NSG - 8011

SUPERRADIANT EFFECTS ON PULSE PROPAGATION IN RESONANT MEDIA

FINAL REPORT

Covering thr Period September 1974 - September 1975

Prepared for

The National Aeronautics and Space Administration

Washington, D. C.

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I. INTRODUCTION

It has been recently recognized that submillimeter waves are most desirable for remote sensing in the atmosphere. The interest in the generation of coherent submillimeter waves has made the study of superradiance under realistic conditions an urgent need [1]. The central problem in this study is the interaction of multimode radiation with extended resonant media. Using the extremely powerful, but not very well known, Dicke-Schwendimann representation [2,3], we should be able to put the coupled superradiance master equations, recently derived by Picard and Willis [4], in an explicit form. With this as a starting point, we can very easily study the effects of superradiance on pulse propagation in resonant media.

In the period of thirteen months. We have had very significant achievements toward this goal. However, the whole project is too big to be finished in one year. We will describe the major results of our studies. The description will be brief since the details of these results has been published or to be be published in open literature.

II. MATRIX ELEMENTS OF COLLECTIVE OPERATORS

One of the chief difficulties encountered in the Dicke-Schwendimann approach is the lack of explicit expressions for the matrix elements of collective operators. We have developed a simple and systematic diagrammatic technique to clear up this difficulty. The detailed description of this technique has been published [5].

III. TRANSITION FROM INCOHERENT TO COHERENT RADIATION

In the collective emission of radiation, the cooperation number r of the atomic system, as defined by Dicke [2], can remain the same; out it can also change. The latter is often ignored. We have found that the emission through r-conserving process is coherent and that through r-nonconserving process is incoherent. The transition

from incoherent to coherent radiation can be best described by considering the number of photons emitted, instead of the time, as the independent variable. The former is a discrete variable while the latter is a continuous variable. Hence, we have a difference equation instead of a differential equation. The result of the numerical solution of this difference equation has been described in detail in a paper to be published [6].

IV. RATE EQUATION FOR SUPERRADIANCE

The difference equation described in the previous section is not suitable for the study of the intensity of superradiant pulse. For this purpose, we have obtained a rate equation as follows:

$$\frac{dP(r,m)}{dt} = \frac{(r+m+2)(r+m+1)}{2r+2}\gamma P(r+1,m+1) + (r+m+1)(r-m)\gamma P(r,m+1)$$

$$- \frac{(r+m)(r+m-1)}{2r}\gamma P(r,m) - (r+m)(r-m+1)\gamma P(r,m),$$

where r is the cooperation number, m is one-half of the difference of the numbers of atoms in the excited state and the ground state, P(r,m) is the probability of the atomic system in the Dicke state $|r,m\rangle$, and γ is the radiation rate of a single atom.

It is convenient to make the following change of variables:

$$k = r+m, \qquad \ell = r-m;$$

and to define the following generating function:

$$f(x,y,t) = \sum_{k=1}^{N} \sum_{\ell=1}^{N} P(k,\ell)x^k y^\ell.$$

Then we can replace the rate equation for P(r,m), a function of t, by a partial differential equation for f(x,y,t) as follows:

$$\frac{\partial f}{\partial t} = \frac{1}{N}(1-x^2)\frac{\partial^2 f}{\partial x^2} + \gamma(y-x)\frac{\partial}{\partial x}\left[y\frac{\partial f}{\partial x} + f\right].$$

The solution to this last equation is yet to be found.

IV. PUBLICATIONS

Two papers have been prepaired based on the results of this research project:

1. "Diagrammatic Technique for Calculating Matrix Elements of Collective Operators in Superradiance"
   Physical Review A, Vol. 12, No. 2, pp. 575-586, August 1975.

2. "Difference Equation for Superradiance"
   To be published.

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Diagrammatic technique for calculating matrix elements of collective operators in superradiance*

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Adopting the so-called "genealogical construction," one can express the eigenstates of collective operators corresponding to a specified mode for an N-atom system in terms of those for an $(N-1)$-atom system. Using these Dicke states as bases and using the Wigner-Eckart theorem, a matrix element of a collective operator of an arbitrary mode can be written as the product of an m-dependent factor and an m-independent reduced matrix element (RME). A set of recursion formulas for the RME is obtained. A graphical representation of the RME on the branching diagram for binary irreducible representations of permutation groups is then introduced. This gives a simple and systematic way of calculating the RME. This method is especially useful when the cooperation number r is close to $N/2$, where almost exact asymptotic expressions can be obtained easily. The result shows explicitly the geometry dependence of superradiance and the relative importance of r-conserving and r-nonconserving processes. This clears up the chief difficulty encountered in the Dicke-Schwendimann approach to the problem of N two-level atoms, spread over large regions, interacting with a multimode radiation field.

