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HIGHER ORDER ACCURATE PARTIAL IMPLICITIZATION: AN UNCONDITIONALLY STABLE FOURTH-ORDER-ACCURATE EXPLICIT NUMERICAL TECHNIQUE

Randolph A. Graves, Jr.

Langley Research Center
Hampton, Va. 23665

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Randolph A. Graves, Jr.

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Randolph A. Graves, Jr.
Langley Research Center

SUMMARY

An unconditionally stable fourth-order-accurate explicit numerical technique is derived, based on the method of partial implicitization. The Von Neumann stability analysis demonstrates the unconditional linear stability. The order of the truncation error is deduced from the Taylor series expansions of the linearized difference equations and is verified by numerical solutions to Burgers' equation. For comparison, results are also presented for Burgers' equation using a second-order-accurate partial-implicitization scheme, as well as the fourth-order scheme of Kreiss.

INTRODUCTION

A great deal of effort has been expended in developing numerical methods to solve fluid-dynamic problems, and much of the recent work is compiled in reference 1. A large part of that work centered on obtaining accurate solutions to the fluid-dynamic equations for transient processes. In many cases where only the steady-state solution is desired, the transient analyses have been applied since the introduced time dependency changes the boundary-value problem to an initial-value problem. The initial-value problem lends itself readily to solution by explicit methods which have highly desirable characteristics for vector processing computers (see ref. 2). The steady-state solutions obtained by marching the transient problem asymptotically to steady state can still be costly in terms of computer resources since the maximum marching step is generally limited by stability considerations. Some effort has gone into speeding up the transient phase, increasing stability, so that the steady state is reached more rapidly; reference 3 typifies this approach. Published in reference 4 was a partial-implicitization technique, an unconditionally stable second-order-accurate explicit scheme, which was shown in reference 2 to be well suited for use on vector processing computers. The unconditional stability of the partial-implicitization technique allows for more rapid calculation of the steady state.

The purpose of the present paper is to present a modification to the procedure of reference 4 which produces a fourth-order-accurate method with only small changes in the second-order-accurate scheme. Maintaining the same form of the solution as reference 2 insures the method will still have the desirable features for vector processing computers.

SYMBOLS

b	intercept
$\overline{E}$	average error (see eq. (20))
G	amplification factor
h	spatial step size, $\Delta\eta$
L	total number of finite-difference points
m	order of accuracy
R_c	cell Reynolds number, $U_0 h/\nu$
R,S	coefficients in scheme 2 finite-difference relations
r,s	coefficients in scheme 1 finite-difference relations
t	time
$\hat{t}$	transformed time
U	velocity
U_0	velocity variable, $U - \overline{U}$
$\overline{U}$	steady-state wave speed, $\frac{1}{2}$
V	amplitude factor
x	coordinate direction
α	parameter in scheme 3 (see eq. (23))

η transformed coordinate

θ phase angle

ν viscosity

Subscripts:

e exact

j finite-difference nodal point index

Superscript:

N time step index

The primes indicate differentiation with respect to the transformed coordinate η.

MATHEMATICAL DEVELOPMENT

Model Equation

Burgers' equation has been used by a number of authors (for example, refs. 5 to 7) as a model equation with which to test numerical techniques. The equation is

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = \nu\frac{\partial^2 U}{\partial x^2} \tag{1}$$

In this form, equation (1) represents a diffuse shock wave through the application of the following boundary conditions:

$$U(x,t) = 1 \qquad (x \to -\infty)$$

$$U(x,t) = 0 \qquad (x \to +\infty)$$

Since only the steady-state solution is desired, the following wave-oriented transformation is applied to equation (1):

$$\eta = x - \overline{U}t$$

$$\hat{t} = t$$

where $\overline{U}$ is the steady-state wave speed. Equation (1) now becomes

$$\frac{\partial U}{\partial \hat{t}} + U_o \frac{\partial U}{\partial \eta} = \nu \frac{\partial^2 U}{\partial \eta^2} \tag{2}$$

where $U_o = U - \overline{U}$. The boundary conditions under the transformations become

$$\left.\begin{aligned} U(\eta,\hat{t}) &= 1 \\ U(\eta,\hat{t}) &= 0 \end{aligned}\right\} \qquad \begin{aligned} &(\eta \to -\infty) \\ &(\eta \to +\infty) \end{aligned} \tag{3}$$

The exact steady-state solution to equation (2) subject to boundary conditions (3) is

$$U(\eta) = \frac{1}{2}\left(1 - \tanh \frac{\eta}{4\nu}\right) \tag{4}$$

Solutions to equation (4) for $\nu = 1/8$, $1/16$, and $1/24$ are given in figure 1 where the decrease in wave thickness with decreasing viscosity is readily apparent.

Partial Implicitization

Scheme 1 (second-order accurate).- The derivation of the second-order-accurate partial-implicitization technique as given in reference 4 will be repeated here in order to provide a base for the development of the fourth-order scheme. In reference 4, three-point equally spaced central finite-difference relations of second-order accuracy were used to express the first and second spatial derivatives of equation (2) at the points $j - 1$, j, and $j + 1$. These finite-difference relations are:

$$\left.\begin{aligned} \left(\frac{\partial U}{\partial \eta}\right)_{j-1} &= \frac{U_j - U_{j-2}}{2\,\Delta\eta} & \left(\frac{\partial^2 U}{\partial \eta^2}\right)_{j-1} &= \frac{U_j - 2U_{j-1} + U_{j-2}}{\Delta\eta^2} \\ \left(\frac{\partial U}{\partial \eta}\right)_{j} &= \frac{U_{j+1} - U_{j-1}}{2\,\Delta\eta} & \left(\frac{\partial^2 U}{\partial \eta^2}\right)_{j} &= \frac{U_{j+1} - 2U_j + U_{j-1}}{\Delta\eta^2} \\ \left(\frac{\partial U}{\partial \eta}\right)_{j+1} &= \frac{U_{j+2} - U_j}{2\,\Delta\eta} & \left(\frac{\partial^2 U}{\partial \eta^2}\right)_{j+1} &= \frac{U_{j+2} - 2U_{j+1} + U_j}{\Delta\eta^2} \end{aligned}\right\} \tag{5}$$

Using the finite-difference relations (5) and expressing the time derivative as a simple backward difference,

$$\frac{\partial U}{\partial \hat{t}} = \frac{U^{N+1} - U^{N}}{\Delta \hat{t}}$$

equation (2) can be written as

$$-\left(r_j + s\right)U_{j-1}^{N+1} + (1 + 2s)U_j^{N+1} - \left(s - r_j\right)U_{j+1}^{N+1} = U_j^N \tag{6}$$

where $r_j = \frac{U_{o,j}\,\Delta \hat{t}}{2\,\Delta\eta}$ and $s = \frac{\nu\,\Delta \hat{t}}{\Delta\eta^2}$. Equation (6) is written at points $j - 1$, j, and $j + 1$; however, the variables at points $j - 2$ and $j + 2$ are considered explicitly. (The explicit consideration of the variables at points $j - 2$ and $j + 2$ does produce an inconsistent technique; that is, the resulting partial-implicitization technique cannot be used on true transient problems. This is of no consequence herein where the interest is in formulating a method to rapidly achieve a steady-state solution.)

The system of three simultaneous equations is of the form

Point $j - 1$

$$(1 + 2s)U_{j-1}^{N+1} - \left(s - r_{j-1}\right)U_{j-1}^{N+1} = U_{j-1}^N + \left(r_{j-1} + s\right)U_{j-2}^N \tag{7a}$$

Point j

$$-\left(r_j + s\right)U_{j-1}^{N+1} + (1 + 2s)U_j^{N+1} - \left(s - r_j\right)U_{j+1}^{N+1} = U_j^N \tag{7b}$$

Point $j + 1$

$$-\left(r_{j+1} + s\right)U_j^{N+1} + (1 + 2s)U_{j+1}^{N+1} = U_{j+1}^N + \left(s - r_{j+1}\right)U_{j+2}^N \tag{7c}$$

