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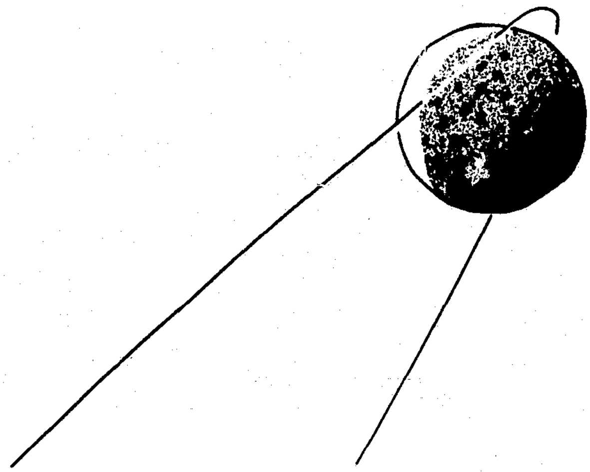

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BY

GIORGIO E. O. GIACAGLIA

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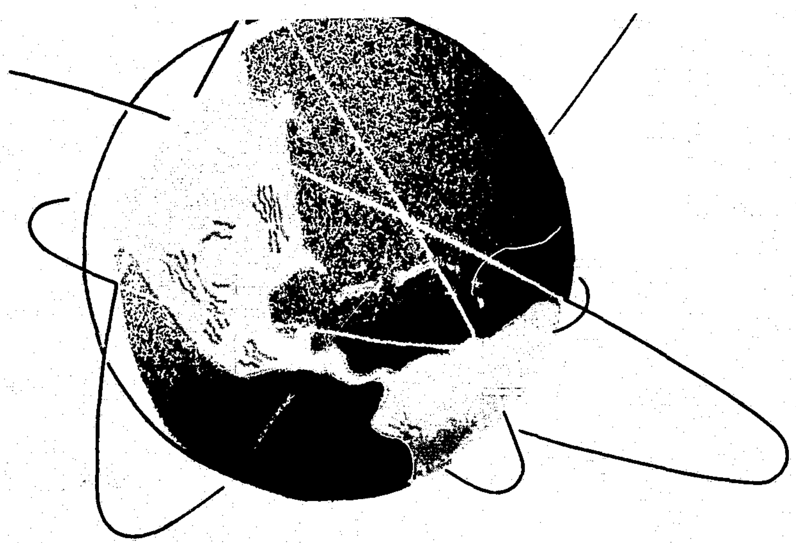

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THE EQUATIONS OF MOTION OF AN ARTIFICIAL SATELLITE IN NONSINGULAR VARIABLES

G. E. O. Giacaglia*
The University of Texas at Austin

Abstract

The equations of motion of an artificial satellite are given in nonsingular variables. Any term in the geopotential is considered as well as luni-solar perturbations up to an arbitrary power of r/r', r' being the geocentric distance of the disturbing body. Resonances with tesseral harmonics and with the moon or sun are also considered. By neglecting the shadow effect, the isturbing function for solar radiation is also developed in nonsingular variables for the long periodic perturbations. Formulas are developed for implementation of the theory in actual computations.

*On leave from the University of Sao Paulo, Brazil.

1. Definition of Nonsingular Elements

For $I \neq \pi$, $e < 1$, the following set of elements is nonsingular

$$
\begin{aligned}
a &= \text{semi major axis} \\
\lambda &= M + \omega + \Omega \\
\xi &= e \cos \tilde{\omega} \quad (\tilde{\omega} = \omega + \Omega) \\
\eta &= e \sin \tilde{\omega} \\
P &= \sin \frac{I}{2} \cos \Omega \\
Q &= \sin \frac{I}{2} \sin \Omega
\end{aligned}
\tag{1.1}
$$

Let $\gamma = \sqrt{1-e^2}$, $c = \cos \frac{I}{2}$, $s = \sin \frac{I}{2}$. For this set of elements, Lagrange's planetary equations are

$$\dot{a} = \frac{2}{na} R_\lambda$$

$$\dot{\lambda} = n - \frac{2}{na} R_a + \frac{\gamma}{2na^2} (\xi R_\xi + \eta R_\eta) + $$

$$+ \frac{1}{2na^2\gamma} (P R_P + Q R_Q)$$

$$\dot{\xi} = - \frac{\gamma}{na^2(1+\gamma)} \xi R_\lambda - \frac{\gamma}{na^2} R_\eta - $$

$$- \frac{1}{2na^2\gamma} \eta (P R_P + Q R_Q)$$

$$\dot{\eta} = - \frac{\gamma}{na^2(1+\gamma)} \eta R_\lambda + \frac{\gamma}{na^2} R_\xi + $$

$$+ \frac{1}{2na^2\gamma} \xi (P R_P + Q R_Q)$$

$$\dot{P} = - \frac{1}{2na^2\gamma} P\, R_\lambda - \frac{1}{4na^2\gamma} R_Q +$$

$$+ \frac{1}{2na^2\gamma} P(\eta\, R_\xi - \xi\, R_\eta)$$

$$\dot{Q} = - \frac{1}{2na^2\gamma} Q\, R_\lambda + \frac{1}{4na^2\gamma} R_P +$$

$$+ \frac{1}{2na^2\gamma} Q\, (\eta\, R_\xi - \xi\, R_\eta) \tag{1.2}$$

If the perturbations are given by a disturbing function R and a disturbing (nonconservative) force $\vec{F}$, in the above equations we must consider

$$R_\alpha = \frac{\partial R}{\partial \alpha} + \vec{F} \cdot \frac{\partial \vec{r}}{\partial \alpha} \tag{1.3}$$

where α is any element and $\vec{r}$ the radius vector. This form is necessary in order to take into account nonconservative perturbations like atmospheric drag, radiation pressure, etc. In rectangular inertial coordinates $\vec{F} = (X, Y, Z)$, $\vec{r} = (x, y, z)$ all we need are the derivatives of the Keplerian x, y, z with respect to the set of elements a, λ, ξ, η, P, Q.

Let $u = \omega + f$. Then we have

$$\frac{\partial x}{\partial a} = \frac{x}{a}, \quad \frac{\partial y}{\partial a} = \frac{y}{a}, \quad \frac{\partial z}{\partial a} = \frac{z}{a}$$

$$\frac{\partial x}{\partial \lambda} = \frac{a}{\gamma} \{\frac{1}{r} \frac{\partial x}{\partial u} - e\, (\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos I)\}$$

$$\frac{\partial y}{\partial \lambda} = \frac{a}{\gamma} \{\frac{1}{r} \frac{\partial y}{\partial u} - e\, (\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos I)\}$$

$$\frac{\partial z}{\partial \lambda} = \frac{a}{\gamma} \{\frac{1}{r}\frac{\partial z}{\partial u} + e \cos \omega \sin I\}$$

$$\frac{\partial x}{\partial u} = -r \{\sin u \cos \Omega + \cos u \sin \Omega \cos I\}$$

$$\frac{\partial y}{\partial u} = -r \{\sin u \sin \Omega - \cos u \cos \Omega \cos I\}$$

$$\frac{\partial z}{\partial u} = r \cos u \sin I$$

$$\frac{\partial x}{\partial \tilde{\omega}} = \frac{\partial x}{\partial u} - \frac{\partial x}{\partial \lambda}, \quad \frac{\partial y}{\partial \tilde{\omega}} = \frac{\partial y}{\partial u} - \frac{\partial y}{\partial \lambda}, \quad \frac{\partial z}{\partial \tilde{\omega}} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial \lambda}$$

$$\frac{\partial x}{\partial \Omega} = - y - \frac{\partial x}{\partial u}, \quad \frac{\partial y}{\partial \Omega} = x - \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial \Omega} = - \frac{\partial z}{\partial u}$$

$$\frac{\partial x}{\partial e} = \frac{\sin f}{\gamma^2}\frac{\partial x}{\partial u} - a (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos I)$$

$$\frac{\partial y}{\partial e} = \frac{\sin f}{\gamma^2}\frac{\partial y}{\partial u} - a (\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos I)$$

$$\frac{\partial z}{\partial e} = \frac{\sin f}{\gamma^2}\frac{\partial z}{\partial u} - a \sin \omega \sin I$$

$$\frac{\partial x}{\partial I} = r \sin u \sin \Omega \sin I$$

$$\frac{\partial y}{\partial I} = - r \sin u \cos \Omega \sin I$$

$$\frac{\partial z}{\partial I} = r \sin u \cos I \qquad (1.4)$$

The above relations are valid if x, y, z are expressed in terms of (a, λ, $\tilde{\omega}$, e, Ω, I). The derivatives with respect to ξ, η, P, Q are easily found by considering that

$$\xi = e \cos \tilde{\omega}$$

$$\eta = e \sin \tilde{\omega}$$

so that

$$\frac{\partial x}{\partial \xi} = \frac{\partial x}{\partial e} \cos \tilde{\omega} - \frac{\partial x}{\partial \tilde{\omega}} \frac{1}{e} \sin \tilde{\omega}$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial x}{\partial e} \sin \tilde{\omega} + \frac{\partial x}{\partial \tilde{\omega}} \frac{1}{e} \cos \tilde{\omega} \qquad (1.5)$$

and analogous for y, z.

Also, from

$$P = \sin \frac{I}{2} \cos \Omega$$

$$Q = \sin \frac{I}{2} \sin \Omega \qquad (c = \cos \frac{I}{2})$$

we find

$$\frac{\partial x}{\partial P} = 2\frac{\partial x}{\partial I} \frac{1}{c} \cos \Omega - \frac{\partial x}{\partial \Omega} \frac{1}{\sin \frac{I}{2}} \sin \Omega$$

$$\frac{\partial x}{\partial Q} = 2\frac{\partial x}{\partial I} \frac{1}{c} \sin \Omega + \frac{\partial x}{\partial \Omega} \frac{1}{\sin \frac{I}{2}} \cos \Omega \quad . \qquad (1.6)$$

2. Development of Geopotential in Nonsingular Variable

We propose a form suitable for both numerical or analytical solution and necessary to recognize possible resonances occuring with tesseral harmonics.

We begin considering the usual form of representation

$$R = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q} R_{\ell mpq} \tag{2.1}$$

where

$$R_{\ell mpq} = \frac{\mu a_e^{\ell}}{a^{\ell+1}} F_{\ell mp}(I)\, G_{\ell pq}(e)\, (A_{\ell m} \cos \psi_{\ell mpq} + B_{\ell m} \sin \psi_{\ell mpq}). \tag{2.2}$$

In the above expression

$\mu = Gm_{\oplus}$

a_e = mean equatorial radius of $\oplus$

$F_{\ell mp}(I)$ = inclination functions (Allan, 1965) (2.3)

$G_{\ell pq}(e) = X_{\ell-2p+q}^{-\ell-1,\, \ell-2p}$ (Plummer, 1960) (2.4)

$$A_{\ell m} = \begin{cases} C_{\ell m}, & \ell-m \text{ even} \\ -S_{\ell m}, & \ell-m \text{ odd} \end{cases}$$

$$B_{\ell m} = \begin{cases} S_{\ell m}, & \ell-m \text{ even} \\ C_{\ell m}, & \ell-m \text{ odd} \end{cases} \tag{2.5}$$

and

$$\psi_{\ell mpq} = (\ell-2p+q)\lambda - q\tilde{\omega} + (m+2p-\ell)\Omega - m\theta \quad (2.6)$$

where θ is Greenwich sidereal time.

The exact D'Alembert characteristics are

$$F_{\ell mp} = s^{|\alpha|} J_{\ell mp}(c) \quad (2.7)$$

$$G_{\ell pq} = e^{|q|} K_{\ell pq}(\gamma) \quad , \quad s = \sin\frac{I}{2} \quad (2.8)$$

where $J_{\ell mp}$ is a polynomial in $c = \cos\frac{I}{2}$, $K_{\ell pq}$ a power series in $\gamma = \sqrt{1 - e^2}$, and $\alpha = m + 2p - \ell$. For $q = 2p - \ell$, $K_{\ell pq}$ are expressible in closed form as functions of γ. We shall give expressions for $J_{\ell mp}$ and $K_{\ell pq}$ in due time. A particular term of R can thus be written in the following form

$$R_{\ell mpq} = \frac{\mu a_e^{\ell}}{a^{\ell+1}} J_{\ell mp}(c) K_{\ell pq}(\gamma) s^{|\alpha|} e^{|q|} (A_{\ell m} \cos\psi_{\ell mpq} +$$

$$+ B_{\ell m} \sin\psi_{\ell mpq}) \quad (2.9)$$

We begin by considering

$$q > 0, \quad \alpha = m + 2p - \ell > 0$$

so that we have

$$e^q \exp(iq\tilde{\omega}) = (\xi + i\eta)^q$$

$$e^q \exp(-iq\tilde{\omega}) = (\xi - i\eta)^q$$

$$s^\alpha \exp(i\alpha\Omega) = (P + iQ)^\alpha$$

$$s^\alpha \exp(-i\alpha\Omega) = (P - iQ)^\alpha$$

Let us consider the expression

$$s^{|\alpha|} e^{|q|} \cos \psi_{\ell mpq}$$

which can be written as

$$\frac{1}{2} e^{|q|-q} s^{|\alpha|-\alpha} \{[(\xi + i\eta)^q (P - iQ)^\alpha +$$

$$+ (\xi - i\eta)^q (P + iQ)^\alpha] \cos \theta_{\ell mpq} -$$

$$- i [(\xi + i\eta)^q (P - iQ)^\alpha - (\xi - i\eta)^q (P + iQ)^\alpha] \sin \theta_{\ell mpq}\}$$

where

$$\theta_{\ell mpq} = (\ell - 2p + q)\lambda - m\theta \quad . \tag{2.10}$$

Obviously for $q > 0$, $\alpha > 0$, the exponents of e and s vanish, but we keep them in order to treat the case $q < 0$, $\alpha < 0$.

Let now

$$\begin{aligned} \mathbf{R}_{\ell mpq} &= \text{Real } \{(\xi + i\eta)^q (P - iQ)^\alpha\} \\ \mathbf{I}_{\ell mpq} &= \text{Imag } \{(\xi + i\eta)^q (P - IQ)^\alpha\} \end{aligned} \tag{2.11}$$

so that

$$e^{|q|} s^{|\alpha|} \cos \psi_{\ell mpq} =$$

$$= \mathbf{R}_{\ell mpq} \cos \theta_{\ell mpq} + \mathbf{I}_{\ell mpq} \sin \theta_{\ell mpq} \quad . \tag{2.12}$$

We now evaluate the expressions for $\mathbf{R}$ and $\mathbf{I}$. We have that

$$(\xi + i\eta)^q (P - iQ)^\alpha =$$

$$= \sum_{n=0}^{k} \sum_{u=u_1}^{u_2} (-1)^{n+u} \binom{q}{u}\binom{\alpha}{2n-u} \xi^{q-u} \eta^u P^{\alpha-2n+u} Q^{2n-u} +$$

$$+ i \sum_{n=0}^{k'} \sum_{u=u_1'}^{u_2'} (-1)^{n+u+1} \binom{q}{u} \binom{\alpha}{2n+1-u} \xi^{q-u} \eta^u P^{\alpha-2n-1+u} Q^{2n+1-u}$$

where $k = [\frac{q+\alpha}{2}]$, $k' = [\frac{q+\alpha-1}{2}]$,

$$u_1 = \max(0, 2n-\alpha), \quad u_2 = \min(2n, q) \tag{2.13}$$

$$u_1' = \max(0, 2n+1-\alpha), \quad u_2' = \min(2n+1, q)$$

Thus

$$\mathbf{R}_{\ell mpq} = \sum_{n=0}^{k} \sum_{u=u_1}^{u_2} (-1)^{n+u} \binom{q}{u}\binom{\alpha}{2n-u} \xi^{q-u} \eta^u P^{\alpha-2n+u} Q^{2n-u} \tag{2.14}$$

and

$$\mathbf{I}_{\ell mpq} = \sum_{n=0}^{k'} \sum_{u=u_1'}^{u_2'} (-1)^{n+u+1} \binom{q}{u}\binom{\alpha}{2n+1-u} \xi^{q-u} \eta^u P^{\alpha-2n-1+u} Q^{2n+1-u} \tag{2.15}$$

By a shift of $-\pi/2$ in the angle $\psi_{\ell mpq}$ we also find

$$c^{|q|} s^{|\alpha|} \sin \psi_{\ell mpq} = \mathbf{R}_{\ell mpq} \sin \theta_{\ell mpq} - \mathbf{I}_{\ell mpq} \cos \theta_{\ell mpq} .$$

Thus, for $q > 0$, $\alpha = m + 2p - \ell > 0$, we have

$$R_{\ell mpq} = \frac{\mu a_e^\ell}{a^{\ell+1}} J_{\ell mpq} (c) K_{\ell pq}(\gamma) e^{|q|-q} s^{|\alpha|-\alpha} .$$

$$\cdot \{\mathbf{R}_{\ell mpq} (A_{\ell m} \cos \theta_{\ell mpq} + B_{\ell m} \sin \theta_{\ell mpq}) +$$

$$+ \mathbf{I}_{\ell mpq} (A_{\ell m} \sin \theta_{\ell mpq} - B_{\ell m} \cos \theta_{\ell mpq})\} \tag{2.16}$$

Consider now q < 0. We may write

$$(\xi + i\eta)^q = (\xi + i\eta)^{-|q|} = (\xi^2 + \eta^2)^{-|q|}(\xi - i\eta)^{|q|} =$$

$$= e^{-2|q|}\ (\xi - i\eta)^{|q|}$$

Thus for q < 0, in the definition of $\mathbf{R}_{\ell mpq}$ and $\mathbf{I}_{\ell mpq}$ we must introduce the change

$$\xi \rightarrow \xi/e^2$$

$$\eta \rightarrow -\eta/e^2$$

$$q \rightarrow -q = |q|$$

The corresponding factor in $R_{\ell mpq}$ is

$$e^{|q|-q} = e^{2|q|}$$

so that, the above changes introducing a factor $e^{-2|q|}$, that factor disappears. Therefore we can leave out the factor $e^{|q|-q}$ for both q > 0 and q < 0 defining for q < 0 the new $\mathbf{R}_{\ell mpq}$ and $\mathbf{I}_{\ell mpq}$ by the changes

$$\xi \rightarrow \xi$$

$$\eta \rightarrow -\eta \qquad (2.17)$$

$$q \rightarrow |q| = -q$$

A similar reasoning applies for α < 0. We see therefore that the expressions valid for any q and α are

$$\mathbb{R}_{\ell mpq} = \sum_{n=0}^{k} \sum_{n=u_1}^{u_2} (-1)^{n+u} \delta_u \binom{|q|}{u} \binom{|\alpha|}{2n-u} \xi^{|q|-u} \eta^{u} P^{|\alpha|-2n+u} Q^{2n-u} \tag{2.18}$$

$$\mathbb{I}_{\ell mpq} = \sum_{n=0}^{k'} \sum_{u=u_1'}^{u_2'} (-1)^{n+u+1} \delta_u \binom{|q|}{u} \binom{|\alpha|}{2n+1-u} \xi^{|q|-u} \eta^{u} P^{|\alpha|-2n-1+u} Q^{2n+1-u} \tag{2.19}$$

where

$$k = \left[\frac{|q| + |\alpha|}{2}\right] , \qquad k' = \left[\frac{|q| + |\alpha| - 1}{2}\right] \tag{2.20}$$

$$u_1 = \max\,(0,\ 2n - |\alpha|), \quad u_2 = \min\,(2n,\ |q|)$$
$$u_1' = \max\,(0,\ 2n + 1 - |\alpha|), \quad u_2' = \min\,(2n + 1,\ |q|) \tag{2.21}$$

$\delta_u = 1$, if q, α are both positive or negative

$\delta_u = (-1)^u$ if q or α is negative.

The final result is

$$R_{\ell mpq} = \frac{\mu a_e^{\ell}}{a^{\ell+1}} J_{\ell mp}\,(c)\; K_{\ell pq}\,(\gamma)\ \cdot$$

$$\cdot\ \{\mathbb{R}_{\ell mpq}\,(A_{\ell m} \cos\theta_{\ell mpq} + B_{\ell m} \sin\theta_{\ell mpq}) +$$

$$+\ \mathbb{I}_{\ell mpq}\,(A_{\ell m} \sin\theta_{\ell mpq} - B_{\ell m} \cos\theta_{\ell mpq})\} \tag{2.22}$$

3. Elimination of Short Periodic Terms

a) No resonance with tesseral harmonics.

In this case, the short periodic terms are eliminated by setting the coefficient of λ equal to zero, that is

$$q = 2p - \ell \tag{3.1}$$

so that

$$\begin{aligned} \theta_{\ell mpq} &= -m\theta \\ k &= \left[\frac{|2p-\ell| + |\alpha|}{2}\right] \\ k' &= \left[\frac{|2p-\ell|+|\alpha|-1}{2}\right] \end{aligned} \tag{3.2}$$

$$u_1 = \max\,(0,\, 2n - |\alpha|), \quad u_2 = \min\,(2n,\, |2p-\ell|)$$
$$u_1' = \max\,(0,\, 2n+1-|\alpha|), \quad u_2' = \min\,(2n+1,\, |2p-\ell|)$$

b) Resonance with tesseral harmonics. We consider two types:

b.1) $s(\dot{M} + \dot{\omega}) = r(\dot{\Omega} - \dot{\theta})$ (Nodal resonance) (3.3)

where s, r are integers mutually primes.

In this case the short periodic terms are eliminated by retaining only integers ℓ, p, m satisfying the conditions

$$\begin{aligned} &0 \le p \le \ell \\ &\ell - 2p = js \qquad (j = \text{integer}) \\ &1 \le m = jr \le \ell \end{aligned} \tag{3.4}$$

so that

$$\theta_{\ell mpq} = (js + q)\lambda - jr\,\theta \tag{3.5}$$

$$|\alpha| = |j(r - s)|$$

$$k = \left[\frac{|q| + |j(r - s)|}{2}\right], \quad k' = \left[\frac{|q| + |j(r - s)| - 1}{2}\right]$$

$$u_1 = \max\,(0,\ 2n - |j(r - s)|), \quad u_2 = \min\,(2n,\ |js|)$$

$$u_1' = \max\,(0,\ 2n + 1 - |j(r - s)|), \quad u_2' = \min\,(2n + 1,\ |js|)$$

b.2) $s\dot{\lambda} = r\dot{\theta}$ (longitude resonance) (3.6)

where s, r are integers mutually primes. The short periodic terms are eliminated by retaining only integers ℓ, p, m satisfying the conditions

$$0 \leq p \leq \ell$$

$$2p - \ell = q - js \qquad (j = \text{integer}) \tag{3.7}$$

$$1 \leq m = jr \leq \ell$$

so that

$$\theta_{\ell mpq} = j(s\lambda - r\theta)$$

$$|\alpha| = |q + j(r - s)| \tag{3.8}$$

In both cases $|q|$ is a free integer indicating the maximum power of e reatined in $R_{\ell mpq}$.

4. Development of Luni-solar Potential in Nonsingular Variables.

The disturbing function due to the moon or sun can be written as

$$R' = \beta' n'^2 r^2 \left(\frac{a'}{r'}\right)^3 \sum_{\ell \geq 2} \left(\frac{r}{r'}\right)^{\ell-2} P_\ell (\cos \psi') \tag{4.1}$$

where

$\beta' = \dfrac{m'}{m' + m_\oplus}$, m' = mass of disturbing body

n' = mean motion in longitude of the disturbing body

a' = mean distance of disturbing body from earth

ψ' = geocentric elongation of the satellite from the perturbing body

r' = geocentric distance of perturbing body

Using equatorial coordinates (α, δ) and (α', δ') for the satellite and the perturbing body, we have

$$\cos \psi' = \sin \delta \sin \delta' + \cos \delta \cos \delta' \cos (\alpha - \alpha')$$

and

$$P_\ell(\cos \psi') = \sum_{m=0}^{\ell} \varepsilon_m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell m} (\sin \delta) P_{\ell m} (\sin \delta') \cos m (\alpha - \alpha')$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ for $m \neq 0$.