I. INTRODUCTION

In 1954 Dicke[1] discussed the spontaneous emission of radiation from an excited system of N identical two-level atoms. By considering the entire collection of atoms as a single quantum-mechanical system, he found that under certain conditions the individual atoms cooperate to emit radiation at a rate proportional to N^2, which is much greater than their incoherent emission rate. This phenomenon is called superradiance.

In the vast literature on superradiance,[2] most of the investigations have been limited to cases in which the emitting atoms either are confined to a region smaller than the wavelength of emitted light or are able to couple with only one radiation mode. This situation permits important mathematical simplifications. However, current attention[3,4] in this field is being focused on the more difficult, but more realistic, problem of N two-level atoms, distributed over a space of dimension much greater than the radiation wavelength, interacting with all the modes of radiation field. Among the works devoted to this problem, the one by Schwendimann[4] elaborates a very elegant approach of the problem initiated by Dicke.[1] One of the chief difficulties encountered in this approach is the lack of explicit expressions for the matrix elements of collective operators. In this paper, we will develop a simple and systematic diagrammatic technique to clear up this difficulty.

II. MODEL HAMILTONIAN

We consider a system of N identical two-level atoms distributed over a space of dimension much larger than the wavelengths of the multimode radiation with which it interacts. In the usual electric dipole and rotating-wave approximations, the system can be described by the following Dicke Hamiltonian:

$$H = \hbar \sum_{\vec{k}} \omega_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} + \hbar\omega_0 R_3 + \sum_{\vec{k}} g_{\vec{k}} (a_{\vec{k}}^{\dagger} R_{\vec{k}}^{-} + a_{\vec{k}} R_{\vec{k}}^{+}), \tag{1}$$

where $\vec{k}$ is the wave vector, $a_{\vec{k}}^{\dagger}$ and $a_{\vec{k}}$ are the corresponding photon creation and annihilation operators and obey the usual boson commutation rules, and $\omega_{\vec{k}}$ is the corresponding frequency. ω_0 is the transition frequency between the two levels of each atom. $R_{\vec{k}}^{+}$, $R_{\vec{k}}^{-}$, and R_3 are collective atomic operators defined in terms of the single-atom flip operators R_i^{+}, R_i^{-}, and R_{3i}, which obey angular-momentum commutation rules as follows:

$$R_{\vec{k}}^{\pm} = \sum_{i=1}^{N} e^{\pm i\vec{k}\cdot\vec{X}_i} R_i^{\pm}, \quad R_3 = \sum_{i=1}^{N} R_{3i}, \tag{2}$$

where $\vec{X}_i$ is the position of the ith atom, and $g_{\vec{k}} = (\hbar\omega_{\vec{k}}/2\epsilon_0 V^*)^{1/2} p_{12}$ are the coupling constants, where p_{12} is the transition dipole moment, and V^* is the volume of the cavity in which the radiation field is quantized.

As one sees from Eq. (2), the $R_{\vec{k}}^{\pm}$ operators depend on the wave vector $\vec{k}$ and on the positions $\vec{X}_i$'s of the atoms. Their commutation relations are

$$[R_{\vec{k}}^{+}, R_{\vec{k}'}^{-}] = \sum_{i=1}^{N} e^{i(\vec{k}-\vec{k}')\cdot\vec{X}_i} R_{3i}. \tag{3}$$

Here we see that the commutator of two operators corresponding to different modes given an operator corresponding to a third mode. Therefore, in contrast to the case of samples smaller

than a wavelength in which the $e^{i\vec{k}\cdot\vec{X}_i}$ factors disappear, the behavior of the atoms cannot be described by a simple set of angular-momentum operators. However, it is possible to factor out the $\vec{k}$ dependence in the operators and confine it to c-number functions. This is the key point of the Dicke-Schwendimann approach, to be discussed in the following.

A. Dicke-Schwendimann approach

Consider the set of operators $\{R^+_{\vec{k}_1}, R^-_{\vec{k}_1}, R_3\}$ as defined by Eqs. (2) with a particular wave vector $\vec{k}_1$ which we will call the principal mode. It can be easily shown that they obey the angular-momentum commutation relations. So we can chose $|+\rangle_i$ and $e^{-i\vec{k}_1\cdot\vec{X}_i}|-\rangle_i$ to be the excited and ground states of the ith atom. Then we can construct eigenstates $|r, m; \alpha\rangle$ of $R^2_{\vec{k}_1}$ and R_3 such that

$$R^2_{\vec{k}_1}|r, m; \alpha\rangle = r(r+1)|r, m; \alpha\rangle,$$
$$R_3|r, m; \alpha\rangle = m|r, m; \alpha\rangle, \tag{4}$$

