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G. E. O. Giacaglia
The University of Texas at Austin
Austin, Texas

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Applied Mechanics Research Laboratory
The University of Texas at Austin
Austin, Texas

Evaluation of Geopotential and Luni-solar Perturbations by a Recursive Algorithm

G. E. O. Giacaglia*
The University of Texas at Austin

Abstract

The disturbing functions due to the geopotential and Luni-solar attractions are linear and bilinear forms in spherical harmonics. Making use of recurrence relations for the solid spherical harmonics and their derivatives, recurrence formulas are obtained for high degree terms as function of lower degree for any term of those disturbing functions and their derivative with respect to any element. The equations obtained are very effective when a numerical integration of the equations of motion is appropriate. In analytical theories, they provide a fast way of obtaining high degree terms starting from initial very simple functions.

*On leave from the University of Sao Paulo.

Recurrence relations for spherical harmonics

The Associate Legendre Functions of the first kind

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \tag{1}$$

satisfy well known recurrence relations, namely

$$(\ell+1)\, P_{\ell+1}^m(x) = (2\ell+1)\, [x\, P_\ell^m(x) + m(1-x^2)^{1/2}\, P_\ell^{m-1}(x)] - \ell P_{\ell-1}^m \tag{2}$$

and

$$xP_\ell^m(x) = P_{\ell-1}^m(x) + (\ell-m+1)(1-x^2)^{1/2}\, P_\ell^{m-1}(x) \tag{3}$$

where $P_\ell^m(x) = 0$ for $m>\ell$, according to definition (1).

Consider now the solid spherical harmonics

$$X_\ell^m = P_\ell^m(\sin\phi)\, e^{im\alpha} r^\ell \tag{4}$$

$$Y_\ell^m = P_\ell^m(\sin\phi)\, e^{im\alpha} \frac{1}{r^{\ell+1}} \tag{5}$$

where

$$r^2 = x^2 + y^2 + z^2 ,$$

$$x + iy = r \cos\phi \; e^{i\alpha} , \tag{6}$$

and $$z = r \sin\phi . \tag{7}$$

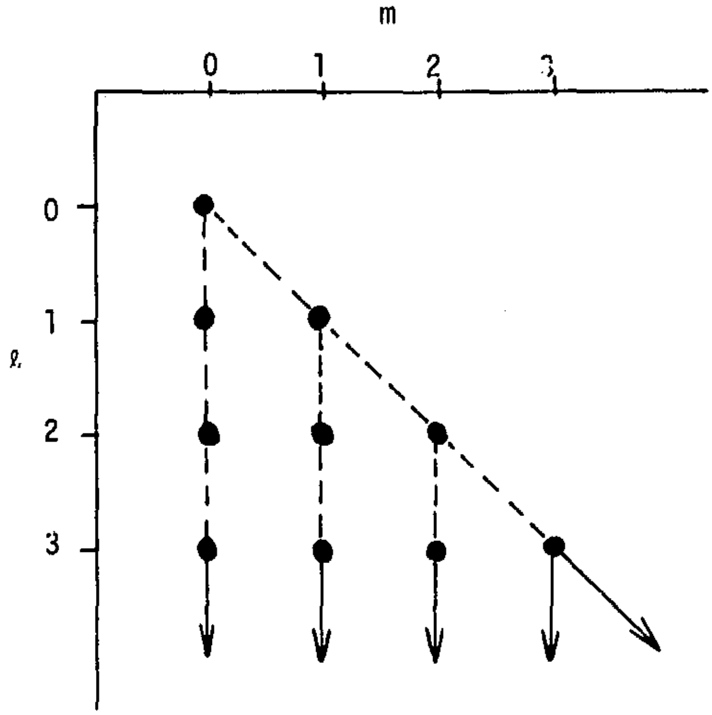
Since $0 \leq m \leq \ell$, for any ℓ, recurrence relations are necessary for

a) A given m, increasing values of ℓ

b) A given pair $(m=\ell,\ell)$, increasing values of ℓ.