In order to obtain the solution at point j, use Cramer's rule (see ref. 8) which gives an equation that will be used at all interior points in the finite-difference mesh except for the two points immediately adjacent to the boundaries. At these points, system (7) is solved to obtain equations for U_{j-1}^{N+1} and U_{j+1}^{N+1}. Note that the equations for U_{j-1}^{N+1} and U_{j+1}^{N+1}

are used only for the two points adjacent to the boundary. The solution using Cramer's rule is obtained from

$$U_j^{N+1} = \frac{\begin{vmatrix} (1+2s) & \overline{A} & 0 \\ -(r_j + s) & \overline{B} & -(s - r_j) \\ 0 & \overline{C} & (1+2s) \end{vmatrix}}{\begin{vmatrix} (1+2s) & -(s - r_{j-1}) & 0 \\ -(r_j + s) & (1+2s) & -(s - r_j) \\ 0 & -(r_{j+1} + s) & (1+2s) \end{vmatrix}} \tag{8}$$

where

$$\overline{A} = \left[U_{j-1}^N + (r_{j-1} + s)U_{j-2}^N\right]$$

$$\overline{B} = U_j^N$$

$$\overline{C} = \left[U_{j+1}^N + (s - r_{j+1})U_{j+2}^N\right]$$

The result is

$$U_j^{N+1} = \overline{D}\Big[(1+2s)U_j^N + (s - r_j)U_{j+1}^N + (s - r_j)(s - r_{j+1})U_{j+2}^N + (r_j + s)U_{j-1}^N + (r_j + s)(r_{j-1} + s)U_{j-2}^N\Big] \tag{9}$$

where

$$\overline{D} = \frac{1}{(1+2s)^2 - \left[(s - r_j)(r_{j+1} + s) + (s - r_{j-1})(r_j + s)\right]}$$

Equation (9) now involves five points from the previous time level only; hence, equation (9) is an explicit equation obtained from the partial implicitization of the difference form of the governing equation. In reference 4 this form was shown to be unconditionally stable for one dimension, and a similar form was shown to be unconditionally stable in the two-dimensional solutions of reference 2. In the present paper this partial-implicitization technique is referred to as scheme 1.

Scheme 2 (fourth-order accurate).- In the following development, five-point finite-difference relations are used to express the first and second derivatives of equation (2) at the points $j - 1$, j, and $j + 1$. The use of five-point relations produce an overall accuracy of fourth order. These finite-difference relations are:

$$\left.\begin{aligned}
\left(\frac{\partial U}{\partial \eta}\right)_{j-1} &= \frac{1}{12\,\Delta\eta}\left[-3U_{j-2} - 10U_{j-1} + 18U_j - 6U_{j+1} + U_{j+2}\right] - \frac{h^4}{20}U^V \\
\left(\frac{\partial U}{\partial \eta}\right)_{j} &= \frac{1}{12\,\Delta\eta}\left[U_{j-2} - 8U_{j-1} + 8U_{j+1} - U_{j+2}\right] + \frac{h^4}{30}U^V \\
\left(\frac{\partial U}{\partial \eta}\right)_{j+1} &= \frac{1}{12\,\Delta\eta}\left[-U_{j-2} + 6U_{j-1} - 18U_j + 10U_{j+1} + 3U_{j+2}\right] - \frac{h^4}{20}U^V \\
\left(\frac{\partial^2 U}{\partial \eta^2}\right)_{j-1} &= \frac{1}{12\,\Delta\eta^2}\left[11U_{j-2} - 20U_{j-1} + 6U_j + 4U_{j+1} - U_{j+2}\right] + \frac{h^3}{12}U^V \\
\left(\frac{\partial^2 U}{\partial \eta^2}\right)_{j} &= \frac{1}{12\,\Delta\eta^2}\left[-U_{j-2} + 16U_{j-1} - 30U_j + 16U_{j+1} - U_{j+2}\right] - \frac{h^4}{90}U^{VI} \\
\left(\frac{\partial^2 U}{\partial \eta^2}\right)_{j+1} &= \frac{1}{12\,\Delta\eta^2}\left[-U_{j-2} + 4U_{j-1} + 6U_j - 20U_{j+1} + 11U_{j+2}\right] - \frac{h^3}{12}U^V
\end{aligned}\right\} \qquad (10)$$

Following a procedure similar to that used in the scheme 1 development, the finite-difference relations (10) with the time derivative expressed as a simple backward difference $\frac{\partial U}{\partial \hat{t}} = \frac{U^{N+1} - U^N}{\Delta \hat{t}}$, equation (2) can be written at points $j - 1$, j, and $j + 1$. In scheme 1 the variables at points $j - 2$ and $j + 2$ were treated explicitly; similarly, the same procedure is followed here. However, it was found that the resulting technique

did not have unconditional stability; hence, some of the values of U at the points $j - 1$, j, and $j + 1$ had also to be treated explicitly in order to obtain unconditional stability. Again, as in scheme 1 the explicit consideration of the variables at the points $j - 2$, $j - 1$, $j + 1$, and $j + 2$ produces an inconsistent technique which is accurate only for the condition of steady state.

When $R_j = \dfrac{U_{o,j}\,\Delta\hat{t}}{12\,\Delta\eta}$ and $S = \dfrac{\nu\,\Delta\hat{t}}{12\,\Delta\eta^2}$, the system of three simultaneous equations is of the form

Point $j - 1$

$$(1 + 20S)U_{j-1}^{N+1} + \left(18R_{j-1} - 6S\right)U_j^{N+1} = U_{j-1}^N + R_{j-1}\left(3U_{j-2}^N - U_{j+2}^N\right) + S\left(11U_{j-2}^N - U_{j+2}^N\right) + U_{j+1}^N\left(6R_{j-1} + 4S\right) + 10R_{j-1}U_{j-1}^N \tag{11a}$$

Point j

$$-\left(8R_j + 16S\right)U_{j-1}^{N+1} + (1 + 30S)U_j^{N+1} + \left(8R_j - 16S\right)U_{j+1}^{N+1} = U_j^N + R_j\left(U_{j+2}^N - U_{j-2}^N\right) - S\left(U_{j+2} + U_{j-2}\right) \tag{11b}$$

Point $j + 1$

$$-\left(18R_{j+1} + 6S\right)U_j^{N+1} + (1 + 20S)U_{j+1}^{N+1} = U_{j+1}^N + R_{j+1}\left(U_{j-2}^N - 3U_{j+2}^N\right) + S\left(11U_{j+2}^N - U_{j-2}^N\right) - U_{j-1}^N\left(6R_{j+1} - 4S\right) - 10R_{j+1}U_{j+1}^N \tag{11c}$$

Use of Cramer's rule, as in the scheme 1 procedure, gives an equation which is used at all interior points in the finite-difference mesh, except for the two points immediately adjacent to the boundaries. At these points, system (11) is solved to obtain equations for U_{j-1}^{N+1} and U_{j+1}^{N+1}. Note that the equations for U_{j-1}^{N+1} and U_{j+1}^{N+1} are used only for the two points adjacent to the boundary. The solution using Cramer's rule is obtained from

$$U_j^{N+1} = \frac{\begin{vmatrix} (1 + 20S) & \overline{\overline{A}} & 0 \\ -(8R_j + 16S) & \overline{\overline{B}} & (8R_j - 16S) \\ 0 & \overline{\overline{C}} & (1 + 20S) \end{vmatrix}}{\begin{vmatrix} (1 + 20S) & (18R_{j-1} - 6S) & 0 \\ -(8R_j + 16S) & (1 + 30S) & (8R_j - 16S) \\ 0 & -(18R_{j+1} + 6S) & (1 + 20S) \end{vmatrix}} \qquad (12)$$

where

$$\overline{\overline{A}} = U_{j-1}^N + R_{j-1}\left(3U_{j-2}^N - U_{j+2}^N\right) + S\left(11U_{j-2}^N - U_{j+2}^N\right)$$

$$+ U_{j+1}^N\left(6R_{j-1} + 4S\right) + 10R_{j-1}U_{j-1}^N$$

$$\overline{\overline{B}} = U_j^N + R_j\left(U_{j+2}^N - U_{j-2}^N\right) - S\left(U_{j+2}^N + U_{j-2}^N\right)$$

$$\overline{\overline{C}} = U_{j+1}^N + R_{j+1}\left(U_{j-2}^N - 3U_{j+2}^N\right) + S\left(11U_{j+2}^N - U_{j-2}^N\right)$$

$$- U_{j-1}^N\left(6R_{j+1} - 4S\right) - 10R_{j+1}U_{j+1}^N$$

The result is

$$U_j^{N+1} = \overline{\overline{D}}\Big\{(1 + 20S)\Big[U_j^N + R_j\left(U_{j+2}^N - U_{j-2}^N\right) - S\left(U_{j+2}^N + U_{j-2}^N\right)\Big] - \left(8R_j - 16S\right)\Big[U_{j+1}^N$$

$$+ R_{j+1}\left(U_{j-2}^N - 3U_{j+2}^N\right) + S\left(11U_{j+2}^N - U_{j-2}^N\right) - U_{j-1}^N\left(6R_{j+1} - 4S\right) - 10R_{j+1}U_{j+1}^N\Big]$$

$$+ \left(8R_j + 16S\right)\Big[U_{j-1}^N + R_{j-1}\left(3U_{j-2}^N - U_{j+2}^N\right) + S\left(11U_{j-2}^N - U_{j+2}^N\right) + U_{j+1}^N$$

$$\times\left(6R_{j-1} + 4S\right) + 10R_{j-1}U_{j-1}^N\Big]\Big\} \qquad (13a)$$

where

$$\overline{\overline{D}} = \frac{1}{(1 + 20S)(1 + 30S) - \left(8R_j - 16S\right)\left(-18R_{j+1} - 6S\right) - \left(18R_{j-1} - 6S\right)\left(-8R_j - 16S\right)} \tag{13b}$$

Equation (13a) involves five points from the previous time level as in scheme 1; however, the difference relations in the present formulation involve all five of those points instead of only three. This technique is hereinafter referred to as scheme 2.

