a multiple of every prime dividing m

3. (a-1) is a multiple of 4, if m is a multiple of 4.

Then the sequence defined by equation (B.2) has a period of length m. Therefore, the numbers

$$\xi_n = (2/m)\eta_n - 1$$

are uniformly distributed in the interval [-1, +1]. To obtain random numbers with a Gaussian distribution, twelve successive values of $\xi_n$ are summed. Therefore,

$$\zeta_r = \sum_{n=12r}^{n=12r+11} \xi_n \tag{B.4}$$

$\zeta_r$ is a Gaussian random variable with zero mean and unit variance.

## Shaping filter

In order to form the displayed process $y(t)$, the white sequence is passed through a second order shaping filter with natural frequency $w_0$ and damping ratio $c_1$. The transfer function of the shaping filter is

$$\frac{y(s)}{w(s)} = \frac{1}{s^2 + 2w_0 c_1 s + w_0^2} \tag{B.5}$$

Let $x_1(t)$ and $x_2(t)$ be the states of this system fined as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{x}_1(t)$$

Then the state space description of the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_0^2 & -2c_1 w_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \tag{B.6}$$

and $y(t) = x_1(t)$ 
$$\tag{B.7}$$

Since the system is time invariant, the state transition matrix can be found with the use of the Laplace transform and is given by:

$$\phi(t,0) = \begin{bmatrix} e^{-c_1 w_0 t}\left[ \cos w_1 t + \frac{c_1}{\sqrt{1-c_1^2}} \sin w_1 t \right] & \frac{e^{-c_1 w_0 t}}{w_0 \sqrt{1-c_1^2}} \sin w_1 t \\ \frac{-w_0 e^{-w_0 c_1 t}}{\sqrt{1-c_1^2}} \sin w_1 t & e^{-c_1 w_0 t}\left[ \cos w_1 t - \frac{c_1}{\sqrt{1-c_1^2}} \sin w_1 t \right] \end{bmatrix}$$

$$w_1 = w_0 \sqrt{1-c_1^2}$$

If $u(t)$ is a zero mean Gaussian white process with covariance Q eq. (B.6) is a stochastic differential equation with the solution

$$\underline{x}(t_n) = \phi(t_n, o)\, \underline{x}(0) + \int_0^{t_n} \phi(t_n, \tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)\, d\tau \qquad (B.9)$$

where

$$\underline{x}^T(t_n) = (x_1(t_n), x_2(t_n))$$

The last term in equation (B.9) is a two dimensional vector random variable $\underline{v}(t_n)$, with mean

$$\overline{\underline{v}}(t_n) = \int_0^{t_n} \phi(t_n, \tau) \binom{0}{1} \overline{w}(\tau)\, d\tau \qquad = 0 \qquad (B.10)$$

and covariance matrix:

$$C_v = E[\underline{v}(t_n)\underline{v}^T(t_n)] = \int_0^{t_n} \phi(t_n,\tau) \binom{0}{1} Q(0\ 1) \phi^T(t_n, \tau)\, d\tau \quad (B.11)$$

Therefore, to form $\underline{v}(t_n)$ two random numbers $\zeta_1$ and $\zeta_2$ are drawn from the generator, in such a way that the correlation between $v_1(t_n)$ and $v_2(t_n)$ will equal $C_{v_{12}}$. Let us define

$$v_1(t_n) = c\zeta_1 \qquad\qquad (B.12)$$
$$v_2(t_n) = a\zeta_1 + b\zeta_2$$

where a, b and c satisfy the following equations:

$$c = \sqrt{C_{v_{11}}}$$
$$a^2 + b^2 = C_{v_{22}}$$
$$ca = C_{v_{12}}$$

therefore:

$$v_1(t_n) = \sqrt{Cv_{11}} \ \zeta_1 \tag{B.14}$$

$$v_2(t_n) = \frac{Cv_{12}}{\sqrt{Cv_{11}}} \ \zeta_1 \ +_{+} \sqrt{C_{v_{22}} - \frac{Cv_{12}^2}{Cv_{11}}} \ \zeta_2$$