These distinct cases are necessary in order to avoid singularities in the equations we are going to obtain. Such a distinction is not necessary in (2) and (3) provided the definition $P_\ell^m = 0$ $(m>\ell)$ is assumed.

Thus we shall provide recurrence relations in the triangular scheme



Consider first the diagonal recurrence

$$(\ell,\ell) \rightarrow (\ell+1,\ell+1)$$

One has

$$X_{\ell+1}^{\ell+1} = r^{\ell+1}\, P_{\ell+1}^{\ell+1}\,(\sin\phi)\, e^{i(\ell+1)\alpha} \tag{8}$$

But from (2), with $x = \sin\phi$

$$P_{\ell+1}^{\ell+1} = (2\ell+1)\cos\phi\, P_\ell^\ell$$

so that

$$X_{\ell+1}^{\ell+1} = r^{\ell+1} (2\ell+1) \cos\phi \, P_{\ell}^{\ell} e^{i\ell\alpha} e^{i\alpha}$$

or, in view of (6),

$$X_{\ell+1}^{\ell+1} = (2\ell+1)(x+iy) X_{\ell}^{\ell} \tag{9}$$

Similarly

$$Y_{\ell+1}^{\ell+1} = (2\ell+1) \frac{1}{r^2}(x+iy) Y_{\ell}^{\ell} . \tag{10}$$

Now, let $m<\ell$ and consider the vertical recurrence

$$(\ell,m) \rightarrow (\ell+1,m) \quad .$$

Combining (2) and (3) one finds:

$$P_{\ell+1}^{m} = \frac{2\ell+1}{\ell-m+1} x P_{\ell}^{m}(x) - \frac{\ell+1}{\ell-m+1} P_{\ell-1}^{m}(x) \tag{11}$$

From (4):

$$X_{\ell+1}^{m} = r^{\ell+1} P_{\ell+1}^{m} e^{im\alpha}$$

and, using (11),

$$X_{\ell+1}^{m} = r^{\ell+1} \left[\frac{2\ell+1}{\ell-m+1} \sin\phi \, P_{\ell}^{m} - \frac{\ell+m}{\ell-m+1} P_{\ell-1}^{m}\right] e^{im\alpha} =$$

$$= \frac{2\ell+1}{\ell-m+1} r \sin\phi \, X_{\ell}^{m} - \frac{\ell+m}{\ell-m+1} r X_{\ell-1}^{m}$$

or, using (7)

$$X_{\ell+1}^{m} = \frac{2\ell+1}{\ell-m+1} z X_{\ell}^{m} - \frac{\ell+m}{\ell-m+1} r X_{\ell-1}^{m} \tag{12}$$

Similarly

$$Y_{\ell+1}^{m} = \frac{2\ell+1}{\ell-m+1} \frac{z}{r^2} Y_{\ell}^{m} - \frac{\ell+m}{\ell-m+1} \frac{1}{r^2} Y_{\ell-1}^{m} \tag{13}$$

Equations (9), (10), (12), (13), are the necessary relations to be used later.

Recurrence relations for the cartesian derivatives of Spherical Harmonics

The solid spherical harmonics can be defined by (Courant-Hilbert, 1973)

$$X_\ell^m = r^{2\ell+1} \frac{(-1)^\ell}{(\ell-m)!} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z}\right)^{\ell-m} \left(\frac{1}{r}\right) \tag{14}$$

and

$$Y_\ell^m = \frac{(-1)^\ell}{(\ell-m)!} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z}\right)^{\ell-m} \left(\frac{1}{r}\right) \tag{15}$$