Stability

The Von Neumann stability analysis of equation (13a) can be performed by substituting Fourier components of the form

$$U_j^N = V^N e^{ij\theta} \tag{14}$$

The phase angle θ is a function of the wave number k_η and the spacing h; that is,

$$\theta = k_\eta \, \Delta\eta$$

The phase angle varies in the range

$$0 \leqq \theta \leqq \pi$$

Substituting the appropriate forms of equation (14) into equation (13a) and assuming R and S to be constant results in

$$V^{N+1} = V^N \overline{\overline{D}}\Big[\left(-2S + 280S^2 + 32R^2\right) \cos 2\theta + \left(32S + 128S^2 + 256R^2\right) \cos \theta + i(2R - 280RS) \sin 2\theta - i(16R + 64RS) \sin \theta + (1 + 20S)\Big] \tag{15a}$$

Looking first at equation (13b),

$$\overline{\overline{D}} = \frac{1}{1 + 50S + 408S^2 + 288R^2} \tag{15b}$$

Since $0 < S < +\infty$ and $-\infty < R < +\infty$, the denominator of equation (15b) is then always greater than zero and, thus, no singularity exists in equation (15a). Defining $G = \frac{V^{N+1}}{V^N}$ and noting that the Von Neumann condition for stability requires $|G| \leq 1$, equation (15a) becomes

$$|G| = \frac{=}{D}\Big\{(1 + 20S)^2 + 2(1 + 20S)\left(-2S + 280S^2 + 32R^2\right)\cos 2\theta + 2(1 + 20S)\left(32S + 128S^2 + 256R^2\right)\cos\theta + \left(-2S + 280S^2 + 32R^2\right)^2\cos^2 2\theta + 2\Big[\left(-2S + 280S^2 + 32R^2\right)\left(32S + 128S^2 + 256R^2\right)\cos\theta\cos 2\theta\Big] + \left(32S + 128S^2 + 256R^2\right)^2\cos^2\theta + (2R - 280RS)^2\sin^2 2\theta + (16R + 64RS)^2\sin^2\theta - 2\Big[(2R - 280RS)(16R + 64RS)\sin 2\theta\sin\theta\Big]\Big\}^{1/2} \qquad (16)$$

Since equation (16) is rather complicated, consider first the simpler limits as in reference 4. For $S = 0$ and $R \neq 0$, equation (16) becomes

$$|G| = \frac{\Big[\Big[1 + 4R^2 + 256R^2 + 64R^2\cos 2\theta + 512R^2\cos\theta + 1024R^4\cos^2 2\theta + 16384R^4\cos\theta\cos 2\theta + 65536R^4\cos^2\theta - 4R^2\cos^2 2\theta - 256R^2\cos^2\theta - 128R^2\cos\theta\left(1 - \cos^2\theta\right)\Big]^{1/2}\Big]}{1 + 288R^2} \qquad (17)$$

The maximum value of equation (17) occurs at $\theta = 0$, which results in

$$|G| = \frac{\left(1 + 576R^2 + 82944R^4\right)^{1/2}}{1 + 288R^2} = 1$$

Equation (16) satisfies the Von Neumann stability criterion in the limit $S = 0$ and $R \neq 0$. Similarly it can be shown that, when $S \neq 0$ and $R = 0$ (the maximum again occurs at $\theta = 0$),

$$|G| = \frac{\left(1 + 100S + 3316S^2 + 40800S^3 + 166464S^4\right)^{1/2}}{1 + 50S + 408S^2} = 1$$

Equation (16) satisfies the Von Neumann stability criterion in both sets of limits, implying equation (13a) is stable for all values of $\Delta\hat{t}$ in the limit conditions. It was determined by numerical evaluation that the maximum of equation (16) occurs at $\theta = 0$; thus,

$$|G| = \frac{\left[\left(1 + 100S + 3316S^2 + 40800S^3 + 166464S^4 + 576R^2 + 82944R^4 + 2880R^2S + 235000R^2S^2\right)^{1/2}\right]}{1 + 50S + 408S^2 + 288R^2}$$

and, finally,

$$|G| = 1$$

Equation (16) thus satisfies the stability criterion for all values of $\Delta\hat{t}$ and, hence, equation (13a) is an unconditionally stable solution for equation (2).

Accuracy

The formal accuracy of the linearized partial-implicitization technique, both schemes 1 and 2, is obtained from the steady-state form of equations (9) and (13a) by expanding each term of the partially implicit-difference scheme in a Taylor series expansion about U_j. The scheme 1 linearized steady-state counterpart to equation (9) is

$$U_j = \frac{(s - r)^2 U_{j+2} + (s + r)^2 U_{j-2}}{2s^2 + 2r^2}$$

Substituting the Taylor series expansions for U_{j+2} and U_{j-2}, making use of the linearized steady-state form of equation (2), and after algebraic simplification, the following result can be obtained:

$$U_o \frac{\partial U}{\partial \eta} - \nu \frac{\partial^2 U}{\partial \eta^2} = -\frac{h^2 U_o}{12} U_j''' - \frac{h^4 U_o}{180} U_j^V + O\left(h^6\right) \tag{18}$$

The scheme 2 linearized steady-state counterpart to equation (13a) is

$$U_j = \frac{\left[\begin{array}{l}\left(140S^2 + 140RS + 16R^2\right)U_{j-2} + \left(128R^2 + 32RS + 64S^2\right)U_{j-1} \\ + \left(128R^2 - 32RS + 64S^2\right)U_{j+1} + \left(140S^2 - 140RS + 16R^2\right)U_{j+2}\end{array}\right]}{408S^2 + 288R^2}$$

Following the previously outlined procedure for scheme 1 results in

$$U_O \frac{\partial U}{\partial \eta} - \nu \frac{\partial^2 U}{\partial \eta^2} = \frac{-34}{1170} h^4 U_O U_j^V + O\left(h^6\right) \tag{19}$$

Note that the third-order truncation errors from the second-derivative finite-difference relations cancel, thereby giving overall fourth-order accuracy for scheme 2. Thus, scheme 1 is of second-order accuracy whereas scheme 2 is fourth-order accurate. The increased accuracy of scheme 2 was accomplished with only a few changes to the scheme 1 procedure.

The formal accuracies given by equations (18) and (19) can be demonstrated numerically by using the following analysis. Defining the average error $\overline{E}$ as

$$\overline{E} = \frac{1}{L - 2} \sum_{j=2}^{L-1} \left| U_{e,j} - U_j \right|$$

and assuming the average error to be a function of the spacing $h = \Delta\eta$ results in the following relation:

$$\overline{E} = \frac{1}{L - 2} \sum_{j=2}^{L-1} \left| U_{e,j} - U_j \right| = bh^m \tag{20}$$

where m is the order of the error term and b is considered constant. Taking the logarithm of both sides of equation (20) gives

$$\ln \overline{E} = m \ln h + \ln b \tag{21}$$

The value of m is determined by the slope of the plot of $\ln \overline{E}$ as a function of $\ln h$.

