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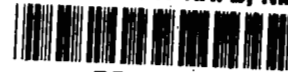
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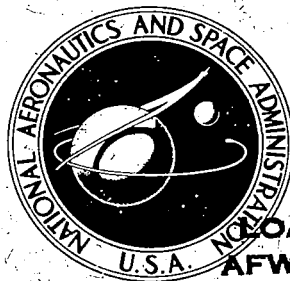
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A THEORETICAL INVESTIGATION OF THE INPUT CHARACTERISTICS OF A RECTANGULAR CAVITY-BACKED SLOT ANTENNA

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16. Abstract

Equations which represent the magnetic and electric stored energies are derived for an infinite section of rectangular waveguide and a rectangular cavity. These representations which are referred to as being "physically observable" are obtained by considering the difference in the volume integrals appearing in the complex Poynting theorem. It is shown that the "physically observable" stored energies are determined by the field components that vanish in a reference plane outside the aperture.

These "physically observable" representations are used to compute the input admittance of a rectangular cavity-backed slot antenna in which a single propagating wave is assumed to exist in the cavity. The slot is excited by a voltage source connected across its center; a sinusoidal distribution is assumed in the slot. Input-admittance calculations are compared with measured data. In addition, input-admittance curves as a function of electrical slot length are presented for several size cavities.

For the rectangular cavity-backed slot antenna, the quality factor and relative bandwidth were computed independently by using these energy relationships. It is shown that the asymptotic relationship which is usually assumed to exist between the quality bandwidth and the reciprocal of relative bandwidth is equally valid for the rectangular cavity-backed slot antenna.

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CONTENTS

iv

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SUMMARY

Equations which represent the magnetic and electric stored energies are derived for an infinite section of rectangular waveguide and a rectangular cavity. These representations which are referred to as being "physically observable" are obtained by considering the difference in the volume integrals appearing in the complex Poynting theorem. It is shown that the "physically observable" stored energies are determined by the field components that vanish in a reference plane outside the aperture.

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For the rectangular cavity-backed slot antenna, the quality factor and relative bandwidth were computed independently by using these energy relationships. It is shown that the asymptotic relationship which is usually assumed to exist between the quality bandwith and the reciprocal of relative bandwidth is equally valid for the rectangular cavity-backed slot antenna.

INTRODUCTION

The aperture (or slot) antenna is one of the most widely used antennas because it is relatively simple to build and can be flush mounted in conducting bodies such as in the surface of aircraft and spacecraft, thus becoming an integral part of the vehicle. For such applications the aperture antenna also meets the requirements of small size and low weight. In addition to its practical usefulness, radiation and impedance characteristics can be investigated theoretically without too much difficulty (refs. 1 to 3). Such investi-

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gations are usually conducted first for an ideal model such as a narrow slot in a perfect conductor of infinite extent.

The input admittance Y_S (or impedance) of a narrow slot in a perfectly conducting infinite sheet can be determined via the Booker relationship (ref. 4) whenever the slot is free to radiate on both sides of the infinite sheet. This relationship, which can be found in many books (refs. 5 to 8), is given by $Y_S = 4Z_d/Z_o^2$ where Z_d is the input impedance of the complementary dipole (planar dipole) and Z_O is the characteristic impedance of the surrounding medium. In practical applications the slot is backed by some sort of cavity, thus destroying the symmetry upon which the Booker relationship depends. The cavity-backed aperture antenna has been the subject of many papers over the past two decades, or longer (refs. 9 to 13).

When the slot is backed by a cavity on one side of the infinite sheet, the radiation pattern and impedance characteristics of the slot antenna are altered; the radiation resistance, the bandwidth, and the stored energy are changed. The impedance properties of the apertures (and slots) which are backed by a rectangular cavity have been investigated by many authors (refs. 9 to 13). In references 9 and 10 the backing rectangular cavity was a shorted waveguide whose cross section was the same as that of the aperture; reference 9 is further restricted to small cavities. In references 11 to 13 the thin slot is backed by a rectangular cavity of different cross section.

In references 9 and 10 the relationship between quality factor and the reciprocal of relative bandwidth, known to exist in nonradiating systems, is assumed equally valid for cavity-backed aperture antennas. Quite often this relationship is assumed for antennas in general (refs. 14 and 15). In reference 16, quality factor and inverse bandwidth are related in an order of magnitude sense. The antenna in references 14 and 15 is assumed to be such that its aperture distribution is frequency independent; hence the frequency derivative of its reactance or susceptance is shown to be proportional to the total stored energies. The reciprocal of relative bandwidth would then be equal to quality factor. For planar antennas in which the aperture distributions are frequency dependent, the reciprocal of relative bandwidth is proportional to the frequency derivative of the difference in stored electric and magnetic energies; whereas the quality factor is proportional to the total stored energy (ref. 17). The "redefined" definition of quality factor given by Rhodes in reference 18 should be used.

The establishment of a relationship between the quality factor and the reciprocal of relative bandwidth for the aperture antenna would be of analytical importance in the area of antenna synthesis (refs. 17 and 18). The evaluation of quality factor is at a single frequency (resonant frequency); whereas the evaluation of relative bandwidth requires a knowledge of the complete frequency behavior of the antenna. The quality factor and relative bandwidth for the planar dipole antenna have been shown to be related in a recip-

rocal manner by Rhodes (ref. 17). He established this relationship by calculating the quality factor and relative bandwidth from independent equations which were based on his time-average "physically observable" stored-energy representations. By considering the difference of the volume integrals that appear in the complex Poynting theorem, Rhodes was able to show that the infinities associated with the individual volume integrals canceled exactly, leaving what he refers to as time-average "physically observable" stored magnetic and electric energies.

In the present paper, expressions which represent the input admittance, the quality factor, and the relative bandwidth of the rectangular cavity-backed slot antenna are derived. For an assumed sinusoidal slot distribution and a single propagating wave in the cavity, input-admittance calculations are compared with available measured data. In addition, input-admittance curves as a function of electrical slot length are given for several size cavities. The primary purpose of this paper, however, is to determine whether the method of Rhodes implies a reciprocal relationship between the quality factor and relative bandwidth for a rectangular cavity-backed slot antenna. Thus Rhodes' concept of time-average "physically observable" stored energies is used in determining this relationship. A pair of time-average "physically observable" stored energies for the internal part of the antenna (cavity or infinite waveguide) are derived in terms of amplitude coefficients at a reference plane. These coefficients are related to the assumed sinusoidal distribution in the narrow slot aperture by applying the appropriate boundary conditions. The cross sectional dimensions of the cavity are chosen so that only one propagating wave exists. The depth of the cavity is assumed to be deep enough so that its back wall will not influence the assumed slot distribution.

The external part of the antenna is the half-space. The solution to this part is given by Rhodes in reference 17 for a slot in an infinite ground plane.

The internal and external parts of the solution are then combined by applying the complex Poynting theorem to the small volume which is formed by the thickness and the openings of the slot. This volume encloses a voltage source which is applied across the center of the slot. As the slot thickness shrinks to zero, the input power equals the sum of the power which enters the cavity (finite volume) and that which enters the half-space (infinite volume). Once the internal and external parts have been combined, the input admittance, the quality factor, and the relative bandwidth are computed numerically for a number of cavity and slot sizes. The only power loss is from radiation since all the metallic surfaces are assumed to be perfect conductors and the region inside the cavity is assumed to be a vacuum.

SYMBOLS

$\left.\begin{array}{l} A_{mn}, B_{mn}, \\ C_{mn}, D_{mn} \end{array}\right\}$ amplitude coefficients

A_1, A_2 slot aperture (see fig. 2)

a height of waveguide (or cavity)

a' width of slot (see fig. 1)

B input susceptance

B.W. relative bandwidth

b width of waveguide (or cavity)

$Cin(x)$ $= \int_0^X \dfrac{1 - \cos u}{u} \, du$

d depth of waveguide (or cavity)

$\vec{E}$ electric field

E_{xa}, E_{ya} aperture electric field

f(y) Fourier transform of $S(\omega)$

G input conductance

$\vec{H}$ magnetic field

I current

$\vec{J}$ current density

j $= \sqrt{-1}$

$K_0(\alpha y)$ modified Bessel function of the second kind

4

k	$= \omega\sqrt{\mu\epsilon}$
k_x, k_y, k_z	directional wave numbers
ℓ	length of slot (see fig. 1)
ℓ_o/λ	resonant slot length
m, n	modal numbers
P_ℓ	power loss in cavity
P_r	radiated power
Q	quality factor
$Si(x)$	$= \int_0^x \dfrac{\sin u}{u}\, du$
$S(\omega)$	Poisson sum
S_1, S_2	surface
t	time, sec
V	voltage
V_o	voltage across center of slot
v	volume of cavity
v'	volume of semi-infinite free space
v''	volume of slot
$\langle\langle W_e \rangle\rangle$	"physically observable" electric stored energy
$\langle\langle W_m \rangle\rangle$	"physically observable" magnetic stored energy
x, y, z	Cartesian coordinates of rectangular waveguide

x',y',z'   Cartesian coordinates of slot

Y   input admittance

Y_s input admittance of slot antenna

Z_d input impedance of complementary dipole antenna

Z_o characteristic impedance of free space

δ depth of slot (see fig. 1)

ϵ permittivity of free space

λ wavelength

μ permeability of free space

ω angular frequency

ω_r angular resonant frequency

Subscripts:

ap   aperture

inf   infinite waveguide

non   nonpropagating wave

pro   propagating wave

Asterisks denote complex conjugates.

GENERAL EXPRESSIONS RELATING INPUT ADMITTANCE, QUALITY FACTOR, AND RELATIVE BANDWIDTH TO STORED ENERGIES

The rectangular cavity-backed slot antenna is shown in figure 1. The antenna is divided into two parts: internal and external. The external part of the solution has already been solved in reference 17. The internal part of the solution is solved in the
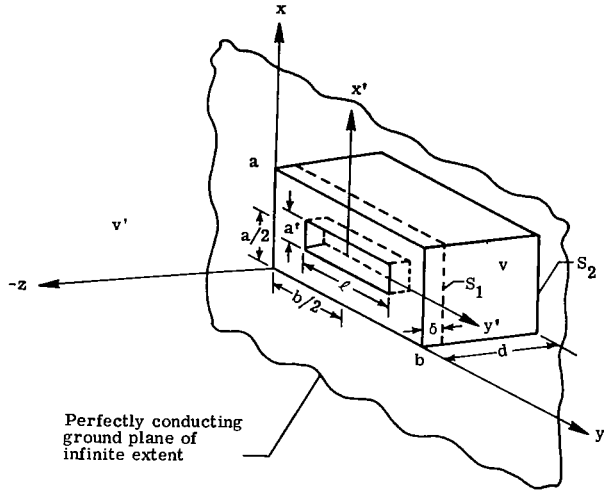
Figure 1.- Slot backed by a rectangular waveguide.
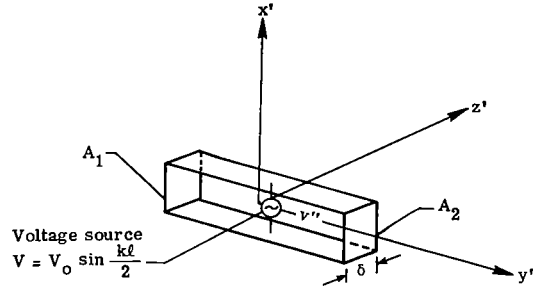
Figure 2.- Slot geometry. (Aperture A_1 empties into half-space and aperture A_2 empties into the rectangular cavity.)

present paper by using the concept of time-average "physically observable" magnetic and electric stored energies.

However, before determining these energies, the internal and external parts of the solution are combined by applying the complex Poynting theorem to the slot volume shown in figure 2. (See ref. 5.) Thus,

$$\frac{1}{2} \oiint \vec{E} \times \vec{H}^* \cdot d\vec{S} = -j2\omega\left(\frac{\mu}{4} \iiint_{v''} \vec{H} \cdot \vec{H}^* \, dv - \frac{\epsilon}{4} \iiint_{v''} \vec{E} \cdot \vec{E}^* \, dv\right) - \frac{1}{2} \iiint_{v''} \vec{E} \cdot \vec{J}^* \, dv \qquad (1)$$

$$-\frac{1}{2} \iiint_{v''} \vec{E} \cdot \vec{J}^* \, dv = \frac{1}{2} \iint_{A_1} \vec{E} \times \vec{H}^* \cdot (-\hat{z} \, dx' \, dy') + \frac{1}{2} \iint_{A_2} \vec{E} \times \vec{H}^* \cdot (+\hat{z} \, dx' \, dy')$$

$$+ j2\omega\left(\frac{\mu}{4} \iiint_{v''} \vec{H} \cdot \vec{H}^* \, dv - \frac{\epsilon}{4} \iiint_{v''} \vec{E} \cdot \vec{E}^* \, dv\right) \qquad (2)$$

The term $-\frac{1}{2} \iiint_{v''} \vec{E} \cdot \vec{J}^* \, dv$ represents input power; therefore,

$$-\frac{1}{2} \iiint_{v''} \vec{E} \cdot \vec{J}^* \, dv = +\frac{1}{2} VI^* \tag{3}$$

The first two terms on the right-hand side of equation (2) are the complex power flow into the half-space and cavity, respectively, and the last term represents the net time-average stored energy in volume v''.

As δ approaches zero, the volume v'' approaches zero, and, hence, the net stored energy in v'' is zero. Therefore,

$$\frac{1}{2} VI^* = \frac{1}{2} \iint_{A_1} \vec{E} \times \vec{H}^* \cdot (-\hat{z} \, dx' \, dy') + \frac{1}{2} \iint_{A_2} \vec{E} \times \vec{H}^* \cdot (+\hat{z} \, dx' \, dy') \tag{4}$$

The complex Poynting theorem will now be applied to the volume v' which is infinite and to the volume v which is finite (cavity). (See fig. 1.) The complex power flow into the half-space z < 0 is given by

$$\frac{1}{2} \iint_{A_1} \vec{E} \times \vec{H}^* \cdot (-\hat{z} \, dx' \, dy') = P_r + j2\omega \left(\frac{\mu}{4} \iiint_{v'} \vec{H} \cdot \vec{H}^* \, dv - \frac{\epsilon}{4} \iiint_{v'} \vec{E} \cdot \vec{E}^* \, dv \right) \tag{5}$$

and the complex power flow into the cavity z > 0 is given by

$$\frac{1}{2} \iint_{A_1} \vec{E} \times \vec{H}^* \cdot (+\hat{z} \, dx' \, dy') = P_l + j2\omega \left(\frac{\mu}{4} \iiint_{v} \vec{H} \cdot \vec{H}^* \, dv - \frac{\epsilon}{4} \iiint_{v} \vec{E} \cdot \vec{E}^* \, dv \right) \tag{6}$$

where P_r is the radiated power and P_l is the power loss in the cavity. The volume integral terms represent the time-average net stored energies in their respective volumes.

Assuming the power loss P_l in the cavity is zero, equations (4), (5), and (6) can be written as

8

$$\frac{1}{2} VI^* = P_r + j2\omega\left(\frac{\mu}{4}\iiint\limits_{v'}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{v'}\vec{E}\cdot\vec{E}^*\,dv\right)$$

$$+ j2\omega\left(\frac{\mu}{4}\iiint\limits_{V}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{V}\vec{E}\cdot\vec{E}^*\,dv\right) \tag{7}$$

Now, let $I = YV$, so that

$$\frac{1}{2} VV^*Y^* = P_r + j2\omega\left(\frac{\mu}{4}\iiint\limits_{v'}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{v'}\vec{E}\cdot\vec{E}^*\,dv\right)$$

$$+ j2\omega\left(\frac{\mu}{4}\iiint\limits_{V}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{V}\vec{E}\cdot\vec{E}^*\,dv\right) \tag{8}$$

Taking the conjugate, equation (8) becomes

$$\frac{1}{2} V^*VY = P_r - j2\omega\left(\frac{\mu}{4}\iiint\limits_{v'}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{v'}\vec{E}\cdot\vec{E}^*\,dv\right)$$

$$- j2\omega\left(\frac{\mu}{4}\iiint\limits_{V}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{V}\vec{E}\cdot\vec{E}^*\,dv\right) \tag{9}$$

The admittance is now given as

$$Y = G + jB$$

$$= \frac{P_r}{\frac{1}{2}V^*V} - \frac{j2\omega\left(\frac{\mu}{4}\iiint\limits_{v'}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{v'}\vec{E}\cdot\vec{E}^*\,dv\right)}{\frac{1}{2}V^*V}$$

$$- \frac{j2\omega\left(\frac{\mu}{4}\iiint\limits_{V}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_{V}\vec{E}\cdot\vec{E}^*\,dv\right)}{\frac{1}{2}V^*V} \tag{10}$$

Therefore,

$$G = \frac{P_r}{\frac{1}{2}VV^*} \qquad (11)$$

$$B = -\frac{2\omega\left(\frac{\mu}{4}\iiint_{v'}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint_{v'}\vec{E}\cdot\vec{E}^*\,dv\right)}{\frac{1}{2}VV^*} - \frac{2\omega\left(\frac{\mu}{4}\iiint_{v}\vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint_{v}\vec{E}\cdot\vec{E}^*\,dv\right)}{\frac{1}{2}VV^*}$$

$$(12a)$$

$$B = -\frac{2\omega\left(<<W_m>>_{v'} - <<W_e>>_{v'}\right)}{\frac{1}{2}VV^*} - \frac{2\omega\left(<<W_m>>_{v} - <<W_e>>_{v}\right)}{\frac{1}{2}VV^*} \qquad (12b)$$

$$B = \frac{-2\omega\left(<<W_m>> - <<W_e>>\right)}{\frac{1}{2}VV^*} \qquad (12c)$$

where

$$<<W_m>> - <<W_e>> = \left(<<W_m>>_{v'} - <<W_e>>_{v'}\right) + \left(<<W_m>>_{v} - <<W_e>>_{v}\right)$$

For a high Q system with a resonance frequency of ω_r,

$$\left(\frac{dG}{d\omega}\right)_{\omega_r} \approx 0$$

$$(13)$$

$$B(\omega_r) = 0$$

As one moves off resonance, the input admittance can be written as

$$Y(\omega) \approx G(\omega_r) + j(\omega - \omega_r)\left(\frac{dB}{d\omega}\right)_{\omega=\omega_r} \qquad (14)$$

10

At the half-power point,

$$Y\left(\omega_r + \frac{\Delta\omega}{2}\right) = \sqrt{2}\, Y(\omega_r) e^{j\frac{\pi}{4}} = \sqrt{2}\, G(\omega_r) e^{j\frac{\pi}{4}} \tag{15}$$

Substituting equation (15) into equation (14) and equating imaginary parts gives

$$(\omega - \omega_r)\left(\frac{dB}{d\omega}\right)_{\omega=\omega_r} \approx G(\omega_r) \tag{16a}$$

or

$$\frac{\Delta\omega}{2}\left(\frac{dB}{d\omega}\right)_{\omega=\omega_r} = G(\omega_r) \tag{16b}$$

Therefore, the relative bandwidth is

$$B.W. = \frac{\Delta\omega}{\omega_r} = \frac{2G(\omega_r)}{\omega_r \dfrac{dB}{d\omega}} \tag{17}$$

Multiplying both numerator and denominator by $\frac{1}{2} VV^*$ gives

$$B.W. = \frac{2\left[\frac{1}{2} VV^* G(\omega_r)\right]}{\omega_r \frac{1}{2} VV^* \dfrac{dB}{d\omega}} = \frac{2P_r}{\omega_r \frac{1}{2} VV^* \dfrac{dB}{d\omega}} \tag{18}$$

From equation (12c),

$$\frac{dB}{d\omega} = \frac{d}{d\omega}\left[\frac{-2\omega\left(<<W_m>> - <<W_e>>\right)}{\frac{1}{2} VV^*}\right]$$

so that

$$\left(\frac{dB}{d\omega}\right)_{\omega=\omega_r} = \frac{1}{\frac{1}{2} VV^*} \frac{d}{d\omega}\left[-2\omega\left(<<W_m>> - <<W_e>>\right)\right]_{\omega=\omega_r} \tag{19}$$

$$\frac{1}{2} VV^* \left(\frac{dB}{d\omega}\right)_{\omega=\omega_r} = -\frac{d}{d\omega}\left[2\omega\left(<<W_m>> - <<W_e>>\right)\right]_{\omega=\omega_r} \tag{20}$$

and, thus,

$$\text{B.W.} = \left[\frac{2P_r}{-\omega_r \dfrac{d}{d\omega} 2\omega\left(<<W_m>> - <<W_e>>\right)}\right]_{\omega=\omega_r} \tag{21a}$$

or

$$\text{B.W.} = \frac{2P_r}{-\omega_r\left\{\dfrac{d}{d\omega}\left[2\omega\left(<<W_m>>_{v'} - <<W_e>>_{v'}\right)\right] + \dfrac{d}{d\omega}\left[2\omega\left(<<W_m>>_{v} - <<W_e>>_{v}\right)\right]\right\}_{\omega=\omega_r}} \tag{21b}$$

The concept of quality factor Q is usually defined as being proportional to the sum of the time-average magnetic and electric stored energies. However, an inconsistency in this classical definition has been asserted by Rhodes in reference 18. This definition includes parts of the magnetic and electric stored energies that can not be observed since these energies are obtained by treating the volume integrals individually in the complex Poynting theorem. This inconsistency has been removed by Rhodes by redefining Q in terms of the time-average "physically observable" stored magnetic and electric energies, determined through differencing the volume integrals in the complex Poynting theorem. This "redefined" definition of Q is

$$Q = \left[\frac{\omega\left(<<W_m>> + <<W_e>>\right)}{P_r}\right]_{\omega=\omega_r} \tag{22}$$

In terms of the volumes considered in the present paper, the quality factor becomes

$$Q = \left[\frac{\omega\left(<<W_m>>_{v'} + <<W_m>>_{v} + <<W_e>>_{v'} + <<W_e>>_{v}\right)}{P_r}\right]_{\omega=\omega_r} \tag{23}$$

12

But at resonance the observable magnetic and electric energies are equal, that is,

$$<<W_m>>_{v'} + <<W_m>>_v = <<W_e>>_{v'} + <<W_e>>_v \tag{24}$$

Hence,

$$Q = \left[\frac{2\omega\left(<<W_{m,e}>>_{v'} + <<W_{m,e}>>_v\right)}{P_r}\right]_{\omega=\omega_r} \tag{25}$$

where the subscript m,e means that either magnetic or electric "physically observable" stored energy is used.

DERIVATION OF TIME-AVERAGE MAGNETIC AND ELECTRIC VOLUME INTEGRALS IN A RECTANGULAR WAVEGUIDE SECTION

The volume integrals appearing in the complex Poynting theorem are given as $\frac{\mu}{4}\iiint_V \vec{H} \cdot \vec{H}^* \, dv$ and $\frac{\epsilon}{4}\iiint_V \vec{E} \cdot \vec{E}^* \, dv$. The fields for a rectangular waveguide will now be substituted into each of these volume integrals. Each volume integral will be carried through in its entirety.

The total magnetic field is

$$\vec{H} = \hat{x}H_x + \hat{y}H_y + \hat{z}H_z \tag{26}$$

where the components are given by equations (A23) in appendix A. Substituting equation (26) into the volume integral results in

$$\frac{\mu}{4}\iiint_V \vec{H} \cdot \vec{H}^* \, dv = \frac{\mu}{4}\iiint_V \left(\hat{x}H_x + \hat{y}H_y + \hat{z}H_z\right) \cdot \left(\hat{x}H_x^* + \hat{y}H_y^* + \hat{z}H_z^*\right)dv \tag{27}$$

where the volume is bounded by the walls of the rectangular waveguide and transverse planes S_1(where $z = \delta$) and S_2(where $z = d$). Therefore,

$$\frac{\mu}{4}\iiint_V \vec{H} \cdot \vec{H}^* \, dv = \frac{\mu}{4}\iiint_V \left(H_x H_x^* + H_y H_y^* + H_z H_z^*\right)dv \tag{28}$$

13

Each term of the integrand (eq. (28)) is written out explicitly as follows:

$$H_x H_x^* = \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[-\frac{1}{k_z \omega \mu} \left(\left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn} \right\} e^{-jk_z z} - \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) B_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] D_{mn} \right\} e^{jk_z z} \right) \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \right] \right\} \left\{ \sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} \left[-\frac{1}{k_z^* \omega \mu} \left(\left\{ \left(\frac{m'\pi}{a}\right)\left(\frac{n'\pi}{b}\right) A_{m'n'}^* \right. \right. \right. \right.$$

$$\left. \left. \left. + \left[\left(\frac{n'\pi}{b}\right)^2 + k_z^2\right] C_{m'n'}^* \right\} e^{jk_z^* z} - \left\{ \left(\frac{m'\pi}{a}\right)\left(\frac{n'\pi}{b}\right) B_{m'n'}^* + \left[\left(\frac{n'\pi}{b}\right)^2 + k_z^2\right] D_{m'n'}^* \right\} e^{-jk_z^* z} \right) \sin\frac{m'\pi}{a}x \cos\frac{n'\pi}{b}y \right] \right\} \qquad (29a)$$

$$H_y H_y^* = \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[+\frac{1}{k_z \omega \mu} \left(\left\{ \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right] A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right\} e^{-jk_z z} - \left\{ \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right] B_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) D_{mn} \right\} e^{jk_z z} \right) \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right] \right\} \left\{ \sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} \left[\frac{1}{k_z^* \omega \mu} \left(\left\{ \left[k_z^2 + \left(\frac{m'\pi}{a}\right)^2\right] A_{m'n'}^* \right. \right. \right. \right.$$

$$\left. \left. \left. + \left(\frac{m'\pi}{a}\right)\left(\frac{n'\pi}{b}\right) C_{m'n'}^* \right\} e^{+jk_z^* z} - \left\{ \left[k_z^2 + \left(\frac{m'\pi}{a}\right)^2\right] B_{m'n'}^* + \left(\frac{m'\pi}{a}\right)\left(\frac{n'\pi}{b}\right) D_{m'n'}^* \right\} e^{-jk_z^* z} \right) \cos\frac{m'\pi}{a}x \sin\frac{n'\pi}{b}y \right] \right\} \qquad (29b)$$

$$H_z H_z^* = \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ -\frac{1}{j\omega\mu} \left[\left(\frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn}\right) e^{-jk_z z} + \left(\frac{m\pi}{a} D_{mn} - \frac{n\pi}{b} B_{mn}\right) e^{jk_z z} \right] \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \right\} \right) \left(\sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} \left\{ \frac{1}{j\omega\mu} \left[\frac{m'\pi}{a} C_{m'n'}^* - \frac{n'\pi}{b} A_{m'n'}^* \right) e^{jk_z z} \right. \right.$$

$$\left. \left. + \left(\frac{m\pi}{a} D_{m'n'}^* - \frac{n\pi}{b} B_{m'n'}^*\right) e^{-jk_z z} \right] \cos\frac{m'\pi}{a}x \cos\frac{n'\pi}{b}y \right\} \right) \qquad (29c)$$