From (14):

$$X_{\ell+1}^{m+1} = r^{2\ell+3} \frac{(-1)^{\ell+1}}{(\ell-m)!} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^{m+1} \left(\frac{\partial}{\partial z}\right)^{\ell-m} \left(\frac{1}{r}\right) =$$

$$= r^{2\ell+3} (-1) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \frac{(-1)^\ell}{(\ell-m)!} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z}\right)^{\ell-m} \left(\frac{1}{r}\right) =$$

$$= r^{2\ell+3} (-1) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) r^{-2\ell-1} X_\ell^m =$$

$$= -r^2 \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) X_\ell^m + (2\ell+1)(x + iy) X_\ell^m$$

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) X_\ell^m = -\frac{1}{r^2} X_{\ell+1}^{m+1} + \frac{1}{r^2} (2\ell+1)(x+iy) X_\ell^m \tag{16}$$

Using (14) again:

$$X_{\ell+1}^{m-1} = r^{2\ell+3} \frac{(-1)^{\ell+1}}{(\ell-m+2)!} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^{m-1} \left(\frac{\partial}{\partial z}\right)^{\ell-m} \frac{\partial^2}{\partial z^2} \left(\frac{1}{r}\right)$$

But one has

$$\frac{\partial^2}{\partial z^2}\ \frac{1}{r} = -\frac{1}{r^3} + 3\,\frac{x^2}{r^5} = \frac{2}{r^3} - 3\,\frac{y^2+z^2}{r^5} =$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{r}\right) = -\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\left(\frac{1}{r}\right)$$

so that

$$X_{\ell+1}^{m-1} = r^{2\ell+3}\ \frac{(-1)^\ell}{(\ell-m+2)!}\ \left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)^m\left(\frac{\partial}{\partial z}\right)^{\ell-m}\left(\frac{1}{r}\right) =$$

$$= r^{2\ell+3}\ \frac{(\ell-m)!}{(\ell-m+2)!}\ \left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\ r^{-2\ell-1}\ X_\ell^m =$$

$$= r^2\ \frac{1}{(\ell-m+2)(\ell-m+1)}\ \left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)X_\ell^m - \frac{(2\ell+1)(x-iy)X_\ell^m}{(\ell-m+2)(\ell-m+1)}$$

$$\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\ X_\ell^m = \frac{(\ell-m+2)(\ell-m+1)}{r^2}\ X_{\ell+1}^{m-1} + \frac{(2\ell+1)}{r^2}\ (x-iy)\ X_\ell^m \tag{17}$$

Adding and subtracting (16) and (17), one has

$$\frac{\partial}{\partial x}\,X_\ell^m = -\frac{1}{2r^2}\,X_{\ell+1}^{m+1} + \frac{1}{2r^2}\,(\ell-m+2)(\ell-m+1)\ X_{\ell+1}^{m-1} +$$

$$+ \frac{x}{r^2}\ (2\ell+1)\ X_\ell^m \tag{18}$$

$$\frac{\partial}{\partial y}\,X_\ell^m = \frac{i}{2r^2}\,X_{\ell+1}^{m+1} + \frac{i}{2r^2}\,(\ell-m+2)(\ell-m+1)\ X_{\ell+1}^{m-1} +$$

$$+ \frac{y}{r^2}\ (2\ell+1)\ X_\ell^m \tag{19}$$

Equations (18) and (19) give the derivatives with respect to x and y. In order to find the z-derivative consider again Eq. (14) with $\ell+1$:

$$X_{\ell+1}^{m} = r^{2\ell+3} \frac{(-1)^{\ell+1}}{(\ell+1-m)!} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^{m} \left(\frac{\partial}{\partial z}\right)^{\ell-m} \frac{\partial}{\partial z} \frac{1}{r} =$$

$$= r^{2\ell+3} \frac{(-1)(\ell-m)!}{(\ell+1-m)!} \frac{\partial}{\partial z} r^{-2\ell-1} X_{\ell}^{m} =$$

$$= -r^{2} \frac{1}{\ell-m+1} \frac{\partial X_{\ell}^{m}}{\partial z} + \frac{2\ell+1}{\ell-m+1} z X_{\ell}^{m}$$

$$\frac{\partial}{\partial z} X_{\ell}^{m} = - \frac{\ell-m+1}{r^{2}} X_{\ell+1}^{m} + (2\ell+1) \frac{z}{r^{2}} X_{\ell}^{m} \tag{20}$$