We define the harmonic coefficients

$$\left.\begin{aligned} C'_{\ell m} &= \left\{\frac{\beta' n'^2}{a'^{\ell-2}} \left(\frac{a'}{r'}\right)^{\ell+1} \varepsilon_m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell m} (\sin \delta')\right\} \cos m \alpha' \\ S'_{\ell m} &= \left\{\frac{\beta' n'^2}{a'^{\ell-2}} \left(\frac{a'}{r'}\right)^{\ell+1} \varepsilon_m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell m} (\sin \delta')\right\} \sin m \alpha' \end{aligned}\right\} \tag{4.2}$$

so that

$$R' = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} R'_{\ell m} \tag{4.3}$$

where

$$R'_{\ell m} = a^{\ell} \left(\frac{r}{a}\right)^{\ell} P_{\ell m} (\sin \delta) \{C'_{\ell m} \cos m \alpha + S'_{\ell m} \sin m \alpha \} \tag{4.4}$$

By the usual transformation to orbital coordinates we have that

$$P_{\ell m} (\sin \delta) \{ C'_{\ell m} \cos m \alpha + S'_{\ell m} \sin m \alpha\} =$$

$$= \sum_{p=0}^{\ell} F_{\ell mp} (I) \{A'_{\ell m} \cos \psi_{\ell mp} + B'_{\ell m} \sin \psi_{\ell mp}\} \tag{4.5}$$

where

$$\psi_{\ell mp} = (\ell - 2p) (\omega + f) + m\Omega \tag{4.6}$$

and for

$$\begin{aligned} &\ell\text{-}m \text{ even:} \quad A'_{\ell m} = C'_{\ell m}, \quad B'_{\ell m} = S'_{\ell m} \\ &\ell\text{-}m \text{ odd :} \quad A'_{\ell m} = -S'_{\ell m}, \quad B'_{\ell m} = C'_{\ell m} \end{aligned} \tag{4.7}$$

so that

$$R' = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} R'_{\ell mp} \tag{4.8}$$

where

$$R'_{\ell mp} = a^{\ell} \left(\frac{r}{a}\right)^{\ell} F_{\ell mp} (I)\{ A'_{\ell m} \cos \psi_{\ell mp} + B'_{\ell m} \sin \psi_{\ell mp} \} \tag{4.9}$$

Using Hansen's coefficients $H_{\ell pq} = X^{\ell,\ell-2p}_{\ell-2p+q}$, we have that

$$\left(\frac{r}{a}\right)^{\ell} \begin{matrix}\cos \\ \sin\end{matrix} \psi_{\ell mp} = \sum_{q} H_{\ell pq} (e) \begin{matrix}\cos \\ \sin\end{matrix} \psi_{\ell mpq} \tag{4.10}$$

where

$$\psi_{\ell mpq} = (\ell - 2p + q)\ \lambda - q\tilde{\omega} + (m + 2p - \ell)\Omega \tag{4.11}$$

and

$$H_{\ell pq} = e^{|q|}\ L_{\ell pq}\ (\gamma)\quad . \tag{4.12}$$

The functions $L_{\ell pq}$ are power series in $\gamma = \sqrt{1-e^2}$ or, in case $q = 2p - \ell$, they can be written in closed form in terms of γ. They will be given later in this work. We finally write

$$R' = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q} R'_{\ell mpq} \tag{4.13}$$

where

$$R'_{\ell mpq} = a^{\ell}\ F_{\ell mp}\ (I)\ e^{|q|}\ L_{\ell pq}\{\ A'_{\ell m}\ \cos\psi_{\ell mpq} + B'_{\ell m}\ \sin\psi_{\ell mpq}\ \} \tag{4.14}$$

Proceeding as in section (2), in terms of the nonsingular variables, one finds

$$R'_{\ell mpq} = a^{\ell}\ J_{\ell mp}(c)\ L_{\ell pq}(\gamma)\ \{\mathbf{R}_{\ell mpq}[A'_{\ell m}\ \cos\ \phi_{\ell pq} + B'_{\ell m}\ \sin\ \phi_{\ell pq}]$$

$$+\ \mathbf{I}_{\ell mpq}[A'_{\ell m}\ \sin\ \phi_{\ell pq} - B'_{\ell m}\ \cos\ \phi_{\ell pq}]\} \tag{4.15}$$

where

$$\phi_{\ell pq} = (\ell - 2p + q)\ \lambda. \tag{4.16}$$

If there is no resonance with the moon or sun, the short periodic terms are easily eliminated by setting $q = 2p - \ell$. In this case the best form of the disturbing function for a numerical integration approach is given by

$$R'_{\ell mp(2p-\ell)} = a^{\ell} J_{\ell mp}\ L_{\ell p(2p-\ell)}\{\ \mathbf{R}_{\ell mp(2p-\ell)}\ A'_{\ell m} - \mathbf{I}_{\ell mp(2p-\ell)} B'_{\ell m}\} \tag{4.17}$$

where $A'_{\ell m}$, $B'_{\ell m}$ depend solely on the coordinates of the perturbing body as defined by Eq. (4.7). In case of longitude resonance with the perturbing body consider the expansions

$$P_{\ell m} (\sin \delta') \cos m \alpha' = \sum_{p'=0}^{\ell} F_{\ell mp'} (I') \{A''_{\ell m} \cos \theta_{\ell mp'} + $$

$$+ B''_{\ell m} \sin \theta_{\ell mp'}\}$$

$$P_{\ell m} (\sin \delta') \sin m \alpha' = \sum_{p'=0}^{\ell} F_{\ell mp} (I') \{-B''_{\ell m} \cos \theta_{\ell mp'} + $$

$$+ A''_{\ell m} \sin \theta_{\ell mp'}\}$$

where

$$\theta_{\ell mp'} = (\ell - 2p') (\omega' + f') + m \Omega'. \tag{4.18}$$

and for

ℓ-m even: $A''_{\ell m} = 1$, $B''_{\ell m} = 0$

ℓ-m odd : $A''_{\ell m} = 0$, $B''_{\ell m} = 1$.

Therefore we find

$$R'_{\ell mpq} = \sum_{p'=0}^{\ell} R'_{\ell mpqp'} \tag{4.19}$$

where

$$R'_{\ell mpqp'} = \beta' n'^2 \left(\frac{a'}{r'}\right)^{\ell+1} \frac{\varepsilon_m}{a'^{\ell-2}} \frac{(\ell - m)!}{(\ell + m)!} a^{\ell} J_{\ell mp} L_{\ell pq} \cdot$$

$$\cdot \{ \mathbb{R}_{\ell mpq} \cos \phi_{\ell pqp'} + \mathbb{I}_{\ell mpq} \sin \phi_{\ell pqp'}\} \tag{4.20}$$

where

$$\phi_{\ell mpqp'} = (\ell -2p + q)\ \lambda - (\ell -2p')\ (\omega' + f') - m\ \Omega' \tag{4.21}$$

We now consider the expression

$$\left(\frac{a'}{r'}\right)^{\ell+1} \begin{matrix}\cos\\ \sin\end{matrix} (\ell -2p')f' = \sum_{q'} G_{\ell p'q'}(e') \begin{matrix}\cos\\ \sin\end{matrix} (\ell -2p' + q')M' \tag{4.22}$$

so that, finally

$$R'_{\ell mpqp'q'} = \beta' n'^2 \frac{a^{\ell}}{a'^{\ell-2}} \varepsilon_m \frac{(\ell - m)!}{(\ell + m)!} J_{\ell mp}(c)\ L_{\ell pq}(\gamma)\ F_{\ell mp'}(I') \cdot$$

$$\cdot\ G_{\ell p'q'}(e')\ \{ \mathbf{R}_{\ell mpq} \cos \phi_{\ell mpqp'q'} +$$

$$+ \ \mathbf{I}_{\ell mpq} \sin \phi_{\ell mpqp'q'}\} \tag{4.23}$$

where

$$\phi_{\ell mpqp'q'} = (\ell -2p + q)\lambda - (\ell -2p' + q')\ \lambda' + q'\ \tilde{\omega}' -$$

$$- (m + 2p' - \ell)\ \Omega' \tag{4.24}$$

and

$$R' = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{p'=0}^{\ell} \sum_{q} \sum_{q'} R'_{\ell mpqp'q'} \tag{4.25}$$

The order of magnitude of any term $R_{\ell mpqp'q'}$ is given by

$$n^2 a^2\ \beta' \left(\frac{n'}{n}\right)^2 \left(\frac{a}{a'}\right)^{\ell-2} \frac{(\ell - m)!}{(\ell + m)!}\ e^{|q|}\ e'^{|q'|} (\sin \frac{I}{2})^{|m+2p-\ell|}$$

$$(\sin \frac{I'}{2})^{|m+2p'-\ell|} \tag{4.26}$$

For a term $R_{\ell mpq}$ in the geopotential, it is

$$n^2 a^2 \left(\frac{a_e}{a}\right)^{\ell} \sqrt{C_{\ell m}^2 + S_{\ell m}^2} \quad e^{|q|} (\sin \frac{I}{2})^{|m+2p-\ell|} \tag{4.27}$$

Note that

$$\beta'_{☾} = 0.0121835$$

$$\beta'_{\odot} = 0.9999967$$

$$n'_{☾} = 13°.1763583/\text{day}$$

$$n'_{\odot} = 0°.9856091/\text{day}$$

$$a'_{☾} = 60.266536\ a_e$$

$$a'_{\odot} = 1.00000129 \text{ A.U.} = 23455.070154\ a_e$$

$$a_e = 6378.160 \text{ km}$$

$$\sin^2 I'_{☾} = 0.16436694 + 0.06513528 \cos N - 0.00127132 \cos 2N$$

$$N = 248°.644 - 0°.0529538\ (t - 1975.0)\ (t \text{ in days})$$

$$I'_{\odot} = 23°.45229444 - 0°.0130125T - 0°.0000016T^2 + 0°.0000005T^3$$

(T in Julian centuries from J. D. 2415020.0)

$$\tfrac{1}{2} n^2 a^2 = \text{mean energy of Keplerian orbit}$$

$$n^2 a^3 = \mu = 3.986009 \times 10^5 \text{ km}^3/\text{sec}^2$$

$$= 2.975539774 \times 10^{15} \text{ km}^3/\text{day}^2$$

$$= 1.1467787471 \times 10^4\ a_e^3/\text{day}^2$$

$$e'_{☾} = 0.054900489$$

$$e'_{\odot} = 0.016751040$$

5. Elimination of Short Periodic Terms

If no resonance occurs with the moon or sun, short periodic terms are eliminated by setting in $R'_{\ell mpqp'q'}$

$$q = 2p - \ell \tag{5.1}$$

If resonance in longitude occurs of the type

$$s\dot{\lambda} = r\dot{\lambda}' \qquad (r, s \text{ integers}) \tag{5.2}$$

then short periodic terms are eliminated by retaining only integers ℓ, p, q, p', q' satisfying the relation

$$(\ell - 2p + q)r = (\ell - 2p' + q')s \tag{5.3}$$

6. The Inclination Functions $F_{\ell mp}$ and $J_{\ell mp}$

They can be defined by

$$F_{\ell mp} = \sum_{j=j_1}^{j_2} F^j_{\ell mp} \; c^{3\ell-m-2p-2j} \; s^{m-\ell+2p+2j} \tag{6.1}$$

where

$$F^j_{\ell mp} = (-1)^k \; \frac{(\ell+m)!}{2^\ell p!(\ell-p)!} \; (-1)^j \binom{2\ell-2p}{j} \binom{2p}{\ell-m-j} \tag{6.2}$$

k = integral part of $[\frac{\ell-m}{2}]$

j_1 = max $(0, -\alpha)$

j_2 = min $(2\ell - 2p, \ell - m)$

α = $m - \ell + 2p$

s = $\sin \frac{I}{2}$, $c = \cos \frac{I}{2}$

Following definition (2.7) and noting that the exponent of c in (6.1) is $2\ell - \alpha - 2j$, we obtain

$$J_{\ell mp} = \sum_{j=j_1}^{j_2} F^j_{\ell mp} \; c^{2\ell-\alpha-2j} \; s^{\alpha-|\alpha|+2j}$$

or

$$J_{\ell mp} = \sum_{j=j_i}^{j_2} F^j_{\ell mp} \; c^{2\ell-\alpha-2j} \; (1 - c^2)^{j + \frac{\alpha-|\alpha|}{2}} \tag{6.4}$$

Note that $J_{\ell mp}$ is a polynomial in $c = \cos \frac{I}{2}$, of degree $2\ell-|\alpha| \leq 2\ell$.

The $F_{\ell mp}$ satisfy the following recurrence relations (Allan, unpublished results)

$$2pF_{\ell mp} = s^2(\ell + m)(\ell + m - 1) F_{\ell-1, m-1, p-1} + $$
$$+ 2cs (\ell + m) F_{\ell-1, m, p-1} - c^2 F_{\ell-1, m+1, p-1} \tag{6.5.1}$$

$$2(\ell - p)\, F_{\ell mp} = c^2(\ell + m)\,(\ell + m - 1)\, F_{\ell-1,\, m-1,\, p} -$$

$$- 2cs\,(\ell + m)\, F_{\ell-1,\, m,\, p} - s^2\, F_{\ell-1,\, m+1,\, p} \qquad (6.5.2)$$

where

$$F_{\ell mp} = (-1)^{\left[\frac{\ell-m}{2}\right]} \frac{(\ell + m)!}{2^{\ell} p!(\ell-p)!} \sum_k (-1)^k \binom{2\ell-2p}{k} \binom{2p}{\ell-m-k} \cdot$$

$$\cdot\, c^{3\ell-m-2p-2k}\; s^{m-\ell+2p+2k} \qquad (6.7)$$

The above relations are the real correspondents to those given by Allan. Setting now

$$F_{\ell mp} = s^{|m+2p-\ell|}\; J_{\ell mp}\,(c) \qquad (6.8)$$

and

$$\alpha = m + 2p - \ell \qquad (6.9)$$

we find

$$2pJ_{\ell mp} = s^{|\alpha-2|}\; s^{-|\alpha|+2}\,(\ell + m)\,(\ell + m - 1)\, J_{\ell-1,\, m-1,\, p-1} +$$

$$+ 2c(\ell + m)\, s^{|\alpha-1|}\; s^{-|\alpha|+1}\; J_{\ell-1,\, m,\, p-1} -$$

$$- c^2\, J_{\ell-1,\, m+1,\, p-1} \qquad (6.10.1)$$

and

$$2(\ell - p)\, J_{\ell mp} = c^2(\ell + m)\,(\ell + m - 1)\, J_{\ell-1,\, m-1,\, p} -$$

$$- 2c(\ell + m)\, s^{|\alpha+1|+1}\; s^{-|\alpha|} J_{\ell-1,\, m,\, p} -$$

$$- s^{|\alpha+2|+2}\; s^{-|\alpha|}\; J_{\ell-1,\, m+1,\, p} \qquad (6.10.2)$$

Eqs. (6.10.1) and (6.10.2) are the final recurrence relations.

The partial derivatives necessary in the integration of Lagrange's equations are

$$\frac{\partial J_{\ell mp}}{\partial P} = \frac{\partial J_{\ell mp}}{\partial c} \frac{\partial c}{\partial P} = -2P \frac{\partial J_{\ell mp}}{\partial c} \tag{6.11}$$

$$\frac{\partial J_{\ell mp}}{\partial Q} = -2Q \frac{\partial J_{\ell mp}}{\partial c} \tag{6.12}$$

If

$$J^{j}_{\ell mp} = (-1)^{k} \frac{(\ell+m)!}{2^{\ell} p!(\ell-p)!} F^{j}_{\ell mp} \tag{6.13}$$

then

$$J_{\ell mp} = \sum_{j=j_1}^{j_2} J^{j}_{\ell mp} c^{2\ell-\alpha-2j} (1 - c^2)^{j} \tag{6.14}$$

and

$$\frac{\partial J_{\ell mp}}{\partial c} = \sum_{j=j_1}^{j_2} J^{j}_{\ell mp} c^{2\ell-\alpha-2j-1} \left[(2\ell-\alpha)s^{2j} - 2j\, s^{2j-2} \right] \tag{6.15}$$

From the recurrence relations (6.10.1) and (6.10.2) we find the following recurrence relations for $J'_{\ell mp} \equiv \partial J_{\ell mp}/\partial c$:

$$2pJ'_{\ell mp} = -(|\alpha-2|-|\alpha|+2)\ cs^{|\alpha-2|-|\alpha|}(\ell+m)\ (\ell+m-1)J_{\ell-1,\ m-1,\ p-1} +$$

$$+ s^{|\alpha-2|-|\alpha|+2}(\ell+m)\ (\ell+m-1)\ J'_{\ell-1,\ m-1,\ p-1} +$$

$$+ 2(\ell+m)\left[s^{|\alpha-1|-|\alpha|+1} - (|\alpha-1|-|\alpha|+1)c^2 s^{|\alpha-1|-|\alpha|-1} \right] \cdot$$

$$\cdot J_{\ell-1,\ m,\ p-1} + 2(\ell+m)\ cs^{|\alpha-1|-|\alpha|+1}\ J'_{\ell-1,\ m,\ p-1} -$$

$$- 2cJ_{\ell-1,\ m+1,\ p-1} - c^2 J'_{\ell-1,\ m+1,\ p-1} \tag{6.16.1}$$

and

$$2(\ell-p)J'_{\ell mp} = 2c(\ell+m)\ (\ell+m-1)J_{\ell-1,\ m-1,\ p} +$$

$$+ c^2(\ell+m)\ (\ell+m-1)J'_{\ell-1,\ m-1,\ p} - 2(\ell+m)\ \left[s^{|\alpha+1|+1-|\alpha|} -\right.$$

$$\left.- c^2(|\alpha+1|+1-|\alpha|)\ s^{|\alpha+1|-|\alpha|-1}\right] J_{\ell-1,\ m,\ p} -$$

$$- 2(\ell+m)\ cs^{|\alpha+1|+1-|\alpha|}J'_{\ell-1,\ m,\ p} +$$

$$+ (|\alpha+2|+2-|\alpha|)\ cs^{|\alpha+2|-|\alpha|}J_{\ell-1,\ m+1,\ p} -$$

$$- s^{|\alpha+2|+2-|\alpha|}\ J'_{\ell-1,\ m+1,\ p} \tag{6.16.2}$$

The Eqs. (6.16.1) and (6.16.2) are the necessary recurrence relations for the derivatives of the J functions.

7. The eccentricity functions $G_{\ell pq}$ and $K_{\ell pq}$.

We begin considering the definition of Hansen's coefficients (Plummer, 1960)

$$\left(\frac{r}{a}\right)^n \exp(imf) = \sum_j X_j^{n,m} \exp(ijM) \tag{7.1}$$

where, for j>m:

$$X_j^{n,m} = (-e)^{|j-m|}\, 2^{-n-1}\, (1+\gamma)^{n+1-|j-m|} \sum_{k=0}^{\infty} \sum_{r=0}^{j-m+k} \sum_{t=0}^{k} \frac{(-1)^r}{r!\,t!} \cdot \binom{n-m+1}{j-m+k-r}\binom{n+m+1}{k-t} \left(\frac{j}{2}\right)^{r+t} (1+\gamma)^{r+t-k} (1-\gamma)^k \tag{7.2}$$

and, for j<m:

$$X_j^{n,m} = (-e)^{|j-m|} 2^{-n-1}\, (1+\gamma)^{n+1-|j-m|} \sum_{k=0}^{\infty} \sum_{r=0}^{-j+m+k} \sum_{t=0}^{k} \frac{(-1)^t}{r!\,t!} \cdot \binom{n+m+1}{m-j+k-r}\binom{n-m+1}{k-t} \left(\frac{j}{2}\right)^{r+t} (1+\gamma)^{r+t-k}\, (1-\gamma)^k \tag{7.3}$$

where

$$\gamma = \sqrt{1 - e^2} \tag{7.4}$$

The definition of the $G_{\ell pq}$ functions is

$$\left(\frac{a}{r}\right)^{\ell+1} \exp[i(\ell-2p)f] = \sum_q G_{\ell pq} \exp[i(\ell-2p+q)M] \tag{7.5}$$

so that

$$G_{\ell pq} = X_{\ell-2p+q}^{-\ell-1,\ell-2p} \tag{7.6}$$

for $\ell=-n-1$, $2p=-n-m-1$, $q=j-m$, in Eq. (7.1), or $n=-\ell-1$, $m=\ell-2p$, $j=\ell-2p+q=m+q$. We see therefore that $G_{\ell pq}$ is factored by $e^{|q|}$ and the remaining factor $(K_{\ell pq})$ is a power series in $\gamma=\sqrt{1-e^2}$.

Making use of the expressions for Hansen's coefficients we find

$$G_{\ell pq} = e^{|g|}K_{\ell pq} \tag{7.7}$$

where, for $q>0$:

$$K_{\ell pq} = (-1)^{|q|}2^{\ell}\,(1+\gamma)^{-\ell-|q|}\sum_{k=0}^{\infty}\sum_{r=0}^{|q|+k}\sum_{t=0}^{k}\frac{(-1)^r}{r!t!}\cdot$$

$$\cdot\binom{2p-2\ell}{|q|+k-r}\binom{-2p}{k-t}\left(\frac{\ell-2p+q}{2}\right)^{r+t}(1+\gamma)^{r+t-k}\,(1-\gamma)^k \tag{7.8}$$

and, for $q<0$:

$$K_{\ell pq} = (-1)^{|q|}\,2^{\ell}\,(1+\gamma)^{-\ell-|q|}\sum_{k=0}^{\infty}\sum_{r=0}^{|q|+k}\sum_{t=0}^{k}\frac{(-1)^t}{r!t!}\cdot$$

$$\cdot\binom{-2p}{|q|+k-r}\binom{2p-2\ell}{k-t}\left(\frac{\ell-2p+q}{2}\right)^{r+t}(1+\gamma)^{r+t-k}\,(1-\gamma)^k \tag{7.9}$$

For $j=0$, $n<0$ and $n+m$ odd, we have:

$$X_o^{n,m} = e^{|j-m|}\,\gamma^{2n+3}\sum_{k=0}^{k'-1}\binom{-n-2}{2k+|m|}\binom{2k+|m|}{k}\cdot$$

$$\cdot\,2^{-2k-|m|}\,(1-\gamma^2)^k\quad,\quad k'=\frac{-n-1-|m|}{2} \tag{7.10}$$

so that, for $q = 2p-\ell$,

$$G_{\ell p(2p-\ell)} = e^{|2p-\ell|}\;K_{\ell p(2p-\ell)} \tag{7.11}$$

where

$$K_{\ell p(2p-\ell)} = \gamma^{-2\ell+1} \sum_{k=0}^{p'-1} \begin{pmatrix} \ell-1 \\ 2k+|2p-\ell| \end{pmatrix} \begin{pmatrix} 2k+|2p-\ell| \\ k \end{pmatrix} \cdot$$

$$\cdot\ 2^{-2k-|2p-\ell|} (1-\gamma^2)^k \tag{7.12}$$

and

$$p' = \frac{\ell - |2p-\ell|}{2} \quad \text{(always integer).} \tag{7.13}$$

The following derivatives are necessary:

$$\frac{\partial K_{\ell pq}}{\partial \xi} = \frac{\partial K_{\ell pq}}{\partial \gamma} \frac{\partial \gamma}{\partial \xi} = -\frac{\xi}{\gamma} \frac{\partial K_{\ell pq}}{\partial \gamma}$$

$$\frac{\partial K_{\ell pq}}{\partial \eta} = -\frac{\eta}{\gamma} \frac{\partial K_{\ell pq}}{\partial \gamma} \tag{7.14}$$

where, for $q>0$:

$$\frac{\partial K_{\ell pq}}{\partial \gamma} = \frac{(-\ell-|q|)}{1+\gamma} K_{\ell pq} + (-1)^{|q|}\, 2^{\ell}\, (1+\gamma)^{-\ell-|q|} \cdot$$

$$\cdot \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \frac{(-1)^r}{r!t!} \begin{pmatrix} 2p-2\ell \\ |q|+k-r \end{pmatrix} \begin{pmatrix} -2p \\ k-t \end{pmatrix} \left(\frac{\ell-2p+q}{2}\right)^{r+t} \cdot$$

$$\cdot\ (1+\gamma)^{r+t-k-1} [(r+t-k)(1-\gamma)^k - k(1+\gamma)(1-\gamma)^{k-1}] \ ; \tag{7.15}$$

for $q<0$:

$$\frac{\partial K_{\ell pq}}{\partial \gamma} = \frac{(-\ell-|q|)}{1+\gamma} K_{\ell pq} + (-1)^{|q|} 2^{\ell} (1+\gamma)^{-\ell-|q|} \cdot$$

$$\cdot \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \frac{(-1)^t}{r!t!} \binom{-2p}{|q|+k-r} \binom{2p-2\ell}{k-t} \left(\frac{\ell-2p+q}{2}\right)^{r+t} \cdot$$

$$\cdot (1+\gamma)^{r+t-k-1} [(r+t-k)(1-\gamma)^k - k(1+\gamma)(1-\gamma)^{k-1}] \quad ; \qquad (7.16)$$

and for $q = 2p - \ell$:

$$\frac{\partial K_{\ell p(2p-\ell)}}{\partial \gamma} = \frac{-2\ell+1}{\gamma} K_{\ell p(2p-\ell)} - 2\gamma^{-2\ell+2} \sum_{k=0}^{p'-1} \binom{\ell-1}{2k+|2p-\ell|} \cdot$$

$$\cdot \binom{2k+|2p-\ell|}{k} 2^{-2k-|2p-\ell|} k(1-\gamma^2)^{k-1} \qquad (7.17)$$

We now obtain recurrence relations for these eccentricity functions.