The integration over the cross section is performed first. Due to orthogonality, the integrations for $m \neq m'$ and $n \neq n'$ are zero. In addition, the resulting integrations are weighted differently depending on m and n. When $m = 0$ in equation (29a) and $n = 0$ in equation (29b), the integrations are identically zero. When $m \neq 0$ and $n = 0$ in equation (29a), $n \neq 0$ and $m = 0$ in equation (29b), and $m \neq 0$, $n = 0$, $n \neq 0$, and $m = 0$ in equation (29c), the integrations result in a multiplying factor of $1/2$. When $m \neq 0$ and $n \neq 0$, the integrations of equations (29) produce a multiplying factor of $1/4$. For $m = 0$ and $n = 0$, the multiplying factor resulting from the integration of equation (29b) is 1. The integrations over the cross section for equations (29) can, therefore, be written as

$$\int_{y=0}^{b} \int_{x=0}^{a} H_x H_x^* \, dx \, dy = \frac{ab}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma_m \epsilon_n}{k_z k_z^* \omega^2 \mu^2} \left(\left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn} \right\} \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn}^* + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn}^* \right\} e^{-j(k_z - k_z^*)z} - \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} \right. \right.$$

$$\left. + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn} \right\} \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) B_{mn}^* + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] D_{mn}^* \right\} e^{-j(k_z + k_z^*)z} - \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) B_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] D_{mn} \right\} \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn}^* \right.$$

$$\left. + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn}^* \right\} e^{j(k_z + k_z^*)z} + \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) B_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] D_{mn} \right\} \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) B_{mn}^* + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] D_{mn}^* \right\} e^{j(k_z - k_z^*)z} \right) \qquad (30a)$$

$$\int_{y=0}^{b}\int_{x=0}^{a} H_y H_y^* \, dx \, dy = \frac{ab}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_n \epsilon_m}{k_z k_z^* \omega^2 \mu^2}\left(\left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right\}\left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}^*+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}^*\right\}e^{-j\left(k_z-k_z^*\right)z} - \left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}\right.$$

$$\left.+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right\}\left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]B_{mn}^*+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}^*\right\}e^{-j\left(k_z+k_z^*\right)z} - \left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]B_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}\right\}\left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}^*\right.$$

$$\left.+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}^*\right\}e^{j\left(k_z+k_z^*\right)z} + \left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]B_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}\right\}\left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]B_{mn}^*+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}^*\right\}e^{j\left(k_z-k_z^*\right)z}\right) \tag{30b}$$

$$\int_{y=0}^{b}\int_{x=0}^{a} H_z H_z^* \, dx \, dy = \frac{ab}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\epsilon_m \epsilon_n}{\omega^2 \mu^2}\left[\left(\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right)\left(\frac{m\pi}{a}C_{mn}^*-\frac{n\pi}{b}A_{mn}^*\right)e^{-j\left(k_z-k_z^*\right)z}+\left(\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right)\left(\frac{m\pi}{a}D_{mn}^*-\frac{n\pi}{b}B_{mn}^*\right)e^{-j\left(k_z+k_z^*\right)z}\right.$$

$$\left.+\left(\frac{m\pi}{a}D_{mn}-\frac{n\pi}{b}B_{mn}\right)\left(\frac{m\pi}{a}C_{mn}^*-\frac{n\pi}{b}A_{mn}^*\right)e^{j\left(k_z+k_z^*\right)z}+\left(\frac{m\pi}{a}D_{mn}-\frac{n\pi}{b}B_{mn}\right)\left(\frac{m\pi}{a}D_{mn}^*-\frac{n\pi}{b}B_{mn}^*\right)e^{+j\left(k_z-k_z^*\right)z}\right] \tag{30c}$$

$$\gamma_{m,n} = \begin{cases} 0 & (m,n=0) \\ 1 & (m,n\neq 0) \end{cases} \tag{31a}$$

$$\epsilon_{m,n} = \begin{cases} 2 & (m,n=0) \\ 1 & (m,n\neq 0) \end{cases} \tag{31b}$$

Now, rewrite equations (30) as

$$\int_{y=0}^{b}\int_{x=0}^{a} H_x H_x^* \, dx \, dy = \frac{ab}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_m \epsilon_n}{k_z k_z^* \omega^2 \mu^2}\left[\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]C_{mn}\right|^2 e^{-j\left(k_z-k_z^*\right)z} - 2\,\mathrm{Re}\left(\left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]C_{mn}\right\}\left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)B_{mn}^*\right.\right.\right.$$

$$\left.\left.\left.+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]D_{mn}^*\right\}e^{-j\left(k_z+k_z^*\right)z}\right)+\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)B_{mn}+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]D_{mn}\right|^2 e^{j\left(k_z-k_z^*\right)z}\right] \tag{32a}$$

$$\int_{y=0}^{b}\int_{x=0}^{a} H_y H_y^* \, dx \, dy = \frac{ab}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_n \epsilon_m}{k_z k_z^* \omega^2 \mu^2}\left[\left|\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2 e^{-j\left(k_z-k_z^*\right)z} - 2\,\mathrm{Re}\left(\left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right\}\left\{\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]B_{mn}^*\right.\right.\right.$$

$$\left.\left.\left.+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}^*\right\}e^{-j\left(k_z+k_z^*\right)z}\right)+\left|\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]B_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}\right|^2 e^{j\left(k_z-k_z^*\right)z}\right] \tag{32b}$$

15

$$\int_{y=0}^{b} \int_{x=0}^{a} H_z H_z^* \, dx \, dy = \frac{ab}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon_m \epsilon_n}{\omega^2 \mu^2} \left\{ \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 e^{-j(k_z - k_z^*)z} + 2 \operatorname{Re}\left[\left(\frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right)\left(\frac{m\pi}{a} D_{mn}^* - \frac{n\pi}{b} B_{mn}^* \right) e^{-j(k_z + k_z^*)z} \right] \right.$$

$$\left. + \left| \frac{m\pi}{a} D_{mn} - \frac{n\pi}{b} B_{mn} \right|^2 e^{j(k_z - k_z^*)z} \right\} \tag{32c}$$

Before integrating on z, the wave number k_z will be written explicitly for propagating waves as

$$k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad \left(k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right) \tag{33a}$$

and for nonpropagating waves as

$$k_z = -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} \qquad \left(k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right) \tag{33b}$$

Substituting for k_z and integrating on z (the integration on z is over the limits z = δ and z = d; the distance δ is then allowed to approach zero), the total magnetic volume integral becomes the following: For propagating waves,

$$\left(\frac{\mu}{4} \iiint_v \bar{H} \cdot \bar{H}^* \, dv \right)_{pro} = \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left\{ \gamma_m \epsilon_n \left[\frac{1}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right] C_{mn} \right|^2 d - 2 \operatorname{Re}\left(\frac{1}{k_z^2} \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right] C_{mn} \right\} \left\{ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) B_{mn}^* \right. \right. \right. $$

$$\left. \left. + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right] D_{mn}^* \right\} \frac{e^{-j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} - 1}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} + \frac{1}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) B_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right] D_{mn} \right|^2 d \right) \right] + \gamma_n \epsilon_m \left[\frac{1}{k_z^2} \left[\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] A_{mn} \right. \right. $$

$$\left. \left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 d - 2 \operatorname{Re}\left(\frac{1}{k_z^2} \left\{ \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right\} \left\{ \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] B_{mn}^* + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) D_{mn}^* \right\} \frac{e^{-j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} - 1}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right. \tag{34}$$

(Equation continued on next page)

$$+ \frac{1}{k_z^2}\left[\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]B_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}\right]^2 d\right)\right] + \epsilon_m\epsilon_n\left\{\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2 d + 2\,\mathrm{Re}\left(\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right)\left(\frac{m\pi}{a}D_{mn}^*\right)\right.$$

$$- \frac{n\pi}{b}B_{mn}^*\right)\frac{e^{-j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} - 1}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} + \left|\frac{m\pi}{a}D_{mn} - \frac{n\pi}{b}B_{mn}\right|^2 d\right\}\right\}_{k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \qquad (34)$$

For nonpropagating waves,

$$\left(\frac{\mu}{4}\iiint_V \vec{H}\cdot\vec{H}^*\,dv\right)_{non} = \frac{ab\mu}{16}\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{1}{\omega^2\mu^2}\left\{\gamma_m\epsilon_n\left[-\frac{1}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 2\,\mathrm{Re}\left(-\frac{1}{k_z^2}\left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}\right.\right.\right.$$

$$\left.+ \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right\}\left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)B_{mn}^* + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]D_{mn}^*\right\}d\right) - \frac{1}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)B_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]D_{mn}\right|^2 \frac{e^{2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right]$$

$$+ \gamma_n\epsilon_m\left[-\frac{1}{k_z^2}\left|\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 2\,\mathrm{Re}\left(-\frac{1}{k_z^2}\left\{\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn}\right.\right.\right.$$

$$\left.+ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}^*\right\}\left\{\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]B_{mn}^* + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}^*\right\}d\right) - \frac{1}{k_z^2}\left|\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]B_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}\right|^2 \frac{e^{2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right] \qquad (35)$$

(Equation continued on next page)

17

$$+ \epsilon_m\epsilon_n \left\{ \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} + 2 \, \mathrm{Re}\left[\left(\frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn}\right)\left(\frac{m\pi}{a} D^*_{mn} - \frac{n\pi}{b} B^*_{mn}\right) d \right] \right.$$

$$\left. + \left| \frac{m\pi}{a} D_{mn} - \frac{n\pi}{b} B_{mn} \right|^2 \frac{e^{2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right\} \Bigg\}_{k_z = -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \tag{35}$$

The total magnetic volume integral is the sum of the volume integrals for propagating and nonpropagating waves; that is,

$$\frac{\mu}{4} \iiint_V \vec{H} \cdot \vec{H}^* \, dv = \left(\frac{\mu}{4} \iiint_V \vec{H} \cdot \vec{H}^* \, dv \right)_{pro} + \left(\frac{\mu}{4} \iiint_V \vec{H} \cdot \vec{H}^* \, dv \right)_{non} \tag{36}$$

Attention is now turned to the volume integral of the electric field. The total electric field is given as

$$\vec{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z \tag{37}$$

where the components E_x, E_y, and E_z are given by equations (A18). Substituting equation (37) into the volume integral results in

$$\frac{\epsilon}{4} \iiint_V \vec{E} \cdot \vec{E}^* \, dv = \frac{\epsilon}{4} \iiint_V \left(E_x E_x^* + E_y E_y^* + E_z E_z^* \, dv \right) \tag{38}$$

where

$$E_x E_x^* = \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn} e^{-jk_z z} + B_{mn} e^{jk_z z} \right) \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \right]$$

$$\times \left[\sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} \left(A_{m'n'}^* e^{jk_z^* z} + B_{m'n'}^* e^{-jk_z^* z} \right) \cos \frac{m'\pi}{a} x \sin \frac{n'\pi}{b} y \right] \tag{39a}$$

18

$$E_y E_y^* = \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(C_{mn} e^{-jk_z z} + D_{mn} e^{jk_z z} \right) \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \right]$$

$$\times \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(C_{m'n'}^* e^{+jk_z^* z} + D_{m'n'}^* e^{-jk_z^* z} \right) \sin \frac{m'\pi}{a} x \cos \frac{n'\pi}{b} y \right] \qquad (39b)$$

$$E_z E_z^* = \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} -\frac{1}{jk_z} \left[\left(\frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right) e^{-jk_z z} - \left(\frac{m\pi}{a} B_{mn} \right. \right. \right.$$

$$\left. \left. \left. + \frac{n\pi}{b} D_{mn} \right) e^{jk_z z} \right] \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \right\}$$

$$\times \left\{ \sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} +\frac{1}{jk_z^*} \left[\left(\frac{m'\pi}{a} A_{m'n'}^* + \frac{n'\pi}{b} C_{m'n'}^* \right) e^{jk_z^* z} - \left(\frac{m'\pi}{a} B_{m'n'}^* \right. \right. \right.$$

$$\left. \left. \left. + \frac{n'\pi}{b} D_{m'n'}^* \right) e^{-jk_z^* z} \right] \sin \frac{m'\pi}{a} x \sin \frac{n'\pi}{b} y \right\} \qquad (39c)$$

As before, integration over the cross section is performed first, i.e.,

$$\int_{y=0}^{b} \int_{x=0}^{a} \vec{E} \cdot \vec{E}^* \, dx \, dy = \frac{ab}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \gamma_n \epsilon_m \left(A_{mn} e^{-jk_z z} + B_{mn} e^{jk_z z} \right) \left(A_{mn}^* e^{jk_z^* z} + B_{mn} e^{-jk_z^* z} \right) \right.$$

$$+ \gamma_m \epsilon_n \left(C_{mn} e^{-jk_z z} + D_{mn} e^{jk_z z} \right) \left(C_{mn}^* e^{jk_z z} + D_{mn}^* e^{-jk_z^* z} \right)$$

$$+ \frac{\gamma_m \gamma_n}{k_z k_z^*} \left[\left(\frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right) e^{-jk_z z} - \left(\frac{m\pi}{a} B_{mn} \right. \right.$$

$$\left. \left. + \frac{n\pi}{b} D_{nm} \right) e^{jk_z z} \right] \left[\left(\frac{m\pi}{a} A_{mn}^* + \frac{n\pi}{b} C_{mn}^* \right) e^{jk_z^* z} \right.$$

$$\left. \left. - \left(\frac{m\pi}{a} B_{mn}^* + \frac{n\pi}{b} D_{mn}^* \right) e^{-jk_z^* z} \right] \right\} \qquad (40)$$

19

where

$$\gamma_{m,n} = \begin{cases} 0 \\ 1 \end{cases} \qquad \epsilon_{m,n} = \begin{cases} 2 & (m,n = 0) \\ 1 & (m,n \neq 0) \end{cases} \tag{41}$$

and orthogonality has been taken into account.

The volume integrals of the electric field for propagating and nonpropagating waves will now be determined in a manner similar to that for the magnetic volume integrals. These electric volume integrals become the following: For propagating waves (eq. (33a)),

$$\left(\frac{\epsilon}{4}\iiint_V \bar{E} \cdot \bar{E}^* \, dv\right)_{pro} = \frac{abc}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\gamma_n \epsilon_m \left\{ |A_{mn}|^2 d + 2 \operatorname{Re} \left[A_{mn} B_{mn}^* \frac{e^{-j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} - 1}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right] + |B_{mn}|^2 d \right\} + \gamma_m \epsilon_n \left\{ |C_{mn}|^2 d + 2 \operatorname{Re} \left[C_{mn} D_{mn}^* \frac{e^{-j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} - 1}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right] \right. \right.$$

$$+ |D_{mn}|^2 d \left. \right\} + \gamma_m \gamma_n \left\{ \frac{1}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 d - 2 \operatorname{Re} \left[\frac{1}{k_z^2} \left(\frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right) \left(\frac{m\pi}{a} B_{mn}^* + \frac{n\pi}{b} D_{mn}^* \right) \frac{e^{-j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} - 1}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right] \right.$$

$$\left. \left. + \frac{1}{k_z^2} \left| \frac{m\pi}{a} B_{mn} + \frac{n\pi}{b} D_{mn} \right|^2 d \right\} \right)_{k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \tag{42}$$

For nonpropagating waves (eq. (33b)),

$$\left(\frac{\epsilon}{4}\iiint_V \bar{E} \cdot \bar{E}^* \, dv\right)_{non} = \frac{abc}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\gamma_n \epsilon_m \left[|A_{mn}|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} + 2 \operatorname{Re}\left(A_{mn} B_{mn}^* d\right) + |B_{mn}|^2 \frac{e^{2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] + \gamma_m \epsilon_n \left[|C_{mn}|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right. \right.$$

$$\left. + 2 \operatorname{Re}\left(C_{mn} D_{mn}^* d\right) + |D_{mn}|^2 \frac{e^{2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] + \gamma_m \gamma_n \left\{ -\frac{1}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right.$$

$$\left. \left. + \frac{2}{k_z^2} \operatorname{Re}\left[\left(\frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn}\right)\left(\frac{m\pi}{a} B_{mn}^* + \frac{n\pi}{b} D_{mn}^*\right)d\right] + \frac{1}{k_z^2} \left| \frac{m\pi}{a} B_{mn} + \frac{n\pi}{b} D_{mn} \right|^2 \frac{e^{2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right\} \right)_{k_z = -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \tag{43}$$

Therefore, the total electric volume integral is

$$\frac{\epsilon}{4} \iiint\limits_{V} \vec{E} \cdot \vec{E}^* \, dv = \left(\frac{\epsilon}{4} \iiint\limits_{V} \vec{E} \cdot \vec{E}^* \, dv \right)_{pro} + \left(\frac{\epsilon}{4} \iiint\limits_{V} \vec{E} \cdot \vec{E}^* \, dv \right)_{non} \qquad (44)$$

Derivation of Time-Average "Physically Observable" Stored Energies
of an Infinite Rectangular Waveguide

When the complex Poynting theorem is applied to radiating systems, or to any closed volume, the volume integrals of $\frac{\mu}{4} \vec{H} \cdot \vec{H}^*$ and $\frac{\epsilon}{4} \vec{E} \cdot \vec{E}^*$ appear in the equation only through their difference. However, in the literature (e.g., refs. 5, 6, and 8) these volume integrals are usually interpreted individually. It has been shown by Rhodes in reference 17 and Collin and Rothschild in reference 14 that by considering an infinite volume (planar antenna) for certain aperture distributions such an interpretation leads to infinite stored energies. Rhodes showed that by treating the difference of the volume integrals the infinities cancelled exactly. From the remaining parts of the difference he determined integral representations of the "physically observable" time-average magnetic and electric stored energies.

An analogous problem to the planar antenna would be the infinite rectangular waveguide which can be identified with figure 1 for the case where d is infinite, a' = a, and ℓ = b. Applying the complex Poynting theorem to the volume v shown in figure 1 produces the volume integrals given by equations (36) and (44). The surface S_2 at z = d is treated as a mathematical boundary; that is, the coefficients B_{mn} and D_{mn} corresponding to no wave traveling in the z-direction are zero. For a source (aperture) that is located at the z = 0 plane, with perfectly conducting walls and no losses within the waveguide, radiation occurs through z = d as d → ∞. Thus, setting B_{mn} and D_{mn} equal to zero in equations (35) and (43) and allowing d to approach infinity, the magnetic and electric volume integrals become

$$\left(\frac{\mu}{4} \iiint\limits_{V} \vec{H} \cdot \vec{H}^* \, dv \right)_{inf} = \lim_{d \to \infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left\{ \frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right) A_{mn} + \left[\left(\frac{m\pi}{a} \right)^2 + k_z^2 \right] C_{mn} \right|^2 d + \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a} \right)^2 \right] A_{mn} + \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right) C_{mn} \right|^2 d + \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} \right. \right.$$

$$\left. - \frac{n\pi}{b} A_{mn} \right|^2 d \Bigg\}_{k_z = \sqrt{k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2}} + \lim_{d \to \infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left\{ - \frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right) A_{mn} + \left[\left(\frac{n\pi}{b} \right)^2 + k_z^2 \right] C_{mn} \right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2}} \right.$$

$$\left. - \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a} \right)^2 \right] A_{mn} + \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right) C_{mn} \right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2}} + \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2}} \right\}_{k_z = -j\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2}} \qquad (45)$$

$$\left(\frac{\epsilon}{4}\iiint_V \vec{E}\cdot\vec{E}^* \, dv\right)_{inf} = \lim_{d\to\infty}\frac{abe}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\gamma_n\epsilon_m\left|A_{mn}\right|^2 d + \gamma_m\epsilon_n\left|C_{mn}\right|^2 d + \frac{\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn}\right|^2 d\right)_{k_z=\sqrt{k^2\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}$$

$$+ \lim_{d\to\infty}\frac{abe}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\gamma_n\epsilon_m\left|A_{mn}\right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} + \gamma_m\epsilon_n\left|C_{mn}\right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} - \frac{\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn}\right.\right.$$

$$\left.\left. + \frac{n\pi}{b}C_{mn}\right|^2 \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]_{k_z=-j\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} \tag{46}$$

It is shown in appendix B that the contributions to the electric and magnetic volume integrals from the propagating waves are identical. Therefore, when the volume integrals are differenced, these contributions cancel identically. In addition, it is also shown in appendix B that further cancellations occur in the volume integrals when the terms are written in a certain form. Before discussing these cancellations, it is essential to offer some physical interpretation which would justify writing the terms in a proper form for carrying out these cancellations.

The coefficients A_{mn} and C_{mn} are determined by matching the aperture distribution with the waveguide fields at $z = \delta$ as δ tends to zero; that is,

$$E_{xa}(x,y,k) = \lim_{\delta\to 0} E_x(x,y,\delta,k) = \lim_{\delta\to 0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,e^{-jk_z\delta} \tag{47a}$$

$$E_{ya}(x,y,k) = \lim_{\delta\to 0} E_y(x,y,\delta,k) = \lim_{\delta\to 0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\,e^{-jk_z\delta} \tag{47b}$$

Multiply equation (47a) by $\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$ and equation (47b) by $\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y$, and integrate over the waveguide cross section. For planar apertures in perfectly conducting metal, E_{xa} and E_{ya} outside the aperture are zero; hence,

$$A_{mn} = \frac{\iint_{ap} E_{xa}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dx\,dy}{ab/4} \qquad (m,n \neq 0) \tag{48a}$$

$$C_{mn} = \frac{\iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \, dx \, dy}{ab/4} \qquad (m,n \neq 0) \qquad (48b)$$

$$A_{0n} = \frac{\iint_{ap} E_{xa} \sin \frac{n\pi}{b} y \, dx \, dy}{ab/2} \qquad (n \neq 0) \qquad (48c)$$

$$C_{m0} = \frac{\iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \, dx \, dy}{ab/2} \qquad (m \neq 0) \qquad (48d)$$

These coefficients are finite for all values of frequencies since the electric-field distributions in the aperture are well-defined functions of frequency. Taking this into account, the volume integrals given by equations (45) and (46) become infinite, in general, as $k_z \to 0$ because the contributions from the E_z, H_x, and H_y components to the volume integrals become infinite. The contributions from the E_x, E_y, and H_z components are finite. When the volume integrals are differenced, these infinities must cancel in some way since the surface integral in the complex Poynting theorem is finite for all frequencies. Following Rhodes (ref. 17), one may argue that since these canceling terms disappear from the complex Poynting theorem for all frequencies they must then have no physical significance and may therefore be neglected in the volume integrals, even when these integrals are considered separately. Those terms which do not so cancel are accordingly referred to as "physically observable." The difference is written as

$$\left(\frac{\mu}{4} \iiint_V \vec{H} \cdot \vec{H}^* \, dv - \frac{\epsilon}{4} \iiint_V \vec{E} \cdot \vec{E}^* \, dv \right)_{inf} = \lim_{d \to \infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left(\left\{ -\frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{m\pi}{a}\right)^2 + k_z^2 \right] C_{mn} \right|^2 - \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 \right.$$

$$\left. - \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 \right\} + 2\epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 \left. \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right) \Bigg|_{k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

$$- \lim_{d \to \infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(2\gamma_n \epsilon_m \left| A_{mn} \right|^2 + 2\gamma_m \epsilon_n \left| C_{mn} \right|^2 + \left\{ -\gamma_n \epsilon_m \left| A_{mn} \right|^2 - \gamma_m \epsilon_n \left| C_{mn} \right|^2 - \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} \right. \right. \right.$$

$$\left. \left. + \frac{n\pi}{b} C_{mn} \right|^2 \right\} \left. \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right) \Bigg|_{k_z = -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \qquad (49)$$

23

It is shown in appendix C that the expressions associated with the two braced terms in equation (49) vanish. Therefore, the "physically observable" stored energies for an infinite rectangular waveguide are

$$<<W_m>>_{inf} = \frac{ab}{16\omega^2\mu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\epsilon_m \epsilon_n \frac{\left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right]_{k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{50a}$$

$$<<W_e>>_{inf} = \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{\gamma_n \epsilon_m |A_{mn}|^2 + \gamma_m \epsilon_n |C_{mn}|^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right]_{k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{50b}$$

Notice the similarity in this result (eqs. (50)) to the result given by Rhodes for the planar antenna (ref. 17), which is another radiating system with infinite volume. The energies stored in a section of uniform waveguide are usually given in the literature as energy per unit length (see ref. 19). To determine the energies stored in a specified section of length one should then multiply by the length. However, for an infinite length such procedure would result in infinite stored energies. The stored energies represented by the preceding equations (50) show that they remain finite even for an infinite section of waveguide.

Derivation of Time-Average "Physically Observable" Stored Energies

of a Rectangular Cavity

When a perfect electric conductor is placed at z = d, the reflected amplitude coefficients B_{mn} and D_{mn} are not zero. Instead, these coefficients are related to the amplitude coefficients A_{mn} and C_{mn}, respectively. This relationship is established by applying the boundary condition at z = 0 and z = d on the transverse electric fields. At z = 0, the coefficients are determined in the same manner as given before; that is,

$$A_{mn} + B_{mn} = \frac{\iint_{ap} E_{xa} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dS}{ab/4} \qquad (m,n \neq 0) \tag{51a}$$

24

$$C_{mn} + D_{mn} = \frac{\iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \, dS}{ab/4} \qquad (m,n \neq 0) \qquad (51b)$$

$$A_{0n} + B_{0n} = \frac{\iint_{ap} E_{xa} \sin \frac{n\pi}{b} y \, dS}{ab/2} \qquad (n \neq 0) \qquad (51c)$$

$$C_{m0} + D_{m0} = \frac{\iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \, dS}{ab/2} \qquad (m \neq 0) \qquad (51d)$$

At $z = d$,

$$\left. \begin{aligned} A_{mn} e^{-jk_z d} + B_{mn} e^{jk_z d} = 0 \\[2mm] C_{mn} e^{-jk_z d} + D_{mn} e^{jk_z d} = 0 \end{aligned} \right\} \qquad (52)$$

Hence,

$$\left. \begin{aligned} B_{mn} = -A_{mn} e^{-j2k_z d} \\[2mm] D_{mn} = -C_{mn} e^{-j2k_z d} \end{aligned} \right\} \qquad (53)$$

where k_z is given by equations (33). By substituting equations (53) for B_{mn} and D_{mn} into equations (34), (35), (42), and (43), the magnetic and electric volume integrals become the following:

$$\frac{\mu}{4}\iiint_V \bar{H}\cdot\bar{H}^*\,dv = \frac{ab\mu}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\omega^2\mu^2}\left\{\frac{2\gamma_m\epsilon_n}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}+\left[\left(\frac{m\pi}{a}\right)^2+k_z^2\right]C_{mn}\right|^2\left[d-\operatorname{Re}\frac{1-e^{j2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}}{j2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+\frac{2\gamma_n\epsilon_m}{k_z^2}\left|\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\frac{m\pi}{a}\left(\frac{n\pi}{b}\right)C_{mn}\right|^2\right.$$

$$\times\left[d-\operatorname{Re}\frac{1-e^{j2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}}{j2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right|^2\left[d+\operatorname{Re}\frac{1-e^{j2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}}{j2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right\}_{k_z=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}+\frac{ab\mu}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\omega^2\mu^2}\left(\gamma_m\epsilon_n\left\{\frac{1}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}\right.\right.\right.$$

$$+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]C_{mn}\Bigg|^2\frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-\frac{1}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]C_{mn}\right|^2\frac{e^{-4d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-\frac{2}{k_z^2}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}$$

$$+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]C_{mn}\Bigg|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\,d\right\}+\gamma_n\epsilon_m\left\{\frac{1}{k_z^2}\left|\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2\frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-\frac{1}{k_z^2}\left|\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}\right.\right.$$

$$+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\Bigg|^2\frac{e^{-4d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-\frac{2}{k_z^2}\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\Bigg|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\,d\right\}+\epsilon_m\epsilon_n\left\{\left|\frac{n\pi}{a}C_{mn}\right.\right.$$

$$-\frac{n\pi}{b}A_{mn}\Bigg|^2\frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}+\left|\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right|^2\frac{e^{-4d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-1}{2\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}-2\left|\frac{m\pi}{a}C_{mn}\right.$$

$$\left.\left.\left.-\frac{n\pi}{b}A_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\,d\right\}\right)_{k_z=-j\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} \tag{54}$$

By expressing the exponentials in their trigonometric and hyperbolic representations, equation (54) is written as follows:

$$\frac{\mu}{4}\iiint_V H\cdot H^*\,dv = \frac{ab\mu}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\omega^2\mu^2}\left\{\frac{2\gamma_m\epsilon_n}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}+\left[\left(\frac{m\pi}{a}\right)^2+k_z^2\right]C_{mn}\right|^2\left[d+\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+\frac{2\gamma_n\epsilon_m}{k_z^2}\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}\right|$$

$$+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\Bigg|^2\left[d+\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right|^2\left[d-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right\}_{k_z=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}} \tag{55}$$

(Equation continued on next page)

$$+ \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2\mu^2} \left\{ \left| \frac{\gamma_m \epsilon_n}{k_z^2} \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{m\pi}{a}\right)^2 + k_z^2\right] C_{mn} \right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[-2d - \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \right.$$

$$+ \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \frac{m\pi^2}{a}\right] A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[-2d - \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] - \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} \right.$$

$$\left. - \frac{n\pi}{b} A_{mn} \right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[2d - \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \right\}_{k_z = -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \qquad (55)$$

$$\frac{\epsilon}{4} \iiint_V \vec{E} \cdot \vec{E}^* \, dv = \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ 2\gamma_n \epsilon_m |A_{mn}|^2 \left[d - \text{Re} \frac{1 - e^{j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right] + 2\gamma_m \epsilon_n |C_{mn}|^2 \left[d - \text{Re} \frac{1 - e^{j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right] + \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 \right.$$

$$\times \left[d + \text{Re} \frac{1 - e^{j2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}}{-j2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right] \right\}_{k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} + \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \gamma_n \epsilon_m |A_{mn}|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[-2d + \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \right.$$

$$+ \gamma_m \epsilon_n |C_{mn}|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[-2d + \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] - \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$\left. \times \left[2d - \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \right\}_{k_z = -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \qquad (56)$$

27

$$\frac{\epsilon}{4}\iiint_V \bar{E}\cdot\bar{E}^*\,dv = \frac{ab\epsilon}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{2\gamma_n\epsilon_m\left|A_{mn}\right|^2\left[d-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+2\gamma_m\epsilon_n\left|C_{mn}\right|^2\left[d-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+\frac{2\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn}\right.\right.$$

$$\left.+\frac{n\pi}{b}C_{mn}\right|^2\left[d+\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right\}_{k_z=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}+\frac{ab\epsilon}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\gamma_n\epsilon_m\left|A_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[-2d+\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]\right.$$

$$+\gamma_m\epsilon_n\left|C_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[-2d+\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]-\frac{\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn}+\frac{n\pi}{b}C_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}$$

$$\times\left[2d-\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]\right\}_{k_z=-j\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} \tag{57}$$

When the volume integrals are differenced, cancellations in terms occur as in the infinite waveguide case. However, before discussing these cancellations, it is necessary to give some physical interpretation leading to these cancellations.

