A MULTILEVEL CONTROL SYSTEM FOR THE LARGE SPACE TELESCOPE

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1. INTRODUCTION

The principal objective of this report is to outline a multilevel control scheme for the Large Space Telescope (LST). The concept and methodology of the scheme is based upon the decomposition-aggregation stability analysis of large-scale systems [1-3], which was used to study structural properties of the control system for a spinning flexible spacecraft [4, 5].

The two-level analysis of the decomposition-aggregation method is ideally suitable for designing a multilevel feedback control [6-10] for dynamic systems composed of interconnected subsystems. Local controllers on the subsystem level are used to stabilize (or optimize) the decoupled subsystems. On the second hierarchial level the global controllers are used to minimize the interactions among the subsystems, and make the control system meet the required performance characteristics for the overall system. This multilevel strategy can solve complex control problems "piece-by-piece" and make the computer use attractive in cases when the direct approach is either not feasible (excessive computer storage), or it is uneconomical (excessive computer time).

The detailed plan of the report is as follows:

In Section 2, we will develop a nonlinear model for the LST which is based upon the linear model described in [11]. The nonlinear representation will serve as a realistic model for evaluating the potentials of the multilevel schemes for control of the LST.

In Section 3, we will outline the general multilevel stabilization algorithm [6-8]. Both local and global controllers are involved. The local controllers are used to stabilize each decoupled subsystem by any of the classical techniques such as pole-shifting, root-locus, parameter plane, etc. The role of the global controllers is to minimize the effect of interactions among the subsystems. Finally, the aggregate system is constructed on the higher hierarchial level to conclude stability of the overall composite system. We will consider a class of dynamic systems [12] which can always be stabilized by the proposed scheme using local controllers only. Since the LST model developed in Section 2 is in that class, we will be able to effectively design the feedback control which stabilizes the LST.

In Section 4, we will present a multilevel optimization scheme for control of large-scale systems [9, 10]. The local controllers are used to optimize the decoupled subsystems with respect to quadratic cost. The global controllers are applied to reduce the subsystem interactions, or entirely decouple the subsystems as is the case of the LST. While this control scheme results in a suboptimal performance when the effective interactions are present, it produces an optimal control when the total decoupling takes place. Thus, the design procedure can effectively be used for constructing an optimal control system for the LST.

Both the stabilization and the optimization multilevel schemes are entirely computerized. The description of the programs is provided in the Appendix.

This report is written under the supervision and with the participation of the Principal Investigator, D. D. Šiljak. Investigator S. K. Sundareshan developed the model of LST in Section 2, and Sections 4 and A.2 on multilevel optimization. Investigator M. B. Vukčević developed the multilevel stabilization scheme presented in Sections 3 and A.1.

2. DEVELOPMENT OF A MODEL FOR THE LST

The Large Space Telescope (LST) is modeled as a rigid body with three orthogonally mounted reaction wheel actuators and is considered to be subject to gravitational and magnetic disturbance torques. Unlike in the earlier analyses [11], nonlinear coupling phenomena are not ignored and a complete three-axes model for the spacecraft is obtained as a nonlinear interconnected system. The interconnections represent the coupling between the motions along the individual axes. Hence this model will be a more accurate description of the LST, which however, is necessary due to the precision pointing requirements demanded of the control system.

The spacecraft's equation of motion can be written down from the Euler equations [11], as

$$I \cdot \dot{\omega} + \omega \times I \circ \omega + \sum_{i=1}^{3} \{\omega \times \omega_i \operatorname{tr} I_i + 2\omega_i \times I_i \cdot \omega + I_i \circ \dot{\omega}_i + \omega_i \times I_i \circ \omega_i\} = M \tag{2.1}$$

and

$$I_i \cdot \dot{\omega}_i + \omega \times I_i \circ \omega + \omega \times \omega_i \operatorname{tr} I_i + 2\omega_i \times I_i \cdot \omega + I_i \circ \dot{\omega}_i + \omega_i \times I_i \cdot \omega_i = M_i \ , \ i = 1, 2, 3 \tag{2.2}$$

where

I is the inertia tensor of the LST given by

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} ,$$

I_x, I_y, I_z denoting the components along the three axes constituting an inertial reference frame I_{rf} ;

I_i , $i = 1, 2, 3$, are the inertia tensors of the three reaction wheels that are mounted orthogonally and parallel to the axes constituting the standard body-fixed reference frame B_{rf} and hence can be expressed as

$$I_1 = \begin{bmatrix} I_{1x} & 0 & 0 \\ 0 & I_{1y} & 0 \\ 0 & 0 & I_{1y} \end{bmatrix} ; \quad I_2 = \begin{bmatrix} I_{2z} & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & I_{2z} \end{bmatrix} ;$$

$$I_3 = \begin{bmatrix} I_{3x} & 0 & 0 \\ 0 & I_{3x} & 0 \\ 0 & 0 & I_{3z} \end{bmatrix}$$

ω is the angular velocity vector of the LST relative to the frame I_{rf} ;

ω_i , $i = 1, 2, 3$, are the angular velocity vectors of the reaction wheels relative to the frame I_{rf} ;

M is the total external torque acting on the LST; and

M_i , $i = 1, 2, 3$, are the internal torques on the reaction wheels.

The angular velocity ω can be expressed in terms of the rates of angular deviations along the three axes of I_{rf} as

$$\omega = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{2.3}$$

where ϕ is the roll angle, θ is the pitch angle and ψ is the yaw angle. Similarly the angular velocities ω_i of the reaction wheels can be expressed in terms of the components as

$$\omega_1 = \begin{bmatrix} \nu_1 \\ 0 \\ 0 \end{bmatrix} ; \; \omega_2 = \begin{bmatrix} 0 \\ \nu_2 \\ 0 \end{bmatrix} ; \; \omega_3 = \begin{bmatrix} 0 \\ 0 \\ \nu_3 \end{bmatrix} . \tag{2.4}$$

Equations (2.1) and (2.2) can now be simplified into the following four sets of scalar equations:

(i) Equations governing the motion of the LST body:

$$\left.\begin{aligned} I_x\ddot\phi + \dot\theta\dot\psi(I_z - I_y) + I_{3z}\nu_3\dot\theta - I_{2y}\nu_2\dot\psi + I_{1x}\dot\nu_1 &= M_x \\ I_y\ddot\theta + \dot\phi\dot\psi(I_x - I_z) + I_{1x}\nu_1\dot\psi - I_{3z}\nu_3\dot\phi + I_{2y}\dot\nu_2 &= M_y \\ I_z\ddot\psi + \dot\phi\dot\theta(I_y - I_x) + I_{2y}\nu_2\dot\phi - I_{1x}\nu_1\dot\theta + I_{3z}\dot\nu_3 &= M_3 \end{aligned}\right\} \tag{2.5}$$

(ii) Equations for the Reaction Wheel mounted parallel to x-axis:

$$\left.\begin{aligned} I_{1x}\ddot\phi + I_{1x}\dot\nu_1 &= M_{1x} \\ I_{1y}\ddot\theta + (I_{1x} - I_{1y})\dot\theta\dot\psi + I_{1x}\nu_1\dot\psi &= M_{1y} \\ I_{1y}\ddot\psi + (I_{1y} - I_{1x})\dot\phi\dot\theta - I_{1x}\nu_1\dot\theta &= M_{1z} \end{aligned}\right\} \tag{2.6}$$

(iii) Equations for the Reaction Wheel mounted parallel to y-axis

$$\left.\begin{aligned} I_{2z}\ddot\phi + (I_{2y} - I_{2z})\dot\phi\dot\psi + I_{2y}\nu_2\dot\phi &= M_{2x} \\ I_{2y}\ddot\theta + I_{2y}\dot\nu_2 &= M_{2y} \\ I_{2z}\ddot\psi + (I_{2z} - I_{2y})\dot\theta\dot\psi - I_{2y}\nu_2\dot\psi &= M_{2z} \end{aligned}\right\} \tag{2.7}$$

(iv) Equations for the Reaction Wheel mounted parallel to z-axis:

$$\left.\begin{aligned} I_{3x}\ddot\phi + (I_{3x} - I_{3z})\dot\phi\dot\theta + I_{3z}\nu_3\dot\theta &= M_{3x} \\ I_{3x}\ddot\theta + (I_{3z} - I_{3x})\dot\phi\dot\psi + I_{3z}\nu_3\dot\phi &= M_{3y} \\ I_{3z}\ddot\psi + I_{3z}\dot\nu_3 &= M_{3z} \end{aligned}\right\} \tag{2.8}$$

For further simplification, we will assume that the reaction wheels are small so that $I_{1x} \ll I_x$, $I_{2y} \ll I_y$, $I_{3z} \ll I_z$ and they have one degree of freedom only. With these, equations (2.5)-(2.8) can be simplified into,

$$\left.\begin{aligned} I_x\ddot{\phi} + \dot{\theta}\dot{\psi}(I_z - I_y) + I_{1x}\dot{\nu}_1 &= M_x \\ I_y\ddot{\theta} + \dot{\phi}\dot{\psi}(I_x - I_z) + I_{2y}\dot{\nu}_2 &= M_y \\ I_z\ddot{\psi} + \dot{\phi}\dot{\theta}(I_y - I_x) + I_{3z}\dot{\nu}_3 &= M_z \end{aligned}\right\} \tag{2.9}$$

and

$$I_{1x}\ddot{\phi} + I_{1x}\dot{\nu}_1 = M_{1x} \tag{2.10}$$

$$I_{2y}\ddot{\theta} + I_{2y}\dot{\nu}_2 = M_{2y} \tag{2.11}$$

$$I_{3z}\ddot{\psi} + I_{3z}\dot{\nu}_3 = M_{3z} \quad . \tag{2.12}$$

Substitution of (2.10)-(2.12) into (2.9) will result in the following three equations describing the motions along the individual axes and their interconnections:

$$\left.\begin{aligned} I_x\ddot{\phi} + (I_z - I_y)\dot{\theta}\dot{\psi} &= (M_x - M_{1x}) \\ I_y\ddot{\theta} + (I_x - I_z)\dot{\phi}\dot{\psi} &= (M_y - M_{2y}) \\ I_z\ddot{\psi} + (I_y - I_x)\dot{\theta}\dot{\phi} &= (M_z - M_{3z}) \end{aligned}\right\} \tag{2.13}$$

It is now necessary to evaluate the various torques. Since the internal torques on the reaction wheels are small, it may be assumed that these are proportional to the control signals actuating the wheels. Hence,

$$\left.\begin{aligned} M_{1x} &= -K_1u_1 \\ M_{2y} &= -K_2u_2 \\ M_{3z} &= -K_3u_3 \end{aligned}\right\} \tag{2.14}$$

Where K_1, K_2 and K_3 are the drive motor constants (the negative signs in (2.14) merely indicate the directions of these torques).

The external torques acting on the body of the LST are mainly environmental disturbance forces and are composed of gravity-gradient, magnetic, aerodynamic and solar pressure torques. The latter two will be negligibly small compared to the others and will usually be accounted for in control system designs by considering them as equivalent zero-mean stationary white noise processes. The gravity-gradient and magnetic torques can be represented as purely deterministic signals involving a constant term and a sinusoidal function of time with twice orbital rate. Hence, following the analysis in [11] the external torques can be obtained as,

$$\left.\begin{aligned} M_x &= \{\gamma_{11} + \gamma_{12}\cos(\omega t + x) + s_1\}\, I_x \\ M_y &= \{\gamma_{21} + \gamma_{22}\cos(\ t + x) + s_2\}\, I_y \\ M_z &= \{\gamma_{31} + \gamma_{32}\cos(\ t + x) + s_3\}\, I_z \end{aligned}\right\} \quad . \tag{2.15}$$

Where γ_{ij}, $i = 1, 2, 3$, are constants that can be determined [11] from the inertia components I_x, I_y, I_z, the magnitude of the LST dipole moment and the earth's magnetic field intensity; and s_i, $i = 1, 2, 3$, are white-noise processes characterizing the aerodynamic and solar pressure torques.

Substitution of (2.14) and (2.15) in (2.13) and further simplification results in the following system of equations:

$$\left.\begin{aligned} \ddot\phi + \alpha_1\dot\theta\dot\psi &= \beta_1 u_1 + M_x \\ \ddot\theta + \alpha_2\dot\phi\dot\psi &= \beta_2 u_2 + M_y \\ \ddot\psi + \alpha_3\dot\phi\dot\theta &= \beta_3 u_3 + M_z \end{aligned}\right\} \tag{2.16}$$

where $\alpha_1 = \dfrac{(I_z - I_y)}{I_x}$, $\alpha_2 = \dfrac{(I_x - I_z)}{I_y}$, $\alpha_3 = \dfrac{(I_y - I_x)}{I_z}$, $\beta_1 = \dfrac{K_1}{I_x}$, $\beta_2 = \dfrac{K_2}{I_y}$,

$\beta_3 = \frac{K_3}{I_z}$ and M_x, M_y, M_z are the external disturbance torques given by (2.15).

It is now simple to obtain a state-space representation of the LST by choosing the state-vector

$$x = [\phi,\ \dot{\phi},\ \theta,\ \dot{\theta},\ \psi,\ \dot{\psi}]^T\ , \tag{2.17}$$

which results in the time-invariant model,

$$\dot{x} = Ax + Bu + h(x) + FM \tag{2.18}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ ; \quad B = \begin{bmatrix} 0 & 0 & 0 \\ \beta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

$$h(x) = \begin{bmatrix} 0 \\ -\alpha_1\dot{\theta}\dot{\psi} \\ 0 \\ -\alpha_2\dot{\phi}\dot{\psi} \\ 0 \\ -\alpha_3\dot{\phi}\dot{\theta} \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The diagonal structure of the matrices A , B and F permits us to partition the state-vector as,

$$x = [x_1, x_2, x_3]^T \tag{2.19}$$

where

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix}, \quad x_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad x_3 = \begin{bmatrix} x_{31} \\ x_{32} \end{bmatrix} = \begin{bmatrix} \psi \\ \dot{\psi} \end{bmatrix}.$$

With this, (2.18) can be described as a set of interconnected subsystems,

$$\dot{x}_i = A_i x_i + b_i u_i + h_i(x) + f_i d_i \ , \ i = 1, 2, 3 \ , \tag{2.20}$$

where

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ \beta_i \end{bmatrix}, \quad f_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad i = 1, 2, 3,$$

and

$$h_1(x) = \begin{bmatrix} 0 \\ -\alpha_1 x_{22} x_{32} \end{bmatrix}, \quad h_2(x) = \begin{bmatrix} 0 \\ -\alpha_2 x_{32} x_{12} \end{bmatrix}, \quad h_3(x) = \begin{bmatrix} 0 \\ -\alpha_3 x_{12} x_{22} \end{bmatrix}$$

with $d_1 = M_x$, $d_2 = M_y$ and $d_3 = M_z$ being the external disturbances.

It may be observed that when $h_i(x) \equiv 0$, $i = 1, 2, 3$, (2.20) represents three decoupled subsystems which describe the motions of the spacecraft along the three axes. However, $h_i(x)$ are not zero and constitute the interconnections among the subsystems, thus making an analysis based on the smaller-dimensional decoupled subsystems alone inaccurate.

The system represented by (2.20), is driven by the disturbance forces d_i in addition to the control signals u_i . However, these external disturbances can be completely cancelled by constructing a disturbance accommodating controller as described in [3, 1]. This involves the determination of a suitable differential equation model for the disturbances and with the augmentation of the disturbance variables with the state variables of the system, designing a feedback controller that counteracts the disturbance forces by feeding back the estimated disturbance variables. Although this analysis is conducted for a single-axis model of the LST (only for the pitch motion control) in [1], a straightforward extension that uses three separate disturbance accommodating controllers can be obtained for the three-axis model presently considered. Due to the above reason, we will ignore the disturbance terms from our model and conduct all further analysis on the system,

$$\dot{x}_i = A_i x_i + b_i u_i + h_i(x) \ , \ i = 1, 2, 3 \ , \tag{2.21}$$

obtained from (2.20) with the substitution $d_i \equiv 0$.

3. STABILIZATION

When a complex dynamic system is given as a number of locally controlled interconnected subsystems, it can be stabilized by a multilevel control scheme [6 - 8] based upon the decomposition-aggregation stability analysis [1 - 3]. In the scheme, the dimensionality problem is resolved by carrying out all operations on the subsystem level. Both local and global controllers can be involved. The local controllers are introduced to stabilize each decoupled subsystem by any of the classical techniques such as the pole-shifting by state feedback, root-locus, parameter plane method, etc. The global controllers minimize the effect of interactions among the subsystems. Finally, the aggregate system is constructed on the higher hierarchical level to conclude stability of the overall composite system.

It is important to note that the proposed stabilization produces large-systems which are connectively stable [1 - 3] . That is, stability is invariant under structural perturbations whereby subsystems are disconnected and connected again in various ways during the operation of the system. Furthermore, the stabilized systems have wide tolerance to nonlinearities in the interactions between the subsystems.

After we outline the multilevel control scheme for stabilization of large-scale systems, we will consider a class of dynamic systems which can be always stabilized by the scheme using local controllers only. Since the LST model developed in the preceeding section falls in that class, we will be able to effectively design the feedback control which stabilizes the LST.

3.1. Multilevel Control

Let us consider a linear dynamic system

$$\dot{x} = Ax + Bu \ , \tag{3.1}$$

where $x(t) \in R^n$ is the state of the system, $u(t) \in R^s$ is the input to the system, and A and B are constant $n \times n$ and $n \times s$ matrices. We assume that the system is brought into the input-decentralized form

$$\dot{x}_i = A_i x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} A_{ij} x_j + b_i u_i \,, \quad i = 1, 2, \ldots, s \tag{3.2}$$

where $x_i(t) \in R^{n_i}$ is the state of the i-th subsystem, and $u_i(t) \in R$ is the corresponding local control, so that

$$R^n = R^{n_1} \times R^{n_2} \times \ldots \times R^{n_s} \,, \tag{3.3}$$

and each pair (A_i, b_i) is controllable.