where $R^2_{\vec{k}_1} = \frac{1}{2}(R^+_{\vec{k}_1}R^-_{\vec{k}_1} + R^-_{\vec{k}_1}R^+_{\vec{k}_1}) + R_3^2$ and $\alpha = 1, 2, \ldots f^N_r$ is a degeneracy index with

$$f^N_r = \frac{N!(2r+1)}{(N/2+r+1)!\,(N/2-r)!}. \tag{5}$$

The eigenstates $|r, m; \alpha\rangle$ will be called Dicke states, r will be called the cooperation number, and m indicates the population inversion or total energy of the atomic system. The trouble is that the eigenstates of $R^2_{\vec{k}_1}$ corresponding to the principal mode are not eigenstates of $R^2_{\vec{k}}$ corresponding to other modes. So we might think that we need to construct a set of Dicke states for each possible mode, which would be an imposing task. Fortunately, this is unnecessary.

If we define[5]

$$R_{3\vec{k}} = \sum_{i=1}^{N} e^{i(\vec{k}-\vec{k}_1)\cdot\vec{X}_i} R_{3i} \tag{6}$$

for an arbitrary mode $\vec{k}$ and consider the set of operators $\{R^+_{\vec{k}}, R^-_{\vec{k}}, R_{3\vec{k}}\}$, they do not obey angular-momentum commutation rules. However, as first pointed out by Dicke,[1] their commutation relations with the established angular-momentum operators $R^\pm_{\vec{k}_1}$ and R_3 are of the following form:

$$[T^1_i, (R^\tau_{\vec{k}_1})_j] = i\epsilon_{ijl}T^1_l, \quad i, j, l = +, -, 3, \tag{7}$$

where we should identify $R^+_{\vec{k}}$, $R^-_{\vec{k}}$, and $R_{3\vec{k}}$ with T^1_+, T^1_-, and T^1_3, respectively. This proves[6] that $\{R^+_{\vec{k}}, R^-_{\vec{k}}, R_{3\vec{k}}\}$ is an irreducible tensor operator set of rank 1, or simply a vector operator. As a consequence, we can discuss them in terms of the eigenstates $|r, m; \alpha\rangle$ of $R^\tau_{\vec{k}_1}$ operators. Furthermore, the Wigner-Eckart theorem can be used to establish the selection rules for the matrix elements $\langle r', m'; \alpha'|R^\pm_{\vec{k}}|r, m; \alpha\rangle$; namely, $r - r' = \pm 1$ or 0 and $m - m' = \pm 1$; and the nonvanishing matrix elements can be factorized as follows:

$$\langle r, m \pm 1; \alpha'|R^\pm_{\vec{k}}|r, m; \alpha\rangle = [(r \mp m)(r \pm m + 1)]^{1/2}(r; \alpha'|R^\pm_{\vec{k}}|r, \alpha),$$
$$\langle r+1, m \pm 1; \alpha'|R^\pm_{\vec{k}}|r, m; \alpha\rangle = \pm[(r \pm m + 1)(r \pm m + 2)]^{1/2}(r+1; \alpha'|R^\pm_{\vec{k}}|r, \alpha),$$
$$\langle r-1, m \pm 1; \alpha'|R^\pm_{\vec{k}}|r, m; \alpha\rangle = \mp[(r \mp m)(r \mp m - 1)]^{1/2}(r-1; \alpha'|R^\pm_{\vec{k}}|r, \alpha), \tag{8}$$

where the reduced matrix elements (RME) are independent of m and are of the following forms:

$$(r; \alpha'|R^\pm_{\vec{k}}|r; \alpha) = \sum_{i=1}^{N} C^{\pm\alpha\alpha'}_{irr}\, e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_i} \equiv \Phi^{\pm\alpha\alpha'}_{rr}(\vec{k}-\vec{k}_1),$$
$$(r \pm 1; \alpha'|R^\pm_{\vec{k}}|r; \alpha) = \sum_{i=1}^{N} C^{\pm\alpha\alpha'}_{irr\pm1}\, e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_i} \equiv \Phi^{\pm\alpha\alpha'}_{rr\pm1}(\vec{k}-\vec{k}_1). \tag{9}$$