Substituting equations (53) for B_{mn} and D_{mn} into equations (51) results in

$$A_{mn} = \frac{\iint_{ap} E_{xa}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dx\,dy}{\frac{ab}{4}\left(1-e^{-j2k_z d}\right)} \tag{58a}$$

$$C_{mn} = \frac{\iint_{ap} E_{ya}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\,dx\,dy}{\frac{ab}{4}\left(1-e^{-j2k_z d}\right)} \tag{58b}$$

$$A_{0n} = \frac{\iint_{ap} E_{xa}\sin\frac{n\pi}{b}y\,dx\,dy}{\frac{ab}{2}\left(1-e^{-j2k_z d}\right)} \tag{58c}$$

28

$$C_{m0} = \frac{\displaystyle\iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \, dx \, dy}{\displaystyle\frac{ab}{2}\left(1 - e^{-j2k_z d}\right)} \tag{58d}$$

For the case of the infinite waveguide, the application of Rhodes' method of identifying the "physically observable" stored energies has been seen to involve the exclusion of certain terms in the stored energy expressions which, when considered separately, tend toward infinity as $1/k_z^2$ when $k_z \to 0$. For the finite-length waveguide (cavity) such $1/k_z^2$ terms also exist, but they are not the only terms which tend to infinity as $k_z \to 0$. The amplitude coefficients of equations (58) also tend to infinity as $k_z \to 0$, with the result that every term in the stored energy expressions tends separately to infinity. In order to apply Rhodes' method to the cavity, one must distinguish between the $1/k_z^2$ type infinities and the infinities due to the amplitude coefficients. The reason for the infinite coefficients is that $k_z = 0$ (or $k_z = n\pi/d$) corresponds to the resonant frequency of the lossless cavity. A physically realizable cavity would necessarily have some ohmic losses so that these coefficients would not in fact become infinite. Alternatively, if the cavity were truly lossless, the aperture fields would have to vanish as $k_z \to n\pi/d$. On this basis, then, the infinities for which cancellation of terms are sought after the manner of Rhodes will be those infinities not caused by the amplitude coefficients.

An examination of the expression for complex power at the aperture shows that it is infinite since the amplitude coefficients A_{mn} and C_{mn} become infinite as $k_z \to 0$. (See discussion following eqs. (62).) However, the infinities noted in the magnetic and electric volume integrals are not due entirely to the infinities caused by A_{mn} and C_{mn}, as discussed earlier in this section; they are also caused by the k_z^2 terms in the denominators resulting from the E_z, H_x, and H_y components. It is these latter contributions to the volume integral which must cancel identically when the volume integrals are differenced. Since the infinite complex power at the aperture is caused by the coefficients becoming infinite, the net infinite stored energy must result from these coefficients only. Therefore, expressing the contributions which are infinite because of the $1/k_z^2$ and the amplitude coefficients in terms of the contributions which depend on the amplitudes only will enable one to define "physically observable" stored energies for a shorted rectangular waveguide.

By subtracting and adding the finite contributions in each volume integral and then grouping the negative finite contributions with the infinite contributions, each volume integral is then written as a new infinite contribution plus twice the finite contribution. This is equivalent to expressing the infinite contribution as a new infinite contribution plus the finite contribution. Thus, by following this procedure the two volume integrals are written as

$$\frac{\mu}{4}\iiint_V \vec{H}\cdot\vec{H}^*\,dv = \frac{ab\mu}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\omega^2\mu^2}\left\{\left\{\left|\frac{2\gamma_m\epsilon_n}{k_z^2}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]C_{mn}\right|^2+\frac{2\gamma_n\epsilon_m}{k_z^2}\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2\right\}\left[d+\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right.$$

$$\left.-2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right|^2\left[d-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+4\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right|^2\left[d-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right\}_{k_z=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}$$

$$+\frac{ab\mu}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\omega^2\mu^2}\left\{\left\{\left|\frac{\gamma_m\epsilon_n}{k_z^2}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}+\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]C_{mn}\right|^2+\frac{\gamma_n\epsilon_m}{k_z^2}\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]A_{mn}+\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2\right\}e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right.$$

$$\left.\times\left[-2d-\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]+\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn}-\frac{n\pi}{b}A_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[2d-\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]-2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn}\right.\right.$$

$$\left.\left.-\frac{n\pi}{b}A_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[2d-\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]\right\}_{k_z=-j\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}$$

$$(59)$$

$$\frac{\epsilon}{4}\iiint_V \vec{E}\cdot\vec{E}^*\,dv = \frac{ab\epsilon}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\left\{\left(-2\gamma_n\epsilon_m\left|A_{mn}\right|^2-2\gamma_m\epsilon_n\left|C_{mn}\right|^2\right)\left[d-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]+\frac{2\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn}+\frac{n\pi}{b}C_{mn}\right|^2\left[d+\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right\}\right.$$

$$\left.+\left(4\gamma_n\epsilon_m\left|A_{mn}\right|^2+4\gamma_m\epsilon_n\left|C_{mn}\right|^2\right)\left[d-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right\}_{k_z=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}+\frac{ab\epsilon}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\left\{\left(-\gamma_n\epsilon_m\left|A_{mn}\right|^2-\gamma_m\epsilon_n\left|C_{mn}\right|^2\right)\right.\right.$$

$$\left.\left.\times e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[-2d+\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]-\frac{\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn}+\frac{n\pi}{b}C_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[2d-\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]\right.$$

$$\left.+2\,\gamma_n\epsilon_m\left|A_{mn}\right|^2+\gamma_m\epsilon_n\left|C_{mn}\right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[-2d+\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]\right\}_{k_z=-j\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}$$

$$(60)$$

30

Taking the difference in the two volume integrals (eqs. (58) and (60)) and regrouping terms results in the following:

$$\frac{\mu}{4}\iiint\limits_v \vec{H}\cdot\vec{H}^*\,dv - \frac{\epsilon}{4}\iiint\limits_v \vec{E}\cdot\vec{E}^*\,dv = \frac{ab}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\{I\} - k^2\{II\} - 4\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2 \frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\left[-4k^2\left(\gamma_n\epsilon_m\left|A_{mn}\right|^2+\gamma_m\epsilon_n\left|C_{mn}\right|^2\right)\right.\right.$$

$$\left.\left.\times\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\right\}_{k_z=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}} + \frac{ab}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\epsilon_{mn}}{\omega^2\mu}\left[\{III\} - k^2\{IV\} + 2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2\right.$$

$$\times e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} - 2k^2\left(\gamma_n\epsilon_m\left|A_{mn}\right|^2+\gamma_m\epsilon_n\left|C_{mn}\right|^2\right)e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}$$

$$\times\left.\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]_{k_z=-j\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} \tag{61}$$

where $\{I\}$, $\{II\}$, $\{III\}$, and $\{IV\}$ are defined in appendix D. The terms $\{I\} - k^2\{II\}$ and $\{III\} - k^2\{IV\}$ are shown to vanish identically in appendix D. Thus, the magnetic and electric stored energies remaining are interpreted as "physically observable" by the following energy equations:

$$<<W_m>>_v = \frac{ab}{16\omega^2\mu}\left(\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2\left[-\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}\right]\right\}_{k^2>\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} + \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2\right.\right.\cdot$$

$$\times e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left.\left.\left[\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\right]\right\}_{k^2<\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right) \tag{62a}$$

and

$$<<W_e>>_v = \frac{ab\epsilon}{16} \left(\left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2\left(\gamma_n \epsilon_m |A_{mn}|^2 + \gamma_m \epsilon_n |C_{mn}|^2\right) - \frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right\}_{k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2\left(\gamma_n \epsilon_m |A_{mn}|^2 + \gamma_m \epsilon_n |C_{mn}|^2\right) \right. \right.$$

$$\times \; e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \Bigg\}_{k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \Bigg) \tag{62b}$$

It should be noted that these "physically observable" stored energies become infinite as $k_z \to 0$ $\left(\text{as well as for } k_z = \frac{p\pi}{d} \text{ where } p = 1, 2, 3, \ldots\right)$ since the amplitude coefficients become infinite (see discussion following eqs. (57)).

These "physically observable" stored energies can also be determined from the surface integral of the complex Poynting theorem as follows:

$$-\frac{1}{2} \iint_{z=0} \vec{E} \times \vec{H}^* \cdot \hat{n} \; dS = j2\omega\left(\frac{\mu}{4} \iiint_V \vec{H} \cdot \vec{H}^* \; dv - \frac{\epsilon}{4} \iiint_V \vec{E} \cdot \vec{E}^* \; dv\right) \tag{63}$$

$$\frac{1}{2} \iint \left(E_x H_y^* - E_y H_x^*\right) dx \, dy = \frac{1}{2} \iint \left[\left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}\left(1 - e^{-j2k_z d}\right)\cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right]\left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k_z \omega\mu}\left\{\left[k^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right\}\left(1 + e^{-j2k_z d}\right)\cos\frac{m\pi}{a} \sin\frac{n\pi}{b}y \right]^* \right.$$

$$\left. - \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn}\left(1 - e^{-j2k_z d}\right)\sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \right]\left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} -\frac{1}{k_z \omega\mu}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\left(1 - e^{-j2k_z d}\right)\sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \right]^* \right] dx \, dy \tag{64a}$$

$$\frac{1}{2} \iint \left(E_x H_y^* - E_y H_x^*\right) dx \, dy = \frac{ab}{2(4)}\left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k_z^* \omega\mu}\left\{\gamma_n \epsilon_m\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]|A_{mn}|^2 + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\left(\gamma_n \epsilon_m A_{mn}C_{mn}^* + \gamma_m \epsilon_n A_{mn}^* C_{mn}\right) \right.\right.$$

$$\left.\left. + \gamma_m \epsilon_n\left[\left(\frac{n\pi}{b}\right)^2 + k^2\right]|C_{mn}|^2\right\}\left(1 - e^{-j2k_z d} + e^{j2k_z^* d} - e^{-j2\left(k_z - k_z^*\right)d}\right)\right) \tag{64b}$$

$$\frac{1}{2} \iint \left(E_x H_y^* - E_y H_x^*\right) dx \, dy = \frac{ab}{8}\left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k_z^* \omega\mu}\left[\gamma_n \epsilon_m k^2 |A_{mn}|^2 - \gamma_n \epsilon_m\left(\frac{n\pi}{b}\right)^2 |A_{mn}|^2 + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\left(\gamma_n \epsilon_m A_{mn}C_{mn}^* + \gamma_m \epsilon_n A_{mn}^* C_{mn}\right)\right. \right.$$

$$\left.\left. + \gamma_m \epsilon_n k^2 |C_{mn}|^2 - \gamma_m \epsilon_n\left(\frac{m\pi}{a}\right)^2 |C_{mn}|^2\right]\left(1 - e^{-j2k_z d} + e^{j2k_z^* d} - e^{-j2\left(k_z - k_z^*\right)d}\right)\right\} \tag{64c}$$

32

$$\frac{1}{2}\iint\left(E_x H_y^* - E_y H_x^*\right)dx\,dy = \frac{ab}{8}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k_z^*\omega\mu}\left\{-\epsilon_m\epsilon_n\left[\left(\frac{n\pi}{b}\right)^2\left|A_{mn}\right|^2 - \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}C_{mn}^* + A_{mn}^*C_{mn} + \left(\frac{m\pi}{a}\right)^2\left(C_{mn}\right)^2\right]\right.\right.$$

$$\left.\left.+\left(\gamma_n\epsilon_m k^2\left|A_{mn}\right|^2 + \gamma_m\epsilon_n k^2\left|C_{mn}\right|^2\right)\right\}\left(1 - e^{-j2k_z d} + e^{j2k_z^* d} - e^{-j2\left(k_z - k_z^*\right)d}\right)\right) \tag{64d}$$

$$\frac{1}{2}\iint\left(E_x H_y^* - E_y H_x^*\right)dx\,dy = \frac{ab}{8}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k_z^*\omega\mu}\left\{\left[-\epsilon_m\epsilon_n\left|\left(\frac{n\pi}{b}\right)A_{mn} - \left(\frac{m\pi}{a}\right)C_{mn}\right|^2\right] + \left(\gamma_n\epsilon_m k^2\left|A_{mn}\right|^2 + {}_m\epsilon_n k^2\left|C_{mn}\right|^2\right)\right\}\left(1 - e^{-j2k_z d} + e^{j2k_z^* d} - e^{-j2\left(k_z - k_z^*\right)d}\right)\right) \tag{64e}$$

$$\frac{1}{2}\iint\left(E_x H_y^* - E_y H_x^*\right)dx\,dy = \frac{ab}{8\omega\mu}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(-\epsilon_m\epsilon_n\left|\frac{n\pi}{b}A_{mn} - \frac{m\pi}{a}C_{mn}\right|^2 + \gamma_n\epsilon_m k^2\left|A_{mn}\right|^2 + \gamma_m\epsilon_n k^2\left|C_{mn}\right|^2\right)\left\{\left[+j2\,\frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right]_{k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right.\right.$$

$$\left.\left.+\left[\frac{1 - e^{-4d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}}{j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right]_{k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right\}\right) \tag{65a}$$

$$\frac{1}{2}\iint\left(E_x H_y^* - E_y H_x^*\right)dx\,dy = \frac{jab}{8\omega\mu}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2\epsilon_m\epsilon_n\left|\frac{n\pi}{b}A_{mn} - \frac{m\pi}{a}C_{mn}\right|^2\left[-\frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}\right]_{k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right\} + \frac{jab}{8\omega\mu}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2\epsilon_m\epsilon_n\left|\frac{n\pi}{b}A_{mn} - \frac{m\pi}{a}C_{mn}\right|^2\right.$$

$$\left.\times e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\left[\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right]_{k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right\} - \frac{jab}{8\omega\mu}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2k^2\left(\gamma_n\epsilon_m\left|A_{mn}\right|^2 + \gamma_m\epsilon_n\left|C_{mn}\right|^2\right)\right.$$

$$\left.\times\left[-\frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}\right]_{k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right\} - \frac{jab}{8\omega\mu}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2k^2\left(\gamma_n\epsilon_m\left|A_{mn}\right|^2 + \gamma_m\epsilon_n\left|C_{mn}\right|^2\right)e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right.$$

$$\left.\times\left[\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right]_{k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right\} \tag{65b}$$

Dividing equation (65b) by 2ω yields the same expressions derived earlier for "physically observable" stored energies. (The nonpropagating contributions are easily identified with magnetic and electric stored energies since they are always positive. However, this is not true of the propagating contributions; identification of these energies is aided by examining which field components contribute.)

Since the expressions given by equations (61) are to be used in determining quality factor Q, it is necessary that each be positive (or zero) at resonance. There is no problem with the contribution from the nonpropagating waves since it is always positive. However, the contribution from the propagating waves can be negative. But what must be considered, in general, is the case when both propagating and nonpropagating wave contributions exist simultaneously; the propagating waves can be limited by the choice of waveguide size, but the nonpropagating waves exist whenever a discontinuity is present in the waveguide, which is the usual case. Since discontinuities produce nonpropagating waves, the existence of propagating waves only can not occur.

For the rectangular cavity-backed slot antenna considered in this paper, losses are inherently included due to the radiating slot. The net stored energy at resonance was always positive when a single propagating wave together with nonpropagating waves were assumed. This result was found to be true for many different size cavities.

AMPLITUDE COEFFICIENTS FOR A NARROW SLOT BACKED
BY A RECTANGULAR CAVITY IN WHICH A SINGLE
PROPAGATING WAVE IS ASSUMED

In this section the amplitude coefficients for a rectangular cavity which backs a narrow slot whose electric-field distribution is sinusoidal are determined. In order to determine the amplitude coefficients A_{mn} and C_{mn} for a narrow slot, the boundary condition at $z = \delta$ (see fig. 1) must be applied. For an aperture opening of some kind at $z = \delta$, the boundary conditions are applied to the transverse electric field. In the aperture it is assumed that the electric-field distributions are given by E_{xa} and E_{ya}. Therefore, at $z = \delta$ (actually the boundary conditions are applied at $z = 0$ to simplify the derivation since later on $\delta \to 0$ anyway),

$$E_{xa} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn} + B_{mn}\right) \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \qquad (n \neq 0) \qquad (66a)$$

$$E_{ya} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(C_{mn} + D_{mn}\right) \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \qquad (m \neq 0) \qquad (66b)$$

34

Multiplying by $\cos \frac{m'\pi}{a} x \sin \frac{n'\pi}{b} y$ and $\sin \frac{m'\pi}{a} x \cos \frac{n'\pi}{b} y$, respectively, and integrating over 0 to a and 0 to b, the coefficients become

$$A_{0n} + B_{0n} = \frac{\iint_{ap} E_{xa} \sin \frac{n\pi}{b} y \, dS}{ab/2} \tag{67a}$$

$$C_{m0} + D_{m0} = \frac{\iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \, dS}{ab/2} \tag{67b}$$

$$A_{mn} + B_{mn} = \frac{\iint_{ap} E_{xa} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dS}{ab/2} \tag{67c}$$

$$C_{mn} + D_{mn} = \frac{\iint_{ap} E_{xa} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \, dS}{ab/4} \tag{67d}$$

Using the boundary condition at $z = d$ for a shorted waveguide results in

$$B_{mn} = -A_{mn} e^{-j2k_z d}$$

$$D_{mn} = -C_{mn} e^{-j2k_z d}$$

where

$$k_z = \begin{cases} \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} & \text{(propagating waves)} \\[2ex] -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} & \text{(nonpropagating waves)} \end{cases}$$

Therefore, equations (67) become

$$A_{0n} = \frac{\frac{2}{ab} \iint_{ap} E_{xa} \sin \frac{n\pi}{b} y \, dS}{1 - e^{-j2k_z d}} \tag{68a}$$

35

$$C_{m0} = \frac{\frac{2}{ab} \iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \, dS}{1 - e^{-j2k_z d}} \tag{68b}$$

$$A_{mn} = \frac{\frac{4}{ab} \iint_{ap} E_{xa} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dS}{1 - e^{-j2k_z d}} \tag{68c}$$

$$C_{mn} = \frac{\frac{4}{ab} \iint_{ap} E_{ya} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \, dS}{1 - e^{-j2k_z d}} \tag{68d}$$

At the plane $\lim_{\delta \to 0^+}$ $(z = \delta)$, a narrow slot is located as shown in figure 1. A reasonable assumption for the electric-field distribution of such a slot is (for relatively deep cavities, ref. 7):

$$\left. \begin{aligned} E_{xa}(y',k) &= \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - |y'|\right) \\[2mm] E_{ya}(y',k) &= 0 \end{aligned} \right\} \tag{69}$$

Using the distributions from equations (69), the amplitude coefficients become

$$A_{0n} = \frac{\frac{2}{ab} \int_{y'=\ell/2}^{\ell/2} \int_{x=a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - |y'|\right) \sin \frac{n\pi}{b} y \, dx' \, dy'}{1 - e^{-j2k_z d}} \tag{70a}$$

$$C_{m0} = 0 \tag{70b}$$

$$A_{mn} = \frac{\frac{4}{ab} \int_{y'=\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - |y'|\right) \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dx' \, dy'}{1 - e^{-j2k_z d}} \tag{70c}$$

$$C_{mn} = 0 \tag{70d}$$

36

where

$$
k_z = \begin{cases} \sqrt{k^2 - \left(\dfrac{m\pi}{a}\right)^2 - \left(\dfrac{n\pi}{b}\right)^2} & \text{(propagating waves)} \\[2em] -j\sqrt{\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2 - k^2} & \text{(nonpropagating waves)} \end{cases}
$$

Equations (70a) and (70c) can be integrated to give (see appendix E)

$$
A_{0n} = \begin{cases} \dfrac{2/ab}{1 - e^{-j2k_z d}} \, 0 & (n = 2, 4, 6, \ldots) \\[2em] \dfrac{2/ab}{1 - e^{-j2k_z d}} (-1)^{\frac{n-1}{2}} \dfrac{2V_0 k\left(\cos \dfrac{n\pi}{b}\dfrac{\ell}{2} - \cos\dfrac{k\ell}{2}\right)}{k^2 - \left(\dfrac{n\pi}{b}\right)^2} & (n = 1, 3, 5, \ldots) \end{cases} \tag{71a}
$$

$$
A_{mn} = \begin{cases} \dfrac{4/ab}{1 - e^{-j2k_z d}} \, 0 & \begin{pmatrix} n = 2, 4, 6, \ldots \\ \text{or} \\ m = 1, 3, 5, \ldots \end{pmatrix} \\[3em] \dfrac{4/ab}{1 - e^{-j2k_z d}} (-1)^{\frac{n-1}{2}} (-1)^{\frac{m}{2}} \dfrac{4V_0 k \sin\left(\dfrac{m\pi a'}{a2}\right)\left(\cos\dfrac{n\pi}{b}\dfrac{\ell}{2} - \cos\dfrac{k\ell}{2}\right)}{a'\left(\dfrac{m\pi}{a}\right)\left[k^2 - \left(\dfrac{n\pi}{b}\right)^2\right]} & \begin{pmatrix} m = 2, 4, 6, \ldots \\ \text{or} \\ n = 1, 3, 5, \ldots \end{pmatrix} \end{cases} \tag{71b}
$$

Assume that only the $m = 0$ and $n = 1$ mode propagates, all others being evanescent $\left(\text{i.e., } \dfrac{\pi}{b} < k < \dfrac{\pi}{a}\right)$, so that

$$A_{01} = \frac{2/ab}{1 - e^{-j2d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}}} \left[\frac{2V_0k\left(\cos \dfrac{\pi}{b}\dfrac{\ell}{2} - \cos \dfrac{k\ell}{2}\right)}{k^2 - \left(\dfrac{\pi}{b}\right)^2} \right] \tag{72a}$$

$$A_{0n} = \begin{cases} \dfrac{2/ab}{1 - e^{-2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}} \cdot 0 & (n = 2,\, 4,\, 6,\, \ldots) \\[4em] \dfrac{2/ab}{1 - e^{-2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}} (-1)^{\frac{n-1}{2}} \dfrac{2V_0k\left(\cos \dfrac{n\pi}{b}\dfrac{\ell}{2} - \cos \dfrac{k\ell}{2}\right)}{k^2 - \left(\dfrac{n\pi}{b}\right)^2} & (n = 3,\, 5,\, 7,\, \ldots) \end{cases} \tag{72b}$$

$$A_{mn} = \begin{cases} \dfrac{4/ab}{1 - e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}} \cdot 0 & \begin{pmatrix} n = 2,\, 4,\, 6,\, \ldots \\ \text{or} \\ m = 1,\, 3,\, 5,\, \ldots \end{pmatrix} \\[5em] \dfrac{4/ab}{1 - e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}} (-1)^{\frac{n-1}{2}} (-1)^{\frac{m}{2}} \dfrac{4V_0k \, \sin\left(\dfrac{m\pi a'}{a2}\right)\left(\cos \dfrac{n\pi}{b}\dfrac{\ell}{2} - \cos \dfrac{k\ell}{2}\right)}{a'\left(\dfrac{m\pi}{a}\right)\left[k^2 - \left(\dfrac{n\pi}{b}\right)^2\right]} & \begin{pmatrix} m = 2,\, 4,\, 6,\, \ldots \\ \text{or} \\ n = 1,\, 3,\, 5,\, \ldots \end{pmatrix} \end{cases} \tag{72c}$$