In (3.2), the matrices A_i, A_{ij}, and the vectors b_i have appropriate dimensions. As shown in reference [6], any linear dynamic system (3.1) can be represented by its input-decentralized form (3.2).

To stabilize the system (3.2), we apply the decentralized feedback control

$$u_i(t) = u_i^{\ell}(t) + u_i^{g}(t) \tag{3.4}$$

where $u_i(t)$ is chosen as a local control law

$$u_i^{\ell} = -k_i^T x_i \,, \tag{3.5}$$

with a constant vector $k_i \in R^{n_i}$, and $u_i^g(t)$ is chosen as a global control law

$$u_i^g = - \sum_{\substack{j=1 \\ j \neq i}}^{s} k_{ij}^T x_j \,, \tag{3.6}$$

where $k_{ij} \in R^{n_j}$ are constant vectors.

By substituting the control (3.4) into (3.2), we get the closed-loop system as

$$\dot{x}_i = (A_i - b_i k_i^T)x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} (A_{ij} - b_i k_{ij}^T)x_j \ , \ i = 1, 2, \ldots , s \ . \quad (3.7)$$

Since each pair (A_i, b_i) is controllable, a simple choice of k_i can be always made [13] to place the eigenvalues of $A_i - B_i k_i^T$ at any desired locations $-\sigma_1^i \pm j\omega_1^i, \ldots, \sigma_p^i \pm j\omega_p^i, -\sigma_{p+1}^i, \ldots, -\sigma_{n_i}^i$ $(\sigma_q^i > 0 \ ; \ q = 1, 2, \ldots, n_i$, and $1 \leq p \leq n_i)$. Then, each uncoupled sybsystem

$$\dot{x}_i = (A_i - b_i k_i^T)x_i \ , \quad i = 1, 2, \ldots , s \quad (3.8)$$

is stabilized with a degree of exponential stability

$$\pi_i = \min_q \sigma_q^i \ . \quad (3.9)$$

To provide a Liapunov function [5 - 8] with the exact estimate of π_i for each decoupled subsystem, we apply to (3.8) the linear nonsingular transformation

$$x_i = T_i \tilde{x}_i \ , \quad (3.10)$$

to get the system (3.8) as

$$\dot{\tilde{x}}_i = \Lambda_i \tilde{x}_i \ , \quad (3.11)$$

where $\Lambda_i = T_i^{-1}(A_i - b_i k_i^T)T_i$ has the quasidiagonal form

$$\Lambda_i = \text{diag}\left\{ \begin{bmatrix} -\sigma_1^i & \omega_1^i \\ -\omega_1^i & -\sigma_1^i \end{bmatrix}, \ldots, \begin{bmatrix} -\sigma_p^i & \omega_p^i \\ -\omega_p^i & -\sigma_p^i \end{bmatrix}, -\sigma_{p+1}^i, \ldots, -\sigma_{n_i}^i \right\}. \quad (3.12)$$

For the system (3.11), we choose the Liapunov function $v_i: R^{n_i} \to R_+$,

$$v_i(\tilde{x}_i) = (\tilde{x}_i^T \tilde{H}_i \tilde{x}_i)^{1/2} , \tag{3.13}$$

where

$$\Lambda_i^T \tilde{H}_i + \tilde{H}_i \Lambda_i = -\tilde{G}_i , \tag{3.14}$$

and

$$\tilde{G}_i = 2\theta_i \operatorname{diag} \{\sigma_1^i, \sigma_1^i, \ldots , \sigma_p^i, \sigma_p^i, \sigma_{p+1}^i, \ldots , \sigma_{n_i}^i\}, \tilde{H}_i = \theta_i I_i . \tag{3.15}$$

In (3.15), $\theta_i > 0$ is an arbitrary constant and I_i is the $n_i \times n_i$ identity matrix.

The aggregate comparison system involving the vector Liapunov function $v: R^n \to R_+^s$,

$$v = (v_1, v_2, \ldots , v_s)^T , \tag{3.16}$$

is obtained for the transformed system (3.7),

$$\dot{\tilde{x}}_i = \Lambda_i \tilde{x}_i + \sum_{\substack{j=1 \\ j\neq i}}^{s} (\tilde{A}_{ij} - \tilde{b}_i \tilde{k}_{ij}^T)\tilde{x}_j , \qquad i = 1, 2, \ldots , s \tag{3.17}$$

where $\tilde{A}_{ij} = T_i^{-1} A_{ij} T_j$, $\tilde{b}_i = T_i^{-1} b_i$, $\tilde{k}_{ij}^T = k_{ij}^T T_j$, and using the Liapunov functions $v_i(\tilde{x}_i)$ defined in (3.13). Using the aggregation method presented in [1 - 5] , we construct the comparison system

$$\dot{v} \leq \tilde{W} v , \tag{3.18}$$

where the constant $s \times s$ matrix $\tilde{W} = (\tilde{w}_{ij})$ has the elements defined as

$$\tilde{w}_{ij} = -\delta_{ij}\pi_i + (1-\delta_{ij})\tilde{\xi}_{ij} , \tag{3.19}$$

where δ_{ij} is the Kronecker symbol, π_i is defined in (3.9), and

$$\tilde{\xi}_{ij} = \lambda_M^{\frac{1}{2}} [(\tilde{A}_{ij} - \tilde{b}_i\tilde{k}_{ij}^T)^T(\tilde{A}_{ij} - \tilde{b}_i\tilde{k}_{ij}^T)] \tag{3.20}$$

where λ_M is the maximum eigenvalue of the indicated matrix.

As known [1 - 3], global asymptotic stability of the system (3.17) and, therefore, original system (3.2), is implied by the Sevastyanov-Kotelyanski conditions [14], which for $\tilde{W} = (\tilde{w}_{ij})$ defined by (3.19) and (3.20) have the following form

$$(-1)^k \begin{vmatrix} -\pi_1 & \tilde{\xi}_{12} & \cdots & \tilde{\xi}_{1k} \\ \tilde{\xi}_{21} & -\pi_2 & \cdots & \tilde{\xi}_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{\xi}_{k1} & \tilde{\xi}_{k2} & \cdots & -\pi_k \end{vmatrix} > 0, \quad k = 1, 2, \ldots, s. \tag{3.21}$$

To satisfy conditions (3.21), we choose the vectors $\tilde{k}_{ij}$ in (3.20) so as to minimize the nonnegative numbers $\tilde{\xi}_{ij}$ which reflect the strength of interconnections among the subsystems in (3.17). Such choice is provided by

$$\tilde{k}_{ij}^* = [(\tilde{b}_i^T\tilde{b}_i)^{-1} \tilde{b}_i^T\tilde{A}_{ij}]^T, \tag{3.22}$$

where $(\tilde{b}_i^T\tilde{b}_i)^{-1} \tilde{b}_i^T$ is the Moore-Penrose generalized inverse of $\tilde{b}_i$ [15]. The choice of $\tilde{k}_{ij}^*$ in (3.22) produces the optimal aggregate matrix $\tilde{W}^*$ in the sense that $\tilde{W}^* \leq \tilde{W}$ (that is, $\tilde{W}^* - \tilde{W} \leq 0$) is valid for all $\tilde{k}_{ij}$. That is equivalent to saying [16] that $\lambda_M(\tilde{W}^*) \leq \lambda_M(\tilde{W})$ for all $\tilde{k}_{ij}$. Since conditions (3.21) are necessary and sufficient for $\lambda_M(\tilde{W}) < 0$, that is, for stability of $\tilde{W}$, the choice $\tilde{k}_{ij} = \tilde{k}_{ij}^*$ is justified.

To conclude stability of the overall system (3.17) with the optimal choice $\tilde{k}_{ij} = \tilde{k}_{ij}^*$, which is

$$\dot{\tilde{x}}_i = \Lambda_i\tilde{x}_i + [I_i - \tilde{b}_i(\tilde{b}_i^T\tilde{b}_i)^{-1} \tilde{b}_i^T] \sum_{\substack{j=1 \\ j\neq i}}^{s} \tilde{A}_{ij}\tilde{x}_j, \quad j = 1, 2, \ldots, s \tag{3.23}$$

we apply the determinantal inequalities (3.21) to the optimal aggregate matrix $\tilde{W}^* = (\tilde{w}^*_{ij})$ defined by (3.19) and $\tilde{\xi}_{ij} = \tilde{\xi}^*_{ij} = \lambda_M^{1/2}\{\tilde{A}^T_{ij}[I_i - \tilde{b}_i(\tilde{b}^T_i\tilde{b}_i)^{-1}\tilde{b}^T_i]\tilde{A}_{ij}\}$.

We arrive at the following:

Theorem 3.1. *The linear control system* (3.2) *is stabilized by the linear control laws*

$$u_i = -k_i^T x_i - \sum_{\substack{j=1 \\ j\neq i}}^{s} k_{ij}^T x_j \,, \quad i = 1, 2, \ldots, s \tag{3.24}$$

where $k^T_{ij} = \tilde{k}^{*T}_{ij} T_j^{-1}$, *if the corresponding* $s \times s$ *aggregate matrix*

$$\tilde{W}^* = [-\delta_{ij}\pi_i + (1-\delta_{ij})\tilde{\xi}^*_{ij}] \tag{3.25}$$

satisfies conditions (3.21).

Successful application of the above theorem depends on appropriate choice of the eigenvalues for the decoupled subsystems (3.8). Once the subsystem eigenvalues are prescribed, the control law (3.24) and, thus, the gain vectors k_i, k_{ij} in (3.24), are computed uniquely using the proposed algorithm. Therefore, if for computed gains k_i, k_{ij} , the conditions (3.21) are not met, a reassignment of the subsystems eigenvalues is required. The search for an appropriate set of subsystems eigenvalues can be aided by the interactive computer program described in Section A.1. The efficiency of the computer program relies on the low order of the subsystems and the simplicity in testing the Sevastyanov-Kotelyanskii conditions (3.21). Furthermore, the computerized procedure provides a considerable freedom to the designer to apply his understanding of the system and the familiarity with the method to come up with a successful design.

3.2 *An Illustrative Example*

Let us consider a system (3.1) described by the equation

$$\dot{x} = \left[\begin{array}{ccc:cc} 1 & 11.50 & 86.50 & 4 & 22.50 \\ 0.45 & 0 & -4.09 & 8.91 & -0.82 \\ 0.18 & 1 & 8.36 & 0.36 & 3.27 \\ \hdashline 0 & 1.25 & 14.75 & 5 & 2.75 \\ 0.18 & 0 & -9.82 & 0.18 & -6.36 \end{array}\right] x + \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ \hdashline 0 & 1 \\ 0 & 1 \end{array}\right] u \, . \tag{3.26}$$

The eigenvalues of the matrix A corresponding to (3.26) are

$$\lambda_{1,2} = 0.76 \pm j\, 1.83, \; \lambda_3 = 11{,}54, \; \lambda_4 = -3.89, \; \lambda_5 = -1.15 \, , \tag{3.27}$$

and the system is unstable.

To stabilize the system (3.26), we start with its input-decentralized representation (3.2) given as

$$\dot{x}_1 = \begin{bmatrix} 1 & 11.50 & 86.50 \\ 0.45 & 0 & -4.09 \\ 0.18 & 1 & 8.36 \end{bmatrix} x_1 + \begin{bmatrix} 4 & 22.50 \\ 8.91 & -0.82 \\ 0.36 & 3.27 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_1$$

$$\dot{x}_2 = \begin{bmatrix} 5 & 2.75 \\ 0.18 & -6.36 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 1.25 & 14.75 \\ 0.18 & 0 & -9.82 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_2 \, . \tag{3.28}$$

and transform each subsystem into its comparison form [13] to get

$$\dot{x}_1^c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8.86 & 8.50 & 9.36 \end{bmatrix} x_1^c + \begin{bmatrix} 3.20 & 1.98 \\ -14.72 & 0.49 \\ -7.92 & 36.01 \end{bmatrix} x_2^c + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_1$$

$$\dot{x}_2^c = \begin{bmatrix} 0 & 1 \\ 32.32 & -1.36 \end{bmatrix} x_2^c + \begin{bmatrix} 1.69 & 1.26 & 0.08 \\ -7.52 & -5.23 & 0.49 \end{bmatrix} x_1^c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 \tag{3.29}$$

The transformation into the comparison form is of no conceptual significance, and is performed on the subsystem level for two practical reasons. First, it is convenient for subsystem stabilization by pole-assignment applying the state feedback and, secondly, the diagonal form (3.17) with no complex roots, can be obtained from the companion form (3.29) using the Vandermonde matrix T_i in (3.10) where x_i is replaced by x_i^c .

Now, by using the local feedback law (3.5) and vectors

$$k_1^T = (1791.14,\ 458.50,\ 46.36)$$
$$k_2^T = (33.82,\ 1.14)\ , \tag{3.30}$$

we allocate the eigenvalues of the uncoupled subsystems (3.27) from

$$\lambda_1^1 = 0.63\ , \qquad \lambda_1^2 = 5.04$$
$$\lambda_2^1 = -1.39\ , \qquad \lambda_2^2 = -6.41$$
$$\lambda_3^1 = 10.12\ , \tag{3.31}$$

to

$$\lambda_1^1 = -10\ , \qquad \lambda_1^2 = -1$$
$$\lambda_2^1 = -12\ , \qquad \lambda_2^2 = -1.5$$
$$\lambda_3^1 = -15\ . \tag{3.32}$$

After the local stabilization, the interconnected subsystems have the quasi-diagonal form

$$\dot{\tilde{x}}_1 = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -15 \end{bmatrix} \tilde{x}_1 + \begin{bmatrix} -23.52 & -43.78 \\ 40.23 & 68.97 \\ -15.49 & -24.96 \end{bmatrix} \tilde{x}_2 + \begin{bmatrix} 0.1 \\ -0.17 \\ 0.07 \end{bmatrix} u_i^g$$

$$\dot{\tilde{x}}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1.5 \end{bmatrix} \tilde{x}_2 + \begin{bmatrix} 179.95 & 247.53 & 367.40 \\ -182.58 & -249.03 & -365.95 \end{bmatrix} \tilde{x}_1 + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u_2^g\ , \tag{3.33}$$

which is not identical to (3.17). For the moment, we did not make use of the global control u_1^g , u_2^g in (3.33). In order to demonstrate the effect of the global controllers, we set $\tilde{k}_{12} = \tilde{k}_{21} = 0$.

From (3.9) and (3.32), we have $\pi_1 = 10$, $\pi_2 = 1$. Using (3.20) and (3.33) we compute $\tilde{\xi}_{12} = 98.51$, $\tilde{\xi}_{21} = 676.68$. The aggregate matrix $\tilde{W}$ in (3.18) is obtained as

$$\tilde{W} = \begin{bmatrix} -10 & 98.51 \\ 676.68 & -1 \end{bmatrix}, \tag{3.34}$$

which does not satisfy the conditions (3.21). Therefore, we cannot conclude stability of the overall system.

Let us use now the global control specified by (3.22),

$$\begin{aligned} \tilde{k}_{12}^{*T} &= (-238.95, -415.34) \\ \tilde{k}_{21}^{*T} &= (90.63, 124.14, 183.33) \end{aligned} \tag{3.35}$$

which yields the subsystems (3.33) as

$$\begin{aligned} \dot{\tilde{x}}_1 &= \begin{bmatrix} -10 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -15 \end{bmatrix} \tilde{x}_1 + \begin{bmatrix} 0.37 & -2.25 \\ 0.40 & -0.25 \\ 0.44 & 2.73 \end{bmatrix} \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \begin{bmatrix} -1 & 0 \\ 0 & -1.5 \end{bmatrix} \tilde{x}_2 + \begin{bmatrix} -1.31 & -0.75 & 0.72 \\ -1.31 & -0.75 & 0.72 \end{bmatrix} \tilde{x}_1 , \end{aligned} \tag{3.36}$$

and the aggregate matrix

$$\tilde{W}^* = \begin{bmatrix} -10 & 3.55 \\ 2.37 & -1 \end{bmatrix}, \tag{3.37}$$

which satisfies the conditions (3.21). Therefore, by theorem 1 the system (3.28) is stabilized by the control law (3.24) determined by the gains (3.30) and (3.35). The eigenvalues of the overall closed-loop system

$$\dot{\tilde{x}} = \left[\begin{array}{ccc|cc} -10 & 0 & 0 & 0.37 & -2.25 \\ 0 & -12 & 0 & 0.40 & -0.25 \\ 0 & 0 & -15 & 0.44 & 2.73 \\ \hline -1.31 & -0.75 & 0.72 & -1 & 0 \\ -1.31 & -0.75 & 0.72 & 0 & -1.5 \end{array}\right] \tilde{x} \tag{3.38}$$

corresponding to (3.36), are

$$\lambda_{1,2} = -1.03 \pm j0.16, \; \lambda_3 = -10.27, \; \lambda_4 = -11.99, \; \lambda_5 = -15.17, \tag{3.39}$$

which have negative real parts.

It is also interesting to note that an upper estimate of the degree π of exponential stability of the system (3.1) is provided by the aggregate matrix $\tilde{W}^*$ since, in general $\pi \leq \min_i \pi_i$. In other words, the degree of exponential stability of the overall system π stabilized by the proposed method, is smaller than the degree of exponential stability of each decoupled subsystem.

3.3 Local Stabilization

In this section, we consider a class of linear input-decentralized large-scale systems which can always be stabilized by only local feedback control applied around each subsystem. This class of systems is characterized by the comparison form of the subsystem matrices and the lower diagonal form of the interconnection matrices.

