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Final Report:

USE OF THE WIGNER REPRESENTATION IN SCATTERING PROBLEMS

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ABSTRACT

The basic equations of quantum scattering are translated into the Wigner representation. This puts quantum mechanics in the form of a stochastic process in phase space. Instead of complex valued wave functions and transition matrices, one now works with real valued probability distributions and source functions--objects more responsive to physical intuition. Aside from writing out certain necessary basic expressions, the main purpose of this paper is to develop and stress the interpretive picture associated with this representation and to derive results used in applications published elsewhere. The quasi-classical guise assumed by the formalism lends itself particularly to approximations of complex multi-particle scattering problems. We hope to be laying the foundation for a systematic application of statistical approximations to such problems. The form of the integral equation for scattering as well as its multiple scattering expansion in this representation are derived. Since this formalism remains unchanged upon taking the classical limit, these results also constitute a general treatment of classical multi-particle collision theory. Quantum corrections to classical propogators are briefly discussed. The basic approximation used in the Monte-Carlo method is derived in a fashion which allows for future refinement and which includes bound state production. The close connection which must exist between inclusive production of a bound state and of its constituents is brought out in an especially graphic way by this formalism. In particular one can see how comparisons between such cross-sections yield direct physical

insight into relevant production mechanisms. Finally, as a simple illustration of some of the formalism we treat scattering by a bound two body system. Simple expressions for single and double scattering contributions to total and differential cross-sections as well as for all necessary shadow corrections thereto, are obtained. These are compared to previous results of Glauber and Goldberger.

I. INTRODUCTION

This paper developes the elements of what may be called the Wigner representation of quantum scattering. It is well known that quantum mechanics can be formulated entirely in terms of density operators and, linear operators (often called superoperators in this context) which act upon the densities.[1,2,9] The Wigner representation of densities puts them in the form of real functions of the coordinates and momenta of the system's particles. In other words, density operator matrix elements are functions defined in the classical phase space of the system. Schroedinger's equation, correspondingly transformed, is a linear equation for the density's time dependence. It looks like the equation governing a Markoffian Stochastic process in phase space. The integral form of Schroedinger's equation, the Lippmann-Schwinger equation in the usual Hilbert space representation of the theory is transformed into an integral form of the equation of continuity. The equation of continuity relates densities, currents and sources. Thus, upon translation into the Wigner representation, the fundamental equations of quantum scattering appear in terms of these physically graphic and transparent objects.

The main purpose of this paper is to develop and stress this picture and, in particular, to indicate how its graphic, intuitively appealing nature may be exploited in the formulation of approximations to complex multi-particle scattering problems.

The formalism encompasses both classical and quantum scattering. In fact the Wigner representation is especially constructed to go over smoothly to the classical limit. In taking this limit none of the equations change their form nor do any of the symbols appearing, in them change in interpretation or role. Schroedinger's equation goes over into the Liouville equation and its integral form similarly emerges as the classical integral equation of motion obtained here in its most general form.[30]

Quantum mechanics assumes a classical world in this formalism; in this lies the source of its intuitive power. Nevertheless, since we have here merely another representation of ordinary quantum mechanics, all of the latter's wave and interference properties must be hidden within. What has happened is that this translation of quantum mechanics necessarily gives birth to a non-classical stochastic process. Densities are not positive definite and in addition exhibit long range oscillatory behavior--just that needed to produce interference. Thus although the total structure of the formalism developed here and the various roles played by its elements (densities, sources, etc.) are isomorphic to classical theory, the particular functions needed to represent these elements will differ considerably in some respects from those of classical stochastic theory.

These qualitative remarks point to the circumstances in which it may or may not be useful to employ this representation. The property being observed should not depend critically on high order interference effects or

concomitantly on the existence of certain long range order or correlations. There also should be a significant advantage in being able to visualize the process quasi-classically. By this we mean that during the time in which the reaction takes place, at least some of its main participants can be usefully pictured as having simultaneously defined positions and momenta while moving along trajectories in classical phase space.

Generally these properties occur in conjunction and typify systems which have been difficult to treat by other methods. Thus one may hope that the approximations engendered by this representation will be complimentary to those previously established.

An example satisfying these conditions occurs in medium and high energy collisions between various projectiles and nuclei. Elastic scattering, most of it diffractive, is treatable by the Glauber approximation, the multiple scattering expansion, the optical model or, various combinations of these. This is a highly coherent process depending critically on high order interference effects. Inelastic scattering, in which the target is left in one or possibly a few well defined final states, may also be treated by such methods. This leaves about half of the possible reactions unaccounted for--the non-elastic collisions--in which many nuclear species may be produced in assorted multiplicities and momenta. Since these final states are so complex, one generally observes averages over them such as inclusive cross-sections and multiplicities. This washes out most high order interference effects. Since this formalism is written directly in terms of the density matrix, such averaging may be automatically done with utmost ease and elegance. In contrast, when calculating in the usual Hilbert space with pure final states, one must enumerate them, square and approximately sum over unobserved final states (e.g.

using closure). Such procedures are finessed by this formalism. Finally, perhaps the most important attribute this formalism brings to such a problem is its quasi-classical guise. These collisions involve many particles, high orders of multiple scattering, possibly collective motion, and large energy and momentum transfer. It is very difficult to see such phenomena in terms of waves in multidimensional position or momentum space. The minds eye is irresistably drawn to hydrodynamic, thermodynamic or transport theory type pictures in which joint average distribution functions in position and momentum play a central role. In contrast to previous ad hoc application of such concepts to such problems of scattering, they occur here as natural and systematic approximation procedures of ordinary quantum mechanics.

Another field which may be mentioned is chemical reactions. Here again complexity is most often the rule, statistical averaging the natural ally, the important steric properties of the compounds is easily representable and, an added positive factor, short wavelengths often make the problem semi-classical as well as quasi-classical. The Wigner representation is especially suited to the semi-classical limit.

There are other potential areas of application[11] which will not be discussed since those already mentioned should serve to illustrate practical reasons and requisite criteria for use of the representation. However, it should be finally noted that the Wigner representation has had a long history of use in, especially, transport theory.[12] What is being developed here is a version suited to collision phenomena on complex yet microscopic systems.

A synopsis of the remainder of this article follows.

Section II: Elementary definitions, examples and theorems connected to the Wigner representation are given. A Bra-Ket notation is introduced to designate elements of the vector space of density operators. The mapping of operators from their usual representation into the new one is given. Schroedinger's equation is then transformed into the new representation.

Section III: The semi-classical limit is very briefly discussed. The propagator for finite time translations on densities (not pure states) is obtained as a perturbation series in terms of its classical, not free, value. Thus, just as one thinks of waves propagating freely between successive scatterings by a potential in the usual representation of perturbation theory, here a particle moves on its classical trajectory between successive quantum jumps. These ideas lead to practical formulae for computing quantum corrections to semi-classical processes but this branch of the subject will not be developed in this paper.

Section IV: The most basic concepts used in scattering are introduced in the context of potential scattering. The time independent integral equation for scattering is derived. The effect of the potential on the incoming stationary flow of particles from the accelerator is expressed by a source distribution function in phase space. This gives the net rate of production of particles being produced by the potential with a certain position and momentum according to the heuristic, quasi-classical interpretation of the symbols in this formalism. Source functions are in many ways analogous to transition matrices but here occupy an even more central role in the theory. The relation between the solution to the integral equation and the observed scattering cross-section is established by noting that the spatial integral

of the local production rate of particles of a certain momentum is the total production rate which in turn is the incident flux times the differential cross-section.

Section V: The results of the previous section are extended to the multiparticle problem where, it is expected, the most effective use of the formalism is to be found. Formulae continue to be analogous to standard results of scattering theory in multiparticle Hilbert space.

Section VI: The multiple scattering expansion is derived. The result is quite similar to that of Watson. Here the role of the fully off shell transition matrix is taken by a 'jump operator'.

Section VII: The general properties of the jump operator for two particles interacting via a phenomenological potential are fully as mysterious as those of its counterpart, the transition matrix. Much energy has been expended but little physical intuition can illuminate the off-shell properties of the latter complex function. Two limiting forms of the jump operator are examined, the classical and the dilute. The relation between a particle moving on a classical trajectory and the concept of sources previously developed is explained. In the dilute limit, in which all particles are very far from each other compared to all other length parameters of the problem, we obtain the basis of the Monte Carlo method.[4] However, having obtained this not as an ad hoc procedure but as a well defined quantum mechanical approximation, one sees immediately how its range of application may be properly extended to include bound final states[5] and in addition, how systematic improvements in the approximation may be made for less dilute systems.

Section VIII: The multiple scattering formalism is developed further and then applied to find expressions for various simple but important

production cross-sections. In addition, the close relation between inclusive production of a bound state and of its unbound constituents is discussed. It is pointed out how comparison between these measurements can provide information on production mechanisms.

Section IX: As a final illustration of the formalism we look in some detail at the simplest problem to which it might usefully be applied; scattering of an elementary projectile by a two body bound state in the dilute limit. The lowest order term in the multiple scattering expansion gives immediately both the differential cross-section formula first derived by Goldberger for quasi-free scattering by a bound particle[6] as well as a corresponding expression for the total cross-section. The second order term gives an obvious shadow correction to Goldberger's formula (the effects of which may be seen quite dramatically in reducing the backward peak in proton-deuteron elastic scattering at intermediate energies),[13] a shadow correction to the total cross-section which is compared to that of Glauber[7] and, the expected double scattering contribution to the differential cross-section. All these results are straight-forward to obtain and transparent in physical meaning. They are written in terms of observed cross-sections. They are extendable in a straight forward fashion to more complex systems and also to non-dilute systems.[8]

II. ELEMENTS OF THE WIGNER REPRESENTATION

We begin with a resume of previous results concerning the Wigner representation. Details of certain standard derivations which may be found in the earlier literature are omitted. Initially, discussion will be confined to the case of one spinless particle. The extension to more than one spinless particle is straight-forward and will be used in later sections. Throughout the paper we ignore spin and indistinguishability of particles.

Let $\mathcal{O}$ be any operator on single particle Hilbert space. Its Wigner representative is defined to be[1]

$$\begin{aligned}\mathcal{O}_W(\underset{\sim}{x},\underset{\sim}{p}) &= \int d\underset{\sim}{y}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{y}} \\ &\quad\cdot\langle \underset{\sim}{x}-\tfrac{1}{2}\underset{\sim}{y}\,|\,\mathcal{O}\,|\,\underset{\sim}{x}+\tfrac{1}{2}\underset{\sim}{y}\rangle \\ &= \int d\underset{\sim}{q}\; e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} \\ &\quad\cdot\langle \underset{\sim}{p}+\tfrac{1}{2}\underset{\sim}{q}\,|\,\mathcal{O}\,|\,\underset{\sim}{p}-\tfrac{1}{2}\underset{\sim}{q}\rangle \qquad (1)\end{aligned}$$

We will give some simple examples. Let $\underset{\sim}{X}$ and $\underset{\sim}{P}$ denote the ordinary position and momentum operators and let $\mathcal{O}(\underset{\sim}{X},\underset{\sim}{P})$ denote some functions of these operators; then

$$(\mathcal{O}(\underset{\sim}{X}))_W(\underset{\sim}{x},\underset{\sim}{p}) = \mathcal{O}(\underset{\sim}{x})$$

and

$$(\mathcal{O}(\underset{\sim}{P}))_W(\underset{\sim}{x},\underset{\sim}{p}) = \mathcal{O}(\underset{\sim}{p}) \qquad (2)$$

When $\mathcal{O}$ depends on products of conjugate operators, then their ordering will have due effect. Another instructive example is the density operator of a normalized Gaussian wave packet state in Hilbert space. The Wigner representative of this operator is

$$\prod_{\alpha=1}^{3} 2 \exp\left[-\tfrac{1}{2}\left((x_\alpha - \bar{x}_\alpha)/\Delta x_\alpha\right)^2\right] \cdot \exp\left[-\tfrac{1}{2}\left((p_\alpha - \bar{p}_\alpha)/\Delta p_\alpha\right)^2\right], \qquad (3)$$

where $\bar{\underline{x}}$ and $\bar{\underline{p}}$ are averages and $\Delta x_\alpha \, \Delta p_\alpha = \frac{1}{2}$.

For any operator $\mathcal{O}$,

$$\langle \underline{x} | \mathcal{O} | \underline{x} \rangle = \int d\underline{p} \, (2\pi)^{-3} \, \mathcal{O}_w(\underline{x}, \underline{p}),$$

$$\langle \underline{p} | \mathcal{O} | \underline{p} \rangle = \int d\underline{x} \, (2\pi)^{-3} \, \mathcal{O}_w(\underline{x}, \underline{p}), \qquad (4)$$

$$|\mathcal{O}| \equiv \int d\underline{x} \, \langle \underline{x} | \mathcal{O} | \underline{x} \rangle = \int d\underline{x} \, d\underline{p} \, (2\pi)^{-3} \, \mathcal{O}_w(\underline{x}, \underline{p}). \qquad (5)$$

In particular, if $\mathcal{O}$ is a density operator ρ, then the normalization condition is

$$|\rho| = 1. \qquad (6)$$

It can be seen that ρ_w appears to be a sort of joint probability distribution function in phase space, the volume element of which is given by

$$d\underline{x} \, d\underline{p} \, (2\pi)^{-3} = d\underline{x} \, d\underline{p} \, h^{-3}$$

(since $h = 1$). We find it convenient to denote a point in phase space by a single symbol

$$\varphi = \{\underline{x}, \underline{p}\}$$

and correspondingly a volume element as

$$d\varphi = d\underset{\sim}{x}\; d\underset{\sim}{p}\; (2\pi)^{-3}$$

It is straight forward to show in general that for any two operators $\mathcal{O}$ and $\mathcal{O}'$,

$$|\mathcal{O}\,\mathcal{O}'| = \int d\varphi\; (\mathcal{O}_\omega(\varphi))^{*}\, \mathcal{O}'_\omega(\varphi) \qquad (7)$$

In particular, the expectation value of any $\mathcal{O}$ in the state described by ρ is given by

$$|\rho\;\mathcal{O}| = \int d\varphi\; \rho_\omega(\varphi)\; (\mathcal{O}_\omega(\varphi)) \qquad (8)$$

This is consistent with a probability density interpretation of ρ_ω. What is inconsistent with this interpretation however, at least in the ordinary sense, is the fact that although ρ_ω is necessarily real, it is not necessarily positive everywhere. Thus

$$\int_{\Delta\varphi} d\varphi\; \rho_\omega(\varphi)$$

cannot refer to the probability of a realizable measurement (measuring whether the particle is in $\Delta\varphi$) for arbitrary regions, $\Delta\varphi$, of phase space. This is consistent with the quantum mechanical fact that to every region of phase space there does not necessarily correspond a physically realizable measurement. In particular, $\Delta\varphi$ might violate the uncertainty principle limit. As we shall see, this fact does not appear to affect the heuristic value in thinking of ρ_ω as a joint probability distribution function in the quantum scattering formalism to be developed.