In order to do this we use Hansen's notation which make relations much easier to handle, and refer to the work of Tisserand (1889).

Recalling the definition of Hansen's coefficients

$$\left(\frac{r}{a}\right)^n \exp(imf) = \sum_k X_k^{n,m} \exp(ikM) \qquad (7.18)$$

where n,m are integers (positive, negative or zero) we observe that

$$X_k^{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n \exp[i(mf-kM)]dM$$

$$= -\frac{1}{2\pi ik} \left[\left(\frac{r}{a}\right)^n \exp[i(mf-kM)]\right]_{-\pi}^{\pi} +$$

$$+ \frac{1}{2\pi ik} \int_{-\pi}^{\pi} \frac{d}{dM} \left[\left(\frac{r}{a}\right)^n \exp(imf) \right] \exp(-ikM)\, dM$$

$$= \frac{1}{2\pi ik} \int_{-\pi}^{\pi} \{ n\left(\frac{r}{a}\right)^{n-1} \exp(imf) \frac{e}{\sqrt{1-e^2}} \sin f +$$

$$+ \left(\frac{r}{a}\right)^n im \exp[i(m-1)f] \left(\frac{a}{r}\right)^2 \sqrt{1-e^2} \} \exp(-ikM)\, dM$$

and writing $\sin f = [\exp(if) - \exp(-if)]/2i$

$$X_k^{n,m} = \frac{-1}{k} \frac{ne}{\sqrt{1-e^2}} \frac{1}{2} \left(X_k^{n-1,m+1} - X_k^{n-1,m-1} \right) +$$

$$+ \frac{m\sqrt{1-e^2}}{k} X_k^{n-2,m}$$

or

$$k\sqrt{1-e^2}\, X_k^{n,m} = m(1-e^2) X_k^{n-2,m} - n \frac{e}{2} \left(X_k^{n-1,m+1} - X_k^{n-1,m-1} \right) \tag{7.19}$$

Since for the G functions $n<0$, the recurrence formula just found must be used in the following order

$$m(1-e^2)\, X_k^{n-2,m} = k\sqrt{1-e^2}\, X_k^{n,m} + n \frac{e}{2} \left(X_k^{n-1,m+1} - X_k^{n-1,m-1} \right) \tag{7.20}$$

which is the final form to be used. This relation is valid for any values of n, m, k. Also, because of the integral definition we see that

$$X_{-k}^{n,-m} = X_k^{n,m} \tag{7.21}$$

For the functions $G_{\ell pq}$ the difference $n-m$ (or $n+m$ or $-n+m$ or $-n-m$) must be odd in order to have an integer value for p, since

$$G_{\ell pq} = X_{\ell-2p+q}^{-\ell-1,\ell-2p}, \quad n-m = -2\ell + 2p - 1. \tag{7.22}$$

Therefore, assume

$$n = -m - 2\alpha + 1$$

so that the recurrence relation yields

$$m(1-e^2)\, X_k^{-m-2\alpha-1,m} = k\sqrt{1-e^2}\, X_k^{-m-2\alpha+1,m} -$$

$$- \quad (m + 2\alpha - 1)\,\frac{e}{2}\,(X_k^{-m-2\alpha,m+1} - X_k^{-m-2\alpha,m-1}$$

Now, let

$$m + 2\alpha = \ell$$

$$m = \ell - 2p$$

$$k = \ell - 2p + q$$

so that we have

$$2\alpha = -m + \ell = 2p$$

$$m = \ell - 2p$$

$$k = \ell - 2p + q$$

and

$$X_k^{-m-2\alpha-1,m} = X_{\ell-2p+q}^{-\ell-1,\ell-2p} = G_{\ell pq}$$

$$X_k^{-m-2\alpha+1,m} = X_{\ell-2p+q}^{-\ell+1,\ell-2p} = G_{\ell-2,p-1,q}$$

$$X_k^{-m-2\alpha,m+1} = X_{\ell-2p+q}^{-\ell,\ell-2p+1} = G_{\ell-1,p-1,q-1}$$

$$X_k^{-m-2\alpha,m-1} = X_{\ell-2p+q}^{-\ell,\ell-2p-1} = G_{\ell-1,p,q+1}$$

The final needed solution is then

$$(\ell-2p)(1-e^2)G_{\ell pq} = (\ell-2p+q)\sqrt{1-e^2}\; G_{\ell-2,p-1,q} -$$

$$- (\ell+1)\frac{e}{2}(G_{\ell-1,p-1,q-1} - G_{\ell-1,p,q+1}) \tag{7.23}$$

When the G's corresponding to $\ell-2$ and $\ell-1$ are known, this relation gives the values of G for ℓ.

Finally, using the definition

$$G_{\ell pq} = e^{|q|}K_{\ell pq} \tag{7.24}$$

we obtain, with $\gamma = \sqrt{1-e^2}$, for $q > 0$:

$$(\ell-2p)\gamma^2\; K_{\ell pq} = (\ell-2p+q)\gamma\, K_{\ell-2,p-1,q} -$$

$$- (\ell+1)\frac{1}{2}[K_{\ell-1,p-1,q-1} - (1-\gamma^2)\, K_{\ell-1,p,q+1}] \tag{7.25}$$

and for $q < 0$:

$$(\ell-2p)\,\gamma^2\, K_{\ell pq} = (\ell-2p+q)\,\gamma\, K_{\ell-2,p-1,q} -$$

$$- (\ell+1)\frac{1}{2}[(1-\gamma^2)\, K_{\ell-1,p-1,q-1} - K_{\ell-1,p,q+1}]\;. \tag{7.26}$$

The symmetry conditions (7.21) are written as

$$G_{\ell pq} = G_{\ell,\ell-p,-q} \tag{7.27}$$

so that we also have

$$K_{\ell pq} = K_{\ell,\ell-p,-q} \tag{7.28}$$

The use of this relation decreases by 50% the number of functions to be evaluated in any case.

For the derivatives of $K_{\ell pq}$ with repsect to γ, it is now easy to derive a recurrence relation, by dividing Eqs. (7.25) and (7.26) by γ^2 and differentiating. We easily find, for $q > 0$:

$$\gamma^2(\ell-2p)\ K'_{\ell pq} = (\ell-2p+q)(-K_{\ell-2,p-1,q} + \gamma K'_{\ell-2,p-1,q}) -$$

$$- \frac{\ell+1}{2} \{[- \frac{2}{\gamma} K_{\ell-1,p-1,q-1} + K'_{\ell-1,p-1,q-1}] +$$

$$+ [\frac{2}{\gamma} K_{\ell-1,p,q+1} - (1-\gamma^2)\ K'_{\ell-1,p,q+1}]\}$$

and for $q < 0$:

$$\gamma^2(\ell-2p)K'_{\ell pq} = (\ell-2p+q)(-K_{\ell-2,p-1,q} + \gamma\ K'_{\ell-2,p-1,q}) +$$

$$+ \frac{\ell+1}{2} \{[- \frac{2}{\gamma} K_{\ell-1,p,q+1} + K'_{\ell-1,p,q+1}]\} +$$

$$+ [\frac{2}{\gamma} K_{\ell-1,\ p-1,\ q-1} - (1 - \gamma^2)\ K'_{\ell-1,\ p-1,\ q-1}]\} \qquad (7.30)$$

For $q=2p-\ell$, the recurrence relation is much simpler, namely for $2p-\ell>0$,

$$(\ell-2p)\ \gamma^2\ K_{\ell p(2p-\ell)} = \frac{1}{2}(\ell+1)\ [(1-\gamma^2)\ K_{\ell-1,p,2p-\ell+1} -$$

$$- K_{\ell-1,p-1,2p-\ell-1}] \qquad (7.31)$$

and for $2p-\ell < 0$,

$$-(\ell-2p)\ \gamma^2\ K_{\ell p(2p-\ell)} = \frac{1}{2}(\ell+1)\ [(1-\gamma^2)\ K_{\ell-1,p-1,2p-\ell-1} -$$

$$- K_{\ell-1,p,2p-\ell+1}]. \qquad (7.32)$$

For the derivatives we have, for $2p - \ell > 0$,

$$\gamma^2(\ell-2p)\ K'_{\ell p(2p-\ell)} = \frac{\ell+1}{2}\{\frac{2}{\gamma}(K_{\ell-1,p-1,2p-\ell-1} -$$

$$- K_{\ell-1,p,2p-\ell+1}) + [(1-\gamma^2)\ K'_{\ell-1,p,2p-\ell+1} -$$

$$- \frac{2}{\gamma} K_{\ell-1,p,2p-\ell+1}]\} \tag{7.33}$$

and for $2p-\ell < 0$:

$$\gamma^2(\ell-2p)\ K'_{\ell p(2p-\ell)} = \frac{\ell+1}{2}\{\frac{2}{\gamma}(K_{\ell-1,p-1,q-1} - K_{\ell-1,p,q+1}) +$$

$$+ [K'_{\ell-1,p,q+1} - (1-\gamma^2)\ K'_{\ell-1,p-1,q-1}]\} \tag{7.34}$$

8. The eccentricity functions $H_{\ell pq}$ and $L_{\ell pq}$

In terms of Hansen's coefficients we have the definition

$$H_{\ell pq} = X_{\ell-2p+q}^{\ell,\ell-2p} \tag{8.1}$$

According to Eq. (4.10):

$$\left(\frac{r}{a}\right)^{\ell} \exp[i(\ell-2p)f] = \sum_q H_{\ell pq} \exp[i(\ell-2p+q)M] \tag{8.2}$$

Also

$$H_{\ell pq} = e^{|q|} L_{\ell pq}(\gamma) \tag{8.3}$$

where

$$\gamma = \sqrt{1-e^2}$$

Comparing with

$$\left(\frac{r}{a}\right)^{n} \exp(imf) = \sum_j X_j^{n,m} \exp(ijM) \tag{8.4}$$

the relationships are in Eq. (8.1):

$$n = \ell$$

$$m = \ell-2p$$

$$j = \ell-2p+q$$

or

$$\ell = n$$

$$2p = n-m$$

$$q = j-m$$

It is seen that, in opposition to the $G_{\ell pq}$ functions $n-m$ should be even and $n>0$. Eqs. (7.2) and (7.3) are still valid, so that for $q>0$:

$$L_{\ell pq} = (-1)^{|q|}\, 2^{-\ell-1}\, (1+\gamma)^{\ell+1-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \frac{(-1)^r}{r!t!} \cdot$$

$$\cdot \binom{2p+1}{|q|+k-r} \binom{2\ell-2p+1}{k-t} \binom{\ell-2p+q}{2}^{r+t} (1+\gamma)^{r+t-k} (1-\gamma)^{k} \qquad (8.5)$$

and for $q < 0$:

$$L_{\ell pq} = (-1)^{|q|}\, 2^{-2\ell-1}\, (1+\gamma)^{\ell+1-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \frac{(-1)^t}{r!t!} \cdot$$

$$\cdot \binom{2\ell-2p+1}{|q|+k-r} \binom{2p+1}{k-t} \left(\frac{\ell-2p+q}{2}\right)^{r+t} (1+\gamma)^{r+t-k} (1-\gamma)^{k} \qquad (8.6)$$

For $j=0$ and $n\geq 0$ we have

$$X_{o}^{nm} = \frac{(-e)^{|m|}(1+\gamma)^{n+1-|m|}}{2^{n+1}} \sum_{k=0}^{n+1-|m|} \binom{n+1-|m|}{k} \binom{n+1+|m|}{|m|+k} \left(\frac{1-\gamma}{1+\gamma}\right)^{k}$$

(8.7)

so that

$$H_{\ell p(2p-\ell)} = \frac{(-e)^{|2p-\ell|}(1+\gamma)^{\ell+1-|2p-\ell|}}{2^{\ell+1}} \cdot$$

$$\cdot \sum_{k=0}^{\ell+1-|2p-\ell|} \binom{\ell-|2p-\ell|+1}{k} \binom{\ell+|2p-\ell|+1}{|2p-\ell|+k} \left(\frac{1-\gamma}{1+\gamma}\right)^{k} \qquad (8.9)$$

and therefore

$$L_{\ell p(2p-\ell)} = \frac{(-1)^{|2p-\ell|}\,(1+\gamma)^{\ell+1-|2p-\ell|}}{2^{\ell+1}} \;\cdot$$

$$\cdot \sum_{k=0}^{\ell+1-|2p-\ell|} \binom{\ell - |2p-\ell|+1}{k} \binom{\ell + |2p-\ell|+1}{|2p-\ell| + k} \left(\frac{1-\gamma}{1+\gamma}\right)^{k} \tag{8.9}$$

For the derivatives, we have, as in Eq. (7.14),

$$\frac{\partial L_{\ell pq}}{\partial \xi} = -\frac{\xi}{\gamma}\,\frac{\partial L_{\ell pq}}{\partial \gamma}$$

$$\frac{\partial L_{\ell pq}}{\partial \eta} = -\frac{\eta}{\gamma}\,\frac{\partial L_{\ell pq}}{\partial \gamma} \quad . \tag{8.10}$$

Indicating by a prime the derivative with respect to γ, we find that,

$$L'_{\ell pq} = \frac{\ell+1-|q|}{1+\gamma}\, L_{\ell pq} + (-1)^{|q|}\, 2^{-\ell-1}\, (1+\gamma)^{\ell+1-|q|} \;\cdot$$

$$\cdot \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} C_{\ell pq}^{krt}\, (1+\gamma)^{r+t+k-1} \Big[(r+t+k)(1-\gamma)^{k} -$$

$$- k(1+\gamma)(1-\gamma)^{k-1}\Big] \tag{8.11}$$

where, for $q > 0$

$$C_{\ell pq}^{krt} = \frac{(-1)^{r}}{r!\; t!} \binom{2p+1}{|q|+k-r} \binom{2\ell-2p+1}{k-t} \left(\frac{\ell-2p+q}{2}\right)^{r+t} \tag{8.12}$$

and for $q < 0$

$$C_{\ell pq}^{krt} = \frac{(-1)^{t}}{r!\; t!} \binom{2\ell-2p+1}{|q|+k-r} \binom{2p+1}{k-t} \left(\frac{\ell-2p+q}{2}\right)^{r+t} \tag{8.13}$$

If $q = 2p - \ell$,

$$L'_{\ell p(2p-\ell)} = \frac{(\ell+1-|2p-\ell|)}{1+\gamma} L_{\ell p(2p-\ell)} - \frac{(-1)^{|2p-\ell|}(1+\gamma)^{\ell-1-|2p-\ell|}}{2^\ell} \cdot$$

$$\cdot \sum_{k=1}^{\ell+1-|2p-\ell|} k \binom{\ell-|2p-\ell|+1}{k} \binom{\ell+|2p-\ell|+1}{|2p+\ell|+k} \left(\frac{1-\gamma}{1+\gamma}\right)^{k-1} \quad (8.14)$$

The recurrence relation appropriate for these functions is given by (7.19) since now $n > 0$. Considering that

$$H_{\ell pq} = X_{\ell-2p+q}^{\ell,\ \ell-2p}$$

we easily find

$$(\ell-2p+q)\gamma\ H_{\ell pq} = (\ell-2p)\gamma^2\ H_{\ell-2,\ p-1,\ q} -$$

$$- \frac{\ell e}{2} (H_{\ell-1,\ p-1,\ q-1} - H_{\ell-1,\ p,\ q+1}) \quad (8.15)$$

Using the definition

$$H_{\ell pq} = e^{|q|}\ L_{\ell pq} \quad (8.16)$$

we have, for $q > 0$:

$$(\ell-2p+q)\gamma\ L_{\ell pq} = (\ell-2p)\gamma^2\ L_{\ell-2,\ p-1,\ q} -$$

$$- \frac{\ell}{2} [L_{\ell-1,\ p-1,\ q-1} - (1-\gamma^2)\ L_{\ell-1,\ p,\ q+1}] \quad (8.17.1)$$

and, for $q < 0$

$$(\ell-2p+q)\gamma L_{\ell pq} = (\ell-2p)\gamma^2\ L_{\ell-2,\ p-1,\ q} -$$

$$- \frac{\ell}{2} [(1-\gamma^2)\ L_{\ell-1,\ p-1,\ q-1} - L_{\ell-1,\ p,\ q+1}] \quad (8.17.2)$$

When $q = 2p - \ell$ we obtain, for $2p - \ell > 0$,

$$\frac{\ell}{2}(1-\gamma^2) L_{\ell-1,\, p,\, 2p-\ell+1} = \frac{\ell}{2} L_{\ell-1,\, p-1,\, 2p-\ell-1} -$$

$$- (\ell-2p)\gamma^2 L_{\ell-2,\, p-1,\, 2p-\ell} \qquad (8.18.1)$$

and for $2p - \ell < 0$,

$$\frac{\ell}{2} L_{\ell-1,\, p,\, 2p-\ell+1} = \frac{\ell}{2}(1-\gamma^2) L_{\ell-1,\, p-1,\, 2p-\ell-1} -$$

$$- (\ell-2p)\gamma^2 L_{\ell-2,\, p-1,\, 2p-\ell} \qquad (8.18.2)$$

Recurrence relations for the derivatives follow from (8.17) and (8.18):

$q > 0$:

$$(\ell-2p+q)\gamma^2 L'_{\ell pq} = (\ell-2p)\gamma^2 L_{\ell-2,\, p-1,\, q} - (\ell-2p)\gamma^3 L'_{\ell-2,\, p-1,\, q} -$$

$$- \frac{\ell}{2}[-L_{\ell-1,\, p-1,\, q-1} + \gamma L'_{\ell-1,\, p-1,\, q-1} +$$

$$+ (1+\gamma^2) L_{\ell-1,\, p,\, q+1} - \gamma(1-\gamma^2) L'_{\ell-1,\, p,\, q+1}] \qquad (8.19.1)$$

$q < 0$:

$$(\ell-2p+q)\gamma^2 L'_{\ell pq} = (\ell-2p)\gamma^2 L_{\ell-2,\, p-1,\, q} - (\ell-2p)\gamma^3 L'_{\ell-2,\, p-1,\, q} -$$

$$- \frac{\ell}{2}[(1+\gamma^2) L_{\ell-1,\, p-1,\, q-1} - \gamma(1+\gamma^2) L'_{\ell-1,\, p-1,\, q-1} -$$

$$- L_{\ell-1,\ p,\ q+1} + \gamma L'_{\ell-1,\ p,\ q+1}] \qquad (8.19.2)$$

$q = 2p - \ell > 0$:

$$\frac{\ell}{2}(1-\gamma^2)^2 L'_{\ell-1,\ p,\ 2p-\ell+1} = \ell\gamma L_{\ell-1,\ p-1,\ 2p-\ell-1} +$$

$$+ \frac{\ell}{2}(1-\gamma^2) L'_{\ell-1,\ p-1,\ 2p-\ell-1} - 2\gamma(\ell-2p) L_{\ell-2,\ p-1,\ 2p-\ell} -$$

$$- (\ell-2p)\gamma^2 (1-\gamma^2) L'_{\ell-2,\ p-1,\ 2p-\ell} \qquad (8.20.1)$$

$q = 2p - \ell < 0$:

$$\frac{\ell}{2} L'_{\ell-1,\ p,\ 2p-\ell+1} = -\ell\gamma L_{\ell-1,\ p-1,\ 2p-\ell-1} +$$

$$+ \frac{\ell}{2}(1-\gamma^2) L'_{\ell-1,\ p-1,\ 2p-\ell-1} - 2(\ell-2p)\gamma L_{\ell-2,\ p-1,\ 2p-\ell} -$$

$$- (\ell-2p)\gamma^2 L'_{\ell-2,\ p-1,\ 2p-\ell} \qquad (8.20.2)$$

We also have the symmetry condition

$$X_k^{n,m} = X_{-k}^{n,-m}$$

giving

$$H_{\ell pq} = H_{\ell,\ \ell-p,\ -q} \qquad (8.21)$$

9. The Functions $\mathbf{R}_{\ell mpq}$ and $\mathbf{I}_{\ell mpq}$

They are defined by (2.18) and (2.19). Expressions for the derivatives are easily found using equations (2.11) in the form

$$2\,\mathbf{R}_{\ell mpq} = (\xi + i\eta)^q\,(P - iQ)^\alpha + (\xi - i\eta)^q\,(P + iQ)^\alpha \tag{9.1}$$

$$2i\,\mathbf{I}_{mpq} = (\xi + i\eta)^q\,(P - iQ)^\alpha - (\xi - i\eta)^q\,(P + iQ)^\alpha \tag{9.2}$$

together with (2.17).

Distinction must be made of the cases $q > 0$, $q < 0$, $\alpha > 0$, $\alpha < 0$. We easily find that

$$\begin{aligned}
\frac{\partial \mathbf{R}_{\ell mpq}}{\partial \xi} &= |q|\,\mathbf{R}_{\ell mpq'} \\
\frac{\partial \mathbf{R}_{\ell mpq}}{\partial \eta} &= -q\,\mathbf{I}_{\ell mpq'} \\
\frac{\partial \mathbf{I}_{\ell mpq}}{\partial \xi} &= |q|\,\mathbf{I}_{\ell mpq'} \\
\frac{\partial \mathbf{I}_{\ell mpq}}{\partial \eta} &= q\,\mathbf{R}_{\ell mpq'}
\end{aligned} \tag{9.3}$$

where

$$q' = \begin{cases} q - 1 & (q > 0) \\ q + 1 & (q < 0) \end{cases}$$

$$\begin{aligned}
\frac{\partial \mathbf{R}_{\ell mpq}}{\partial P} &= |\alpha|\,\mathbf{R}_{\ell m'pq} \\
\frac{\partial \mathbf{R}_{\ell mpq}}{\partial Q} &= -\alpha\,\mathbf{I}_{\ell m'pq}
\end{aligned} \tag{9.4}$$

$$\frac{\partial \mathbf{I}_{\ell mpq}}{\partial P} = |\alpha| \ \mathbf{I}_{\ell m'pq}$$

$$\frac{\partial \mathbf{I}_{\ell mpq}}{\partial Q} = \alpha \ \mathbf{R}_{\ell m'pq}$$

where

$$m' = \begin{cases} m - 1 & (\alpha > 0) \\ m + 1 & (\alpha < 0) \end{cases}$$

$$\alpha = m + 2p - \ell \ .$$

In order to obtain recurrence relations we observe that Eqs. (9.1) and (9.2) may also be written as

$$\begin{aligned} (\xi + i\eta)^q \ (P - iQ)^\alpha &= \mathbf{R}_{q\alpha} + i \ \mathbf{I}_{q\alpha} \\ (\xi - i\eta)^q \ (P + iQ)^\alpha &= \mathbf{R}_{q\alpha} - i \ \mathbf{I}_{q\alpha} \end{aligned} \tag{9.5}$$

where the indices of $\mathbf{R}$, $\mathbf{I}$ have been compressed for simplicity of notation, and when q and/or α are negative the changes (2.17) must be considered.