NUMERICAL RESULTS

The solution to Burgers' equation was obtained over the interval $-5 \leqq \eta \leqq 5$ with $\nu = 1/8$, $1/16$, and $1/24$ for a range of values of h. For the partial-implicitization techniques, the time step $\Delta\hat{t} = 1000$ was used because this value exceeds the value of $\Delta\hat{t}$ necessary for convergence in the minimum number of iterations. For Kreiss' method (see ref. 9), $\Delta\hat{t}$ was taken to be a value which did not exceed the stability criterion. The solutions were considered to be converged when the following criteria were met:

$$\left| \frac{U_j^{N+1} - U_j^N}{U_j^{N+1}} \right| \leqq 1 \times 10^{-8}$$

The order of accuracy of schemes 1 and 2, as well as Kreiss' method included for comparison purposes, was obtained from figures 2 to 4 where, from the analysis of equation (21), the order m was taken as the slope of the corresponding curves. The partial-implicitization results indicate that the scheme 1 technique is of second-order accuracy and the scheme 2 technique is of fourth-order accuracy, thus verifying the formal accuracies given by equations (18) and (19). The fourth-order Kreiss technique is more accurate in terms of the average error than the fourth-order scheme 2; but, the explicit nature of scheme 2 offers a number of advantages, particularly for vector processing computers. In tables 1 to 3 representative results are presented which reflect the relative accuracies seen in figures 2 to 4 for scheme 1, scheme 2, and Kreiss' method. Both of the fourth-order accurate techniques give considerably better results, compared with the exact solution, than the second-order accurate method of scheme 1. This is particularly true for the smaller values of ν and larger values of h which imply larger cell Reynolds numbers.

From the solutions presented in tables 1 to 3, it can be observed that scheme 1 generally produces a steeper wave than does the exact solution, whereas scheme 2 produces a more diffuse wave. This observation leads to the possibility of combining the difference relations of schemes 1 and 2 so that a wave more closely resembling the exact solution could be calculated. Assuming new finite-difference relations of the linearly combined form results in a solution referred to as scheme 3. Since

$$\text{Scheme 3} = \alpha(\text{Scheme 1}) + (1 - \alpha)(\text{Scheme 2})$$

then the error terms in equations (18) and (19) involve α linearly. Next, setting α such that the derivative error terms through the fifth are zero results in the functional form

$$\alpha = \frac{R_c^2}{\beta_1 + \beta_2 R_c^2} \tag{22}$$

where β_1 and β_2 are constants to be determined from numerical solutions. By varying α over a range of values, the resulting average errors can be plotted as a function of α to determine that value of α which gives the minimum error for each R_c. Figure 5 demonstrates just such a procedure. By calculating a number of α values corresponding to the minimum errors, the constants in equation (22) can be evaluated. The best-fit results are given by $\beta_1 = 7$ and $\beta_2 = 0.88$ so that

$$\alpha = \frac{R_c^2}{7 + 0.88 R_c^2} \tag{23}$$

Equation (23) is plotted in figure 6 along with values of α determined from error minimums. Equation (23) was used to determine the appropriate values of α and the test cases were rerun; these results are given in the last column of tables 1 to 3. It is immediately apparent that scheme 3 is always more accurate than schemes 1 and 2, particularly at larger cell Reynolds numbers. At these large cell Reynolds numbers the combination of second-order-accurate and fourth-order-accurate schemes gives a fourth-order result, indicating that the lower order truncation errors may not be a true measure of what is occurring. The approximate fourth-order accuracy can be seen in figures 7 to 9.

The number of iterations to convergence varied from a minimum of 26 for scheme 1 with $\nu = 1/24$ and $h = 0.1042$ to a maximum of 1159 for scheme 3 with $\nu = 1/8$ and $h = 0.0521$. For the conditions of tables 1 to 3, scheme 1 consistently took the fewest iterations whereas scheme 3 took the most. However, a more meaningful relationship is to compare the iterations to convergence for a given error level. From table 3, scheme 3 took 114 iterations for $h = 0.2083$ to converge to an accuracy of 1.3860×10^{-4} and scheme 1 took 660 iterations for $h = 0.0521$ to converge to a comparable accuracy of 1.2621×10^{-4}. Scheme 2 took only 89 iterations for $h = 0.2083$ to converge to an accuracy of 2.2395×10^{-4} which is in the range of the just-mentioned two cases. Thus, since schemes 2 and 3 do not require six times as many operations per step as does scheme 1, it appears that for a given level of accuracy, the higher order techniques are more efficient.

CONCLUDING REMARKS

The second-order-accurate partial-implicitization numerical technique has been modified with little complication to achieve fourth-order accuracy yet retain the unconditionally stable explicit feature of the method. The resulting fourth-order method still retains the desirable features for application to vector processing computers. In addition, an observation was made that at coarse grid spacings a linear combination of the second- and fourth-order schemes produces a more accurate result.

Langley Research Center
National Aeronautics and Space Administration
Hampton, Va. 23665
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TABLE 1.- SOLUTIONS TO BURGERS' EQUATION FOR $\nu = 1/24$

$$\left[\frac{\partial U}{\partial \hat{t}} + U_0 \frac{\partial U}{\partial \eta} = \nu \frac{\partial^2 U}{\partial \eta^2}\right]$$

(a) $h = 0.2083$; $\alpha = 0.500$; $R_c = 2.5$

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	1.00000	1.00000	1.00013	0.99919	1.00000
-1.0417	1.00000	1.00000	1.00056	1.00162	1.00008
-.8333	.99995	.99985	1.00153	.99691	1.00057
-.6250	.99945	1.00137	1.00090	1.00590	1.00208
-.4167	.99331	.98780	.98113	.98180	.99884
-.2083	.92414	1.12500	.86568	.95055	.92383
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	8.8688×10^{-3}	3.1649×10^{-3}	2.1574×10^{-3}	3.9081×10^{-4}

(b) $h = 0.1042$; $\alpha = 0.186$; $R_c = 1.25$

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	1.00000	1.00000	1.00000	1.00000	1.00000
-1.0417	1.00000	1.00000	1.00000	1.00000	1.00000
-.8333	.99995	.99999	.99998	.99995	.99998
-.6250	.99945	.99985	.99965	.99944	.99969
-.4167	.99331	.99717	.99402	.99322	.99473
-.2083	.92414	.94944	.91887	.92415	.92413
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	1.6221×10^{-3}	3.6103×10^{-4}	5.5105×10^{-5}	1.7380×10^{-4}

(c) $h = 0.0521$; $\alpha = 0.0532$; $R_c = 0.6250$

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	1.00000	1.00000	1.00000	1.00000	1.00000
-1.0417	1.00000	1.00000	1.00000	1.00000	1.00000
-.8333	.99995	.99997	.99996	.99995	.99996
-.6250	.99945	.99957	.99946	.99945	.99946
-.4167	.99331	.99436	.99335	.99330	.99340
-.2083	.92414	.92999	.92391	.92415	.92423
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	3.8364×10^{-4}	2.2888×10^{-5}	3.3731×10^{-6}	1.4873×10^{-5}

TABLE 2.- SOLUTIONS TO BURGERS' EQUATION FOR $\nu = 1/16$

$$\left[\frac{\partial U}{\partial \hat{t}} + U_O \frac{\partial U}{\partial \eta} = \nu \frac{\partial^2 U}{\partial \eta^2}\right]$$

(a) h = 0.2083; α = 0.294; R_c = 1.667

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	0.99995	1.00000	1.00009	0.99990	1.00003
-1.0417	.99976	1.00000	1.00032	.99987	1.00018
-.8333	.99873	.99993	1.00018	.99840	1.00029
-.6250	.99331	.99925	.99442	.99381	.99701
-.4167	.96555	.99180	.95750	.96359	.96803
-.2083	.84113	.91667	.81639	.84835	.83507
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	4.6476×10^{-3}	1.5360×10^{-3}	4.3483×10^{-4}	6.3338×10^{-4}

(b) h = 0.1042; α = 0.0912; R_c = 0.8333

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	0.99995	0.99997	0.99996	0.99995	0.99996
-1.0417	.99976	.99986	.99978	.99976	.99978
-.8333	.99873	.99917	.99878	.99872	.99882
-.6250	.99331	.99515	.99344	.99329	.99360
-.4167	.96555	.97206	.96544	.96553	.96604
-.2083	.84113	.85503	.83925	.84137	.84059
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	1.0409×10^{-3}	1.1081×10^{-4}	1.6336×10^{-5}	6.8578×10^{-5}

(c) h = 0.0521; α = 0.0243; R_c = 0.4167

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	0.99995	0.99996	0.99995	0.99995	0.99995
-1.0417	.99976	.99979	.99976	.99976	.99976
-.8333	.99873	.99885	.99873	.99873	.99873
-.6250	.99331	.99378	.99331	.99331	.99333
-.4167	.96555	.96716	.96555	.96555	.96559
-.2083	.84113	.84441	.84102	.84114	.84110
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	2.5364×10^{-4}	6.6485×10^{-6}	9.8238×10^{-7}	4.4013×10^{-6}