B. Transformed Hamiltonian

We now expand $R^\pm_{\vec{k}}$ in terms of Dicke states

$$R^\pm_{\vec{k}} = \sum_r \sum_{m'} \sum_{\alpha'} \sum_r \sum_m \sum_\alpha \langle r', m'; \alpha'|R^\pm_{\vec{k}}|r, m; \alpha\rangle |r', m'; \alpha'\rangle\langle r, m; \alpha|$$
$$= \sum_r \sum_\alpha \sum_{\alpha'} [\Phi^{\pm\alpha\alpha'}_{rr}(\vec{k}-\vec{k}_1)(R^\pm)^{\alpha\alpha'}_{rr} + \Phi^{\pm\alpha\alpha'}_{rr+1}(\vec{k}-\vec{k}_1)(P^\pm)^{\alpha\alpha'}_{rr+1} + \Phi^{\pm\alpha\alpha'}_{rr-1}(\vec{k}-\vec{k}_1)(P^\pm)^{\alpha\alpha'}_{rr-1}], \tag{10}$$

where we have used Eqs. (8) and (9) and have defined

$$(R^{\pm})^{\alpha\alpha'}_{rr} = \sum_{m}[(r \mp m)(r \pm m + 1)]^{1/2}|r, m \pm 1; \alpha'\rangle\langle r, m; \alpha|,$$

$$(P^{\pm})^{\alpha\alpha'}_{rr+} = \pm\sum_{m}[(r \pm m + 1)(r \pm m + 2)]^{1/2}|r+1, m \pm 1; \alpha'\rangle\langle r, m; \alpha|, \tag{11}$$

$$(P^{\pm})^{\alpha\alpha'}_{rr-1} = \mp\sum_{m}[(r \mp m)(r \mp m - 1)]^{1/2}|r-1, m \pm 1; \alpha'\rangle\langle r, m; \alpha|.$$

Using these results, the Hamiltonian in Eq. (1) can be rewritten as

$$H = \hbar\sum_{\vec{k}}\omega_{\vec{k}}a^{\dagger}_{\vec{k}}a_{\vec{k}} + \hbar\omega_0 R_3 + H_{\mathrm{RC}} + H_{\mathrm{RNC}},$$

$$H_{\mathrm{RC}} = \sum_{\vec{k}} g_{\vec{k}}\left(a^{\dagger}_{\vec{k}}\sum_{r}\sum_{\alpha}\sum_{\alpha'}\Phi^{-\alpha\alpha'}_{rr}(\vec{k}-\vec{k}_1)(R^{-})^{\alpha\alpha'}_{rr} + \mathrm{H.c.}\right), \tag{12}$$

$$H_{\mathrm{RNC}} = \sum_{\vec{k}} g_{\vec{k}}\left[\left(a^{\dagger}_{\vec{k}}\sum_{r}\sum_{\alpha}\sum_{\alpha}\Phi^{-\alpha\alpha'}_{rr+1}(\vec{k}-\vec{k}_1)(P^{-})^{\alpha\alpha'}_{rr+1} + \mathrm{H.c.}\right) + \left(a^{\dagger}_{\vec{k}}\sum_{r}\sum_{\alpha}\sum_{\alpha'}\Phi^{-\alpha\alpha'}_{rr-1}(\vec{k}-\vec{k}_1)(P^{-})^{\alpha\alpha'}_{rr-1} + \mathrm{H.c.}\right)\right],$$

where H_{RC} describes processes in which the cooperation number r is conserved, and H_{RNC} describes processes in which r is not conserved.

III. EVALUATION OF REDUCED MATRIX ELEMENTS

In the Dicke-Schwendimann approach that we have elaborated in Sec. II, we notice the important role played by the reduced matrix elements which we have written as $\Phi^{\pm\alpha\alpha'}_{rr'}(\vec{k}-\vec{k}_1)$, a c-number function of the wave vectors. All the geometry and mode dependence of our problem are determined by these functions. They are independent of time, and hence, describe the static effects of the system. However, these functions appear as coefficients of dynamical operators in the expansion of the Hamiltonian. Therefore, the usefulness of this approach depends on whether we can obtain an explicit expression for each of them.

A. "Genealogical construction"

From the group-theoretical point of view, the set of the so-called symmetrized states[7] $|r, m; \alpha\rangle$, $\alpha = 1, 2, \ldots, f^{N}_{r}$ forms a basis for an irreducible representation of the permutation group S_N corresponding to a certain binary partition of N. There is a one-to-one correspondence between a binary irreducible representation and the cooperation number r. Because the matrix representation of a group is fixed only within a similarity transformation, the explicit expressions for RME depend on how the Dicke states are constructed. So we must chose a definite way to construct them. A very natural and convenient scheme is the so-called "genealogical construction."[8] We will use it here.