Let us initially consider q, α positive. Then

$$\begin{aligned} \mathbf{R}_{q+1,\alpha} + i \ \mathbf{I}_{q+1,\alpha} &= (\xi + i\eta) \ (\mathbf{R}_{q\alpha} + i \ \mathbf{I}_{q\alpha}) \\ &= (\xi \ \mathbf{R}_{q\alpha} - \eta \ \mathbf{I}_{q\alpha}) + i(\eta \ \mathbf{R}_{q\alpha} + \xi \ \mathbf{I}_{q\alpha}) \end{aligned}$$

so that

$$\begin{aligned} \mathbf{R}_{q+1,\alpha} &= \xi \ \mathbf{R}_{q\alpha} - \eta \ \mathbf{I}_{q\alpha} \\ \mathbf{I}_{q+1,\alpha} &= \eta \ \mathbf{R}_{q\alpha} + \xi \ \mathbf{I}_{q\alpha} \end{aligned} \qquad (q \rightarrow q + 1) \tag{9.6}$$

If $q < 0$, we must change $q \to -q$ and $\eta \to -\eta$ in the expressions for $\mathbb{R}$ and $\mathbb{I}$, that is, for $q < 0$,

$$(\xi - i\eta)^{-q} (P - iQ)^{\alpha} = \mathbb{R}_{q\alpha} + i\,\mathbb{I}_{q\alpha}$$

Multiplication by $(\xi - i\eta)$ gives

$$\mathbb{R}_{q+1,\alpha} + i\,\mathbb{I}_{q+1,\alpha} = (\xi - i\eta)\,(\mathbb{R}_{q\alpha} + i\,\mathbb{I}_{q\alpha})$$

$$= (\xi\,\mathbb{R}_{q\alpha} + \eta\,\mathbb{I}_{q\alpha}) + i(-\eta\,\mathbb{R}_{q\alpha} + \xi\,\mathbb{I}_{q\alpha})$$

so that, for $q < 0$

$$\begin{aligned} \mathbb{R}_{q+1,\alpha} &= \xi\,\mathbb{R}_{q\alpha} + \eta\,\mathbb{I}_{q\alpha} \\ \mathbb{I}_{q+1,\alpha} &= -\eta\,\mathbb{R}_{q\alpha} + \xi\,\mathbb{I}_{q\alpha} \end{aligned} \qquad (q \to q + 1) \quad (9.7)$$

For the recurrence relations in α the same reasoning holds but we shall obtain opposite results because of the definitions (9.5), so that, for $\alpha > 0$

$$\begin{aligned} \mathbb{R}_{q,\alpha+1} &= \xi\,\mathbb{R}_{q\alpha} + \eta\,\mathbb{I}_{q\alpha} \\ \mathbb{I}_{q,\alpha+1} &= -\eta\,\mathbb{R}_{q\alpha} + \xi\,\mathbb{I}_{q\alpha} \end{aligned} \qquad (m \to m + 1) \quad (9.8)$$

and for $\alpha < 0$

$$\begin{aligned} \mathbb{R}_{q,\alpha+1} &= \xi\,\mathbb{R}_{q\alpha} - \eta\,\mathbb{I}_{q\alpha} \\ \mathbb{I}_{q,\alpha+1} &= \eta\,\mathbb{R}_{q\alpha} + \xi\,\mathbb{I}_{q\alpha} \end{aligned} \qquad (m \to m + 1) \quad (9.9)$$

On the other hand we note that if ℓ increases by one unit α decreases by one unit so that relations (9.8) and (9.9) are useful when m increases by one unit. For the ℓ case we have, solving for $\mathbf{R}_{q\alpha}$, $\mathbf{I}_{q\alpha}$ in (9.8) and (9.9) and changing from α to $\alpha - 1$, for $\alpha > 0$

$$(\xi^2 + \eta^2)\,\mathbf{R}_{q,\alpha-1} = \xi\,\mathbf{R}_{q\alpha} - \eta\,\mathbf{I}_{q\alpha}$$
$$(\xi^2 + \eta^2)\,\mathbf{I}_{q,\alpha-1} = \eta\,\mathbf{R}_{q\alpha} + \xi\,\mathbf{I}_{q\alpha} \qquad (\ell \rightarrow \ell + 1) \quad (9.10)$$

and for $\alpha < 0$

$$(\xi^2 + \eta^2)\,\mathbf{R}_{q,\alpha-1} = \xi\,\mathbf{R}_{q\alpha} + \eta\,\mathbf{I}_{q\alpha}$$
$$(\xi^2 + \eta^2)\,\mathbf{I}_{q,\alpha-1} = -\eta\,\mathbf{R}_{q\alpha} + \xi\,\mathbf{I}_{q\alpha} \qquad (\ell \rightarrow \ell + 1) \quad (9.11)$$

When q is negative it is important to have a recurrence relation for the change $q \rightarrow q - 1$. We shall find, from (9.11) changing η to $-\eta$,

$$(\xi^2 + \eta^2)\,\mathbf{R}_{q-1,\alpha} = \xi\,\mathbf{R}_{q\alpha} - \eta\,\mathbf{I}_{q\alpha}$$
$$(\xi^2 + \eta^2)\,\mathbf{I}_{q-1,\alpha} = \eta\,\mathbf{R}_{q\alpha} + \xi\,\mathbf{I}_{q\alpha} \qquad (q \rightarrow q - 1) \quad (9.12)$$

Finally, when p increases by one unit, α increases by two units. The relations (9.8) and (9.9) should be used but by increasing α two steps. For $\alpha+1>0$ one has from (9.8)

$$\mathbf{R}_{q,\alpha+2} = \xi\,\mathbf{R}_{q,\alpha+1} + \eta\,\mathbf{I}_{q,\alpha+1}$$
$$\mathbf{I}_{q,\alpha+2} = -\eta\,\mathbf{R}_{q,\alpha+1} + \xi\,\mathbf{I}_{q,\alpha+1}$$

and using (9.8) in the right-hand members of these we have, for

$\alpha > 0$ $(\alpha + 1 > 0)$:

$$\mathbf{R}_{q,\alpha+2} = (\xi^2 - \eta^2)\,\mathbf{R}_{q\alpha} + 2\xi\eta\,\mathbf{I}_{q\alpha}$$

$$\mathbf{I}_{q,\alpha+2} = (\xi^2 - \eta^2)\,\mathbf{I}_{q\alpha} - 2\xi\eta\,\mathbf{R}_{q\alpha} \qquad (p \to p + 1) \quad (9.13)$$

For $\alpha + 1 < 0$, from (9.9)

$$\mathbf{R}_{q,\alpha+2} = \xi\,\mathbf{R}_{q,\alpha+1} - \eta\,\mathbf{I}_{q,\alpha+1}$$

$$\mathbf{I}_{q,\alpha+2} = \eta\,\mathbf{R}_{q,\alpha+1} + \xi\,\mathbf{I}_{q,\alpha+1}$$

Considering $\alpha < -1$ $(\alpha < 0)$, the use of (9.9) in the right-hand members of these, gives for $\alpha < -1$,

$$\mathbf{R}_{q,\alpha+2} = (\xi^2 - \eta^2)\,\mathbf{R}_{q\alpha} - 2\xi\eta\,\mathbf{I}_{q\alpha}$$

$$\mathbf{I}_{q,\alpha+2} = (\xi^2 - \eta^2)\,\mathbf{I}_{q\alpha} + 2\xi\eta\,\mathbf{R}_{q\alpha} \qquad (p \to p + 1) \quad (9.14)$$

The only cases not considered by (9.13) and (9.14) are $\alpha=0$ and $\alpha=-1$, when α increases by two units. The case $\alpha = 0$ is covered by (9.13). Let us analyse the case $\alpha = -1$ when α increases by two units. In the first step, Eqs. (9.9) should be used. In the second step $\alpha = 0$ changes to $\alpha = 1$. In this case either relations (9.8) or (9.9) can be used, so that this case is covered by (9.14) which is valid for $\alpha \leq -1$ therefore, while 9.13 is valid for $\alpha \geq 0$.

Putting together these results we have the following recurrence relations:

$$\mathbf{R}_{\ell,m,p,q+1} = \xi\,\mathbf{R}_{\ell mpq} - \eta\,\mathbf{I}_{\ell mpq}$$
$$\mathbf{I}_{\ell,m,p,q+1} = \eta\,\mathbf{R}_{\ell mpq} + \xi\,\mathbf{I}_{\ell mpq} \qquad (q \geq 0) \qquad (9.15.1)$$

$$(\xi^2 + \eta^2)\,\mathbf{R}_{\ell,m,p,q-1} = \xi\,\mathbf{R}_{\ell mpq} - \eta\,\mathbf{I}_{\ell mpq}$$
$$(\xi^2 + \eta^2)\,\mathbf{I}_{\ell,m,p,q-1} = \eta\,\mathbf{R}_{\ell mpq} + \xi\,\mathbf{I}_{\ell mpq} \qquad (q < 0) \qquad (9.15.2)$$

$$(\xi^2 + \eta^2)\,\mathbf{R}_{\ell+1,m,p,q} = \xi\,\mathbf{R}_{\ell mpq} - \eta\,\mathbf{I}_{\ell mpq}$$
$$(\xi^2 + \eta^2)\,\mathbf{I}_{\ell+1,m,p,q} = \eta\,\mathbf{R}_{\ell mpq} + \xi\,\mathbf{I}_{\ell mpq} \qquad (\alpha \geq 0) \qquad (9.15.3)$$

$$(\xi^2 + \eta^2)\,\mathbf{R}_{\ell+1,m,p,q} = \xi\,\mathbf{R}_{\ell mpq} + \eta\,\mathbf{I}_{\ell mpq}$$
$$(\xi^2 + \eta^2)\,\mathbf{I}_{\ell+1,m,p,q} = -\eta\,\mathbf{R}_{\ell mpq} + \xi\,\mathbf{I}_{\ell mpq} \qquad (\alpha < 0) \qquad (9.15.4)$$

$$\mathbf{R}_{\ell,m+1,p,q} = \xi\,\mathbf{R}_{\ell mpq} + \eta\,\mathbf{I}_{\ell mpq}$$
$$\mathbf{I}_{\ell,m+1,p,q} = -\eta\,\mathbf{R}_{\ell mpq} + \xi\,\mathbf{I}_{\ell mpq} \qquad (\alpha \geq 0) \qquad (9.15.5)$$

$$\mathbf{R}_{\ell,m+1,p,q} = \xi\,\mathbf{R}_{\ell mpq} - \eta\,\mathbf{I}_{\ell mpq}$$
$$\mathbf{I}_{\ell,m+1,p,q} = \eta\,\mathbf{R}_{\ell mpq} + \xi\,\mathbf{I}_{\ell mpq} \qquad (\alpha < 0) \qquad (9.15.6)$$

$$\mathbf{R}_{\ell,m,p+1,q} = (\xi^2 - \eta^2)\,\mathbf{R}_{\ell mpq} + 2\xi\eta\,\mathbf{I}_{\ell mpq}$$
$$\mathbf{I}_{\ell,m,p+1,q} = (\xi^2 - \eta^2)\,\mathbf{I}_{\ell mpq} - 2\xi\eta\,\mathbf{R}_{\ell mpq} \qquad (\alpha \geq 0) \qquad (9.15.7)$$

$$\mathbf{R}_{\ell,m,p+1,q} = (\xi^2 - \eta^2)\,\mathbf{R}_{\ell mpq} - 2\xi\eta\,\mathbf{I}_{\ell mpq}$$
$$\mathbf{I}_{\ell,m,p+1,q} = (\xi^2 - \eta^2)\,\mathbf{I}_{\ell mpq} + 2\xi\eta\,\mathbf{R}_{\ell mpq} \qquad (\alpha < 0) \qquad (9.15.8)$$

Equations (9.15.1) through (9.15.8) are all recurrence relations necessary for $\mathbb{R}$ and $\mathbb{I}$.

For the derivatives, no recurrence relations are necessary in view of Eqs. (9.3) and 9.4).

10. Short Table for $J_{\ell mp}$

ℓ	m	p	$\lvert\alpha\rvert = \lvert m+2p-\ell\rvert$	$J_{\ell mp}(c)$, $c = \cos\frac{I}{2}$
2	0	0	2	$-\frac{3}{2}c^2$
2	0	1	0	$-\frac{1}{2} + 3c^2 - 3c^4$
2	0	2	2	$-\frac{3}{2}c^2$
2	1	0	1	$3c^3$
2	1	1	1	$3c - 6c^3$
2	1	2	3	$-3c$
2	2	0	0	$3c^4$
2	2	1	2	$6c^2$
2	2	2	4	3
3	0	0	3	$-\frac{5}{2}c^3$
3	0	1	1	$-\frac{3}{2}c + \frac{15}{2}c^3 - \frac{15}{2}c^5$
3	0	2	1	$\frac{3}{2}c - \frac{15}{2}c^3 + \frac{15}{2}c^5$
3	0	3	3	$\frac{5}{2}c^3$
3	1	0	2	$-\frac{15}{2}c^4$
3	1	1	0	$-9c^2 + 30c^4 - \frac{45}{2}c^6$
3	1	2	2	$-\frac{3}{2} + \frac{15}{4}c^2 - \frac{45}{8}c^4$
3	1	3	4	$-\frac{15}{2}c^2$
3	2	0	1	$15c^3$

ℓ	m	p	$\lvert\alpha\rvert = \lvert m+2p-\ell\rvert$	$J_{\ell mp}(c)$, $c = \cos\frac{I}{2}$
3	2	1	1	$30\ c^3 - 45\ c^5$
3	2	2	3	$15\ c - 45\ c^3$
3	2	3	5	$-\ 15\ c$
3	3	0	0	$15\ c^6$
3	3	1	2	$45\ c^4$
3	3	2	4	$45\ c^2$
3	3	3	6	15
4	0	0	4	$\frac{35}{8}\ c^4$
4	0	1	2	$\frac{15}{4}\ c^2 - \frac{35}{2}\ c^4 + \frac{35}{2}\ c^6$
4	0	2	0	$\frac{3}{8} - \frac{15}{2}\ c^2 + \frac{135}{4}\ c^4 - \frac{105}{2}\ c^6$
4	0	3	2	$\frac{15}{4}\ c^2 - \frac{35}{2}\ c^4 + \frac{35}{2}\ c^6$
4	0	4	4	$\frac{35}{8}\ c^4$
4	1	0	3	$-\ \frac{35}{2}\ c^5$
4	1	1	1	$-\ 25\ c^3 + \frac{175}{2}\ c^5 - 70\ c^7$
4	1	2	1	$-\ \frac{15}{2}\ c + \frac{135}{2}\ c^3 - \frac{315}{2}\ c^5 + 105\ c^7$
4	1	3	3	$\frac{15}{2}\ c - \frac{105}{2}\ c^3 + 70\ c^5$
4	1	4	5	$\frac{35}{2}\ c^3$
4	2	0	2	$-\ \frac{105}{2}\ c^6$
4	2	1	0	$-\ \frac{45}{2}\ c^4 + 45\ c^6 - 30\ c^8$
4	2	2	2	$-\ \frac{75}{2}\ c^2 + 315\ c^4 - 315\ c^6$

ℓ	m	p	$\lvert\alpha\rvert = \lvert m+2p-\ell\rvert$	$J_{\ell mp}(c), \quad c = \cos \frac{I}{2}$
4	2	3	4	$-\frac{15}{2} + 105\ c^2 - 210\ c^4$
4	2	4	6	$-\frac{105}{2}\ c^2$
4	3	0	1	$105\ c^7$
4	3	1	1	$315\ c^5 - 420\ c^7$
4	3	2	3	$315\ c^3 - 630\ c^5$
4	3	3	5	$-105\ c + 420\ c^3$
4	3	4	7	$-105\ c$
4	4	0	0	$105\ c^8$
4	4	1	2	$420\ c^6$
4	4	2	4	$630\ c^4$
4	4	3	6	$420\ c^2$
4	4	4	8	105

11. Short Table for $K_{\ell pq}$

$(K_{\ell pq} = K_{\ell,\ell-p,-q})$

ℓ	p	q	ℓ	p	q	$K_{\ell pq}$
2	0	-2	2	2	2	0
2	0	-1	2	2	1	$-\frac{1}{2} + \frac{1}{16} e^2 + \ldots$
2	0	0	2	2	0	$1 - \frac{5}{2} e^2 + \frac{13}{16} e^4 + \ldots$
2	0	1	2	2	-1	$\frac{7}{2} - \frac{123}{16} e^2 + \ldots$
2	0	2	2	2	-2	$\frac{17}{2} - \frac{115}{6} e^2 + \ldots$
2	1	-2	2	1	2	$\frac{9}{4} + \frac{7}{4} e^2 + \ldots$
2	1	-1	2	1	1	$\frac{3}{2} + \frac{27}{16} e^2 + \ldots$
2	1	0	2	1	0	$(1 - e^2)^{-3/2}$
3	0	-2	3	3	2	$\frac{1}{8} + \frac{1}{48} e^2 + \ldots$
3	0	-1	3	3	1	$-1 + \frac{5}{4} e^2 + \ldots$
3	0	0	3	3	0	$1 - 6 e^2 + \frac{423}{64} e^4 + \ldots$
3	0	1	3	3	-1	$5 - 22 e^2 + \ldots$
3	0	2	3	3	-2	$\frac{127}{8} - \frac{3065}{48} e^2 + \ldots$
3	1	-2	3	2	2	$\frac{11}{8} + \frac{49}{16} e^2 + \ldots$
3	1	-1	3	2	1	$(1 - e^2)^{-5/2}$
3	1	0	3	2	0	$1 + 2 e^2 + \frac{239}{64} e^4 + \ldots$
3	1	1	3	2	-1	$3 + \frac{11}{4} e^2 + \ldots$
3	1	2	3	2	-2	$\frac{53}{8} + \frac{39}{16} e^2 + \ldots$

ℓ	p	q	ℓ	p	q	$K_{\ell pq}$
4	0	-2	4	4	2	$\frac{1}{2} - \frac{1}{3} e^2 + \ldots$
4	0	-1	4	4	1	$-\frac{3}{2} + \frac{75}{16} e^2 + \ldots$
4	0	0	4	4	0	$1 - 11 e^2 + \frac{199}{8} e^4 + \ldots$
4	0	1	4	4	-1	$\frac{13}{2} - \frac{765}{16} e^2 + \ldots$
4	0	2	4	4	-2	$\frac{51}{2} - \frac{321}{2} e^2 + \ldots$
4	1	-2	4	3	2	$\frac{3}{4} (1 - e^2)^{-7/2}$
4	1	-1	4	3	1	$\frac{1}{2} + \frac{33}{16} e^2 + \ldots$
4	1	0	4	3	0	$1 + e^2 + \frac{65}{16} e^4 + \ldots$
4	1	1	4	3	-1	$\frac{9}{2} - \frac{3}{16} e^2 + \ldots$
4	1	2	4	3	-2	$\frac{53}{4} - \frac{179}{24} e^2 + \ldots$
4	2	-2	4	2	2	$5 + \frac{155}{12} e^2 + \ldots$
4	2	-1	4	2	1	$\frac{5}{2} + \frac{135}{16} e^2 + \ldots$
4	2	0	4	2	0	$(1 + \frac{3}{2} e^2)(1 - e^2)^{-7/2}$

12. Short Table for $L_{\ell pq}$

$(L_{\ell pq} = L_{\ell,\ell-p,-q})$

ℓ	p	q	ℓ	p	q	$L_{\ell pq}$
2	0	-2	2	2	2	$\frac{5}{2}$
2	0	-1	2	2	1	$-3 + \frac{39}{24} e^2 + \ldots$
2	0	0	2	2	0	$1 - \frac{5}{2} e^2 + \frac{23}{16} e^4 + \ldots$
2	0	1	2	2	-1	$1 - \frac{19}{8} e^2 + \ldots$
2	0	2	2	2	-2	$1 - \frac{5}{2} e^2 + \ldots$
2	1	-2	2	1	2	$-\frac{1}{4} + \frac{1}{12} e^2 + \ldots$
2	1	-1	2	1	1	$-1 + \frac{1}{8} e^2 + \ldots$
2	1	0	2	1	0	$1 + \frac{3}{2} e^2$
3	0	-2	3	3	2	$\frac{57}{8} - \frac{65}{16} e^2 + \ldots$
3	0	-1	3	3	1	$-\frac{9}{2} + \frac{33}{4} e^2 + \ldots$
3	0	0	3	3	0	$1 - 6 e^2 + \frac{591}{64} e^4 + \ldots$
3	0	1	3	3	-1	$\frac{3}{2} - \frac{57}{8} e^2 + \ldots$
3	0	2	3	3	-2	$\frac{15}{8} - \frac{135}{16} e^2 + \ldots$
3	1	-2	3	2	2	$\frac{11}{8} + \frac{7}{48} e^2 + \ldots$
3	1	-1	3	2	1	$-\frac{5}{2} - \frac{15}{8} e^2$
3	1	0	3	2	0	$1 + 2 e^2 - \frac{123}{192} e^4 + \ldots$
3	1	1	3	2	-1	$-\frac{1}{2} + e^2 + \ldots$

ℓ	p	q	ℓ	p	q	$K_{\ell pq}$
3	1	2	3	2	-2	$-\frac{3}{8} + \frac{11}{16} e^2 + \ldots$
4	0	-2	4	4	2	$14 - \frac{137}{6} e^2 + \ldots$
4	0	-1	4	4	1	$-6 - \frac{93}{4} e^2 + \ldots$
4	0	0	4	4	0	$1 - 11 e^2 + \frac{253}{8} e^4 + \ldots$
4	0	1	4	4	-1	$2 - \frac{63}{4} e^2 + \ldots$
4	0	2	4	4	-2	$3 - 21 e^2 + \ldots$
4	1	-2	4	3	2	$\frac{21}{4} + \frac{21}{8} e^2$
4	1	-1	4	3	1	$-4 - 3 e^2 + \ldots$
4	1	0	4	3	0	$1 + e^2 - \frac{129}{48} e^4 + \ldots$
4	1	1	4	3	-1	$\frac{3}{2} e^2 - \frac{9}{4} e^4 + \ldots$
4	1	2	4	3	-2	$-\frac{1}{4} + \frac{37}{24} e^2 + \ldots$
4	2	-2	4	2	2	$\frac{1}{2} - \frac{7}{12} e^2 + \ldots$
4	2	-1	4	2	1	$-2 - \frac{9}{4} e^2 + \ldots$
4	2	0	4	2	0	$1 + 5 e^2 + \frac{15}{8} e^4$