The last variable to be determined is the value of $Q$. Let the covariance matrix for the state be defined as

$$E[\underline{x}(t) \ \underline{x}^T(t)] = P(t) \tag{B.15}$$

this matrix is the solution of the following differential equation

$$\dot{P}(t) = FP + PF^T + GQG^T \qquad\qquad P(t_0) = P_0 \tag{B.16}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ -w_0^2 & -2c_1 w_0 \end{bmatrix} \qquad\qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$Q$ is now chosen in such a way to produce in the steady state a displayed output which is a zero mean Gaussian random variable with variance $\sigma_{ss}$ ($\sigma_{ss}$ is known). Using equation (B.7)

$$P_{11_{ss}} = \sigma_{ss} \tag{B.17}$$

also, for the steady state, equation (B.16) is

$$FP + PF^T + GQG^T = 0$$

Solving for the above F matrix

$$P_{12_{ss}} = P_{22_{ss}} = 0 \tag{B.18}$$

$$P_{22_{ss}} = w_0^2 P_{11_{ss}} \qquad\qquad P_{11_{ss}} = Q/4c_1 w_0^3 \tag{B.19}$$

Therefore

$$Q = 4\sigma_{ss} c_1 w_0^3 \tag{B.20}$$

The integral in (B.11) is computed numerically using rectangular integration with step size $t_n/20$ and with Q taken from (B.20). Then two Gaussian random numbers are drawn and $\underline{v}(t_n)$ is found using (B.14). Since the system is time invariant, the state transition matrix can be computed apriori to the integration from (B.8). Therefore, the integration of (B.6) is given by the following iteration scheme:

$$\underline{x}(t_n) = \phi(t_n, \ t_{n-1}) \ \underline{x}(t_{n-1}) + \underline{v}(t_n) \tag{B.21}$$

# REFERENCES

1. Abrahamson, I. G., Levitt, H., "Statistical Analysis of Data from Experiments in Human Signal Detection", J. Math Psyc., Vol. 6, No. 3, 1969, pp. 391-417.

2. Atans, M., Editor, "Special Issue on LQG Problem", IEEE Tran. on Control Theory, Dec., 1971.

3. Birdsall, T. G., Roberts, R. A., "On the Theory of Signal Detectibility: An Optimum Nonsequential Observation Decision Procedure", IEEE Tran. on Information Theory, April 1965, pp. 195-204.

4. Birdsall, T. G., Roberts, R. A., "Theory of Signal Detectibility: Differed Decision Theory", J. Acous. Soc. Amer., Vol. 37, No. 6, 1965, pp. 1064-1074.

5. Blackwell, H. R., "The Influence of Data Collections Procedures upon Psychophysical Measurement of Two Sensory Functions", J. of Exp. Psyc., Vol. 44, No. 4, 1952, pp. 306-315.

6. Blacklock, J. H., "Automatic Control of Aircraft and Missiles", J. Wiley, 1965.

7. Bryson, A. E., Ho, Y. C., "Applied Optimal Control", Blaisdell Pub. Comp. 1969.

8. Chein, T. T., "An Adaptive Technique for a Redundant Sensor Navigation System", C. S. Draper Report, T-S60, 1972.

9. Cohen, H. S., Ferrell, W. R., "Human Operator Decision Making in Manual Control", IEEE Tran. MMS, Vol. 10, No. 2, 1969, pp. 41-47.

10. Curry, R. E., "Investigation of Pilot's Role and Display Requirements in Automatic Landing", Man-Vehicle Lab., Semi-Annual Progress Report, N A S A Grant NGR22-009-733, March 31, 1973.

11. Curry, R. E., "A General Multinominal Maximum Likelihood Program for Behavioral Research", In preparation.

12. Curry, R. E., Nagel, D. C., Gai, E. G., "Decision Behaviour with Changing Signal Strength", Proc. of the 10th Ann. Conf. on Mannual Control, Wright Patterson A F B, 1974, pp. 1-38.