Equation 1 maps Hilbert space operators onto phase space functions. The inverse mapping may be performed with the help of the operator

$$R(\underset{\sim}{x}, \underset{\sim}{p}) = \int d\underset{\sim}{q}\; e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}}\; |\underset{\sim}{p} - \tfrac{1}{2}\underset{\sim}{q}\rangle\langle\underset{\sim}{p} + \tfrac{1}{2}\underset{\sim}{q}| \qquad (9)$$

which has Wigner representative

$$\left(R(\underset{\sim}{x}, \underset{\sim}{p})\right)_\omega (\underset{\sim}{x}', \underset{\sim}{p}') = (2\pi)^3\, \delta(\underset{\sim}{x} - \underset{\sim}{x}')\; \delta(\underset{\sim}{p} - \underset{\sim}{p}') \qquad (10)$$

or

$$(R(\varphi))_W(\varphi') = \delta(\varphi, \varphi') \tag{11}$$

Thus $R(\varphi)$ maps into the Dirac delta density in phase space. Although it is not a true quantum mechanical density operator, since it does not describe a physically realizable state of the system, it will be a convenient heuristic device to speak of it as a density.

We now note some useful properties of $R(\varphi)$;

$$\int d\varphi \; R(\varphi) = 1 \,, \tag{12}$$

$$| R(\varphi) | = 1 \,, \tag{13}$$

$$\mathcal{O}_W(\varphi) = | \mathcal{O} R(\varphi) | \,, \tag{14}$$

$$| R(\varphi) R(\varphi') | = \delta(\varphi, \varphi') \,. \tag{15}$$

The above equations imply

$$\mathcal{O} = \int d\varphi \; | \mathcal{O} R(\varphi) | R(\varphi) \tag{16}$$

In fact it is easy to see that this integral has the same Wigner representative as $\mathcal{O}$. It therefore must equal $\mathcal{O}$ as long as distinct Hilbert space operators map into necessarily distinct phase space functions. The latter statement is true because the Wigner representation is accomplished via a Fourier transformation on the matrix elements of $\mathcal{O}$.

Another bit of notation is now introduced which serves, among other things, to enhance the analogy between the usual quantum scattering formalism and the representation of it to be developed here. $\mathcal{O}_W$ is considered as an element of a linear vector space[2] and written in bra-ket notation as $|\mathcal{O}\rangle$.

$$|\vartheta\rangle \equiv \vartheta_W \tag{17}$$

$$\langle \vartheta_1 | \vartheta_2 \rangle = |\vartheta_1^\dagger \vartheta_2| = \int d\varphi \; \vartheta_{1W}^*(\varphi) \; \vartheta_{2W}(\varphi) \tag{18}$$

We also write

$$|R(\varphi)\rangle \equiv |\varphi\rangle, \tag{19}$$

from which one gets

$$\langle \varphi | \vartheta \rangle = \vartheta_W(\varphi) \tag{20}$$

$$\langle \varphi | \varphi' \rangle = \delta(\varphi, \varphi') \tag{21}$$

$$\int d\varphi \; |\varphi\rangle\langle\varphi| = 1 \tag{22}$$

Quantum operators appear in this formalism in two distinct ways: as kets, of the type just introduced, or as operators on kets. Generally, density operators are mapped into kets. Now let $\mathcal{O}$ and ρ be any Hilbert space operators (ρ generally being a density operator), we write

$$\mathcal{O}_L |\rho\rangle \equiv |\mathcal{O}\rho\rangle,$$

$$\mathcal{O}_R |\rho\rangle \equiv |\rho\mathcal{O}\rangle. \tag{23}$$

Previous authors have shown,[3] in effect, that the matrix elements of these operators are given by

$$\langle \varphi | \mathcal{O}_L | \varphi' \rangle = \vartheta_W(\varphi) \left(\exp \frac{\Lambda}{2i}\right) \langle \varphi | \varphi' \rangle$$

$$\langle \varphi | \mathcal{O}_R | \varphi' \rangle = \vartheta_W(\varphi) \left(\exp -\frac{\Lambda}{2i}\right) \langle \varphi | \varphi' \rangle. \tag{24}$$

where Λ is the Poisson Bracket operator,

$$f \Lambda g \equiv (\partial/\partial p f)(\partial/\partial x g) - (\partial/\partial x f)(\partial/\partial p g) \quad (25)$$

Alternatively we note

$$\langle \varphi | \mathcal{O}_L | \varphi' \rangle = \langle \varphi | \mathcal{O} R(\varphi') \rangle$$
$$= | R(\varphi) \mathcal{O} R(\varphi') | . \quad (26)$$

Similarly it is easy to see that

$$\langle \varphi | \mathcal{O}_R | \varphi' \rangle = | R(\varphi') \mathcal{O} R(\varphi) | . \quad (27)$$

Thus they are related by transposition.

$$\langle \varphi | \mathcal{O}_L^T | \varphi' \rangle \equiv \langle \varphi' | \mathcal{O}_L | \varphi \rangle$$
$$= \langle \varphi | \mathcal{O}_R | \varphi' \rangle \quad (28)$$

We are now prepared to consider the form which dynamics takes in the Wigner representation. Schroedinger's equation is

$$\partial/\partial t \, \rho = -i (H\rho - \rho H) \quad (29)$$

as written in the usual Hilbert space. In terms of Wigner representatives this becomes

$$\partial/\partial t \, | \rho \rangle = -i \, | H\rho - \rho H \rangle$$
$$= -i (H_L - H_R) | \rho \rangle$$
$$\equiv D | \rho \rangle \quad (30)$$

It is easy to see from the preceeding that the hermeticity of H implies

$$H_L^* = H_R \quad (31)$$

from which are obtained

$$D = D^{*} \quad . \tag{32}$$

In addition we note

$$D = - D^{T}$$

The antisymmetry of the time evolution operator is sufficient to prove conservation of probability and time reversal invariance.

By choosing to think of $\langle \varphi | \rho \rangle$ as a probability, this representation of Schroedinger's equation appears to describe the time evolution of a 'stochastic process', $|\rho(t)\rangle$. Schroedinger's equation becomes its differential equation. The procedure leading from this equation to the master equation and thence to regions of statistical physics has been under investigation for a long time. This paper is not an attempt to contribute to such lines of research. Rather, we wish to show that the analogy to statistical physics implied by the probability density interpretation of $\langle \varphi | \rho \rangle$ may be carried through to all the basic equations of quantum scattering theory. In fact quantum scattering theory falls most naturally into the mold of a stochastic process describable in classical phase space.

III. QUANTUM CORRECTIONS TO CLASSICAL DYNAMICS

The dynamical equation can be re-expressed as

$$\partial/\partial t \langle \varphi | \rho \rangle = H(\varphi) \left(-2 \sin \frac{\Lambda}{2}\right) \langle \varphi | \rho \rangle \tag{34}$$

where $H(\varphi)$ is just the classical Hamiltonian. The first term in the expansion of the sine gives the classical Liouville equation

$$\partial/\partial t \langle \varphi | \rho \rangle = - H(\varphi) \Lambda \langle \varphi | \rho \rangle \tag{35}$$

Let us call D_c the operator which generates classical time evolution;

$$\langle \varphi | D_c | \rho \rangle \equiv - H(\varphi) \Lambda \langle \varphi | \rho \rangle \tag{36}$$

It is clear that the solution to the classical problem is given formally by

$$e^{D_c t} | \varphi \rangle = | \varphi_c(\varphi, t) \rangle \tag{37}$$

where φ_c is the function which gives the coordinates of the phase point of the classical trajectory at time t such that $\varphi_c(\varphi, 0) = \varphi$.

Let $\Delta(t)$ be the operator which corrects the classical propagator,

$$e^{Dt} = \Delta(t)\, e^{D_c t} \tag{38}$$

It satisfies

$$\begin{aligned} \partial/\partial t\, \Delta(t) &= e^{Dt} (D - D_c) e^{-D_c t} \\ &= \Delta\, e^{D_c t} (D - D_c) e^{-D_c t} \end{aligned} \tag{39}$$

Its perturbative solution with initial condition $\Delta(0) = 1$, is

$$\begin{aligned} \Delta(t) = 1 &+ \int_0^t dt'\; e^{D_c t'} (D - D_c) e^{-D_c t'} \\ &+ \int_0^t dt' \int_0^{t'} dt''\; e^{D_c t''} (D - D_c) e^{-D_c t''} \\ &\cdot\, e^{D_c t'} (D - D_c) e^{-D_c t'} \\ &+ \cdots \end{aligned} \tag{40}$$

The first quantum correction to the matrix element of the time evolution operator is thus given by

$$\langle \varphi' | e^{Dt} - e^{D_c t} | \varphi \rangle = \int_0^t dt' \langle \varphi_c(\varphi', -t') | D - D_c | \varphi_c(\varphi, t-t') \rangle + \cdots \quad (41)$$

using equation (37).

The interpretation of this equation is that a particle may travel from φ to φ' in a time t via an infinite variety of paths in phase space. Each path is itself made up of some number of classical path segments, the first beginning at φ and the last ending at φ'. The particle moves classically along the first segment, performs a quantum jump through phase space to the second when it reaches the end of the first and, proceeds in like manner until the end. Each quantum jump, say that going from point φ_1 to φ_2, occurs with a probability $\langle \varphi_2 | D - D_c | \varphi_1 \rangle$. This probability may be negative. In fact, due to the antisymmetry of both D and D_c, the time-reversed jump always occurs with a sign opposite that of the time-direct jump.

It is interesting at this point to exhibit in more detail the jump probability operator's matrix elements for potential scattering. If $H = \underline{p}^2/2m + V(\underline{x})$ then it is easy to see that only V can contribute to $D - D_c$. We have

$$\begin{aligned} \langle \varphi | V_c | \varphi' \rangle &= | R(\varphi) V R(\varphi') | \\ &= \int d\underline{y}\, d\underline{y}'\, e^{i \underline{p} \cdot \underline{y} + i \underline{p}' \cdot \underline{y}'} \\ &\quad \cdot | | \underline{x} + \tfrac{1}{2}\underline{y} \rangle \langle \underline{x} - \tfrac{1}{2}\underline{y} | V | \underline{x}' + \tfrac{1}{2}\underline{y}' \rangle \langle \underline{x}' - \tfrac{1}{2}\underline{y}' | | \\ &= \delta(\underline{x} - \underline{x}') \int d\underline{y}\, e^{i(\underline{p} - \underline{p}') \cdot \underline{y}}\, V(\underline{x} - \tfrac{1}{2}\underline{y}) \end{aligned} \quad (42)$$

so that

$$\langle \varphi | D - D_c | \varphi' \rangle$$
$$= \delta(\underline{x} - \underline{x}') \left\{ \int d\underline{y}\, e^{i(\underline{p}-\underline{p}')\cdot\underline{y}} \right.$$
$$\left. \cdot\, i\left(V(\underline{x}+\tfrac{1}{2}\underline{y}) - V(\underline{x}-\tfrac{1}{2}\underline{y}) \right) - (2\pi)^3 \left(\partial/\partial \underline{x}\, V(\underline{x})\ \partial/\partial \underline{p}\, \delta(\underline{p}-\underline{p}') \right) \right\} \tag{43}$$

This shows that a particle may experience a finite jump in momentum but not in position, which is what is expected for a particle subject to a random force. One can show that the average of this random force is just the classical force.

IV. CURRENTS, SOURCES AND CROSS-SECTIONS IN POTENTIAL SCATTERING

In this section we wish to develop the essential ideas of this paper in the simplified context of potential scattering. These results will subsequently be extended to the general multiparticle, multichannel case. The plan here is first to derive the integral equation for scattering--the Lippmann-Schwinger equation in the Wigner representation. Next to relate the solution of this equation to the cross-section. Finally to interpret the symbols appearing in these results in a way which brings out clearly the analogy to statistical physics.

Let

$$D_0 = -i\left[\left(\underset{\sim}{P}^2/2M\right)_L - \left(\underset{\sim}{P}^2/2M\right)_R\right] \tag{44}$$

be the time evolution operator in the absence of interaction. Explicitly, using the expansion in terms of Poisson Bracket operators, it is

$$\langle\varphi|D_0 = -\underset{\sim}{v}\cdot\partial/\partial\underset{\sim}{x}\langle\varphi| \tag{45}$$

where

$$\underset{\sim}{v} = \underset{\sim}{p}/M$$

Let $|\rho(t)\rangle$ represent any density of freely moving particles,

$$\partial/\partial t\,|\rho(t)\rangle = D_0|\rho(t)\rangle \tag{46}$$

and consider a density $|\rho^{(+)}(t)\rangle$ satisfying

$$|\rho^{(+)}(t)\rangle = |\rho(t)\rangle + \int_{-\infty}^{t} dt'\, e^{D_0(t-t')}(D-D_0)\,|\rho^{(+)}(t')\rangle \tag{47}$$

then clearly

$$\partial/\partial t \, |\rho^{(+)}(t)\rangle = D \, |\rho^{(+)}(t)\rangle \tag{48}$$

and

$$\lim_{t \to -\infty} \left(|\rho^{(+)}(t)\rangle - |\rho(t)\rangle \right) = 0 . \tag{49}$$

Thus $|\rho^{(+)}(t)\rangle$ is that solution to the potential scattering problem with boundary condition that it approach a specific free density $|\rho(t)\rangle$ as $t \to -\infty$.