In order to obtain the derivatives for the Y_{ℓ}^{m} harmonics, one operates as for the X_{ℓ}^{m} and find

$$\frac{\partial}{\partial x} Y_{\ell}^{m} = - \frac{1}{2} Y_{\ell+1}^{m+1} + \frac{1}{2} (\ell-m+1)(\ell-m+2) Y_{\ell+1}^{m-1} \tag{21}$$

$$\frac{\partial}{\partial y} Y_{\ell}^{m} = \frac{i}{2} Y_{\ell+1}^{m+1} + \frac{i}{2} (\ell-m+1)(\ell-m+2) Y_{\ell+1}^{m-1} \tag{22}$$

$$\frac{\partial}{\partial z} Y_{\ell}^{m} = - (\ell-m+1) Y_{\ell+1}^{m} \tag{23}$$

<u>Disturbing function for geopotential and derivatives</u>

It is given by

$$R = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} \frac{\mu a_e^{\ell}}{r^{\ell+1}} P_\ell^m (\sin \phi) \, [C_\ell^m \cos m\lambda + S_\ell^m \sin m\lambda] \tag{24}$$

where

$\lambda = \alpha - \theta$

$\alpha =$ right ascension

$\theta =$ Greenwich right ascension

$\phi =$ declination

and the other quantities have the usual meaning.

The cartesian coordinates satisfy the relations

$$\begin{aligned} x + iy &= r \cos\phi \, e^{i\alpha} \\ z &= r \sin \phi \\ x - iy &= r \cos \phi \, e^{-i\alpha} \end{aligned} \tag{25}$$

Let

$$R_\ell^m = \frac{\mu a_e^{\ell}}{r^{\ell+1}} P_\ell^m (\sin\phi) \, (C_\ell^m \cos m\lambda + S_\ell^m \sin m\lambda)$$

so that

$$R = \sum_{\ell \leq 2} \sum_{m=0}^{n} R_\ell^m \, .$$

Considering that $\lambda = \alpha - \theta$, one finds

$$C_\ell^m \cos m\lambda + S_\ell^m \sin m\lambda = (C_\ell^m \cos m\theta - S_\ell^m \sin m\theta) \cos m\alpha +$$
$$+ (C_\ell^m \sin m\theta + S_\ell^m \cos m\theta) \sin m\alpha =$$
$$= A_\ell^m \cos m\alpha + B_\ell^m \sin m\alpha$$

where

$$\begin{aligned} A_\ell^m &= C_\ell^m \cos m\theta - S_\ell^m \sin m\theta \\ B_\ell^m &= C_\ell^m \sin m\theta + S_\ell^m \sin m\theta \end{aligned} \tag{26}$$

Also, let us define , from Eq. (5)

$$Y_\ell^m = U_\ell^m + i\, V_\ell^m \quad . \tag{27}$$

Thus

$$R_\ell^m = \mu a_e^\ell \, (A_\ell^m U_\ell^m + B_\ell^m V_\ell^m) \tag{28}$$

Terms R_ℓ^m can now be generated in succession starting from $R_0^0 = \frac{\mu}{r}$, by the diagonal and vertical relations (10) and (13),

From (10) one finds

$$U_{\ell+1}^{\ell+1} = \frac{2\ell+1}{r^2} (x\, U_\ell^\ell - y\, V_\ell^\ell)$$

and (29)

$$V_{\ell+1}^{\ell+1} = \frac{2\ell+1}{r^2} (x\, V_\ell^\ell + y\, U_\ell^\ell)$$

where, using orbital coordinates, one has

$$x = r \cos\phi \cos\alpha = r\,[\cos(\omega+f) \cos\Omega - \cos I \sin(\omega+f) \sin\Omega]$$

and (30)

$$y = r \cos\phi \cos\alpha = r\,[\cos(\omega+f) \sin\Omega + \cos I \sin(\omega+f) \cos\Omega]$$