In this construction, the eigenstates of N atoms with eigenvalues (r, m) are constructed from those of $N-1$ atoms with eigenvalues $(r-\frac{1}{2}, m-\frac{1}{2})$ and $(r-\frac{1}{2}, m+\frac{1}{2})$, or $(r+\frac{1}{2}, m-\frac{1}{2})$ and $(r+\frac{1}{2}, m+\frac{1}{2})$, by adding the "spin" functions for the Nth atom as follows:

$$|N, r, m; \alpha\rangle = [(r+m)/2r]^{1/2}|N-1, r-\tfrac{1}{2}, m-\tfrac{1}{2}; \alpha\rangle|+\rangle_N + [(r-m)/2r]^{1/2}|N-1, r-\tfrac{1}{2}, m+\tfrac{1}{2}; \alpha\rangle|-\rangle_N e^{-i\vec{k}_1\cdot\vec{X}_N},$$

$$\text{for } \alpha = 1, 2, \ldots, f^{N-1}_{r-1/2};$$

$$|N, r, m; \alpha\rangle = [(r-m+1)/(2r+2)]^{1/2}|N-1, r+\tfrac{1}{2}, m-\tfrac{1}{2}; \beta\rangle|+\rangle_N - [(r+m+1)/(2r+2)]^{1/2}|N-1, r+\tfrac{1}{2}, m+\tfrac{1}{2}; \beta\rangle|-\rangle_N e^{-i\vec{k}_1\cdot\vec{X}_N},$$

$$\text{for } \alpha = f^{N-1}_{r-1/2} + \beta \text{ and } \beta = 1, 2, \ldots, f^{N-1}_{r+1/2}; \tag{13}$$

and we have the relation

$$f^{N}_{r} = f^{N-1}_{r-1/2} + f^{N-1}_{r+1/2},$$

which can be easily verified with the use of Eq. (5).

Starting from the eigenstates for one atom, we can construct the eigenstates for N atoms by repeated application of Eqs. (13). The Dicke states obtained in this way form an orthonormal set automatically. In this construction, an eigenstate is specified, in addition to r and m, by a series of "partial-cooperation numbers" $r_1, r_2, \ldots, r_{N-1}$, which will replace the single index α. So we will adopt a new notation

$$|N, r, m; \alpha\rangle \equiv |r, m; r_1 r_2 \cdots r_{N-1}\rangle \tag{14}$$

for the eigenstates. r_1 is always $\frac{1}{2}$; r_2 can be either 0 or 1; if r_2 is 0, r_3 can only be $\frac{1}{2}$, but if r_2 is 1, r_3 can be either $\frac{1}{2}$ or $\frac{3}{2}$; and so on. Anyway, we must have

$$r_i \geq 0 \quad \text{and} \quad r_{i+1} - r_i = \pm \tfrac{1}{2}. \tag{15}$$

B. Recursion formula for reduced matrix elements

We can write a collective operator for the N-atom system as that for the first $N-1$ atoms plus that for the Nth atom,

$$R_k^\pm(N) = R_k^\pm(N-1) + e^{\pm i(\vec{k}\cdot\vec{X}_N)} R_N^\pm. \tag{16}$$

Using the Dicke states constructed by the repeated application of Eqs. (13), we can reduce the matrix element of a collective operator of N atoms to that of $N-1$ atoms very easily. Let us work out the following example as an illustration:

$$\begin{aligned}\langle r-1, m\pm 1; r_1' \cdots r_{N-2}' r-\tfrac{1}{2} | R_k^\pm(N) | r, m; r_1 \cdots r_{N-2} r-\tfrac{1}{2}\rangle \\ = (1/2r)\{[(r+m)(r-m\mp 1)]^{1/2}\langle r-\tfrac{1}{2}, m\pm 1-\tfrac{1}{2}; r_1' \cdots r_{N-2}' | R_k^\pm(N-1) | r-\tfrac{1}{2}, m-\tfrac{1}{2}; r_1 \cdots r_{N-2}\rangle \\ - [(r-m)(r+m\pm 1)]^{1/2}\langle r-\tfrac{1}{2}, m\pm 1+\tfrac{1}{2}; r_1' \cdots r_{N-2}' | R_k^\pm(N-1) | r-\tfrac{1}{2}, m+\tfrac{1}{2}; r_1 \cdots r_{N-2}\rangle \\ \pm [(r\mp m)(r\mp m-1)]^{1/2}\langle r-\tfrac{1}{2}, m\pm\tfrac{1}{2}; r_1' \cdots r_{N-2}' | r-\tfrac{1}{2}, m\pm\tfrac{1}{2}; r_1 \cdots r_{N-2}\rangle e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_N}\}.\end{aligned} \tag{17}$$

We can use the orthonormal property of the Dicke states to express the last term in angular brackets in the above equation as the product of Kronecker δ's:

$$\begin{aligned}\langle r-\tfrac{1}{2}, m\pm\tfrac{1}{2}; r_1' \cdots r_{N-2}' | r-\tfrac{1}{2}, m\pm\tfrac{1}{2}; r_1 \cdots r_{N-2}\rangle \\ = \delta(r_1, r_1')\delta(r_2, r_2') \cdots \delta(r_{N-2}, r_{N-2}') \\ = \delta_{N-2},\end{aligned} \tag{18}$$

where

$$\delta(r_i, r_i') = \begin{cases} 1 & \text{for } r_i = r_i' \\ 0 & \text{for } r_i \neq r_i' \end{cases}. \tag{19}$$