We next specialize to the case of stationary flow. If $\rho(t)$ is chosen to be time independent then $\rho^{(+)}(t)$ must also be stationary and the integral equation becomes

$$|\rho^{(+)}\rangle = |\rho\rangle + \left[\int_{-\infty}^{0} dt \, e^{-D_0 t} \right] (D - D_0) \, |\rho^{(+)}\rangle \tag{50}$$

The upper limit on the integral, being arbitary, is chosen for convenience at $t = 0$. Now since

$$e^{D_0 t} \, |\underset{\sim}{x}, \underset{\sim}{p}\rangle = |\underset{\sim}{x} + \underset{\sim}{v} t, \underset{\sim}{p}\rangle \tag{51}$$

then

$$\langle \underset{\sim}{x}, \underset{\sim}{p} | \, e^{-D_0 t} = \langle \underset{\sim}{x} + \underset{\sim}{v} t, \underset{\sim}{p} | \tag{52}$$

Therefore the integral equation can be rewritten as

$$\langle \underset{\sim}{x}, \underset{\sim}{p} | \rho^{(+)}\rangle = \langle \underset{\sim}{x}, \underset{\sim}{p} | \rho \rangle + \int_{-\infty}^{0} dt \, \langle \underset{\sim}{x} + \underset{\sim}{v} t, \underset{\sim}{p} | \, D - D_0 \, | \rho^{(+)}\rangle . \tag{53}$$

As we shall see shortly, this equation has a very simple interpretation. We note in passing that

$$\int_{-\infty}^{0} dt \, e^{-D_0 t} = \int_{-\infty}^{0} dt \, e^{-(D_0 - \eta) t}$$

$$= -(D_0 - \eta)^{-1} \tag{54}$$

in the limit $\eta \to 0$. Thus we can also write

$$|\rho^{(+)}\rangle = |\rho\rangle - (D_0 - \eta)^{-1}(D - D_0)|\rho^{(+)}\rangle \qquad (55)$$

To obtain an expression for the cross-section we first consider the standard incoming beam of one particle per unit volume. Thus we set

$$|\rho\rangle = |E\rangle = \int d\underline{x}\, |\underline{x}, \underline{p}_E\rangle \qquad (56)$$

The state represented by this density provides an entering flux $= v_E$. One denotes by $|E^{(+)}\rangle$ the corresponding solution to the stationary scattering problem. Then

$$\underline{J}_E(\underline{x}, \underline{p}) = \underline{v} \langle \underline{x}, \underline{p} | E^{(+)}\rangle \qquad (57)$$

is the current density of particles of momentum $\underline{p}$ set up by the interaction and

$$(2\pi)^{-3} d\underline{p} \int_{S'} \underline{J}_E \cdot d\underline{S}'$$

where S' is a surface totally enclosing the region of interaction, is the total outgoing flux of particles within the interval $d\underline{p}$ about $\underline{p}$. Let $\sigma(\underline{p}, \underline{p}_E) d\underline{p}$ be the probability that an incident particle of momentum $\underline{p}_E$ be scattered by the potential into the corresponding interval. Thus

$$\sigma(\underline{p}, \underline{p}_E)\, v_E = \int_{S'} \underline{J}_E \cdot d\underline{S}'\, (2\pi)^{-3} \qquad (58)$$

Applying Gauss's theorem we get

$$\sigma(\underline{p}, \underline{p}_E)\, v_E = (2\pi)^{-3} \int d\underline{x}\, S_E(\underline{x}, \underline{p}) \qquad (59)$$

where the integral extends over all space and the source function S_E is defined by

$$\begin{aligned} S_E(\underline{x}, \underline{p}) &= \partial/\partial \underline{x} \cdot \underline{J}_E(\underline{x}, \underline{p}) \\ &= \underline{v} \cdot \partial/\partial \underline{x} \langle \underline{x}, \underline{p} | E^{(+)}\rangle \\ &= -\langle \underline{x}, \underline{p} | D_0 | E^{(+)}\rangle \\ &= \langle \underline{x}, \underline{p} | D - D_0 | E^{(+)}\rangle \end{aligned} \qquad (60)$$

The equation of continuity when applied to a stationary distribution shows that the divergence of the current, the source function as defined here, does indeed give the local rate of production of particles within the element $d\varphi = dx\, dp\, (2\pi)^{-3}$. Our equation for the cross-section therefore merely states that the total rate of production of particles in interval dp, is the integral over all space of the local production rate. It is easy to see that the usual differential cross-section is given by

$$\sigma(p, p_E) = |p|^{-2}\, \delta(|p| - |p_E|)\, d\sigma/d\Omega \qquad (61)$$

The equation for the source function in terms of the density function can be rewritten as

$$S_E(\varphi) = \int d\varphi' \langle\varphi|D - D_0|\varphi'\rangle\langle\varphi'|E^{(+)}\rangle \qquad (62)$$

Now $\langle\varphi|D|\varphi'\rangle$ is the total probability per unit time that a particle at φ' will jump to φ, while $\langle\varphi|D_0|\varphi'\rangle$ is the contribution to that probability due to simple free particle streaming. Thus by the above equation, since $d\varphi'\langle\varphi'|E^{(+)}\rangle$ is the number of particles in steady state in $d\varphi'$, $S_E(\varphi)\, d\varphi$ is the rate at which particles are jumping into $d\varphi$ due to the potential. This is equal and opposite to the rate at which particles are jumping into $d\varphi$ due to streaming. The net rate at which particles are jumping into $d\varphi$ is zero since we are in steady state i.e.

$$\int d\varphi' \langle\varphi|D|\varphi'\rangle\langle\varphi'|E^{(+)}\rangle = 0. \qquad (63)$$

The Wigner representation of the Lippmann-Schwringer equation for the time independent wave function of the scattering problem can be rewritten as

$$\langle x, p|E^{(+)}\rangle = \langle x, p|E\rangle + \int_{-\infty}^{0} dt\; S_E(x + vt, p) \qquad (64)$$

This expresses the number of particles to be found in $d\varphi$ as being equal to those which would be there in the absence of the potential plus the net number (possibly negative) of arrivals of those streaming in to this interval after jumping in somewhere upstream because of having undergone an interaction.

In the usual Hilbert space representation of scattering, an equation for the transition matrix may be derived from the equation for the wave function by operating on the latter with the potential operator. A similar maneuver here yields a similar result;

$$S_E(\varphi) = \langle \varphi | D - D_0 | E \rangle + \int d\varphi' \int_{-\infty}^{0} dt \, \langle \varphi | D - D_0 | \varphi' \rangle \, S_E(\varphi_0(\varphi', t)) ; \tag{65}$$

$$\varphi_0(\varphi', t) \equiv \underline{x}' + \underline{v}' t, \underline{p}'.$$

In summary, what we have shown is that the elementary relations of scattering theory translate, under the Wigner representation, into elementary relations of transport theory. The only thing which distinguishes quantum from classical theory in all of this lies in the details of the jump probability function $\langle \varphi' | D - D_0 | \varphi \rangle$!

V. CURRENTS, SOURCES AND CROSS-SECTIONS FOR MULTIPARTICAL SCATTERING

Our first step will be to generalize the 'density' operator, $R(\varphi)$, describing a completely localized elementary particle to obtain a corresponding operator for bound systems of elementary particles. Consider as a simplest example a bound state, D , of particles 1 and 2. Let $|\mathcal{P}_D\rangle$ be the Hilbert space ket corresponding to a momentum eigenstate of D . Then

$$|\mathcal{P}_D\rangle = \int d\mathcal{P}_1\, d\mathcal{P}_2\, \langle \mathcal{P}|D\rangle\, \delta(\mathcal{P}_1 + \mathcal{P}_2 - \mathcal{P}_D) \qquad (66)$$

where

$$\mathcal{P} = (M_2\mathcal{P}_1 - M_1\mathcal{P}_2)/(M_1 + M_2)$$

Thus $\langle \mathcal{P}|D\rangle$ is the momentum space wave function describing the internal structure of this system and

$$\langle D|D\rangle = \int d\mathcal{P}\, \langle D|\mathcal{P}\rangle\langle \mathcal{P}|D\rangle = 1 \qquad (67)$$

We find it useful to define, as before,

$$R_D(\varphi_D) = \int d\mathcal{Q}\, e^{i\mathcal{Q}\cdot\mathcal{X}_D}\, |\mathcal{P}_D + \tfrac{1}{2}\mathcal{Q}\rangle\langle \mathcal{P}_D - \tfrac{1}{2}\mathcal{Q}| \qquad (68)$$

This is now a two body operator but it has much the same properties as before. In particular

$$\int d\varphi_D\, R_D(\varphi_D) = \mathcal{P}_D \qquad (69)$$

$$| R_D(\varphi_D) | = 1 \,, \tag{70}$$

$$| R_D(\varphi_D) R_D(\varphi_D') | = \delta(\varphi_D, \varphi_D') \,, \tag{71}$$

where $\varphi_D = \underset{\sim}{x}_0, \underset{\sim}{p}_D$ and $\mathcal{P}_D$ is the projection operator for D. Using as obvious extension of the expansion theorem written in terms of the elementary particle density operator $R(\varphi)$ which was previously discussed, we have

$$R_D(\varphi_D) = \int d\varphi_1 \, d\varphi_2 \cdot | R_D(\varphi_D) R_1(\varphi_1) R_2(\varphi_2) | R_1(\varphi_1) R_2(\varphi_2) \tag{72}$$

Carrying through the algebra yields

$$R_D(\varphi_D) = \int d\varphi \, w_D(\varphi) \; R_1(\varphi_1) R_2(\varphi_2) \tag{73}$$

where $\underset{\sim}{x}_1$, $\underset{\sim}{p}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{p}_2$ are the linear combinations of $\underset{\sim}{x}_D$, $\underset{\sim}{p}_D$, $\underset{\sim}{x}$, $\underset{\sim}{p}$ appropriate to the center of mass transformation and

$$w_D(\underset{\sim}{x}, \underset{\sim}{p}) = \int d\underset{\sim}{q} \; e^{i \underset{\sim}{q} \cdot \underset{\sim}{x}} \cdot \langle \underset{\sim}{p} - \tfrac{1}{2}\underset{\sim}{q} | D \rangle \langle D | \underset{\sim}{p} + \tfrac{1}{2}\underset{\sim}{q} \rangle \tag{74}$$

is just the Wigner representative of the internal structure wave function of D.

The set of internal phase coordinates of a bound system will be given the generic symbol ξ. Thus in the simple case we have been discussing

$$\{ \underline{\chi}_D, \underline{p}_D ; \underline{x}, \underline{p} \} = \{ \varphi_D ; \xi_D \},$$

$$(2\pi)^{-3} d\underline{\chi}_D \, d\underline{p}_D \, (2\pi)^{-3} d\underline{x} \, d\underline{p} = d\varphi_D \, d\xi_D$$

etc. If the system D were to be instead, a 3 body system, the set ξ_D would denote two internal phase points and so on. In all cases of N elementary particles bound into a state D we write

$$d\varphi_1 \cdots d\varphi_N = d\varphi_D \, d\xi_D \tag{75}$$

$$R_D(\varphi_D) = \int d\xi_D \, \omega_D(\xi_D) \cdot R_1(\varphi_1) \cdots R_N(\varphi_N) \tag{76}$$

Consider next the integral equation of the stationary density describing a beam of particles B scattered by a target A in the laboratory frame. Both A and B may in general be bound states. Assume the whole system to be made up of N elementary particles. The incoming channel will be denoted by the subscript AB. The incoming channel Hamiltonian is H_{AB}. In the absence of any interaction between A and B the incoming density operator satisfies the Schroedinger equation

$$\partial/\partial t \, \rho_{AB}(t) = -i \left(H_{AB} \rho_{AB}(t) - \rho_{AB}(t) H_{AB} \right)$$

or

$$\partial/\partial t \, | \rho_{AB}(t) \rangle = D_{AB} | \rho_{AB}(t) \rangle \tag{77}$$

Proceeding as before we obtain

$$| \rho_{AB}^{(+)} \rangle = | \rho_{AB} \rangle + \left[\int_{-\infty}^{0} dt \, e^{-D_{AB} t} \right] (D - D_{AB}) \, | \rho_{AB}^{(+)} \rangle \tag{78}$$

for the stationary scattering state.