Combining (29) and (28), one has the diagonal recurrence

$$R_{\ell+1}^{\ell+1} = \mu a_e^{\ell+1} \frac{2\ell+1}{r^2} \left[A_{\ell+1}^{\ell+1} (xU_\ell^\ell - yV_\ell^\ell) + B_{\ell+1}^{\ell+1} (xV_\ell^\ell + yU_\ell^\ell)\right]$$

or, more conveniently,

$$R_{\ell+1}^{\ell+1} = \mu a_e^{\ell+1} \frac{2\ell+1}{r^2} \left[(xA_{\ell+1}^{\ell+1} + yB_{\ell+1}^{\ell+1})\, U_\ell^\ell + (xB_{\ell+1}^{\ell+1} - yA_{\ell+1}^{\ell+1})\, V_\ell^\ell\right] \tag{31}$$

where x,y are defined by (30), when necessary, and $A_{\ell+1}^{\ell+1}$, $B_{\ell+1}^{\ell+1}$ by (26). Note that, in general

$$\frac{1}{r}(xA_\ell^m + y\, B_\ell^m) = C_\ell^m\, [\cos(\omega+f)\, \cos(\Omega-m\theta) - \cos I\, \sin(\omega+f)\, \sin(\Omega-m\theta)] +$$

$$+ S_\ell^m\, [\cos(\omega+f)\, \sin(\Omega-m\theta) + \cos I\, \sin(\omega+f)\, \cos(\Omega-m\theta)] \tag{32}$$

and

$$\frac{1}{r}(xB_\ell^m - yA_\ell^m) = -C_\ell^m\, [\cos(\omega+f)\, \sin(\Omega-m\theta) + \cos I\, \sin(\omega+f)\, \cos(\Omega-m\theta)] +$$

$$+ S_\ell^m\, [\cos(\omega+f)\, \cos(\Omega-m\theta) - \cos I\, \sin(\omega+f)\, \sin(\Omega-m\theta)] \tag{33}$$

The vertical recurrence relations are also easily found. From (13) it follows that

$$\begin{Bmatrix} U_{\ell+1}^m \\ V_{\ell+1}^m \end{Bmatrix} = \frac{2\ell+1}{\ell-m+1} \frac{z}{r^2} \begin{Bmatrix} U_\ell^m \\ V_\ell^m \end{Bmatrix} - \frac{\ell+m}{\ell-m+1} \frac{1}{r^2} \begin{Bmatrix} U_{\ell-1}^m \\ V_{\ell-1}^m \end{Bmatrix} \tag{34}$$

where we should remember that $U_{\ell-1}^m$, $V_{\ell-1}^m = 0$ if $m>\ell-1$. Considering (28), one finds

$$R^m_{\ell+1} = \mu a_e^{\ell+1} \left(A^m_{\ell+1}\ U^m_{\ell+1} + B^m_{\ell+1}\ V^m_{\ell+1}\right) \tag{35}$$

where $U^m_{\ell+1}$, $V^m_{\ell+1}$ are given by (34). Also, when orbital elements are used,

$$z = r \sin \phi = r \sin I \sin (\omega+f). \tag{36}$$

Where using cartesian coordinates, the cartesian derivatives of R^m_ℓ give directly the equations of motion, that is,

$$\ddot{x} = -\frac{\mu x}{r^3} + \frac{\partial R}{\partial x} + X \tag{37}$$

where X is a nonconservative force (X, Y, Z), and similarly for y, z. When using a set of elements α_i (i=1,2,3) and β_i(i=1,2,3), let

$$w = col\ (x, y, z, \dot{x}, \dot{y}, \dot{z})$$

$$\gamma = col\ (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$$

and J the state transition matrix

$$J = \frac{\partial \gamma}{\partial w} \quad , \quad J_{ij} = \frac{\partial \gamma_i}{\partial w_j} \tag{38}$$

Define

$$E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where 0, I are the 3x3 null and identity matrices respectively.