13. Decelles, J. L., Burke, E. J., Burroughs, K. L., "The Fail Safe Landing", Proc. of the ALPA 17th Air Safety Forum, San Francisco 1970.

14. Donaldson, W., Murdock, B. B., "Criterion Change in Continuous Recognition Memory", J. of Exp. Psyc., Vol. 76, 1968, pp. 325-330.

15. Dorfman, D. C., Alf, E., "Maximum Likelihood Estimation of Parameters of SDT and Determination of Confidence Intervals", J. of Math. Psyc., Vol. 6, No. 4, 1969, pp. 487-496.

16. Draper, N., Smith, H., Applied Regression Analysis, John Wiley, 1966.

17. Deyst, J. J., "Class Notes for Course 16.371" Dep. of Aero., M.I.T., 1973.

18. Ephrath, A., "Investigating the Problem of Failure Detection by Pilots", Ph.D. Thesis, Dep. of Aero., M.I.T. (to be published).

19. Fletcher, R., and Powell, M. D. J., "A Rapidly Convergent Descent Method for Minimization", The Computer Journal, Vol. 6, No. 2, July 1963, pp. 163-173.

20 Gai, E. G., Curry, R. E., "Signal Detection with Time Varying Signal to Noise Ratio", Proc. of the Ninth Annual Conference on Manual Control, Cambridge, Mass., 1973, p. 401.

21. Gaussin, J., Hupet, B., "An Experimental Validation of the SDT for Active Touch", Psyc. Sci., Vol. 27, No. 6, 1972, pp. 339-341.

22. Glorioso, R. M., Lav , R. M., "Operator Behaviour in a Dynamic Visual Signal Detection Task", Perception and Psychophysics, Vol. 4, No. 1, 1968, pp. 7-9.

23. Gourevitch, V., and Galanter, E., "A Significant Test for One Parameter Isosensitivity Functions", Psychometrics, Vol. 32, No. 1, 1967, pp. 25-33.

24. Green, D. M., "Detection of Multi-Component Auditory Signals in Noise", J. Acous. Soc. Amer., Vol. 29, 1957, p. 1257.

25. Green, D. M., and Swets, J. A., "Signal Detection Theory and Psychophysics", J. Wiley, 1966.

26. Grey, D. R., Morgan, B. J. T., "Some Aspects of ROC Curve Fitting Normal and Logistic Models", J. Math. Psyc., Vol. 9, No. 1, 1972, pp. 128-139.

27. Hammerton, M., Altman, P. M. E., "A Non-Parametric Alternative to d'", Nature, Vol. 234, 24 Dec. 1971, pp. 487-488.

28. Healy, A. J., Jones, C., "Criterion Shifts in Recall", Psyc. Bull., Vol. 79, No. 5, 1973, pp. 335-340.

29. Hoel, P. G., "Elementary Statistics", J. Wiley, 1960.

30. Hoel, P. G., "Introduction to Mathematical Statistics", J. Wiley, 1971.

31. Howard, R. A., "Dynamic Probabilistic Systems", Vol. I, J. Wiley, 1971.

32. Jacobson, D. H., Oksman, W., "An Algorithm that Minimizes Homogeneous Functions of N Variables in N + 2 Iterations", Division of Eng. and Applied Physics, Harvard University Technical Report, No. 618, Oct. 1970.

33. Jazwinsky, A. H., "Stochastic Processes and Filtering Theory", Academic Press, 1970.

34. Kailath, T., "The Innovation Approach to Detection and Estimation Theory", Proc. IEEE, Vol. 58, No. 5, 1970, pp. 680-695.

35. Kailath, T., "A View of Three Decades of Linear Filtering Theory", IEEE, Tran. Information Theory, Vol. 20, No. 2, 1974, pp. 146-181.

36. Kalman, R. E., Bucy, R. S., "New Results in Linear Filtering and Prediction Theory", Trans. of the ASME, J. Basic Eng., 1961, pp. 95-108.

37. Kinchala, R. A., Allen, L. G., "A Theory of Visual Movement Detection" Psyc. Rev., Vol. 76, No. 6, 1969, pp. 537-558.