Let us consider again the system

$$\dot{x}_i = A_i x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} A_{ij} x_j + b_i u_i \,, \quad i = 1, 2, \ldots, s \tag{3.2}$$

where the $n_i \times n_i$ matrix A_i and the n_i vector b_i are

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ -a_1^i & -a_2^i & \cdots & -a_{n_i}^i \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (3.40)$$

and the $n_i \times n_j$ matrices $A_{ij} = (a_{pq}^{ij})$ are such that

$$a_{pq}^{ij} = 0 \, , \quad p < q \qquad (3.41)$$

where $p = 1, 2, \ldots, n_i$ and $q = 1, 2, \ldots, n_j$.

In order to stabilize system (3.2) characterized by (3.40) and (3.41), we apply the local control

$$u_i^{\ell} = -k_i^T x_i \, , \qquad (3.5)$$

and get (3.2) as

$$\dot{x}_i = (A_i - b_i k_i^T) x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} A_{ij} x_j \, , \quad i = 1, 2, \ldots, s \, . \qquad (3.42)$$

Gain vectors k_i are chosen so that each matrix $A_i - b_i k_i^T$ has a set L_i of distinct real eigenvalues λ_p^i defined by

$$L_i = \{\lambda_p^i : \lambda_p^i = -\alpha \sigma_p^i \, ; \quad \alpha \geq 1, \; \sigma_p^i > 0, \; p = 1, 2, \ldots, n_i\}$$
$$i = 1, 2, \ldots, s \, . \qquad (3.43)$$

The positive constant α is to be determined, so that the overall system (3.2) is stabilized.

Following the development in Section 3.1, we transform (3.42) into

$$\dot{\tilde{x}}_i = \Lambda_i \tilde{x}_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} \tilde{A}_{ij} \tilde{x}_j \, , \quad i = 1, 2, \ldots, s \qquad (3.44)$$

where the transformation (3.10) is used to get

$$\Lambda_i = T_i^{-1}(A_i - b_i k_i^T)T_i \ , \quad \tilde{A}_{ij} = T_i^{-1}A_{ij}T_j \ . \tag{3.45}$$

with Λ_i in the quasidiagonal form

$$\Lambda_i = \text{diag}\ \{-\alpha\sigma_1^i, -\alpha\sigma_2^i, \ldots , -\alpha\sigma_{n_i}^i\} \ . \tag{3.46}$$

In this case, the transformation matrix T_i can be factorized as

$$T_i = R_i \hat{T}_i \ , \tag{3.47}$$

where

$$R_i = \text{diag}\ \{1, \alpha, \ldots , \alpha^{n_i - 1}\} \ , \tag{3.48}$$

and $\hat{T}_i$ is the Vandermonde matrix

$$\hat{T}_i = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -\sigma_1^i & -\sigma_2^i & \cdots & -\sigma_{n_i}^i \\ \vdots & \vdots & & \vdots \\ (-\sigma_1^i)^{n_i - 1} & (-\sigma_2^i)^{n_i - 1} & \cdots & (-\sigma_{n_i}^i)^{n_i - 1} \end{bmatrix} \tag{3.49}$$

For the moment, we consider the free uncoupled subsystems

$$\dot{\tilde{x}}_i = \Lambda_i \tilde{x}_i \ , \quad i = 1, 2, \ldots , s \ . \tag{3.50}$$

Each subsystem (3.50) is stabilized with a degree of exponential stability

$$\pi_i = \alpha\hat{\pi}_i \tag{3.51}$$

where

$$\hat{\pi}_i = \min_p \sigma_p^i \ . \tag{3.52}$$

Now, we choose again the Liapunov function $v: R^{n_i} \to R_+$,

$$v_i(\tilde{x}_i) = (\tilde{x}_i^T \tilde{H}_i \tilde{x}_i)^{\frac{1}{2}} \tag{3.13}$$

where

$$A_i^T \tilde{H}_i + \tilde{H}_i A_i = -\tilde{G}_i \tag{3.14}$$

and

$$\tilde{G}_i = 2\theta_i \operatorname{diag} \{\alpha\sigma_1^i, \alpha\sigma_2^i, \dots, \alpha\sigma_{n_i}^i\}, \quad \tilde{H}_i = \theta_i I_i \quad . \tag{3.53}$$

The aggregate system

$$\dot{v} \leq \tilde{W} v \tag{3.18}$$

is formed as in Section 3.1 computing the elements $\tilde{w}_{ij}$ of the aggregate matrix $\tilde{W}$ with

$$\tilde{\xi}_{ij} = \lambda_M^{\frac{1}{2}} (\tilde{A}_{ij}^T \tilde{A}_{ij}) \tag{3.54}$$

and

$$\tilde{A}_{ij} = \hat{T}_i^{-1} R_i^{-1} A_{ij} R_j \hat{T}_j \quad . \tag{3.55}$$

Our ability to stabilize the system depends ultimately on satisfying the Sevastyanov-Kotelyanskii conditions (3.21) by the aggregate matrix $\tilde{W} = (\tilde{w}_{ij})$ defined by

$$\tilde{w}_{ij} = -\delta_{ij}\pi_i + (1-\delta_{ij})\tilde{\xi}_{ij} \quad . \tag{3.19}$$

Since the matrix $\tilde{W}$ has nonnegative off-diagonal elements, it is a well-known fact [16] that the conditions (3.21) are equivalent to the quasidominant diagonal property of $\tilde{W}$,

$$d_j |\tilde{w}_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{s} d_i |\tilde{w}_{ij}| , \quad j = 1, 2, \dots, s \tag{3.56}$$

where d_i's are positive numbers. Apparently, we can make the matrix $\tilde{W}$ satisfy conditions (3.20), if we can increase the diagonal elements $\tilde{w}_{ii}$ sufficiently large while keeping the off-diagonal elements $\tilde{w}_{ij}$ bounded. This is exactly the case with the class of systems under consideration. We notice that the diagonal elements $(i = j)$,

$$\tilde{w}_{ii} = -\alpha\hat{\pi}_i, \tag{3.57}$$

depend linearily on the adjustable parameter α. The off-diagonal elements $(i \neq j)$,

$$\tilde{w}_{ij} = \tilde{\xi}_{ij}(\alpha) \quad , \tag{3.58}$$

are bounded functions of α. To see this, we note that the elements $\alpha^{q-p}\, a_{pq}^{ij}$ of the matrices $R_i^{-1}A_{ij}R_j$ are either zero for $p < q$ due to (3.41), or they are bounded for $p \geq q$ due to nonpositive powers of α. We have

$$\lim_{\alpha\to+\infty} R_i^{-1}A_{ij}R_j = D_{ij} \ , \tag{3.59}$$

where the matrix $D_{ij} = (d_{pq}^{ij})$ is defined by: $d_{pq}^{ij} = a_{pq}^{ij}$, when $p = q$, and $d_{pq}^{ij} = 0$, when $p \neq q$. From (3.55) and (3.59), we define $\tilde{D}_{ij} = \hat{T}_i^{-1}D_{ij}\hat{T}_j$ and conclude from

$$\lim_{\alpha\to+\infty} \tilde{\xi}_{ij}(\alpha) = \lambda_M^{1/2}(\tilde{D}_{ij}^T\tilde{D}_{ij}), \tag{3.60}$$

that the off-diagonal elements $\tilde{w}_{ij}$ are bounded in α.

Therefore, for the selected class of dynamic systems we can always choose a sufficiently large parameter α, and use local linear feedback control to stabilize the systems. From (3.43), we see that by increasing the value of α, we move the subsystem eigenvalues away from the origin, thus, increasing the degree of exponential stability of each subsystem. This, however, requires an increase of the local feedback gains in the course of stabilization.

3.4. *An Illustrative Example*

Let us illustrate the local stabilization procedure using the following example:

$$\dot{x} = \left[\begin{array}{ccc:cc} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ -2 & -1 & -1 & 2 & 1 \\ \hdashline 4 & 0 & 0 & 0 & 1 \\ 5 & 6 & 0 & -3 & -2 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 1 \end{array}\right] u \,. \tag{3.61}$$

The eigenvalues of the system matrix A corresponding to (3.61), are

$$\lambda_1 = 1.7244,\ \lambda_2 = 5.1042,\ \lambda_3 = -1.2633,\ \lambda_{4,5} = -4.2826 \pm j1.7755 \tag{3.62}$$

and the system (3.61) is unstable.

The system (3.61) can be decomposed as

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 2 & 1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_1 \tag{3.63a}$$

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 4 & 0 & 0 \\ 5 & 6 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 \tag{3.63b}$$

The eigenvalues of the subsystem (3.63a) are moved from

$$\lambda_1^1 = -1.3532,\ \lambda_{2,3}^1 = 0.1766 \pm j1.2028 \tag{3.64}$$

to the new locations

$$\lambda_1^1 = -\sigma_1^1 = -1,\ \lambda_2^1 = -\sigma_2^1 = -2,\ \lambda_3^1 = -\sigma_3^1 = -3 \tag{3.65}$$

applying the local control (3.5) and

$$k_1^T = (4,\ 10,\ 5) \,. \tag{3.66}$$

Similarly, the eigenvalues of the subsystem (3.63b) are changed from

$$\lambda^2_{1,2} = -1 \pm j1.4142 , \tag{3.67}$$

to

$$\lambda^2_1 = -\sigma^2_1 = -1, \ \lambda^2_2 = -\sigma^2_2 = -2 \tag{3.68}$$

applying the local control (3.5) and

$$k^T_2 = (-1, 1) . \tag{3.69}$$

Referring to (3.46), we see that in (3.65) and (3.68), the parameter $\alpha = 1$.

We construct the transformation matrices R_1, R_2, $\hat{T}_1$, $\hat{T}_2$ for $\alpha > 1$ as

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix} , \quad \hat{T}_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} , \quad \hat{T}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} . \tag{3.70}$$

The numbers π_1, π_2 are both set to one. Then, the aggregation matrix of (3.18) defined by (3.57) is given as

$$\tilde{W} = \begin{bmatrix} -\alpha & \tilde{\xi}_{12} \\ \tilde{\xi}_{21} & -\alpha \end{bmatrix} , \tag{3.71}$$

which for $\alpha = 1$ takes the form

$$\tilde{W} = \begin{bmatrix} -1 & 17.0011 \\ 12.2936 & -1 \end{bmatrix} \tag{3.72}$$

where

$$\tilde{\xi}_{12} = \lambda^{\frac{1}{2}}_M (\hat{T}^{-1}_1 A_{12} \hat{T}_2) , \ \tilde{\xi}_{21} = \lambda^{\frac{1}{2}}_M (\hat{T}^{-1}_2 A_{21} \hat{T}_1)$$

and A_{12}, A_{21} are specified in (3.63).

It is obvious that the matrix $\tilde{W}$ in (3.72) does not satisfy the inequalities (3.21).

From (3.63) and (3.59), we find that

$$D_{12} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} \tag{3.73}$$

and for $\alpha > 15$, we have $\tilde{\xi}_{12} \approx 32.55$, $\tilde{\xi}_{21} \approx 18.98$. Thus, for $\alpha = 25$, we have the aggregate matrix

$$\tilde{W} = \begin{bmatrix} -25 & 32.55 \\ 18.98 & -25 \end{bmatrix}, \tag{3.75}$$

which satisfies the conditions (3.21), and the overall system is stable. The corresponding eigenvalues of the overall closed-loop system are

$$\lambda_1 = -36.0364, \ \lambda_{2,3} = -25.9599 \pm j3.5219, \ \lambda_{4,5} = -68.5213 \pm j6.0474. \tag{3.76}$$

For the chosen value of $\alpha = 25$, we have the eigenvalue sets L_1 and L_2 defined in (3.43) given as

$$L_1 = \{-\alpha\sigma_1^1, -\alpha\sigma_2^1, -\alpha\sigma_3^1\} = \{-25, -50, -75\}$$

$$L_2 = \{-\alpha\sigma_1^2, -\alpha\sigma_2^2\} = \{-25, -50\}. \tag{3.77}$$

The locations of the subsystem eigenvalues specified by L_1, L_2 of (3.77), are achieved by the local state-variable feedback defined by (3.5) and

$$k_1^T = (93748, 6874, 149)$$

$$k_2^T = (1247, 73) \tag{3.78}$$

The gains in (3.78) are relatively high which is due to the use of local controllers only. The gains can be considerably reduced by applying global controllers in the multilevel scheme outlined in Section 3.1 and illustrated in Section 3.2.

3.5 Application to LST

In this section, we design a control system for the nonlinear model of LST described in Section 2, by using only the local linear controllers as proposed in Section 3.3. This necessitates an application of the results obtained by Weissenberger [17] which are concerned with the finite regions of stability of large-scale systems rather than their global stability properties.

We notice that the LST model (2.20) belongs to a general class of systems described by the equations

$$\dot{x}_i = A_i x_i + a_i x_\ell^T \sum_{\substack{j=1 \\ j\neq i,\ell}}^{s} A_{ij} x_j + b_i u_i$$

$$i = 1, 2, \ldots s; \; \ell = \begin{cases} 1, & i = s \\ i+1, & i \neq s \end{cases} \tag{3.79}$$

where A_i are constant $n_i \times n_i$ matrices, A_{ij} are $n_\ell \times n_j$ constant matrices, a_i and b_i are n_i constant vectors.

To stabilize the system (3.79), we choose the local control

$$u_i = -k_i^T x_i \,, \quad i = 1, 2, \ldots , s \tag{3.80}$$

so that each uncoupled subsystem

$$\dot{x}_i = (A_i - b_i k_i^T) x_i \,, \quad i = 1, 2, \ldots , s \tag{3.81}$$

has a prescribed set of distinct eigenvalues

$$L_i = \{-\sigma_1^i \pm j\omega_1^i, \ldots , \sigma_p^i \pm j\omega_p^i \,, -\sigma_{p+1}^i, \ldots , -\sigma_{n_i}^i;$$

$$\sigma_q^i > 0;\; p,\, q = 1, 2, \ldots , n_i\}, \; i = 1, 2, \ldots , s. \tag{3.82}$$

By using the transformation (3.10), the closed-loop system corresponding to (3.79) is obtained as

$$\dot{\tilde{x}}_i = \Lambda_i\tilde{x}_i + \tilde{a}_i\tilde{x}_\ell^T \sum_{\substack{j=1\\ j\neq i,\ell}}^{s} \tilde{A}_{ij}\tilde{x}_j , \qquad i = 1, 2, \ldots , s , \ \ell = \begin{cases} 1, & i = s \\ i+1, & i \neq s , \end{cases} \tag{3.83}$$

where $\Lambda_i = T_i^{-1}(A_i - b_ik_i^T)T_i$ has the quasidiagonal form (3.12), $\tilde{A}_{ij} = T_\ell^TA_{ij}T_j$, and $\tilde{a}_i = T_i^{-1}a_i$.

We define the interaction function $h: T \times R^n \to R^{n_i}$ among the subsystems of (3.83) as

$$h_i(\tilde{x}) = \tilde{a}_i\tilde{x}_\ell^T \sum_{\substack{j=1\\ j\neq i,\ell}}^{s} \tilde{A}_{ij}\tilde{x}_j . \tag{3.84}$$

The interactions $h_i(\tilde{x})$ can be bounded as

$$||h_i(\tilde{x})|| \leq v_{0\ell} \sum_{\substack{j=1\\ j\neq i,\ell}}^{s} \xi_{ij}||\tilde{x}_j|| , \quad \forall\tilde{x} \in \Gamma \tag{3.85}$$

on the region

$$\Gamma = \{\tilde{x} \in R^n: ||\tilde{x}_i|| < v_{0i} , i = 1, 2, \ldots , s\} , \tag{3.86}$$

where v_{0i} are positive yet unspecified constants, and $\tilde{\xi}_{ij} = (\tilde{a}_i^T\tilde{a}_i)^{\frac{1}{2}} \lambda_M^{\frac{1}{2}}(\tilde{A}_{ij}^T\tilde{A}_{ij})$.

The aggregate $s \times s$ matrix $\tilde{W} = (\tilde{w}_{ij})$ which corresponds to the system (3.83) and constraints (3.85), is obtained following reference [17],

$$\tilde{W} = D\bar{W} \tag{3.87}$$

where

$$D = \text{diag}\ \{v_{02}, v_{03}, \ldots , v_{0s}, v_{01}\} \tag{3.88}$$

and the $s \times s$ matrix $\tilde{W} = (\tilde{w}_{ij})$ is defined by

$$\tilde{w}_{ij} = -\delta_{ij} v_{0\ell}^{-1} \pi_i + (1-\delta_{ij}) \tilde{\xi}_{ij} \tag{3.89}$$

with π_i defined in (3.9).