To obtain an expression for the cross-section we again construct a standard density describing the unperturbed system.

$$|\rho_{AB}\rangle = |AB\rangle = |R_A(0,0)\int d\underline{x}_B\, R_B(\underline{x}_B, \underline{p}_B)\rangle \qquad (79)$$

At this point the objection might well be raised that since this operator does not really represent a physically realizable quantum mechanical state of the system, an error may be made in its use as the unperturbed initial state in a quantum scattering problem. A more careful and lengthy derivation not reproduced here, shows this not to be the case. It is yet another example of a phenomenon noted earlier; the formalism is indifferent to the quantum nature of the problem. This density describes a target particle A localized at the origin with zero momentum while at the same time a steady stream of bombarding particles B flows in with momentum $\underline{p}_B$ and flux

$$v_B = |\underline{v}_B| = |\underline{p}_B/m_B|.$$

Due to the interaction, this distribution is altered to become

$$|AB^{(+)}\rangle = |AB\rangle + \left[\int_{-\infty}^{0} dt\, e^{-D_{AB}t}\right](D-D_B)\,|AB^{(+)}\rangle \qquad (80)$$

Suppose one were interacted in the inclusive cross-section, $A+B \rightarrow C+$ anything, where C is some, in general, bound state of the system. Asymptotically far from the origin, the components of C, if it is a bound state, will have negligable probability of being close together unless in fact they are in their bound state. Thus, at large distances $\langle\varphi_c|AB^{(+)}\rangle$, where

$$|\varphi_c\rangle = |R_c(\varphi_c)\rangle$$

is the prob lity density for finding C within $d\varphi_c$. Note that this inner product, ce it is in fact the Hilbert space trace of $R_c(\varphi_c)$ with the

stationary density describing the collision, includes an integration over unobserved final state particles and an integration over ξ_C, the internal coordinates of C. The current of particles C far from the region of interaction is

$$\underset{\sim}{J}_{AB}(\varphi_c) = \underset{\sim}{v}_c \langle \varphi_c | AB^{(+)} \rangle \qquad (82)$$

Proceeding exactly as in potential scattering yields, now for the inclusive production rate

$$\sigma_{C;AB}(\underset{\sim}{p}_c)\, v_B = \int d\underset{\sim}{x}_c\, (2\pi)^{-3}\, S_{C;AB}(\varphi_c) \qquad (83)$$

where

$$S_{C;AB}(\varphi_c) = \partial/\partial \underset{\sim}{x}_c \cdot \underset{\sim}{J}_{AB}(\varphi_c)$$

$$= \underset{\sim}{v}_c \cdot \partial/\partial \underset{\sim}{x}_c \langle \varphi_c | AB^{(+)} \rangle$$

Note next that, if H_C is the Hilbert space Hamiltonian acting on the subspace of praticles in C such that

$$H_C | \underset{\sim}{p}_c \rangle = (\underset{\sim}{p}_c^2 / 2M_c) | \underset{\sim}{p}_c \rangle \qquad (84)$$

where $|\underset{\sim}{p}_c\rangle$ is the Hilbert space eigenket of momentum $\underset{\sim}{p}_c$, then it is straightforward to prove that

$$-i(H_C R_C(\varphi_c) - R_C(\varphi_c) H_C) = \underset{\sim}{v}_c \cdot \partial/\partial \underset{\sim}{x}_c R_C(\varphi_c) \qquad (85)$$

This implies

$$D_C | \varphi_c \rangle = \underset{\sim}{v}_c \cdot \partial/\partial \underset{\sim}{x}_c | \varphi_c \rangle \qquad (86)$$

$$\langle \varphi_c | D_C = -\underset{\sim}{v}_c \cdot \partial/\partial \underset{\sim}{x}_c \langle \varphi_c |$$

$$e^{D_C t} | \varphi_c \rangle = | e^{-iH_C t} R_C(\varphi_c) e^{+iH_C t} \rangle$$

$$= | R_C(\underset{\sim}{x}_c + \underset{\sim}{v}_c t, \underset{\sim}{p}_c) \rangle$$

$$= | \underset{\sim}{x}_c + \underset{\sim}{v}_c t, \underset{\sim}{p}_c \rangle \qquad (87)$$

Further note that

$$\langle \varphi_c | D_{\bar{c}} = 0 \qquad (88)$$

where $D_{\bar{c}}$ is the generator for all particles not included in the cluster C. This is true because the symbol $\langle \varphi_c |$ as used here contains an implicit integration over all the phase coordinates of these unobserved particles; this fact, in conjunction with equation 12, yields the result.

Combining equations 87, 88 and 83 yields

$$S_{c;AB}(\varphi_c) = \langle \varphi_c | - D_c | AB^{(+)} \rangle$$

$$= \langle \varphi_c | D - D_c - D_{\bar{c}} | AB^{(+)} \rangle$$

or

$$S_{c;AB}(\varphi_c) = \langle \varphi_c | I_c' | AB^{(+)} \rangle \qquad (89)$$

where I_c' is the sum of all interactions between members of C and members of $\bar{C}$ --the set of all unobserved particles in the final state.

It might be noted that this equation is very similar to its analog in the ordinary representation of scattering theory except that it is simpler having eliminated all reference to unobserved final state channels by means of the closure idea.

Cross-sections for two or more particles in the final state may be derived in a similar manner. For example, consider the cross-section for $A + B \rightarrow C + D +$ anything. The pertinent two particle phase space probability density is $\langle \varphi_c \varphi_D | AB^{(+)} \rangle$. Consider surfaces S_C' and S_D', exterior to and enclosing the region of interaction. The probability that C be found in $d\underset{\sim}{x}_c\, d\underset{\sim}{p}_c$ about $\underset{\sim}{x}_c$, $\underset{\sim}{p}_c$ and, at the same time, that D be found anywhere in the volume V_D surrounded by S_D' with momentum in $d\underset{\sim}{p}_D$ about $\underset{\sim}{p}_D$ is

$$d\,\underset{\sim}{p}_c\, d\,\underset{\sim}{p}_D\, d\underset{\sim}{x}_c\,(2\pi)^{-3} \int_{V_D} d\underset{\sim}{x}_D\,(2\pi)^{-3} \langle \varphi_c \varphi_D \mid A\, B^{(+)} \rangle \qquad (90)$$

The total number of such pairs of particles being produced per unit time is

$$\sigma_{cD;AB}(\underset{\sim}{p}_c\, \underset{\sim}{p}_D ; \underset{\sim}{p}_B)\,|\underset{\sim}{v}_B|\, d\,\underset{\sim}{p}_c\, d\,\underset{\sim}{p}_D$$

$$= d\,\underset{\sim}{p}_c\, d\,\underset{\sim}{p}_D \int_{S'_c} (2\pi)^{-3}\, d\underset{\sim}{S}'_c \cdot \underset{\sim}{v}_c$$

$$\cdot \lim_{V_D \to \infty} \int_{V_D} d\underset{\sim}{x}_D\,(2\pi)^{-3} \langle \varphi_c \varphi_D \mid A\, B^{(+)} \rangle . \qquad (91)$$

This can be converted by use of Gauss's theorem to

$$\sigma_{cD;AB}(\underset{\sim}{p}_c\, \underset{\sim}{p}_D ; \underset{\sim}{p}_B)\, v_B =$$

$$= \lim_{V_c \to \infty} \lim_{V_D \to \infty} \int_{V_c} d\underset{\sim}{x}_c\,(2\pi)^{-3}\, \underset{\sim}{v}_c \cdot \partial/\partial \underset{\sim}{x}_c$$

$$\cdot \int_{V_D} d\underset{\sim}{x}_D\,(2\pi)^{-3} \langle \varphi_c \varphi_D \mid A\, B^{(+)} \rangle . \qquad (92)$$

The order in which the limits are to be taken when the integrand is written in this form is important. In fact, were the order to be reversed, the integral would vanish since for fixed $\underset{\sim}{x}_D$, $\langle \varphi_c \varphi_D \mid A\, B^{(+)} \rangle$ must rapidly vanish as $|\underset{\sim}{x}_c| \to \infty$, and vice versa, due to the necessarily correlated nature of the particles' emission. We can take advantage of this fact and add to the above integrand the perfect differential

$$\underset{\sim}{v}_D \cdot \partial/\partial \underset{\sim}{x}_D \langle \varphi_c \varphi_D \mid A\, B^{(+)} \rangle$$

without changing the value of the integral. This allows us to write

$$\sigma_{cD;AB}(\underset{\sim}{p}_c, \underset{\sim}{p}_D ; \underset{\sim}{p}_B)\, v_B = \int d\underset{\sim}{x}_c\,(2\pi)^{-3}\, d\underset{\sim}{x}_D\,(2\pi)^{-3}$$

$$\cdot (\underset{\sim}{v}_c \cdot \partial/\partial \underset{\sim}{x}_c + \underset{\sim}{v}_D \cdot \partial/\partial \underset{\sim}{x}_D) \langle \varphi_c \varphi_D \mid A\, B^{(+)} \rangle .$$

(93)

We have deleted the symbols

$$\lim_{V_c \to \infty} \qquad \lim_{V_D \to \infty}$$

because in this form the order is clearly immaterial. Further, using

$$(\underset{\sim}{v}_c \cdot \partial/\partial \underset{\sim}{x}_c + \underset{\sim}{v}_D \cdot \partial/\partial \underset{\sim}{x}_D) < \varphi_c \varphi_D |$$

$$= < \varphi_c \varphi_D | (- D_c - D_D) , \qquad (94)$$

$$< \varphi_c \varphi_D | D_{\overline{CD}} \qquad (95)$$

where $\overline{CD}$ is the set of particles neither in C nor D, one obtains, in complete analogy to the one particle inclusive case,

$$\sigma_{CD;AB} (\underset{\sim}{p}_c, \underset{\sim}{p}_D ; \underset{\sim}{p}_B) v_B$$

$$= \int d\underset{\sim}{x}_c (2\pi)^{-3} \, d\underset{\sim}{x}_D (2\pi)^{-3} \; S_{CD;AB} (\varphi_c \varphi_D) \qquad (96)$$

where

$$S_{CD;AB} (\varphi_c \varphi_D) = < \varphi_c \varphi_D | I'_{CD} | A B^{(+)} > \qquad (97)$$

I'_{CD} is the sum of all interaction between members of the set CD and the unobserved set $\overline{CD}$ as well as interactions between members of C and members of D.

Thus we have obtained the general integral equation for and shown how to write down the expression for any desired cross-section. The formulae are simple generalizations of those obtained in potential scattering and analogous to those of ordinary scattering theory.

VI. MULTIPLE SCATTERING EQUATIONS

The analog of the Lippman Schwinger equation for the multiparticle problem, given in equation 80, may be rewritten as

$$|AB^{(+)}\rangle = |AB\rangle + G_{AB}\, I_{AB}\, |AB^{(+)}\rangle \qquad (98)$$

where

$$G_{AB} = \int_{-\infty}^{0} dt\; e^{-D_{AB}t}$$

$$= -(D_{AB} - \eta)^{-1} \quad ; \quad \eta \to +0 \qquad (99)$$

and

$$I_{AB} = D - D_{AB} \qquad (100)$$

are the Green's function and interaction operators respectively. As in the case of the Hilbert space representation of the problem, there is an alternative form for this equation:

$$|AB^{(+)}\rangle = |AB\rangle + G\, I_{AB}\, |AB\rangle \qquad (101)$$

where

$$G = -(D - z)^{-1}$$

The full Green's function G may be expanded in the usual manner. We write the interaction as a sum over pairwise parts

$$D = D_0 + I \qquad (102)$$

$$I = \sum_{\alpha} I_{\alpha} \qquad (103)$$

where α runs over all pairs of particles in the system $A+B$. Now define a 'jump' operator by

$$J_\alpha = I_\alpha + I_\alpha G_0 J_\alpha \tag{104}$$

where

$$G_0 = -(D_0 - \gamma)^{-1} \tag{105}$$

Thus the two body jump operators take the place of the two body transition operators in the usual development. There is one interesting difference between the two representations however; neither the Green's function or consequently the jump operators used here have the energy of the $A+B$ system as a parameter as is the case with the transition matrices in the Watson expansion. Thus there is no possibility of an 'off-shell' or 'half off-shell' type jump operator and, in fact, the same jump operator is valid for any energy and any system in which the corresponding two particles appear. This fact does not effect the algebraic structure of the equations which is the same here as in the usual analysis so that we get finally

$$G = G_0 + G_0 J G_0 , \tag{106}$$

$$J = I + I G_0 J , \tag{107}$$

$$J^{(\alpha)} = I_\alpha + J G_0 I_\alpha , \tag{108}$$

$$\sum_\alpha J^{(\alpha)} = J , \tag{109}$$

$$J^{(\alpha)} = J_\alpha + \sum_{\beta \neq \alpha} J^{(\beta)} G_0 J_\alpha \tag{110}$$

$$\begin{aligned} G I_\alpha &= G_0 I_\alpha + G_0 J G_0 I_\alpha \\ &= G_0 I_\alpha + G_0 [J^{(\alpha)} - I_\alpha] \\ &= G_0 J^{(\alpha)} \end{aligned} \tag{111}$$

The last result inserted into equation (101) yields

$$|AB^{(+)}\rangle = |AB\rangle + \sum_{\alpha}{}' G_0 J^{(\alpha)} |AB\rangle \qquad (112)$$

where the prime on the summation indicates that α is restricted to all pairs not interacting in the AB channel. The perturbation expansion of equation (110) inserted into equation (112) is the multiple scattering equation:

$$\begin{aligned} |AB^{(+)}\rangle &= |AB\rangle \\ &+ \sum_{\alpha}{}' G_0 T_\alpha |AB\rangle \\ &+ \sum_{\alpha}{}' \sum_{\beta \neq \alpha} G_0 T_\beta G_0 T_\alpha |AB\rangle \\ &+ \cdots \end{aligned} \qquad (113)$$

VII. PROPERTIES OF THE JUMP OPERATOR

Before extending further the multiple scattering theory begun in the previous section, it will be useful to examine the properties of the elementary operators appearing in the expansion given in equation 113.