38

or

$$A_{01} = \cfrac{1}{jab\, e^{-jd\sqrt{k^2-\left(\frac{\pi}{b}\right)^2}} \sin d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}} \left[\frac{2V_ok\left(\cos\frac{\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{k^2 - \left(\frac{\pi}{b}\right)^2}\right] \tag{73a}$$

$$A_{0n} = \cfrac{1}{ab\, e^{-d\sqrt{\left(\frac{n\pi}{b}\right)^2-k^2}} \sinh d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \left[(-1)^{\frac{n-1}{2}} \frac{2V_ok\left(\cos\frac{m\pi}{a}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{k^2 - \left(\frac{n\pi}{b}\right)^2}\right]$$

$$(n = 3, 5, 7, \ldots) \tag{73b}$$

$$A_{mn} = \cfrac{2}{ab\, e^{-d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} \sinh d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$\times \left\{(-1)^{\frac{n-1}{2}}(-1)^{\frac{m}{2}} \frac{4V_ok\sin\left(\frac{m\pi}{a}\frac{a'}{2}\right)\left(\cos\frac{n\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{a'\left(\frac{m\pi}{a}\right)\left[k^2 - \left(\frac{n\pi}{b}\right)^2\right]}\right\} \left(\begin{array}{c} m = 2, 4, 6, \ldots \\ \text{or} \\ n = 1, 3, 5, \ldots \end{array}\right) \tag{73c}$$

$$C_{mn} = 0 \tag{73d}$$

Substituting the coefficients of equations (73) into the "physically observable" stored energy expressions given by energy equations (62) gives for $<<W_m>>_v$ and $<<W_e>>_v$ the following:

$$<<W_m>>_v = \frac{ab}{16\omega^2\mu}\left((2)2\left(\frac{\pi}{b}\right)^2 \frac{1}{a^2b^2 \sin^2 d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}}\left[\frac{2V_0 k\left(\cos\frac{\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{k^2 - \left(\frac{\pi}{b}\right)^2}\right]\right.$$

$$\times \left[-\frac{\sin 2d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}}{\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}}\right] + \sum_{n=3,5}^{\infty} (2)2\left(\frac{n\pi}{b}\right)^2 \frac{1}{a^2b^2 e^{-2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}\sinh^2 d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$\times e^{-2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}\left[\frac{\sinh 2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}\right]\left[\frac{2V_0 k\left(\cos\frac{n\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{k^2 - \left(\frac{n\pi}{b}\right)^2}\right]^2 + \sum_{m=2,4}^{\infty}\sum_{n=1,3}^{\infty}$$

$$\times (2)\left(\frac{n\pi}{b}\right)^2 \frac{4}{a^2b^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\sinh^2 d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$\times e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\left[\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right]$$

$$\times \left.\left\{\frac{4V_0 k \sin\left(\frac{m\pi}{a}\frac{a'}{2}\right)\left(\cos\frac{n\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{a'\left(\frac{m\pi}{a}\right)\left[k^2 - \left(\frac{n\pi}{b}\right)^2\right]}\right\}^2\right) \tag{74}$$

and

$$\langle\langle W_e \rangle\rangle_v = \frac{ab\epsilon}{16} \left((2)2 \frac{1}{a^2 b^2 \sin^2 d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}} \left[\frac{2V_o k \left(\cos\frac{\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{k^2 - \left(\frac{\pi}{b}\right)^2} \right]^2 \left[- \frac{\sin 2d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}}{\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}} \right] \right.$$

$$+ \sum_{n=3,5}^{\infty} (2)2 \frac{1}{a^2 b^2 e^{-2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \sinh^2 d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} e^{-2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$\times \left[\frac{\sinh 2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \left[\frac{2V_o k\left(\cos\frac{n\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{k^2 - \left(\frac{n\pi}{b}\right)^2} \right]^2 + \sum_{m=2,4}^{\infty} \sum_{n=1,3}^{\infty} (2)$$

$$\times \frac{4}{a^2 b^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \sinh^2 d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$\times \left. \left[\frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \left\{ \frac{4V_o k \sin\left(\frac{m\pi}{a}\frac{a'}{2}\right)\left(\cos\frac{n\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)}{a'\left(\frac{m\pi}{a}\right)\left[k^2 - \left(\frac{n\pi}{b}\right)^2\right]} \right\}^2 \right) \quad (75)$$

41

Now, making use of the identities

$$\sin 2x = 2 \cos x \sin x$$

and

$$\sinh 2x = 2 \cosh x \sinh x$$

gives finally

$$<<W_m>>_v = \frac{2|V_o|^2 k^2}{ab\omega^2 \mu} \left\{ \frac{\left(\frac{\pi}{b}\right)^2 \left(\cos \frac{\pi}{b}\frac{\ell}{2} - \cos \frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{5/2}} \left[-\cot d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\right] + \sum_{n=3,5}^{\infty} \right.$$

$$\times \frac{\left(\frac{n\pi}{b}\right)^2 \left(\cos \frac{n\pi}{b}\frac{\ell}{2} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{5/2}} \left[\coth d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\right] + 2 \sum_{m=2,4}^{\infty} \sum_{n=1,3}^{\infty} \left(\frac{\sin \frac{m\pi}{a}\frac{a'}{2}}{\frac{m\pi}{a}\frac{a'}{2}}\right)^2$$

$$\times \frac{\left(\frac{n\pi}{b}\right)^2 \left(\cos \frac{n\pi}{b}\frac{\ell}{2} - \cos \frac{k\ell}{2}\right)^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^2} \left[\coth d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\right] \right\} \qquad (76)$$

and

$$<<W_e>>_v = \frac{2|V_o|^2 k^2 \epsilon}{ab} \left\{ \frac{\left(\cos \frac{\pi}{b}\frac{\ell}{2} - \cos \frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{5/2}} \left[-\cot d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\right] + \sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi}{b}\frac{\ell}{2} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{5/2}} \right.$$

$$(77)$$

(Equation continued on next page)

42

$$\times \left[\coth d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\right] + 2 \sum_{m=2,4}^{\infty} \sum_{n=1,3}^{\infty} \left(\frac{\sin \frac{m\pi}{a}\frac{a'}{2}}{\frac{m\pi}{a}\frac{a'}{2}}\right)^2$$

$$\times \frac{\left(\cos \frac{n\pi}{b}\frac{\ell}{2} - \cos \frac{k\ell}{2}\right)^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^2} \left[\coth d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\right] \Bigg\} \tag{77}$$

In conclusion, equations (76) and (77) represent "physically observable" magnetic and electric stored energies for a section of rectangular waveguide which is shorted at depth d and bounded by a slot at $\lim_{\delta \to 0^+} (z = \delta)$ in which the electric-field distribution is assumed to be

$$\left.\begin{aligned} E_{xa} &= \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - |y'|\right) \\[2ex] E_{ya} &= 0 \end{aligned}\right\} \tag{78}$$

where ℓ is the length of the slot and a' is its width. The transverse dimensions of the rectangular waveguide are chosen such that only one field propagates ($m = 0$ and $n = 1$); all other fields are evanescent.

RESULTS FOR THE RECTANGULAR CAVITY-BACKED SLOT ANTENNA
IN WHICH A SINGLE PROPAGATING WAVE IS ASSUMED

General

When $m = 0$ and $n = 1$ are assumed for the propagating wave with all others taken as nonpropagating waves, the differences in the stored energies in the exterior region (for a slot, ref. 17) and in the interior region (eqs. (76) and (77)) are given, respectively, as

$$2\omega\left(<<W_m>>_{v'} - <<W_e>>_{v'}\right) = -\frac{8|V_o|^2}{(2\pi)^2 Z_o}\frac{\pi}{4}\left\{ Si(k\ell) + \left[Si(k\ell) - \frac{1}{2}Si(2k\ell)\right]\cos k\ell \right.$$

$$\left. + \left[Cin(k\ell) - \frac{1}{2}Cin(2k\ell) - \ln\frac{e^{3/2}\ell}{2a'}\right]\sin k\ell\right\} \tag{79}$$

and

$$2\omega\left(<<W_m>>_v - <<W_e>>_v\right) = 2\omega\frac{2|V_o|^2 k^2}{ab}\left\{\left[\epsilon - \frac{\left(\frac{\pi}{b}\right)^2}{\omega^2\mu}\right]\cot d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\frac{\left(\cos\frac{\pi}{b}\frac{\ell}{2} - \cos\frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{5/2}}\right.$$

$$+ \sum_{n=3,5}^{\infty}\left[\frac{\left(\frac{n\pi}{b}\right)^2}{\omega^2\mu} - \epsilon\right]\coth d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\frac{\left(\cos\frac{n\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{5/2}}$$

$$+ 2\sum_{m=2,4}^{\infty}\sum_{n=1,3}^{\infty}\left(\frac{\sin\frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2\left[\frac{\left(\frac{n\pi}{b}\right)^2}{\omega^2\mu} - \epsilon\right]$$

$$\left. \times \coth d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\frac{\left(\cos\frac{n\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^2}\right\}$$

$$\tag{80a}$$

44

$$2\omega\left(<<W_m>>_v - <<W_e>>_v\right) = \frac{4\left|V_o\right|^2 k^2 \omega}{ab\omega^2\mu}\left\{\frac{\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{3/2}}\cot d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} + \sum_{n=3,5}^{\infty}\right.$$

$$\times \frac{\left(\cos\frac{n\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{3/2}}\coth d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} + 2\sum_{m=2,4}^{\infty}\sum_{n=1,3}^{\infty}$$

$$\times \left(\frac{\sin\frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{\left(\cos\frac{n\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]}$$

$$\left.\times \coth d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\right\} \tag{80b}$$

Therefore, the total difference in the stored energies becomes

$$2\omega\left(<<W_m>> - <<W_e>>\right) = \frac{4\left|V_o\right|^2 k}{abZ_o}\left\{\frac{\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{3/2}}\cot d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} + \sum_{n=3,5}^{\infty}\right.$$

$$\times \frac{\left(\cos\frac{n\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{3/2}}\coth d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} + 2\sum_{m=2,4}^{\infty}\sum_{n=1,3}^{\infty} \tag{81}$$

45

$$\times \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{\left(\cos \frac{n\pi \ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \frac{\coth d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\left(\frac{n\pi}{b}\right)^2 - k^2}\Bigg\}$$

$$- \frac{8\left|V_0\right|^2}{(2\pi)^2 Z_0} \frac{\pi}{4}\Bigg\{Si(k\ell) + \left[Si(k\ell) - \frac{1}{2}Si(2k\ell)\right]\cos k\ell$$

$$+ \left[Ci\dot{n}(k\ell) - \frac{1}{2}Cin(2k\ell) - \ln \frac{e^{3/2}\ell}{2a'}\right]\sin k\ell\Bigg\} \tag{81}$$

The total stored electric energy at resonance is needed for computing Q; i.e., the sum of electric stored energies from the half-space and the cavity regions. The half-space electric stored energy is given by Rhodes in reference 17 for a slot, and the cavity electric stored energy is given by equation (77). Therefore,

$$2\omega\left(<<W_e>>\right) = \frac{8\left|V_0\right|^2}{(2\pi)^2 Z_0} \frac{\pi}{8}\Bigg(Si(k\ell) - \frac{k\ell}{2}Cin(k\ell) + (k\ell - \sin k\ell)\ln \frac{e\ell}{a'} + \Bigg\{\left[Si(k\ell) - \frac{1}{2}Si(2k\ell)\right]$$

$$- \frac{k\ell}{2}\left[Cin(k\ell) - Cin(2k\ell)\right] - k\ell \ln 2\Bigg\}\cos k\ell + \Bigg\{\left[Cin(k\ell) - \frac{1}{2}Cin(2k\ell)\right]$$

$$+ \frac{k\ell}{2}\left[Si(k\ell) - Si(2k\ell)\right] + \ln \frac{2}{e}\Bigg\}\sin k\ell\Bigg) + \frac{4\omega\left|V_0\right|^2 k^2\epsilon}{ab}\Bigg\{\frac{\left(\cos \frac{\pi \ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{5/2}}$$

$$\times \left[-\cot d\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\right] + \sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi \ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{5/2}}\coth d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \tag{82}$$

(Equation continued on next page)

$$+ 2 \sum_{m=2,4}^{\infty} \sum_{n=1,3}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}} \right)^2 \frac{\left(\cos \frac{n\pi \ell}{2b} - \cos \frac{k\ell}{2} \right)^2}{\sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2} \left[\left(\frac{n\pi}{b} \right)^2 - k^2 \right]}$$

$$\times \coth d \sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2} \Bigg\} \qquad (82)$$

For relative bandwidth the angular frequency derivative of equation (81) is needed, that is,

$$\omega \frac{d}{d\omega} \left[2\omega \left(<<W_m>> - <<W_e>> \right) \right] = \frac{4 |V_0|^2 k}{(2\pi)^2 ab Z_0} \left(\sum_{n=3,5}^{\infty} \frac{\cos \frac{n\pi \ell}{2b} - \cos \frac{k\ell}{2}}{\left[\left(\frac{n\pi}{b} \right)^2 - k^2 \right]^{3/2}} \left\{ k\ell \sin \frac{k\ell}{2} \right. \right.$$

$$\left. + \left(\cos \frac{n\pi \ell}{2b} - \cos \frac{k\ell}{2} \right) \left[\frac{3k^2}{\left(\frac{n\pi}{b} \right)^2 - k^2} + 1 \right] \right\} + 2 \sum_{m=2,4}^{\infty} \sum_{n=1,3}^{\infty}$$

$$\times \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}} \right)^2 \frac{\cos \frac{n\pi \ell}{2b} - \cos \frac{k\ell}{2}}{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2} \left\{ k\ell \sin \frac{k\ell}{2} \right.$$

$$\left. \left. + \left(\cos \frac{n\pi \ell}{2b} - \cos \frac{k\ell}{2} \right) \left[\frac{2k^2}{\left(\frac{n\pi}{b} \right)^2 - k^2} + \frac{k^2}{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2} + 1 \right] \right\} \right)$$

$$(83)$$

$$- \frac{8|V_o|^2}{(2\pi)^2 Z_o} \frac{\pi}{4} k\ell \left(-\left[Si(k\ell) - \frac{1}{2} Si(2k\ell)\right] \sin k\ell \right.$$

$$\left. + \left\{ \left[Cin(k\ell) - \frac{1}{2} Cin(2k\ell)\right] - \ln \frac{e^{3/2}\ell}{2a'} \right\} \cos k\ell + \frac{\sin k\ell}{k\ell} \right) \quad (83)$$

The radiated power is given by (see ref. 17)

$$P_r = \frac{8|V_o|^2}{(2\pi)^2} \frac{1}{Z_o} \frac{\pi}{2} \frac{1}{2} \left\{ Cin(k\ell) + \left[Cin(k\ell) - \frac{1}{2} Cin(2k\ell)\right] \cos k\ell - \left[Si(k\ell) - \frac{1}{2} Si(2k\ell)\right] \sin k\ell \right\}$$

$$(84)$$

The quality factor, relative bandwidth, and the input admittance given by equations (25), (21a), and (10), respectively, are repeated here for completeness

$$Q = \left(\frac{2\omega <<W_e>>}{P_r} \right)_{\omega = \omega_r}$$

$$B.W. = \left[\frac{2P_r}{-\omega_r \frac{d}{d\omega} 2\omega \left(<<W_m>> - <<W_e>> \right)} \right]_{\omega = \omega_r}$$

and

$$Y = G + jB$$

where

$$G = \frac{P_r}{\frac{1}{2}|V_o|^2 \sin^2 k\ell} \quad (85a)$$

48

$$B = \frac{-2\omega\left(<<W_m>> - <<W_e>>\right)}{\frac{1}{2}\left|V_o\right|^2 \sin^2 k\ell} \tag{85b}$$

Approximate Solutions for Moderate Cavity Depths and Narrow Slots

Since the summation on m in equations (81) and (83) is slowly convergent for a' << a, the numerical evaluation of the double summation in these equations is cumbersome. Adding further to the complications of evaluating these equations is the presence of the hyperbolic terms. The evaluation of equations (81) and (83) is greatly facilitated by making the following approximations:

$$\coth d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \approx 1 \qquad\qquad (n = 3, 5, 7, \ldots) \tag{86a}$$

$$\coth d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} \approx 1 \qquad\qquad \left(\begin{array}{c} m = 2, 4, 6, \ldots \\ \text{or} \\ n = 1, 3, 5, \ldots \end{array}\right) \tag{86b}$$

These approximations (eqs. (86)) place additional restrictions on the cavity size. From tables given in reference 20, the hyperbolic cotangent becomes very large for small argument and approaches unity as the argument increases (around 3 or 4). Therefore, the approximation becomes less valid for decreasing argument. Physically, this means the deeper the cavity the more valid the approximation (coth $\approx$ 1). This is also true for the assumed aperture distribution; that is, the sinusoidal distribution assumed in the slot is valid so long as the back wall of the cavity is not too close.

By indicating a criterion that the hyperbolic cotangent functions must satisfy, the depth of the cavity is restricted, e.g., the arguments of equations (86) may be chosen so that

$$d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} > 4 \qquad\qquad (n = 3, 5, 7, \ldots) \tag{87a}$$

$$d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} > 4 \qquad\qquad \left(\begin{array}{c} m = 2, 4, 6, \ldots \\ \text{or} \\ n = 1, 3, 5, \ldots \end{array}\right) \tag{87b}$$

or

$$d > \frac{4}{\sqrt{\left(\frac{3\pi}{b}\right)^2 - k^2}} \tag{88a}$$

$$d > \frac{4}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \qquad \left(\begin{array}{c} m = 2,\ 4,\ 6,\ .\ .\ . \\ \text{or} \\ n = 1,\ 3,\ 5,\ .\ .\ . \end{array} \right) \tag{88b}$$

where in equation (88a) only $n = 3$ is necessary since for all other values of n the depth d is also greater. However, in equation (88b) the size of height a and width b will enter into determining which m and n must be used for d to be greater for all m and n. In addition to this restriction, the restriction resulting from the assumption of a single propagating wave must be satisfied; that is,

or

$$\left. \begin{array}{c} k^2 > \left(\frac{\pi}{b}\right)^2 \\[3em] kb > \pi \\[3em] \frac{b}{\lambda} > \frac{1}{2} \end{array} \right\} \tag{89}$$

Even with the approximations (87) to (89) (coth ≈ 1), the summation on m is still slowly converging for $a' \ll a$; hence, more approximations are needed to make the summation converge faster. (The details of the $a' \ll a$ approximation are given in appendix F.) Finally, substituting the approximations into equations (81) to (83), yields the following solutions for moderate cavity depths and narrow slots:

$$2\omega\left(<<W_m>> - <<W_e>>\right) \approx \frac{4|V_o|^2}{(ka)(kb)Z_o}\left(\frac{\left(\cos\frac{\pi k\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{1 - \left(\frac{\pi}{kb}\right)^2}\cot kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\right.$$

$$+ \frac{ka}{\pi}\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{1 - \left(\frac{\pi}{kb}\right)^2}\left[\ln\left(\frac{\pi ka'}{ka}\right) + \ln 2 - \frac{3}{2}\right] + \frac{ka}{\pi}\sum_{n=3,5}^{\infty}$$

$$\times \frac{\left(\cos\frac{n\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{kb}\right)^2 - 1}\left\{\frac{3}{2} - \gamma - \ln\left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1}\frac{ka'}{2}\right]\right\}\right)$$

$$- \frac{8|V_o|^2}{(2\pi)^2 Z_o}\frac{\pi}{4}\left\{Si(k\ell) + \left[Si(k\ell) - \frac{1}{2}Si(2k\ell)\right]\cos k\ell\right.$$

$$\left.+ \left[Cin(k\ell) - \frac{1}{2}Cin(2k\ell) - \ln\frac{e^{3/2}k\ell}{2ka'}\right]\sin k\ell\right\} \tag{90}$$

$$2\omega\left(<<W_e>>\right) \approx \frac{4|V_o|^2}{Z_o(ka)(kb)}\left(\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{5/2}}\left[-\cot kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\right] - \frac{ka}{\pi}\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2}\right.$$

$$\tag{91}$$

$$\times \left[\ln\left(\frac{\pi ka'}{ka}\right) + \ln 2 - \frac{3}{2}\right] + \frac{ka}{\pi} \sum_{n=3,5}^{\infty} \frac{\left(\cos\frac{n\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{kb}\right)^2 - 1\right]^2} \left\{\frac{3}{2} - \gamma - \ln\right.$$

$$\left.\times \left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1}\,\frac{ka'}{2}\right]\right\}\right) + \frac{8|V_o|^2}{(2\pi)^2 Z_o}\frac{\pi}{8}\left(\text{Si}(k\ell) - \frac{k\ell}{2}\text{Cin}(k\ell) + (k\ell - \sin k\ell)\ln\frac{ek\ell}{ka'}\right.$$

$$+ \left\{\left[\text{Si}(k\ell) - \frac{1}{2}\text{Si}(2k\ell)\right] - \frac{k\ell}{2}\left[\text{Cin}(k\ell) - \text{Cin}(2k\ell)\right] - k\ell\ln 2\right\}\cos k\ell$$

$$\left.+ \left\{\text{Cin}(k\ell) - \frac{1}{2}\text{Cin}(2k\ell) + \frac{k\ell}{2}\left[\text{Si}(k\ell) - \text{Si}(2k\ell)\right] + \ln\frac{2}{e}\right\}\sin k\ell\right) \tag{91}$$

$$\omega\frac{d}{d\omega}\left[2\omega\left(<<W_m>> - <<W_e>>\right)\right] \approx \frac{4|V_o|^2}{(ka)(kb)}\frac{1}{Z_o}\left(-\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2}\,kd\cos^2 kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\right.$$

$$+ \cot kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\left\{k\ell\sin\frac{k\ell}{2}\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{3/2}}\right.$$

$$\left.\left.- \frac{2\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{5/2}} + \frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{3/2}}\right\}\right. \tag{92}$$

(Equation continued on next page)

52

$$+ \frac{ka}{\pi} \left[\ln\left(\frac{\pi ka'}{ka}\right) - \frac{3}{2} + \ln 2 \right] \left\{ k\ell \sin\frac{k\ell}{2} \frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)}{1 - \left(\frac{\pi}{kb}\right)^2} \right.$$

$$\left. - \frac{2\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2} + \frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{1 - \left(\frac{\pi}{kb}\right)^2} \right\}$$

$$+ \frac{ka}{\pi} \sum_{n=3,5}^{\infty} \left(\frac{\left(\cos\frac{n\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)}{\left(\frac{n\pi}{kb}\right)^2 - 1} + \left\{ \frac{3}{2} - \gamma \right. \right.$$

$$\left. - \ln\left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1}\,\frac{ka'}{2}\right]\right\} \left\{ k\ell \sin\frac{k\ell}{2} \frac{\left(\cos\frac{n\pi k\ell}{2k\ell} - \cos\frac{k\ell}{2}\right)}{\left(\frac{n\pi}{kb}\right)^2 - 1} \right.$$

$$\left. \left. + 2 \frac{\left(\cos\frac{n\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{kb}\right)^2 - 1\right]^2} + \frac{\left(\cos\frac{n\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{kb}\right)^2 - 1} \right\} \right) \tag{92}$$

$$-\frac{8\left|V_0\right|^2}{(2\pi)^2 Z_0}\frac{\pi}{4}k\ell\left\{-\left[\text{Si}(k\ell)-\frac{1}{2}\text{Si}(2k\ell)\right]\sin k\ell\right.$$

$$\left.+\left[\text{Cin}(k\ell)-\frac{1}{2}\text{Cin}(2k\ell)-\ln\left(\frac{e^{3/2}k\ell}{2ka'}\right)\right]\cos k\ell+\frac{\sin k\ell}{k\ell}\right\}\quad(92)$$

Input Admittance Calculations

The input admittance of the rectangular cavity-backed slot is represented by $Y = G + jB$ where the power loss in the cavity is assumed negligible. Explicitly, the conductance and the susceptance are given as

$$G\approx\frac{8}{(2\pi)^2 Z_0 \frac{1}{2}\sin^2\frac{k\ell}{2}}\frac{\pi}{2}\frac{1}{2}\left\{\text{Cin}(k\ell)+\left[\text{Cin}(k\ell)-\frac{1}{2}\text{Cin}(2k\ell)\right]\cos k\ell\right.$$

$$\left.-\left[\text{Si}(k\ell)-\frac{1}{2}\text{Si}(2k\ell)\right]\sin k\ell\right\}\quad(93)$$

and

$$B\underset{a'\to 0}{\approx}-\frac{1}{\frac{1}{2}Z_0\sin^2\frac{k\ell}{2}}\left[\frac{4}{(ka)(kb)}\left(\frac{\left(\cos\frac{\pi k\ell}{2kb}-\cos\frac{k\ell}{2}\right)^2\cot kd\sqrt{1-\left(\frac{\pi}{kb}\right)^2}}{\left[1-\left(\frac{\pi}{kb}\right)^2\right]^{3/2}}+\frac{ka}{\pi}\right.\right.$$

$$\times\frac{\left(\cos\frac{\pi k\ell}{2kb}-\cos\frac{k\ell}{2}\right)^2}{1-\left(\frac{\pi}{kb}\right)^2}\left[\ln\left(\frac{\pi ka'}{ka}\right)+\ln 2-\frac{3}{2}\right]+\frac{ka}{\pi}\sum_{n=3,5}^{\infty}\frac{\left(\cos\frac{n\pi k\ell}{2kb}-\cos\frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{kb}\right)^2-1}\quad(94)$$

$$\times \left\{\frac{3}{2} - \gamma - \ln\left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1}\,\frac{ka'}{2}\right]\right\}\right) - \frac{8}{(2\pi)^2}\frac{\pi}{4}\left\{Si(k\ell) + \left[Si(k\ell) - \frac{1}{2}Si(2k\ell)\right]\cos k\ell\right.$$

$$\left.\left.+ \left[Cin(k\ell) - \frac{1}{2}Cin(2k\ell) - \ln\left(\frac{e^{3/4}k\ell}{2ka'}\right)\right]\sin k\ell\right\}\right] \tag{94}$$

respectively, where equation (84) has been substituted into equation (85a).

The input admittance can be represented as a function of any of the cavity or slot dimensions in wavelengths, as well as a function of frequency. For instance, a family of curves (G and B versus ℓ/λ) can be generated for fixed cavity size and varying slot widths in wavelengths. The representation of these curves is similar to the input imped-ance curves of the planar dipole antenna considered by Rhodes in reference 21 and of the monopole antenna considered by Jordan and Balmain in reference 5. Such curves would enable the user to design rectangular cavity-backed slot antennas. Once the operating frequency (resonant frequency, B = 0) is chosen, the physical slot length and width and the physical dimensions of the cavity can be determined from these curves. However, in order to compare such curves with measurements, either the frequency must be held con-stant and the physical length of slot allowed to vary while the remaining dimensions are held fixed or all dimensions except the slot length must vary in such a manner that the electrical lengths remain constant as the frequency is changed. Obviously, the first method of performing the measurements would be more desirable.

Before presenting any design admittance curves, the input admittance represented by equations (93) and (94) is compared with experimental results (Long, ref. 22). In ref-erence 22, measured input impedance data for a slot (1 by 25 cm) backed by a rectangular cavity cross section (10 by 35 cm) with variable cavity depths over a frequency range of approximately 500 to 750 MHz are presented. Actually, only half the impedance was mea-sured since an imaging plane was used to bisect the slot lengthwise; the measured imped-ance must then be doubled.