Lagrange's equations are

$$\dot{\gamma} = J\ E \left[\left(\frac{\partial F}{\partial w}\right)^T - \binom{\psi}{0}\right] \tag{39}$$

where

$$F = \frac{\mu}{r} + R - \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{40}$$

and

$$\binom{\psi}{0} = col\,(X, Y, Z, 0, 0) \quad . \tag{41}$$

One sees that given the cartesian derivatives, Lagrange's equations are easily written for any set of elements, provided the state transition matrix J is known, for a Keplerian orbit. Or else, one can consider that

$$\frac{\partial R}{\partial \alpha_i} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial \alpha_i} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial \alpha_i} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial \alpha_i}$$

and similar equations for β_i. Whatever the form used, if one knows the cartesian derivatives of R, the equations are easily constructed by straight matrix multiplication.

Using Eqs. (21), (22), and (23) one finds

$$\frac{\partial}{\partial x}\begin{Bmatrix} U_\ell^m \\ V_\ell^m \end{Bmatrix} = -\frac{1}{2}\begin{Bmatrix} U_{\ell+1}^{m+1} \\ V_{\ell+1}^{m+1} \end{Bmatrix} + \frac{1}{2}(\ell-m+1)(\ell-m+2)\begin{Bmatrix} U_{\ell+1}^{m-1} \\ V_{\ell+1}^{m-1} \end{Bmatrix} \tag{42}$$

$$\frac{\partial}{\partial y}\begin{Bmatrix} U_\ell^m \\ V_\ell^m \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} -V_{\ell+1}^{m+1} \\ U_{\ell+1}^{m+1} \end{Bmatrix} + \frac{1}{2}(\ell-m+1)(\ell-m+2)\begin{Bmatrix} -V_{\ell+1}^{m-1} \\ U_{\ell+1}^{m-1} \end{Bmatrix} \tag{43}$$

$$\frac{\partial}{\partial z}\begin{Bmatrix} U_\ell^m \\ V_\ell^m \end{Bmatrix} = -(\ell-m+1)\begin{Bmatrix} U_{\ell+1}^m \\ V_{\ell+1}^m \end{Bmatrix} \tag{44}$$

Thus,

$$\frac{\partial R_\ell^m}{\partial(x,y,z)} = \mu a_e^\ell \left[A_\ell^m \frac{\partial U_\ell^m}{\partial(x,y,z)} + B_\ell^m \frac{\partial V_\ell^m}{\partial(x,y,z)}\right] \tag{45}$$

can be written by recurrence using equations (42), (43), and (44) together with (29) and (34). For the geopotential, the recurrence scheme is therefore complete.

Disturbing function for Luni-solar perturbations and derivatives

For lunar or solar attractions the disturbing function is given by

$$R' = \sum_{\ell>2} \sum_{m=0}^{\ell} R_\ell'^m \tag{46}$$

where

$$R_\ell'^m = Gm' \frac{r^\ell}{r'^{\ell+1}} \varepsilon_m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\sin\phi)\, P_\ell^m(\sin\phi') \cos m(\alpha-\alpha') \tag{47}$$

where G = gravitational constant

m' = mass of the disturbing body

ε_m = 1 (m=0) or 2 (m≠0)

ϕ',α' = declination and right ascension of the disturbing body

r' = earth-moon or earth-sun distance.