13. Short Tables for $R_{\ell mpq}$, $I_{\ell mpq}$

(α = m+2p-ℓ)

q	α	$R_{\ell mpq}$	$I_{\ell mpq}$
0	0	1	0
0	1	P	$-Q$
0	2	$P^2 - Q^2$	$-2PQ$
1	0	ξ	η
1	1	$\xi P + \eta Q$	$\eta P - \xi Q$
1	2	$\xi(P^2-Q^2) + 2\eta PQ$	$\eta(P^2-Q^2) - 2\xi PQ$
2	0	$\xi^2 - \eta^2$	$2\xi\eta$
2	1	$P(\xi^2-\eta^2) + 2Q\xi\eta$	$Q(\xi^2-\eta^2) + 2P\xi\eta$
2	2	$(\xi^2-\eta^2)(P^2-Q^2) + 4\xi\eta PQ$	$-2(\xi^2-\eta^2)PQ + 2(P^2-Q^2)\xi\eta$
0	-1	P	Q
0	-2	$P^2 - Q^2$	$2PQ$
1	-1	$\xi P - \eta Q$	$\eta P + \xi Q$
1	-2	$\xi(P^2-Q^2) - 2\eta PQ$	$\eta(P^2-Q^2) + 2\xi PQ$
2	-1	$P(\xi^2-\eta^2) - 2Q\xi\eta$	$-Q(\xi^2-\eta^2) + 2P\xi\eta$
2	-2	$(\xi^2-\eta^2)(P^2-Q^2) - 4\xi\eta PQ$	$2(\xi^2-\eta^2) PQ + 2(P^2-Q^2)\xi\eta$
-2	0	$\xi^2 - \eta^2$	$-2\xi\eta$
-2	1	$P(\xi^2-\eta^2) - 2Q\xi\eta$	$Q(\xi^2-\eta^2) - 2P\xi\eta$
-2	2	$(\xi^2-\eta^2)(P^2-Q^2) - 4\xi\eta PQ$	$-2(\xi^2-\eta^2)PQ - 2(P^2-Q^2)\xi\eta$
-1	0	ξ	$-\eta$
-1	1	$\xi P - \eta Q$	$-\eta P - \xi Q$
-1	2	$\xi(P^2-Q^2) - 2\eta PQ$	$-\eta(P^2-Q^2) - 2\xi PQ$
-2	-1	$P(\xi^2-\eta^2) + 2Q\xi\eta$	$-Q(\xi^2-\eta^2) - 2P\xi\eta$

q	α	$\mathbf{R}_{\ell mpq}$	$\mathbf{I}_{\ell mpq}$
-2	-2	$(\xi^2-\eta^2)(P^2-Q^2) + 4\xi\eta PQ$	$2(\xi^2-\eta^2)PQ - 2(P^2-Q^2)\xi\eta$
-1	-1	$\xi P + \eta Q$	$-\eta P + \xi Q$
-1	-2	$\xi(P^2-Q^2) + 2\eta PQ$	$-\eta(P^2-Q^2) + 2\xi PQ$

Note that the $H_{\ell pq}$, $G_{\ell pq}$ were obtained directly from Cayley's tables considering that

$$H_{\ell pq} = X_j^{n,m} \begin{cases} n = \ell \\ m = \ell - 2p \\ j = m + q \end{cases}$$

$$G_{\ell pq} = X_j^{n,m} \begin{cases} m = -\ell - 1 \\ m = \ell - 2p \\ j = m + q \end{cases}$$

14. Solar Radiation

Let the force per unit mass of the satellite be σ and the unit vector from the earth to the sun's position be $\hat{r}_\odot$. The force due to radiation pressure is then, per unit mass,

$$\vec{F} = -\sigma \, \hat{r}_\odot$$

and the disturbing function, in absence of shadow is

$$R^p = -\sigma \, \hat{r}_\odot \cdot \vec{r}$$

where $\vec{r}$ is the radius vector of the satellite. If ψ' is the geocentric elongation of this from the sun

$$R^p = -\sigma \, r \cos \psi \qquad (14.1)$$

Transforming to orbital elements and observing that the average value of (r/a) cosf with respect to the satellite's mean anomaly is $-\frac{3}{2}$ e, we find, for the long periodic part of R^p,

$$\overline{R} = \frac{3}{2} \sigma a \, \{2\sqrt{1-P^2-Q^2} \; (\eta P - \xi Q) \sin \varepsilon \sin u' +$$

$$+ [\xi(1-2Q^2) + 2\eta PQ] \cos u' + [\eta(1-2P^2) + 2\xi PQ] \cos \varepsilon \sin u'\} \qquad (14.2)$$

where

$$u' = f_\odot + \omega_\odot$$

ε = obliquity of the ecliptic

If the eccentricity of the sun is neglected, $\lambda_{\odot}$ can be substituted for u' and

$$\lambda_{\odot} = 279°.041 + 0°.985\,647\,4\ (t - 1975.0)$$

where t is days.

The partial derivatives are easily found

$$\frac{\partial\overline{R}}{\partial a} = \frac{1}{a}\,\overline{R}$$

$$\frac{\partial\overline{R}}{\partial\lambda} = 0$$

$$\frac{\partial\overline{R}}{\partial\xi} = \frac{3}{2}\,\sigma a\ \{-2\sqrt{1-P^2-Q^2}\ Q \sin\varepsilon \sin u' + $$

$$+ (1-2Q^2) \cos u + 2\,PQ \cos\varepsilon \sin u'\}$$

$$\frac{\partial\overline{R}}{\partial\eta} = \frac{3}{2}\,\sigma a\ \{2\sqrt{1-P^2-Q^2}\ P \sin\varepsilon \sin u' + $$

$$+ 2\,PQ \cos u' + (1-2\,P^2) \cos\varepsilon \sin u'\}$$

$$\frac{\partial\overline{R}}{\partial P} = 3\,\sigma a\ \{-(1-P^2-Q^2)^{-1/2}\ [\eta(1-Q^2) - \xi\,PQ] \sin\varepsilon \sin u' + $$

$$+ \eta Q \cos u' + (\xi Q - 2\eta P) \cos\varepsilon \sin u'\}$$

$$\frac{\partial\overline{R}}{\partial Q} = 3\,\sigma a\ \{-(1-P^2-Q^2)^{-1/2}\ [\xi(1-P^2) + \eta\,PQ] \sin\varepsilon \sin u' + $$

$$+ (\eta P - 2\xi Q) \cos u' + \xi P \cos\varepsilon \sin u'\} \qquad (14.3)$$

If short periodic terms are to be included, the elliptic expansions necessary to develop (14.1) are simply given by

$$\frac{r}{a} \cos f = -\frac{3}{2} e + \sum_{k=1}^{\infty} \frac{1}{k} [J_{k-1}(ke) - J_{k+1}(ke)] \cos kM$$

$$\frac{r}{a} \sin f = 2 \sqrt{1-e^2} \sum_{k=1}^{\infty} \frac{1}{k} [J_{k-1}(ke) + J_{k+1}(ke)] \sin kM$$

where $J_k(ke)$ is Bessel's function. Recurrence relations for these are also simple and classical

$$J_k(x) = \frac{2(k-1)}{x} J_{k-1}(x) - J_{k-2}(x) \quad .$$

APPENDIX A

Because of the recurrence relations for the G's and H's it is seen that initial large values of q may be necessary to evaluate low values of q for large degrees and orders. The functions necessary to start the recurrence relations are the following:

$G_{\ell pq}$			$X_j^{n,m}$		
ℓ	p	q	n	m	j
2	0	q	-3	2	2+q
2	1	q	-3	0	q
3	0	q	-4	3	3+q
3	1	q	-4	1	1+q

$H_{\ell pq}$			$X_j^{n,m}$		
2	0	q	2	2	2+q
2	1	q	2	0	q
3	0	q	3	3	3+q
3	1	q	3	1	1+q

The $X_j^{n,m}$ corresponding to the above (n, m, j) sets for $-20 \leq j \leq 20$, are listed in the following tables reproduced from Vol. XVIII of the Astronomical Papers of the Nautical Almanac Office, U. S. Naval Observatory. Note that the exponent of (R/A) is n, the argument of Exp (MIF) is m and each line gives the coefficients $X_j^{n,m}$, $X_{-j}^{n,m}$.

$(R/A)^{+2}$ EXP(0iF)

```
EXP(0iM)
   (+1/1) E(0)  +(+3/2) E(2)
EXP(+1iM), EXP(-1iM)
   ((-1, -1)/1) E(1)  +((+1, +1)/8) E(3)  +((-1, -1)/192) E(5)  +((+1, +1)/9216) E(7)
   +((-1, -1)/7 37280) E(9)  +((+1, +1)/884 73600) E(11)  +((-1, -1)/1 48635 64800) E(13)
   +((+1, +1)/332 94385 15200) E(15)  +((-1, -1)/95887 82923 77600) E(17)
   +((+1, +1)/345 19618 52559 36000) E(19)
EXP(+2iM), EXP(-2iM)
   ((-1, -1)/4) E(2)  +((+1, +1)/12) E(4)  +((-1, -1)/96) E(6)  +((+1, +1)/1440) E(8)
   +((-1, -1)/34560) E(10)  +((+1, +1)/12 09600) E(12)  +((-1, -1)/580 60800) E(14)
   +((+1, +1)/36578 30400) E(16)  +((-1, -1)/29 26264 32000) E(18)
   +((+1, +1)/2897 00167 68000) E(20)
EXP(+3iM), EXP(-3iM)
   ((-1, -1)/8) E(3)  +((+9, +9)/128) E(5)  +((-81, -81)/5120) E(7)
   +((+81, +81)/40960) E(9)  +((-729, -729)/45 87520) E(11)
   +((+6561, +6561)/7340 03200) E(13)  +((-2187, -2187)/58720 25600) E(15)
   +((+19683, +19683)/164 41671 68000) E(17)  +((-1 77147, -1 77147)/57874 68431 36000) E(19)
EXP(+4iM), E   (-4iM)
   ((-1, -1)/12) E(4)  +((+1, +1)/15) E(6)  +((-1, -1)/45) E(8)  +((+4, +4)/945) E(10)
   +((-1, -1)/1890) E(12)  +((+2, +2)/42525) E(14)  +((-2, -2)/6 37875) E(16)
   +((+8, +8)/491 16375) E(18)  +((-1, -1)/1473 49125) E(20)
EXP(+5iM), EXP(-5iM)
   ((-25, -25)/384) E(5)  +((+625, +625)/9216) E(7)  +((-15625, -15625)/5 16096) E(9)
   +((+3 90625, +3 90625)/495 45216) E(11)  +((-97 65625, -97 65625)/71345 11104) E(13)
   +((+97 65625, +97 65625)/5 70760 88832) E(15)
   +((-2441 40625, -2441 40625)/1506 80874 51648) E(17)
   +((+61035 15625, +61035 15625)/5 06287 73837 53728) E(19)
EXP(+6iM), EXP(-6iM)
   ((-9, -9)/160) E(6)  +((+81, +81)/1120) E(8)  +((-729, -729)/17920) E(10)
   +((+243, +243)/17920) E(12)  +((-2187, -2187)/7 16800) E(14)
   +((+19683, +19683)/394 24000) E(16)  +((-19683, -19683)/3153 92000) E(18)
   +((+1 77147, +1 77147)/2 87006 72000) E(20)
EXP(+7iM), EXP(-7iM)
   ((-2401, -2401)/46080) E(7)  +((+1 17649, +1 17649)/14 74560) E(9)
   +((-57 64801, -57 64801)/1061 68320) E(11)
   +((+2824 75249, +2824 75249)/1 27401 98400) E(13)
   +((-1 38412 87201, -1 38412 87201)/224 22749 18400) E(15)
   +((+67 82230 72849, +67 82230 72849)/53814 59804 16000) E(17)
   +((-3323 29305 69601, -3323 29305 69601)/167 90154 58897 92000) E(19)
EXP(+8iM), EXP(-8iM)
   ((-16, -16)/315) E(8)  +((+256, +256)/2835) E(10)  +((-1024, -1024)/14175) E(12)
   +((+16384, +16384)/4 67775) E(14)  +((-16384, -16384)/14 03325) E(16)
   +((+2 62144, +2 62144)/912 16125) E(18)  +((-10 48576, -10 48576)/19155 38625) E(20)
EXP(+9iM), EXP(-9iM)
   ((-59049, -59049)/11 46880) E(9)  +((+47 82969, +47 82969)/458 75200) E(11)
   +((-3874 20489, -3874 20489)/40370 17600) E(13)
   +((+34867 84401, +34867 84401)/6 45922 81600) E(15)
   +((-28 24295 36481, -28 24295 36481)/1343 51945 72800) E(17)
   +((+2287 67924 54961, +2287 67924 54961)/3 76185 44803 84000) E(19)
EXP(+10iM), EXP(-10iM)
   ((-15625, -15625)/2 90304) E(10)  +((+3 90625, +3 90625)/31 93344) E(12)
   +((-97 65625, -97 65625)/766 40256) E(14)  +((+2441 40625, +2441 40625)/29889 69984) E(16)
   +((-61035 15625, -61035 15625)/16 73823 19104) E(18)
   +((+61035 15625, +61035 15625)/50 21469 57312) E(20)
EXP(+11iM), EXP(-11iM)
   ((-2143 58881, -2143 58881)/37158 91200) E(11)
   +((+2 59374 24601, +2 59374 24601)/17 83627 77600) E(13)
   +((-313 84283 76721, -313 84283 76721)/1854 97288 70400) E(15)
   +((+37974 98335 83241, +37974 98335 83241)/3 11635 44502 27200) E(17)
   +((-45 94972 98635 72161, -45 94972 98635 72161)/747 92506 80545 28000) E(19)
EXP(+12iM), EXP(-12iM)
   ((-243, -243)/3850) E(12)  +((+4374, +4374)/25025) E(14)
   +((-39366, -39366)/1 75175) E(16)  +((+1 57464, +1 57464)/8 75875) E(18)
   +((-1 77147, -1 77147)/17 51750) E(20)
EXP(+13iM), EXP(-13iM)
   ((-13 78584 91849, -13 78584 91849)/196 19905 53600) E(13)
   +((+2329 80851 22481, +2329 80851 22481)/10987 14710 01600) E(15)
   +((-3 93737 63856 99289, -3 93737 63856 99289)/13 18457 65201 92000) E(17)
   +((+665 41660 91831 79841, +665 41660 91831 79841)/2531 43869 18768 64000) E(19)
EXP(+14iM), EXP(-14iM)
   ((-2824 75249, -2824 75249)/35582 97600) E(14)
```

$(R/A)^{+2}$ EXP(OiF) CONTINUED

```
EXP(+14iM), EXP(-14iM) CONTINUED
    +((+1 38412 87201, +1 38412 87201)/5 33744 64000) E(16)
    +((-67 82230 72849, -67 82230 72849)/170 79828 48000) E(18)
    +((+3323 29305 69601, +3323 29305 69601)/8710 71252 48000) E(20)
EXP(+15iM), EXP(-15iM)
    ((-2 13574 21875, -2 13574 21875)/23 51159 05024) E(15)
    +((+480 54199 21875, +480 54199 21875)/1504 74179 21536) E(17)
    +((-1 08121 94824 21875, -1 08121 94824 21875)/2 04644 88373 28896) E(19)
EXP(+16iM), EXP(-16iM)
    ((-671 08864, -671 08864)/6385 12875) E(16)
    +((+42949 67296, +42949 67296)/1 08547 18875) E(18)
    +((-6 87194 76736, -6 87194 76736)/9 76924 69875) E(20)
EXP(+17iM), EXP(-17iM)
    ((-168 37782 65594 00929, -168 37782 65594 00929)/1371 19595 80999 68000) E(17)
    +((+48661 19187 56668 68481, +48661 19187 56668 68481)/98726 10898 31976 96000) E(19)
EXP(+18iM), EXP(-18iM)
    ((-28 24295 36481, -28 24295 36481)/195 16456 96000) E(18)
    +((+2287 67924 54961, +2287 67924 54961)/3708 12682 24000) E(20)
EXP(+19iM), EXP(-19iM)
    ((-2 88441 41356 76211 67681, -2 88441 41356 76211 67681)/16 78343 85271 43608 32000) E(19)
EXP(+20iM), EXP(-20iM)
    ((-61035 15625, -61035 15625)/2 96985 10842) E(20)
```

$(R/A)^{+2}$ EXP(2iF)

```
EXP(0 lM)
    (+5/2) E(2)
EXP(+1 lM), EXP(-1 lM)
    ((-3, +0)/1) E(1)  +((+39, -7)/24) E(3)  +((+10, -47)/384) E(5)
    +((+1135, -1091)/15360) E(7)  +((+1 03047, -1 03150)/22 11840) E(9)
    +((+207 00211, -206 99915)/6193 15200) E(11)
    +((+5035 62396, -5035 62463)/1 98180 86400) E(13)
    +((+60 34703 91705, -60 34703 91181)/2996 49466 36800) E(15)
    +((+7892 91238 51345, -7892 91238 51676)/4 79439 14618 88000) E(17)
    +((+17 44220 78782 68437, -17 44220 78782 68165)/1265 71934 59384 32000) E(19)
EXP(+2 lM), EXP(-2 lM)
    ((+1, +0)/1) E(0)  +((-5, +0)/2) E(2)  +((+23, -1)/16) E(4)  +((-325, -33)/1440) E(6)
    +((+425, -173)/11520) E(8)  +((+7049, -7965)/8 06400) E(10)
    +((+20510, -20397)/29 03040) E(12)  +((+32 15307, -32 15884)/6096 38400) E(14)
    +((+4025 25525, -4025 23849)/9 75421 44000) E(16)
    +((+19 30571 35975, -19 30571 49987)/5794 00335 36000) E(18)
    +((+79 90203 37852, -79 90203 37065)/28970 01676 80000) E(20)
EXP(+3 lM), EXP(-3 lM)
    ((+1, +0)/1) E(1)  +((-19, +0)/8) E(3)  +((+1070, -17)/640) E(5)  +((-479, -7)/1024) E(7)
    +((+45647, -3446)/5 73440) E(9)  +((-1 45838, -1 80025)/458 75200) E(11)
    +((+24 25284, -20 81925)/7340 03200) E(13)
    +((+4343 34405, -4389 31479)/20 55208 96000) E(15)
    +((+2 41527 86691, -2 41410 35940)/1446 86710 78400) E(17)
    +((+77 68879 89119, -77 69013 93242)/57874 68431 36000) E(19)
EXP(+4 lM), EXP(-4 lM)
    ((+1, +0)/1) E(2)  +((-5, +0)/2) E(4)  +((+1515, -11)/720) E(6)
    +((-8239, -17)/10080) E(8)  +((+15113, -265)/80640) E(10)
    +((-5 40423, -42745)/217 72800) E(12)  +((+3 66135, -1 28567)/870 91200) E(14)
    +((+613 21095, -751 00039)/6 70602 24000) E(16)
    +((+86065 76319, -84914 95103)/965 66722 56000) E(18)
    +((+53 34148 52347, -53 38282 53435)/75322 04359 68000) E(20)
EXP(+5 lM), EXP(-5 lM)
    ((+25, +0)/24) E(3)  +((-1075, +0)/384) E(5)  +((+58975, -225)/21504) E(7)
    +((-83 32100, +3975)/61 93152) E(9)  +((+593 28750, -3 44375)/1486 35648) E(11)
    +((-1830 19425, -25 27450)/23781 70368) E(13)
    +((+2 23134 88175, -16689 56925)/188 35109 31456) E(15)
    +((-10 21716 55850, -14 17248 28525)/21095 32243 23072) E(17)
    +((+2752 86417 55225, -2335 99405 83350)/43 87827 06591 98976) E(19)
EXP(+6 lM), EXP(-6 lM)
    ((+9, +0)/8) E(4)  +((-261, +0)/80) E(6)  +((+16317, -36)/4480) E(8)
    +((-19215, +18)/8960) E(10)  +((+1 40580, -369)/1 79200) E(12)
    +((-7 65567, -2070)/39 42400) E(14)  +((+112 89267, -1 88055)/3153 92000) E(16)
    +((-2526 71913, -250 15851)/5 74013 44000) E(18)
    +((+4014 43965, -1612 32633)/45 92107 52000) E(20)
EXP(+7 lM), EXP(-7 lM)
    ((+2401, +0)/1920) E(5)  +((-12005, +0)/3072) E(7)  +((+649 13436, -88837)/132 71040) E(9)
    +((-17771 60175, +16 01467)/5308 41600) E(11)
    +((+13 60754 14475, -2048 19706)/9 34281 21600) E(13)
    +((-1475 13936 96501, -34184 59765)/3363 41237 76000) E(15)
    +((+13696 72417 37326, -64 44040 94677)/1 39917 95490 81600) E(17)
    +((-91411 99586 41617, -1640 20973 07211)/55 96718 19632 64000) E(19)
EXP(+8 lM), EXP(-8 lM)
    ((+64, +0)/45) E(6)  +((-1504, +0)/315) E(8)  +((+31260, -28)/4725) E(10)
    +((-4 81646, +365)/93555) E(12)  +((+48 54553, -4889)/18 71100) E(14)
    +((-2236 89557, +80725)/2432 43000) E(16)
    +((+1 49079 14567, -265 23975)/6 12972 36000) E(18)
    +((-3 03502 61345, -1171 62911)/61 29723 60000) E(20)
EXP(+9 lM), EXP(-9 lM)
    ((+59049, +0)/35840) E(7)  +((-13 58127, +0)/2 29376) E(9)
    +((+45370 29915, -27 75303)/5046 27200) E(11)
    +((-6 31884 52998, +389 13291)/80740 35200) E(13)
    +((+376 15280 01543, -27744 17265)/83 96996 60800) E(15)
    +((-85870 99733 70249, +40 66120 04490)/47023 18100 48000) E(17)
    +((+2 10658 63199 85945, -192 14141 29533)/3 76185 44803 84000) E(19)
EXP(+10 lM), EXP(-10 lM)
    ((+15625, +0)/8064) E(8)  +((-3 59375, +0)/48384) E(10)
    +((+4697 34375, -2 03125)/383 20128) E(12)  +((-1 17440 78125, +57 96875)/9963 23328) E(14)
    +((+42 11380 46875, -2396 09375)/5 57941 06368) E(16)
    +((-349 28023 59375, +15914 21875)/100 42939 14624) E(18)
```