TABLE 3.- SOLUTIONS TO BURGERS' EQUATION FOR $\nu = 1/8$

$$\left[\frac{\partial U}{\partial \hat{t}} + U_o \frac{\partial U}{\partial \eta} = \nu \frac{\partial^2 U}{\partial \eta^2}\right]$$

(a) $h = 0.2083$; $\alpha = 0.0912$; $R_c = 0.83333$

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	0.99331	0.99515	0.99344	0.99329	0.99360
-1.0417	.98473	.98830	.98484	.98471	.98517
-.8333	.96555	.97206	.96544	.96553	.96604
-.5250	.92414	.93474	.92333	.92418	.92434
-.4167	.84113	.85503	.83925	.84137	.84059
-.2083	.69706	.70833	.69506	.69748	.69613
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	2.1041×10^{-3}	2.2395×10^{-4}	3.3019×10^{-5}	1.3860×10^{-4}

(b) $h = 0.1042$; $\alpha = 0.0243$; $R_c = 0.4167$

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss' method	Scheme 3 (linearly combined form)
-1.2500	0.99331	0.99378	0.99331	0.99331	0.99333
-1.0417	.98473	.98564	.98474	.98473	.98476
-.8333	.96555	.96716	.96555	.96555	.96559
-.6250	.92414	.92670	.92411	.92414	.92416
-.4167	.84113	.84441	.84102	.84114	.84110
-.2083	.69706	.69967	.69694	.69708	.69700
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	5.0996×10^{-4}	1.3367×10^{-5}	1.9743×10^{-6}	8.8487×10^{-6}

(c) $h = 0.0521$; $\alpha = 0.00617$; $R_c = 0.2083$

η	U				
	Exact steady-state solution	Scheme 1 (second-order accurate)	Scheme 2 (fourth-order accurate)	Kreiss'	Scheme 3 (linearly combined form)
-1.2500	0.99331	0.99343	0.99331	0.99331	0.99331
-1.0417	.98473	.98496	.98473	.98473	.98473
-.8333	.96555	.96596	.96555	.96555	.96556
-.6250	.92414	.92478	.92414	.92414	.92414
-.4167	.84113	.84194	.84112	.84113	.84113
-.2083	.69706	.69770	.69705	.69706	.69706
0	.50000	.50000	.50000	.50000	.50000
$\overline{E}$	------	1.2621×10^{-4}	8.1648×10^{-7}	1.2053×10^{-7}	5.5739×10^{-7}

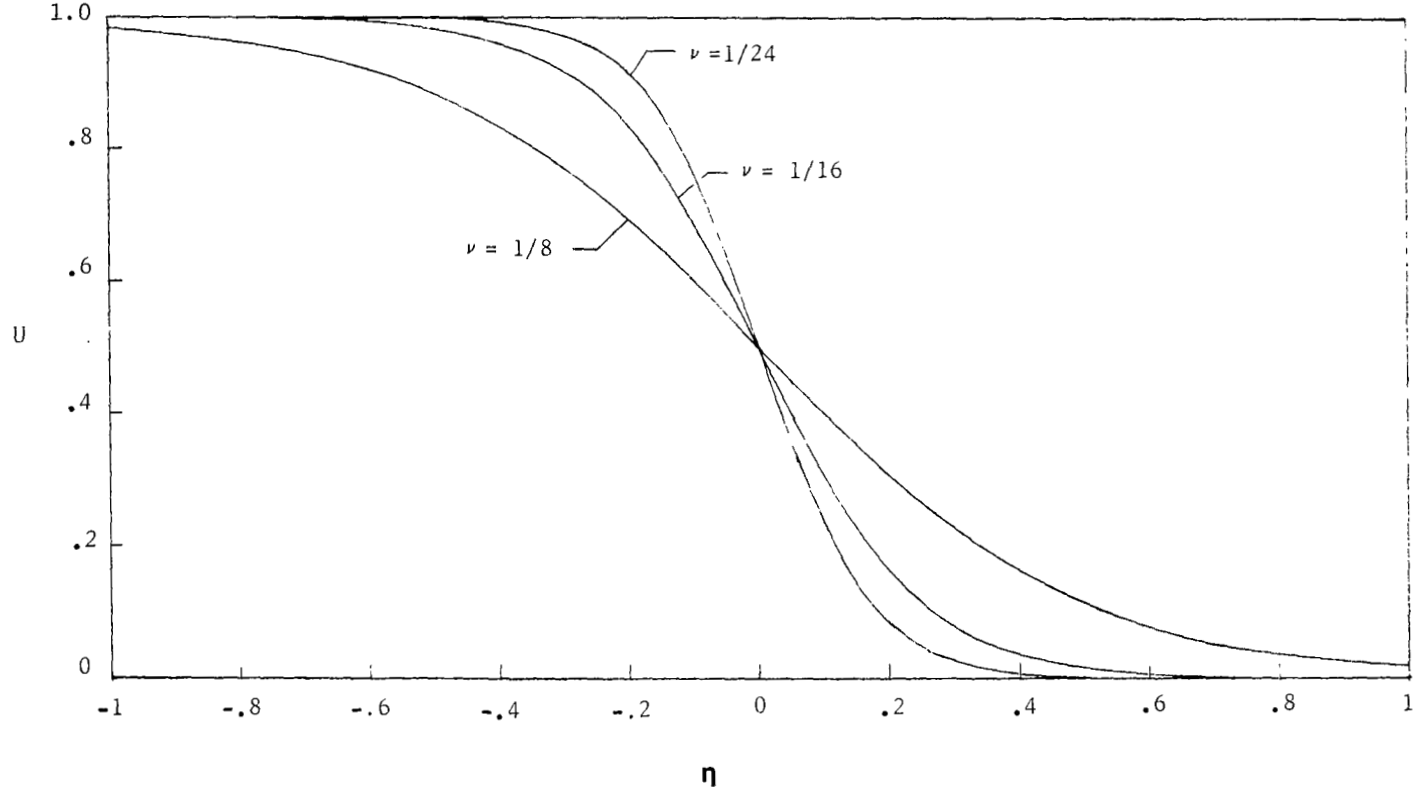

Figure 1.- Exact solutions to Burgers' equation.