Consider first po tential scattering and the matrix element

$$\langle \varphi' | \sigma | \varphi \rangle = \langle \varphi' | G_0^{-1} G | \varphi \rangle$$

$$= \int_0^\infty dt \; \underset{\sim}{v}' \cdot \partial/\partial \underset{\sim}{x}' \langle \varphi' | e^{\mathcal{D} t} | \varphi \rangle \qquad (114)$$

According to the probability interpretation of the symbols developed in this paper, $\langle \varphi' | e^{\mathcal{D} t} | \varphi \rangle$ is the probability density of a particle to be at φ' at a time t after it was localized at φ. The integrand in 114 is thus the divergence of the current set up at t at φ'. Applying the equation of continuity we get

$$\langle \varphi' | \sigma | \varphi \rangle = \int_0^\infty dt \; [\, S(\varphi', t) - \partial/\partial t \, \rho(\varphi', t) \,]$$

$$= \int_0^\infty dt \; S(\varphi', t) + \langle \varphi' | \varphi \rangle$$

where $S(\varphi', t)$ describes the sources due to the interaction and $\rho(\varphi', t)$, the particle probability density at any time t. Since $S(\varphi', t)$ is a time rate of production, its time integral equals the net production. We can now describe the situation as follows: An external source pumps a net of 1 particle into the system at φ. The system over a period of time reacts and redistributes this source distribution from $\langle \varphi' | \varphi \rangle$ to $\langle \varphi' | \sigma | \varphi \rangle$.

Since σ is in the nature of a probability one must have

$$\int d\varphi' \langle \varphi' | \sigma | \varphi \rangle = 1 \qquad (115)$$

This is in fact satisfied by the theory already since, using equation 106,

$$\sigma = 1 + J G_0 \qquad (116)$$

and

$$\int d\varphi' \langle \varphi' | J = 0 \qquad (117)$$

identically.

The relation between the operator σ and the function $S_E(\varphi)$ already introduced in the discussion of time independent potential scattering (equation 60) is,

$$S_E(\varphi) = \int d\underset{\sim}{b}_E \langle \varphi | \sigma | \varphi_E \rangle \nu_E \qquad (118)$$

where $\underset{\sim}{\chi}_E = (\underset{\sim}{b}_E, z_E)$,

$\underset{\sim}{b}_E$ is the impact parameter two-vector perpendicular to the entering momentum $\underset{\sim}{p}_E$, and $z_E \to -\infty$, along the beam line, where the accelerator is situated. The equation says that the accelerator is a source distributed randomly over the plane $z_E = -\infty$ with an average density of one per unit area. The system reacts to this 'external probe, by redistributing this density. Thus a particle introduced at φ_E will reappear at φ with probability density $\langle \varphi | \sigma | \varphi_E \rangle$. The cross-section is

$$\sigma(\underset{\sim}{p}, \underset{\sim}{p}_E) = \int d\underset{\sim}{\chi} (2\pi)^{-3} d\underset{\sim}{b}_E \langle \varphi | \sigma | \varphi_E \rangle \qquad (119)$$

In principle σ could be computed using

$$\langle \varphi' | \sigma | \varphi \rangle = \underset{\sim}{v}' \cdot \partial/\partial \underset{\sim}{\chi}' \int_0^\infty dt \cdot | R(\varphi') e^{-iHt} R(\varphi) e^{+iHt} | \qquad (120)$$

if the spectral decomposition of H is known. This seems to be a difficult task. It will be seen that one can often avoid such an explicit computation.

The classical limit is of practical interest and can be discussed in some generality. We have, as discussed in Section III,

$$D \cong D_c$$

$$\langle \varphi' | \sigma_c | \varphi \rangle = \int_0^\infty dt \; \underline{v}' \cdot \partial/\partial \underline{x}' \langle \varphi' | e^{D_c t} | \varphi \rangle$$

$$= \int_0^\infty dt \; \langle \varphi' | -D_0 \, e^{D_c t} | \varphi \rangle \qquad (121)$$

To see what this says imagine a numerical integration of the classical equation of motion. Time is segmented; the first interval is $0 \le t < t_1$; the next is $t_1 \le t < t_2$, etc. During the i^{th} interval the trajectory is approximated by a straight line :

$$\underline{x}_i(t) = \underline{x}_i(0) + \underline{v}_i (t - t_{i-1})$$

$$\underline{x}_i(t_i) = \underline{x}_{i+1}(0) \qquad (122)$$

while the momentum $\underline{p}_i$ is constant. The momentum changes discontinuously, as if by an impulsive force, at the instants $t_1, t_2 \cdots$. Thus

$$\langle \varphi' | \sigma_c | \varphi \rangle \cong \sum_{i=1} \int_{t_{i-1}}^{t_i} dt \; \langle \varphi' | -D_0 \, e^{D_0 t} | \varphi_i \rangle$$

$$= \langle \varphi' | \varphi \rangle$$

$$+ \sum_{i=1} \left[- \langle \varphi' | \underline{x}_{i+1}, \underline{p}_i \rangle + \langle \varphi' | \underline{x}_{i+1}, \underline{p}_{i+1} \rangle \right] \qquad (123)$$

having used,

$$-D_0 \, e^{D_0 t} = + \frac{d}{dt} e^{D_0 t}$$

$$(\underline{x}_1, \underline{p}_1) = (\underline{x}, \underline{p}) = \varphi$$

Thus the reaction of the system in the classical limit to an imposed unit source at φ is a line distribution, along the classical trajectory, of sources and sinks. At each spatial point subsequent to $\underset{\sim}{x}$ there is a unit sink for the old momentum going into the point and a unit source for the new momentum leaving the point.

Another limit of considerable practical and conceptual interest which can be discussed in some generality may be termed 'dilute'. This will be applied to a multiple scattering problem when the distance between successive collisions is large compared to other characteristic lengths of the problem--wavelengths and ranges of potentials of particles. In particular this limit implies no necessary ordering between wavelengths and ranges whereas the classical limit requires wavelengths to be smaller than all other characteristic lengths.

Consider, for simplicity, scattering by a system of fixed scatterers centered at points $\underset{\sim}{x}_\alpha$. The multiple scattering series is still given by equation 113; the subscripts now range over fixed scatterers' indices. In any one of the terms of the series in which the α^{th} jump operator T_α appears it operates on some particle density function; call it $|\rho\rangle$. In the dilute limit, $|\rho\rangle$ is set up by sources centered far away from $\underset{\sim}{x}_\alpha$. This leads to the conjecture that any such density must be slowly varying over the scale of distance set by the size of the α^{th} scatterer, the range of its potential. We could then, expand $|\rho\rangle$ about $\underset{\sim}{x}_\alpha$ and, in the most favorable case, keep only the lowest order term; thus

$$T_\alpha |\rho\rangle = \int d\varphi \; T_\alpha |\varphi\rangle\langle\varphi|\rho\rangle$$

$$\cong \int d\varphi \; T_\alpha |\varphi\rangle \langle \underset{\sim}{x}_\alpha, \underset{\sim}{p}|\rho\rangle \qquad (124)$$

However, Quantum Mechanics implies the existence, in principle, of substantial oscillations in $|\rho\rangle$ associated with interference. These would vary on the scale of a wavelength and contradict the conjecture. This question is resolved in the further quantum mechanical requirement that sources be distributed over a certain minimum region in phase space. Oscillations due to distributed sources wash out increasingly with distance--in the dilute limit they disappear.

When the condition of diluteness is not strong enough to allay all worries associated with the approximation of equation 124, other factors may assist. Randomness is often invoked to accomplish phase averaging. A relatively density packed medium such as a liquid or a nucleus will possess only short range order. The approximation in equation 124 might be good provided only that $|\rho\rangle$ contains no contribution in which a near neighbor of α was the last scatterer. Terms in the multiple scattering series which describe successive scatterings by near neighbors might be treated separately, leading to an expansion in powers of correlation functions of the medium.

Another escape hatch opens if the wavelength of the particle(s) being scattered is short compared to a long range component of the potentials doing the scattering. It is straight forward to show that the multiple scattering series can be re-written so that the classical propagator G_{α} replaces G_0. Concurrently the jump operator is replaced by one which represents the stochastic quantum jump correction to the classical motion as discussed in Section III. This correction is due mainly to the shortrange component of the potential. Such cores may be spaced widely enough apart for the diluteness condition to apply.

$J_\alpha |\rho\rangle$ is the source distribution created by the α^{th} scatterer when $|\rho\rangle$ impinges. This must always be integrated over another density, say $\langle\rho'|$ to calculate an observable probability, $\langle\rho'|J_\alpha|\rho\rangle$. But the arguments applied to $J_\alpha|\rho\rangle$, apply to $\langle\rho'|J_\alpha$, using time reversal invariance. Thus one can write

$$\langle\rho'|J_\alpha = \int d\varphi\ \langle\rho'|\varphi\rangle\langle\varphi|J_\alpha$$
$$\cong \int d\varphi\ \langle\rho'|\underline{\chi}_\alpha\ \underline{p}\rangle\langle\varphi|J_\alpha \qquad (125)$$

Combining equations 124 and 125,

$$\langle\varphi'|J_\alpha|\varphi\rangle \cong \delta(\underline{x}'-\underline{\chi}_\alpha)\ \delta(\underline{x}-\underline{\chi}_\alpha) \cdot \int d\underline{x}'\ d\underline{x}\ \langle\varphi'|J_\alpha|\varphi\rangle \qquad (126)$$

In contrast with the general expression for σ given in equation 120, this approximation for the closely related jump operator is easy to evaluate and interpert. Note that

$$\int d\underline{x}\ |\underline{x}, \underline{p}\rangle$$

is the density for a unit concerning plane wave state. It can therefore be written as

$$(2\pi)^3 |((|p\rangle\langle p|)\rangle \qquad (127)$$

where $|p\rangle$ is a Hilbert space ket. Next note that

$$J_\alpha = G_0^{-1}\ G\ T_\alpha \qquad (128)$$

which, in combination with our integral equation for scattering, equation 101 yields

$$J_\alpha \int d\underline{x}\ |\underline{x}\ \underline{p}\rangle$$
$$= (2\pi)^3\ G_0^{-1}\left[|((|p^{(+)}\rangle\langle p^{(+)}|)\rangle - |((|p\rangle\langle p|)\rangle\right] \qquad (129)$$

Hence,

$$\int d\underset{\sim}{x}'\, d\underset{\sim}{x}\, \langle \underset{\sim}{x}', \underset{\sim}{k}' | T_\alpha | \underset{\sim}{x}, \underset{\sim}{k} \rangle$$

$$= (2\pi)^3 \int d\underset{\sim}{x}' \langle \underset{\sim}{x}', \underset{\sim}{k}' | G_0^{-1} \left[| (| p^{(+)} \rangle \langle p^{(+)} |) \rangle \right.$$

$$\left. - | (| p \rangle \langle p |) \rangle \right]$$

$$= (2\pi)^3 \int d\underset{\sim}{x}'\ \underset{\sim}{v}' \cdot \partial/\partial \underset{\sim}{x}' \left[\langle \underset{\sim}{k}^{(+)} | R(\underset{\sim}{x}', \underset{\sim}{k}') | \underset{\sim}{k}^{(+)} \rangle \right.$$

$$\left. - \langle \underset{\sim}{k} | R(\underset{\sim}{x}', \underset{\sim}{k}') | \underset{\sim}{k} \rangle \right], \qquad (130)$$

using equations 5, 7, 18, and 19. Inserting an expansion for $R(\underset{\sim}{x}', \underset{\sim}{k}')$ such as given in equation 9 and writing

$$\langle \underset{\sim}{k}' + \tfrac{1}{2}\underset{\sim}{q}' | \underset{\sim}{k}^{(+)} \rangle$$

$$= \langle \underset{\sim}{k}' + \tfrac{1}{2}\underset{\sim}{q}' | \underset{\sim}{k} \rangle$$

$$+ \left(\epsilon(\underset{\sim}{k}) + i0 - \epsilon(\underset{\sim}{k}' + \tfrac{1}{2}\underset{\sim}{q}') \right)^{-1} T_\alpha(\underset{\sim}{k}' + \tfrac{1}{2}\underset{\sim}{q}'; \underset{\sim}{k}), \qquad (131)$$

$$\langle \underset{\sim}{k}^{(+)} | \underset{\sim}{k}' - \tfrac{1}{2}\underset{\sim}{q}' \rangle$$

$$= \langle \underset{\sim}{k} | \underset{\sim}{k}' - \tfrac{1}{2}\underset{\sim}{q}' \rangle$$

$$+ \left(\epsilon(\underset{\sim}{k}) - i0 - \epsilon(\underset{\sim}{k}' - \tfrac{1}{2}\underset{\sim}{q}') \right)^{-1} T_\alpha^*(\underset{\sim}{k}' - \tfrac{1}{2}\underset{\sim}{q}'; \underset{\sim}{k}),$$

$$(132)$$

for the usual expression relating outgoing wave solutions to Green's function and transition matrices, one gets,

$$\int d\underline{x}'\, d\underline{x}\ \langle \underline{x}', \underline{p}' | J_\alpha | \underline{x}, \underline{p} \rangle$$

$$= (2\pi)^3 \int d\underline{x}'\, d\underline{q}'\ \underline{v}' \cdot \partial/\partial \underline{x}'\ e^{i \underline{q}' \cdot \underline{x}'}$$

$$\cdot \Big\{ \langle \underline{p} | \underline{p}' - \tfrac{1}{2}\underline{q}' \rangle \left(\epsilon(\underline{p}) + i0 - \epsilon(\underline{p}' + \tfrac{1}{2}\underline{q}') \right)^{-1} T_\alpha(\underline{p}' + \tfrac{1}{2}\underline{q}'; \underline{p})$$

$$+ \langle \underline{p}' + \tfrac{1}{2}\underline{q}' | \underline{p} \rangle \left(\epsilon(\underline{p}) - i0 - \epsilon(\underline{p}' - \tfrac{1}{2}\underline{q}') \right)^{-1} T_\alpha^*(\underline{p}' - \tfrac{1}{2}\underline{q}'; \underline{p})$$

$$+ \left(\epsilon(\underline{p}) + i0 - \epsilon(\underline{p}' + \tfrac{1}{2}\underline{q}') \right)^{-1} \left(\epsilon(\underline{p}) - i0 - \epsilon(\underline{p}' - \tfrac{1}{2}\underline{q}') \right)^{-1}$$

$$\cdot T_\alpha(\underline{p}' + \tfrac{1}{2}\underline{q}'; \underline{p})\, T_\alpha^*(\underline{p}' - \tfrac{1}{2}\underline{q}'; \underline{p}) \Big\} \qquad (133)$$