$(R/A)^{+2}$ EXP(2iF) CONTINUED

```
EXP(+10iM), EXP(-10iM) CONTINUED
   +((+487 96125 31250, -29035 46875)/401 71756 58496) E(20)
EXP(+11iM), EXP(-11iM)
   ((+2143 58881, +0)/928 97280) E(9)  +((-3 49404 97603, +0)/37158 91200) E(11)
   +((+1294 68476 94638, -40728 18739)/77 29053 69600) E(13)
   +((-6 86332 01441 75495, +271 57126 63389)/38954 43062 78400) E(15)
   +((+388 66004 31026 08585, -17596 33469 53042)/31 16354 45022 72000) E(17)
   +((-6389 60150 01089 30884, +2 58904 90724 00505)/997 23342 40727 04000) E(19)
EXP(+12iM), EXP(-12iM)
   ((+486, +0)/175) E(10)  +((-4617, +0)/385) E(12)  +((+321 30189, -7533)/14 01400) E(14)
   +((-3666 93075, +1 16883)/140 14000) E(16)  +((+22733 56125, -3 35677)/1121 12000) E(18)
   +((-21 89795 96529, +769 22865)/1 90590 40000) E(20)
EXP(+13iM), EXP(-13iM)
   ((+13 78584 91849, +0)/4 08748 03200) E(11)
   +((-1006 36699 04977, +0)/65 39968 51200) E(13)
   +((+51 77758 16611 06665, -923 65189 53883)/1 64807 20650 24000) E(15)
   +((-815 24769 74737 89944, +21078 56340 37121)/21 09532 24323 07200) E(17)
   +((+4 66598 30373 89784 10012, -141 26070 15693 41471)/14344 81925 39688 96000) E(19)
EXP(+14iM), EXP(-14iM)
   ((+2824 75249, +0)/684 28800) E(12)  +((-14123 76245, +0)/711 65952) E(14)
   +((+613 25376 55790, -8474 25747)/14 23319 04000) E(16)
   +((-2 47400 24083 66202, +52 28616 85899)/4355 35626 24000) E(18)
   +((+53 91740 46804 98457, -1359 12966 05635)/1 04528 55029 76000) E(20)
EXP(+15iM), EXP(-15iM)
   ((+2 13574 21875, +0)/41984 98304) E(13)  +((-604 41503 90625, +0)/23 51159 05024) E(15)
   +((+15 12430 10156 25000, -164 45214 84375)/25580 61046 66112) E(17)
   +((-170 14148 79785 15625, +2962 27441 40625)/2 04644 88373 28896) E(19)
EXP(+16iM), EXP(-16iM)
   ((+2684 35456, +0)/425 67525) E(14)  +((-71135 39584, +0)/2128 37625) E(16)
   +((+792 91974 94272, -6878 65856)/9 76924 69875) E(18)
   +((-22502 00894 66880, +3 25226 33216)/185 61569 27625) E(20)
EXP(+17iM), EXP(-17iM)
   ((+168 37782 65594 00929, +0)/21 42493 68453 12000) E(15)
   +((-11954 82508 57174 65959, +0)/274 23919 16199 93600) E(17)
   +((+696 87709 72292 51782 91733, -4882 95697 02226 26941)/6 25265 35689 35854 08000) E(19)
EXP(+18iM), EXP(-18iM)
   ((+28 24295 36481, +0)/2 87006 72000) E(16)
   +((-5563 86186 86757, +0)/97 58228 48000) E(18)
   +((+113 52114 00408 55545, -649 58793 39063)/74162 53644 80000) E(20)
EXP(+19iM), EXP(-19iM)
   ((+2 88441 41356 76211 67681, +0)/23310 33128 76994 56000) E(17)
   +((-83 64800 99346 10138 62749, +0)/1 11889 59018 09573 88800) E(19)
EXP(+20iM), EXP(-20iM)
   ((+1 22070 31250, +0)/7815 39759) E(18)  +((-145 87402 34375, +0)/1 48492 55421) E(20)
EXP(+21iM), EXP(-21iM)
   ((+1962 37131 72043 69449, +0)/99 19012 22174 72000) E(19)
EXP(+22iM), EXP(-22iM)
   ((+5559 91731 34922 31481, +0)/221 17290 98342 40000) E(20)
```

$(R/A)^{+3}$ EXP(1iF)

```
EXP(0iM)
   (-5/2) E(1)  +(-15/8) E(3)
EXP(+1iM), EXP(-1iM)
   ((+1, +0)/1) E(0)  +((+16, +11)/8) E(2)  +((-123, +28)/192) E(4)
   +((-592, +795)/9216) E(6)  +((-36535, +36184)/7 37280) E(8)
   +((-28 95936, +28 96475)/884 73600) E(10)  +((-3537 10931, +3537 10164)/1 48635 64800) E(12)
   +((-6 09259 90112, +6 09259 91147)/332 94385 15200) E(14)
   +((-1403 51370 09903, +1403 51370 08560)/95887 82923 77600) E(16)
   +((-4 16084 70403 13680, +4 16084 70403 15371)/345 19618 52559 36000) E(18)
   +((-140 15921 18794 57889, +140 15921 18794 57700)/13807 84741 02374 40000) E(20)
EXP(+2iM), EXP(-2iM)
   ((-1, +0)/2) E(1)  +((+6, +1)/6) E(3)  +((-35, -1)/96) E(5)  +((+115, +21)/2880) E(7)
   +((-381, +161)/69120) E(9)  +((-1393, +1555)/12 09600) E(11)
   +((-43187, +42963)/580 60800) E(13)  +((-16 68105, +16 68401)/36578 30400) E(15)
   +((-1724 13055, +1724 12299)/58 52528 64000) E(17)
   +((-1 13963 73989, +1 13963 74929)/5794 00335 36000) E(19)
EXP(+3iM), EXP(-3iM)
   ((-3, +0)/8) E(2)  +((+88, +7)/128) E(4)  +((-1755, -72)/5120) E(6)
   +((+3012, +147)/40960) E(8)  +((-6237, +0)/6 55360) E(10)
   +((+5 28048, +31095)/7340 03200) E(12)  +((+1 94151, -4 45656)/58720 25600) E(14)
   +((+1925 13780, -1895 76639)/164 41671 68000) E(16)
   +((+7 48368 31311, -7 48701 15264)/57874 68431 36000) E(18)
   +((+121 67753 60904, -121 67617 20585)/9 25994 94901 76000) E(20)
EXP(+4iM), EXP(-4iM)
   ((-7, +0)/24) E(3)  +((+275, +13)/480) E(5)  +((-2103, -73)/5760) E(7)
   +((+27293, +867)/2 41920) E(9)  +((-40495, -977)/19 35360) E(11)
   +((+4 61007, -2255)/1741 82400) E(13)  +((-13 64125, -10 60707)/1 04509 44000) E(15)
   +((+279 25117, -243 86173)/22 99207 68000) E(17)
   +((+81239 43309, -81855 47149)/7725 33780 48000) E(19)
EXP(+5iM), EXP(-5iM)
   ((-95, +0)/384) E(4)  +((+4920, +155)/9216) E(6)  +((-2 13395, -5980)/5 16096) E(8)
   +((+82 15040, +2 06835)/495 45216) E(10)  +((-2931 56775, -64 52600)/71345 11104) E(12)
   +((+4001 47160, +41 49715)/5 70760 88832) E(14)
   +((-1 18611 98195, -12540 36180)/1506 80874 51648) E(16)
   +((+77 00513 99520, -35 72096 02645)/5 06287 73837 53728) E(18)
   +((+1228 03193 85035, -1343 38830 38160)/191 46881 74219 59168) E(20)
EXP(+6iM), EXP(-6iM)
   ((-9, +0)/40) E(5)  +((+1197, +27)/2240) E(7)  +((-4356, -99)/8960) E(9)
   +((+2142, +45)/8960) E(11)  +((-53415, -1017)/7 16800) E(13)
   +((+79794, +1125)/49 28000) E(15)  +((-15 99561, -53811)/6307 84000) E(17)
   +((+52 46748, -6 21243)/1 43503 36000) E(19)
EXP(+7iM), EXP(-7iM)
   ((-9947, +0)/46080) E(6)  +((+8 28688, +14063)/14 74560) E(8)
   +((-622 49355, -11 63456)/1061 68320) E(10)
   +((+43438 27460, +777 74907)/1 27401 98400) E(12)
   +((-28 75843 64683, -47775 18032)/224 22749 18400) E(14)
   +((+1833 02045 99448, +25 86441 11495)/53814 59804 16000) E(16)
   +((-1 12607 80992 78587, -2079 71169 09072)/167 90154 58897 92000) E(18)
   +((+1 43606 35429 78540, -2603 77745 25469)/1343 21236 71183 36000) E(20)
EXP(+8iM), EXP(-8iM)
   ((-68, +0)/315) E(7)  +((+1737, +23)/2835) E(9)  +((-40835, -637)/56700) E(11)
   +((+18 07069, +27939)/37 42200) E(13)  +((-191 18541, -2 80115)/898 12800) E(15)
   +((+7823 96641, +103 26815)/1 16756 64000) E(17)
   +((-1 55541 84289, -2163 98751)/98 07557 76000) E(19)
EXP(+9iM), EXP(-9iM)
   ((-2 55879, +0)/11 46880) E(8)  +((+314 92800, +3 34611)/458 75200) E(10)
   +((-36207 46899, -477 90324)/40370 17600) E(12)
   +((+4 39604 84592, +5928 71643)/6 45922 51600) E(14)
   +((-462 25014 72615, -6 02499 77928)/1343 51945 72800) E(16)
   +((+47186 58456 96336, +572 25004 93035)/3 76185 44803 84000) E(18)
   +((-5 22224 72109 63675, -6229 18461 32316)/150 47417 92153 60000) E(20)
EXP(+10iM), EXP(-10iM)
   ((-34375, +0)/1 45152) E(9)  +((+25 09375, +21875)/31 93344) E(11)
   +((-865 21875, -9 78125)/766 40256) E(13)  +((+57134 59375, +677 90625)/59779 39968) E(15)
   +((-2 60628 59375, -3043 28125)/4 78235 19744) E(17)
   +((+22 69725 84375. +25196 03125)/100 42939 14624) E(19)
EXP(+11iM), EXP(-11iM)
   ((-1364 10197, +0)/5308 41600) E(10)
   +((+16 32245 44296, +11887 17431)/17 83627 77600) E(12)
```

$(R/A)^{+3}$ EXP(1iF) CONTINUED

```
EXP(+11IM), EXP(-11IM) CONTINUED
    +((-2658 50417 54843, -26 01927 07152)/1854 97268 70400) E(14)
    +((+4 17104 24213 49100, +4386 66885 87115)/3 11635 44502 27200) E(16)
    +((-635 47004 22051 83955, -6 68692 63840 76576)/747 92506 80545 28000) E(18)
    +((+94567 78610 06236 14944, +961 70228 57429 07775)/2 39336 02177 74489 60000) E(20)
EXP(+12IM), EXP(-12IM)
    ((-2187, +0)/7700) E(11)  +((+4 32783, +2673)/4 00400) E(13)
    +((-205 39575, -1 75689)/112 11200) E(15)  +((+2096 64045, +19 67571)/1121 12000) E(17)
    +((-11729 48607, -111 80673)/8968 96000) E(19)
EXP(+13IM), EXP(-13IM)
    ((-62 56654 63007, +0)/196 19905 53600) E(12)
    +((+14193 06196 08232, +75 29194 55483)/10987 14710 01600) E(14)
    +((-31 05522 33913 33635, -23410 49281 58348)/13 18457 65201 92000) E(16)
    +((+6602 48415 21505 36832, +55 61931 60669 60867)/2531 43869 18768 64000) E(18)
    +((-13 71503 84465 96879 82919, -11897 28583 21429 06520)/6 88551 32419 05070 03000) E(20)
EXP(+14IM), EXP(-14IM)
    ((-403 53607, +0)/1111 96800) E(13)  +((+16 64586 28875, +7667 18533)/10 67489 28000) E(15)
    +((-259 86108 76372, -1 73924 04617)/85 39914 24000) E(17)
    +((+15817 98039 23701, +120 26181 95814)/4355 35626 24000) E(19)
EXP(+15IM), EXP(-15IM)
    ((-9 82441 40625, +0)/23 51159 05024) E(14)
    +((+2856 76875 00000, +11 53300 78125)/1504 74179 21536) E(16)
    +((-8 07764 60566 40625, -4831 90312 50000)/2 04644 88373 28896) E(18)
    +((+82 68375 52265 62500, +57014 49199 21875)/16 37159 06986 31168) E(20)
EXP(+16IM), EXP(-16IM)
    ((-3103 78496, +0)/6385 12875) E(15)  +((+2 52769 73056, +901 77536)/1 08547 18875) E(17)
    +((-50 19590 98368, -26995 58912)/9 76924 69875) E(19)
EXP(+17IM), EXP(-17IM)
    ((-782 46166 45995 69023, +0)/1371 19595 80999 68000) E(16)
    +((+2 83825 58811 09525 30672, +901 31660 09944 40267)/98726 10898 31976 96000) E(18)
    +((-91 52473 89997 72246 73905, -44491 36452 38134 69004)/13 64215 32413 14590 72000) E(20)
EXP(+18IM), EXP(-18IM)
    ((-9 41431 78827, +0)/13 94032 64000) E(17)
    +((+6618 26547 15381, +18 82863 57654)/1854 06341 12000) E(19)
EXP(+19IM), EXP(-19IM)
    ((-13 51120 30565 88570 48611, +0)/16 78343 85271 43608 32000) E(18)
    +((+5981 36404 97706 70529 80600,
    +15 33293 83001 73546 28199)/1342 67508 21714 88665 60000) E(20)
EXP(+20IM), EXP(-20IM)
    ((-5 73730 46875, +0)/5 93970 21684) E(19)
EXP(+21IM), EXP(-21IM)
    ((-841 01627 88018 72621, +0)/721 38270 70361 60000) E(20)
```

$(R/A)^{+3}$ EXP(3iF)

```
EXP(0iM)
   (-35/8) E(3)
EXP(+1iM), EXP(-1iM)
   ((+57, +0)/8) E(2)  +((-520, +75)/128) E(4)  +((-95, +1432)/5120) E(6)
   +((-21092, +20117)/1 22880) E(8)  +((-44 57831, +44 63232)/412 87680) E(10)
   +((-5131 77232, +5131 68275)/66060 28800) E(12)
   +((-56262 19665, +56262 20584)/9 51268 14720) E(14)
   +((-1874 74033 46260, +1874 74033 26183)/39953 20218 24000) E(16)
   +((-5 39808 86009 63399, +5 39808 86009 91424)/140 63548 28820 48000) E(18)
   +((-650 99751 90874 13128, +650 99751 90874 00521)/20251 50953 50149 12000) E(20)
EXP(+2iM), EXP(-2iM)
   ((-9, +0)/2) E(1)  +((+33, +0)/4) E(3)  +((-665, +13)/160) E(5)  +((+165, +8)/240) E(7)
   +((-4403, +1007)/53760) E(9)  +((-12341, +17635)/16 12800) E(11)
   +((-4 05639, +3 99095)/580 60800) E(13)  +((-93 04845, +93 10427)/20321 28000) E(15)
   +((-68668 60495, +68667 53243)/214 59271 68000) E(17)
   +((-2 98665 05977, +2 98665 15053)/1287 55630 08000) E(19)
EXP(+3iM), EXP(-3iM)
   ((+1, +0)/1) E(0)  +((-6, +0)/1) E(2)  +((+591, +0)/64) E(4)  +((-27920, +137)/5120) E(6)
   +((+8 39895, +5016)/5 73440) E(8)  +((-104 72448, +2 58915)/458 75200) E(10)
   +((+33 17237, +5 77810)/1835 00800) E(12)  +((-1333 09152, +762 08769)/4 11041 79200) E(14)
   +((-7 57885 94145, +8 05783 27536)/7234 33553 92000) E(16)
   +((-19 59885 26360, +19 52940 51071)/28937 34215 68000) E(18)
   +((-12169 54865 76807, +12171 60673 38120)/300 94835 84307 20000) E(20)
EXP(+4iM), EXP(-4iM)
   ((+3, +0)/2) E(1)  +((-57, +0)/8) E(3)  +((+11, +0)/1) E(5)
   +((-1 01675, +171)/13440) E(7)  +((+29155, +29)/10752) E(9)
   +((-56 77287, +24295)/96 76800) E(11)  +((+7 89967, +12849)/96 76800) E(13)
   +((-1629 31989, +139 68661)/1 78827 26400) E(15)
   +((+3736 15825, +9609 59279)/214 59271 68000) E(17)
   +((-5 69072 00207, +4 97037 45231)/20085 87829 24800) E(19)
EXP(+5iM), EXP(-5iM)
   ((+15, +0)/8) E(2)  +((-135, +0)/16) E(4)  +((+13975, +0)/1024) E(6)
   +((-73 75760, +5135)/6 88128) E(8)  +((+1173 77505, +13120)/247 72608) E(10)
   +((-5236 20840, +5 73965)/3963 61728) E(12)
   +((+5 23294 25115, +1373 71760)/20 92789 92384) E(14)
   +((-245 25541 45800, +2 88920 36425)/7031 77414 41024) E(16)
   +((+4902 60097 73455, +339 56210 85920)/14 62609 02197 32992) E(18)
   +((-21892 77963 42190, +6536 94467 32815)/526 53924 79103 87712) E(20)
EXP(+6iM), EXP(-6iM)
   ((+9, +0)/4) E(3)  +((-81, +0)/8) E(5)  +((+2781, +0)/160) E(7)
   +((-27468, +9)/1792) E(9)  +((+14 33295, -81)/1 79200) E(11)
   +((-214 81713, +8289)/78 84800) E(13)  +((+102 29967, +5193)/157 69600) E(15)
   +((-65609 47107, +143 02575)/5 74013 44000) E(17)
   +((+6996 66903, +63 22941)/4 59210 75200) E(19)
EXP(+7iM), EXP(-7iM)
   ((+343, +0)/128) E(4)  +((-7889, +0)/640) E(6)  +((+27 68353, +0)/1 22880) E(8)
   +((-12982 41280, +2 19863)/589 82400) E(10)
   +((+41 04904 96185, -309 00184)/3 11427 07200) E(12)
   +((-395 45049 34072, +6988 73819)/74 74249 72800) E(14)
   +((+16069 78415 57071, +1 33441 33740)/10599 84506 88000) E(16)
   +((-6 07264 58481 06656, +322 75691 80955)/18 65572 73210 88000) E(18)
   +((+484 50110 57383 11935, +78038 91235 31696)/8954 74911 41222 40000) E(20)
EXP(+8iM), EXP(-8iM)
   ((+16, +0)/5) E(5)  +((-76, +0)/5) E(7)  +((+3104, +0)/105) E(9)
   +((-65 81355, +619)/2 07900) E(11)  +((+532 18605, -3373)/24 94800) E(13)
   +((-63806 88457, +6 27785)/6486 48000) E(15)
   +((+4 48397 08277, -5 01365)/1 36216 08000) E(17)
   +((-2 10086 78559, +36 04127)/2 51475 84000) E(19)
EXP(+9iM), EXP(-9iM)
   ((+19683, +0)/5120) E(6)  +((-13 58127, +0)/71680) E(8)
   +((+3590 76969, +0)/91 75040) E(10)  +((-36 76823 76600, +203 32539)/80740 35200) E(12)
   +((+51 95559 12525, -250 36776)/1 52672 66560) E(14)
   +((-4 16014 25542 61148, +25 88321 38905)/23511 59050 24000) E(16)
   +((+25 45245 49719 64293, -75 24329 06016)/3 76185 44803 84000) E(18)
   +((-477 35990 88448 61088, +3563 42720 99823)/240 75868 67445 76000) E(20)
EXP(+10iM), EXP(-10iM)
   ((+3125, +0)/672) E(7)  +((-1 28125, +0)/5376) E(9)  +((+50 46875, +0)/96768) E(11)
   +((-2 17444 90625, +7 40625)/3321 07776) E(13)
   +((+24 94619 96875, -88 71875)/46495 08864) E(15)
```

$(R/A)^{+3}$ EXP(3iF) CONTINUED

```
EXP(+10iM), EXP(-10iM) CONTINUED
   +((-346 01080 71875, +1471 34375)/11 15882 12736) E(17)
   +((+892 23811 90625, -2620 50000)/66 95292 76416) E(19)
EXP(+11iM), EXP(-11iM)
   ((+194 87171, +0)/34 40640) E(8)  +((-194 87171, +0)/6 45120) E(10)
   +((+69 21648 26749, +0)/99090 43200) E(12)
   +((-81411 56987 64368, +1 77528 12781)/865 65401 39520) E(14)
   +((+870 05468 52681 23355, -2270 91019 04456)/10 38784 81674 24000) E(16)
   +((-17685 10900 97729 40736, +53900 96349 90607)/332 41114 13575 68000) E(18)
   +((+8 60108 59650 25250 48459, -21 55291 61813 38861)/33905 93641 84719 36000) E(20)
EXP(+12iM), EXP(-12iM)
   ((+243, +0)/35) E(9)  +((-26973, +0)/700) E(11)  +((+1 80792, +0)/1925) E(13)
   +((-75634 19955, +1 09107)/560 56000) E(15)  +((+2 90696 73255, -5 59143)/2242 24000) E(17)
   +((-136 76094 25947, +308 06811)/1 52472 32000) E(19)
EXP(+13iM), EXP(-13iM)
   ((+1 06044 99373, +0)/12386 30400) E(10)  +((-134 67714 20371, +0)/2 72498 68800) E(12)
   +((+6024 41609 38013, +0)/47 56340 73600) E(14)
   +((-6796 98017 22827 95040, +6671 29055 55543)/35 15887 07205 12000) E(16)
   +((+2 38124 11491 62926 81505, -3 40323 83567 80652)/1195 40160 44974 08000) E(18)
   +((-2564 73889 79784 83802 46992, +4362 37020 29236 69885)/17 21378 31047 62675 20000) E(20)
EXP(+14iM), EXP(-14iM)
   ((+403 53607, +0)/38 01600) E(11)  +((-403 53607, +0)/6 33600) E(13)
   +((+20 30997 04031, +0)/11860 99200) E(15)
   +((-1 33861 38867 64700, +91602 68789)/483 92847 36000) E(17)
   +((+5 29765 72399 02195, -5 68178 78656)/1742 14250 49600) E(19)
EXP(+15iM), EXP(-15iM)
   ((+42714 84375, +0)/3229 61408) E(12)  +((-34 59902 34375, +0)/41984 98304) E(14)
   +((+3966 07324 21875, +0)/17 09933 85472) E(16)
   +((-808 85702 92500 00000, +394 25800 78125)/2 04644 88373 28896) E(18)
   +((+2 87129 89973 58925 78125, -2 33520 34218 75000)/622 12044 65479 84384) E(20)
EXP(+16iM), EXP(-16iM)
   ((+335 54432, +0)/20 27025) E(13)  +((-15183 38048, +0)/141 89175) E(15)
   +((+6 69746 46272, +0)/2128 37625) E(17)
   +((-2326 37726 72000, +823 13216)/4 12479 31725) E(19)
EXP(+17iM), EXP(-17iM)
   ((+9 90457 80329 05937, +0)/47610 97076 73600) E(14)
   +((-663 60672 82046 97779, +0)/4 76109 70767 36000) E(16)
   +((+15 63051 36395 09188 23907, +0)/3656 52255 49332 48000) E(18)
   +((-4 02034 46472 80621 73050 98920,
   +1 05018 24088 29016 50011)/500 21228 55148 68326 40000) E(20)
EXP(+18iM), EXP(-18iM)
   ((+9 41431 78827, +0)/35875 84000) E(15)  +((-1044 98928 49797, +0)/5 74013 44000) E(17)
   +((+8727 07267 72629, +0)/15 01265 92000) E(19)
EXP(+19iM), EXP(-19iM)
   ((+15181 12702 98747 98299, +0)/457 06531 93666 56000) E(16)
   +((-9 26048 74882 23626 96239, +0)/3885 05521 46165 76000) E(18)
   +((+35411 95234 36968 50756 19637, +0)/44 75583 60723 82955 52000) E(20)
EXP(+20iM), EXP(-20iM)
   ((+12207 03125, +0)/289 45917) E(17)  +((-10 86425 78125, +0)/3473 51004) E(19)
EXP(+21iM), EXP(-21iM)
   ((+280 33875 96006 24207, +0)/5 22053 27482 88000) E(18)
   +((-81578 57904 37816 44237, +0)/198 38024 44349 44000) E(20)
EXP(+22iM), EXP(-22iM)
   ((+505 44702 84992 93771, +0)/7 37243 03278 08000) E(19)
EXP(+23iM), EXP(-23iM)
   ((+7 46154 70927 59071 05619 08487, +0)/8503 60885 37527 61548 80000) E(20)
```