From (3.87), it follows that $\tilde{W}$ satisfies inequalities (3.21) if and only if $\bar{W}$ does. Inequalities (3.21) applied to $\tilde{W}$ determine the constants $v_{01}, v_{02}, \dots, v_{0s}$ in (3.85). It is possible to calculate these constants recursively. To see this, we note that the k-th leading principal $k \times k$ submatrix $\tilde{W}_k$ can be expressed as

$$\tilde{W}_k = \left[\begin{array}{c|c} \tilde{W}_{k-1} & f_k \\ \hline g_k^T & \tilde{w}_{kk} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline g_k^T \tilde{W}_{k-1}^{-1} & 1 \end{array}\right] \left[\begin{array}{c|c} W_{k-1} & 0 \\ \hline 0 & \tilde{w}_{kk} - g_k^T \tilde{W}_{k-1}^{-1} f_k \end{array}\right] \left[\begin{array}{c|c} I & \tilde{W}_{k-1}^{-1} f_k \\ \hline 0 & 1 \end{array}\right] \tag{3.90}$$

Therefore, the k-th leading principle minor of $\tilde{W}$ is

$$\det \tilde{W}_k = \det \tilde{W}_{k-1} \, (\tilde{w}_{kk} - g_k^T \tilde{W}_{k-1}^{-1} f_k) \quad . \tag{3.91}$$

For the inequalities (3.91) to be satisfied by $\tilde{W}$, it is necessary and sufficient that

$$-\tilde{w}_{kk} + g_k^T \tilde{W}_{k-1}^{-1} f_k > 0 \;, \quad k = 1, 2, \dots, s \;. \tag{3.92}$$

From (3.89), we have

$$f_k^T = (\tilde{\xi}_{1k}, \tilde{\xi}_{2k}, \dots, \tilde{\xi}_{sk}) \;, \; g_k^T = (\tilde{\xi}_{k1}, \tilde{\xi}_{k2}, \dots, \tilde{\xi}_{ks}) \tag{3.93}$$

and from (3.89) and (3.92), we get the constants $v_{0\ell}$ as

$$v_{0\ell} < -\pi_k (g_k^T \tilde{W}_{k-1}^{-1} f_k)^{-1} \;, \; \ell = \begin{cases} 1 & , \; k = s \\ k+1, & k \neq s \end{cases} \quad . \tag{3.94}$$

Once the constants $v_{0\ell}$ are calculated by (3.94), the region $\tilde{\Omega}$ of (3.86) is determined. Now, it remains to imbed a Liapunov function $V: R^n \to R_+$ inside the region Ω and determine a region of stability [17]

$$\tilde{\Omega} = \{\tilde{x} \in R^n: V(\tilde{x}) < \gamma\} \quad . \tag{3.95}$$

In (3.95), we choose

$$V(\tilde{x}) = \sum_{i=1}^{s} d_i |v_i| \ , \tag{3.96}$$

where d_i are positive numbers, and $v_i = v_i(\tilde{x}_i) = ||\tilde{x}_i||$. Following [17], we calculate the positive constant γ in (3.95) using (3.94) and

$$\gamma = \min_i \ d_i v_{0i} \ , \quad i = 1, 2, \ldots , s \tag{3.97}$$

where the positive vector $d^T = (d_1, d_2, \ldots , d_s)$ is computed by

$$d^T = -c^T \tilde{W}^{-1} \ , \tag{3.98}$$

where c is any positive s vector $(c > 0)$.

Since $\tilde{x}_i = T_i^{-1} x_i$, and $||\tilde{x}_i|| \leq ||T_i^{-1}|| \ ||x_i||$, from (3.96) and (3.97), we get finally the region of stability Ω in the original state space, which is

$$\Omega = \{x \in R^n: \sum_{i=1}^{s} d_i ||T_i^{-1}|| \ ||x_i|| < \gamma\} \ . \tag{3.99}$$

Now, we consider the nonlinear model of the LST given in Section 2, which belongs to the class of systems described by (2.79) with

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \ A_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \ a_i = \begin{bmatrix} 0 \\ -\alpha_i \end{bmatrix} , \ b_i = \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} ,$$

$$i = 1, 2, 3 \ . \tag{3.100}$$

Applying the control law

$$u_i = -k_i^T x_i \ , \quad i = 1, 2, 3 \tag{3.101}$$

where

$$k_i^T = \beta_i^{-1} \bar{k}_i^T \ , \quad i = 1, 2, 3 \tag{3.102}$$

and $\bar{k}_i^T = (\bar{k}_{i1}, \bar{k}_{i2})$, we obtain the closed-loop uncoupled subsystems (3.81) with

$$A_i - b_i k_i^T = \begin{bmatrix} 0 & 1 \\ -\bar{k}_{i1} & -\bar{k}_{i2} \end{bmatrix} , \quad i = 1, 2, 3 \ . \tag{3.103}$$

The gains $\bar{k}_i$ are chosen so that each subsystem has a set of eigenvalues

$$L_i = \{-\sigma_1^i, -\sigma_2^i\} \ , \quad i = 1, 2, 3 \ . \tag{3.104}$$

To get the transformed system corresponding to (3.83), we use the transformation matrix

$$T_i = \begin{bmatrix} 1 & 1 \\ -\sigma_1^i & -\sigma_2^i \end{bmatrix} , \quad i = 1, 2, 3 \tag{3.105}$$

and get

$$\Lambda_i = \begin{bmatrix} -\sigma_1^i & 0 \\ 0 & -\sigma_2^i \end{bmatrix} , \quad h_i(\tilde{x}_i) = \tilde{a}_i \tilde{x}_\ell^T \tilde{A}_{ij} \tilde{x}_j$$
$$i, j = 1, 2, 3, \ i \neq j; \ \ell = \begin{cases} i+1, & i \neq 3 \\ 1 \ , & i = 3 \end{cases} \tag{3.106}$$

To compute $\tilde{\xi}_{ij}$, we choose $\sigma_1^i = \sigma_1$, $\sigma_2^i = \sigma_2$, $i = 1, 2, 3$, and calculate $||A_{ij}|| = (\sigma_1)^2 + (\sigma_2)^2$, $(\tilde{a}_i^T \tilde{a}_i)^{1/2} = \sqrt{2}\ |\alpha_i|(|\sigma_1 - \sigma_2|)^{-1}$. We can minimize the numbers $\tilde{\xi}_{ij}$ with respect to the distance $\rho = \sigma_2 - \sigma_1$ between the two subsystem eigenvalues. This yields

$$\tilde{\xi}_{ij} = \sqrt{2}\ |\alpha_i|\ \rho^{-1}[(\sigma_1)^2 + (\sigma_1 + \rho)^2] \ , \tag{3.107}$$

and we get the minimal values $\tilde{\xi}^m_{ij}$ for $\tilde{\xi}_{ij}$ as

$$\tilde{\xi}^m_{ij} = (4 + 2\sqrt{2}\,)\ |\alpha_i|\sigma_1 \ , \tag{3.108}$$

which is obtained for $\rho = \sqrt{2}\,\sigma_1$.

The corresponding matrix $\tilde{W}$ in (3.87), is

$$\tilde{W} = \begin{bmatrix} -\dfrac{\sigma_1}{v_{02}} & 0 & \tilde{\xi}_{13} \\ \tilde{\xi}_{21} & -\dfrac{\sigma_1}{v_{03}} & 0 \\ 0 & \tilde{\xi}_{32} & -\dfrac{\sigma_1}{v_{01}} \end{bmatrix} . \tag{3.109}$$

From (3.94) and (3.109), we get

$$v_{01}\ v_{02}\ v_{03} < \frac{(\sigma_1)^3}{\tilde{\xi}_{13}\ \tilde{\xi}_{21}\ \tilde{\xi}_{32}} . \tag{3.110}$$

Choosing $v_{01} = v_{02} = v_{03} = v_0$, and using (3.108) and (3.110), we compute $v_0 < 0.584$. Selecting $v_0 = 0.574$, $\sigma_1 = 10$, and choosing $c = (1, 1, 1)^T$, we further compute from (3.98) the vector

$$d = (4.8,\ 13.7614,\ 4.2963)^T . \tag{3.111}$$

From (3.97), we calculate

$$\gamma = 2.4663 \ , \tag{3.112}$$

and the region $\tilde{\Omega}$ in the transformed state space as

$$\tilde{\Omega} = \{\tilde{x} \in R^n\colon 4.8||\tilde{x}_1|| + 13.7614||\tilde{x}_2|| + 4.2963||\tilde{x}_3|| < 2.4663\} . \tag{3.113}$$

In the original space, the stability region Ω is finally obtained as

$$\Omega = \{x \in R^n: 4.8||x_1|| + 13.7614||x_2|| + 4.2963||x_3|| < 1.3331\} \tag{3.114}$$

where we used $||T_i^{-1}|| = 1.8500$, $i = 1, 2, 3$.

The feedback gains that yield the region Ω are computed from (3.102) and

$$\bar{k}_i^T = (\sigma_1\sigma_2, \sigma_1 + \sigma_2)^T = (141.4213, 34.1421)^T , \quad i = 1, 2, 3 \tag{3.115}$$

as

$$\begin{aligned} k_1^T &= (1.6517, \quad 0.3988)^T \\ k_2^T &= (10.3303, \quad 2.4939)^T \\ k_3^T &= (10.7056, \quad 2.5846)^T \quad . \end{aligned} \tag{3.116}$$

This completes the design of the LST control system.

4. OPTIMAL CONTROL

In this section we will describe the application of a recently developed multilevel optimal control scheme [9] for the decentralized regulation of the LST. Such multilevel control schemes are quite efficient in the analysis of large-scale systems that may be decomposed into a number of interconnected subsystems of smaller dimensions. Since our model for the LST, described in Section 2, is a nonlinear interconnected system composed of three linear subsystems describing the motions of the spacecraft along the three axes, generation of the necessary control scheme basing the analysis on the subsystems is highly desirable in view of the complexities involved in the optimization of a nonlinear system of a large dimension. In the sequel, we will describe the general theory for the multilevel optimal control of interconnected systems, which will be followed by the specific application to the LST.

4.1. Problem Formulation

Let us consider a continuous dynamic system described by the differential equation

$$\dot{x} = f(x, u) \tag{4.1}$$

where $x(t) \in R^n$ is the state and $u(t) \in R^m$ is the control function of the system at time $t \in T$. The function $f: R^n \times R^m \to R^n$ is continuous on a bounded region $\mathcal{D} \subset R^n$ and is locally Lipschitzian with respect to x in $\mathcal{D}$ so that for every fixed control function $u(t)$, a unique solution $x(t; t_0, x_0)$ exists for all initial conditions $(t_0, x_0) \in R \quad \mathcal{D}$ and all $t \in T$, T being an interval $[t_0, \infty)$ of R.

We assume that system (4.1) can be decomposed into s interconnected subsystems

$$\dot{x}_i = A_i x_i + B_i u_i + h_i(x) , \quad i = 1, 2, \dots , s . \tag{4.2}$$

where, $x_i \in R^{n_i}$ is the state of the i-th subsystem so that

$$R^n = R^{n_1} \times R^{n_2} \times \dots \times R^{n_s} ;$$

$u_i \in R^{m_i}$ is the control function of the i-th subsystem so that

$$R^m = R^{m_1} \times R^{m_2} \times \dots \times R^{m_s} ;$$

$A_i \in R^{n_i \times n_i}$ and $B_i \in R^{n_i \times n_i}$ are constant matrices; and $h_i : R^n \to R^{n_i}$ is the function which represents the interconnection of the i-th subsystem inside the overall system.

The multilevel control scheme [9] used for the optimization of system (4.2) can be developed by considering the control function $u_i(t)$ as consisting of two parts, the local control $u_i^{\ell}(t)$ and the global control $u_i^{g}(t)$,

$$u_i(t) = u_i^{\ell}(t) + u_i^{g}(t) . \tag{4.3}$$

The local control $u_i^{\ell}(t)$ is chosen as a linear control law

$$u_i^{\ell}(t) = -K_i^{\ell} x_i(t) \tag{4.4}$$

to optimize isolated subsystems, and the global control law $u_i^{g}(t)$ is chosen as a suitable function of the state

$$u_i^{g}(t) = -K_i^{g}(x(t)) \tag{4.5}$$

to minimize the performance deviation from the optimum due to the presence of interconnections among the subsystems.

With the application of the control (4.3), the equations (4.2) governing the system under consideration take the form,

$$\dot{x}_i = A_i x_i + B_i u_i^{\ell} + h_i(x) + B_i u_i^{g} , \; i = 1, 2, \dots , s . \tag{4.6}$$

Since, as described earlier, the global control functions $u_i^g(t)$ are assigned only the task of reducing the effects of interconnections $h_i(x)$ the terms

$$h_{e_i}(x, u_i^g) = h_i(x) + B_i u_i^g \; , \; i = 1, 2, \dots , s \; , \tag{4.7}$$

may be regarded as the "effective interconnections" among the s isolated subsystems

$$\dot{x}_i = A_i x_i + B_i u_i^\ell \; , \; i = 1, 2, \dots , s \; . \tag{4.8}$$

We shall assume that all s-pairs (A_i, B_i) are completely controllable, and that with each isolated subsystem (4.8) a quadratic performance index

$$J_i(t_0, x_{i0}, u_i^\ell) = \int_{t_0}^{\infty} \{||x_i(t)||_{Q_i}^2 + ||u_i^\ell(t)||_{R_i}^2\} \, dt \tag{4.9}$$

is associated. In (4.9) $Q_i \in R^{n_i \times n_i}$ is a symmetric nonnegative definite matrix and $R_i \in R^{m_i \times m_i}$ is a symmetric positive definite matrix.

The local control $u_i^\ell(t)$ in (4.4) can now be chosen to minimize the performance index $J_i(t_0, x_{i0}, u_i)$ in (4.9). From linear-quadratic regulator theory [18], the optimal control $u_i^{\ell *}(t)$ is given by

$$u_i^{\ell *}(t) = -K_i^{\ell *} x_i(t) \tag{4.10}$$

where

$$K_i^{\ell *} = R_i^{-1} B_i^T P_i \; . \tag{4.11}$$

In (4.11), $P_i \in R^{n_i \times n_i}$ is symmetric and is the positive definite solution of the algebraic Riccati equation

$$P_i A_i + A_i^T P_i - P_i B_i R_i^{-1} B_i^T P_i + Q_i = 0 \; . \tag{4.12}$$

The optimal cost $J_i^*(t_0, x_{i0}) = J_i(t_0, x_{i0}, u_i^{\ell *})$ can in this case be calculated as

$$J_i^*(t_0, x_{i0}) = ||x_{i0}||^2_{P_i} \quad . \tag{4.13}$$

Furthermore, under the assumption that Q_i can be factored as $Q_i = C_i C_i^T$, where $C_i \in R^{n_i \times n_i}$ such that the pair (A_i, C_i) is completely observable, each closed-loop subsystem

$$\dot{x}_i = (A_i - B_i R_i^{-1} B_i^T P_i)\, x_i \ , \ i = 1, 2, \ldots , s \ , \tag{4.14}$$

is globally asymptotically stable.

The controls $u_i^{\ell *}(t)$, $i = 1, 2, \ldots , s$, will not, in general, be optimal for the composite system (4.6) and will not result in the optimal cost

$$J^*(t_0, x_0) = \sum_{i=1}^{s} J_i^*(t_0, x_0) \tag{4.15}$$

unless the effective interconnection functions $h_{e_i}(x, u_i^g) \equiv 0$. When $h_{e_i}(x, u_i^g) \not\equiv 0$, the controls $u_i^{\ell *}(t)$ produce a value of the performance index for the composite system given by

$$\tilde{J}(t_0, x_0) = \sum_{i=1}^{s} \tilde{J}_i(t_0, x_{i0}) \tag{4.16}$$

where

$$\tilde{J}_i(t_0, x_{i0}) = \tilde{J}_i(t_0, x_{i0}, u_i^{\ell *}) \ . \tag{4.16}$$

It is obvious that

$$\tilde{J}(t_0, x_0) \geq J^*(t_0, x_0) \ \forall (t_0, x_0) \in T \times R^n \tag{4.17}$$

and the local control law $u_i^{\ell *}(t)$ in (4.10) can only be a suboptimal policy for the composite system (4.6), with an index of suboptimality $\varepsilon > 0$ defined by the inequality

$$\tilde{J}(t_0, x_0) \leq (1+\varepsilon)\ J^*(t_0, x_0) \ \forall (t_0, x_0) \in T \times R^n \ . \tag{4.18}$$

The suboptimality index ε for the system with the optimal local control,

$$\dot{x}_i = (A_i - B_i R_i^{-1} B_i^T P_i)\, x_i + h_{e_i}(x, u_i^g) \;, \; i = 1, 2, \ldots , s \qquad (4.19)$$

depends on the size of the effective interactions $h_{e_i}(x, u_i^g)$ and hence is a measure of the performance deterioration due to these.

We can now give a formal definition of this concept.

Definition. The system (4.19) *with the optimal local control law* (4.10) *is said to be suboptimal with the index* ε *if there exists a number* $\varepsilon > 0$ *for which inequality* (4.18) *is satisfied.*

As described earlier, the suboptimality index ε is a function of the interactions $h_{e_i}(x, u_i^g)$ and the following problem is of interest:

Problem 1. Establish conditions on $h_{e_i}(x, u_i^g)$ *to guarantee a prescribed value of the suboptimality index* ε .

It is important to note that in Problem 1, the rate of the global control function $u_i^g(t)$ is ignored as it is taken together with the existing interconnections $h_i(x)$ in the system to yield the effective interconnection function $h_{e_i}(x, u_i^g)$. However, as we shall see later, the solution to Problem 1 indicates a method of choosing the global control $u_i^g(t)$ so as to reduce the size of $h_{e_i}(x, u_i^g)$ and, hence, minimize the suboptimality index ε . In other words, we consider the index $\varepsilon = \varepsilon[||h_e(x, u^g)||]$ where $h_e: R^n \times R^m \to R^n$ is $h_e = [h_{e_1}^T, h_{e_2}^T, \ldots , h_{e_s}^T]^T$ and $u^g \in R^m$ is $u^g = [(u_1^g)^T, (u_2^g)^T, \ldots , (u_s^g)^T]^T$ and solve the following:

Problem 2. Find a control law of the form (4.5) *or equivalently,*

$$u^g(t) = -K^g(x(t)) \qquad (4.20)$$

for which

$$\varepsilon^{*} = \inf_{u^{g}(t)} \varepsilon\{[||h_{e}(x, u^{g})||]\} \ \forall x \in \mathcal{D} \tag{4.21}$$

is attained.

We will now provide the solutions to the above problems.

4.2. Multilevel Optimization

A solution to Problem 1 may be obtained by using the classical Hamilton-Jacobi theory. Since in our optimization procedure, we chose the local control laws (4.10) to optimize the decoupled subsystems, the optimal indices satisfy the corresponding Hamilton-Jacobi equations. When the subsystems are interconnected, the equations are not satisfied by the respective performance indices and the overall system is not optimal. However, a majorization procedure is possible to provide an estimate of the performance deviation from the optimum due to the interactions.