r,ϕ,α = spherical equatorial coordinates of the satellite

Let, from Eq. (4),

$$X_\ell^m = W_\ell^m + iZ_\ell^m \tag{48}$$

so that

$$W_\ell^m = r^\ell \, P_\ell^m \, (\sin\phi) \cos m\,\alpha$$

and

$$Z_\ell^m = r^\ell \, P_\ell^m \, (\sin\phi) \sin m\,\alpha .$$

It is easily seen that

$$R_\ell^{\prime m} = Gm' \, \varepsilon_m \, \frac{(\ell-m)!}{(\ell+m)!} \left[W_\ell^m \, U_\ell^{\prime m} + Z_\ell^m \, V_\ell^{\prime m} \right] \tag{49}$$

where $U_\ell^{\prime m}$ and $V_\ell^{\prime m}$ are the U_ℓ^m, V_ℓ^m functions defined over the coordinates of the disturbing body, that is

$$U_\ell^{\prime m} = \frac{1}{r'^{\,\ell+1}} \, P_\ell^m \, (\sin\phi') \cos m\,\alpha'$$

and (50)

$$V_\ell^{\prime m} = \frac{1}{r'^{\,\ell+1}} \, P_\ell^m \, (\sin\phi') \sin m\,\alpha' .$$

Again, any term $R_\ell^{\prime m}$ can be generated by lower degree terms by recurrence. In the equations of motion for the satellite, only cartesian derivatives of W_ℓ^m and Z_ℓ^m are used. Recurrence relations for $U_\ell^{\prime m}$, $V_\ell^{\prime m}$ are the same than for the U_ℓ^m, V_ℓ^m, that is, given by equations (29) and (34) by "priming" all variables.

The recurrence relations for W_ℓ^m, Z_ℓ^m are easily obtained from Eqs. (9) and (12), by considering definition (48). One finds, from (9)

$$W_{\ell+1}^{\ell+1} = (2\ell+1)(x \, W_\ell^\ell - y \, Z_\ell^\ell)$$

$$Z_{\ell+1}^{\ell+1} = (2\ell+1)(x \, Z_\ell^\ell + y \, W_\ell^\ell)$$

where x, y are given by (30) when orbital elements are used. Also, from (12),

$$\begin{pmatrix} W_{\ell+1}^m \\ Z_{\ell+1}^m \end{pmatrix} = \frac{2\ell+1}{\ell-m+1}\, z \begin{pmatrix} W_{\ell+1}^m \\ Z_{\ell+1}^m \end{pmatrix} - \frac{\ell+m}{\ell-m+1}\, r \begin{pmatrix} W_{\ell-1}^m \\ Z_{\ell-1}^m \end{pmatrix} \tag{52}$$

where z is given by (36) when orbital elements are used.

The cartesian derivatives of W_ℓ^m, Z_ℓ^m are found from (18), (19), and (20), that is,

$$\frac{\partial}{\partial x}\begin{pmatrix} W_\ell^m \\ Z_\ell^m \end{pmatrix} = -\frac{1}{2r^2}\begin{pmatrix} W_{\ell+1}^{m+1} \\ Z_{\ell+1}^{m+1} \end{pmatrix} + \frac{1}{2r^2}(\ell-m+2)(\ell-m+1)\begin{pmatrix} W_{\ell+1}^{m-1} \\ Z_{\ell+1}^{m-1} \end{pmatrix} + \frac{x}{r^2}(2\ell+1)\begin{pmatrix} W_\ell^m \\ Z_\ell^m \end{pmatrix} \tag{53}$$

$$\frac{\partial}{\partial y}\begin{pmatrix} W_\ell^m \\ Z_\ell^m \end{pmatrix} = \frac{1}{2r^2}\begin{pmatrix} -Z_{\ell+1}^{m+1} \\ W_{\ell+1}^{m+1} \end{pmatrix} + \frac{1}{2r^2}(\ell-m+2)(\ell-m+1)\begin{pmatrix} -Z_{\ell+1}^{m-1} \\ W_{\ell+1}^{m-1} \end{pmatrix} + \frac{y}{r^2}(2\ell+1)\begin{pmatrix} W_\ell^m \\ Z_\ell^m \end{pmatrix} \tag{54}$$

$$\frac{\partial}{\partial z}\begin{pmatrix} W_\ell^m \\ Z_\ell^m \end{pmatrix} = -\frac{\ell-m+1}{r^2}\begin{pmatrix} W_{\ell+1}^m \\ Z_{\ell+1}^m \end{pmatrix} + \frac{z}{r^2}(2\ell+1)\begin{pmatrix} W_\ell^m \\ Z_\ell^m \end{pmatrix} \tag{55}$$