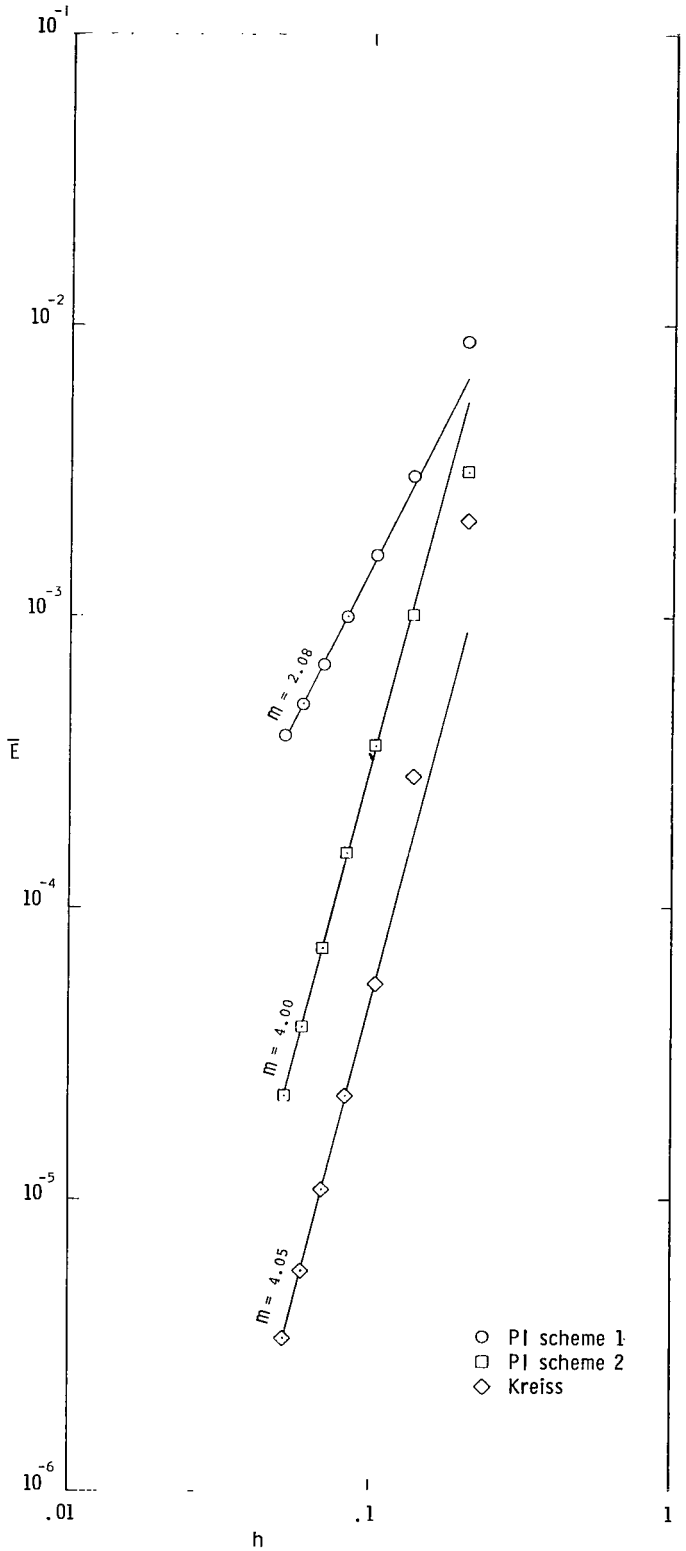

Figure 2.- Average errors from solutions to Burgers' equation for $\nu = 1/24$. (The abbreviation PI indicates partial implicitization.)

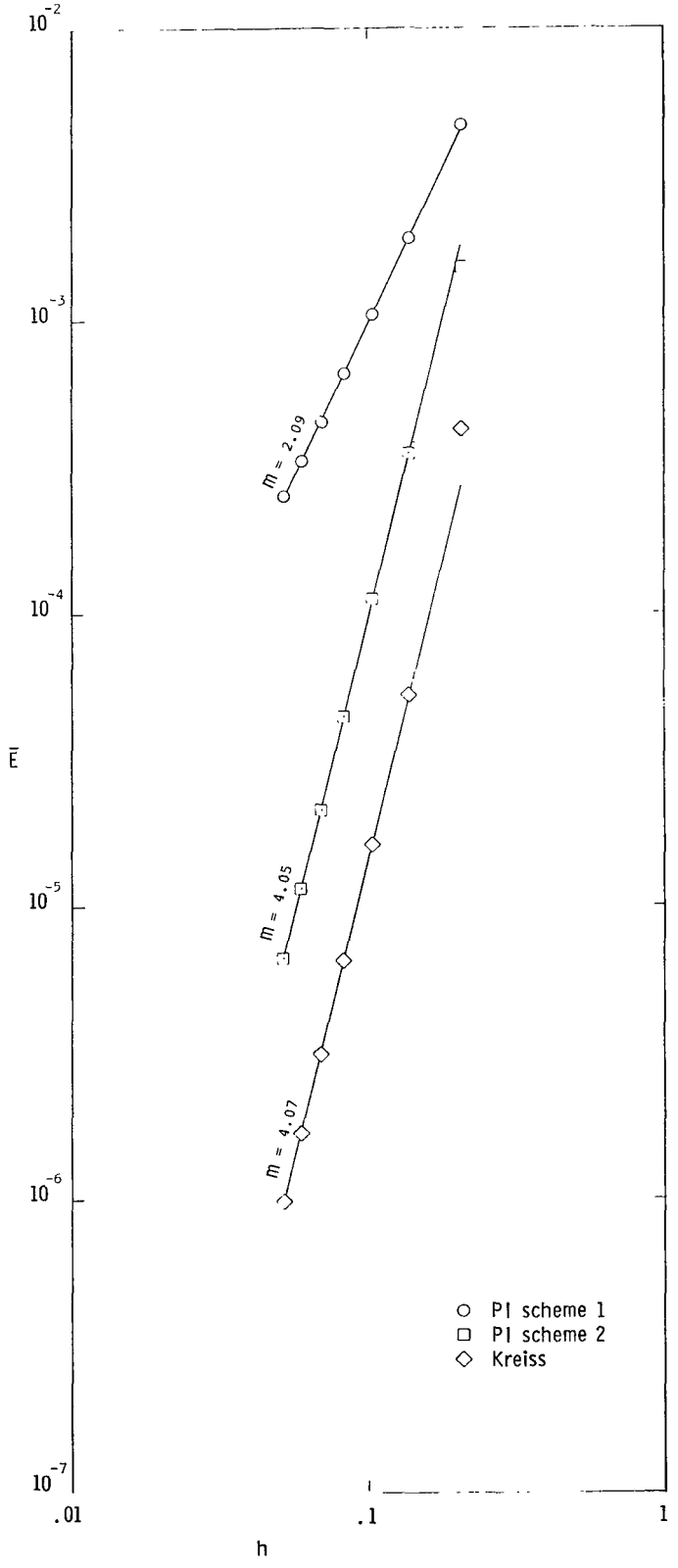

Figure 3.- Average errors from solutions to Burgers' equation for $\nu = 1/16$. (The abbreviation PI indicates partial implicitization.)

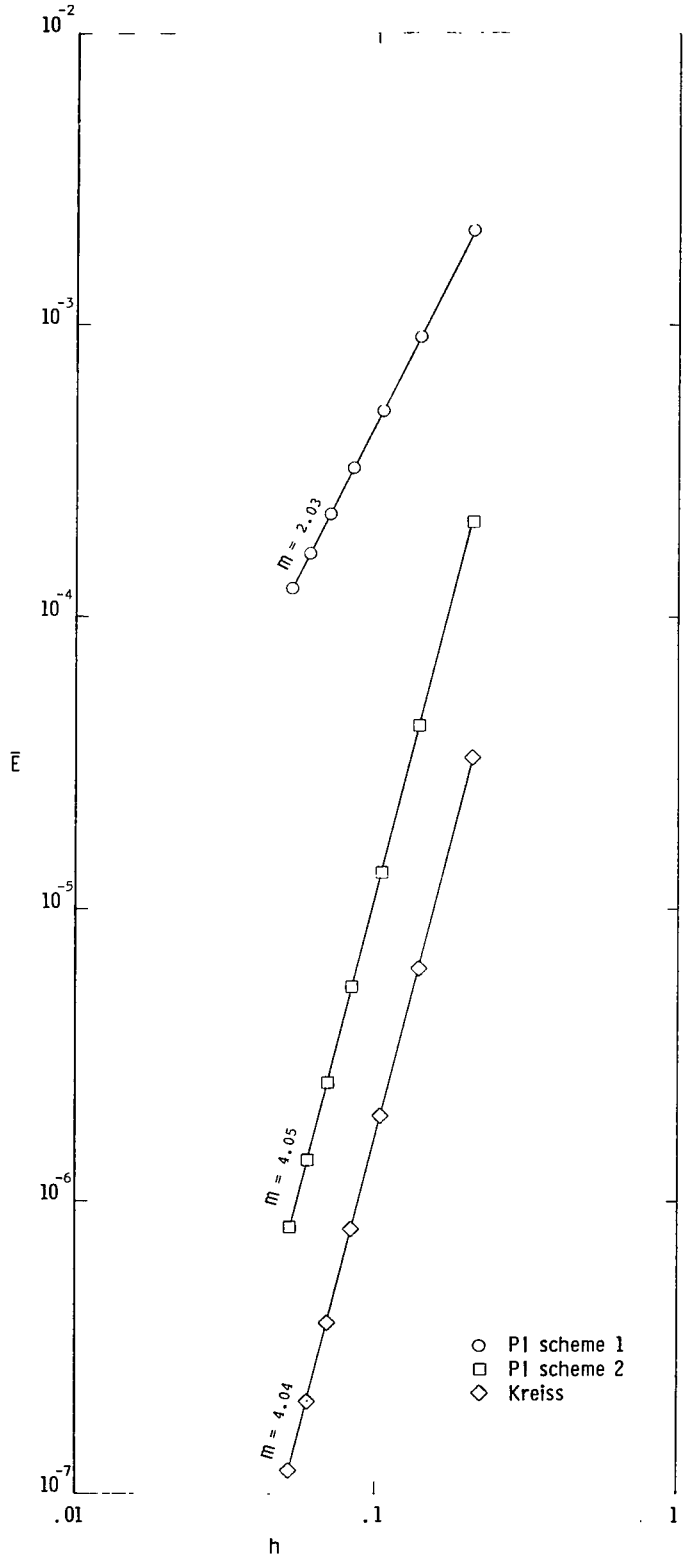

Figure 4.- Average errors from solutions to Burgers' equation for $\nu = 1/8$. (The abbreviation PI indicates partial implicitization.)