Now, we provide a solution to Problem 1 by the following:

Theorem 4.1. *Let there exist nonnegative numbers* ξ_{ij} *such that the function* $h_{e_i}(x, u_i^g)$ *in* (4.19) *satisfy the constraints*

$$||h_{e_i}(x, u_i^g)|| \le \sum_{j=1}^{s} \xi_{ij}||x_j|| \ , \ \forall x \in R^n \ , \ \forall i = 1, 2, \ldots , s \tag{4.22}$$

and

$$\xi \le \frac{\varepsilon}{1+\varepsilon} \ \frac{\lambda_m(W)}{2\lambda_M(P)} \tag{4.23}$$

where $\xi = \sum_{i=1}^{s} \sum_{j=1}^{s} \xi_{ij}$, $P = \mathrm{diag}\{P_1, P_2, \ldots , P_s\}$, $W = \mathrm{diag}\{W_1, W_2, \ldots , W_s\}$, P_i *being defined by* (4.12) *and* $W_i = Q_i + P_iB_iR_i^{-1}B_i^TP_i$, *and* $\lambda_M(P)$ *and* $\lambda_m(W)$ *are the maximum and minimum eigenvalues of* P *and* W *respectively. Then the composite system* (4.19) *is*

(i) suboptimal with index ε

and

(ii) globally asymptotically stable.

Proof. Since the decoupled subsystems (4.14) are optimal, the functions $v_i(x_i) = ||x_i||^2_{P_i}$, $i = 1, 2, \ldots, s$, satisfy individually the Hamilton-Jacobi equations

$$[\text{grad } v_i(x_i)]^T[(A_i - B_iK_i^{\ell *})x_i] + ||x_i||^2_{Q_i} + ||K_i^{\ell *} x_i||^2_{R_i} = 0 , \quad \forall x_i \in R^{n_i} , \; i = 1, 2, \ldots , s . \tag{4.24}$$

Now, the time-derivative $\dot{v}_i(x_i)$ can be calculated along the trajectories $\tilde{x}_i(t)$ of the composite system (4.19) as

$$\dot{v}_i(\tilde{x}_i) = [\text{grad } v_i(\tilde{x}_i)]^T \{(A_i - B_iR_i^{-1}B_i^TP_i)\,\tilde{x}_i + h_{e_i}(\tilde{x}, u_i^g)\} \tag{4.25}$$

where $\tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T, \ldots , \tilde{x}_s^T]^T$.

Substitution of (4.25) in (4.24) and rearrangement of terms gives

$$||\tilde{x}_i||^2_{W_i} = -\,(1+\varepsilon)\;\dot{v}_i(x_i) + (1+\varepsilon)\;[\text{grad } v_i(\tilde{x}_i)]^T\, h_{e_i}(\tilde{x}, u_i^g) - \varepsilon||\tilde{x}_i||^2_{W_i} \quad \forall\tilde{x} \in R^n , \; i = 1, 2, \ldots , s , \tag{4.26}$$

where the simplification $||\tilde{x}_i||^2_{Q_i} + ||K_i^{*}x_i||^2_{R_i} = ||\tilde{x}_i||^2_{W_i}$ with $W_i = Q_i + P_iB_iR_i^{-1}B_i^TP_i$ is made.

Denoting $v(\tilde{x}) = \sum_{i=1}^{s} v_i(\tilde{x}_i)$ and summing the s-equations in (4.26) we get,

$$||\tilde{x}||^2_W = -\,(1+\varepsilon)\;\dot{v}(\tilde{x}) + (1+\varepsilon)[\text{grad } v(\tilde{x})]^T\, h_e(\tilde{x}, u^g) - \varepsilon||x||^2_W, \;\forall\tilde{x} \in R^n . \tag{4.27}$$

Now, integrating (4.27) from t_0 to ∞ we obtain

$$\tilde{J}(t_0, x_0) = (1+\varepsilon)\, J^*(t_0, x_0) + (1+\varepsilon) \int_{t_0}^{\infty} \{[\text{grad } v(\tilde{x})]^T h_e(\tilde{x}, u^g) - \tfrac{\varepsilon}{1+\varepsilon} ||x||_W^2\}\, dt \,, \tag{4.28}$$

where $\tilde{J}$ and J^* are defined in (4.15) and (4.16).

It is now simple to observe from (4.18) and (4.28) that the system is suboptimal with index ε if

$$\int_{t_0}^{\infty} \{[\text{grad } v(\tilde{x})]^T h_e(\tilde{x}, u^g) - \tfrac{\varepsilon}{1+\varepsilon} ||\tilde{x}||_W^2\}\, dt \geq 0 \,, \quad \forall \tilde{x} \in R^n \,. \tag{4.29}$$

For further simplification of (4.29) we note that

$$v(\tilde{x}) = \sum_{i=1}^{s} v_i(\tilde{x}_i) = \sum_{i=1}^{s} ||\tilde{x}_i||_{P_i}^2 = ||\tilde{x}||_P^2 \,. \tag{4.30}$$

Also, since $||h_{e_i}(\tilde{x}, u_i^g)|| \leq \sum_{j=1}^{s} \xi_{ij} ||\tilde{x}_j||, \forall x \in R^n$ we have the inequality

$$||h_e(\tilde{x}, u^g)|| \leq \xi ||x|| \,, \forall \tilde{x} \in R^n \tag{4.31}$$

where $\xi = \sum_{i=1}^{s} \sum_{j=1}^{s} \xi_{ij}$.

Using (4.30) and (4.31) it can be easily shown that a sufficient condition for the inequality (4.29) to hold is

$$2\xi P\tilde{x} ||\tilde{x}|| \leq \tfrac{\varepsilon}{1+\varepsilon} ||\tilde{x}||_W^2 \,, \ \forall x \in R^n \tag{4.32}$$

which, however, is implied by the main inequality (4.23) of the Theorem.

To complete the proof of the Theorem, we demonstrate the global asymptotic stability of the system (4.19) by using the function $v(\tilde{x}) = ||\tilde{x}||_P^2$ as a

Liapunov function. Note that $v(\tilde{x})$ is positive definite since P is a diagonal matrix formed from the positive definite solutions of the s Riccati equations (4.12). Further, the time-derivative of $v(\tilde{x})$ along the solutions of (4.19) gives

$$\dot{v}(\tilde{x}) = - ||\tilde{x}||_W^2 + 2\tilde{x}^T P h_e(\tilde{x}, u^g) \leq 0 \quad \forall \tilde{x} \in R^n , \tag{4.33}$$

from (4.31) and (4.23). This completes the proof of the Theorem.

It is important to note that the above theorem provides an explicit algebraic constraint on the interactions that is easy to check. Inequality (4.23) involves calculation of eigenvalues of block-diagonal matrices P and W, and since $\lambda_M(P) = \max_i \{\lambda_M(P_i)\}$ and $\lambda_m(W) = \min_i \{\lambda_m(W_i)\}$, the calculation can be carried out on the subsystem level.

In the context of the above Theorem, it is of interest to consider Problem 2 of determining the global control $u^g(t)$ so as to minimize the suboptimality index ε. From (4.23) and (4.31), it is evident that ε is a non-decreasing function of $||h_e(x, u^g)||$ and hence, Problem 2 reduces to one of choosing $u^g(t)$ to minimize $||h_e(x, u^g)||$. This function minimization problem is particularly simple in the present case since, from (4.7)

$$h_e(x, u^g) = h(x) + Bu^g \tag{4.34}$$

which, on using the control law (4.5) reduces to

$$h_e(x, u^g) = h(x) - BK^g(x) \quad . \tag{4.35}$$

A perfect neutralization of the effects of interconnections occurs if a choice of $K^g(x)$ results in

$$BK^g(x) = -h(x) \tag{4.36}$$

and, in this case, $\varepsilon = 0$. In the special case, when B is square and nonsingular, the explicit expression for $K^g(x)$ is available as

$$K^g(x) = -B^{-1}h(x) . \tag{4.37}$$

In general, a perfect neutralization of the interaction effects mentioned above, is not possible and one may attempt to minimize $||h(x) - BK^g(x)||$ by the proper choice of $K^g(x)$ in order to solve Problem 2. This is admittedly a complex minimization problem and a general solution is difficult to obtain. However, in the particular case of linear interconnections, the problem can be simplified and an elegant solution can be provided. This is, we assume

$$\hat{h}(t, x) = Hx \tag{4.38}$$

where $H \in R^{n\times n}$. In this case, the global control can also be chosen as a linear law

$$K^g(x) = K^g x \tag{4.39}$$

where $K^g \in R^{m\times n}$. With (52) and (53), Problem 2 simplifies to:

Problem 2'. Choose the matrix K^g such that $\inf_{K^g} ||(H-BK^g)x||$ is achieved for all $x \in R^n$.

Remembering that $||(H-BK^g)x|| \leq ||H-BK^g|| \; ||x||$ holds for all $x \in R^n$, Problem 2' actually reduces to finding $\min_{K^g} ||H-BK^g||$. When rank $B = m$, the solution to this latter problem is well-known and K^g is given by

$$K^g = (B^TB)^{-1} B^TH \tag{4.40}$$

where $(B^TB)^{-1} B^T$ is the Moore-Penrose generalized inverse of B [15]. It is interesting to note that in the particular case when

$$\text{Rank } [B \,|\, H] = \text{Rank } B \tag{4.41}$$

the choice (4.40) leads to a perfect neutralization of interaction effects and $\varepsilon = 0$.

4.3. *An Illustrative Example*

For the purpose of illustrating the multilevel control scheme presented here, let us consider the following example.

The system is described by

$$\dot{x} = Ax + Bu \tag{4.42}$$

where

$$A = \begin{bmatrix} -5 & 6 & 0 & -0.0095 \\ 4 & -4 & 0.003 & 0 \\ -0.00332 & 0 & -3 & 1 \\ 0 & 0.00995 & 8 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and is required to be optimized with respect to the performance index

$$J = \int_{t_0}^{\infty} \{||x||^2 + ||u||^2\}\, dt \ . \tag{4.43}$$

In this particular case, the dimension of the system $(n = 4)$ is small and hence the problem is amenable for a direct analysis and the required control can be obtained from solving the associated Riccati equation (of fourth order). However, as our interest here is to provide an illustration of the decentralized optimal control scheme*, let us consider system (4.42) as being

---

*Besides the advantage of permitting an analysis based on the subsystems of small orders, the decentralized control scheme presented results in important connectivity properties of the system. The suboptimality and stability of the system remain invariant under structural perturbations caused by the on-off participation of the parts of the system. This property, however, does not result when direct optimization of the system is carried out [10].

composed of two subsystems

$$\dot{x}_1 = \begin{bmatrix} -5 & 6 \\ 4 & -4 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 \tag{4.44}$$

and

$$\dot{x}_2 = \begin{bmatrix} -3 & 1 \\ 8 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 \tag{4.45}$$

with the interconnection matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & -0.0095 \\ 0 & 0 & 0.003 & 0 \\ -0.00332 & 0 & 0 & 0 \\ 0 & 0.00995 & 0 & 0 \end{bmatrix} . \tag{4.46}$$

By splitting the control functions u_1 and u_2 into a local component and a global component, the decoupled subsystems (4.44) and (4.45) can be optimized with respect to the performance indices

$$J_1 = \int_{t_0}^{\infty} \{||x_1||^2 + ||u_1||^2\}\, dt \quad \text{and} \quad J_2 = \int_{t_0}^{\infty} \{||x_2||^2 + ||u_2||^2\}\, dt. \tag{4.47}$$

The solutions of the associated Riccati equations can be obtained as

$$P_1 = \begin{bmatrix} 1.1910 & 1.5411 \\ 1.5411 & 2.1397 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 5.5591 & 2.3746 \\ 2.3746 & 1.1224 \end{bmatrix}$$

and the local control laws are,

$$u_1 = -[1.1910 \quad 1.5411]\, x_1$$

$$u_2 = -[2.3746 \quad 1.1224]\, x_2 \tag{4.48}$$

In the absence of the global controls, the functions (4.48) will only be suboptimal policies for the overall system (4.42) with the index of suboptimality ε given by

$$||H|| \leq \frac{\varepsilon}{\varepsilon+1} \cdot \frac{\min\{\lambda_m(W_1)\ ,\ \lambda_m(W_2)\}}{2\ \max\{\lambda_M(P_1),\ \lambda_M(P_2)\}} \tag{4.49}$$

where $W_i = Q_i + P_iB_iR_i^{-1}B_i^TP_i$, $i = 1, 2$, are

$$W_1 = \begin{bmatrix} 2.4185 & 1.8354 \\ 1.8354 & 3.3748 \end{bmatrix} \text{ and } W_2 = \begin{bmatrix} 6.6386 & 2.6651 \\ 2.6651 & 2.2597 \end{bmatrix} .$$

Inequality (4.49) is satisfied with $\varepsilon = 2$ and hence the performance degradation from the optimum is 200% .

In order to improve the performance, we now use the global controls u_1^g and u_2^g given by

$$u^g = -(B^TB)^{-1}B^THx \tag{4.50}$$

where $u^g = \begin{bmatrix} u_1^g \\ u_2^g \end{bmatrix}$. (4.50) can be simplified to yield

$$u_1^g = -[\ 0 \quad -0.0095\ ]\ x_2$$

$$u_2^g = -[\ 0 \quad 0.00995]\ x_1 \quad . \tag{4.51}$$

The effective interaction matrix $\tilde{H}$ with the application of the global control is

$$\tilde{H} = [\ I - B(B^TB)^{-1}B^T]\ H$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0.003 & 0 \\ -0.00332 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.52}$$

and the suboptimality inequality (4.49) can now be satisfied with a value $\varepsilon = 0.2$. Hence the performance degradation is reduced from the original 200% to only 20%, thus illustrating the effectiveness of the global controls.

4.4. Application to LST

The results developed in the earlier parts of this section may be directly used for the multilevel optimization of the LST. As described in Section 2, the model for the LST is a set of three interconnected subsystems, described by (2.20) as,

$$\dot{x}_i = A_i x_i + b_i u_i + h_i(x) \ , \quad i = 1,\ 2,\ 3 \ , \tag{4.53}$$

where $x_i \in R^2$, $A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b_i = \begin{bmatrix} 0 \\ \beta_i \end{bmatrix}$ and $h_1(x) = \begin{bmatrix} 0 \\ -\alpha_2 x_{32} x_{12} \end{bmatrix}$,

$h_2(x) = \begin{bmatrix} 0 \\ -\alpha_2 x_{32} x_{12} \end{bmatrix}$, $h_3(x) = \begin{bmatrix} 0 \\ -\alpha_3 x_{12} x_{22} \end{bmatrix}$, x being the composite state-vector $x = [x_1^T, x_2^T, x_3^T]^T$ and $x_i = [x_{i1}, x_{i2}]^T$, $i = 1, 2, 3$.

Following our multilevel control policy, we split each of the control functions u_i into a local component u_i^ℓ and a global component u_i^g and optimize the decoupled subsystems

$$\dot{x}_i = A_i x_i + b_i u_i^\ell \ , \quad i = 1,\ 2,\ 3 \ , \tag{4.54}$$

with respect to the performance indices

$$J_i = \int_{t_0}^{\infty} \{||x_i||^2 + ||u_i^\ell||^2\}\, dt \ , \quad i = 1,\ 2,\ 3 \ , \tag{4.55}$$

obtained with the choice $Q_i = I_{2\times 2}$ and $R_i = 1$ $\forall i = 1, 2, 3$. The solution of this linear-quadratic optimal control problem is simple and involves the solution of the associated Riccati equations,

$$A_i^T P_i + P_i A_i - P_i b_i b_i^T P_i + Q_i = 0 \ , \ i = 1, 2, 3 \ . \tag{4.56}$$

With the specified structure of A_i and b_i, the solution of (4.56) can be obtained as,

$$P_i = \begin{bmatrix} (1 + \frac{2}{\beta_i})^{\frac{1}{2}} & \frac{1}{\beta_i} \\ \frac{1}{\beta_i} & \frac{1}{\beta_i}(1 + \frac{2}{\beta_i})^{\frac{1}{2}} \end{bmatrix} \tag{4.57}$$

and the local optimal controls are

$$u_i^{\ell} = -b_i^T P_i x_i = - [\, 1 \quad (1 + \frac{2}{\beta_i})^{\frac{1}{2}}] \, x_i \ , \quad i = 1, 2, 3 \ . \tag{4.58}$$

However, in the absence of a suitable choice of the global control functions u_i^g, (4.58) will only be suboptimal for the composite system (4.53), with the index of suboptimality ε determined by the size of the effective interconnections,

$$h_{e_i}(x, u_i^g) = h_i(x) + b_i u_i^g \ , \quad i = 1, 2, 3 \ . \tag{4.59}$$

(4.59) can be simplified to yield

$$h_{e_1}(x, u_1^g) = \begin{bmatrix} 0 \\ -\alpha_1 x_{22} x_{32} + \beta_1 u_1^g \end{bmatrix}$$

$$h_{e_2}(x, u_2^g) = \begin{bmatrix} 0 \\ -\alpha_2 x_{12} x_{32} + \beta_2 u_2^g \end{bmatrix}$$

$$h_{e_3}(x, u_3^g) = \begin{bmatrix} 0 \\ -\alpha_3 x_{12} x_{22} + \beta_3 u_3^g \end{bmatrix} \tag{4.60}$$

It is now simple to observe that the choice of

$$u_1^g(x) = \frac{\alpha_1}{\beta_1} x_{22}x_{32}$$

$$u_2^g(x) = \frac{\alpha_2}{\beta_2} x_{12}x_{32}$$

$$u_3^g(x) = \frac{\alpha_3}{\beta_3} x_{12}x_{22} \tag{4.61}$$

will make the functions $h_{e_i}(x, u_1^g) \equiv 0$ and hence $\varepsilon = 0$, thus resulting in no degradation of the performance from the optimum.