We can also use Eqs. (8) to factorize the matrix elements in Eq. (17) and drop common factors on both sides; then this equation will reduce to

$$\begin{aligned}(r-1; r_1' \cdots r_{N-2}' r-\tfrac{1}{2} | R_k^\pm(N) | r; r_1 \cdots r_{N-2} r-\tfrac{1}{2}) \\ = (1/2r)[(r-\tfrac{1}{2}; r_1' \cdots r_{N-2}' | R_k^\pm(N-1) \\ \times | r-\tfrac{1}{2}; r_1 \cdots r_{N-2}) - e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_N}\delta_{N-2}].\end{aligned} \tag{20}$$

This is a recursion formula for one of the RME. We notice that m does not appear in this formula.

Following similar procedures and keeping in mind that the selection rules $r_i - r_i' = \pm 1$ or 0 apply to any of the (r_i, r_i') pairs, we can obtain a set of all the nonvanishing recursion formulas like Eq. (20) for RME. To save space we will drop the $r_1 \cdots r_{N-2}$ and $r_1' \cdots r_{N-2}'$ parts of the indices.

$$(r; r-\tfrac{1}{2} | R_k^\pm(N) | r; r-\tfrac{1}{2}) = [(2r-1)/2r](r-\tfrac{1}{2} | R_k^\pm(N-1) | r-\tfrac{1}{2}) + (1/2r)\delta_{N-2}e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_N}, \tag{21a}$$

$$(r; r+\tfrac{1}{2} | R_k^\pm(N) | r; r+\tfrac{1}{2}) = [(2r+3)/(2r+2)](r+\tfrac{1}{2} | R_k^\pm(N-1) | r+\tfrac{1}{2}) - [1/(2r+2)]\delta_{N-2}e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_N}, \tag{21b}$$

$$(r; r\pm\tfrac{1}{2} | R_k^\pm(N) | r; r\mp\tfrac{1}{2}) = -[r(r+1)]^{-1/2}(r\pm\tfrac{1}{2} | R_k^\pm(N-1) | r\mp\tfrac{1}{2}), \tag{21c}$$

$$(r+1; r+\tfrac{1}{2} | R_k^\pm(N) | r; r+\tfrac{1}{2}) = [1/(2r+2)](r+\tfrac{1}{2} | R_k^\pm(N-1) | r+\tfrac{1}{2}) - [1/(2r+2)]\delta_{N-2}e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_N}, \tag{21d}$$

$$(r+1; r+\tfrac{1}{2} | R_k^\pm(N) | r; r-\tfrac{1}{2}) = [r/(r+1)]^{1/2}(r+\tfrac{1}{2} | R_k^\pm(N-1) | r-\tfrac{1}{2}), \tag{21e}$$

$$(r+1; r+\tfrac{3}{2} | R_k^\pm(N) | r; r+\tfrac{1}{2}) = [(r+2)/(r+1)]^{1/2}(r+\tfrac{3}{2} | R_k^\pm(N-1) | r+\tfrac{1}{2}), \tag{21f}$$

$$(r-1; r-\tfrac{1}{2} | R_k^\pm(N) | r; r-\tfrac{1}{2}) = [1/2r](r-\tfrac{1}{2} | R_k^\pm(N-1) | r-\tfrac{1}{2}) - [1/2r]\delta_{N-2}e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_N}, \tag{21d'}$$

$$(r-1; r-\tfrac{3}{2} | R_k^\pm(N) | r; r-\tfrac{1}{2}) = [(r-1)/r]^{1/2}(r-\tfrac{3}{2} | R_k^\pm(N-1) | r-\tfrac{1}{2}), \tag{21e'}$$

$$(r-1; r-\tfrac{1}{2} | R_k^\pm(N) | r; r+\tfrac{1}{2}) = [(r+1)/r]^{1/2}(r+\tfrac{1}{2} | R_k^\pm(N-1) | r+\tfrac{1}{2}). \tag{21f'}$$

C. Evaluation of $C_{ir'}^{\pm\alpha r'}$

From Eqs. (8), we can see that the evaluation of the matrix elements of collective operators $R_k^\pm$ is reduced to the evaluation of the RME; which, in turn, is reduced to the evaluation of the coefficients $C_{ir'}^{\pm\alpha r'}$, as can be seen from Eqs. (9). An examination of Eqs. (21) leads to the following conclusions:

(i) The recursion formulas can be generalized

by replacing N and r by i and r_i, respectively, with $i = 1, 2, \ldots, N$, and can be summarized by one general formula

$$\begin{aligned}(r_i'; r_{i-1}'|R_k^{\dagger}(i)|r_i; r_{i-1}) &= f(r_i, r_i'; r_{i-1}, r_{i-1}')(r_{i-1}'|R_k^{\dagger}(i-1)|r_{i-1}) \\ &+ g(r_i, r_i'; r_{i-1}, r_{i-1}')\delta_{i-1}e^{i(\vec{k}-\vec{k}_1)\cdot\vec{x}_i}.\end{aligned} \tag{22}$$

By repeated application of this general formula, we can obtain a general expression for $C_{rr'}^{\dagger\alpha\beta'}$ as

$$C_{rr'}^{\dagger\alpha\beta'} = \delta_{i-1}g(r_i, r_i'; r_{i-1}, r_{i-1}') \times \prod_{n=i+1}^{N} f(r_n, r_n'; r_{n-1}, r_{n-1}'). \tag{23}$$

(ii) The δ_{i-1} in Eq. (23) implies that the coefficient $C_{rr'}^{\dagger\alpha\beta'}$ vanishes unless the α and α' are such that when expressed in the notation of Eq. (14) give $r_j = r_j'$ for all $j = 1, 2, \ldots,$ up to at least $j = i-1$. In other words, if we have

$$\Delta r_j \equiv r_j - r_j' \begin{cases} = 0 & \text{for } j = 1, 2, \ldots, l-1 \\ \neq 0 & \text{for } j = l, \end{cases} \tag{24}$$

then we will have

$$C_{rr'}^{\dagger\alpha\beta'} \begin{cases} \neq 0 & \text{for } i \leq l \text{ and } r' = r \text{ or } r \pm 1 \\ = 0 & \text{otherwise} \end{cases}. \tag{25}$$

This interesting result is, of course, a consequence of the genealogical construction.

(iii) The two functions f and g in Eq. (22) are the same for both operators $R_k^{\dagger}$ and $R_k^{\dagger}$, so we have

$$C_{rr'}^{\dagger\alpha\beta'} = C_{rr'}^{\dagger\alpha\beta'} \equiv C_{rr'}^{\alpha\alpha'}. \tag{26}$$

(iv) If we replace the r in Eqs. (21d')–(21f') by $r+1$ and then take Hermitian conjugate, they will be exactly the same as Eqs. (21d)–(21f). This kind of argument leads to the following relation:

$$C_{rr'}^{\alpha\alpha'} = C_{r'r}^{\alpha'\alpha}. \tag{27}$$

However, using Eq. (23) to evaluate $C_{rr'}^{\alpha\alpha'}$ for large systems may still be rather tedious. In Sec. III D, we will introduce a diagrammatic technique which will help us to visualize the situations much more clearly.

D. Branching-diagram representation

The branching diagram[8] is usually used to determine the dimension of an irreducible representation of a permutation group S_N corresponding to a binary partition of N. When this diagram is used in the discussion of a system of electrons, the abscissa and the ordinate represent the number of electrons and the total spin, respectively. In our case, they will represent the number of atoms and the cooperation number, respectively. We will represent a Dicke state $|r, m; r_1 \cdots r_{N-1}\rangle$