In the above equations the recurrence relations (51) and (52) should be used in order to obtain all the derivatives, up to any degree and order, by recurrence.

Therefore,

$$\frac{\partial R_\ell'^m}{\partial(x,y,z)} = G\, m'\, \epsilon_m \frac{(\ell-m)!}{(\ell+m)!} \left[U_\ell'^m \frac{\partial W_\ell^m}{\partial(x,y,z)} + V_\ell'^m \frac{\partial Z_\ell^m}{\partial(x,y,z)} \right] \tag{56}$$

can be written be recurrence using Equations (53), (54), (55) together with (51) and (52) for the satellite terms and using Eqs. (29) and (36) for the lunar and solar terms. Obviously, the disturbing function (49) can be obtained by recurrence using Eqs. (51), (52), (29), (34).

<u>Calculation procedure for geopotential</u>

a) <u>Disturbing function</u>

$$R_0^0 = \frac{\mu}{r}$$

Find

$R_1^1,\ R_2^2,\ R_3^3,\ \ldots$ by (31)

Find

$R_1^0,\ R_2^0,\ R_3^0,\ R_4^0,\ \ldots$

$R_2^1,\ R_3^1,\ R_4^1,\ \ldots$

$R_3^2,\ R_4^2,\ R_5^2,\ \ldots$

$\ldots\ldots\ldots\ldots$ by (35)

b) Derivatives

$$\frac{\partial R_0^0}{\partial(x,y,z)} = \left[-\frac{\mu x}{r^3}, -\frac{\mu y}{r^3} - \frac{\mu z}{r^3} \right]$$

Find gradient of R_ℓ^m using (45) plus (42), (43), (44), plus (29), (34).

Calculation procedure for luni-solar

a) Disturbing function

$$R'^0_0 = \frac{Gm'}{r'}$$

Find R'^1_1, R'^2_2, R'^3_3, ... by (51) and (29).

Find

$$R'^0_1,\ R'^0_2,\ R'^0_3,\ R'^0_4,\ \ldots$$

$$R'^1_2,\ R'^1_3,\ R'^1_4,\ \ldots$$

$$R'^2_3,\ R'^2_4,\ \ldots$$

.... by (52) and (34).

b) Derivatives

$$\frac{\partial R'^0_0}{\partial(x,y,z)} = [0,0,0]$$

Find gradient of R'^m_ℓ using (56) plus (53), (54), (55), plus (51), (52), (29), (34), for recurrence.

General comments

If numerical integration is to be used, the best equations are those in cartesian coordinates. The right-hand numbers of the equations

$$\ddot{x} = -\frac{\mu x}{r^3} + \frac{\partial R}{\partial x} + \frac{\partial R^{☾}}{\partial x} + \frac{\partial R^{\odot}}{\partial x}$$

etc., can be computed numerically and sequentially by recurrence. Note that repeated application of the formulas developed for the cartesian derivatives allow partial derivatives up to any order to be quickly computed by recurrence. These might be necessary either for construction of variational equations or for the integrating process to be used if it requires values of derivatives up to a certain order, as in a Runge-Kutta method.

If an analytical theory is to be developed, the recurrence relations can be used with great advantage to generate both the disturbing function and its cartesian derivatives by recurrence. If a computer algebraic formula processor is available then the right hand member to very high order in analytic form can be evaluated very quickly and stored in tape for further manipulation.

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