It is of interest to evaluate the control functions for a representation set of values of the parameters of the LST. For the values of the inertia components* $I_x = 14656\ Kg_m^2$, $I_y = 91772\ Kg_m^2$ and $I_z = 95027\ Kg_m^2$ and typical reaction wheel constants $K_1 = K_2 = K_3 = 12.57 \times 10^5$ N-m/rad., the values of α_i, β_i, $i = 1, 2, 3$ can be calculated as

$$\begin{array}{lcl} \alpha_1 = 0.2221 & & \beta_1 = 85.62 \\ \alpha_2 = -0.08754 & \text{and} & \beta_2 = 13.69 \\ \alpha_3 = 0.8112 & & \beta_3 = 13.21 \end{array} .$$

Hence, the control components can be evaluated from (4.58) and (4.61) as,

$$u_1^\ell = -[1 \quad 1.012] \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$u_2^\ell = -[1 \quad 1.061] \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$u_3^\ell = -[1 \quad 1.07\ ] \begin{bmatrix} x_{31} \\ x_{32} \end{bmatrix} \tag{4.62}$$

---

*These values correspond to the on-orbit configuration of the LST with extended light shield and solar wings, with the corresponding mass of the body totalling 9380 Kg [11].

and

$$u_1^g = 0.0026 \quad x_{22}x_{32}$$

$$u_2^g = -0.064 \quad x_{12}x_{32}$$

$$u_3^g = 0.0613 \quad x_{12}x_{22} \quad . \tag{4.63}$$

This completes the multilevel optimization of the LST control system.

5. CONCLUSIONS

A multilevel scheme was proposed for control of Large Space Telescope modeled by a three-axis-six-order nonlinear equation. Local controllers were used on the subsystem level to stabilize motions corresponding to the three axes. Global controllers were applied to reduce (and sometimes nullify) the interactions among the subsystems. A multilevel optimization method was developed whereby local quadratic optimizations were performed on the subsystem level, and global control was again used to reduce (nullify) the effect of interactions.

The proposed multilevel stabilization and optimization methods are presented as general tools for design and then used in the design of the LST Control System. Furthermore, the methods are entirely computerized (Appendices A.1 and 2), so that they can accommodate higher order LST models with both conceptual and numerical advantages over the standard straightforward design techniques.

6. REFERENCES

1. Šiljak, D. D., "Stability of Large-Scale Systems", Proc. 5th IFAC Congr., Paris, 1972, pp. C-32:1-11.

2. Šiljak, D. D., "Stability of Large-Scale Systems Under Structural Perturbations", IEEE Trans., SMC-2(1972), 657-663.

3. Šiljak, D. D., "On Stability of Large-Scale Systems Under Structural Perturbations", IEEE Trans., SMC-3(1973), 415-417.

4. Šiljak, D. D., S. Weissenberger, and S. M. Ćuk, "Decomposition-aggregation Stability Analysis", NASA Contract Report, No. 2196, 1973.

5. Šiljak, D. D., and S. M. Ćuk, "Stability Region Maximization by Decomposition-Aggregation Method", NASA Contract Report, No. 2428, 1974.

6. Šiljak, D. D., and M. B. Vukčević, "On Hierarchic Stabilization of Linear Large-Scale Systems", Proc. 8th Asilomar Conf. Circuits, Systems, Computers, Pacific Grove, Calif., 1974, pp. 503-507.

7. Šiljak, D. D., "Stabilization of Large-Scale Systems: A Spinning Flexible Spacecraft", Proc. 6th IFAC Congr., Boston, Mass., 1975, pp. 35.1:1-10.

8. Šiljak, D. D., and M. B. Vukčević, "Large-Scale Systems: Stability, Complexity, Reliability:, IEEE Proc., Special Issue on Recent Advances in System Science, Edited by W. A. Porter (to appear in December, 1975).

9. Šiljak, D. D., and S. K. Sundareshan, "On Hierarchic Optimal Control of Large-Scale Systems", Proc. 8th Asilomar Conf. Circuits, Systems, Computers, Pacific Grove, Calif., 1974, pp. 495-502.

10. Šiljak, D. D., and S. K. Sundareshan, "Large-Scale Systems: Optimality vs. Reliability", Conference on Directions in Decentralized Control, Many-Person Optimization, and Large-Scale Systems, Cambridge, Mass., Sept. 1-3, 1975.

11. Schiehlen, W. O., "A Fine Pointing System for the Large-Space Telescope", NASA Technical Note, No. TN D-7500, 1973.

12. Davison, E. J., "The Decentralized Stabilization and Control of a Class of Unknown Non-Linear Time-Varying Systems", Automatica, 10(1974), 309-316.

13. Chen, C. T., "Introduction to Linear System Theory", Holt, Rinehart, and Winston, New York, 1970.

14. Gantmacher, F. R., "The Theory of Matrices", Vol. II, Chelsea, New York, 1960.

15. Langenhop, C. E., "On Generalized Inverses of Matrices", SIAM J. Appl. Math., 15(1967), 1239-1246.

16. Fiedler, M., and V. Pták, "On Matrices With Non-Positive Off-Diagonal Elements and Positive Principal Minors", Czech. Math. J., 12(1962), 382-400.

17. Weissenberger, S., "Stability Regions of Large-Scale Systems", Automatica, 9(1973), 653-663.

18. Anderson, B.D.O., and J. B. Moore, "Linear Optimal Control", Prentice-Hall, Englewood Cliffs, New Jersey, 1971.

19. Kleinman, D. L., "On An Iterative Technique for Riccati Equation Computations", IEEE Trans., AC-13(1968), 114-115.

APPENDIX

COMPUTER APPLICATION

A.1. Stabilization Program

The entire stabilization method is computerized. In this section we present the computer program for the stabilization of a class of large-scale systems by local state feedback, according to Section 3.3. The program is written in FORTRAN IV, for HP2100 computer. It is, basically, an interactive user oriented program.

Designers can enter the program from VDU (Visual Display Unit) and thus freely alter the course of computation, according to the nature of the problem. Program accepts input data from a logical unit that has to be previously assigned. As a result of computation, it prints out stabilizing parameter α , corresponding aggregation matrix, stabilizing set of subsystem eigenvalues, and enables the designer to reenter the program with so computed new set. The program finally prints out the corresponding subsystem feedback gains. The name of the main program is PP1. Its function is to coordinate the sequence of actions during the course of the stabilization and to enable the designer to access the program at various points during its operation. The program PP1 calls subprograms, DECP, PPL, TRF, AGR and MINV. The processing of variables between the main program and subroutines is realized via COMMON block.

Program PP1

Purpose:

Local stabilization of a class of large-scale linear systems.

Description of input parameters:

A - N by N system matrix.

B - N by M input matrix.

II - one dimensional integer array. It stores dimensions of each subsystem. The other parameters are working variables.

User has to specify integers N and M and number of subsystems IS, into which system matrix A , and input matrix B are decomposed. During the course of stabilization, user has to enter the program with subsystem eigenvalues, and specify an increment delta by which α is increased during the process of iterations.

At the very beginning of the program, the user has to assign input-output units. Also, during the operation of the program, user communicates with the program by specifying commands, by which the sequence of calculations is controlled. These commands are in the "question-answer" form. For example, program prints out the question:

"DO YOU WANT TO CONTINUE, YES OR NO". The user then types either "YES" or "NO" accordingly. Other commands are self explanatory, and are not going to be discussed here.

Subroutine DECP

Purpose:

Decomposes system matrix A and input matrix B into subsystems. The product of the decomposition is stored in A2 and B2.

Usage:

CALL DECP (IS, M, N)

Description of parameters:

IS - Number of subsystems.

M - Number of inputs.

N - Order of the overall system.

The following parameters are passed via COMMON block as;

COMMON A, B, A2, B2, II

A - N by N system matrix.

B - N by M input matrix.

A2 - Three dimensional array which contains the product of the decomposition of the matrix A .

B2 - Three dimensional array which contains the product of the decomposition of the matrix B .

II - One dimensional integer array which contains the dimensions of each subsystem.

Subroutines required: None.

Subroutine PPL

Purpose:

Pole shifting using state feedback.

Usage:

CALL PPL (N, IW)

Description of parameters:

N - Order of the system.

IW - Integer for the output logical unit.

The following parameters are passed via COMMON block as:

COMMON A1, B1, B2, II A, Q1, Q, I11, B, R1, R2, D, SK

A - N by N system matrix

B - N-th dimensional input vector

R1 - One dimensional array which contains real parts of eigenvalues of matrix A .

R2 - One dimensional array which contains imaginary parts of eigenvalues of matrix A .

D - N-th dimensional gain vector.

All other parameters are working variables, which are placed in COMMON block in order to make it consistent with the COMMON block of the main program PP1.

Subroutines required: ALAM, DISP, KBAR.

As a result of the pole shifting, the subroutine passes back matrix A of the closed loop system, the gain vector D , and the new eigenvalues. The subroutine itself is written as a user-oriented interactive program. The user enters the desired eigenvalues from VDU. The commands for controlling a sequence of computations, are self explanatory.

Subroutine TRF

Purpose:

Transforms subsystems by similarity transformation.

Usage:

CALL TRF (1S)

Description of parameters:

IS - Number of subsystems.

The following parameters are passed via COMMON block as:

COMMON A, B, A2, B2, II, A1, Q1, Q, I11, B3

A2 - Three dimensional array. It contains the product of the decomposition of the system matrix A .

B2 - Three dimensional array. It contains the product of the decomposition of the input matrix B .

II - One dimensional integer array that contains dimensions of each subsystem.

Q - Three dimensional array that contains transformation matrices.

All other parameters are working variables.

Subroutines required: MINV

The product of transformation is in A2 and B2. The array Q is unchanged.

Subroutine AGR

Purpose:

Forms an aggregate matrix.

Usage:

CALL AGR (IS)

Description of parameters:

IS - Number of subsystems.

The following parameters are passed via COMMON block as:

COMMON A, B, A2, B2, II, A4, A3, Q, I11, B3, R1, R2, KB, SK

A2 - Three dimensional array which contains the product of the decomposition of the matrix A .

B2 - Three dimensional array which contains the product of the decomposition of the matrix B .

II - Integer array that contains the dimensions of each subsystem.

R1 - One dimensional array that contains real parts of subsystem eigenvalues.

A3 - Matrix that contains the aggregate model.

All other parameters are working variables.

Subroutine required: ALAM, BIG1, SMAL1.

Subroutine KBAR

Purpose:

Computes gain vector for state feedback control.

Usage:

CALL KBAR (A, N, Z, 1Z, D, B)

Description of parameters:

A - N by N system matrix.

N - Dimension of the system.

Z - One dimensional array that contains the desired characteristic polynomial.

IZ - Its dimension.

D - One dimensional array that contains resultant gain vector.

B - Input vector.

Subroutine required: COEF1, SCALU, VECPR, MINV.

Method:

Described in reference [13].

Subroutine ALAM

Purpose:

Calculates eigenvalues of general N by N matrix.

Usage:

CALL ALAM (A, N, D, COF, R1, R2)

Description of parameters:

A - N by N system matrix.

N - Dimension of the system.

D - N+1 dimensional working vector.

COF - N+1 dimensional working vector.

R1 - One dimensional array of real parts of eigenvalues of matrix A .

R2 - One dimensional array of imaginary parts of eigenvalues of matrix A .

Subroutines required: COEF1, POLRT

Method:

Computes coefficients of characteristic polynomial, and calculates its zeros.

Subroutine COEF1 (A, N, D)

Purpose:

Calculates coefficients of the characteristic polynomial of matrix A .

Usage:

CALL COEF1 (A, N, D)

Description of parameters:

A - N by N system matrix.

N - Dimensions of the system.

D - One dimensional array of coefficients of characteristic polynomial.

Subroutines required: UNIT1, PROD1, TRAC1, SCM1, ADD1

Method:

Uses Souriau-Frame-Faddeev algorithm.

Subroutine DISP

Purpose:

Form polynomial from its zeros.

Usage:

CALL DISP (R1, R2, N, Z)

Descriptions of parameters:

R1 - One dimensional array of real parts of roots of a given polynomial.

R2 - One dimensional array of imaginary parts of roots of a given polynomial.

N - Order of a polynomial.

Z - One dimensional array that contains computed coefficients of the polynomial.

Subroutine required: PMPY

All other subroutines, listed in the Appendix are self explanatory and are not going to be explained here. Subroutines PMPY, POLRT and MINV are IBM-SSP subroutines.

```
:PR,FFTN4,99
FTN4,L
      PROGRAM PP1
C
C   INTERACTIVE PROGRAM TO STABILIZE A CLASS OF LARGE SCALE SYSTEMS
C
C
C
C
C   AUTHOR :    MIROSLAV B. VUKCEVIC,          DATE /7/20/1975/
C
C
C
      REAL KB
      INTEGER A10
      INTEGER DD
      DIMENSION A(10,10),B(10,3),A2(9,10,10),
     1 B2(3,10,3),II(5),AK(5,5),Q1(5,5),Q(3,5,5),I11(5,5),B3(5),R1(6),
     1 R2(6),KB(6),SK(6,10),A1(3,10),T1(6,6),Y2(6,6),MM(25),LL(25),C(25)
      COMMON A,B,A2,B2,II,AK,Q1,Q,I11,B3,R1,R2,KB,SK,A1
      DATA DD/2HYE/
      WRITE(1,1000)
 1000 FORMAT(10X,"STABILIZATION OF A CLASS OF LARGE SCALE SYSTEMS")
      WRITE(1,145)
  145 FORMAT(1X,"ASSIGN LOGICAL UNITS"/1X,"2I2")
C
C     READ THE DATA
C
      READ(1,111)IRD,IW
      WRITE(IW,146)
  146 FORMAT(1X,"SPECIFY ORDER OF THE SYSTEM AND NUMBER OF INPUTS"
     11X,"2I2")
      READ(IRD,111)N,M
  111 FORMAT(5I2)
      WRITE(IW,150)
  150 FORMAT(1X,"ENTER SYSTEM MATRIX")
      DO 8 K=1,N
      WRITE(IW,151)
  151 FORMAT(1X,",          ,",",          ,",",          ,",",
     1 ,",",       ,")
    8 READ(IRD,100)(A(K,J),J=1,N)
      WRITE(IW,152)
  152 FORMAT(1X,"ENTER INPUT MATRIX")
      DO 9 K=1,N
      WRITE(IW,151)
    9 READ(IRD,100)(B(K,J),J=1,M)
  100 FORMAT(5F10.0)
C
C  WRITE THE DATA
C
      WRITE(IW,530)N,M
  530 FORMAT(1X,"ORDER OF SYSTEM = ",I2,"NUMBER OF INPUTS = ",I2)
      WRITE(IW,532)
  532 FORMAT(15X,"SYSTEM MATRIX")
      DO 371 K=1,N
  371 WRITE(IW,105)(A(K,J),J=1,N)
      WRITE(IW,537)
  537 FORMAT(15X,"INPUT MATRIX")
```

```
      DO 372 K=1,N
  372 WRITE(IW,105)(B(K,J),J=1,M)
  105 FORMAT(1X,5F14.6)
      WRITE(IW,147)
  147 FORMAT(1X,"SPECIFY NUMBER OF SUBSYSTEMS"/1X,"I2")
      READ(IRD,111)IS
      WRITE(IW,544)IS
  544 FORMAT(1X,"NUMBER OF SUBSYSTEMS = ",I2)
      WRITE(IW,149)
  149 FORMAT(1X,"SPECIFY ORDER OF EACH SUBSYSTEM"/1X,"5I2")
      READ(IRD,111)(II(K),K=1,IS)
      DO 800 K=1,IS
  800 WRITE(IW,5545)K,II(K)
 5545 FORMAT(1X,"ORDER OF SUBSYSTEM",I2,"=",I2)
C
C   START DECOMPOSITION
C
  215 CONTINUE
C
C   DECOMPOSE SYSTEM INTO SUBSYSTEMS
C
      CALL DECP(IS,M,N)
C
C    START STABILIZATION
C
      DO 10 K=1,IS
      L1=II(K)
      IP=(K-1)*IS+K
      DO 20 L=1,L1
      DO 20 J=1,L1
   20 AK(L,J)=A2(IP,L,J)
      DO 30 L=1,L1
   30 B3(L)=B2(K,L,K)
C
C    LOCATE POLES OF EACH SUBSYSTEM
C
      CALL PPL(L1,IW)
      DO 15 L=1,L1
   15 T1(L,K)=R1(L)
   10 CONTINUE
      WRITE(IW,500)
  500 FORMAT(1X,"DO YOU WANT TO CONTINUE,YES OR NO")
      READ(1,501)A10
  501 FORMAT(A2)
      IF(A10.NE.DD)GO TO 266
C
C   START ITERATION FOR ALFA PARAMETER
C
      WRITE(IW,600)
  600 FORMAT(1X,"SPECIFY INCREME   DELT"/1X,"F10.0")
      READ(1,200)DELT
  200 FORMAT(F10.0)
      ALF=1.
  211 CONTINUE
C
C   FORM VANDERMONDE MATRIX
C
      DO 112 K=1,IS
      L1=II(K)
```

```
      DO 112 J=1,L1
      SK(J,K)=T1(J,K)
  112 SK(J,K)=ALF*SK(J,K)
      DO 12 K=1,IS
      L1=II(K)
      DO 88 L=1,L1
      DO 88 J=1,L1
   88 Q(K,L,J)=SK(J,K)**(L-1)
   12 CONTINUE
C
C     FORM AGGREGATE MODEL
C
      CALL DECP(IS,M,N)
      CALL TRF(IS)
      CALL AGR(IS)
      WRITE(IW,531)
  531 FORMAT(1X,"AGGREGATION MATRIX")
      DO 281 K=1,IS
  281 WRITE(6,105)(Q1(K,J),J=1,IS)
      ALF=ALF+DELT
C
C    CHECK THE SIGN OF THE K-TH MINOR
C
      IZ=1
      DO 282 K=1,IS
      IF=K
      DO 283 I=1,IF
      DO 283 J=1,IF
      IP=(I-1)*K+J
  283 C(IP)=Q1(J,I)
      CALL MINV(C,IF,D,LL,MM)
      IZ=-IZ
      D=IZ*D
      WRITE(6,270)D
  270 FORMAT(1X,F14.6)
      IF(D)211,211,282
C
C     PRINT CORRESPONDING EIGENVALUES
C
  282 CONTINUE
      DO 11 K=1,IS
      L1=II(K)
       WRITE(IW,700)K
  700 FORMAT(1X,"EIGENVALUES OF SUBSYSTEM",I2)
      DO 18 J=1,L1
   18 WRITE(6,701)J,SK(J,K)
  701 FORMAT(1X,"LAMBDA(",I2,") = ",F15.6)
   11 CONTINUE
      GO TO 215
  266 CONTINUE
      END
      END$
```

```
PAGE  0001        FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE BIG1(R1,L,N)
C
C    PROGRAM TO CALCULATE THE LARGEST ELEMENT OF AN ARRAY
C
      DIMENSION R1(6)
      K=1
      J=K+1
   10 IF(R1(K).GE.R1(J))GO TO 8
      K=J
    8 J=J+1
      IF(J.LE.N)GO TO 10
      L=K
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00055     COMMON = 00000
```

```
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE TRAN1(Q1,AG,LI,LS)
C
C    PROGRAM TO TRANSPOZE A MATRIX
C
      DIMENSION Q1(5,5 ,AG(5,5)
      DO 8 K=1,LI
      DO 8 J=1,LS
    8 AG(J,K)=Q1(K,J)
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00056      COMMON = 00000
```

```
      SUBROUTINE SMAL1(R1,L,N)
C
C   PROGRAM TO CALCULATE THE SMALLEST ELEMENT OF AN ARRAY
C
      DIMENSION R1(6)
      K=1
      J=K+1
   10 IF(R1(K).LE.R1(J))GO TO 8
    9 K=J
    8 J=J+1
      IF(J.LE.N)GO TO 10
      L=K
      RETURN
      END
```

```
**  NO ERRORS**    PROGRAM = 00055     COMMON = 00000
```

```
      SUBROUTINE KBAR(A,N,Z,IZ,D,B)
C
C     PROGRAM TO CALCULATE GAIN VECTOR
C
      DIMENSION A(5,5),Z(1),D(6),B(5),Q(5,5),R(5),P1(25),LL(5),MM(5)
     1,C(5)
      CALL COEF1(A,N,D)
      DO 8 K=1,N
    8 Z(K)=D(K)-Z(K)
      DO 9 I=1,N
    9 Q(I,N)=B(I)
      L=N-1
      DO 99 K=1,L
      L1=N-K+1
      L2=N-K
      F=D(L1)
      DO 7 I=1,N
    7 B(I)=Q(I,N)
      CALL SCALU(B,N,F)
      DO 77 I=1,N
   77 R(I)=Q(I,L1)
      CALL VECPR(A,R,N,C)
      DO 10 I=1,N
   10 Q(I,L2)=C(I)+B(I)
   99 CONTINUE
      DO 200 K=1,N
      DO 200 J=1,N
      I=(K-1)*N+J
  200 P1(I)=Q(J,K)
      DO 105 K=1,N
  105 B(K)=Q(K,N)
      CALL MINV(P1,N,SD,LL,MM)
      IF(SD)5,4,5
    4 WRITE(6,102)
  102 FORMAT(1X,13HSINGULAR CASE)
    5 CONTINUE
      DO 201 K=1,N
      DO 201 J=1,N
      I=(K-1)*N+J
  201 Q(J,K)=P1(I)
      DO 91 K=1,N
      S=0.
      DO 92 J=1,N
   92 S=S+Z(J)*Q(J,K)
   91 D(K)=S
      RETURN
      END
```