$(R/A)^{-3}$ EXP(0iF)

```
EXP(0\M)
    (+1/1) E(0)  +(+3/2) E(2)  +(+15/8) E(4)  +(+35/16) E(6)  +(+315/128) E(8)  +(+693/256) E(10)
    +(+3003/1024) E(12)  +(+6435/2048) E(14)  +(+1 09395/32768) E(16)  +(+2 30945/65536) E(18)
    +(+9 69969/2 62144) E(20)
EXP(+1\M), EXP(-1\M)
    ((+3, +3)/2) E(1)  +((+27, +27)/16) E(3)  +((+261, +261)/128) E(5)
    +((+14309, +14309)/6144) E(7)  +((+4 23907, +4 23907)/1 63840) E(9)
    +((+554 89483, +554 89483)/196 60800) E(11)
    +((+3 01169 27341, +3 01169 27341)/99090 43200) E(13)
    +((+239 85984 68863, +239 85984 68863)/73 98752 25600) E(15)
    +((+73142 26651 09147, +73142 26651 09147)/21308 40649 72800) E(17)
    +((+831 50569 13108 70329, +831 50569 13108 70329)/230 13079 01706 24000) E(19)
EXP(+2\M), EXP(-2\M)
    ((+9, +9)/4) E(2)  +((+7, +7)/4) E(4)  +((+141, +141)/64) E(6)  +((+197, +197)/80) E(8)
    +((+62401, +62401)/23040) E(10)  +((+2 62841, +2 62841)/89600) E(12)
    +((+90 10761, +90 10761)/28 67200) E(14)  +((+81421 35359, +81421 35359)/24385 53600) E(16)
    +((+3 27396 16891, +3 27396 16891)/92897 28000) E(18)
    +((+2382 23873 87983, +2382 23873 87983)/643 77815 04000) E(20)
EXP(+3\M), EXP(-3\M)
    ((+53, +53)/16) E(3)  +((+393, +393)/256) E(5)  +((+24753, +24753)/10240) E(7)
    +((+2 11557, +2 11557)/81920) E(9)  +((+37 02771, +37 02771)/13 10720) E(11)
    +((+44638 53417, +44638 53417)/14680 06400) E(13)
    +((+3 80852 21651, +3 80852 21651)/1 17440 51200) E(15)
    +((+1128 99407 50071, +1128 99407 50071)/328 83343 36000) E(17)
    +((+4 18293 33148 69191, +4 18293 33148 69191)/1 15749 36862 72000) E(19)
EXP(+4\M), EXP(-4\M)
    ((+77, +77)/16) E(4)  +((+129, +129)/160) E(6)  +((+899, +899)/320) E(8)
    +((+1 07279, +1 07279)/40320) E(10)  +((+4 21681, +4 21681)/1 43360) E(12)
    +((+45 07393, +45 07393)/14 33600) E(14)  +((+58181 75177, +58181 75177)/17418 24000) E(16)
    +((+40936 27549, +40936 27549)/11612 16000) E(18)
    +((+3176 99708 64593, +3176 99708 64593)/858 37086 72000) E(20)
EXP(+5\M), EXP(-5\M)
    ((+1773, +1773)/256) E(5)  +((-4987, -4987)/6144) E(7)
    +((+4 18315, +4 18315)/1 14688) E(9)  +((+41 02765, +41 02765)/15 72864) E(11)
    +((+1 46288 73205, +1 46288 73205)/47563 40736) E(13)
    +((+4 11140 19031, +4 11140 19031)/1 26835 75296) E(15)
    +((+1150 24815 90811, +1150 24815 90811)/334 84638 78144) E(17)
    +((+12 20136 17638 26065, +12 20136 17638 26065)/3 37525 15891 69152) E(19)
EXP(+6\M), EXP(-6\M)
    ((+3167, +3167)/320) E(6)  +((-2187, -2187)/560) E(8)  +((+1 95147, +1 95147)/35840) E(10)
    +((+78881, +78881)/35840) E(12)  +((+94 27827, +94 27827)/28 67200) E(14)
    +((+2619 67071, +2619 67071)/788 48000) E(16)  +((+3180 20577, +3180 20577)/901 12000) E(18)
    +((+21 25170 28629, +21 25170 28629)/5 74013 44000) E(20)
EXP(+7\M), EXP(-7\M)
    ((+4 32091, +4 32091)/30720) E(7)  +((-30 73843, -30 73843)/3 27680) E(9)
    +((+2154 87317, +2154 87317)/235 92960) E(11)
    +((+79279 62371, +79279 62371)/84934 65600) E(13)
    +((+16 98542 42467, +16 98542 42467)/4 52984 83200) E(15)
    +((+40033 91124 03797, +40033 91124 03797)/11958 79956 48000) E(17)
    +((+406 23233 71081 16071, +406 23233 71081 16071)/111 93436 39265 28000) E(19)
EXP(+8\M), EXP(-8\M)
    ((+1 78331, +1 78331)/8960) E(8)  +((-90 09241, -90 09241)/4 83840) E(10)
    +((+175 69129, +175 69129)/10 75200) E(12)  +((-176 59127, -176 59127)/78 84800) E(14)
    +((+3 74606 81513, +3 74606 81513)/76640 25600) E(16)
    +((+15 16259 77559, +15 16259 77559)/4 74439 68000) E(18)
    +((+10491 96292 18697, +10491 96292 18697)/2789 70531 84000) E(20)
EXP(+9\M), EXP(-9\M)
    ((+643 70707, +643 70707)/22 93760) E(9)  +((-4419 11811, -4419 11811)/131 07200) E(11)
    +((+24 08366 15967, +24 08366 15967)/80740 35200) E(13)
    +((-120 99970 17913, -120 99970 17913)/12 91845 63200) E(15)
    +((+20651 50610 78643, +20651 50610 78643)/2687 03891 45600) E(17)
    +((+19 07519 29907 90427, +19 07519 29907 90427)/7 52370 89607 68000) E(19)
EXP(+10\M), EXP(-10\M)
    ((+76 35529, +76 35529)/1 93536) E(10)  +((-136 72591, -136 72591)/2 36544) E(12)
    +((+683 62955, +683 62955)/12 61568) E(14)
    +((-4 84094 59705, -4 84094 59705)/19926 46656) E(16)
    +((+7 57760 91835, +7 57760 91835)/53137 24416) E(18)
    +((+54416 40509, +54416 40509)/1 01443 82976) E(20)
EXP(+11\M), EXP(-11\M)
    ((+65265 03251, +65265 03251)/1179 64800) E(11)
```

$(R/A)^{-3}$ EXP(0iF) CONTINUED

```
EXP(+11IM), EXP(-11IM) CONTINUED
   +((-1136 18224 72981, -1136 18224 72981)/11 89085 18400) E(13)
   +((+39949 93423 49057, +39949 93423 49057)/412 21619 71200) E(15)
   +((-37 40767 67108 48707, -37 40767 67108 48707)/69252 32111 61600) E(17)
   +((+14436 20704 90157 58871, +14436 20704 90157 58871)/498 61671 20363 52000) E(19)
EXP(+12IM), EXP(-12IM)
   ((+6104 33321, +6104 33321)/78 84800) E(12)
   +((-3 15465 70047, -3 15465 70047)/2050 04800) E(14)
   +((+24 40450 01913, +24 40450 01913)/14350 33600) E(16)
   +((-22 73898 84987, -22 73898 84987)/20500 48000) E(18)
   +((+503 54875 69569, +503 54875 69569)/8 34928 64000) E(20)
EXP(+13IM), EXP(-13IM)
   ((+14143 76106 52069, +14143 76106 52069)/130 79937 02400) E(13)
   +((-5 93275 40765 34017, -5 93275 40765 34017)/2441 58824 44800) E(15)
   +((+857 84218 25805 42163, +857 84218 25805 42163)/2 92990 58933 76000) E(17)
   +((-3 65316 04996 31386 74871, -3 65316 04996 31386 74871)/1687 62579 45645 76000) E(19)
EXP(+14IM), EXP(-14IM)
   ((+88320 07763, +88320 07763)/585 72800) E(14)
   +((-1344 37817 23441, -1344 37817 23441)/3 55829 76000) E(16)
   +((+18796 19283 02173, +18796 19283 02173)/37 95517 44000) E(18)
   +((-7 88210 12132 17217, -7 88210 12132 17217)/1935 71389 44000) E(20)
EXP(+15IM), EXP(-15IM)
   ((+897 55665 12641, +897 55665 12641)/4 27483 46368) E(15)
   +((-17 46095 58032 88969, -17 46095 58032 88969)/3009 48358 43072) E(17)
   +((+3373 81979 24673 27855, +3373 81979 24673 27855)/4 09289 76746 57792) E(19)
EXP(+16IM), EXP(-16IM)
   ((+4 07280 37230 77897, +4 07280 37230 77897)/1394 85265 92000) E(16)
   +((-19 91647 19285 49961, -19 91647 19285 49961)/2258 33287 68000) E(18)
   +((+7697 92095 37781 74313, +7697 92095 37781 74313)/5 69099 88495 36000) E(20)
EXP(+17IM), EXP(-17IM)
   ((+1 23587 91532 00261 02523, +1 23587 91532 00261 02523)/304 71021 29111 04000) E(17)
   +((-874 74306 15216 32298 89671, -874 74306 15216 32298 89671)/65817 40593 87984 64000) E(19)
EXP(+18IM), EXP(-18IM)
   ((+2853 01090 89289, +2853 01090 89289)/5 06920 96000) E(18)
   +((-147 45489 10591 11513, -147 45489 10591 11513)/7416 25364 48000) E(20)
EXP(+19IM), EXP(-19IM)
   ((+8730 10514 85962 95201 05271,
   +8730 10514 85962 95201 05271)/11 18895 90180 95738 88000) E(19)
EXP(+20IM), EXP(-20IM)
   ((+1869 76601 50045 08449, +1869 76601 50045 08449)/1 73006 36502 58944) E(20)
```

$(R/A)^{-3}$ EXP(2iF)

```
EXP(0iM)
   0
EXP(+1iM), EXP(-1iM)
   ((-1, +0)/2) E(1)  +((+3, +1)/48) E(3)  +((-10, +11)/768) E(5)
   +((-715, +939)/92160) E(7)  +((-27291, +33550)/44 23680) E(9)
   +((-61 52713, +72 97445)/12386 30400) E(11)
   +((-4866 84724, +5629 87977)/11 89085 18400) E(13)
   +((-20 62127 16855, +23 41794 16303)/5992 98932 73600) E(15)
   +((-2821 13011 39285, +3158 43970 01468)/9 58878 29237 76000) E(17)
   +((-19 37760 46742 77333, +21 44999 37513 74985)/7594 31607 56305 92000) E(19)
EXP(+2iM), EXP(-2iM)
   ((+1, +0)/1) E(0)  +((-5, +0)/2) E(2)  +((+39, +2)/48) E(4)  +((-175, +42)/1440) E(6)
   +((-50, +131)/5760) E(8)  +((-16464, +21515)/12 09600) E(10)
   +((-1 30375, +1 65702)/116 12160) E(12)  +((-233 03607, +285 84374)/24385 53600) E(14)
   +((-12026 25675, +14379 06764)/14 63132 16000) E(16)
   +((-20 71449 56425, +24 28355 20014)/2897 00167 68000) E(18)
   +((-104 04995 83706, +120 07761 59145)/16554 29529 60000) E(20)
EXP(+3iM), EXP(-3iM)
   ((+7, +0)/2) E(1)  +((-123, +0)/16) E(3)  +((+4890, +81)/1280) E(5)
   +((-1763, +31)/2048) E(7)  +((+94689, +38718)/11 46880) E(9)
   +((-25 25166, +25 00875)/917 50400) E(11)  +((-239 44372, +329 60925)/14680 06400) E(13)
   +((-8612 44545, +11037 53871)/5 87202 56000) E(15)
   +((-37 16903 76123, +46 25507 86740)/2893 73421 56800) E(17)
   +((-1312 91470 37897, +1595 44562 60286)/1 15749 36862 72000) E(19)
EXP(+4iM), EXP(-4iM)
   ((+17, +0)/2) E(2)  +((-115, +0)/6) E(4)  +((+9015, +64)/720) E(6)
   +((-1423, +16)/360) E(8)  +((+3 40333, +21120)/4 83840) E(10)
   +((-24 80283, +7 74080)/217 72800) E(12)  +((-23 55855, +52 19456)/1741 82400) E(14)
   +((-9895 90085, +12817 75152)/5 02951 68000) E(16)
   +((-16 45490 96151, +21 18993 22432)/965 66722 56000) E(18)
   +((-574 99488 14779, +719 66045 19840)/37661 02179 84000) E(20)
EXP(+5iM), EXP(-5iM)
   ((+845, +0)/48) E(3)  +((-32525, +0)/768) E(5)  +((+43 72725, +15625)/1 29024) E(7)
   +((-1714 36300, +5 15625)/123 86304) E(9)  +((+10164 45750, +160 15625)/2972 71296) E(11)
   +((-12497 53635, +866 56250)/20384 31744) E(13)
   +((+17 62001 88175, +13 65023 59375)/376 70218 62912) E(15)
   +((-114 54434 31050, +121 41641 09375)/3835 51316 95104) E(17)
   +((-5 35806 01010 42025, +7 25428 47144 68750)/263 26962 39551 93856) E(19)
EXP(+6iM), EXP(-6iM)
   ((+533, +0)/16) E(4)  +((-13827, +0)/160) E(6)  +((+7 28889, +1458)/8960) E(8)
   +((-7 30085, +486)/17920) E(10)  +((+91 61295, +48114)/7 16800) E(12)
   +((-445 05483, +7 50870)/157 69600) E(14)  +((+660 73711, +67 84560)/1576 96000) E(16)
   +((-23474 23299, +10700 68482)/2 87006 72000) E(18)
   +((-86418 96120, +1 50516 79767)/45 92107 52000) E(20)
EXP(+7iM), EXP(-7iM)
   ((+2 28347, +0)/3840) E(5)  +((-30 71075, +0)/18432) E(7)
   +((+47694 80852, +57 64801)/265 42080) E(9)
   +((-11 30096 49375, -57 64801)/10616 83200) E(11)
   +((+2266 24434 47875, +4 94274 03774)/56 05687 29600) E(13)
   +((-72921 93541 91097, +332 04965 51995)/6726 82475 52000) E(15)
   +((+45580 96003 90634, +1058 92303 66477)/21525 83921 66400) E(17)
   +((-121 30515 24974 54107, +14 21701 19051 78859)/335 80309 17795 84000) E(19)
EXP(+8iM), EXP(-8iM)
   ((+73369, +0)/720) E(6)  +((-7 75727, +0)/2520) E(8)
   +((+3387 25845, +2 62144)/9 07200) E(10)  +((-15267 34319, -3 93216)/59 87520) E(12)
   +((+10 88676 82781, +1200 61952)/9580 03200) E(14)
   +((-12 95992 56603, +1589 24800)/35925 12000) E(16)
   +((+6711 19839 90961, +44 26137 60000)/784 60462 08000) E(18)
   +((-6403 27394 15405, +182 97972 98176)/3923 02310 40000) E(20)
EXP(+9iM), EXP(-9iM)
   ((+121 44273, +0)/71680) E(7)  +((-2523 61629, +0)/4 58752) E(9)
   +((+74 69687 25855, +3874 20489)/10092 54400) E(11)
   +((-131 90864 86278, -3874 20489)/23068 67200) E(13)
   +((+49105 95949 35351, +32 34961 08315)/167 93993 21600) E(15)
   +((-101 08719 30928 99923, +2342 73169 69830)/94046 36200 96000) E(17)
   +((+224 79334 50359 49165, +50676 57580 56939)/7 52370 89607 68000) E(19)
EXP(+10iM), EXP(-10iM)
   ((+6 34085, +0)/2304) E(8)  +((-2778 61775, +0)/2 90304) E(10)
   +((+10 83061 97175, +390 62500)/766 40256) E(12)
   +((-242 58697 76525, -6640 62500)/19926 46656) E(14)
   +((+5877 52185 02000, +2 62207 03125)/8 36911 59552) E(16)
```

$(R/A)^{-3}$ EXP(2iF) CONTINUED

```
EXP(+10iM), EXP(-10iM) CONTINUED
   +((-14746 88440 97610, -1 14941 40625)/50 21469 57312) E(18)
   +((+13678 64765 59225, +12 85351 56250)/146 07911 48544) E(20)
EXP(+11iM), EXP(-11iM)
   ((+8 15403 06287, +0)/1857 94560) E(9)  +((-1210 78620 68291, +0)/74317 82400) E(11)
   +((+12 13690 39604 93598, +313 84283 76721)/463 74322 17600) E(13)
   +((-276 62774 95943 89765, -6590 69959 11141)/11129 83732 22400) E(15)
   +((+99481 58471 26755 37295, +32 72125 42556 93146)/62 32708 90045 44000) E(17)
   +((-44 78780 30810 01268 92204, -751 79482 55016 31945)/5983 40054 44362 24000) E(19)
EXP(+12iM), EXP(-12iM)
   ((+617 73797, +0)/89600) E(10)  +((-5367 01569, +0)/1 97120) E(12)
   +((+339 05013 70893, +6449 72544)/7175 16800) E(14)
   +((-439 62530 87975, -8868 37248)/8958 96000) E(16)
   +((+39801 46860 18675, +10 19056 61952)/11 48026 88000) E(18)
   +((-22842 30409 77879, -4 19232 15360)/12 67302 40000) E(20)
EXP(+13iM), EXP(-13iM)
   ((+8744 53103 00933, +0)/8 17496 06400) E(11)
   +((-17 57684 47926 44837, +0)/392 39811 07200) E(13)
   +((+27523 86320 52120 24105, +3 93737 63856 99289)/3 29614 41300 48000) E(15)
   +((-3 96150 64821 85954 18408, -66 93539 85568 87913)/42 19064 48646 14400) E(17)
   +((+367 10773 70999 70710 07076, +7571 57478 96997 32747)/5062 87738 37537 28000) E(19)
EXP(+14iM), EXP(-14iM)
   ((+44 95246 74151, +0)/2737 15200) E(12)  +((-207 03325 80533, +0)/2846 63808) E(14)
   +((+12 37170 08548 71015, +135 64461 45698)/85 39914 24000) E(16)
   +((-1528 39833 32916 37549, -21567 49371 65982)/8710 71252 48000) E(18)
   +((+30702 36746 13014 00319, +5 21553 54302 08810)/2 09057 10059 52000) E(20)
EXP(+15iM), EXP(-15iM)
   ((+2098 44044 36665, +0)/83969 96608) E(13)
   +((-5 48794 44141 21525, +0)/47 02318 10048) E(15)
   +((+12654 73390 86280 08600, +1 08121 94824 21875)/51161 22093 32224) E(17)
   +((-1 31360 08421 73225 85025, -15 49747 92480 46875)/4 09289 76746 57792) E(19)
EXP(+16iM), EXP(-16iM)
   ((+32895 97021 10873, +0)/8 71782 91200) E(14)
   +((-6 06029 54385 05719, +0)/32 69165 92000) E(16)
   +((+16664 58100 33213 68657, +1 12589 99068 42624)/40014 83566 08000) E(18)
   +((-62583 29132 71688 07405, -6 19244 94876 34432)/1 08611 69679 36000) E(20)
EXP(+17iM), EXP(-17iM)
   ((+4949 04385 88943 83777, +0)/87448 72181 76000) E(15)
   +((-14 54980 74551 65831 26743, +0)/49 86167 12036 35200) E(17)
   +((+25 98268 89087 49630 54611 49803,
   +140 63084 45206 77249 91009)/37 51592 14136 15124 48000) E(19)
EXP(+18iM), EXP(-18iM)
   ((+3725 65896 43389, +0)/44154 88000) E(16)
   +((-88 92751 83115 15449, +0)/195 16456 96000) E(18)
   +((+1 68967 25825 65906 35015, +7 41208 07554 07364)/1 48325 07289 60000) E(20)
EXP(+19iM), EXP(-19iM)
   ((+448 77721 52823 29002 37939, +0)/3586 20481 34922 24000) E(17)
   +((-4 74232 64674 41388 65302 46479, +0)/6 71337 54108 57443 32800) E(19)
EXP(+20iM), EXP(-20iM)
   ((+1892 28998 81543 35435, +0)/10243 79792 91648) E(18)
   +((-2 11767 54887 46618 66325, +0)/1 94632 16065 41312) E(20)
EXP(+21iM), EXP(-21iM)
   ((+3 16865 09767 16213 93629, +0)/11 66942 61432 32000) E(19)
EXP(+22iM), EXP(-22iM)
   ((+351 77078 02890 78076 63759, +0)/884 69163 93369 60000) E(20)
```

$(R/A)^{-4}$ EXP(1iF)

```
EXP(0iM)
    (+1/1) E(1)  +(+3/2) E(3)  +(+35/8) E(5)  +(+105/16) E(7)  +(+1155/128) E(9)
   +(+3003/256) E(11)  +(+15015/1024) E(13)  +(+36465/2048) E(15)  +(+6 92835/32768) E(17)
   +(+16 16615/65536) E(19)
EXP(+1iM), EXP(-1iM)
    ((+1, +0)/1) E(0)  +((+16, +11)/8) E(2)  +((+239, +196)/64) E(4)
   +((+53168, +46995)/9216) E(6)  +((+59 69465, +54 76024)/7 37280) E(8)
   +((+3150 53888, +2953 30425)/294 91200) E(10)
   +((+20 07276 18469, +19 07924 36964)/1 48635 64800) E(12)
   +((+5508 07423 51648, +5285 61887 47187)/332 94385 15200) E(14)
   +((+6 32383 68884 86779, +6 11034 84183 68720)/31962 60974 59200) E(16)
   +((+8014 55869 24656 06320, +7784 04944 24409 29931)/345 19618 52559 36000) E(18)
   +((+3 70469 87219 19987 89911, +3 61251 36818 14269 22100)/13807 84741 02374 40000) E(20)
EXP(+2iM), EXP(-2iM)
    ((+3, +0)/1) E(1)  +((+33, +23)/12) E(3)  +((+245, +178)/48) E(5)
   +((+6945, +5663)/960) E(7)  +((+1 67574, +1 44431)/17280) E(9)
   +((+300 05941, +267 67715)/24 19200) E(11)  +((+2967 85889, +2708 92054)/193 53600) E(13)
   +((+27 03321 11925, +25 07645 51051)/1 46313 21600) E(15)
   +((+319 16967 33695, +299 60908 58704)/14 63132 16000) E(17)
   +((+48935 38642 05283, +46354 16313 14037)/1931 33445 12000) E(19)
EXP(+3iM), EXP(-3iM)
    ((+53, +0)/8) E(2)  +((+312, +343)/128) E(4)  +((+35205, +22552)/5120) E(6)
   +((+3 57268, +2 77403)/40960) E(8)  +((+74 47867, +61 28960)/6 55360) E(10)
   +((+1 04084 96752, +89348 68055)/7340 03200) E(12)
   +((+10 11067 74999, +8 93193 33256)/58720 25600) E(14)
   +((+3363 78766 94020, +3033 84542 23049)/164 41671 68000) E(16)
   +((+13 82647 43827 89079, +12 66543 66113 82464)/57874 68431 36000) E(18)
   +((+254 67313 06190 63336, +236 10177 56174 43815)/9 25994 94901 76000) E(20)
EXP(+4iM), EXP(-4iM)
    ((+77, +0)/6) E(3)  +((-125, +899)/240) E(5)  +((+4751, +2441)/480) E(7)
   +((+6 05297, +4 67493)/60480) E(9)  +((+63 51695, +50 33692)/4 83840) E(11)
   +((+3098 25863, +2581 05655)/193 53600) E(13)
   +((+10 00843 84225, +8 60972 21439)/52254 72000) E(15)
   +((+5 87597 14114, +5 17682 29709)/26127 36000) E(17)
   +((+8373 06982 02881, +7511 91561 20084)/321 88907 52000) E(19)
EXP(+5iM), EXP(-5iM)
    ((+2955, +0)/128) E(4)  +((-83112, +48203)/9216) E(6)
   +((+83 89045, +29 05460)/5 16096) E(8)  +((+1704 35200, +1454 77705)/165 15072) E(10)
   +((+10 81163 41065, +8 20590 10600)/71345 11104) E(12)
   +((+101 94308 08760, +83 07000 47635)/5 70760 88832) E(14)
   +((+10618 06919 40631, +8936 14951 03156)/502 26958 17216) E(16)
   +((+124 37955 55141 10880, +107 44631 79985 84355)/5 06287 73837 53728) E(18)
   +((+59348 55523 44443 17465, +52304 82011 26785 56240)/2106 15699 16415 50848) E(20)
EXP(+6iM), EXP(-6iM)
    ((+3167, +0)/80) E(5)  +((-62993, +16337)/2240) E(7)  +((+2 73019, +52226)/8960) E(9)
   +((+2 68793, +3 62255)/35840) E(11)  +((+129 55385, +90 42878)/7 16800) E(13)
   +((+30923 16623, +24963 14225)/1576 96000) E(15)
   +((+36579 21796, +30205 10321)/1576 96000) E(17)
   +((+153 18884 44567, +130 16719 37153)/5 74013 44000) E(19)
EXP(+7iM), EXP(-7iM)
    ((+30 24637, +0)/46080) E(6)  +((-979 69328, +149 75567)/14 74560) E(8)
   +((+21815 07495, +1888 52992)/353 89440) E(10)
   +((-4 77528 70060, +15 08163 80883)/1 27401 98400) E(12)
   +((+485 77094 16343, +278 57314 94672)/20 38431 74400) E(14)
   +((+3 72396 81934 12264, +3 08179 67174 52285)/17938 19934 72000) E(16)
   +((+4266 49088 79745 48597, +3451 92351 70796 56752)/167 90154 58897 92000) E(18)
   +((+38733 73108 03805 81980, +32473 38241 63384 39499)/1343 21236 71183 36000) E(20)
EXP(+8iM), EXP(-8iM)
    ((+1 78331, +0)/1680) E(7)  +((-1003 42917, +102 50507)/7 25760) E(9)
   +((+9226 46255, +260 86711)/72 57600) E(11)
   +((-1 12793 88923, +46042 38917)/3193 34400) E(13)
   +((+21 65419 00749, +8 31435 51355)/57480 19200) E(15)
   +((+2942 95088 20047, +2788 50306 97255)/149 44349 92000) E(17)
   +((+58902 31317 01977, +46017 31604 84993)/2092 27898 88000) E(19)
EXP(+9iM), EXP(-9iM)
    ((+1931 12121, +0)/11 46880) E(8)  +((-1 22423 96800, +8999 93651)/458 75200) E(10)
   +((+104 47461 81941, -11208 46084)/40370 17600) E(12)
   +((-725 32517 69488, +120 32666 79923)/6 45922 81600) E(14)
   +((+97412 39459 33385, +19612 80668 45592)/1343 51945 72800) E(16)
   +((+44 02609 57712 01936, +76 78345 98371 37435)/3 76185 44803 84000) E(18)
   +((+4920 14923 74416 77725, +3523 87562 01963 49044)/150 47417 92153 60000) E(20)
EXP(+10iM), EXP(-10iM)
    ((+381 77645, +0)/1 45152) E(9)  +((-31271 27113, +1738 47457)/63 86688) E(11)
```