38. Kinchala, R. A., Smyzer, F., "A Diffusion Model of Perceptual Memory", Perception and Psychophysics, Vol. 2, No. 6, 1967, pp. 219-229.

39. Kinchala, R. A., "A Comparison of Sequential Effects in Detection and Recognition". Paper presented at the 1965 Meeting of the American Psychological Association.

40. Klienman, D. L., Baron, S., "Manned Vehicle Systems Analysis by Means of Modern Control Theory", BBN Report, No. 1967; BNN 1970.

41. Knuth, D. C., "Seminumerical Algorithms", Addison Wesley Publishing Company, 1969.

42. Lee, R. C. K., "Optimal Estimation Identification and Control", M. I. T. Press, 1964.

43. Lee, W., "Choosing Among Confusably Distributed Stimuli with Specified Likelihood Ratios", Percp. and Motor Skills, Vol. 16, 1963, pp. 445-467.

44. Levison, W. H., "A Control Theory Model for Human Decision Making", Proc. of the 7th Ann. Conf. on Manual Control, 1971, pp. 23-32.

45. Levison, W. H., Tanner, R. B., "A Control Theory Model for Human Decision Making" N A S A Report CR-1953, 1971.

46. Levison, W. H., Baron, S. and Klienman, D. L., "A Model for the Human Controller Remnant", IEEE Trans. on MMS, Vol. 10, No. 4, 1969, pp. 101-107.

47. Luce, R. D., Bush, R. R., Galanter, E., "Handbook of Mathmatical Psychology", Volume I, J. Wiley, 1963.

48. Marascuilo, L. A., "Extensions of the Significance Test for One Parameter SD Hypothesis", Psychometrica, Vol. 35, No. 2, 1970, pp. 237-248.

49. Morgan, B. J. T., "Measurement of Signal Detectability", Nature, Volume 239, Sept. 15, 1972.

50. McGill, W. J., "Serial Effects in Auditory Threshold Judgements", J. of Exp. Psy., Vol. 53, No. 5, 1957, pp. 297-303.

51. Neyman, J. and Pearson, E. S., "On the Problem of Most Efficients Tests of Statistical Hypothesis", Phil. Trans. Roy. Soc. London, Ser. A, 1933, pp. 289-297.

52. Niemela, R. J., Krendel, E. S., "Detection of A Change in Plant Dynamics in a Man Machine System", Proc. of the 10th Annual Conf. on Mannual Control, Wright Patterson A F B, 1974, pp. 97-112.

53. Ogilvie, J. C., Creelman, C. D., "Maximum Likelihood Estimation of ROC Curve Parameters", J. Math. Psyc., Vol. 5, No. 3, 1968, pp. 377-391.

54. Parks, T. E., "Signal Detectability Theory of Recognition Memory Performance", Psyc. Rev., Vol. 73, No. 2, 1966, pp. 44-58.

55. Peterson, W. W., Birdsall, T. G., Fox, W. C., "The Theory of Signal Detectability", I R E Trans. Prof. Group Information Theory", PGIT-4, 171, 1954, pp. 171-212.

56. Phatak, A. V., Bekey, G. A., "Decision Processes in the Adaptive Behavior of Human Controllers", IEEE, Tran. SMC, Vol. 5, No. 4, 1969, pp. 339-351.

57. Phatak, A. V., Klienman, D. L., "Current Status of Models for the Human Operator as a Controller and Decision Maker in Manned Areospace Systems", Proc. of AGARD Conf., No. 114, Oct. 1972, pp. 16.1-16.10.

58. Robinson, G. H., "Continuous Estimation of a Time Varying Probability", Erganomics, Vol. 7, 1964, pp. 7-21.

59. Schweppe, F. C., "Uncertain Dynamic Systems", Prentice Hall, 1973.

60. Senders, W. J., Elkind, J. I., Grignette, M. C., Smallwood, R., "An Investigation of the Visual Sampling Behavior of Human Observers", N A S A, Report CR-434, 1966.