** NO ERRORS** PROGRAM = 00545 COMMON = 00000

```
      SUBROUTINE DECP(IS,M,N)
C
C     PROGRAM TO DECOMPOSE SYSTEM INTO SUBSYSTEMS
C
      DIMENSION A(10,10),B(10,3),A2(9,10,10),B2(3,10,3),II(5)
      COMMON A,B,A2,B2,II
      IL=1
      L1=0
      DO 10 K=1,IS
      IP=1
      L11=II(K)
      L1=L1+L11
      L2=0
      DO 11 J=1,IS
      NN=(K-1)*IS+J
      L22=II(J)
      L2=L2+L22
      IR=0
      DO 12 JJ=IL,L1
      IR=IR+1
      IC=0
      DO 12 KK=IP,L2
      IC=IC+1
   12 A2(NN,IR,IC)=A(JJ,KK)
   11 IP=IP+L22
      IR=0
      DO 13 JJ=IL,L1
      IR=IR+1
      DO 13 KK=1,M
   13 B2(K,IR,KK)=B(JJ,KK)
      IL=IL+L11
   10 CONTINUE
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00187      COMMON = 02245
```

```
      SUBROUTINE PPL(N,IW)
C
C     POLE SHIFTING PROGRAM
C
      INTEGER DD
      INTEGER YES
      REAL K1
      DIMENSION A1(10,10),B1(10,3),A2(9,10,10),B2(3,10,3),II(5),A(5,5),
     1Q1(5,5),Q(3,5,5),I11(5,5),B(5),R1(6),R2(6),D(6),SK(6,10),COF(6)
     1,Z(8)
      COMMON A1,B1,A2,B2,II,A,Q1,Q,I11,B,R1,R2,D,SK
      DATA YES/2HYE/
      WRITE(IW,153)
  153 FORMAT(15X,"POLES OF THE SYSTEM")
      CALL ALAM(A,N,D,COF,R1,R2)
      DO 19 K=1,N
   19 WRITE(IW,100)K,R1(K),R2(K)
  100 FORMAT(1X,8HLAMBDA (,I1,4H)  =,F14.6,4H + J,F14.6)
      WRITE(IW,200)
  200 FORMAT(1X,"DO YOU WANT TO ALTER THE POLES ?,YES OR NO?")
      READ(1,330)DD
  330 FORMAT(A2)
      IF(YES.NE.DD)GO TO 340
      WRITE(IW,500)
  500 FORMAT(1X,"ENTER DESIRED EIGENVALUES")
      WRITE(IW,510)
  510 FORMAT(1X,"ASSIGN INPUT UNIT"/1X,"I2")
      READ(1,111)IRD
  111 FORMAT(I2)
      DO 209 JI=1,N
      WRITE(IW,501)
  501 FORMAT(1X,",             ,",",             ,")
  209 READ(IRD,208)R1(JI),R2(JI)
  208 FORMAT(2F14.6)
      CALL DISP(R1,R2,N,Z)
      CALL KBAR(A,N,Z,IZ,D,B)
      DO 8 I=1,N
      DO 8 J=1,N
    8 A(I,J)=A(I,J)+B(I)*D(J)
      WRITE(IW,919)
  919 FORMAT(20X,8HVECTOR K//)
      WRITE(IW,109)(D(K),K=1,N)
  109 FORMAT(1X,5F14.6)
  340 CONTINUE
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00434     COMMON = 02686
```

```
      SUBROUTINE AGR(IS)
C
C   PROGRAM TO FORM AN AGGREGATE MATRIX
C
      REAL KB
      DIMENSION A(10,10),B(10,3),A2(9,10,10),B2(3,10,3),II(5),A4(5,5)
     1,A3(5,5),Q(3,5,5),I11(5,5),B3(5),R1(6),R2(6),KB(6),SK(6,10),
     1COF(6),D(6),Q1(5,5)
      COMMON A,B,A2,B2,II,A4,A3,Q,I11,B3,R1,R2,KB,SK
      DO 5 K=1,IS
      L1=II(K)
      DO 5 J=1,IS
      L2=II(J)
      IT=(K-1)*IS+J
      IF(K.EQ.J)GO TO 13
      DO 8 L=1,L1
      DO 8 I=1,L2
    8 A4(I,L)=A2(IT,L,I)
      DO 9 L=1,L2
      DO 9 I=1,L2
      S=0.
      DO 10 IR=1,L1
   10 S=S+A4(I,IR)*A2(IT,IR,L)
    9 Q1(I,L)=S
      CALL ALAM(Q1,L2,D,COF,R2,KB)
      DO 11 I=1,L2
      IF(R2(I).LE.0.)GO TO 11
      R2(I)=SQRT(R2(I))
   11 CONTINUE
      CALL BIG1(R2,L,L2)
      GO TO 14
   13 CONTINUE
      DO 20 I=1,L1
   20 R2(I)=-SK(I,K)
      CALL SMAL1(R2,L,L1)
      R2(L)=-R2(L)
   14 A3(K,J)=R2(L)
    5 CONTINUE
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00383      COMMON = 02686
```

```
      SUBROUTINE TRF(IS)
C
C     PROGRAM TO TRANSFORM SUBSYSTEM BY SIMILARITY TRANSFORMATION
C
      DIMENSION A(10,10),B(10,3),A2(9,10,10),B2(3,10,3),II(5),A1(5,5)
     1,Q1(5,5),Q(3,5,5),I11(5),B3(5)
     1,C(25),LL(5),MM(5)
      COMMON A,B,A2,B2,II,A1,Q1,Q,I11,B3
      DO 25 K=1,IS
      L1=II(K)
      DO 26 J=1,L1
      DO 26 L=1,L1
      IP=(J-1)*L1+L
   26 C(IP)=Q(K,L,J)
      CALL MINV(C,L1,D,LL,MM)
      DO 27 J=1,L1
      DO 27 L=1,L1
      IP=(J-1)*L1+L
   27 Q1(L,J)=C(IP)
      DO 33 JJ=1,IS
      IT=(K-1)*IS+JJ
      L2=II(JJ)
      DO 28 J=1,L1
      DO 28 L=1,L2
      S=0.
      DO 29 IZ=1,L1
   29 S=S+Q1(J,IZ)*A2(IT,IZ,L)
   28 A1(J,L)=S
      DO 31 J=1,L1
      DO 31 L=1,L2
      S=0.
      DO 32 IZ=1,L2
   32 S=S+A1(J,IZ)*Q(JJ,IZ,L)
   31 A2(IT,J,L)=S
   33 CONTINUE
      DO 39 J=1,L1
      S=0.
      DO 34 L=1,L1
   34 S=S+Q1(J,L)*B2(K,L,K)
   39 B3(J)=S
      DO 20 J=1,L1
   20 B2(K,J,K)=B3(J)
   25 CONTINUE
      DO 72 K=1,IS
      L1=II(K)
      DO 73 J=1,L1
      DO 73 L=1,L1
      IP=(J-1)*L1+L
   73 C(IP)=Q(K,L,J)
      CALL MINV(C,L1,D,LL,MM)
      DO 74 J=1,L1
      DO 74 L=1,L1
      IP=(J-1)*L1+L
   74 Q(K,L,J)=C(IP)
   72 CONTINUE
      RETURN
```

```
0422        END

**  NO ERRORS**    PROGRAM = 00561      COMMON = 02510
```

```
      SUBROUTINE ALAM(A,N,D,COF,R1,R2)
C
C    PROGRAM TO CALCULATE EIGENVALUES OF GENERAL N BY N MATRIX
C
      DIMENSION A(5,5),D(6),COF(6),R1(6),R2(6)
      CALL COEF1(A,N,D)
      M1=N+1
      CALL POLRT(D,COF,N,R1,R2,IER)
      IF(IER.EQ.0)GO TO 10
      WRITE(6,100)IER
  100 FORMAT(1X,13HERROR CODE = ,I2)
   10 CONTINUE
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00064       COMMON = 00000
```

```
      SUBROUTINE UNIT1(R,N)
C
C   SUBROUTINE TO FORM UNIT MATRIX
C
      DIMENSION R(5,5)
      DO 9 K=1,N
      DO 9 J=1,N
    9 R(K,J)=0.
      DO 8 K=1,N
    8 R(K,K)=1.
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00072      COMMON = 00000
```

```
      SUBROUTINE COEF1(A,N,D)
C
C     SUBROUTINE TO CALCULATE COEFFITIENTS OF CHARACTERISTIC POLYNOMIAL
C   OF MATRIX A
C
      DIMENSION A(5,5),B(5,5),C(5,5),D(6)
      CALL UNIT1(B,N)
      DO 9 K=1,N
      CALL PROD1(A,B,N,N,N)
      CALL TRAC1(B,N,S)
      L=N-K+1
      D(L)=-(1./K)*S
      CALL UNIT1(C,N)
      D1=D(L)
      CALL SCM1(C,D1,N)
    9 CALL ADD1(C,B,N)
      L=N+1
      D(L)=1.
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00211     COMMON = 00000
```

```
      SUBROUTINE SCALU(B,N,F)
C
C    SUBROUTINE TO MULTIPLY MATRIX BY A SCALAR
C
      DIMENSION B(5)
      DO 8 K=1,N
    8 B(K)=F*B(K)
      RETURN
      END
```

** NO ERRORS** PROGRAM = 00041 COMMON = 00000

```
      SUBROUTINE VECPR(A,B,N,C)
C
C   PROGRAM TO MULTIPLY MATRIX BY A VECTOR
C
      DIMENSION A(5,5),B(5),C(5)
      DO 9 K=1,N
      S=0.
      DO 8 J=1,N
    8 S=S+A(K,J)*B(J)
    9 C(K)=S
      RETURN
      END

** NO ERRORS**    PROGRAM = 00075     COMMON = 00000
```

```
      SUBROUTINE DISP(R1,R2,N,Z)
C
C    FORM POLYNOMIAL FROM ITS ZEROS
C
      DIMENSION R1(1),R2(1),Z(1),X(8),Y(8)
      K=1
      Y(1)=1.
      Y(2)=0.
      IY=2
   20 IF(R2(K).EQ.0.)GO TO 10
      X(1)=R1(K)**2+R2(K)**2
      X(2)=-2.*R1(K)
      X(3)=1.
      IX=3
      K=K+1
       GO TO 11
   10 X(1)=-R1(K)
      X(2)=1.
      IX=2
   11 CALL PMPY(Z,IZ,X,IX,Y,IY)
      DO 8 L=1,IZ
    8 Y(L)=Z(L)
      IY=IZ
      K=K+1
      IF(K.LE.N)GO TO 20
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00230      COMMON = 00000
```

```
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      SUBROUTINE PROD1(A,B,N,I,M)
C
C    SUBROUTINE TO MULTIPLY TWO MATRICES
C
      DIMENSION A(5,5),B(5,5),C(5,5)
      S=0.
      DO 9 K=1,N
      DO 9 L=1,M
      S=0.
      DO 10 J=1,I
   10 S=S+A(K,J)*B(J,L)
    9 C(K,L)=S
      DO 8 K=1,N
      DO 8 L=1,M
    8 B(K,L)=C(K,L)
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00183      COMMON = 00000
```

```
      SUBROUTINE ADD1(A,B,N)
C
C    SUBROUTINE TO ADD TWO MATRICES
C
      DIMENSION A(5,5),B(5,5)
      DO 8 K=1,N
      DO 8 J=1,N
    8 B(K,J)=B(K,J)+A(K,J)
      RETURN
      END

** NO ERRORS**    PROGRAM = 00066      COMMON = 00000
```

```
      SUBROUTINE TRAC1(A,N,S)
C
C    SUBROUTINE TO CALCULATE TRACE OF A MATRIX
C
      DIMENSION A(5,5)
      S=0.
      DO 8 K=1,N
    8 S=S+A(K,K)
      RETURN
      END
```

** NO ERRORS** PROGRAM = 00044 COMMON = 00000

```
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE SCM1(A,D,N)
C
C     SUBROUTINE TO MULTIPLY MATRIX BY A SCALAR
C
      DIMENSION A(5,5)
      DO 9 K=1,N
      DO 9 J=1,N
    9 A(K,J)=D*A(K,J)
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00057      COMMON = 00000
```

```
0564        END$
```

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OF POOR QUALITY

A.2. Optimization Program

The only step that may introduce some computational complexities in the optimization scheme described in Section 4 is the solution of the matrix Riccati equation for the evaluation of the local controls. Despite the fact that this computation is performed at the subsystem level and hence involves matrices of small orders, simulation on a digital computer will invariably be necessary. Although many different methods for the solution of the Riccati equation exist in the literature, the particular method that is adopted here is the iterative technique due to Kleinman [19]. In addition to determining the symmetric positive-definite solution P of the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 ,$$

the program described here also computes the eigenvalues of the matrices P and $W = Q + PBR^{-1}B^T P$ that is necessary in the evaluation of the suboptimality index ε .

The simulation analysis was conducted on the HP 2100 digital computing system (32K memory) in FORTRAN language. In the following description, only the subroutines MINV, SIMQ and POLRT are to be supplied externally (from IBM Scientific Subroutine Package), while the rest are contained internally. Since the computation involves only the subsystems that result from a suitable decomposition of the overall system and hence are necessarily of small dimensions, the program is prepared to handle subsystems of dimension up to five.

DESCRIPTION OF THE EXTERNAL SUBROUTINES (From SSP)

Subroutine MINV

Purpose:

Invert a matrix.

Usage:

CALL MINV (A, N, D, L, M)

Description of parameters:

A - Input matrix, destroyed in computation and replaced by resultant inverse.

N - Order of matrix A .

D - Resultant determinant.

L - Work vector of length N .

M - Work vector of length N .

Remarks:

Matrix A must be a general (nonsingular) matrix.

Subroutines and function subprograms required: None.

Method:

The standard Gauss-Jordan method is used. The determinant is also calculated. A determinant with absolute value less than 10**(-20) indicates singularity.

Subroutine SIMQ

Purpose:

Obtain solution of a set of simultaneous linear equations $AX = b$.

Usage:

CALL SIMQ (A, B, N, KS)

Description of parameters:

A - Matrix of coefficients stored solummwise. These are destroyed in the computation. The size of matrix A is N by N .

B - Vector of original constants (length N). These are replaced by final solution values, vector X .

N - Number of equations and variables. N must be greater than 1.