$(R/A)^{-4}$ EXP(1iF) CONTINUED

```
EXP(+10IH), EXP(-10IH) CONTINUED
    +((+1 31159 10995, -1953 12500)/255 46752) E(13)
    +((-340 74092 70655, +30 92038 49535)/1 19558 79936) E(15)
    +((+188 79934 03865, +15 79560 75680)/1 19558 79936) E(17)
    +((-523 56428 61935, +762 99646 85895)/33 47646 38208) E(19)
EXP(+11IH), EXP(-11IH)
    ((+7 17915 35761, +0)/1769 47200) E(10)
    +((-15443 33366 87544, +672 96445 38791)/17 83627 77600) E(12)
    +((+18 35957 74365 79717, -38210 70308 72152)/1854 97288 70400) E(14)
    +((-675 95402 09199 00324, +40 02351 68354 72305)/1 03878 48167 42400) E(16)
    +((+2 67838 56110 53664 68245, +6449 37282 59329 50304)/747 92506 80545 28000) E(18)
    +((-222 30042 68062 08751 04416,
    +63 67476 65745 77944 64575)/2 39336 02177 74489 60000) E(20)
EXP(+12IH), EXP(-12IH)
    ((+6104 33321, +0)/9 85600) E(11)  +((-15 25730 69883, +53557 42997)/1025 02400) E(13)
    +((+266 38558 99635, -6 07092 12563)/14350 33600) E(15)
    +((-499 29522 99190, +21 77647 14463)/35875 84000) E(17)
    +((+9258 74687 30341, -27 01966 21576)/11 48026 88000) E(19)
EXP(+13IH), EXP(-13IH)
    ((+1 83868 89384 76897, +0)/196 19905 53600) E(12)
    +((-274 91731 63531 05272, +7 94338 29090 74507)/10987 14710 01600) E(14)
    +((+14923 99037 71696 33495, -340 67327 33004 16724)/4 39485 88400 64000) E(16)
    +((-71 91839 83426 22318 19392, +2 50884 88496 25969 96843)/2531 43869 18768 64000) E(16)
    +((+12190 45923 01905 66054 77609,
    -176 03100 00704 67040 10360)/6 88551 32419 05070 08000) E(20)
EXP(+14IH), EXP(-14IH)
    ((+6 18240 54341, +0)/439 29600) E(13)
    +((-88174 31294 30775, +2134 10991 30551)/21 34978 56000) E(15)
    +((+5 18882 18297 40533, -11384 97655 81662)/85 39914 24000) E(17)
    +((-324 84932 11373 72017, +9 56527 26086 83337)/5807 14168 32000) E(19)
EXP(+15IH), EXP(-15IH)
    ((+4487 78325 63205, +0)/2 13741 73184) E(14)
    +((-101 02359 61486 65248, +2 07817 99300 27757)/1504 74179 21536) E(16)
    +((+21818 12569 38157 63855, -450 60034 15226 87440)/2 04644 88373 28896) E(18)
    +((-1 75104 47296 59783 00140, +4504 20138 27561 71275)/16 37159 06986 31168) E(20)
EXP(+16IH), EXP(-16IH)
    ((+4 07260 37250 77897, +0)/130 76743 68000) E(15)
    +((-7664 08178 48309 30069, +135 65539 58390 44811)/71137 48561 92000) E(17)
    +((+78408 33847 64846 58173, -1509 12893 87031 15043)/4 26824 91371 52000) E(19)
EXP(+17IH), EXP(-17IH)
    ((+21 00994 56044 04437 42891, +0)/457 06531 93666 56000) E(16)
    +((-16879 91217 00272 21001 30192,
    +259 79400 11162 07322 26443)/98726 10898 31976 96000) E(18)
    +((+46 87931 16914 75417 16047 99725,
    -83449 55043 80202 27213 16756)/150 06368 56544 60497 92000) E(20)
EXP(+18IH), EXP(-18IH)
    ((+8559 03272 67867, +0)/1 26730 24000) E(17)
    +((-1992 98060 22083 01919, +26 91538 80868 15079)/7416 25364 48000) E(19)
EXP(+19IH), EXP(-19IH)
    ((+1 65871 99782 33296 08820 00149, +0)/16 78343 85271 43608 32000) E(18)
    +((-562 31981 42067 65188 60545 24200,
    +6 71712 88325 60958 42192 76919)/1342 67508 21714 88665 60000) E(20)
EXP(+20IH), EXP(-20IH)
    ((+9348 83007 50225 42245, +0)/64877 38688 47104) E(19)
EXP(+21IH), EXP(-21IH)
    ((+151 06879 06586 54933 76659, +0)/721 38270 70361 60000) E(20)
```

$(R/A)^{-4}$ EXP(3iF)

```
EXP(0IM)
   0
EXP(+1IM), EXP(-1IM)
   ((+1, +0)/8) E(2)  +((+8, +1)/384) E(4)  +((+55, +8)/3072) E(6)
   +((+4738, +823)/3 68640) E(8)  +((+12 10867, +2 31480)/1238 63040) E(10)
   +((+1545 13552, +311 57705)/1 38180 86400) E(12)
   +((+91443 08289, +19091 96720)/142 69022 20800) E(14)
   +((+129 21061 23460, +27 63834 52457)/23971 95730 94400) E(16)
   +((+1 94768 12411 32451, +42405 29248 79296)/421 90644 86461 44000) E(18)
   +((+243 75388 12035 52136, +53 78885 89628 48859)/60754 52860 50447 36000) E(20)
EXP(+2IM), EXP(-2IM)
   ((-1, +0)/1) E(1)  +((+5, +0)/4) E(3)  +((-35, +2)/240) E(5)  +((+115, +13)/1440) E(7)
   +((+1708, +337)/40320) E(9)  +((+1 64171, +35650)/48 38400) E(11)
   +((+9 53121, +2 23798)/348 36480) E(13)  +((+2766 78105, +684 00437)/1 21927 68000) E(15)
   +((+3 09318 95335, +79311 63506)/160 94453 76000) E(17)
   +((+95 90641 91162, +25 26267 18873)/5794 00335 36000) E(19)
EXP(+3IM), EXP(-3IM)
   ((+1, +0)/1) E(0)  +((-6, +0)/1) E(2)  +((+423, +0)/64) E(4)  +((-10000, +81)/5120) E(6)
   +((+52731, +1944)/1 14688) E(8)  +((+23 22432, +7 54515)/458 75200) E(10)
   +((+128 72237, +27 54810)/1835 00800) E(12)
   +((+22739 34624, +5527 11033)/4 11041 79200) E(14)
   +((+335 72811 81375, +86 71818 56688)/7234 33553 92000) E(16)
   +((+229 07972 01352, +61 92835 89939)/5787 46843 13600) E(18)
   +((+10 31797 61646 72033, +2 88590 51499 46920)/300 94835 84307 20000) E(20)
EXP(+4IM), EXP(-4IM)
   ((+5, +0)/1) E(1)  +((-22, +0)/1) E(3)  +((+607, +0)/24) E(5)  +((-3430, +8)/315) E(7)
   +((+2 29271, +2048)/80640) E(9)  +((-98049, +9280)/3 62880) E(11)
   +((+131 26539, +20 79232)/870 91200) E(13)  +((+362 48883, +91 45976)/4191 26400) E(15)
   +((+12 22928 94295, +3 18263 89504)/160 94453 76000) E(17)
   +((+612 91125 46843, +168 43147 21824)/9415 25544 96000) E(19)
EXP(+5IM), EXP(-5IM)
   ((+127, +0)/8) E(2)  +((-3065, +0)/48) E(4)  +((+2 43805, +0)/3072) E(6)
   +((-894 72880, +78125)/20 64384) E(8)  +((+10402 80675, +25 00000)/743 17824) E(10)
   +((-31322 29656, +421 09375)/11890 85184) E(12)
   +((+35 24427 25395, +2 10012 50000)/62 78369 77152) E(14)
   +((+1785 79818 37960, +655 19042 96875)/21095 32243 23072) E(16)
   +((+45141 84261 26135, +11374 21775 00000)/3 98893 36962 90816) E(18)
   +((+149 20450 21821 81550, +41 19960 66569 53125)/1579 61774 37311 63136) E(20)
EXP(+6IM), EXP(-6IM)
   ((+163, +0)/4) E(3)  +((-2577, +0)/16) E(5)  +((+1089, +0)/5) E(7)
   +((-12 71837, +486)/8960) E(9)  +((+19 96815, +1458)/35840) E(11)
   +((-1114 99641, +3 63042)/78 84800) E(13)  +((+227 03131, +3 42387)/78 84800) E(15)
   +((-29460 14299, +5364 87790)/1 43503 36000) E(17)
   +((+2 15598 02238, +43518 52251)/11 48026 88000) E(19)
EXP(+7IM), EXP(-7IM)
   ((+35413, +0)/384) E(4)  +((-7 09471, +0)/1920) E(6)  +((+400 12175, +0)/73728) E(8)
   +((-21 65193 81120, +403 53607)/5308 41600) E(10)
   +((+1766 51899 58955, +41967 75128)/9 34281 21600) E(12)
   +((-1203 50730 68312, +1 18236 06851)/20 38431 74400) E(14)
   +((+48 36349 15575 06119, +18708 33574 12700)/3 49794 88727 04000) E(16)
   +((-24 10268 99903 31680, +57198 69968 06197)/11 19313 63926 52800) E(18)
   +((+13754 56387 46923 00205, +1282 86630 77956 47568)/26864 24734 23667 20000) E(20)
EXP(+8IM), EXP(-8IM)
   ((+23029, +0)/120) E(5)  +((-35614, +0)/45) E(7)  +((+126 72577, +0)/10080) E(9)
   +((-26589 26325, +2 62144)/24 94800) E(11)  +((+5 45679 66563, +41 94304)/958 00320) E(13)
   +((-40 83600 14491, +1412 39080)/19459 44000) E(15)
   +((+7516 70812 69007, +8 25334 16960)/130 76743 68000) E(17)
   +((-2295 79530 99351, +12 10844 44672)/196 15115 52000) E(19)
EXP(+9IM), EXP(-9IM)
   ((+3 85095, +0)/1024) E(6)  +((-1153 01691, +0)/71680) E(8)
   +((+2 52405 02097, +0)/91 75040) E(10)
   +((-2091 01342 38840, +11622 61467)/80740 35200) E(12)
   +((+1 31493 52328 27931, +2 78342 75208)/83 96996 60800) E(14)
   +((-6 23895 78036 01068, +86 82093 15849)/940 46352 00960) E(16)
   +((+789 35781 57603 06429, +26804 53885 65408)/3 76185 44803 84000) E(18)
   +((-12274 85328 83651 96064, +17 56440 79376 38623)/240 75868 67445 76000) E(20)
EXP(+10IM), EXP(-10IM)
   ((+44377, +0)/63) E(7)  +((-505 94255, +0)/16128) E(9)
   +((+33385 07815, +0)/5 80608) E(11)  +((-591 93467 56895, +1953 12500)/9963 23328) E(13)
   +((+2793 31427 35225, +488 28125)/69742 63296) E(15)
   +((-16032 74870 73026, +1 02050 78125)/8 36911 59552) E(17)
```

$(R/A)^{-4}$ EXP(3iF) CONTINUED

```
EXP(+10iM), EXP(-10iM) CONTINUED
   +((+2 76745 74376 17965, +30 40039 06250)/401 71756 58496) E(19)
EXP(+11iM), EXP(-11iM)
   ((+1 31467 55961, +0)/103 21920) E(8)  +((-6 87022 50353, +0)/116 12160) E(10)
   +((+34406 42501 89231, +0)/2 97271 29600) E(12)
   +((-1868 81658 02403 41552, +3452 27121 43931)/12984 81020 92800) E(14)
   +((+60221 14241 91537 21005, -27618 16971 51446)/6 23270 89004 54400) E(16)
   +((-51 31671 22876 29638 19776, +169 26485 76416 93693)/997 23342 40727 04000) E(18)
   +((+2110 75395 54293 47244 87257, +7312 98063 61610 47661)/1 01717 80925 54158 08000) E(20)
EXP(+12iM), EXP(-12iM)
   ((+200 59301, +0)/8960) E(9)  +((-2431 09203, +0)/22400) E(11)
   +((+1 77991 97193, +0)/7 88480) E(13)  +((-307 02612 51785, +403 10784)/1121 12000) E(15)
   +((+1 27486 71460 84455, -77396 70528)/5 74013 44000) E(17)
   +((-6 35588 12424 75633, +12 18998 10816)/48 79114 24000) E(19)
EXP(+13iM), EXP(-13iM)
   ((+1428 79335 02359, +0)/37158 91200) E(10)
   +((-1 50998 53230 74733, +0)/8 17496 06400) E(12)
   +((+672 79928 69429 60437, +0)/1569 59244 28800) E(14)
   +((-58 85390 94894 73470 00800, +51 18589 30140 90757)/105 47661 21615 36000) E(16)
   +((+351 59437 96583 46870 98037, -204 74357 20563 63028)/717 24096 26984 44800) E(18)
   +((-16 20607 51926 38489 59133 15376,
   +19 91899 02664 33418 08655)/51 64134 93142 88025 60000) E(20)
EXP(+14iM), EXP(-14iM)
   ((+2 95484 65037, +0)/45 61920) E(11)  +((-935 18503 76827, +0)/2737 15200) E(13)
   +((+56604 30928 80643, +0)/71165 95200) E(15)
   +((-1604 64465 38456 06355, +949 51230 19886)/1451 78542 08000) E(17)
   +((+54622 72122 85661 81397, -27535 85675 76694)/52264 27514 88000) E(19)
EXP(+15iM), EXP(-15iM)
   ((+346 64772 41959, +0)/3229 61408) E(12)  +((-2251 65711 07545, +0)/3816 81664) E(14)
   +((+272 09535 05115 91305, +0)/188 09272 40192) E(16)
   +((-4 37242 23946 05173 52320, +1 80203 24707 03125)/2 04644 88373 26896) E(18)
   +((+1345 38362 49517 23989 03725, -562 23413 08593 75000)/622 12044 65479 84364) E(20)
EXP(+16iM), EXP(-16iM)
   ((+10920 65465 02373, +0)/62270 20800) E(13)  +((-27335 13819 06871, +0)/27243 21600) E(15)
   +((+338 04657 52006 12309, +0)/130 76743 68000) E(17)
   +((-7 68396 70177 41800 16741, +2 25179 98136 85248)/1 90070 46936 86000) E(19)
EXP(+17iM), EXP(-17iM)
   ((+40431 17948 03715 58867, +0)/1 42832 91230 20800) E(14)
   +((-72 16252 15097 68845 98663, +0)/42 84987 36906 24000) E(16)
   +((+907 12131 66921 50143 82551, +0)/199 44668 48145 40800) E(18)
   +((-11263 77493 27267 18590 31227 99960,
   +2390 72435 68515 13248 47153)/1500 63685 65446 04979 20000) E(20)
EXP(+18iM), EXP(-18iM)
   ((+32433 03882 60189, +0)/71751 68000) E(15)
   +((-16 03491 49656 43317, +0)/5 74013 44000) E(17)
   +((+3080 34119 89387 47987, +0)/390 32913 92000) E(19)
EXP(+19iM), EXP(-19iM)
   ((+196 08461 02363 24919 93309, +0)/274 23919 16199 93600) E(16)
   +((-53440 57721 33000 01159 95621, +0)/11655 16564 38497 28000) E(18)
   +((+1815 74814 07842 15604 65761 88343, +0)/134 26750 82171 48866 56000) E(20)
EXP(+20iM), EXP(-20iM)
   ((+638 27132 04347 20807, +0)/569 09988 49536) E(17)
   +((-19091 82968 01670 56745, +0)/2560 94948 22912) E(19)
EXP(+21iM), EXP(-21iM)
   ((+9 11509 73621 99267 65439, +0)/5 22053 27482 88000) E(18)
   +((-2384 06612 31025 98084 67857, +0)/198 38024 44349 44000) E(20)
EXP(+22iM), EXP(-22iM)
   ((+119 42364 46743 17087 20597, +0)/44 23458 19668 48000) E(19)
EXP(+23iM), EXP(-23iM)
   ((+1 05844 02520 51576 02971 11802 94181, +0)/25510 82656 12582 84646 40000) E(20)
```

APPENDIX B

Linear Perturbations by Tesseral Harmonics

It is supposed that mean values $\bar{\xi}$, $\bar{\eta}$, $\dot{\bar{\xi}}$, $\dot{\bar{\eta}}$, $\bar{P}$, $\bar{Q}$, $\dot{\bar{P}}$, $\dot{\bar{Q}}$, $\bar{a}$, $\dot{\bar{\lambda}}$ are available either from observations or as given by the leading zonal harmonics. Under these circumstances one can easily write the perturbations in the nonsingular elements due to a particular term $R_{\ell mpq}$ pertaining to a tesseral harmonic (ℓ, m). This is readily achieved by transforming the well known results in Keplerian elements to the nonsingular elements.

Let us define:

$$\bar{a} = \text{mean semimajor axis}$$

$$\bar{n} = \text{mean motion}$$

$$\alpha = m + 2p - \ell$$

$$K'_{\ell pq} = \partial K_{\ell pq}/\partial\gamma \quad , \quad \gamma = \sqrt{1 - e^2}$$

$$J'_{\ell mp} = \partial J_{\ell mp}/\partial c \quad , \quad c = \cos\frac{I}{2}$$

$$\theta_{\ell mpq} = (\ell - 2p + q)\lambda - m\theta$$

and

$$S_{\ell mpq} = \mathbb{R}_{\ell mpq}(A_{\ell m} \cos\theta_{\ell mpq} + B_{\ell m} \sin\theta_{\ell mpq}) +$$

$$+ \mathbb{I}_{\ell mpq}(A_{\ell m} \sin\theta_{\ell mpq} - B_{\ell m} \cos\theta_{\ell mpq}) ,$$

$$T_{\ell mpq} = \mathbf{R}_{\ell mpq}(A_{\ell m} \sin\theta_{\ell mpq} - B_{\ell m}\cos\theta_{\ell mpq}) -$$

$$- \mathbf{I}_{\ell mpq}(A_{\ell m}\cos\theta_{\ell mpq} + B_{\ell m}\sin\theta_{\ell mpq}) ,$$

$$D_{\ell mpq} = (\ell-2p+q)\dot{\overline{\lambda}} - q\, \frac{\overline{\xi}\,\dot{\overline{\eta}} - \eta\,\dot{\overline{\xi}}}{\overline{\xi}^2 + \overline{\eta}^2} + \alpha\, \frac{\overline{P}\,\dot{\overline{Q}} - \overline{Q}\,\dot{\overline{P}}}{\overline{P}^2 - \overline{Q}^2} - m\dot{\theta} .$$

The linear perturbations due to $R_{\ell mpq}$ are given by

$$\delta\lambda_{\ell mpq} = \overline{n} \left(\frac{a_e}{\overline{a}}\right)^{\ell} D^{-1}_{\ell mpq} K_{\ell pq} \{[\frac{|\alpha|}{2\gamma} + 2(\ell+1)]\, J_{mp} -$$

$$- \frac{1-c^2}{c} J'_{\ell mp} \}\; T_{\ell mpq}$$

$$\delta a_{\ell mpq} = 2\overline{n}\,\overline{a} \left(\frac{a_e}{\overline{a}}\right)^{\ell} (\ell-2p+q)\, D^{-1}_{\ell mpq} J_{\ell mp} K_{\ell pq} S_{\ell mpq}$$

$$\overline{\xi}\delta\xi_{\ell mpq} + \overline{\eta}\,\delta\eta_{\ell mpq} = \overline{n} \left(\frac{a_e}{\overline{a}}\right)^{\ell} D^{-1}_{\ell mpq}\, \gamma\, [(\ell-2p+q)\gamma - (\ell-2p)] \cdot$$

$$\cdot J_{\ell mp} K_{\ell pq} S_{\ell mpq}$$

$$\overline{P}\delta P_{\ell mpq} + \overline{Q}\delta Q_{\ell mpq} = \frac{1}{4}\overline{n} \left(\frac{a_e}{\overline{a}}\right)^{\ell} D^{-1}_{\ell mpq} \frac{1}{\gamma} [(\ell-2p)(c^2-s^2)-m] \cdot$$

$$\cdot J_{\ell mp} K_{\ell pq} S_{\ell mpq}$$

$$\bar{\xi}\delta\dot{\eta}_{\ell mpq} - \bar{\eta}\delta\xi_{\ell mpq} = \bar{n} \left(\frac{a_e}{\bar{a}}\right)^{\ell} D^{-1}_{\ell mpq} \left[(\gamma|q| + \frac{e^2|\alpha|}{2\gamma})J_{\ell mp}K_{\ell pq} - \right.$$

$$\left. - e^2 J_{\ell mp} K'_{\ell pq} - \frac{e^2(1-c^2)}{2\gamma c} J'_{\ell mp} K_{\ell pq}\right] T_{\ell mpq}$$

$$\bar{P}\delta Q_{\ell mpq} - \bar{Q}\delta P_{\ell mpq} = \frac{1}{4}\bar{n} \left(\frac{a_e}{\bar{a}}\right)^{\ell} D^{-1}_{\ell mpq} \frac{1}{\gamma}(|\alpha| J_{\ell mp} -$$

$$- \frac{1-c^2}{c} J'_{\ell mp}) K_{\ell pq} T_{\ell mpq}$$

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