Admittance as a function of frequency using the representation given by equa-tions (93) and (94) for the same cavities and slot size with cavity depths of 13.395, 17.86, 22.325, and 35.72 cm is presented in figure 3, and measured data are presented for com-

parison. The measured susceptance is in good agreement with calculations, particularly around resonance. The slope of the susceptance (which is related to bandwidth) at resonance becomes steeper as the depth of the cavity increases; the bandwidth becomes narrower. Also, the resonance frequency decreases as the depth of the cavity increases. The measured conductance is always greater than the calculated conductance; this may be caused by the losses in the cavity and ground plane, which have been neglected in the calculations.
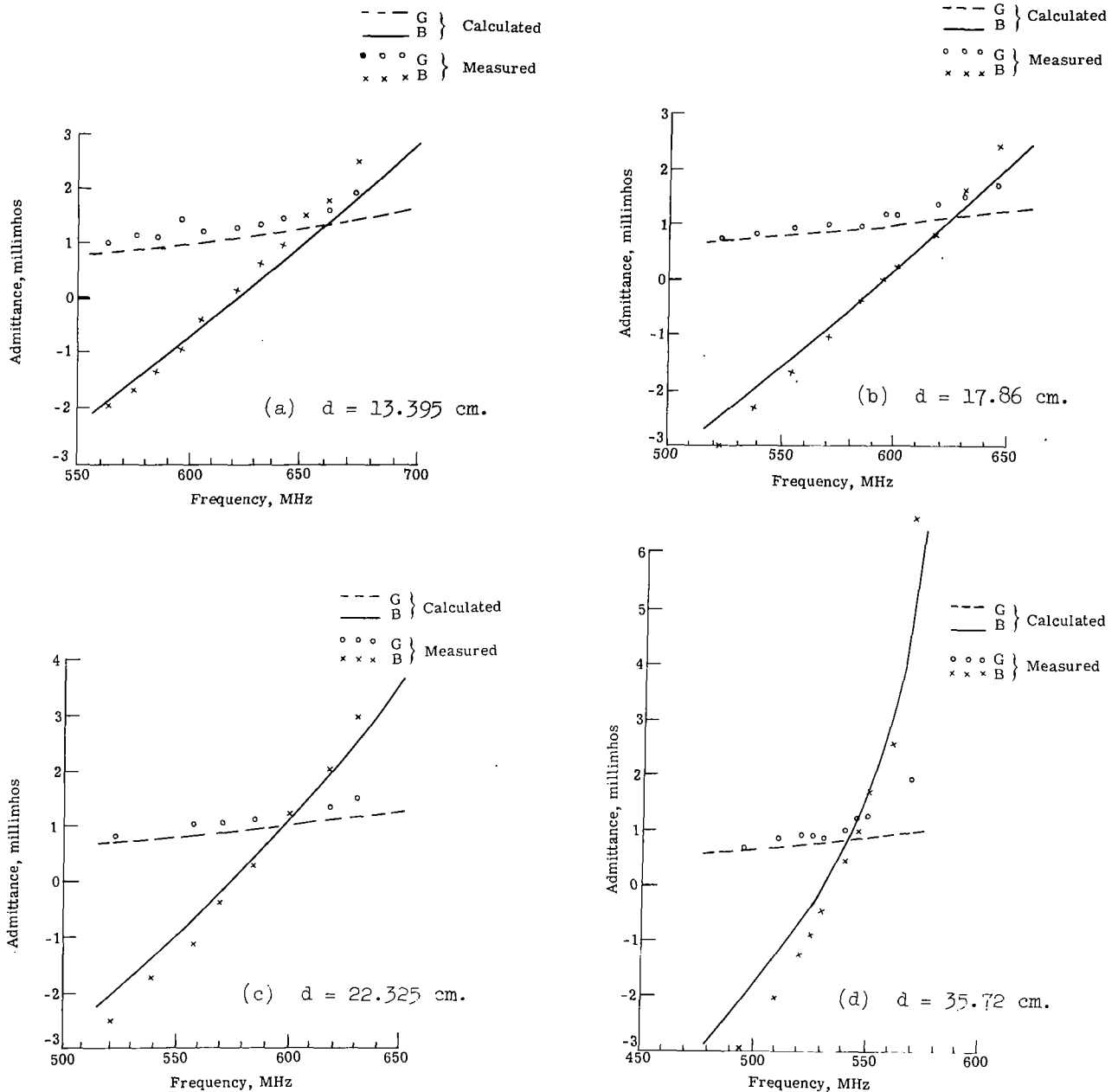
Figure 3.- Input admittance as a function of frequency for a' = 1 cm, ℓ = 25 cm, a = 10 cm, and b = 35 cm with varying cavity depths.

Since the representation of the input admittance has been shown to agree reasonably well with measured data, theoretical curves, based on this representation, were evaluated numerically and plotted as a function of slot length in wavelengths. In figure 4 the input admittance as a function of electrical slot length for three electrical slot widths (0.1, 0.01, and 0.001) are shown for increasing electrical cavity depths, respectively; the electrical cross section of cavity for these figures is rectangular (0.3 by 0.6). Similar curves for a square cavity cross section (0.6 by 0.6 wavelengths) are given in figure 5. Obviously, numerous sets of admittance curves for many different choices of slots and cavities could be computed. However, the calculated curves presented in figures 4 and 5 represent typical sets of curves that can be generated by equations (93) and (94). Such curves do offer invaluable design information; that is, once the operating frequency (resonant frequency, $B = 0$) is selected, the slot length and width and the cavity dimensions can be determined. Admittance curves as a function of frequency can also be generated for known physical dimensions, as was shown in figure 3.
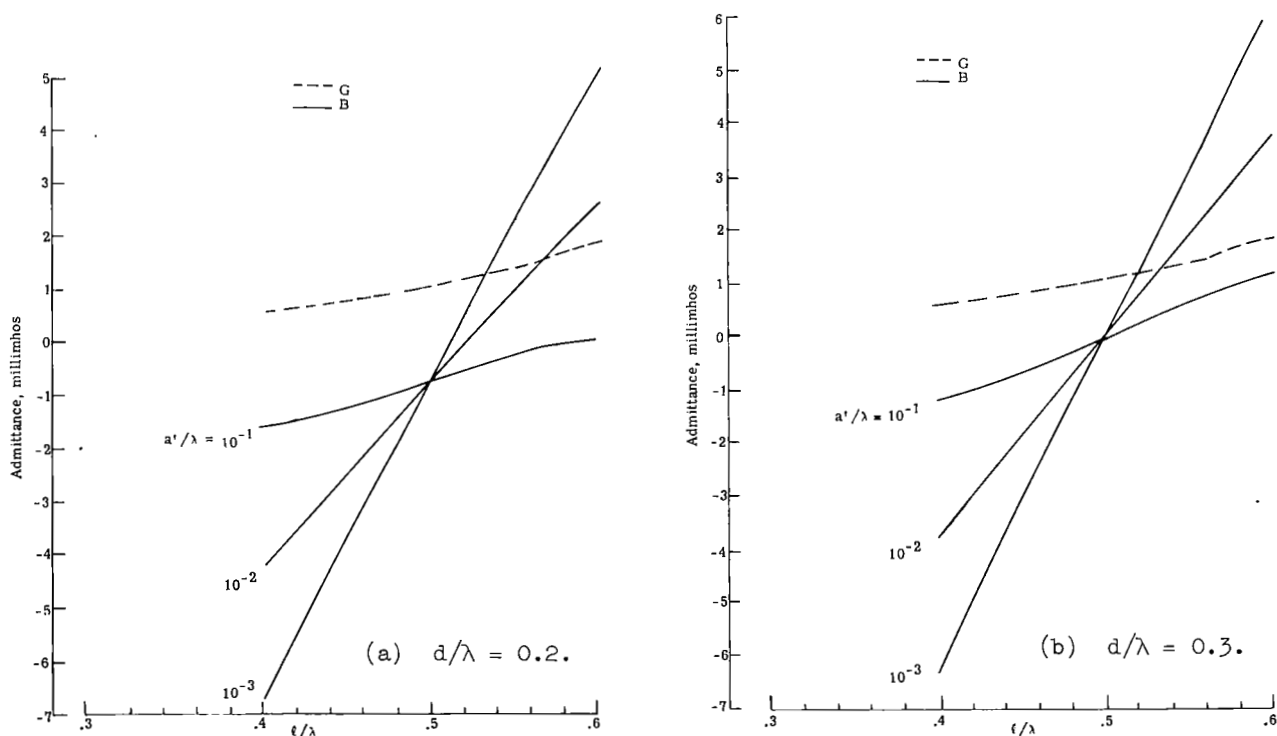
Figure 4.- Input admittance as a function of ℓ/λ
for $a/\lambda = 0.3$ and $b/\lambda = 0.6$.

(c) $d/\lambda = 0.4.$

(d) $d/\lambda = 0.5.$

(e) $d/\lambda = 0.6.$
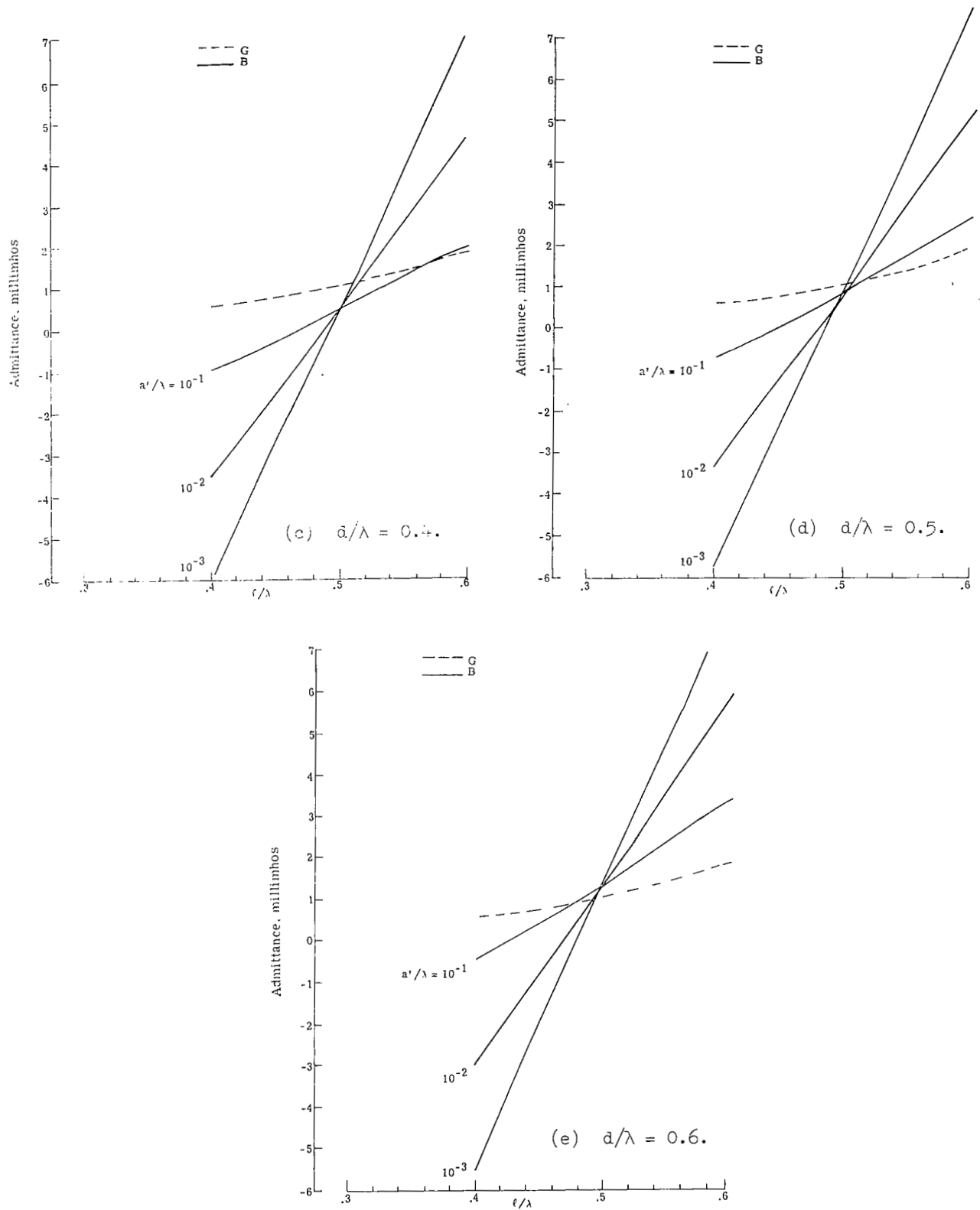
Figure 4.- Concluded.
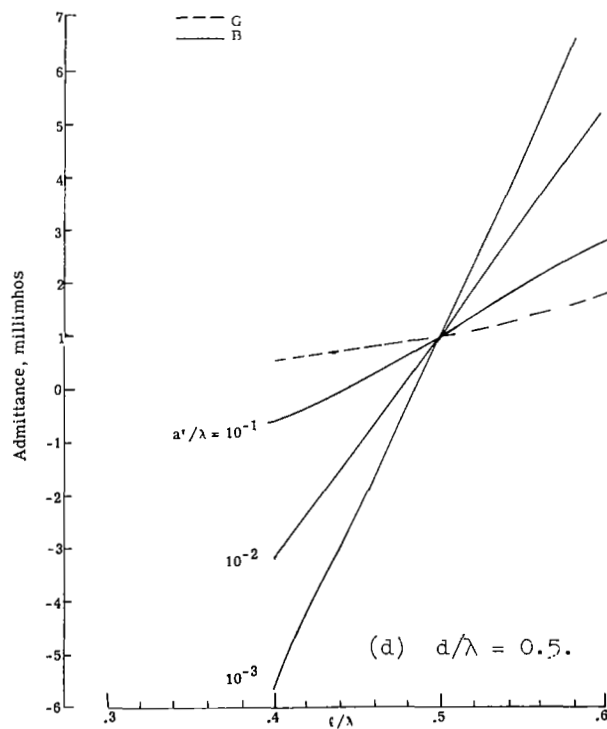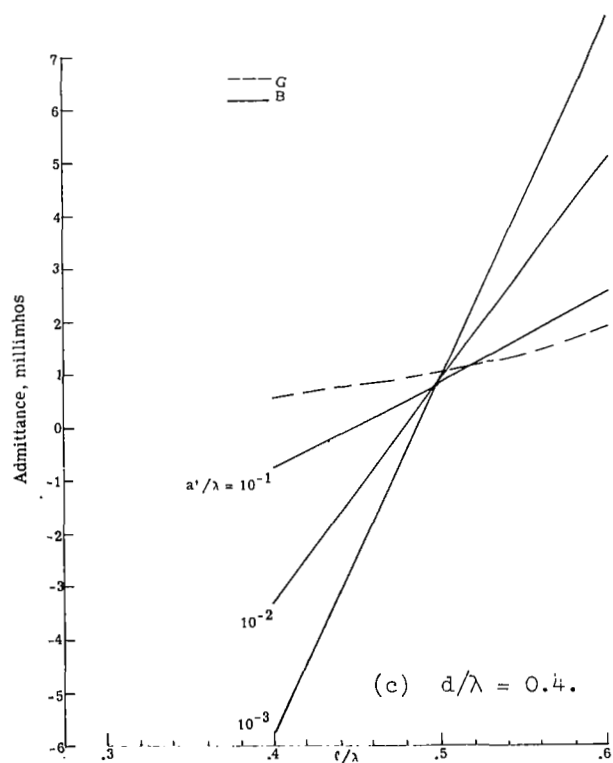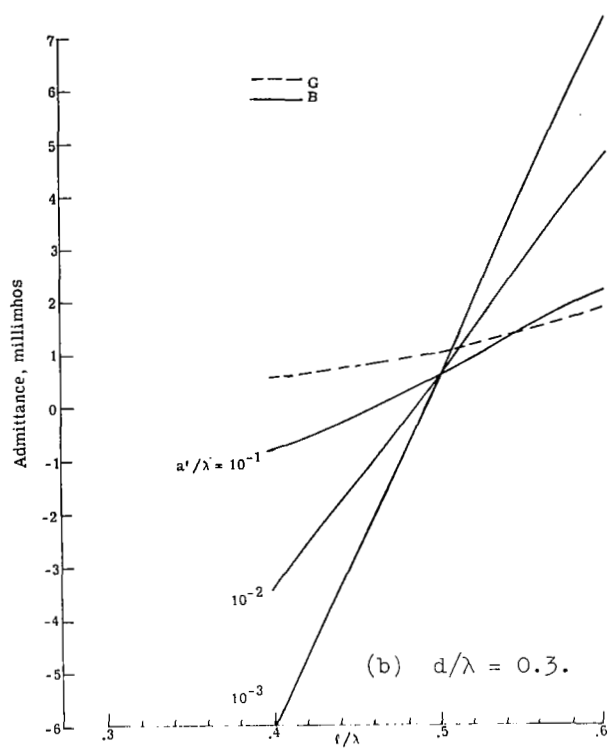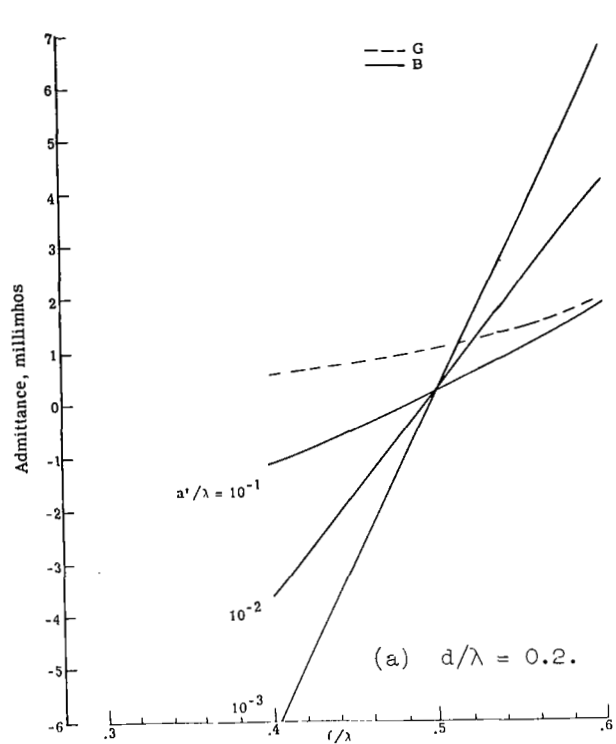
Figure 5.- Input admittance as a function of ℓ/λ for $a/\lambda = 0.6$ and $b/\lambda = 0.6$.
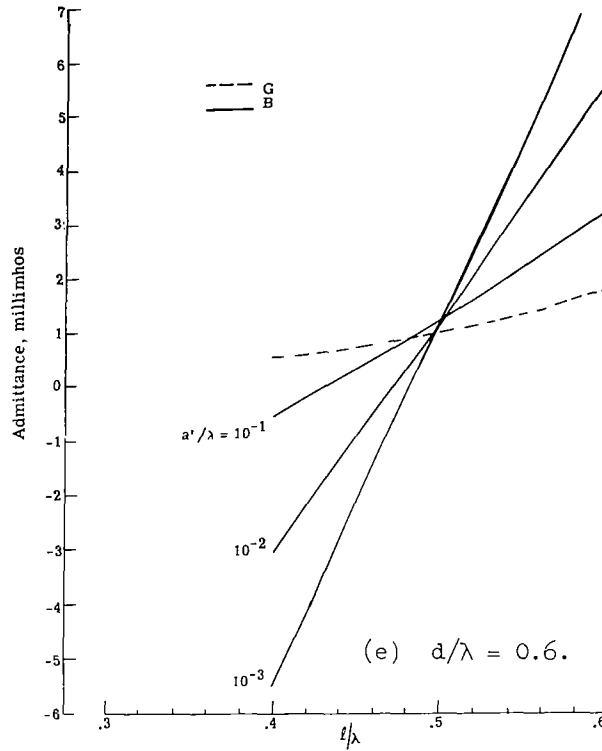
Figure 5.- Concluded.

An examination of equation (94), which represents the susceptance as seen at the input terminals of the slot, shows that the contribution from the propagating wave (first term in eq. (94)) to this susceptance is zero whenever $kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2} = \frac{n\pi}{2}$ for odd values of n; the contribution of this term is infinite whenever $kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2} = n\pi$ for all values of n. Its contribution to the total susceptance for other values of $kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}$ is finite and may be either inductive or capacitive; its contribution is inductive whenever

$$n\pi < kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2} < (2n + 1)\frac{\pi}{2} \quad \text{for} \quad n = 0, 1, 2, \ldots \quad \text{and is capacitive whenever}$$

$$(2n - 1)\frac{\pi}{2} < kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2} < n\pi \quad \text{for} \quad n = 1, 2, 3, \ldots \quad \text{The contribution that the second}$$

term in equation (94) makes to the total susceptance is capacitive (this is the case for

60

n = 1 and m ≠ 0 in the original double summation given by eq. (F1)). The contribution from the third term in equation (94) is inductive (this is the case for n > 1 in the summations given by eq. (F1)), and the contribution from the external region, given by the last term, is either inductive or capacitive.

As was noted earlier, the calculated conductance of the input admittance will probably be slightly lower than its actual value since the analysis does not include cavity and ground plane losses. The calculated conductance, which is given by equation (93), is one-half the conductance of the slot when it is located in an infinite perfectly conducting ground plane and is free to radiate on both sides.

Quality Factor and Relative Bandwidth Calculations

Before the quality factor Q and relative bandwidth B.W. can be computed, the resonant frequency (or slot length) must be determined for a given set of cavity and slot dimensions (or slot width and cavity dimensions in wavelengths). The resonant slot length rather than the resonant frequency will be determined because it is more convenient to work with electrical lengths and because it gives a more general description of the slot length. This resonant slot length ℓ_0/λ is found numerically by setting equation (90) to zero for a given set of electrical slot width a'/λ and cavity size a/λ, b/λ, and d/λ. The computer subroutine for determining the resonant lengths was verified graphically for the cross-over (B = 0) frequencies 620 MHz, 593 MHz, 574 MHz, and 530 MHz shown in figures 3(a), 3(b), 3(c), and 3(d), respectively.

Once ℓ_0/λ is determined, its value along with the corresponding values of a'/λ, a/λ, b/λ, and d/λ are used in the following equations to determine Q and B.W.:

$$Q = \left[\frac{2\omega\left(<<W_e>>\right)}{P_r} \right]_{\omega=\omega_r} \tag{95}$$

$$B.W. = \left\{ \frac{2P_r}{-\omega_r \frac{d}{d\omega}\left[2\omega\left(<<W_m>> - <<W_e>>\right)\right]} \right\}_{\omega=\omega_r} \tag{96}$$

where

$$2\omega\left(<<W_e>>\right) \approx \frac{4\left|V_o\right|^2}{Z_o(ka)(kb)} \left(\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{5/2}} \left[-\cot kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\right] - \frac{ka}{\pi} \frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2} \right.$$

$$\times \left[\ln\left(\frac{\pi ka'}{ka}\right) + \ln 2 - \frac{3}{2}\right] + \frac{ka}{\pi} \sum_{n=3,5}^{\infty} \frac{\left(\cos\frac{n\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{kb}\right)^2 - 1\right]^2} \left\{\frac{3}{2} - \gamma - \ln \right.$$

$$\left. \left. \times \left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1} \frac{ka'}{2}\right]\right\}\right) + \frac{8\left|V_o\right|^2}{(2\pi)^2 Z_o} \frac{\pi}{8} \left(Si(k\ell) - \frac{k\ell}{2} Cin(k\ell) + (k\ell - \sin k\ell)\ln\frac{ek\ell}{ka'} \right.$$

$$+ \left\{\left[Si(k\ell) - \frac{1}{2} Si(2k\ell)\right] - \frac{k\ell}{2}\left[Cin(k\ell) - Cin(2k\ell)\right] - k\ell \ln 2\right\} \cos k\ell$$

$$+ \left\{\left[Cin(k\ell) - \frac{1}{2} Cin(2k\ell)\right] + \frac{k\ell}{2}\left[Si(k\ell) - Si(2k\ell)\right] + \ln\frac{2}{e}\right\} \sin k\ell \right) \tag{97}$$

$$\omega\frac{d}{d\omega}\left[2\omega\left(<<W_m>> - <<W_e>>\right)\right] \approx \frac{4\left|V_o\right|^2}{(ka)(kb)} \frac{1}{Z_o} \left[-\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2} kd\cos^2 kd\sqrt{1 - \left(\frac{\pi}{kb}\right)^2} \right.$$

$$\tag{98}$$

$$+ \cot kd \sqrt{1 - \left(\frac{\pi}{kb}\right)^2} \left\{ k\ell \sin \frac{k\ell}{2} \frac{\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{3/2}} \right.$$

$$- \frac{2\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{5/2}} + \left. \frac{\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{3/2}} \right\}$$

$$+ \frac{ka}{\pi} \left[\ln\left(\frac{\pi ka'}{ka}\right) - \frac{3}{2} + \ln 2 \right] \left\{ k\ell \sin \frac{k\ell}{2} \frac{\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)}{1 - \left(\frac{\pi}{kb}\right)^2} \right.$$

$$- \frac{2\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2} + \left. \frac{\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{1 - \left(\frac{\pi}{kb}\right)^2} \right\} + \frac{ka}{\pi} \sum_{n=3,5}^{\infty}$$

$$\times \left(\frac{\left(\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)}{\left(\frac{n\pi}{kb}\right)^2 - 1} + \left\{ \frac{3}{2} - \gamma - \ln\left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1} \, \frac{ka'}{2} \right] \right\} \right.$$

$$\times \left\{ k\ell \sin \frac{k\ell}{2} \frac{\left(\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)}{\left(\frac{n\pi}{kb}\right)^2 - 1} + 2 \frac{\left(\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{kb}\right)^2 - 1\right]^2} \right.$$

(98)

(Equation continued on next page)

$$+ \frac{\left(\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{kb}\right)^2 - 1}\Bigg\}\Bigg)\Bigg] - \frac{8\left|V_o\right|^2}{(2\pi)^2 Z_O} \frac{\pi}{4} k\ell \Bigg\{ -\Bigg[Si(k\ell)$$

$$- \frac{1}{2} Si(2k\ell)\Bigg] \sin k\ell + \Bigg[Cin(k\ell) - \frac{1}{2} Cin(2k\ell)$$

$$- \ln\left(\frac{e^{3/2}k\ell}{2ka'}\right)\Bigg] \cos k\ell + \frac{\sin k\ell}{k\ell}\Bigg\} \tag{98}$$

and

$$P_r = \frac{8\left|V_o\right|^2}{(2\pi)^2} \frac{1}{Z_O} \frac{\pi}{2} \frac{1}{2} \Bigg\{ Cin(k\ell) + \Bigg[Cin(k\ell) - \frac{1}{2} Cin(2k\ell)\Bigg] \cos k\ell - \Bigg[Si(k\ell) - \frac{1}{2} Si(2k\ell)\Bigg] \sin k\ell \Bigg\} \tag{99}$$

The quality factor Q and the reciprocal of the relative bandwidth $1/B.W.$ using equations (103) and (104) were computed as a function of a'/λ for a number of different size cavities. A typical set of such curves is shown in figure 6 in graphical form.