KS - Output digit: 0 for a normal solution; 1 for a singular set of equations.

Remarks:

Matrix A must be general. If matrix is singular, solution values are meaningless.

Subroutines and function subprograms required: None.

Method:

Method of solution is by elimination using largest pivotal divisor.

Subroutine POLRT

Purpose:

Computes the real and complex roots of a real polynomial.

Usage:

CALL POLRT (XCOF, COF, M, ROOTR, ROOTI, IER)

Description of parameters:

XCOF - Vector of M+1 coefficients of the polynomial ordered from smallest to largest power.

COF - Working vector of length M+1 .

M - Order of polynomial.

ROOTR - Resultant vector of length M containing real roots of the polynomial.

ROOTI - Resultant vector of length M containing the corresponding imaginary roots of the polynomial.

IER - Error code where

IER = 0 No error

IER = 1 M less than one

IER = 2 M greater than 36

IER = 3 Unable to determine root with 500 iterations on 5 starting values.

IER = 4 High order coefficient is zero.

Remarks:

Limited to 36-th order polynomial or less. Floating point overflow may occur for high order polynomials but will not affect the accuracy of the results.

Subroutines and function subprograms required: None

Method:

Newton-Raphson iterative technique.

```
FTN4,L
      PROGRAM RICAT
C     SOLUTION OF ALGEBRAIC RICCATI EQUATION BY KLEINMAN ITERATION
      DIMENSION A(5,5),B(5,5),Q(5,5),R(5,5),SO(5,5),S(5,5),
     1BS(5,5),AM(5,5),TS(5,5),AUX(5,5),D(5,5),P(5,5),W(5,5),
     2RIN(5,5),SF(5,5),LV(5),MV(5),VR(25),TB(5,5),TA(5,5),
     3AUXA(5,5),AUXB(5,5),AUXC(5,5),AUXD(5,5),AUXE(5,5),AUXF(5,5),
     4AUXG(5,5),AUXH(5,5),VD(6),COF(6),R1(6),R2(6)
C     READ THE STATE DIMENSION N
      READ(5,1)N
    1 FORMAT(I5)
C     READ THE CONTROL DIMENSION M
      READ(5,1)M
C      READ THE SYSTEM MATRIX A
       READ(5,2)((A(I,J),J=1,N),I=1,N)
    2 FORMAT(8F10.5)
C     READ THE STATE-CONTROL MATRIX B
      READ(5,2)((B(I,J),J=1,M),I=1,N)
C     READ THE COST MATRICES Q AND R
      READ(5,2)((Q(I,J),J=1,N),I=1,N)
      READ(5,2)((R(I,J),J=1,M),I=1,M)
      WRITE(6,20)
   20 FORMAT(3X,"SOLUTION OF RICCATI EQUATION-KLEINMAN ITRATIVE METHOD"
     1,////)
      WRITE(6,21)
   21 FORMAT(5X,"SYSTEM MATRIX A",//)
      DO 22 I=1,N
   22 WRITE(6,23)(A(I,J),J=1,N)
   23 FORMAT(1H0,10(2X,E13.5))
      WRITE(6,24)
   24 FORMAT(///)
      WRITE(5,25)
   25 FORMAT(5X,"STATE-CONTROL MATRIX B",//)
      DO 25 I=1,N
   26 WRITE(5,23)(B(I,K),K=1,M)
      WRITE(6,24)
      WRITE(5,27)
   27 FORMAT(5X,"COST MATRICES Q AND R",//)
      WRITE(5,28)
   28 FORMAT(1H0,"Q MATRIX",//)
      DO 29 I=1,N
   29 WRITE(5,23)(Q(I,J),J=1,N)
      WRITE(6,30)
   30 FORMAT(//,1X,"R MATRIX",//)
      DO 31 I=1,M
   31 WRITE(6,23)(R(I,J),J=1,M)
      WRITE(5,24)
C     READ THE STARTING APPROXIMANT SO
      READ(5,2)((SO(I,J),J=1,N),I=1,M)
      DO 3 I=1,M
      DO 3 J=1,N
    3 S(I,J)=SO(I,J)
      CALL MAVE(R,M,VR)
      CALL MINV(VR,M,DET,LV,MV)
      CALL VEMA(VR,M,RIN)
C     FIRST ITERATION COMMENCES HERE
```

```
   32 WRITE(6,33)
   33 FORMAT(10X,"START OF THE FIRST ITERATION",///)
      WRITE(6,34)
   34 FORMAT(5X,"STARTING APPROXIMANT=S MATRIX",//)
      DO 35 I=1,M
   35 WRITE(6,23)(S(I,J),J=1,N)
      WRITE(6,24)
      DO 200 NVAR=1,50
       CALL MAMUL(B,S,BS,N,M,N)
      DO 4 I=1,N
      DO 4 J=1,N
    4 BS(I,J)=-BS(I,J)
C     THE MODIFIED SYSTEM MATRIX AM IS OBTAINED NOW
       CALL MASUM(A,BS,AM,N,N)
      WRITE(6,36)
   36 FORMAT(5X,"MODIFIED SYSTEM MATRIX--AM=A-B*S",//)
      DO 37 I=1,N
   37 WRITE(6,23)(AM(I,J),J=1,N)
      WRITE(6,24)
  141 CALL EIVAL(AM,N,VD,COF,R1,R2)
      WRITE(6,142)
  142 FORMAT(5X,"EIGENVALUES OF CLOSED LOOP MATRIX",//)
      DO 143 J=1,N
  143 WRITE(6,23)R1(J),R2(J)
      WRITE(6,24)
C     THE LYAPUNOV MATRIX EQUATION IS NOW FORMED
        CALL MATRN(S,TS,M,N)
      CALL MAMUL(TS,R,AUXG,N,M,M)
      CALL MAMUL(AUXG,S,AUXH,N,M,N)
      CALL MASUM(Q,AUXH,D,N,N)
C     THE LYAPUNOV MATRIX EQUATION IS NOW P*AM+TR(AM)*P=-D
         CALL LYAPU(D,AM,P,N)
    7 WRITE(6,8)
    8 FORMAT(5X,"P MATRIX-SOLUTION OF LYAPUNOV EQUATION",//)
      DO 9 I=1,N
    9 WRITE(6,10)(P(I,J),J=1,N)
   10 FORMAT(1H0,10(2X,E13.5))
      WRITE(6,24)
C     COMPUTE THE APPROXIMANT FOR THE NEXT ITERATION
      CALL MATRN(B,TB,N,M)
       CALL MAMUL(RIN,TB,AUX,M,M,N)
       CALL MAMUL(AUX,P,SF,M,N,N)
      CALL MATRN(A,TA,N,N)
      CALL MAMUL(TA,P,AUXA,N,N,N)
      CALL MAMUL(P,A,AUXB,N,N,N)
      CALL MAMUL(B,SF,AUXC,N,M,N)
      CALL MAMUL(P,AUXC,AUX,N,N,N)
C     COMPUTE THE W MATRIX , W=Q+P*B*RIN*TB*P
      CALL MASUM(Q,AUX,W,N,N)
      WRITE(6,131)
  131 FORMAT(5X,"W MATRIX   ---   W=Q+P*B*RIN*TB*P",//)
      DO 132 I=1,N
  132 WRITE(6,23)(W(I,J),J=1,N)
      WRITE(6,24)
      DO 112 I=1,N
      DO 112 J=1,N
```

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```
  112 AUX(I,J)=-AUX(I,J)
      CALL MASUM(AUX,AUXA,AUXF,N,N)
      CALL MASUM(AUXF,AUXB,AUXD,N,N)
      CALL MASUM(AUXD,Q,AUXE,N,N)
C     CHECKING THE SOLUTION
      WRITE(6,111)
  111 FORMAT(30X,"CHECKING THE SOLUTION",//)
  113 WRITE(6,114)
  114 FORMAT(5X,"VALUE OF RICCATI MATRIX",//)
      DO 115 I=1,N
  115 WRITE(6,23)(AUXE(I,J),J=1,N)
      WRITE(6,24)
C     MATRIX SF IS THE STARTING APPROXIMANT FOR NEXT ITERATION
C     THE STEADY VALUE OF SF IS THE FEEDBACK GAIN MATRIX FOR THE OPTIMAL
C     REGULATOR K=RIN*TB*P
      DO 6 I=1,M
      DO 6 J=1,N
    6 S(I,J)=SF(I,J)
C     ADDITIONAL COMPUTATIONS--CALCULATION OF EIGENVALUES OF
C     MATRICES W AND P--THIS PART MAY BE OMITTED WHEN THE SOLUTION
C     OF A RICCATI EQUATION ONLY IS DESIRED
      CALL EIVAL(P,N,VD,COF,R1,R2)
      WRITE(6,133)
  133 FORMAT(5X,"EIGENVALUES OF P",//)
      DO 134 J=1,N
  134 WRITE(6,23)R1(J),R2(J)
      WRITE(6,24)
      CALL EIVAL(W,N,VD,COF,R1,R2)
      WRITE(6,135)
  135 FORMAT(5X,"EIGENVALUES OF W",//)
      DO 136 I=1,N
  136 WRITE(6,23)R1(I),R2(I)
      WRITE(6,24)
      WRITE(6,38)
   38 FORMAT(10X,"NEXT ITERATION COMMENCES HERE",///)
      WRITE(6,106)
  106 FORMAT(5X,"MODIFIED APPROXIMANT-NEW S MATRIX",//)
      DO 39 I=1,M
   39 WRITE(6,23)(S(I,J),J=1,N)
      WRITE(6,24)
  200 CONTINUE
   99 STOP
      END

**  NO ERRORS**    PROGRAM = 02921      COMMON = 00000
```

```
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE MAMUL(F,G,H,N1,N2,N3)
      DIMENSION F(5,5),G(5,5),H(5,5)
      DO 51 I=1,N1
      DO 51 J=1,N3
      SUM=0.0
      DO 52 K=1,N2
   52 SUM=SUM+F(I,K)*G(K,J)
      H(I,J)=SUM
   51 CONTINUE
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00091      COMMON = 00000
```

```
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE MASUM(F,G,H,N1,N2)
      DIMENSION F(5,5),G(5,5),H(5,5)
      DO 61 I=1,N1
      DO 61 J=1,N2
      H(I,J)=F(I,J)+G(I,J)
   61 CONTINUE
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00067      COMMON = 00000
```



```
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE MATRN(F,G,N1,N2)
      DIMENSION F(5,5),G(5,5)
      DO 71 I=1,N1
      DO 71 J=1,N2
      G(J,I)=F(I,J)
   71 CONTINUE
      RETURN
      END

**  NO ERRORS**    PROGRAM = 00055      COMMON = 00000
```

PAGE 0001 FTN4 COMPILER: HP24177 (SEPT. 1974)

```
      SUBROUTINE LYAPU(G,A,H,N)
      DIMENSION G(5,5),A(5,5),H(5,5),P(15),L(5,5),U(15,15),P1(225)
      M=(N+1)*N/2
      CALL TRIN(G,P,N,M)
      CALL MATL(N,M,L)
      CALL LP(L,A,N,M,U)
      DO 10 I=1,M
   10 P(I)=-P(I)
      CALL MAVEC(U,M,P1)
      CALL SIMQ(P1,P,M,KS)
      IF(KS-1)11,12,11
   12 WRITE(6,100)
  100 FORMAT(1X,13HSINGULAR CASE)
   11 CALL ATRI(P,H,M,N)
      RETURN
      END
```

** NO ERRORS** PROGRAM = 01076 COMMON = 00000

PAGE 0001 FTN4 COMPILER: HP24177 (SEPT. 1974)

```
      SUBROUTINE TRIN(A,P,N,M)
      DIMENSION A(5,5),P(15)
      M=0
      DO 8 K=1,N
      L=K
      DO 8 I=L,N
      M=M+1
    8 P(M)=A(L,I)
      RETURN
      END
```

** NO ERRORS** PROGRAM = 00064 COMMON = 00000

PAGE 0001 FTN4 COMPILER: HP24177 (SEPT. 1974)

```
      SUBROUTINE ATRI(P,A,M,N)
      DIMENSION A(5,5),P(15)
      M=0
      DO 8 K=1,N
      L=K
      DO 8 I=L,N
      M=M+1
    8 A(L,I)=P(M)
      DO 9 J=1,N
      L=J
      DO 9 I=L,N
    9 A(I,J)=A(J,I)
      RETURN
      END
```

** NO ERRORS** PROGRAM = 00104 COMMON = 00000

PAGE 0001 FTN4 COMPILER: HP24177 (SEPT. 1974)

```
      SUBROUTINE MATL(N,M,L)
      DIMENSION L(5,5),L1(15)
      DO 8 K=1,M
    8 L1(K)=K
      M1=0
      DO 18 K=1,N
      LS=K
      DO 18 I=LS,N
      M1=M1+1
   18 L(LS,I)=L1(M1)
      DO 9 J=1,N
      LS=J
      DO 9 I=LS,N
    9 L(I,J)=L(J,I)
      RETURN
      END
```

** NO ERRORS** PROGRAM = 00128 COMMON = 00000

PAGE 0001 FTN4 COMPILER: HP24177 (SEPT. 1974)

```
      SUBROUTINE MAVEC(A,N,P1)
      DIMENSION A(15,15),P1(225)
      DO 6 K=1,N
      DO 8 J=1,N
      L=N*(K-1)+J
    8 P1(L)=A(J,K)
      RETURN
      END
```

** NO ERRORS** PROGRAM = 00061 COMMON = 00000

```
      PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE MAVE(A,N,P1)
      DIMENSION A(5,5),P1(25)
      DO 8 K=1,N
      DO 8 J=1,N
      L=N*(K-1)+J
    8 P1(L)=A(J,K)
      RETURN
      END

 ** NO ERRORS**    PROGRAM = 00061      COMMON = 00000

      PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE VEMA(P1,N,A)
      DIMENSION A(5,5),P1(25)
      DO 8 K=1,N
      DO 8 J=1,N
      L=N*(K-1)+J
    8 A(J,K)=P1(L)
      RETURN
      END

 ** NO ERRORS**    PROGRAM = 00062      COMMON = 00000

      PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

      SUBROUTINE LP(L,A,N,M,U)
      DIMENSION L(5,5),A(5,5),U(15,15)
      DO 11 I=1,M
      DO 11 J=1,M
   11 U(I,J)=0.
      DO 12 I=1,N
      DO 12 J=1,N
      DO 12 K=1,N
      II=L(I,K)
      IJ=L(J,K)
   12 U(II,IJ)=A(J,I)+U(II,IJ)
      DO 13 I=1,N
      DO 13 J=1,M
      IL=L(I,I)
   13 U(IL,J)=2.*U(IL,J)
      RETURN
      END

 ** NO ERRORS**    PROGRAM = 00181      COMMON = 00000
```

```
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

            SUBROUTINE EIVAL(A,N,D,COF,R1,R2)
            DIMENSION A(5,5),D(6),COF(6),R1(6),R2(6)
            CALL COEF1(A,N,D)
            M1=N+1
            CALL POLRT(D,COF,N,R1,R2,IER)
            IF(IER.EQ.0)GO TO 10
            WRITE(6,100)IER
        100 FORMAT(1X,13HERROR CODE = ,I2)
         10 CONTINUE
            RETURN
            END

**  NO ERRORS**    PROGRAM = 00064       COMMON = 00000

PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

            SUBROUTINE UNIT1(R,N)
            DIMENSION R(5,5)
            DO 9 K=1,N
            DO 9 J=1,N
          9 R(K,J)=0.
            DO 8 K=1,N
          8 R(K,K)=1.
            RETURN
            END

**  NO ERRORS**    PROGRAM = 00072       COMMON = 00000

PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

            SUBROUTINE COEF1(A,N,D)
            DIMENSION A(5,5),B(5,5),C(5,5),D(6)
            CALL UNIT1(B,N)
            DO 9 K=1,N
            CALL PROD1(A,B,N,N,N)
            CALL TRAC1(B,N,S)
            L=N-K+1
            D(L)=-(1./K)*S
            CALL UNIT1(C,N)
            D1=D(L)
            CALL SCM1(C,D1,N)
          9 CALL ADD1(C,B,N)
            L=N+1
            D(L)=1.
            RETURN
            END

**  NO ERRORS**    PROGRAM = 00211       COMMON = 00000
```

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PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

       SUBROUTINE SUM1(A,D,N)
       DIMENSION A 5,5)
       DO 9 K=1,N
       DO 9 J=1,N
     9 A(K,J)=D*A(K,J)
       RETURN
       END

** NO ERRORS**    PROGRAM = 00057      COMMON = 00000
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

       SUBROUTINE TRAC1(A,N,S)
       DIMENSION A(5,5)
       S=0.
       DO 8 K=1,N
     8 S=S+A(K,K)
       RETURN
       END

** NO ERRORS**    PROGRAM = 00044      COMMON = 00000

PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

       SUBROUTINE ADD1(A,B,N)
       DIMENSION A(5,5),B(5,5)
       DO 8 K=1,N
       DO 8 J=1,N
     8 B(K,J)=B(K,J)+A(K,J)
       RETURN
       END

** NO ERRORS**    PROGRAM = 00066      COMMON = 00000
PAGE  0001          FTN4 COMPILER: HP24177  (SEPT. 1974)

       SUBROUTINE PROD1(A,B,N,I,M)
       DIMENSION A(5,5),B(5,5),C(5,5)
       S=0.
       DO 9 K=1,N
       DO 9 L=1,M
       S=0.
       DO 10 J=1,I
    10 S=S+A(K,J)*B(J,L)
     9 C(K,L)=S
       DO 8 K=1,N
       DO 8 L=1,M
     8 B(K,L)=C(K,L)
       RETURN
       END

** NO ERRORS**    PROGRAM = 00183      COMMON = 00000

PAGE  0002  PROD1   FTN4 COMPILER: HP24177  (SEPT. 1974)

ENDS
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