61. Sheridan, T. B., Ferrell, W. R., "Man Machine Systems", M.I.T. Press, 1974.

62. Speeth, S. D., Matthews, M. V., "Sequential Effects in Signal Detection Situation", J. Acous. Soc. Amer., Vol. 33, No. 8, 1961, pp. 1046-1054.

63. Swets, J. A., Tanner, W. P., Birdsall, T. G., "Decision Processes in Perception", Psyc. Rev., Vol. 68, No. 5, 1961, pp. 301-340.

64. Swets, J. A., "Signal Detection and Recognition by Human Observers", J. Wiley, 1964.

65. Swets, J. A., Review of G. T. Fechner book "Elemente der Psychophysik", Science, Vol. 154, 1966, p. 1532.

66. Swets, J. A., "The Relative Operating Characteristic in Psychology", Science, Vol. 182, 7 Dec. 1973, pp. 990-1000.

67. Swets, J. A., Birdsall, T. G., "Deferred Decision in Human Signal Detection: A Preliminary Experiment", Perception and Psychophysics, Vol. 2, No. 1, 1967, pp. 15-28.

68. Tanner, T. A., Haller, R. W., Atkinson, R. C., "Signal Recognition by Presentation Schedules", Perception and Psychophysics, Vol. 2, No. 2, 1967, pp. 349-358.

69. Tanner, T. A., Rouk, J. A., Atkinson, R. C., "Signal Recognition as Influenced by Information Feedback", J. Math. Psyc., Vol. 4, No. 7, 1970, pp. 259-274.

70. Tanner, W. P., Swets, J. A., "A Decision Making Theory of Visual Detection", Psyc. Rev., Vol. 61, 1954, pp. 401-409.

71. Tanner, W. P., Swets, J. A., "A New Theory of Visual Detection" University of Michigan, Electronic Defense Group, Technical Report, No. 18, 1953.

72. Teper, G. L., "Aircraft Stability and Control Data", System Technology Report - STI TR - 176-1, 1969.

73. Thomas, E. A. C., Meyers, J. L., "Implications of Latency Data for Threshold and Non-threshold Models of Signal Detection", J. Math. Psyc., Vol. 9, No. 2, 1972, pp. 253-285.

74. Thomas, E. A. C., Legge, D., "Probability Matching as a Basis for the Detection and Recognition Decisions", Psy. Rev., Vol. 77, No. 1, 1970, pp. 65-72.

75. Thurmond, J. B., Binford, J. R., Lock, M., "Effects of Signal to Noise Variability over Repeated Sessions in an Auditory Vigilance Task", Perception and Psychophysics, Vol. 7, No. 2, 1970, pp. 100-102.

76. Thurstone, L. L., "A Law of Comparative Judgement", Psych. Rev., Vol. 34, 1927, pp. 273-286.

77. Treisman, M., Watts, T. R., "Relation Between Signal Detectability Theory and the Traditional Procedures for Measuring Sensory Thresholds", Psyc. Bull., Vol. 66, No. 6, 1966, pp. 438-454.

78. Vander Velde, W. E., "Class Notes for Course 19.39", Dept. of Aero., M.I.T., 1973.

79. Vander Velde, W. E., "Class Notes for Course 19.37", Dept. of Aero., M.I.T., 1972.

80. Van Trees, H. L., "Detection, Estimation and Modulation Theory", J. Wiley, 1968.

81. Wald, A., "Statistical Decision Functions", J. Wiley, 1950.

82. Wald, A., "Sequential Analysis", J. Wiley, 1947.

83. Wiener, N., "Time Series", M.I.T. Press, 1949.

84. Weir, D. H., Klein, R. H., "The Measurement and Analysis of Pilot Scanning and Control Behaviour during Simulated Instrument Approaches", N A S A Report, CR-1535, 1970.

85. Yntema, D. B., "Keeping Track of Several Things at Once", Human Factors, Feb. 1963, pp. 7-17.

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Eliezer Gai was born on ████████████████████████.
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