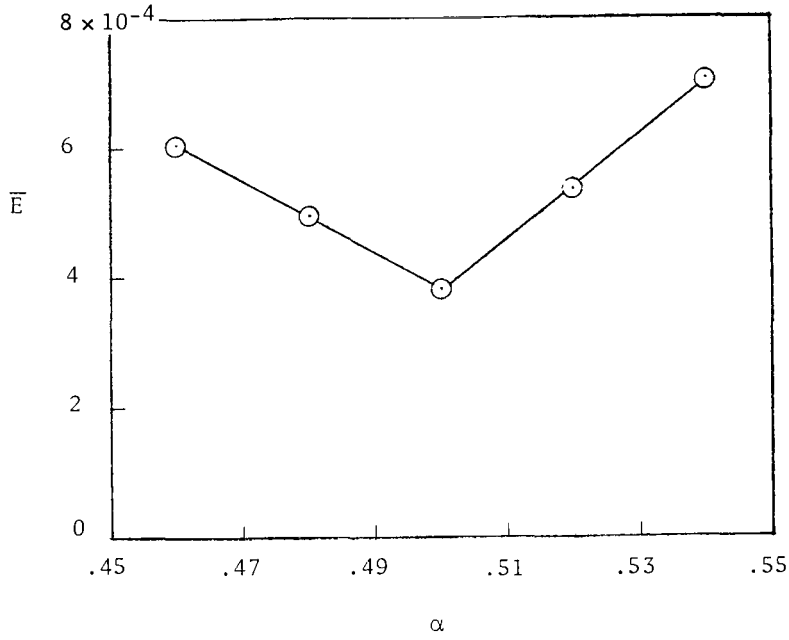

Figure 5.- Average error as a function of the free parameter from solutions to Burgers' equation. $\nu = 1/24$; $L = 49$; $R_c = 2.5$.

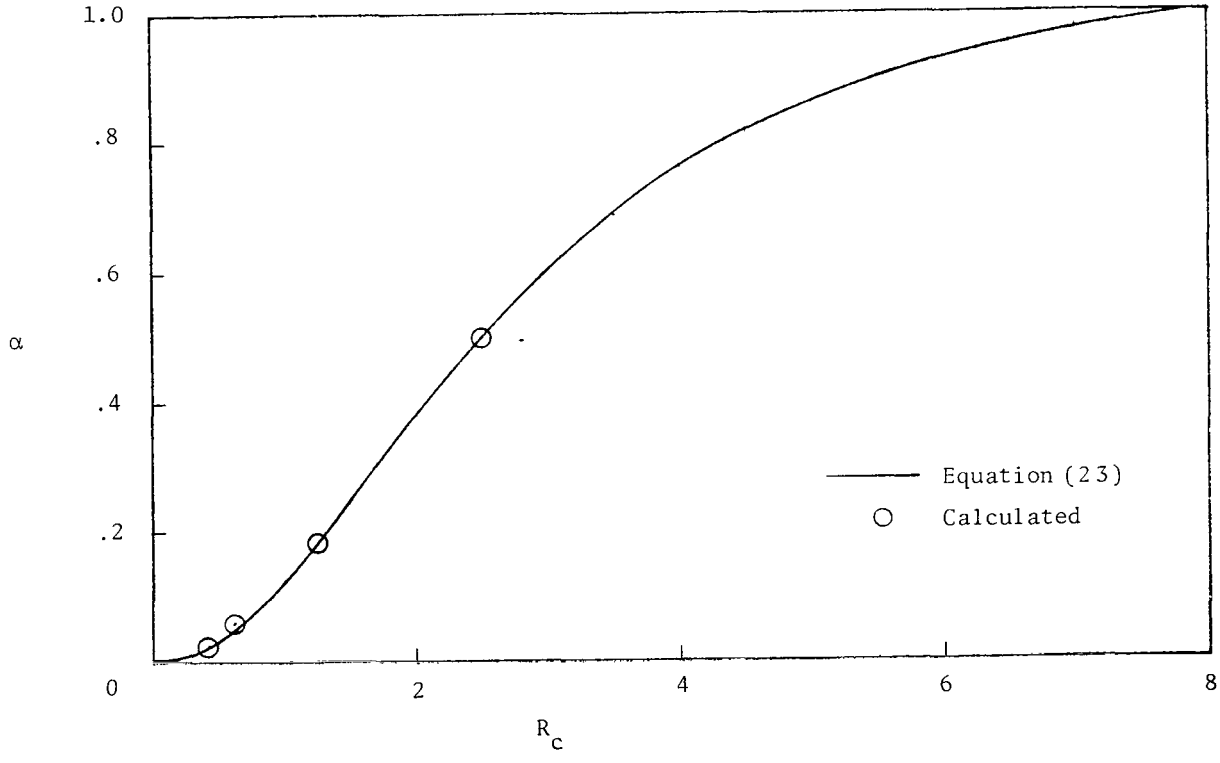

Figure 6.- Variation of the free parameter with cell Reynolds number.

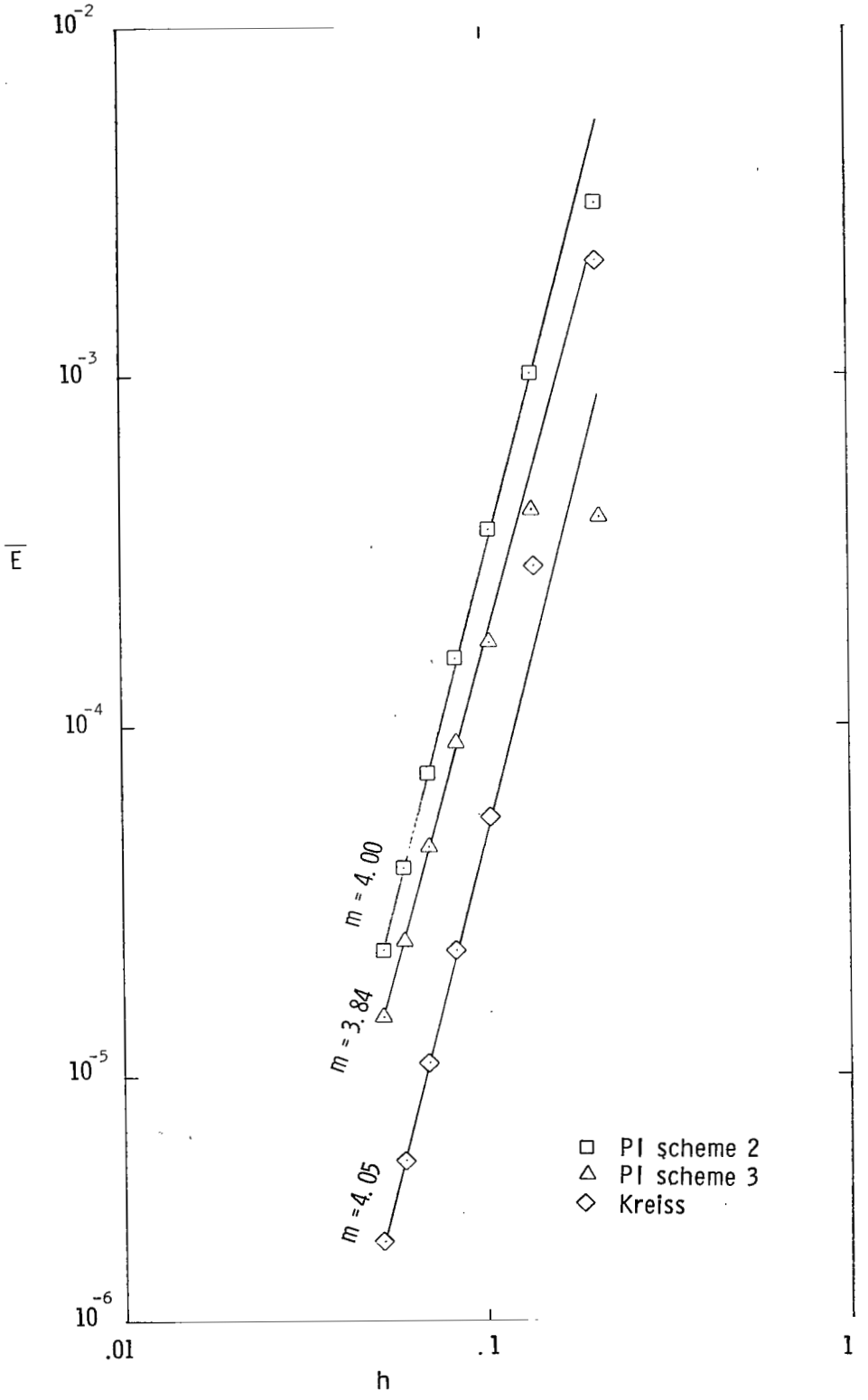

Figure 7.- Average errors from solutions to Burgers' equation for $\nu = 1/24$. (The abbreviation PI indicates partial implicitization.)