From the curves shown in figure 6, it can be seen that both curves vary linearly with the log of a'/λ; furthermore, their slopes are approximately the same. These characteristics were found to be true for the cavity sizes considered in this paper. Therefore,

$$\left.\begin{array}{l} Q \approx h \log \frac{a'}{\lambda} + C_1 \\[2em] \frac{1}{B.W.} \approx h \log \frac{a'}{\lambda} + C_2 \end{array}\right\} \tag{100}$$
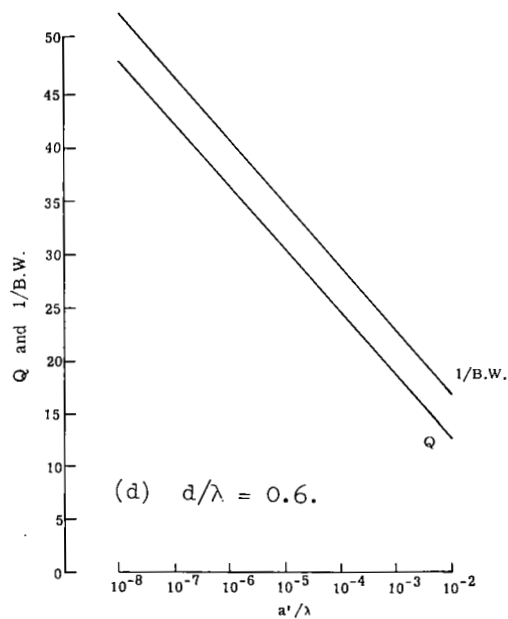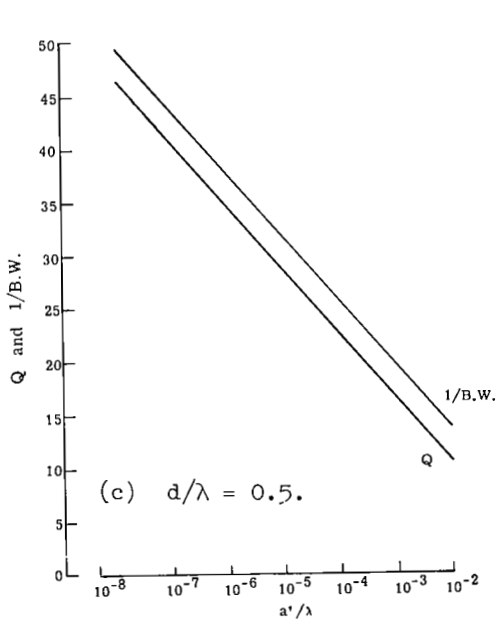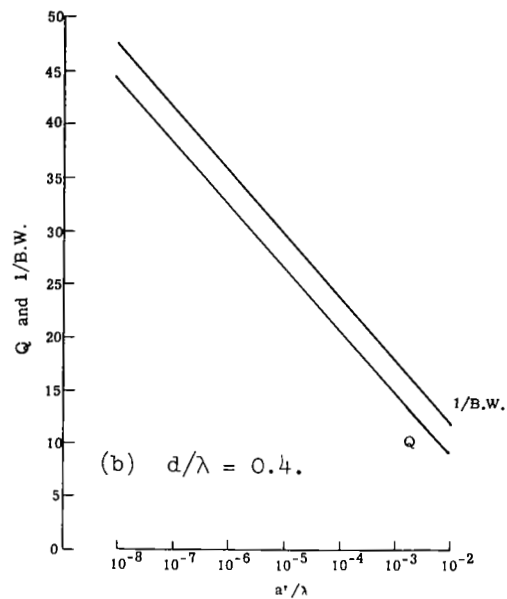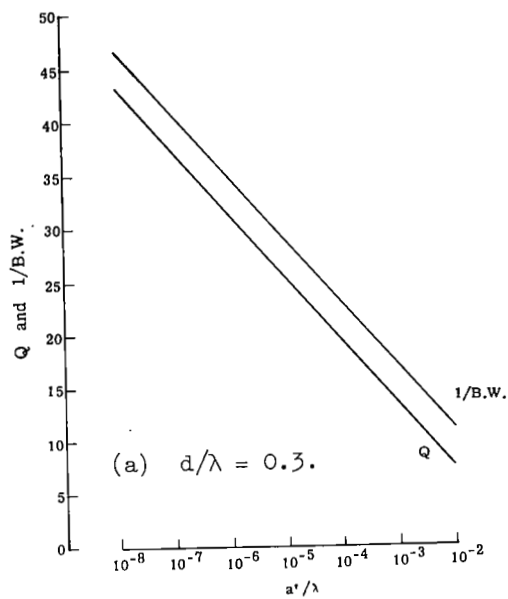
Figure 6.- Quality factor and reciprocal of relative bandwidth as a
function of a'/λ for $a/\lambda = 0.3$ and $b/\lambda = 0.6$.

where h is the slope (which appears to be independent of cavity dimensions, $h \approx -6$) and C_1 and C_2 are constants which depend on the cavity and slot size. These constants, which are the intercepts of Q and $1/B.W.$ axis, are quite large. However, for the cases considered in this research, the difference in $1/B.W.$ and Q is finite; that is,

$$\frac{1}{B.W.} - Q \approx C_2 - C_1 = C \tag{101}$$

where C is a finite constant which depends on the cavity and slot dimensions. Therefore,

$$\frac{1}{B.W.} \approx Q + C \tag{102}$$

The magnitude of $1/B.W.$ and Q determines the importance of the constant C; that is, if C is of the same order of magnitude as Q, then the reciprocal of relative bandwidth $1/B.W.$ is related to Q by equation (102). However, as Q becomes large, its relative magnitude compared with that of C is many orders of magnitude greater (C compared with Q is very small). Hence,

$$\lim_{Q \to \infty} \frac{1}{B.W.} = Q \tag{103}$$

which is the asymptotic relationship usually assumed in the literature. It is concluded that according to the application of Rhodes' method of identifying "physically observable" stored energies the reciprocal relationship, usually associated with nonradiating systems, is equally valid for the rectangular cavity-backed slot antenna.

Since Q and $1/B.W.$ were found to approximately satisfy equation (100) for the many cavity dimensions investigated, results are given in tabular form. Table I shows the computed resonant slot lengths for given electrical slot widths and cavity dimensions. The intercepts C_1 and C_2 which are obtained from equation (100) are also given. From this table, therefore, the Q and the $1/B.W.$ (and hence, relative bandwidth) can be computed with very little effort.

TABLE I.- COMPUTED RESONANT SLOT LENGTHS FOR GIVEN ELECTRICAL SLOT WIDTHS AND DIFFERENT CAVITY DIMENSIONS

$$\left[Q = -6 \log \frac{a'}{\lambda} + C_1; \quad \frac{1}{B.W.} \approx -6 \log \frac{a'}{\lambda} + C_2 \right]$$

d/λ	C_1	C_2	ℓ_0/λ for a'/λ of —						
			10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
$a/\lambda = 0.3$; $b/\lambda = 0.6$									
0.3	-4.5	-1.5	0.500	0.500	0.500	0.500	0.500	0.500	0.500
.4	-3.2	0	.489	.493	.495	.495	.495	.497	.497
.5	-1.1	2.0	.479	.487	.496	.492	.493	.494	.495
.6	1.0	5.0	.469	.480	.485	.488	.490	.491	.493
$a/\lambda = 0.4$; $b/\lambda = 0.6$									
0.3	-3.5	-1.0	0.493	0.495	0.496	0.497	0.498	0.498	0.498
.4	-2.7	-.3	.484	.490	.493	.494	.495	.495	.495
.5	-1.5	1.0	.478	.486	.489	.491	.493	.494	.494
.6	.5	3.5	.470	.481	.486	.488	.490	.491	.493
$a/\lambda = 0.5$; $b/\lambda = 0.6$									
0.3	-2.7	-0.7	0.489	0.493	0.494	0.495	0.495	0.497	0.497
.4	-2.0	.5	.482	.488	.491	.493	.494	.494	.495
.5	-1.0	1.5	.477	.485	.489	.490	.492	.493	.494
.6	.5	3.0	.471	.481	.486	.488	.490	.492	.493
$a/\lambda = 0.6$; $b/\lambda = 0.6$									
0.3	-2.7	-0.5	0.487	0.492	0.494	0.496	0.497	0.497	0.498
.4	-2.0	.3	.480	.481	.490	.492	.494	.494	.495
.5	-1.5	.8	.475	.484	.488	.491	.492	.493	.494
.6	-.4	2.5	.471	.481	.486	.488	.491	.492	.493
$a/\lambda = 0.3$; $b/\lambda = 0.7$									
0.3	-2.5	-0.3	0.497	0.498	0.498	0.498	0.499	0.504	0.499
.4	-1.5	1.2	.483	.488	.492	.493	.494	.495	.495
.5	.2	4.5	.467	.478	.484	.487	.489	.491	.492

TABLE I.- COMPUTED RESONANT SLOT LENGTHS FOR GIVEN
ELECTRICAL SLOT WIDTHS AND DIFFERENT
CAVITY DIMENSIONS – Concluded

$$\left[Q = -6 \log \frac{a'}{\lambda} + C_1; \quad \frac{1}{B.W.} \approx -6 \log \frac{a'}{\lambda} + C_2 \right]$$

d/λ	C_1	C_2	ℓ_0/λ for a'/λ of –						
			10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
$a/\lambda = 0.3$; $b/\lambda = 0.8$									
0.3	-2.0	-0.4	0.496	0.497	0.498	0.498	0.498	0.498	0.498
.4	-.5	2.0	.419	.486	.489	.491	.493	.493	.495
.5	1.5	8.0	.457	.471	.478	.482	.485	.487	.489
$a/\lambda = 0.4$; $b/\lambda = 0.7$									
0.3	-2.0	0.0	0.491	0.494	0.495	0.496	0.497	0.497	0.497
.4	-1.0	1.2	.480	.487	.491	.492	.493	.494	.495
.5	.0	3.3	.469	.480	.485	.488	.490	.491	.492
$a/\lambda = 0.4$; $b/\lambda = 0.8$									
0.3	-1.5	-0.5	0.490	0.493	0.494	0.496	0.496	0.497	0.497
.4	-1.0	1.2	.478	.489	.489	.491	.492	.493	.494
.5	1.0	5.5	.461	.474	.481	.485	.487	.489	.490

CONCLUSIONS

It is concluded that the time-average "physically observable" stored energies in an infinite rectangular waveguide and in a rectangular cavity can be expressed in a form which requires only the transverse electric and longitudinal magnetic field components. This same conclusion had been reached earlier for the planar aperture antenna. For the infinite waveguide case, the propagating fields do not contribute to these stored energies whereas the nonpropagating fields do. However, both propagating and nonpropagating fields were found to contribute to the stored energies for the rectangular cavity case. The representations of these energies were found to be analogous to the representations given for the planar aperture antenna; although, in the cavity case the representations are modified by the presence of the propagating field contributions and functions which depend on the cavity depths. It is concluded that the method of Rhodes can be adapted to the infinite waveguide and finite waveguide (cavity), etc.

Input susceptance calculations were found to agree with measurements, particularly around resonance. The slope of the susceptance at resonance became steeper as the depth of the cavity increased; the bandwidth became narrower. Also, the resonant frequency decreased as the depth of the cavity increased. The calculated input conductance was always less than the measured input conductance. This difference was attributed to neglecting the losses in the walls of the cavity and ground plane in the analysis.

The asymptotic relationship which is usually assumed to exist between the quality factor Q and the reciprocal of relative bandwidth for nonradiating systems was found to be equally applicable for the rectangular cavity-backed slot antenna. For lower values of Q, the reciprocal of relative bandwidth and Q are approximately related by constants which depend on the slot and cavity dimensions.

Langley Research Center,
 National Aeronautics and Space Administration,
  Hampton, Va., March 31, 1975.

DERIVATION OF FIELDS INSIDE A RECTANGULAR WAVEGUIDE SECTION

From Maxwell's equations, assuming an $e^{j\omega t}$ time convention, a wave equation is obtained in terms of the electric field $\vec{E}$,

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \tag{A1}$$

where $k = \omega\sqrt{\mu\epsilon}$. Expressing $\vec{E}$ in its rectangular components, equation (A1) for the E_x and E_y components becomes

$$\frac{\partial^2}{\partial x^2} E_x(x,y,z,k) + \frac{\partial^2}{\partial y^2} E_x(x,y,z,k) + \frac{\partial^2}{\partial z^2} E_x(x,y,z,k) + k^2 E_x(x,y,z,k) = 0 \tag{A2a}$$

$$\frac{\partial^2}{\partial x^2} E_y(x,y,z,k) + \frac{\partial^2}{\partial y^2} E_y(x,y,z,k) + \frac{\partial^2}{\partial z^2} E_y(x,y,z,k) + k^2 E_y(x,y,z,k) = 0 \tag{A2b}$$

By separation of variables, a solution to equations (A2) can be written as

$$E_x(x,y,z,k) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)\left(F e^{-jk_z z} + G e^{jk_z z}\right) \tag{A3a}$$

$$E_y(x,y,z,k) = (A' \cos k_x x + B' \sin k_x x)(C' \cos k_y y + D' \sin k_y y)\left(F' e^{-jk_z z} + G' e^{jk_z z}\right) \tag{A3b}$$

where $k_x^2 + k_y^2 + k_z^2 = k^2$ and A, B, etc., and A', B', etc., are arbitrary constants.

The coefficients will now be determined for a rectangular waveguide (see fig. 1). The boundary conditions on E_y at $x = 0,a$ are

$$E_y = 0 \tag{A4}$$

that is, for all y and z within the cavity,

$$A'(C' \cos k_y y + D' \sin k_y y)\left(F' e^{-jk_z z} + G' e^{jk_z z}\right) = 0 \tag{A5a}$$

$$(A' \cos k_x a + B' \sin k_x a)(C' \cos k_y y + D' \sin k_y y)\left(F' e^{-jk_z z} + G' e^{jk_z z}\right) = 0 \tag{A5b}$$

Equation (A5a) implies that $A' = 0$. Hence from equation (A5b) for $B' \neq 0$, it follows that $\sin k_x a = 0$, or that

$$k_x a = m\pi \qquad\qquad (m = 0, 1, 2, \ldots) \qquad\qquad (A6)$$

Therefore, equation (A3b) becomes

$$E_y = B' \sin \frac{m\pi}{a} x (C' \cos k_y y + D' \sin k_y y)\left(F'e^{-jk_z z} + G'e^{jk_z z}\right) \qquad (A7)$$

Applying the boundary condition $E_x = 0$ at $y = 0,b$ yields

$$E_x = (A \cos k_x x + B \sin k_x x) D \sin \frac{n\pi}{b} y \left(Fe^{-jk_z z} + Ge^{jk_z z}\right) \qquad (A8)$$

where k_y was found to have the value $n\pi/b$ for $n = 0, 1, 2, \ldots$. Rewriting equations (A7) and (A8) with $k_x = \frac{m\pi}{a}$ and $k_y = \frac{n\pi}{b}$

$$E_x = D\left(A \cos \frac{m\pi}{a} x + B \sin \frac{m\pi}{a} x\right)\sin \frac{n\pi}{b} y\left(Fe^{-jk_z z} + Ge^{jk_z z}\right) \qquad (A9a)$$

$$E_y = B' \sin \frac{m\pi}{a} x\left(C' \cos \frac{n\pi}{b} y + D' \sin \frac{n\pi}{b} y\right)\left(F'e^{-jk_z z} + G'e^{jk_z z}\right) \qquad (A9b)$$

At this point Maxwell's divergence equation $\nabla \cdot D = 0$ is used to determine the remaining electric field component E_z

$$\nabla \cdot D = 0 \qquad\qquad (A10)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad\qquad (A11)$$

$$\frac{\partial E_z}{\partial z} = -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y}$$

$$= -D \frac{m\pi}{a}\left(-A \sin \frac{n\pi}{b} x + B \cos \frac{m\pi}{a} x\right)\sin \frac{n\pi}{b} y\left(Fe^{-jk_z z} + Ge^{jk_z z}\right)$$

$$- B' \frac{n\pi}{b} \sin \frac{m\pi}{a} x\left(-C' \sin \frac{n\pi}{b} y + D' \cos \frac{n\pi}{b} y\right)\left(F'e^{-jk_z z} + G'e^{jk_z z}\right) \quad (A12)$$

Integrating on z, equation (A12) becomes

$$E_z = -D \frac{m\pi}{a}\left(-A \sin \frac{m\pi}{a} x + B \cos \frac{m\pi}{a} x\right)\sin \frac{n\pi}{b} y\left(\frac{F}{-jk_z} e^{-jk_z z} + \frac{G}{jk_z} e^{jk_z z}\right)$$

$$- B' \frac{n\pi}{b} \sin \frac{m\pi}{a} x\left(-C' \sin \frac{n\pi}{b} y + D' \cos \frac{n\pi}{b} y\right)\left(\frac{F'}{jk_z} e^{-k_z z} + \frac{G'}{jk_z} e^{jk_z z}\right) \qquad (A13)$$

The constant of integration has been set equal to zero since it would not represent a traveling wave in the z direction. Applying the boundary conditions that $E_z = 0$ for $x = 0, a$ and $y = 0, b$,

$$-D \frac{m\pi}{a} B \sin \frac{n\pi y}{b}\left(\frac{F}{-jk_z} e^{-jk_z z} + \frac{G}{jk_z} e^{jk_z z}\right) + 0 = 0 \qquad (x = 0) \qquad (A14)$$

which implies $B = 0$ and

$$0 - B'D' \frac{n\pi}{b} \sin \frac{m\pi}{a} x\left(\frac{F'}{-jk_z} e^{-jk_z z} + \frac{G}{jk_z} e^{jk_z z}\right) = 0 \qquad (y = 0) \qquad (A15)$$

which implies $D' = 0$. The boundary conditions at $x = a$ and $y = b$ give redundant information. Substituting these conditions into the equation for the electric field components yields

$$E_x = DA \cos \frac{m\pi x}{a} \sin \frac{n\pi}{b} y\left(Fe^{-jk_z z} + Ge^{jk_z z}\right) \qquad (A16a)$$

$$E_y = B'C' \sin \frac{m\pi x}{b} \cos \frac{n\pi}{b} y\left(F'e^{-jk_z z} + G'e^{jk_z z}\right) \qquad (A16b)$$

$$E_z = DA \frac{m\pi}{a} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y\left(\frac{F}{jk_z} e^{-jk_z z} + \frac{G}{jk_z} e^{jk_z z}\right)$$

$$+ B'C' \frac{n\pi}{b} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y\left(\frac{F'}{-jk_z} e^{-jk_z z} + \frac{G'}{jk_z} e^{jk_z z}\right) \qquad (A16c)$$

Simplifying equations (A16) with the substitutions $DAF = A_{mn}$, $DAG = B_{mn}$, $B'C'F' = C_{mn}$, and $B'C'G' = D_{mn}$ and recognizing the dependence of E_x, E_y, and E_z upon m and n these quantities (eqs. (A16)) may be written as

$$(E_x)_{mn} = \left(A_{mn} e^{-jk_z z} + B_{mn} e^{jk_z z}\right) \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \tag{A17a}$$

$$(E_y)_{mn} = \left(C_{mn} e^{-jk_z z} + D_{mn} e^{jk_z z}\right) \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \tag{A17b}$$

$$(E_z)_{mn} = \frac{1}{-jk_z} \left[\left(\frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn}\right) e^{-jk_z z} - \left(\frac{m\pi}{a} B_{mn} + \frac{n\pi}{b} D_{mn}\right) e^{jk_z z}\right] \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \tag{A17c}$$

Thus the most general solution of equation (A3) will be given by

$$E_x = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (E_x)_{mn} \tag{A18a}$$

$$E_y = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (E_y)_{mn} \tag{A18b}$$

$$E_z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (E_z)_{mn} \tag{A18c}$$

From Maxwell's curl equation

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{A19}$$

the magnetic field can be determined. Hence, the magnetic field components are determined as follows:

73

$$(H_x)_{mn} = -\frac{1}{j\omega\mu}\left[\frac{\partial(E_z)_{mn}}{\partial y} - \frac{\partial(E_y)_{mn}}{\partial z}\right]$$

$$= -\frac{1}{j\omega\mu}\left\{\genfrac{}{}{0pt}{}{n\pi/b}{-jk_z}\left[\left(\frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn}\right)e^{-jk_z z} - \left(\frac{m\pi}{a}B_{mn} + \frac{n\pi}{b}D_{mn}\right)e^{jk_z z}\right]\right.$$

$$\left. \times \sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y - \left(-jk_z C_{mn}e^{-jk_z z} + jk_z D_{mn}e^{jk_z z}\right)\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\right\}$$

$$= -\frac{1}{j\omega\mu}\left(\frac{1}{-jk_z}\right)\left(\left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right\}e^{-jk_z z} - \left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)B_{mn}\right.\right.$$

$$\left.\left. + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]D_{mn}\right\}e^{jk_z z}\right)\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y$$

$$= -\frac{1}{k_z\omega\mu}\left(\left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right\}e^{-jk_z z} - \left\{\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)B_{mn}\right.\right.$$

$$\left.\left. + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]D_{mn}\right\}e^{jk_z z}\right)\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \tag{A20}$$

$$(H_y)_{mn} = -\frac{1}{j\omega\mu}\left[\frac{\partial(E_x)_{mn}}{\partial z} - \frac{\partial(E_z)_{mn}}{\partial x}\right]$$

$$= -\frac{1}{j\omega\mu}\left\{\left(-jk_z A_{mn}e^{-jk_z z} + jk_z B_{mn}e^{jk_z z}\right)\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y - \left(\frac{1}{-jk_z}\right)\left[\left(\frac{m\pi}{a}A_{mn}\right.\right.\right.$$

$$\tag{A21}$$

$$+ \frac{n\pi}{b} C_{mn}\Bigg)e^{-jk_z z} - \left(\frac{m\pi}{a} B_{mn} + \frac{n\pi}{b}\right)D_{mn}\,e^{jk_z z}\Bigg]\frac{m\pi}{a}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\Bigg\}$$

$$= \frac{1}{k_z\omega\mu}\left(\Bigg\{\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\Bigg\}e^{-jk_z z} - \Bigg\{\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]B_{mn}\right.$$

$$\left.+ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)D_{mn}\Bigg\}e^{jk_z z}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\right) \qquad (A21)$$

$$(H_z)_{mn} = -\frac{1}{j\omega\mu}\left[\frac{\partial(E_y)_{mn}}{\partial x} - \frac{\partial(E_x)_{mn}}{\partial y}\right]$$

$$= -\frac{1}{j\omega\mu}\left[\frac{m\pi}{a}\left(C_{mn}\,e^{-jk_z z} + D_{mn}\,e^{jk_z z}\right)\cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y - \frac{n\pi}{b}\left(A_{mn}\,e^{-jk_z z}\right.\right.$$

$$\left.\left.+ B_{mn}\,e^{jk_z z}\right)\cos\frac{m\pi}{b}x\cos\frac{n\pi}{a}y\right]$$

$$= -\frac{1}{j\omega\mu}\left[\left(\frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn}\right)e^{-jk_z z} + \left(\frac{m\pi}{a} D_{mn} - \frac{n\pi}{b} B_{mn}\right)e^{jk_z z}\right]$$

$$\times \cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \qquad (A22)$$

75

Hence,

$$H_x = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (H_x)_{mn} \tag{A23a}$$

$$H_y = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (H_y)_{mn} \tag{A23b}$$

$$H_z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (H_z)_{mn} \tag{A23c}$$

PROOF THAT THE CONTRIBUTIONS FROM THE PROPAGATING WAVES TO THE STORED ENERGIES CANCEL WHEN THE VOLUME INTEGRALS ARE DIFFERENCED

That the contributions from the propagating waves to the stored energies cancel when the volume integrals are differenced is proved as follows:

From equations (45) and (46),

$$
\lim_{d\to\infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{\omega^2\mu^2}\left\{ \frac{\gamma_m\epsilon_n}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 d + \frac{\gamma_n\epsilon_m}{k_z^2}\left|\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn}\right.\right.
$$

$$
\left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2 d + \epsilon_m\epsilon_n\left|\frac{m\pi}{a}\,C_{mn} - \frac{n\pi}{b}\,A_{mn}\right|^2 d\right\} - \lim_{d\to\infty}\frac{ab\epsilon}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\gamma_n\epsilon_m\left|A_{mn}\right|^2 d\right.
$$

$$
+ \gamma_m\epsilon_n\left|C_{mn}\right|^2 d + \frac{\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}\,A_{mn} + \frac{n\pi}{b}\,C_{mn}\right|^2 d\right) = \lim_{d\to\infty}\frac{ab\mu}{16\omega^2\mu^2}\left(\sum_{m=1}^{\infty}\left(2k_{z1}^2\left|C_{m0}\right|^2\right.\right.
$$

$$
+ 2\left|\frac{m\pi}{a}\,C_{m0}\right|^2 - 2k^2\left|C_{m0}\right|^2\right)d + \sum_{n=1}^{\infty}\left(2k_{z2}^2\left|A_{0m}\right|^2 + 2\left|\frac{n\pi}{b}\,A_{0n}\right|^2 - 2k^2\left|A_{0n}\right|^2\right)d
$$

$$
+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{1}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 d + \frac{1}{k_z^2}\left|\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2 d\right.
$$

$$
+ \left|\frac{m\pi}{a}\,C_{mn} - \frac{n\pi}{b}\,A_{mn}\right|^2 d - k^2\left(\left|A_{mn}\right|^2 d + \left|C_{mn}\right|^2 d + \frac{1}{k_z^2}\left|\frac{m\pi}{a}\,A_{mn} + \frac{n\pi}{b}\,C_{mn}\right|^2 d\right)\right\}\right) \tag{B1}
$$

where

$$
\left.\begin{aligned}
k_{z1}^2 &= k^2 - \left(\frac{m\pi}{a}\right)^2 \\[2mm]
k_{z2}^2 &= k^2 - \left(\frac{n\pi}{b}\right)^2 \\[2mm]
k_z^2 &= k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2
\end{aligned}\right\}
\tag{B2}
$$

Substituting equations (B2) into the right-hand side of equation (B1) yields

$$
\lim_{d\to\infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2\mu^2} \left\{ \frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn} \right|^2 d + \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right] A_{mn} \right.\right.
$$

$$
\left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 d + \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 d \right\} - \lim_{d\to\infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\gamma_n \epsilon_m \left| A_{mn} \right|^2 d \right.
$$

$$
\left. + \gamma_m \epsilon_n \left| C_{mn} \right|^2 d + \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 d \right) = 0 + 0 + \lim_{d\to\infty} \frac{ab}{16\omega^2\mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k_z^2}
$$

$$
\times \left(\left| A_{mn} \right|^2 \left\{ \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]^2 + k_z^2\left(\frac{n\pi}{b}\right)^2 - k^2 k_z^2 - \left(\frac{m\pi}{a}\right)^2 k^2 \right\} + 2\mathrm{Re}\!\left(A_{mn} C_{mn}^*\right) \right.
$$

$$
\times \left[\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\right]\left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 + k_z^2 + \left(\frac{m\pi}{a}\right)^2 - k_z^2 - k^2\right] + \left| C_{mn} \right|^2 \left\{ \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]^2 + \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \right.
$$

$$
\left.\left. + \left(\frac{m\pi}{a}\right)^2 k_z^2 - k^2 k_z^2 - k^2\left(\frac{n\pi}{b}\right)^2 \right\} \right) d
\tag{B3}
$$

Rewriting equation (B3) yields

$$\lim_{d\to\infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left\{ \frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn} \right|^2 d + \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right] A_{mn} \right.\right.$$

$$\left.\left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 d + \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 d \right\} - \lim_{d\to\infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\gamma_n \epsilon_m \left| A_{mn} \right|^2 d \right.$$

$$\left. + \gamma_m \epsilon_n \left| C_{mn} \right|^2 d + \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 d \right) = \lim_{d\to\infty} \frac{ab}{16\omega^2 \mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k_z^2} \left\{ \left| A_{mn} \right|^2 \right.$$

$$\times \left[\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + k^4 - 2k^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 + k^2\left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 - \left(\frac{n\pi}{b}\right)^4 - k^4 + k^2\left(\frac{n\pi}{b}\right)^2 \right.$$

$$\left. + k^2\left(\frac{m\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2 k^2 \right] + 2\mathrm{Re}\left(A_{mn} C_{mn}^*\right)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\left[k^2 - \left(\frac{m\pi}{a}\right)^2 + k^2 - \left(\frac{n\pi}{b}\right)^2 - k^2 \right.$$

$$\left. + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2 \right] + \left| C_{mn} \right|^2 \left[k^4 - 2k^2\left(\frac{m\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^4 + \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + k^2\left(\frac{m\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \right.$$

$$\left.\left. - \left(\frac{m\pi}{a}\right)^4 - k^4 + k^2\left(\frac{m\pi}{a}\right)^2 + k^2\left(\frac{n\pi}{b}\right)^2 - k^2\left(\frac{n\pi}{b}\right)^2 \right] \right\} d \qquad \text{(B4)}$$

Hence,

$$
\lim_{d\to\infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2\mu^2} \left\{ \frac{\gamma_m\epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn} \right|^2 d + \frac{\gamma_n\epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} \right. \right.
$$

$$
\left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn} \right|^2 d + \epsilon_m\epsilon_n \left| \frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn} \right|^2 d \right\} - \lim_{d\to\infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\gamma_n\epsilon_m \left| A_{mn} \right|^2 d \right.
$$

$$
\left. + \gamma_m\epsilon_n \left| C_{mn} \right|^2 d + \frac{\gamma_m\gamma_n}{k_z^2} \left| \frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn} \right|^2 d \right) \equiv 0 \tag{B5}
$$

PROOF THAT THE BRACED TERMS GIVEN IN EQUATION (49) CANCEL

That the braced terms given in equation (49) cancel is proved as follows:

From equation (49)

$$
\lim_{d \to \infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left(\left\{ \left| -\frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right] C_{mn} \right|^2 - \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] A_{mn} \right. \right. \right. \right.
$$

$$
\left. \left. \left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 - \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 \right\} \right) \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}
$$

$$
- \lim_{d \to \infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\left(-\gamma_n \epsilon_m \left| A_{mn} \right|^2 - \gamma_m \epsilon_n \left| C_{mn} \right|^2 - \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 \right) \right.
$$

$$
\left. \times \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] = \lim_{d \to \infty} \frac{ab}{16\omega^2 \mu} \left(\sum_{m=1}^{\infty} \left(-2k_{z1}^2 \left| C_{m0} \right|^2 - 2\left(\frac{m\pi}{a}\right)^2 \left| C_{m0} \right|^2 \right) \right. \tag{C1}
$$

$$+ 2k^2 \left|C_{m0}\right|^2\right) \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 - k^2}} + \sum_{n=1}^{\infty}\left[-2k_{z2}^2\left|A_{0n}\right|^2 - 2\left(\frac{n\pi}{b}\right)^2\left|A_{0n}\right|^2 - 2k^2\left|A_{0n}\right|^2\right]$$

$$\times \frac{e^{-2d\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{1}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 - \frac{1}{k_z^2}\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn}\right.$$

$$\left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2 - \left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2 + k^2\left|A_{mn}\right|^2 + k^2\left|C_{mn}\right|^2 + \frac{1}{k_z^2}\left|\frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn}\right|^2\right\}$$

$$\times \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right) \tag{C1}$$

where

$$\left.\begin{array}{l} k_{z1}^2 = k^2 - \left(\frac{m\pi}{a}\right)^2 \\[2em] k_{z2}^2 = k^2 - \left(\frac{n\pi}{b}\right)^2 \\[2em] k_z^2 = k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \end{array}\right\} \tag{C2}$$

$$\lim_{d \to \infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left(\left\{ -\frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn} \right|^2 - \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] A_{mn} \right. \right. \right.$$

$$\left. \left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 - \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 \right\} \right) \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$- \lim_{d \to \infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\left(-\gamma_n \epsilon_m \left| A_{mn} \right|^2 - \gamma_m \epsilon_n \left| C_{mn} \right|^2 - \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 \right) \right]$$

$$\times \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} = 0 + 0 + \lim_{d \to \infty} \frac{1}{16\omega^2 \mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -\frac{1}{k_z^2} \left(\left| A_{mn} \right|^2 \left\{ \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 \right. \right.$$

$$\left. + \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right]^2 + k_z^2 \left(\frac{n\pi}{b}\right)^2 - k^2 k_z^2 - k^2 \left(\frac{m\pi}{a}\right)^2 \right\} + \left| C_{mn} \right|^2 \left\{ \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right]^2 + \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 \right.$$

$$\left. + \left(\frac{m\pi}{a}\right)^2 k_z^2 - k^2 k_z^2 - k^2 \left(\frac{n\pi}{b}\right)^2 \right\} - 2\mathrm{Re} \left\{ \left(A_{mn} C_{mn}^* \right) \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 + k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right. \right.$$

$$\left. \left. \left. - k_z^2 - k^2 \right\} \right) \right) \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - 1} } \tag{C3}$$

$$\lim_{d\to\infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2\mu^2} \left(\left\{-\frac{\gamma_m\epsilon_n}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 - \frac{\gamma_n\epsilon_m}{k_z^2}\left|\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn}\right.\right.\right.$$

$$\left.\left.\left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2 - \epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2\right\}\frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right.$$

$$- \lim_{d\to\infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\left(-\gamma_n\epsilon_m\left|A_{mn}\right|^2 - \gamma_m\epsilon_n\left|C_{mn}\right|^2 - \frac{\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn}\right|^2\right)\right.$$

$$\left.\times \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right. = \lim_{d\to\infty} \frac{ab}{16\omega^2\mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -\frac{1}{k_z^2}\left(\left|A_{mn}\right|^2\left[\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + k^4 - 2k^2\left(\frac{n\pi}{b}\right)^2\right.\right.$$

$$\left. + \left(\frac{n\pi}{b}\right)^4 + k^2\left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 - \left(\frac{n\pi}{b}\right)^4 - k^4 + k^2\left(\frac{m\pi}{a}\right)^2 + k^2\left(\frac{n\pi}{b}\right)^2 - k^2\left(\frac{m\pi}{a}\right)^2\right] + \left|C_{mn}\right|^2$$

$$\times \left[k^4 - 2k^2\left(\frac{m\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^4 + \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + k^2\left(\frac{m\pi}{a}\right)^2 - k^4 + k^2\left(\frac{m\pi}{a}\right)^2 + k^2\left(\frac{n\pi}{b}\right)^2 - k^2\left(\frac{n\pi}{b}\right)^2\right]$$

$$- 2\text{Re}\left\{\left(A_{mn}C_{mn}^*\right)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\left[\left(\frac{n\pi}{b}\right)^2 + k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2 - k^2\right]\right\}\right) \tag{C4}$$

$$\lim_{d \to \infty} \frac{ab\mu}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\omega^2 \mu^2} \left(\left\{ -\frac{\gamma_m \epsilon_n}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right] C_{mn} \right|^2 - \frac{\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right] A_{mn} \right. \right. \right.$$

$$\left. \left. \left. + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right|^2 - \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 \right\} \right) \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$- \lim_{d \to \infty} \frac{ab\epsilon}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\left(-\gamma_n \epsilon_m \left| A_{mn} \right|^2 - \gamma_m \epsilon_n \left| C_{mn} \right|^2 - \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 \right) \right]$$

$$\times \frac{e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} - 1}{-2\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \equiv 0 \tag{C5}$$

PROOF THAT TERMS IN EQUATION (61) CANCEL

In order to show that all terms in equation (61) cancel, first let

$$
\left\{ I \right\} = \left(\left\{ \left| \frac{2\gamma_m \epsilon_n}{k_z^2} \right| \frac{m\pi}{a} \frac{n\pi}{b} A_{mn} + \left[\left(\frac{n\pi}{b} \right)^2 + k_z^2 \right] C_{mn} \right|^2 + \frac{2\gamma_n \epsilon_m}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a} \right)^2 \right] A_{mn} \right. \right.
$$

$$
\left. \left. + \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right) C_{mn} \right\}^2 \left[d + \frac{\sin 2d \sqrt{k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2}} \right] - 2\epsilon_{mn}\epsilon_n \left| \frac{m\pi}{a} C_{mn} \right. \right.
$$

$$
\left. - \frac{n\pi}{b} A_{mn} \right|^2 \left[d - \frac{\sin 2d \sqrt{k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2}} \right] + 4\epsilon_{mn}\epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 d \right) \quad (D1)
$$

Then, let

$$
\left\{ II \right\} = \left\{ -2\gamma_n \epsilon_m \left| A_{mn} \right|^2 - 2\gamma_m \epsilon_n \left| C_{mn} \right|^2 \left[d + \frac{\sin 2d \sqrt{k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2}} \right] \right.
$$

$$
(D2)
$$

(Equation continued on next page)

$$+ \frac{2\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn}\right|^2\left[d + \frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}\right]$$

$$\left. + \left(4\gamma_n\epsilon_n\left|A_{mn}\right|^2 + 4\gamma_n\epsilon_n\left|C_{mn}\right|^2\right)d\right\} \tag{D2}$$

Therefore, from equation (61),

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{I\right\} - k^2\left\{II\right\} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\left[d + \frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}\right]\left\{\frac{2\gamma_m\epsilon_n}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}\right.\right.\right.$$

$$\left.+ \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 + \frac{2\gamma_n\epsilon_m}{k_z^2}\left|\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2$$

$$\left.+ 2\epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}^*\right|^2\right\} - k^2\left[d + \frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}\right]$$

$$\tag{D3}$$

87

$$\times \left(2\gamma_n\epsilon_m\left|A_{mn}\right|^2 + 2\gamma_m\epsilon_n\left|C_{mn}\right|^2 + \frac{2\gamma_m\gamma_n}{k_z^2}\left|\frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn}\right|^2\right)$$

(D3)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{I\} - k^2\{II\} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2\left[d + \frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}\right]\left\{\frac{\gamma_m\epsilon_n}{k_z^2}\left|\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)A_{mn}\right.\right.$$

$$+ \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 + \frac{\gamma_n\epsilon_m}{k_z^2}\left|\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2$$

$$+ \epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn} - \frac{n\pi}{b}A_{mn}\right|^2 - \gamma_n\epsilon_n k^2\left|A_{mn}\right|^2 - \gamma_m\epsilon_n k^2\left|C_{mn}\right|^2$$

$$- \frac{\gamma_m\gamma_n k^2}{k_z^2}\left|\frac{m\pi}{a}A_{mn} + \frac{n\pi}{b}C_{mn}\right|^2\right\}$$

(D4)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{I\} - k^2\{II\} = \sum_{m=1}^{\infty}2\left[d + \frac{\sin 2d\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2}}{2\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2}}\right]\left(2k_{z1}^2\left|C_{m0}\right|^2 + 2\left|\frac{m\pi}{a}C_{m0}\right|^2\right)$$

(D5)

88

$$
- 2k^2 \left| C_{m0} \right|^2 \Biggr) + \sum_{n=1}^{\infty} 2 \left[d + \frac{\sin 2d \sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2}}{2 \sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2}} \right] \left(2k_{z2}^2 \left| A_{0n} \right|^2 \right.
$$

$$
+ 2 \left| \frac{n\pi}{b} A_{0n} \right|^2 - 2k^2 \left| A_{0n} \right|^2 \Biggr) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2 \left[d + \frac{\sin 2d \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{2 \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \right]
$$

$$
\times \left\{ \frac{1}{k_z^2} \left| \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right] C_{mn} \right|^2 + \frac{1}{k_z^2} \left| \left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] A_{mn} \right.\right.
$$

$$
+ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \Biggr|^2 + \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 - k^2 \left| A_{mn} \right|^2 - k^2 \left| C_{mn} \right|^2
$$

$$
- \frac{k^2}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 \right\} \tag{D5}
$$

where

$$
\left. \begin{aligned}
k_{z1}^2 &= k^2 - \left(\frac{m\pi}{a}\right)^2 \\[2ex]
k_{z2}^2 &= k^2 - \left(\frac{n\pi}{b}\right)^2 \\[2ex]
k_z^2 &= k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2
\end{aligned} \right\} \tag{D6}
$$

89

Substituting equations (D6) into the right-hand side of equation (D5) yields

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\mathrm{I}\right\}-k^2\left\{\mathrm{II}\right\}=0+0+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}2\left[d+\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]$$

$$\times\left(\left|A_{mn}\right|^2\left\{\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left[k_z^2+\left(\frac{m\pi}{a}\right)^2\right]^{-2}+k_z^2\left(\frac{n\pi}{b}\right)^2-k^2k_z^2\right.\right.$$

$$\left.-\left(\frac{m\pi}{a}\right)^2k^2\right\}+2\mathrm{Re}\left(A_{mn}C^*_{mn}\right)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\left[\left(\frac{n\pi}{b}\right)^2+k_z^2+k_z^2\right.$$

$$\left.+\left(\frac{m\pi}{a}\right)^2-k_z^2-k^2\right]+\left|C_{mn}\right|^2\left\{\left[\left(\frac{n\pi}{b}\right)^2+k_z^2\right]^2+\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\right.$$

$$\left.\left.+k_z^2\left(\frac{m\pi}{a}\right)^2-k^2k_z^2-k^2\left(\frac{n\pi}{b}\right)^2\right\}\right) \tag{D7}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\mathrm{I}\right\}-k^2\left\{\mathrm{II}\right\}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2\left[d+\frac{\sin 2d\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{2\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}\right]\left\{\left|A_{mn}\right|^2\left[\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\right.\right.$$

$$\tag{D8}$$

$$+ k^4 - 2k^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 + k^2\left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 - \left(\frac{n\pi}{b}\right)^4 - k^4$$

$$+ k^2\left(\frac{m\pi}{a}\right)^2 + k^2\left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2 k^2 + \left|C_{mn}\right|^2\left[k^4 - 2k^2\left(\frac{m\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^4\right.$$

$$+ \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + k^2\left(\frac{m\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^4 - \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 - k^4 + k^2\left(\frac{m\pi}{a}\right)^2$$

$$+ k^2\left(\frac{n\pi}{b}\right)^2 - k^2\left(\frac{n\pi}{b}\right)^2\right] \tag{D8}$$

Hence,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{I\right\} - k^2\left\{II\right\} = 0 \tag{D9}$$

Now let

$$\left\{III\right\} = \left(\left\{\left|\frac{\gamma_m\epsilon_n}{k_z^2}\right|\frac{m\pi}{a}\frac{n\pi}{b}A_{mn} + \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2\right]C_{mn}\right|^2 + \left|\frac{\gamma_n\epsilon_m}{k_z^2}\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_{mn}\right|^2\right\}\right.$$

$$\times e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2-k^2}}\left[-2d - \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}\right] + \epsilon_m\epsilon_n\left|\frac{m\pi}{a}C_{mn}\right.$$

$$- \frac{n\pi}{b} \left| A_{mn} \right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[2d - \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] - 4\epsilon_m \epsilon_n \left| \frac{m\pi}{a} \right| C_{mn}$$

$$- \frac{n\pi}{b} \left| A_{mn} \right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \qquad\qquad (D10)$$

and

$$\left\{ IV \right\} = \left\{ \left(-\gamma_n \epsilon_m \left| A_{mn} \right|^2 - \gamma_m \epsilon_n \left| C_{mn} \right|^2 \right) e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right.$$

$$\times \left[-2d + \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] - \frac{\gamma_m \gamma_n k^2}{k_z^2} \left| \frac{m\pi}{a} \right| A_{mn}$$

$$+ \frac{n\pi}{b} \left| C_{mn} \right|^2 e^{-2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \left[2d - \frac{\sinh 2d\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right]$$

$$(D11)$$

$$- 4k^2 \left(\gamma_n \epsilon_m \left| A_{mn} \right|^2 + \gamma_m \epsilon_n \left| C_{mn} \right|^2 \right) e^{-2d \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \Biggr\} d \tag{D11}$$

Then, from equation (61),

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ III \right\} - k^2 \left\{ IV \right\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[-2d - \frac{\sinh 2d \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] \Biggl\{ \left| \frac{\gamma_m \epsilon_n}{k_z^2} \left| \frac{m\pi}{a} \frac{n\pi}{b} A_{mn} \right. \right.$$

$$+ \left[\left(\frac{n\pi}{b}\right)^2 + k_z^2 \right] C_{mn} \Biggr|^2 + \left| \frac{\gamma_n \epsilon_n}{k_z^2} \left[\left[k_z^2 + \left(\frac{m\pi}{a}\right)^2 \right] A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) C_{mn} \right] \right|^2$$

$$+ \epsilon_m \epsilon_n \left| \frac{m\pi}{a} C_{mn} - \frac{n\pi}{b} A_{mn} \right|^2 - k^2 \left(\gamma_n \epsilon_n \left| A_{mn} \right|^2 + \gamma_n \epsilon_n \left| C_{mn} \right|^2 \right.$$

$$\left. + \frac{\gamma_m \gamma_n}{k_z^2} \left| \frac{m\pi}{a} A_{mn} + \frac{n\pi}{b} C_{mn} \right|^2 \right) \Biggr\} e^{-2d \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \tag{D12}$$

Since the term $\left\{ \left| \frac{\gamma_m \epsilon_n}{k_z^2} \left| \frac{m\pi}{a} \frac{n\pi}{b} \ldots + \frac{n\pi}{b} C_{mn} \right|^2 \right. \right\}$ in equation (D12) is identical to the

same term in equation (D4), it follows that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ III \right\} - k^2 \left\{ IV \right\} = 0$$

APPENDIX E

EVALUATION OF THE INTEGRALS SHOWN IN EQUATIONS (70)

Consider the integral given in equation (70a) as

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - |y'|\right) \sin \frac{n\pi y}{b} \, dx' \, dy'$$

$$= V_o \int_{-\ell/2}^{\ell/2} \left(\sin \frac{k\ell}{2} \cos k|y'| - \cos \frac{k\ell}{2} \sin k|y'|\right)\left[\sin \frac{n\pi}{b}\left(y' + \frac{b}{2}\right)\right] dy'$$

$$= V_o \int_{-\ell/2}^{\ell/2} \left(\sin \frac{k\ell}{2} \cos k|y'| - \cos \frac{k\ell}{2} \sin k|y'|\right)\left(\sin \frac{n\pi y'}{b} \cos \frac{n\pi}{2} + \cos \frac{n\pi y'}{b} \sin \frac{n\pi}{b}\right) dy'$$

$$= V_o \int_{-\ell/2}^{0} \left(\sin \frac{k\ell}{2} \cos ky' + \cos \frac{k\ell}{2} \sin ky'\right)\left(\sin \frac{n\pi y'}{b} \cos \frac{n\pi}{2} + \cos \frac{n\pi y'}{b} \sin \frac{n\pi}{2}\right) dy'$$

$$+ \int_{0}^{\ell/2} \left(\sin \frac{k\ell}{2} \cos ky' - \cos \frac{k\ell}{2} \sin ky'\right)\left(\sin \frac{n\pi y'}{b} \cos \frac{n\pi}{2} + \cos \frac{n\pi y'}{b} \sin \frac{n\pi}{2}\right) dy' \qquad (E1)$$

Now let $y' = -y'$ in the first integral so that

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - |y'|\right) \sin \frac{n\pi y}{b} \, dx' \, dy'$$

$$= V_o \int_{0}^{\ell/2} \left[\left(\sin \frac{k\ell}{2} \cos ky' - \cos \frac{k\ell}{2} \sin ky'\right)\left(-\sin \frac{n\pi y'}{b} \cos \frac{n\pi}{2} + \cos \frac{n\pi y'}{b} \sin \frac{n\pi}{2}\right)\right.$$

$$+ \left(\sin \frac{k\ell}{2} \cos ky' - \cos \frac{k\ell}{2} \sin ky'\right)\left(\sin \frac{n\pi y'}{b} \cos \frac{n\pi}{2} + \cos \frac{n\pi y'}{b} \sin \frac{n\pi}{2}\right)\right] dy'$$

$$= V_o \int_{0}^{\ell/2} \left(-\sin \frac{k\ell}{2} \cos \frac{n\pi}{2} \cos ky' \sin \frac{n\pi y'}{b} + \sin \frac{k\ell}{2} \sin \frac{n\pi}{2} \cos ky' \cos \frac{n\pi y'}{b}\right.$$

$$+ \cos \frac{k\ell}{2} \cos \frac{n\pi}{2} \sin ky' \sin \frac{n\pi y'}{b} - \cos \frac{k\ell}{2} \sin \frac{n\pi}{2} \sin ky' \cos \frac{n\pi y'}{b} + \sin \frac{k\ell}{2} \cos \frac{n\pi}{2} \qquad (E2)$$

(Equation continued on next page)

$$\times \cos ky' \, \sin \frac{n\pi y'}{b} + \sin \frac{k\ell}{2} \sin \frac{n\pi}{2} \cos ky' \cos \frac{n\pi y'}{b} - \cos \frac{k\ell}{2} \cos \frac{n\pi}{2} \sin ky' \sin \frac{n\pi y'}{b}$$

$$\left. - \cos \frac{k\ell}{2} \sin \frac{n\pi}{2} \sin ky' \, \cos \frac{n\pi y'}{b} \right) dy'$$

$$= V_o 2 \sin \frac{k\ell}{2} \sin \frac{n\pi}{2} \int_0^{\ell/2} \cos ky' \, \cos \frac{n\pi y'}{b} \, dy' \qquad (E2)$$

Since

$$\int_0^{\ell/2} \cos ky' \, \cos \frac{n\pi y'}{b} \, dy' = \frac{\sin\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2}}{2\left(k - \frac{n\pi}{b}\right)} + \frac{\sin\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2}}{2\left(k + \frac{n\pi}{b}\right)}$$

$$= \frac{1}{2}\left[\frac{\left(k + \frac{n\pi}{b}\right)\sin\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2} + \left(k - \frac{n\pi}{b}\right)\sin\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2}}{k^2 - \left(\frac{n\pi}{b}\right)^2}\right]$$

$$= -2 \cos \frac{k\ell}{2} \sin \frac{n\pi}{2} \int_0^{\ell/2} \sin ky' \, \cos \frac{n\pi y'}{b} \, dy' \qquad (E3)$$

and

$$\int_0^{\ell/2} \sin ky' \, \cos \frac{n\pi y'}{b} \, dy' = -\frac{1}{2}\left[\frac{\cos\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2}}{k - \frac{n\pi}{b}} + \frac{\cos\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2}}{k + \frac{n\pi}{b}} - \frac{1}{k - \frac{n\pi}{b}} - \frac{1}{k - \frac{n\pi}{b}}\right]$$

$$= \frac{\left(k + \frac{n\pi}{b}\right)\cos\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2} + \left(k - \frac{n\pi}{b}\right)\cos\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2}}{k^2 - \left(\frac{n\pi}{b}\right)^2} \quad \frac{-2k}{k^2 - \left(\frac{n\pi}{b}\right)^2} \qquad (E4)$$

Making use of equations (E3) and (E4) combined with equation (E2) results in

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \sin \frac{n\pi y}{b} \, dx' \, dy'$$

$$= \frac{V_o \sin \frac{n\pi}{2}}{k^2 - \left(\frac{n\pi}{b}\right)^2} \left(\sin \frac{k\ell}{2} \left\{ k\left[\sin\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2} + \sin\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2} \right] + \frac{n\pi}{b}\left[\sin\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2} - \sin\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2} \right] \right\} \right.$$

$$\left. + \cos \frac{k\ell}{2} \left\{ k\left[\cos\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2} + \cos\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2} \right] + \frac{n\pi}{b}\left[\cos\left(k - \frac{n\pi}{b}\right)\frac{\ell}{2} - \cos\left(k + \frac{n\pi}{b}\right)\frac{\ell}{2} \right] - 2k \right\} \right) \quad \text{(E5)}$$

Using trigonometric identities,

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \sin \frac{n\pi y}{b} \, dx' \, dy'$$

$$= \frac{V_o \sin \frac{n\pi}{2}}{k^2 - \left(\frac{n\pi}{b}\right)^2} 2\left[k\left(\sin^2 \frac{k\ell}{2} \cos \frac{n\pi}{b} \frac{\ell}{2} + \cos^2 \frac{k\ell}{2} \cos \frac{n\pi}{b} \frac{\ell}{2} \right) - \frac{n\pi}{b}\left(\sin \frac{k\ell}{2} \cos \frac{k\ell}{2} \sin \frac{n\pi}{b} \frac{\ell}{2} \right. \right.$$

$$\left. \left. - \sin \frac{k\ell}{2} \cos \frac{k\ell}{2} \sin \frac{n\pi}{b} \frac{\ell}{2} \right) - k \cos \frac{k\ell}{2} \right]$$

$$= \frac{V_o \sin \frac{n\pi}{2}}{k^2 - \left(\frac{n\pi}{b}\right)^2} 2\left(k \cos \frac{n\pi}{b} \frac{\ell}{2} - k \cos \frac{k\ell}{2} \right) \quad \text{(E6)}$$

or for n even,

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \sin \frac{n\pi y}{b} \, dx' \, dy' = 0 \quad \text{(E7a)}$$

or for n odd,

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \sin \frac{n\pi y}{b} \, dx' \, dy' = (-1)^{\frac{n-1}{2}} 2V_o k \, \frac{\cos \frac{n\pi}{b}\frac{\ell}{2} - \cos \frac{k\ell}{2}}{k^2 - \left(\frac{n\pi}{b}\right)^2}$$

(E7b)

Now consider the integral given in equation (70c) with $y = \frac{b}{2} + y'$ and $x = \frac{a}{2} + x'$

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx' \, dy'$$

$$= \int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_o}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \cos \frac{m\pi}{a}\left(\frac{a}{2} + x'\right) \sin \frac{n\pi}{b}\left(\frac{b}{2} + y'\right) \, dx' \, dy'$$

$$= \frac{V_o}{a'} \int_{y'=-\ell/2}^{\ell/2} \left(\sin \frac{k\ell}{2} \cos k\left|y'\right| - \cos \frac{k\ell}{2} \sin k\left|y'\right|\right)\left(\sin \frac{n\pi}{2} \cos \frac{n\pi y'}{b} + \cos \frac{n\pi}{2} \sin \frac{n\pi y'}{b}\right)$$

$$\times \int_{x'=-a'/2}^{a'/2} \left(\cos \frac{m\pi}{2} \cos \frac{m\pi x'}{a} - \sin \frac{m\pi}{2} \sin \frac{m\pi x'}{a}\right) dx' \, dy'$$

(E8)

Since

$$\int_{x'=-a'/2}^{a'/2} dx' = \left.\frac{\cos \frac{m\pi}{2} \sin \frac{m\pi x'}{a}}{\frac{m\pi}{a}}\right|_{-a'/2}^{a'/2} + \left.\frac{\sin \frac{m\pi}{2} \cos \frac{m\pi x}{a}}{\frac{m\pi}{a}}\right|_{-a'/2}^{a'/2}$$

$$\int_{x'=-a'/2}^{a'/2} dx' = \frac{2 \cos \frac{m\pi}{2} \sin \frac{m\pi a'}{2a}}{\frac{m\pi}{a}}$$

(E9)

Hence, equation (E8) can be rewritten as

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_0}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx' \, dy'$$

$$= \frac{2 \cos \dfrac{m\pi}{2} \sin \dfrac{m\pi a'}{2a}}{\dfrac{m\pi}{a}} \frac{V_0}{a'} \, 2k \sin \frac{n\pi}{2} \frac{\cos \dfrac{n\pi}{b}\dfrac{\ell}{2} - \cos \dfrac{k\ell}{2}}{k^2 - \left(\dfrac{n\pi}{b}\right)^2} \tag{E10}$$

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_0}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx' \, dy' = 0 \qquad (n \text{ even or } m \text{ odd})$$

$$\tag{E11a}$$

$$\int_{y'=-\ell/2}^{\ell/2} \int_{x'=-a'/2}^{a'/2} \frac{V_0}{a'} \sin k\left(\frac{\ell}{2} - \left|y'\right|\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx' \, dy'$$

$$= (-1)^{\frac{n-1}{2}} (-1)^{\frac{m}{2}} 4V_0 k \frac{\sin \dfrac{m\pi a'}{2a}}{a' \dfrac{m\pi}{a}} \frac{\cos \dfrac{n\pi}{b}\dfrac{\ell}{2} - \cos \dfrac{k\ell}{2}}{k^2 - \left(\dfrac{n\pi}{b}\right)^2} \qquad (n \text{ odd or } m \text{ even})$$

$$\tag{E11b}$$

where the y' integration has been evaluated as in equation (E8).