The terms linear in T are the complex conjugates of each other. The first of these integrates over $d\underline{q}'$ to become

$$(2\pi)^3 \int d\underline{x}'\, e^{i \underline{q}' \cdot \underline{x}'} (\delta i\, \underline{v}' \cdot \underline{q}') \left(\epsilon(\underline{p}) + i0 - \epsilon(\underline{p} + \underline{q}') \right)^{-1}$$

$$\cdot T_\alpha(\underline{p} + \underline{q}'; \underline{p}) \qquad (134)$$

with $\underline{q}' = 2(\underline{p}' - \underline{p})$. The integral over $d\underline{x}'$ will give a δ-function distribution in $\underline{q}'$ so that we can write inside the integral

$$\epsilon(\underline{p}) - \epsilon(\underline{p} + \underline{q}') \doteq -\underline{q}' \cdot \partial\epsilon/\partial \underline{p} = -\underline{q}' \cdot \underline{v}. \qquad (135)$$

Integration over $d\underline{x}'$ now gives

$$-i\, \delta(\underline{p}' - \underline{p})\, (2\pi)^6\, T_\alpha(\underline{p}; \underline{p}) \qquad (136)$$

The sum of the linear terms in T is

$$2\, (2\pi)^6\, \delta(\underline{p}' - \underline{p})\ \mathrm{Im}\, T_\alpha(\underline{p}; \underline{p}). \qquad (137)$$

The quadratic term may be easily evaluated by noting again that the $\underset{\sim}{\chi}'$ integration restricts the integrand to the neighborhood $\underset{\sim}{q}' \approx 0$. Thus we get

$$(2\pi)^6 \, |T_\alpha(\underset{\sim}{p}';\underset{\sim}{p})|^2 \lim_{\underset{\sim}{q}'\to 0} i\,\underset{\sim}{v}\cdot\underset{\sim}{q}' \cdot \left(\epsilon(\underset{\sim}{p}) + i0 - \epsilon(\underset{\sim}{p}'+\tfrac{1}{2}\underset{\sim}{q}')\right)^{-1} \left(\epsilon(\underset{\sim}{p}) - i0 - \epsilon(\underset{\sim}{p}'-\tfrac{1}{2}\underset{\sim}{q}')\right)^{-1}$$

$$= (2\pi)^6 \, (2\pi)\, \delta(\epsilon(\underset{\sim}{p}) - \epsilon(\underset{\sim}{p}')) \, |T_\alpha(\underset{\sim}{p}';\underset{\sim}{p})|^2 \qquad (138)$$

Hence we obtain in the dilute limit,

$$\langle \varphi' | J_\alpha | \varphi \rangle$$

$$\cong \delta(\underset{\sim}{\chi}' - \underset{\sim}{\chi}_\alpha)\,(2\pi)^6\,\delta(\underset{\sim}{\chi} - \underset{\sim}{\chi}_\alpha)$$

$$\cdot \left\{ \delta(\underset{\sim}{p}'-\underset{\sim}{p})\, 2\,\mathrm{Im}\, T_\alpha(\underset{\sim}{p};\underset{\sim}{p}) + \delta(\epsilon(\underset{\sim}{p}') - \epsilon(\underset{\sim}{p}))\, 2\pi\, |T_\alpha(\underset{\sim}{p}';\underset{\sim}{p})|^2 \right\} \qquad (139)$$

Using the optical theorem and the relation between the transition matrix and the differential cross-section

$$v\,\sigma_\alpha^{\text{TOTAL}}(p) = -2\,(2\pi)^3\, T(\underset{\sim}{p};\underset{\sim}{p}) \qquad (140)$$

$$v\,\sigma_\alpha(\underset{\sim}{p}';\underset{\sim}{p}) = (2\pi)^4\,\delta(\epsilon(\underset{\sim}{p}') - \epsilon(\underset{\sim}{p}))\,|T(\underset{\sim}{p}';\underset{\sim}{p})|^2 \qquad (141)$$

where $\sigma_\alpha(\underset{\sim}{p}';\underset{\sim}{p})$ is related to the differential cross-section by equation 61, this result can be rewritten as

$$\langle \varphi' | J_\alpha | \varphi \rangle$$

$$\cong \delta(\underset{\sim}{\chi}' - \underset{\sim}{\chi}_\alpha)\,\delta(\underset{\sim}{\chi} - \underset{\sim}{\chi}_\alpha)$$

$$\cdot (2\pi)^3 \left\{ -\delta(\underset{\sim}{p}'-\underset{\sim}{p})\, v\,\sigma_\alpha^{\text{TOTAL}}(p) + v\,\sigma_\alpha(\underset{\sim}{p}';\underset{\sim}{p}) \right\} \qquad (142)$$

This equation says something which could have been written down right away using elementary considerations. In particular, a density $|\rho\rangle$ impinging upon the α^{th} fixed scatterer gives rise to a source distribution function

$$\langle \varphi' | \bar{J}_\alpha | \rho \rangle = (2\pi)^3 \delta(\underline{x}' - \underline{x}_\alpha) \cdot \int d\underline{p} \left\{ (2\pi)^{-3} v \langle \underline{x}_\alpha, \underline{p} | \rho \rangle \right\} \cdot \left\{ -\delta(\underline{p}' - \underline{p}) \sigma_\alpha^{TOTAL}(p) + \sigma_\alpha(\underline{p}'; \underline{p}) \right\} \tag{143}$$

The first bracket is the incoming flux per unit area of particles between $\underline{p}$ and $\underline{p} + d\underline{p}$ at $\underline{x}_\alpha$. This is multiplied by two factors in the second bracket; the first gives the number produced per unit $d\underline{p}'$ at $\underline{p}'$--a positive source term; the second gives the number lost to the incoming beam--a negative sink term.

This result can be immediately generalized to include non-fixed scatterers. Let α now refer to a pair of particles, e.g. particles 1 and 2. Then we have

$$\langle \varphi_1' \varphi_2' | \bar{J}_{12} | \varphi_1 \varphi_2 \rangle \equiv (2\pi)^6 \delta(\underline{X}_{12}' - \underline{X}_{12}) \delta(\underline{P}_{12}' - \underline{P}_{12}) \delta(\underline{x}_{12}') \delta(\underline{x}_{12}) \cdot \left\{ -|\underline{v}_1 - \underline{v}_2| \delta(\underline{p}_{12}' - \underline{p}_{12}) \sigma_{12}^{TOTAL}(\underline{p}_{12}) + |\underline{v}_1 - \underline{v}_2| \sigma_{12}(\underline{p}_{12}', \underline{p}_{12}) \right\} \tag{144}$$

where $\underset{\sim}{X}_{12}$, $\underset{\sim}{P}_{12}$ refer to phase coordinates of the center of mass while $\underset{\sim}{x}_{12}$, $\underset{\sim}{p}_{12}$ apply in the center of mass. This result may be obtained from the formalism but can be written down directly on physical grounds.

The advantage gained in having derived this approximation via this formalism is that it is now part of quantum mechanical scattering theory. The possibility of systematic refinements to any approximation is thereby created. We have already discussed quantum corrections to the classical limit in Section III. The possibility of combining the classical approximation (for motion through the long range part of a potential) with the dilute approximation (for transitions induced by short range potential cores) has been mentioned. In addition, finite size corrections to the dilute approximation, arising when the range of the potential is not negligably small compared to the distance between scatterers, have been calculated.[8] The relation between the dilute approximation and Monte Carlo calculations has been discussed elsewhere.[5]

VIII. MULTIPLE SCATTERING EXPANSIONS OF INCLUSIVE CROSS-SECTIONS

The previous section developed certain limiting cases which are useful for future applications and serve to illustrate and clarify the meaning of the basic jump operators that appear in the multiple scattering expansion. We now return to more general developments of the theory. Formulae for the simplest inclusive production processes are rewritten in a multiple scattering form which facilitates their interpertations and computation in many cases of interest.

It is useful to define a symbol, ${}^{(\beta)}J^{(\alpha)}$, for the sum of all scatterings beginning with the α and ending with the β pair. Simple expansion in powers of the Jump operators will verify that it satisfies,

$$
\begin{aligned}
{}^{(\beta)}J^{(\alpha)} &= \bar{J}_\alpha \delta_{\alpha\beta} + \sum_\gamma \bar{J}_\beta (1-\delta_{\beta\gamma}) G_0 {}^{(\gamma)}J^{(\alpha)} \\
&= J_\alpha \delta_{\alpha\beta} + \sum_\gamma {}^{(\beta)}J^{(\gamma)} (1-\delta_{\gamma\alpha}) G_0 J_\alpha
\end{aligned}
\tag{145}
$$

Furthermore, using equation 106, 107 and some strictly algebraic manipulations, one can show that

$$
{}^{(\beta)}J^{(\alpha)} = \bar{I}_\alpha \delta_{\alpha\beta} + \bar{I}_\beta G \bar{I}_\alpha .
$$

Particle B impinges on a composite target A. Consider first the inclusive source distribution function for B observed in the final state. Application of equation 89 shows,

$$S_{B;AB}(\varphi_B') = \langle \varphi_B' | T_B' | AB^{(+)} \rangle$$

$$= \langle \varphi_B' | T_B' (1 + G T_{AB}) | AB \rangle . \qquad (147)$$

Since $T_B' = T_{AB}$, the sum of all interactions between B and constituents of A, one immediately obtains

$$S_{B;AB}(\varphi_B') = \sum_{a,a'} \langle \varphi_B' | {}^{(Ba')}T^{(Ba)} | AB \rangle \qquad (148)$$

where a and a' run over all constituents of A. If, for example, A were a two body bound state composed of particles labelled 1 and 2, the multiple scattering expansion of equation 148 yields

$$S_{B;AB}(\varphi_B') = \langle \varphi_B' | T_{B1} + T_{B2}$$

$$+ T_{B2} G_0 T_{B1} + T_{B1} G_0 T_{B2}$$

$$+ \cdots | AB \rangle \qquad (149)$$

These expressions will be evaluated in the dilute limit in the next section.

Expressions for the inclusive production of a single particle other than the incident one (knockout), follow in a similar manner. These source functions are, of course, to be integrated over space to give the total production rate or, equivalently, the cross-section.

Several examples of practical interest involve two or more particles observed in the final state. They may be in bound or unbound condition. Our intuition would like to relate, for example, a two particle inclusive production rate to that of a bound state of the same two particles. The present formalism brings this out in an especially graphic manner.

The inclusive source function for the bound state production of D, made up of constituents 1 and 2 originally in A is given by equation 89:

$$S_{D;AB}(\varphi_D') = \langle \varphi_D' | I_D' | AB^{(+)} \rangle \qquad (150)$$

that for 1 and 2 unbound is given by equation 97:

$$S_{12;AB}(\varphi_1' \varphi_2') = \langle \varphi_1' \varphi_2' | I_{12}' | AB^{(+)} \rangle \qquad (151)$$

Now

$$I_D' = I_{1B} + \sum_{a \neq 1,2} I_{1a} + I_{2B} + \sum_{a \neq 1,2} I_{2a} \qquad (152)$$

while

$$I_{12}' = I_D' + I_{12} \qquad (153)$$

Thus we see that the basic source functions required to calculate bound and unbound production differ only in the term

$$\langle \varphi_1' \varphi_2' | I_{12} | AB^{(+)} \rangle$$

This term represents final state interaction between the observed constituents. It is the sum of all contributions to the multiple scattering series in which the very last interaction is between the constituents of the observed final state. This final state interaction must be deleted in the computation of bound state production--included in the unbound.

Except for kinematics which especially emphasize the role of the final state interaction (e.g. at low energies and low relative momenta $\vec{p}_1' - \vec{p}_2'$ of the observed pair), this term is unimportant. It is useful to define a source function which neglects this term;

$$\tilde{S}_{12;AB}(\varphi_1' \varphi_2') = \langle \varphi_1' \varphi_2' | I_D' | AB^{(+)} \rangle . \qquad (154)$$

Then

$$\sigma_{D;AB}(\underline{p}_D')\, v_B = \int d\underline{x}_D'\,(2\pi)^{-3} \int d\underline{x}'\, d\underline{p}'\,(2\pi)^{-3} \cdot w_D(\underline{x}', \underline{p}')\; \tilde{S}_{12;AB}(\varphi_1'\varphi_2') \tag{155}$$

$$\tilde{\sigma}_{12;AB}(\underline{p}_1', \underline{p}_2')\, v_B = \int d\underline{x}_1'\,(2\pi)^{-3}\, d\underline{x}_2'\,(2\pi)^{-3} \cdot \tilde{S}_{12;AB}(\varphi_1'\varphi_2') \tag{156}$$

where $\tilde{\sigma}_{12;AB}$ is the cross-section less final state interaction contribution. In the above formulae we have used:

$$\langle \varphi_D' | = \int d\underline{x}'\, d\underline{p}'\,(2\pi)^{-3}\, w_D(\underline{x}', \underline{p}') \langle \varphi_1'\varphi_2' |\,,$$

$$d\underline{x}_1'\, d\underline{x}_2' = d\underline{x}'\, d\underline{x}_D'\,,$$

$$\underline{p}_D' = \underline{p}_1' + \underline{p}_2'\,,$$

$$\underline{p}' = (M_2\,\underline{p}_1' - M_1\,\underline{p}_2') / (M_1 + M_2)\,. \tag{157}$$

These equations say that the source function $\tilde{S}$, giving the spatial probability distribution of the origin of the particles produced in the interaction already contains all the information necessary to compute the bound state production rate. One just convolutes $\tilde{S}$ with the Wigner density w_D describing the bound state. Thus, for example, a Monte-Carlo calculation approximating the multiple scattering series and hence providing an estimate of $\tilde{S}$, can, without further effort, provide an approximation to the bound state production rate.