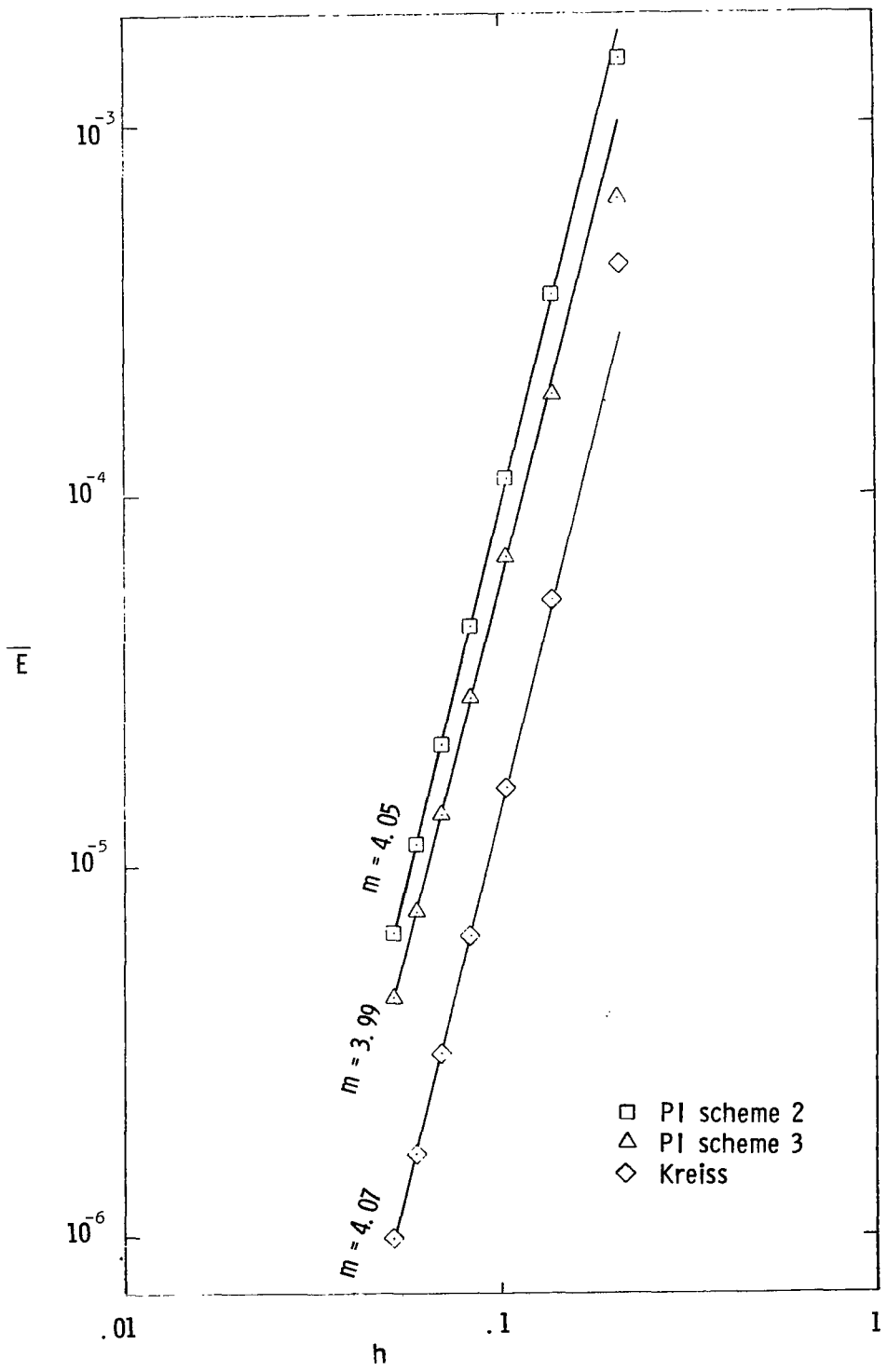

Figure 8.- Average errors from solutions to Burgers' equation for $\nu = 1/16$. (The abbreviation PI indicates partial implicitization.)

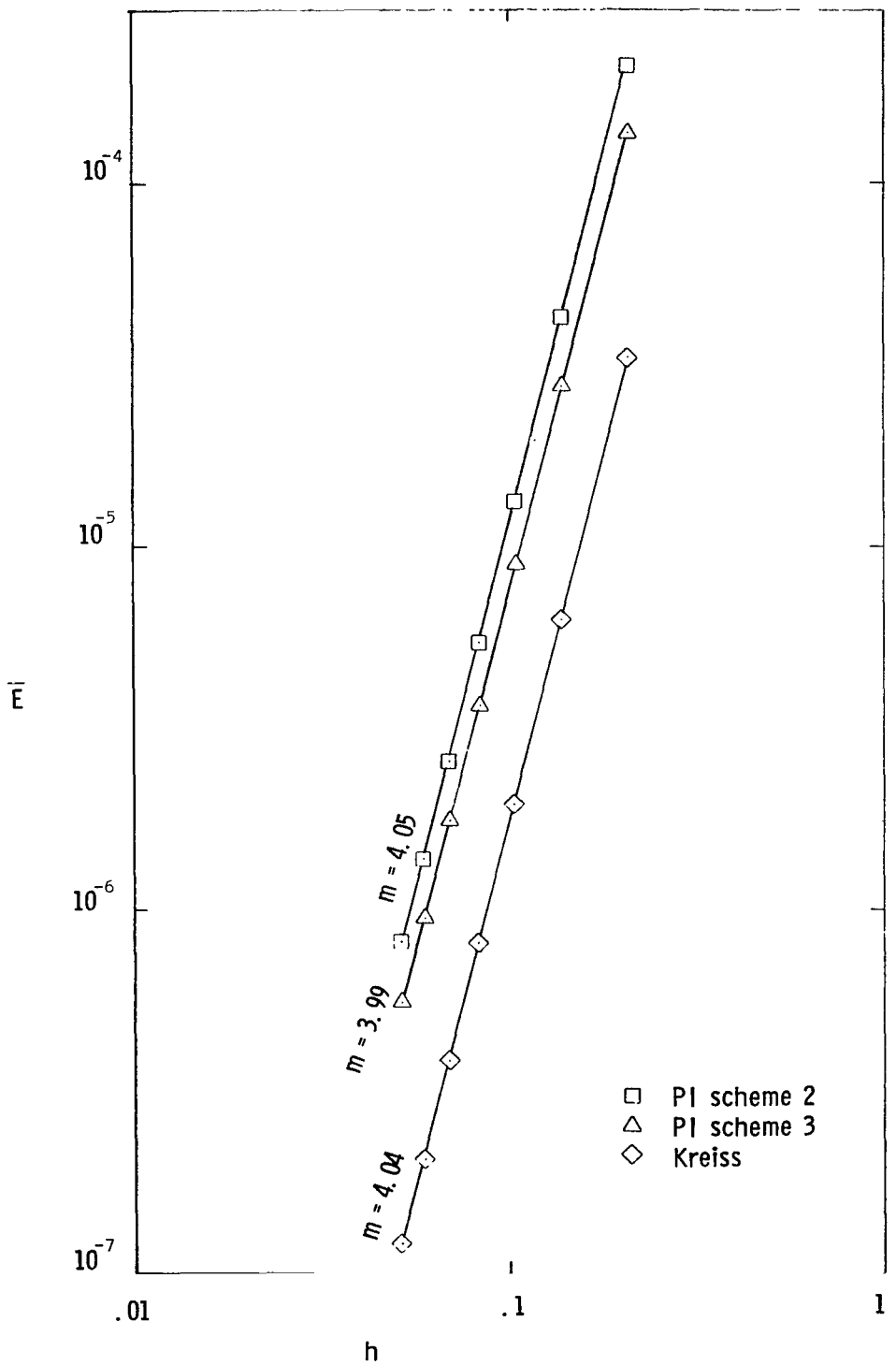

Figure 9.- Average errors from solutions to Burgers' equation for $\nu = 1/8$. (The abbreviation PI indicates partial implicitization.)

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