APPROXIMATION FOR $a' \ll a$ IN EQUATIONS (81), (82), AND (83)

Rewrite the summation terms given in the expression for the difference in stored energies beginning with equation (81) (coth ≈ 1) as

$$\sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{3/2}} + 2 \sum_{m=2,4}^{\infty} \sum_{n=1,3}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \frac{1}{\left(\frac{n\pi}{b}\right)^2 - k^2}$$

$$= \sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{b}\right)^2 - k^2} \left[\frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} + 2 \sum_{m=2,4}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}\right]$$

$$+ \frac{2\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)}{\left(\frac{\pi}{b}\right)^2 - k^2} \sum_{m=2}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 - k^2}} \tag{F1}$$

Write the function $\left(\dfrac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2$ in its integral form (the integral from whence it came)

as

$$\left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 = \frac{1}{a'} \int_{-a'/2}^{a'/2} \cos \frac{m\pi}{a}\left(\frac{a}{2} + x\right) dx \frac{1}{a'} \int_{-a'/2}^{a'/2} \cos \frac{m\pi}{a}\left(\frac{a}{2} + x'\right) dx'$$

$$= \frac{1}{(a')^2} \int_{-a'/2}^{a'/2} \int_{-a'/2}^{a'/2} \cos \frac{m\pi}{a}\left(\frac{a}{2} + x\right) \cos \frac{m\pi}{a}\left(\frac{a}{2} + x'\right) dx\, dx' \tag{F2}$$

Let $x = \dfrac{a'x}{2}$ and $x' = \dfrac{a'x'}{2}$, so that

$$\left(\frac{\sin\frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\cos\frac{m\pi}{2a}(a + a'x)\cos\frac{m\pi}{2a}(a + a'x')\,dx\,dx' \qquad (F3)$$

Expand the cosine product

$$\cos\frac{m\pi}{2a}(a + a'x)\cos\frac{m\pi}{2a}(a + a'x') = \left(\cos\frac{m\pi}{2}\cos\frac{m\pi a'x}{2a} - \sin\frac{m\pi}{2}\sin\frac{m\pi a'x}{2a}\right)\left(\cos\frac{m\pi}{2}\right.$$

$$\left.\times\cos\frac{m\pi a'x'}{2a} - \sin\frac{m\pi}{2}\sin\frac{m\pi a'x'}{2a}\right)$$

$$= \cos^2\frac{m\pi}{2}\cos\frac{m\pi a'x}{2a}\cos\frac{m\pi a'x'}{2a} - \sin\frac{m\pi}{2}\cos\frac{m\pi}{2}$$

$$\times\left(\cos\frac{m\pi a'x}{2a}\sin\frac{m\pi a'x'}{2a} + \cos\frac{m\pi x'}{2a}\sin\frac{m\pi a'x}{2a}\right)$$

$$+ \sin^2\frac{m\pi}{2}\sin\frac{m\pi a'x}{2a}\sin\frac{m\pi a'x'}{2a} \qquad (F4)$$

Substituting equation (F4) into equation (F3) gives

$$\left(\frac{\sin\frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\cos^2\frac{m\pi}{2}\cos\frac{m\pi a'x}{2a}\cos\frac{m\pi a'x}{2a}\,dx\,dy'$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\cos\frac{m\pi a'x}{2a}\cos\frac{m\pi a'x}{2a}\,dx\,dx'$$

$$= \int_{0}^{1}\int_{0}^{1}\cos\frac{m\pi a'x}{2a}\cos\frac{m\pi a'x'}{2a}\,dx\,dx' \qquad (m\ \text{even}) \qquad (F5)$$

Consider the summation in which   n = 1   (last term in equation (F1))

$$\frac{2\left(\cos\frac{\pi\ell}{2} - \cos\frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \sum_{m=2,4}^{\infty} \left(\frac{\sin\frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 - k^2}}$$

$$= \frac{2\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \int_0^1 \int_0^1 \left[\sum_{m=2,4}^{\infty} \frac{\cos\frac{m\pi a'x}{2a}\cos\frac{m\pi a'x'}{2a}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 - k^2}}\right] dx\, dx' \qquad (F6)$$

The summation on the right-hand side of equation (F6) is approximated by replacing

$$\frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 - k^2}} \quad \text{with} \quad \frac{1}{\frac{m\pi}{a}} \quad \text{since} \quad \left(\frac{m\pi}{a}\right)^2 > \left|\left(\frac{\pi}{b}\right)^2 - k^2\right|; \text{ therefore,}$$

$$\frac{2\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \sum_{m=2,4}^{\infty} \left(\frac{\sin\frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2 - k^2}}$$

$$\approx \frac{2\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \int_0^1 \int_0^1 \left(\sum_{m=2,4}^{\infty} \frac{\cos\frac{m\pi a'x}{2a}\cos\frac{m\pi a'x'}{2a}}{\frac{m\pi}{a}}\right) dx\, dx' \qquad (F7)$$

The summation on the right-hand side of equation (F7) can be summed in closed form by rewriting as

$$\sum_{m=2,4}^{\infty} \frac{\cos\frac{m\pi a'x}{2a}\cos\frac{m\pi a'x'}{2a}}{\frac{m\pi}{a}} = \frac{a}{\pi}\sum_{m=2,4}^{\infty} \frac{1}{2}\frac{\cos\frac{m\pi a'}{2a}(x + x') + \cos\frac{m\pi a'}{2a}(x - x')}{m}$$

$$= \frac{a}{2\pi}\, \text{Re} \sum_{m=2,4}^{\infty} \frac{e^{j\frac{m\pi a'}{2a}(x+x')} + e^{j\frac{m\pi b'}{2}(x-x')}}{m} \qquad (F8)$$

101

Consider the following sums (see ref. 19):

$$\sum_{m=1}^{\infty} \frac{e^{jmx}}{m} = -\ln\left(1 - e^{jx}\right) \tag{F9a}$$

and

$$\sum_{m=1}^{\infty} \frac{e^{jm(x+\pi)}}{m} = -\ln\left(1 + e^{jx}\right) \tag{F9b}$$

where the last and first are written, respectively, as

$$\sum_{m=1}^{\infty} \frac{e^{jm(x+\pi)}}{m} = -\sum_{m=1,3}^{\infty} \frac{e^{jmx}}{m} + \sum_{m=2,4}^{\infty} \frac{e^{jmx}}{m} = -\ln\left(1 + e^{jx}\right) \tag{F10a}$$

and

$$\sum_{m=1}^{\infty} \frac{e^{jmx}}{m} = \sum_{m=1,3}^{\infty} \frac{e^{jmx}}{m} + \sum_{m=2,4}^{\infty} \frac{e^{jmx}}{m} = -\ln\left(1 - e^{jx}\right) \tag{F10b}$$

Now, add the two series of equations (F10)

$$2\sum_{m=2,4}^{\infty} \frac{e^{jmx}}{m} = -\ln\left(1 + e^{jx}\right)\left(1 - e^{jx}\right) = -\ln\left[2e^{j\frac{x}{2}}\cos\frac{x}{2}\, 2(-j)e^{j\frac{x}{2}}\sin\frac{x}{2}\right]$$

$$= -\left[\ln\left(\cos\frac{x}{2}\sin\frac{x}{2}\right) + 2\ln 2 + j\left(x - \frac{\pi}{2}\right)\right] = -\left[\ln\left(\frac{1}{2}\sin x\right) + 2\ln 2 + j\left(x - \frac{\pi}{2}\right)\right]$$

$$\tag{F11}$$

$$\sum_{m=2,4}^{\infty} \frac{e^{jmx}}{m} = -\frac{1}{2}\left[\ln\sin x + \ln 2 + j\left(x - \frac{\pi}{2}\right)\right] \tag{F12}$$

$$\text{Re}\sum_{m=2,4}^{\infty} \frac{e^{jmx}}{m} = -\frac{1}{2}\left(\ln\sin x + \ln 2\right) \tag{F13}$$

Therefore, replacing x by $\frac{\pi a'}{2a}(x + x')$ and $\frac{\pi a'}{2a}(x - x')$, equation (F8) becomes

$$
\sum_{m=2,4}^{\infty} \frac{\cos \frac{m\pi a'x}{2a} \cos \frac{m\pi a'x'}{2a}}{\frac{m\pi}{a}} = \frac{a}{2\pi}\left\{-\frac{1}{2}\left[\ln \sin \frac{\pi a'}{2a}(x + x') + \ln 2\right]\right.
$$

$$
-\frac{1}{2}\left[\ln \sin \frac{\pi a'}{2a}(x - x') + \ln 2\right]\right\}
$$

$$
= -\frac{a}{4\pi}\left\{2 \ln 2 + \ln\left[\sin \frac{\pi a'}{2a}(x + x')\right] + \ln\left[\sin \frac{\pi a'}{2a}(x - x')\right]\right.
$$

$$
= -\frac{a}{4\pi}\left[2 \ln 2 + \ln \frac{1}{2}\left(\cos \frac{\pi a'x}{a} - \cos \frac{\pi a'x'}{a}\right)\right]\right\} \qquad \text{(F14)}
$$

For small values of y $\left(\cos y \approx 1 - \frac{y^2}{2}\right)$,

$$
\sum_{m=2,4}^{\infty} \frac{\cos \frac{m\pi a'x}{2a} \cos \frac{m\pi a'x'}{2a}}{\frac{m\pi}{a}} \underset{a'\to 0}{\approx} -\frac{a}{4\pi}\left\{2 \ln 2 + \ln \frac{1}{2}\left[\frac{1}{2}\left(\frac{\pi a'}{a}\right)^2 |x'^2 - x^2|\right]\right\}
$$

$$
\underset{a'\to 0}{\approx} \frac{a}{4\pi}\left[2 \ln\left(\frac{\pi a'}{a}\right) + \ln|x'^2 - x^2|\right] \qquad \text{(F15)}
$$

Therefore,

$$
2 \frac{\left(\cos \frac{\pi \ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \sum_{m=2,4}^{\infty}\left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 - k^2}}
$$

<div align="right">(F16)</div>

$$\underset{a' \to 0}{\approx} 2 \frac{\left(\cos \frac{\pi \ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \int_0^1 \int_0^1 \left\{ -\frac{a}{4\pi} \left[2 \ln \left(\frac{\pi a'}{a}\right) + \ln \left| x'^2 - x^2 \right| \right] \right\} dx \; dx'$$

$$\underset{a' \to 0}{\approx} -\frac{a}{2\pi} \frac{\left(\cos \frac{\pi \ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \left[2 \ln \left(\frac{\pi a'}{a}\right) + 2 \ln 2 - 3 \right] \tag{F16}$$

Next, consider the case when $n > 1$; the summation on m is

$$2 \sum_{m=2,4}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}} \right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$= 2 \int_0^1 \int_0^1 \left[\sum_{m=2,4}^{\infty} \frac{\cos \frac{m\pi a'x}{2a} \cos \frac{m\pi x'}{2a}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k}} \right] dx \; dx'$$

$$= \int_0^1 \int_0^1 \left[\sum_{m=2,4}^{\infty} \frac{\cos \frac{m\pi a'}{2a} (x + x') + \cos \frac{m\pi a'}{2a} (x - x')}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \right] dx \; dx' \tag{F17}$$

The approximation made for the case when $n = 1$ is not valid here since $\left(\frac{m\pi}{a}\right)^2$ is not always greater than $\left| \left(\frac{n\pi}{b}\right)^2 - k^2 \right|$; hence, other approximations must be sought. In order

to convert this summation to a more rapidly converging series, use is made of the Fourier transform (ref. 19):

$$\int_{-\infty}^{\infty} \frac{e^{j\omega(y+y')}}{\sqrt{\omega^2 + \alpha^2}} \, d\omega = 2K_0\left[\alpha(y + y')\right]$$

(F18)

where $K_0(\alpha y)$ is the modified Bessel function of the second kind, and of the definition of the Poisson summation formula

$$\sum_{-\infty}^{\infty} S\left(\frac{m\pi}{a}\right) = \frac{a}{\pi} \sum_{m=-\infty}^{\infty} f(2ma)$$

(F19)

where $f(y)$ is the Fourier transform of $S(\omega)$. (See ref. 19.) Convert the series to be summed by means of Euler's identity as

$$\sum_{m=2,4}^{\infty} \frac{\cos\frac{m\pi a'}{2a}(x+x) + \cos\frac{m\pi a'}{2a}(x-x')}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} = \frac{1}{2} \sum_{m=\pm2,\pm4}^{\infty} \frac{e^{j\frac{m\pi a'}{2a}(x+x')} + e^{j\frac{m\pi a'}{2a}(x-x')}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$= \frac{1}{2} \sum_{m=0,\pm2}^{\infty} \frac{e^{j\frac{m\pi a'}{2a}(x+x')} + e^{j\frac{m\pi a'}{2a}(x-x)}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$- \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}$$

(F20)

Then from the Poisson summation formula

$$\sum_{-\infty}^{\infty} \frac{e^{j\frac{m\pi y}{a}}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \alpha^2}} = \frac{2a}{\pi} \sum_{-\infty}^{\infty} K_0(\alpha y + 2ma)$$

(F21)

By replacing   a   with   a/2, even terms are obtained thusly

$$\sum_{m=0,\pm1}^{\infty} \frac{e^{j\frac{2m\pi y}{a}}}{\sqrt{\left(\frac{2m\pi}{a}\right)^2 + \alpha^2}} = \sum_{n=0,\pm2}^{\infty} \frac{e^{j\frac{n\pi y}{b}}}{\sqrt{\left(\frac{n\pi}{b}\right)^2 + \alpha^2}} = \frac{a}{\pi} \sum_{-\infty}^{\infty} K_0(\alpha y + ma) \tag{F22}$$

Applying this result to the case at hand, equation (F20) becomes

$$\sum_{m=2,4}^{\infty} \frac{\cos\frac{m\pi a'}{2a}(x + x') + \cos\frac{m\pi a'}{2a}(x - x')}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} = \frac{a}{2\pi} \sum_{-\infty}^{\infty} \left\{ K_0\left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\,\frac{a'}{2}(x + x') + ma\right] \right.$$

$$+ K_0\left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\,\frac{a'}{2}(x - x') + ma\right] \Bigg\}$$

$$- \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \tag{F23}$$

The modified Bessel function of the second kind decays rapidly as the argument increases; therefore,   m = 0   should be sufficient so that

$$\sum_{m=2,4}^{\infty} \frac{\cos\frac{m\pi a'}{2a}(x + x') + \cos\frac{m\pi a'}{2a}(x - x')}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} = \frac{a}{2\pi} \left\{ K_0\left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\,\frac{a'}{2}(x + x')\right] \right.$$

$$+ K_0\left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\,\frac{a'}{2}(x - x')\right] \Bigg\} - \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$\tag{F24}$$

Hence, equation (F17) becomes

$$2 \sum_{m=2,4}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}} \right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} = \int_0^1 \int_0^1 \left(\frac{a}{2\pi} \left\{ K_0 \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \, \frac{a'}{2}(x + x') \right] \right. \right.$$

$$\left. + K_0 \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \, \frac{a'}{2}(x - x') \right] \right\}$$

$$\left. - \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \right) dx \, dx' \qquad (F25)$$

For small argument, $\quad K_0(x) \approx -\left(\gamma + \ln \frac{x}{2} \right),$

$$2 \sum_{m=2,4}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}} \right)^2 \frac{1}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}} \approx \int_0^1 \int_0^1 \left(\frac{a}{2\pi} \left\{ -2\gamma + 2 \ln 2 \right. \right.$$

$$- \ln \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \, \frac{a'}{2}(x + x') \right]$$

$$\left. - \ln \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \, \frac{a'}{2}(x - x') \right] \right\} - \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \right) dx \, dx'$$

$$\approx \int_0^1 \int_0^1 \left(\frac{a}{2\pi} \left\{ -2\gamma - 2 \ln \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \, a' \right] \right. \right.$$

$$(F26)$$

$$\left. - \ln \left| x^2 - x'^2 \right| \right\} - \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \right) dx\ dx$$

$$\approx \frac{a}{2\pi} \left\{ -2\gamma - 2\ \ln \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\ \frac{a'}{2} \right] + 3 \right\} - \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}}$$

$$(F26)$$

where γ is Euler's constant = 0.5772157. Therefore, the summation terms in the expressions for the difference in stored energies become

$$\sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{3/2}} + 2 \sum_{m=2,4}^{\infty} \sum_{n=1,3}^{\infty} \left(\frac{\sin \frac{m\pi a'}{2a}}{\frac{m\pi a'}{2a}}\right)^2 \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}\ \left(\frac{n\pi}{b}\right)^2 - k^2}$$

$$\approx \sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{b}\right)^2 - k^2} \left(\frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} + \frac{a}{2\pi} \left\{ 3 - 2\gamma - 2\ \ln \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\ \frac{a'}{2} \right] \right\} \right.$$

$$\left. - \frac{1}{\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}} \right) - \frac{a}{2\pi} \frac{\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{\pi}{b}\right)^2 - k^2} \left[2\ \ln \left(\frac{\pi a'}{a}\right) + 2\ \ln 2 - 3 \right]$$

$$(F27)$$

$$\approx \frac{a}{2\pi} \sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{b}\right)^2 - k^2} \left\{ 3 - 2\gamma - 2 \ln \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \, \frac{a'}{2} \right] \right\}$$

$$+ \frac{a}{2\pi} \frac{\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{k^2 - \left(\frac{\pi}{b}\right)^2} \left[2 \ln \left(\frac{\pi a'}{a}\right) + 2 \ln 2 - 3 \right] \tag{F27}$$

Hence, equation (81) is now written as

$$2\omega \left(<<W_m>> - <<W_e>> \right) \approx \frac{4 \left|V_0\right|^2 k}{ab Z_0} \left(\frac{\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2 \right]^{3/2}} \cot \sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} \, d \right.$$

$$+ \frac{a}{\pi} \frac{\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{k^2 - \left(\frac{\pi}{b}\right)^2} \left[\ln \left(\frac{\pi a'}{a}\right) + \ln 2 - \frac{3}{2} \right]$$

$$+ \frac{a}{\pi} \sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{b}\right)^2 - k^2} \left\{ \frac{3}{2} - \gamma - \ln \left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \, \frac{a'}{2} \right] \right\} \right)$$

$$- \frac{8 \left|V_0\right|^2}{(2\pi)^2 Z_0} \frac{\pi}{4} \left\{ Si(k\ell) + \left[Si(k\ell) - \frac{1}{2} Si(2k\ell) \right] \cos k\ell \right.$$

$$\left. + \left[Cin(k\ell) - \frac{1}{2} Cin(2k\ell) - \ln \frac{e^{3/2} \ell}{2a'} \right] \sin k\ell \right\} \tag{F28}$$

Using the same approximation, the representation for $2\omega\left(<<W_e>>\right)$ (eq. (82)) is

$$2\omega\left(<<W_e>>\right) \approx \frac{4\left|V_o\right|^2 k^3}{abZ_o}\left(\frac{\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{5/2}}\left[-\cot\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\,d\right] - \frac{a}{\pi}\frac{\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^2}\right.$$

$$\times\left[\ln\left(\frac{\pi a'}{a}\right) + \ln 2 - \frac{3}{2}\right] + \frac{a}{\pi}\sum_{n=3,5}^{\infty}\frac{\left(\cos\frac{n\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^2}$$

$$\times\left\{\frac{3}{2} - \gamma - \ln\left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\,d\,\frac{a'}{2}\right]\right\}\right) + \frac{8\left|V_o\right|^2}{(2\pi)^2 Z_o}\frac{\pi}{8}\left(Si(k\ell) - \frac{k\ell}{2}Cin(k\ell)\right.$$

$$+ (k\ell - \sin k\ell)\ln\frac{e\ell}{a'} + \left\{Si(k\ell) - \frac{1}{2}Si(2k\ell) - \frac{k\ell}{2}\left[Cin(k\ell) - Cin(2k\ell)\right]\right.$$

$$\left. - k\ell\ln 2\right\}\cos k\ell + \left\{Cin(k\ell) - \frac{1}{2}Cin(2k\ell) + \frac{k\ell}{2}\left[Si(k\ell) - Si(2k\ell)\right]\right.$$

$$\left. + \ln\frac{2}{e}\right\}\sin k\ell\right) \tag{F29}$$

Equation (83) is therefore written as

$$\omega\frac{d}{d\omega}\left[2\omega\left(<<W_m>> - <<W_e>>\right)\right] \approx \frac{4\left|V_o\right|^2}{abZ_o}\left[-\frac{\left(\cos\frac{\pi\ell}{2b} - \cos\frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^2} - k^2 d\cos^2\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\,d\right.$$

$$\tag{F30}$$

$$+ \cot \sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} \; d \left\{ k\ell \sin \frac{k\ell}{2} \frac{\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{3/2}} \right.$$

$$\left. - 3k^2 \frac{\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{3/2}} + \frac{\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^{3/2}} \right\}$$

$$+ \frac{a}{\pi} \left[\ln\left(\frac{\pi a'}{a}\right) - \frac{3}{2} + \ln 2 \right] \left\{ k\ell \sin \frac{k\ell}{2} \frac{\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}}{k^2 - \left(\frac{\pi}{b}\right)^2} \right.$$

$$\left. - 2k^2 \frac{\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[k^2 - \left(\frac{\pi}{b}\right)^2\right]^2} + \frac{\left(\cos \frac{\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{k^2 - \left(\frac{\pi}{b}\right)^2} \right\}$$

$$+ \frac{a}{\pi} \sum_{n=3,5}^{\infty} \left(\frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^2} + \left\{ \frac{3}{2} - \gamma - \ln\left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} \; \frac{a'}{2} \right] \right\} \right.$$

$$\times \left\{ k\ell \sin \frac{k\ell}{2} \frac{\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}}{\left(\frac{n\pi}{b}\right)^2 - k^2} + 2k^2 \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^2} \right.$$

(F30)

(Equation continued on next page)

$$+ \frac{\left(\cos \frac{n\pi\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{b}\right)^2 - k^2}\Biggr\}\Biggr)\Biggr] - \frac{8|V_0|^2}{(2\pi)^2 Z_0} \frac{\pi}{4} k\ell \Biggl\{-\Bigl[\mathrm{Si}(k\ell)$$

$$- \frac{1}{2}\mathrm{Si}(2k\ell)\Bigr] \sin k\ell + \Bigl[\mathrm{Cin}(k\ell) - \frac{1}{2}\mathrm{Cin}(2k\ell) - \ln \frac{e^{3/4}\ell}{2a'}\Bigr]$$

$$\times \cos k\ell + \frac{\sin k\ell}{k\ell}\Biggr\} \tag{F30}$$

Rewrite equations (F28), (F29), and (F30) as

$$2\omega\Bigl(<<W_m>> - <<W_e>>\Bigr) \approx \frac{4|V_0|^2}{(ka)(kb)Z_0}\Biggl(\frac{\left(\cos \frac{\pi k\ell}{2b} - \cos \frac{k\ell}{2}\right)^2}{1 - \left(\frac{\pi}{kb}\right)^2} \cot \sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\, kd$$

$$+ \frac{ka}{\pi} \frac{\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{1 - \left(\frac{\pi}{kb}\right)^2}\Bigl[\ln \left(\frac{\pi ka'}{ka}\right) + \ln 2 - \frac{3}{2}\Bigr] +$$

$$+ \frac{ka}{\pi} \sum_{n=3,5}^{\infty} \frac{\left(\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{kb}\right)^2 - 1}\Biggl\{\frac{3}{2} - \gamma - \ln \Bigl[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1}\, \frac{ka'}{2}\Bigr]\Biggr\}\Biggr)$$

$$- \frac{8|V_0|^2}{(2\pi)^2 Z_0} \frac{\pi}{4}\Biggl\{\mathrm{Si}(k\ell) + \Bigl[\mathrm{Si}(k\ell) - \frac{1}{2}\mathrm{Si}(2k\ell)\Bigr] \cos k\ell$$

$$+ \Bigl[\mathrm{Cin}(k\ell) - \frac{1}{2}\mathrm{Cin}(2k\ell) - \ln \frac{e^{3/2}k\ell}{2ka'}\Bigr] \sin k\ell\Biggr\} \tag{F31}$$

$$
2\omega\left(<<W_e>>\right) \approx \frac{4\left|V_o\right|^2}{(ka)(kb)}\left(\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{5/2}}\left[-\cot\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\;kd\right] - \frac{ka}{\pi}\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2}\right.
$$

$$
\times\left[\ln\left(\frac{\pi ka'}{ka}\right) + \ln 2 - \frac{3}{2}\right] + \frac{ka}{\pi}\sum_{n=3,5}^{\infty}\frac{\left(\cos\frac{n\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{kb}\right)^2 - 1\right]^2}
$$

$$
\times\left\{\frac{3}{2} - \gamma - \ln\left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1}\;\frac{ka'}{2}\right]\right\}\right) + \frac{8\left|V_o\right|^2}{(2\pi)^2 Z_o}\frac{\pi}{8}\left(Si(k\ell) - \frac{k\ell}{2}Cin(k\ell)\right.
$$

$$
+ (k\ell - \sin k\ell)\ln\frac{ek\ell}{ka'} + \left\{Si(k\ell) - \frac{1}{2}Si(2k\ell) - \frac{k\ell}{2}\left[Cin(k\ell) - Cin(2k\ell)\right]\right.
$$

$$
\left. - k\ell\ln 2\right\}\cos k\ell + \left\{Cin(k\ell) - \frac{1}{2}Cin(2k\ell) + \frac{k\ell}{2}\left[Si(k\ell) - Si(2k\ell)\right]\right.
$$

$$
\left.\left. + \ln\frac{2}{e}\right\}\sin k\ell\right) \tag{F32}
$$

$$
\omega\frac{\partial}{\partial\omega}\left[2\omega\left(<<W_m>> - <<W_e>>\right)\right] \approx \frac{4\left|V_o\right|^2}{(ka)(kb)}\frac{1}{Z_o}\left[-\frac{\left(\cos\frac{\pi k\ell}{2kb} - \cos\frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2}kd\cos^2\sqrt{1 - \left(\frac{\pi}{kb}\right)^2}\;kd\right.
$$

<div align="right">(F33)</div>

$$+ \cot \sqrt{1 - \left(\frac{\pi}{kb}\right)^2} \; kd \left\{ k\ell \sin \frac{k\ell}{2} \; \frac{\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{3/2}} \right.$$

$$- \frac{2\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{5/2}} + \left. \frac{\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^{3/2}} \right\}$$

$$+ \frac{ka}{\pi} \left[\ln\left(\frac{\pi ka'}{ka}\right) - \frac{3}{2} + \ln 2\right] \left\{ k\ell \sin \frac{k\ell}{2} \; \frac{\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}}{1 - \left(\frac{\pi}{kb}\right)^2} \right.$$

$$- \frac{2\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[1 - \left(\frac{\pi}{kb}\right)^2\right]^2} + \left. \frac{\left(\cos \frac{\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{1 - \left(\frac{\pi}{kb}\right)^2} \right\}$$

$$+ \frac{ka}{\pi} \sum_{n=3,5}^{\infty} \left(\frac{\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}}{\left(\frac{n\pi}{kb}\right)^2 - 1} + \left\{ \frac{3}{2} - \gamma - \ln \left[\sqrt{\left(\frac{n\pi}{kb}\right)^2 - 1} \; \frac{ka'}{2}\right] \right\} \right)$$

<div style="text-align:right">(F33)</div>

$$\times \left\{ k\ell \sin \frac{k\ell}{2} \frac{\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}}{\left(\frac{n\pi}{kb}\right)^2} + 2 \frac{\left(\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left[\left(\frac{n\pi}{kb}\right)^2 - 1\right]^2} \right.$$

$$+ \frac{\left(\cos \frac{n\pi k\ell}{2kb} - \cos \frac{k\ell}{2}\right)^2}{\left(\frac{n\pi}{kb}\right)^2 - 1} \Bigg\}\Bigg] - \frac{8|V_o|^2}{(2\pi)^2 Z_o} \frac{\pi}{4} k\ell \left\{ -\left[Si(k\ell) \right.\right.$$

$$- \frac{1}{2} Si(2k\ell)\right] \sin k\ell + \left[Cin(k\ell) - \frac{1}{2} Cin(2k\ell) - \ln\left(\frac{e^{3/2}k\ell}{2ka'}\right)\right]$$

$$\times \cos k\ell + \frac{\sin k\ell}{k\ell} \Bigg\} \qquad\qquad (F33)$$

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