If one further approximation, often valid in practice, is made, these relations become even more striking. The dependence of $\tilde{S}$ on $\underline{p}'$ is

assumed to be much less rapid than that of w_D which is peaked at $\underline{p}' = 0$. In this case one has

$$\sigma_{D;AB}(\underline{p}_0')\, v_B \cong \int d\underline{x}_0' (2\pi)^{-3} \int d\underline{x}' \, |\psi_D(\underline{x}')|^2 \cdot \tilde{S}_{12;AB}(\underline{x}_1', \tfrac{\underline{p}_0'}{2}; \underline{x}_2', \tfrac{\underline{p}_0'}{2}) \qquad (158)$$

$$\tilde{\sigma}_{12;AB}(\underline{p}_1', \underline{p}_2')\, v_B \cong \int d\underline{x}_1' (2\pi)^{-3} \, d\underline{x}_2' (2\pi)^{-3} \cdot \tilde{S}_{12;AB}(\underline{x}_1', \tfrac{\underline{p}_0'}{2}; \underline{x}_2', \tfrac{\underline{p}_0'}{2}) \qquad (159)$$

The integrands differ only in the substitution of the spatial probability

$$|\psi_D(\underline{x}')|^2 = \int d\underline{p}' (2\pi)^{-3} \, w_D(\underline{x}', \underline{p}') \qquad (160)$$

in one for a factor of $(2\pi)^{-3}$ in the other. Thus the ratio

$$\sigma_{D;AB}(\underline{p}_0') \,/\, \sigma_{12;AB}(\tfrac{\underline{p}_0'}{2}, \tfrac{\underline{p}_0'}{2})$$

is related to the volume from which the particles 1 and 2 emanate and hence the mechanism of their production.

IX. MULTIPLE SCATTERING BY A TWO BODY BOUND STATE IN THE DILUTE LIMIT

The multiple scattering expression for inclusive scattering of B by a two body bound state A is now evaluated in the dilute limit. The methods and results of this formalism may be connected with others in the context of this well known example.

The contribution first order in T_{B1} to this process is given by (see equaiton 149),

$$\int d\underline{x}'_B\, (2\pi)^{-3} \langle \varphi'_B \mid T_{B1} \mid A\, B \rangle$$
$$= \int d\underline{x}'_B\, (2\pi)^{-3}\, d\varphi_1'\, d\varphi_2'\, d\varphi\, d\underline{x}_B\, \omega_A(\varphi)$$
$$\cdot \langle \varphi_1' \varphi_2' \varphi_B' \mid T_{B1} \mid \varphi_1 \varphi_2 \varphi_B \rangle . \qquad (161)$$

Since we are in the laboratory system $\underline{p}_1 + \underline{p}_2 = 0$. As before

$$\varphi = \underline{x},\ \underline{p}$$

are the relative coordinates and momenta of 1 and 2.

The dilute expression for T_{B1} can be written in the present notation, using equation 144, as

$$\langle \varphi_1' \varphi_2' \varphi_B' \mid T_{B1} \mid \varphi_1 \varphi_2 \varphi_B \rangle$$
$$= \langle \varphi_2' \mid \varphi_2 \rangle \langle \Phi'_{B1} \mid \Phi_{B1} \rangle\, \delta(\underline{x}_{B1})\, \delta(\underline{x}'_{B1})$$
$$\cdot (2\pi)^{-3} \mid v_B - v_1 \mid \{ -\delta(\underline{p}'_{B1} - \underline{p}_{B1})\, \sigma^T_{B1}(\underline{p}_B, \underline{p}_1)$$
$$+ \sigma_{B1}(\underline{p}'_B \underline{p}'_1; \underline{p}_B \underline{p}_1) \} \qquad (162)$$

$$\underline{\Phi}_{B1} = \underline{X}_{B1}, \underline{P}_{B1}$$

are the phase coordinates of the $B1$ center of mass;

$$\varphi_{B1} = \chi_{B1}, \; \rho_{B1}$$

refer to relative coordinates, specifically

$$\underline{\chi}_{B1} = \underline{\chi}_B - \underline{\chi}_1$$

and

$$\underline{\rho}_{B1} = (M_1 \underline{\rho}_B - M_B \underline{\rho}_1) / (M_1 + M_B) ;$$

$\sigma^T_{B1}(\rho_B, \rho_1)$, which really only depends on ρ_{B1}, is the total cross section for B on 1; $\sigma_{B1}(\rho_B', \rho_1'; \rho_B, \rho_1)$, which really on depends on ρ_{B1} and ρ_{B1}', is the differential cross-section for scattering into $d\rho_{B1}'$. Its relation to the usual center of mass differential cross-section is given by equation 61. Combining these equations and integrating over spatial functions yields

$$\begin{aligned}
&\int d\rho_1' \, d\rho \, d\chi \, (2\pi)^{-3} \, w_A(\chi, \rho) \, \delta(\underline{\rho}_{B1}' - \underline{\rho}_{B1}) \\
&\cdot |\underline{v}_B - \underline{v}_1| \Big\{ - \delta(\rho_{B1}' - \rho_{B1}) \, \sigma^T_{B1}(\rho_B, \rho_1) \\
&\qquad + \sigma_{B1}(\rho_B', \rho_1'; \rho_B, \rho_1) \Big\} \\
&= - \delta(\rho_B' - \rho_B) \int d\rho \, |\psi_A(\rho)|^2 \, |\underline{v}_B - \underline{v}_1| \, \sigma^T_{B1}(\rho_B, \rho_1) \\
&+ \int d\rho \, |\psi_A(\rho)|^2 \, |\underline{v}_B - \underline{v}_1| \, \sigma_{B1}(\rho_B', \rho_1'; \rho_B, \rho_1)
\end{aligned} \tag{163}$$

where now

$$\underline{P}_1 = \underline{P}$$

$$\underline{P}_1' = \underline{P}_B - \underline{P}_B' + \underline{P}$$

The first term gives the contribution to the total cross-section due to collisions of B with 1 only. If we add to this the corresponding contribution from 2, the first order estimate of the total cross-section is

$$\int d\underline{P} \, |\psi_A(\underline{P})|^2 \left\{ \left| \frac{\underline{v}_B - \underline{v}}{v_B} \right| \sigma^T_{B1}(\underline{P}_B, \underline{P}) + \left| \frac{\underline{v}_B + \underline{v}}{v_B} \right| \sigma^T_{B2}(\underline{P}_B, -\underline{P}) \right\} \qquad (164)$$

The second term in equation 163 is the well known single scattering term originally obtained by Goldberger,[6]

$$\int d\underline{P} \, |\psi_A(\underline{P})|^2 \left| \frac{\underline{v}_B - \underline{v}}{v_B} \right| \sigma_{B1}(\underline{P}_B', \underline{P}_1'; \underline{P}_B, \underline{P}_1) \qquad (165)$$

to which one should add the corresponding term for scattering of B on 2.

The contributions second order in the jump operators provide shadow corrections to the first order results as well as double scattering. The contribution from the $\bar{J}_{B2} G_0 \bar{J}_{B1}$ term in equation 149 is

$$\int d\underline{x}_B' (2\pi)^{-3} d\varphi_1' d\varphi_2' d\varphi \, d\underline{x}_B \, \psi_A(\varphi) \cdot \langle \varphi_1' \varphi_2' \varphi_B' | \bar{J}_{B2} G_0 \bar{J}_{B1} | \varphi_1 \varphi_2 \varphi_B \rangle . \qquad (166)$$

Expanding over intermediate states and using

$$G_0 \,|\, \varphi_1, \varphi_2, \cdots \rangle = \int_0^\infty dt \,|\, \varphi_0(\varphi_1, t), \varphi_0(\varphi_2, t), \cdots \rangle$$

where (167)

$$\varphi_0(\varphi, t) = \underset{\sim}{x} + \underset{\sim}{v} t, \; \mathcal{P}$$

we get

$$\langle \varphi_1' \varphi_2' \varphi_B' \,|\, T_{B2} \, G_0 \, T_{B1} \,|\, \varphi_1 \varphi_2 \varphi_B \rangle$$

$$= \int_0^\infty dt \int d\varphi_1'' \, d\varphi_2'' \, d\varphi_B'' \, \langle \varphi_1' \varphi_2' \varphi_B' \,|\, T_{B2} \,|\, \varphi_0(\varphi_1'', t) \cdots \rangle$$

$$\cdot \langle \varphi_1'' \varphi_2'' \varphi_B'' \,|\, T_{B1} \,|\, \varphi_1 \varphi_2 \varphi_B \rangle$$

$$= \int_0^\infty dt \int d\varphi_B'' \, \langle \varphi_2' \varphi_B' \,|\, T_{B2} \,|\, \varphi_0(\varphi_2, t), \varphi_0(\varphi_B'', t) \rangle$$

$$\cdot \langle \varphi_0(\varphi_1', -t), \varphi_B'' \,|\, T_{B1} \,|\, \varphi_1 \varphi_B \rangle$$

(168)

Inserting this result into equation 166 and integrating over the spatial functions, we get

$$\int_0^\infty dt \int d\underset{\sim}{p}_1' \, d\underset{\sim}{p}_2' \, d\underset{\sim}{p} \, d\underset{\sim}{p}_B'' \, d\underset{\sim}{x} \, (2\pi)^{-3} w_A(\underset{\sim}{x}, \underset{\sim}{p})$$

$$\cdot \delta(\underset{\sim}{x} - \underset{\sim}{v}_2 t + \underset{\sim}{v}_B'' t) \Big\{ |\underset{\sim}{v}_B'' - \underset{\sim}{v}_2| \, \delta(\underset{\sim}{p}_B' + \underset{\sim}{p}_2' - \underset{\sim}{p}_B'' - \underset{\sim}{p}_2)$$

$$\cdot \left[-\sigma_{B2}^T(\underset{\sim}{p}_B'', \underset{\sim}{p}_2) \, \delta(\underset{\sim}{p}_{B2}'' - \underset{\sim}{p}_{B2}') + \sigma_{B2}(\underset{\sim}{p}_B', \underset{\sim}{p}_2'; \underset{\sim}{p}_B'', \underset{\sim}{p}_2) \right]$$

$$\cdot |\underset{\sim}{v}_B - \underset{\sim}{v}_1| \, \delta(\underset{\sim}{p}_B'' + \underset{\sim}{p}_1' - \underset{\sim}{p}_B - \underset{\sim}{p}_1)$$

$$\cdot \left[-\sigma_{B1}^T(\underset{\sim}{p}_B, \underset{\sim}{p}_1) \, \delta(\underset{\sim}{p}_{B1}'' - \underset{\sim}{p}_{B1}) + \sigma_{B1}(\underset{\sim}{p}_B'', \underset{\sim}{p}_1'; \underset{\sim}{p}_B, \underset{\sim}{p}_1) \right] \Big\}$$

(169)

The spatial δ function requires that at some time t after collision of B with 1, the position of B, $\underset{\sim}{x}_1 + \underset{\sim}{v}_B'' t$, coincide with the position of 2, $\underset{\sim}{x}_2 + \underset{\sim}{v}_2 t$. The only new notation used in equation 169 is $\underset{\sim}{p}_{B2}''$ for the relative momentum of $\underset{\sim}{p}_B''$ and $\underset{\sim}{p}_2'$, $\underset{\sim}{p}_{B2}'$ for that of $\underset{\sim}{p}_B'$ and $\underset{\sim}{p}_2'$, $\underset{\sim}{p}_{B1}''$ for that of $\underset{\sim}{p}_B''$ and $\underset{\sim}{p}_1'$ and, $\underset{\sim}{p}_{B1}$ for that of $\underset{\sim}{p}_B$ and $\underset{\sim}{p}_1$.

The term quadratic in σ^T represents a shadow correction to the total cross-section; the terms linear in σ^T represent shadow corrections to single scattering; the term quadratic in the differential cross-sections represent double scattering. We consider each in turn.

The shadow correction to the total cross-section given by this expression is

$$-\int d\underset{\sim}{p}\,(2\pi)^{-3} \int_0^\infty dt\; |\underset{\sim}{v}_B + \underset{\sim}{v}|\; w_A(-(\underset{\sim}{v}_B + \underset{\sim}{v})t, \underset{\sim}{p}) \cdot \sigma^T_{B2}(\underset{\sim}{p}_B, -\underset{\sim}{p}) \left|\frac{\underset{\sim}{v}_B - \underset{\sim}{v}}{v_B}\right| \sigma^T_{B1}(\underset{\sim}{p}_B, \underset{\sim}{p}) \qquad (170)$$

which must be added to a similar term coming from collisions in the reverse order to get the total shadow correction to second order. This expression simplifies somewhat in the high energy limit where it can be compared to the corresponding correction due to Glauber. Using

$$|\underset{\sim}{v}_B \pm \underset{\sim}{v}| \cong v_B$$

$$v_B\, dt = dz$$

$$\int d\underset{\sim}{p}\,(2\pi)^{-3}\, w_A(\underset{\sim}{x}, \underset{\sim}{p}) = |\tilde{\psi}_A(x)|^2,$$

for the bound state spatial density, we get,

$$-\int_0^\infty dz\, |\tilde{\psi}_A(0,0,z)|^2\, \sigma_{B1}^T(\underline{p}_B,\underline{p})\, \sigma_{B2}^T(\underline{p}_B,\underline{p}) \qquad (171)$$

Taking into account that the contribution to the shadow correction coming from the reverse order of collision, one finds that the total shadow correction in the dilute limit is twice that obtained by Glauber[7] in his simple approximation to black sphere scattering. The reason for the difference in these results may be traced to the fact that they each hold true in different, non-overlapping, regimes. Our result is accurate in the dilute limit in which the Glauber approximation is invalid. It should be re-emphasized at this point that the basic formalism--the Wigner representation of scattering--is not limited to the dilute limit but is susceptible to other approximations which may lead to simple formulas valid in other regimes.

The shadow correction to single scattering is composed of two parts. The particle casting the shadow may lie either between the accelerator or between the detector and the single scatterer. In either case one gets a reduction in the single scattering--a 'unitarity correction' since it is merely a manifestation of probability conservation. The shadow correction to single scattering by 1, to be added to the uncorrected term given in equation 164 is obtained by straightforward application of the equations just discussed yielding again in the dilute limit,

$$-\int_0^\infty dt \int d\underline{p}\,(2\pi)^{-3}$$

$$\cdot\Big\{ |\underline{v}_B' + \underline{v}|\, w_A(-(\underline{v}_B'+\underline{v})t, \underline{p})\, \sigma_{B2}^T(\underline{p}_B', -\underline{p})$$

$$\cdot \left|\frac{\underline{v}_B - \underline{v}}{v_B}\right| \sigma_{B1}(\underline{p}_B', \underline{p}_1'; \underline{p}_B, \underline{p})$$

$$+ |\underline{v}_B' - \underline{v}|\, w_A((\underline{v}_B'-\underline{v})t, \underline{p})\, \sigma_{B1}(\underline{p}_B', \underline{p}_1'; \underline{p}_B \underline{p})$$

$$\cdot \left|\frac{\underline{v}_B + \underline{v}}{v_B}\right| \sigma_{B2}^T(\underline{p}_B, -\underline{p}) \,. \qquad (172)$$

where $\underline{p}_1'$ is again given as in equation 164. A similar term reduces single scattering on the other particle. The physical origin of these terms is transparent.

We might note that the effect of such corrections is quite striking in reducing the backward peak observed in proton-deutron elastic scattering ($\sim$ 50% correction).[13] It may be that a simple formula such as derived here (perhaps somewhat refined to take into account the not completely dilute nature of nuclear matter) will be quite accurate in calculating such shadow corrections for many nuclear targets, including the deuteron. This would be especially helpful at those intermediate energies at which the Glauber approximation begins to break down. We also note that, as before, this formula simplifies considerable when v_B is much greater than v, the Fermi velocity.

Finally, the double scattering term obtained from equation 169 is

$$\int d\underline{p}\,(2\pi)^{-3}\, d\underline{p}_B''\int_0^\infty dt\;\; v_B^{-1}\, w_A(-(\underline{v}_B''+\underline{v})t,\underline{p})$$
$$\cdot\,|\underline{v}_B''+\underline{v}|\,\sigma_{B2}(\underline{p}_B'\,\underline{p}_2';\,\underline{p}_B'',-\underline{p})$$
$$\cdot\,|\underline{v}_B-\underline{v}|\,\sigma_{B1}(\underline{p}_B''\,\underline{p}_1';\,\underline{p}_B,\underline{p}) \qquad (173)$$

where now momentum conservation gives

$$\underline{p}_1' = \underline{p}_B + \underline{p} - \underline{p}_B''$$
$$\underline{p}_2' = \underline{p}_B'' - \underline{p} - \underline{p}_B'$$

To this must be added the corresponding term from the reversed order of collision to get the total double scattering.

We have here a 5-dimensional integral since there are two energy conserving δ-functions implied in the elastic cross-sections. When v_0 is large compared to v, we can again effect a considerable simplification by integrating analytically over $d\varphi$. One is then left with a 2-dimensional integral. In either case this integral is especially suited to Monte-Carlo integration techniques.

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FIGURE CAPTIONS

Figure 1: This illustrates the first order correction to the classical propogator given in Equation 41. A particle is known to be at φ initially and one asks for the probability of it being at φ' at a time t later. In addition to the classical value of this probability there is a contribution corresponding to the particle traveling a time $t - t'$ along its initial classical trajectory and then performing a quantum jump to a new classical trajectory which brings it to φ' in the required time t. The trajectories are pictured in position space emphasizing the fact that, for a local potential, there is a jump in momentum only as with a classical stochastic impulse. The probability that the particle makes the jump is given by Equation 41.

Figure 2: This illustrates the classical limit of scattering in terms of source functions as given in Equation 123 and discussion subsequent to it. A circled portion of the spatial trajectory φ_c is enlarged. The continuous force is approximated by a sequence of inpulses. A particle starts at $\underline{x}_1$ with momentum $\underline{p}_1$. Thus at $\underline{x}_1$ there is a unit source of such particles. At $\underline{x}_2$ it receives an impulse, changing its momentum from $\underline{p}_1$ to $\underline{p}_2$. Thus at $\underline{x}_2$ there is a unit sink for $\underline{p}_1$ and source for $\underline{p}_2$, and so on. In this picture, classical motion is a consequence of the potential's creation of a line of sources and sinks in response to the externally imposed source at $\underline{x}_1, \underline{p}_1$.

Figure 3: This illustrates two phenomena generated by the second order term of the multiple scattering expansion for scattering of projectile $\mathcal{B}$ by a two particle bound state A (See Equation 149). When $\mathcal{B}$ hits constituent 1 it happens, in this instance, to have momentum $\underset{\sim}{p}$ and be displaced from constituent 2 by $\underset{\sim}{x}$. The solid line shows the trajectory of $\mathcal{B}$. Between collisions $\mathcal{B}$, having gotten intermediate momentum $\underset{\sim}{p}_{\mathcal{B}}''$, displaces itself along $\underset{\sim}{v}_{\mathcal{B}}'' t$ while constituent 2, since A was initally at rest in the laboratory frame, has momentum $-\underset{\sim}{p}$ and has moved $-\underset{\sim}{v} t$. The collision between $\mathcal{B}$ and 2 at a time t after the first collision has two consequences: First it reduces the number of (single scattering) events into $\underset{\sim}{p}_{\mathcal{O}}''$, i.e. particle 2 shadow events shining from particle 1. This phenomenon is described by the first bracketed term in Equation 172. Second, it increases the number of (double scattering) events into $\underset{\sim}{p}_{\mathcal{O}}'$. This is described by Equation 173.

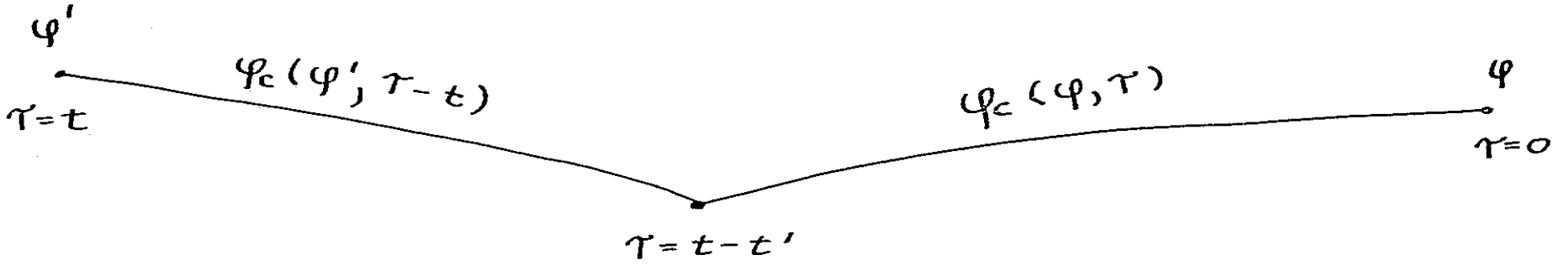

Fig. 1.

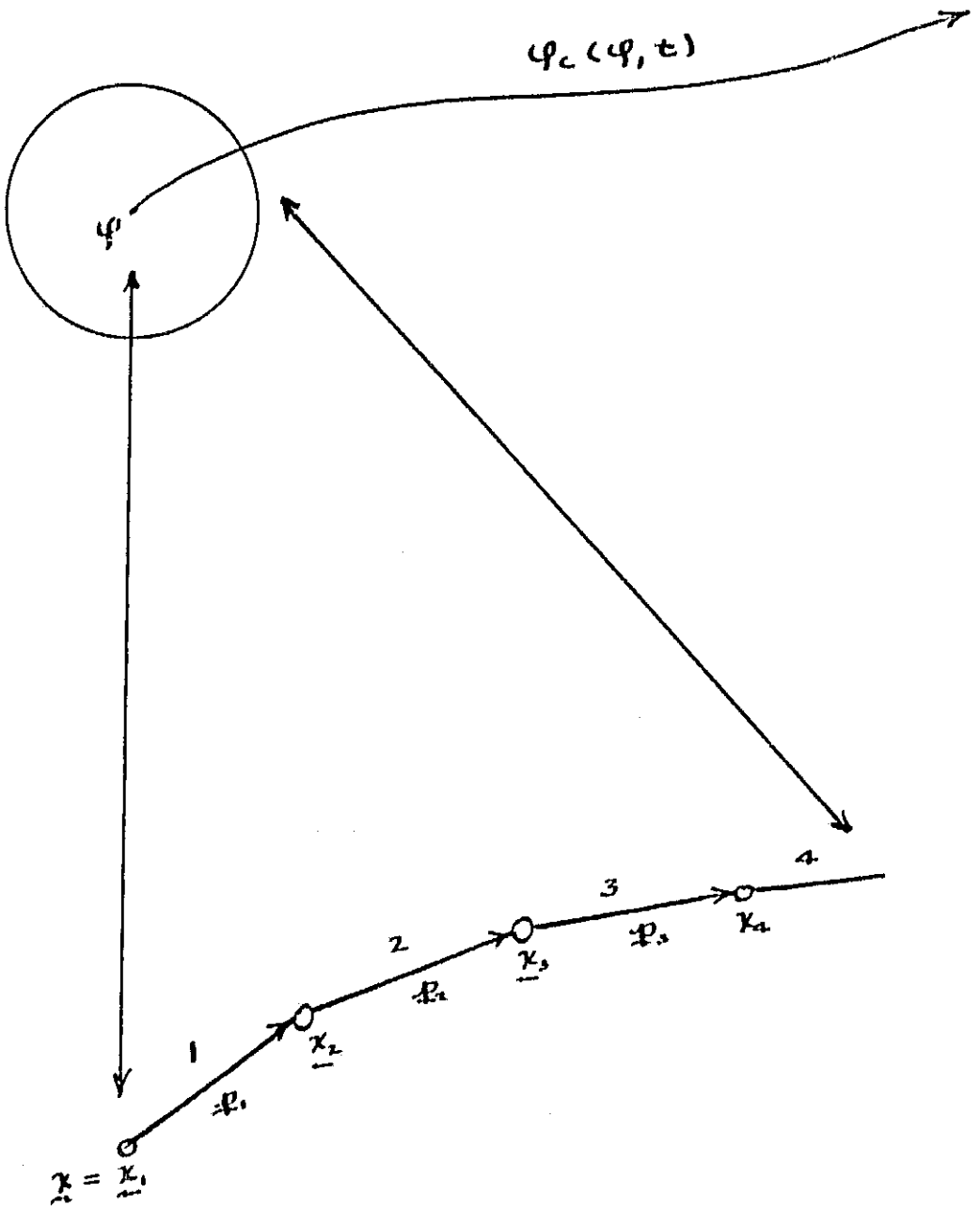
$\varphi_c(\varphi, t)$
φ
1
2
3
4
$\underline{x} = \underline{x}_1$
$\underline{x}_2$
$\underline{x}_3$
$\underline{x}_4$
$\underline{p}_1$
$\underline{p}_2$
$\underline{p}_3$

Fig. 2

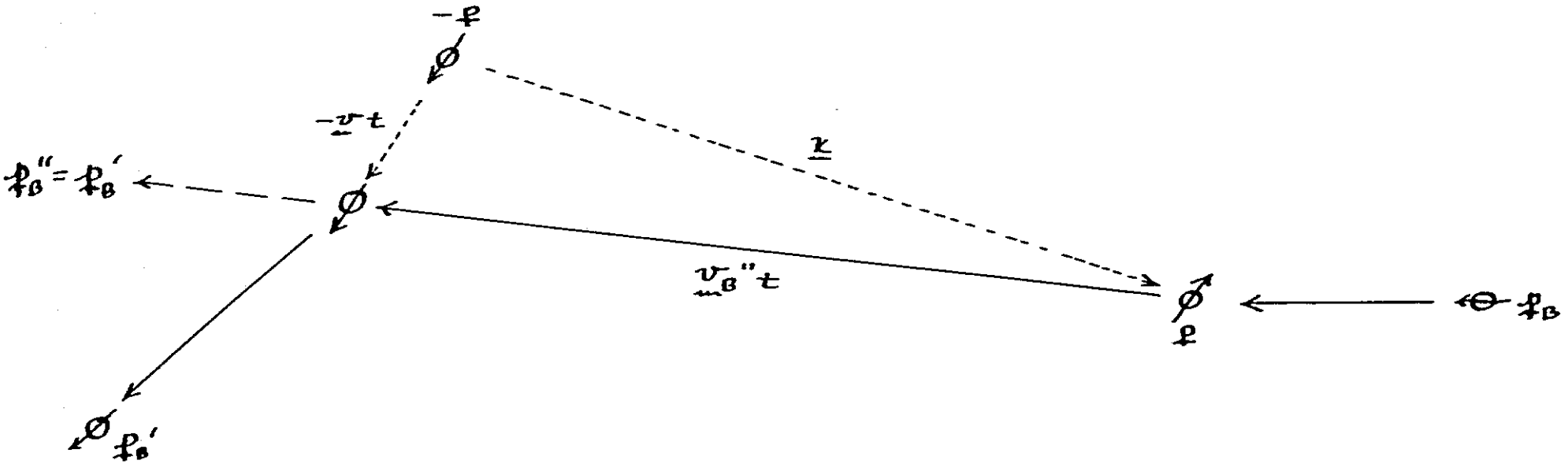
$-\underline{r}$
$-\underline{v}t$
$\underline{x}$
$\underline{r}_B'' = \underline{r}_B'$
$\underline{v}_{mB}''t$
$\underline{r}$
$\underline{r}_B$
$\underline{r}_B'$

Fig. 3