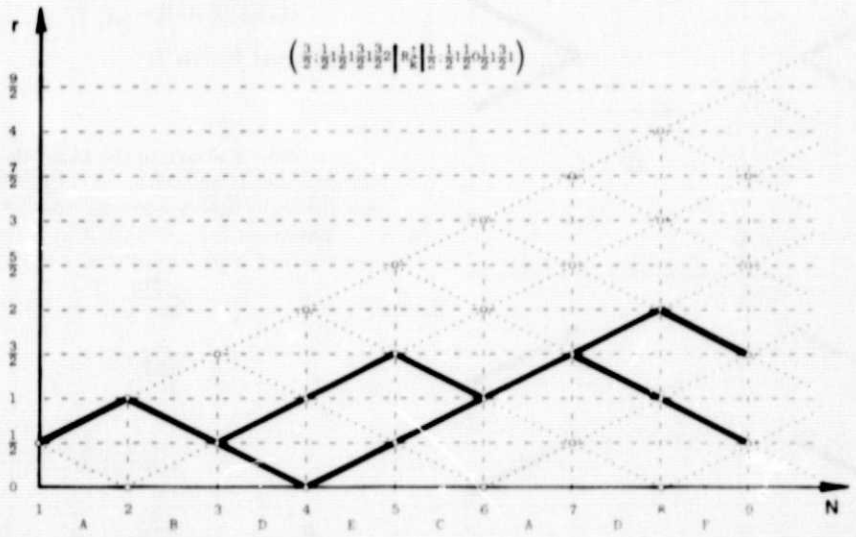


FIG. 1. Graphical representation of an example of reduced matrix element $(\frac{9}{2}; \frac{1}{2}1\frac{1}{2}1\frac{9}{2}1\frac{9}{2}2|R_{\vec{k}}^{\dagger}|\frac{1}{2}; \frac{1}{2}1\frac{1}{2}0\frac{1}{2}1\frac{9}{2}1)$ of a collective operator of a nine-atom system corresponding to an arbitrary mode $\vec{k}$. The background is the branching diagram usually used to determine the dimensions of irreducible representations of permutation groups S_N corresponding to binary partitions of N. The abscissa denotes the number of atoms in the system or a subsystem, and the ordinate represents the cooperation number. The letters at the bottom indicate the types of sections, defined in Fig. 2, of this example diagram which is of type $NC3$.

by a zig-zag line on this branching diagram, and a pair of such lines will represent a reduced matrix element as shown in Fig. 1. In this diagram representation, there is no way to tell the m value of an eigenstate. Fortunately, a reduced matrix element is independent of m; so we need not worry about it. Because of Eqs. (26) and (27), we also need not distinguish whether the operator is R^+_ξ or R^-_ξ and which line represents the state on the right of the operator the ket vector, and which line represents the state on the left, the bra vector.

From Eqs. (21) and (27), we can see that the reduction of a RME of N atoms to that of $N-1$ atoms can occur in six different situations. They are represented by a section of lines, which represents this RME, between two vertical lines corresponding to N and $N-1$. The six different sections are shown in Fig. 2. The factors $f(r_i, r'_i; r_{i-1}, r'_{i-1})$ and $g(r_i, r'_i; r_{i-1}, r'_{i-1})$ can also be considered as functions of the type of a section and the r value of one of the four corners, say the upper-left one, of this section such as $f(A, r)$ and $g(D, r)$. These functions are listed in Table I.

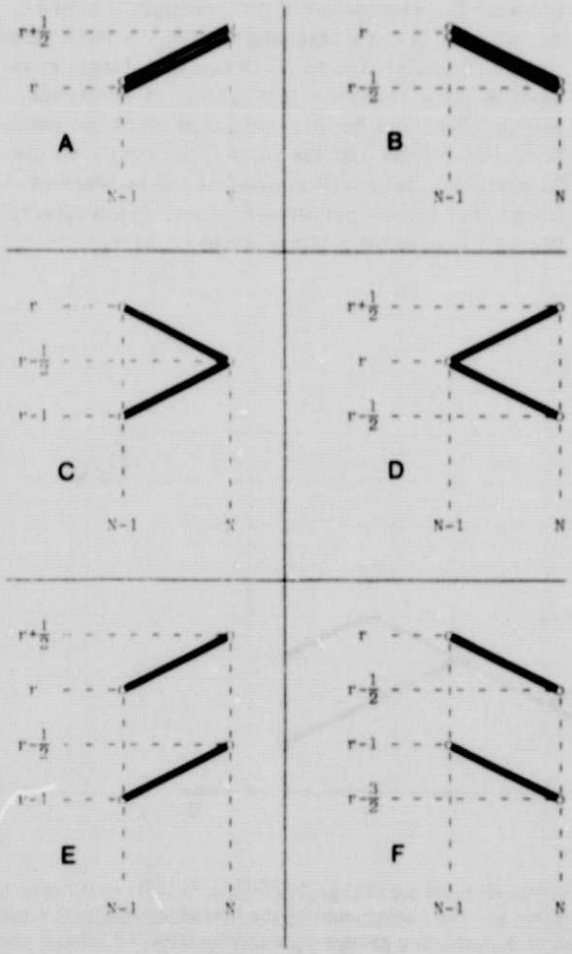


FIG. 2. Six distinct types of sections of the graphical representation of a reduced matrix element of a collective operator.

E. Exampl

With Fig. 2 and Table I as guides, we can now easily use Eq. (23) to evaluate, as an illustration, all the coefficients $C^{\alpha\alpha'}_{lrr'}$ of the RME of nine atoms represented by the diagram in Fig. 1 as follows:

The first nonvanishing value of $(r_i - r'_i)$ occurs at $i = 4$; so we have

$$C_5 = C_6 = C_7 = C_8 = C_9 = 0,$$

where, as well as in the following, we have dropped the other subscripts and superscripts in the notation of the coefficients.

A common factor of all the nonvanishing coefficients is

$$f_{4\to9} = f(E,1)f(C,\tfrac{3}{2})f(A,1)f(D,\tfrac{3}{2})f(F,2)$$
$$= (1/\sqrt{3})[-2/(4\times2)^{1/2}](\tfrac{2}{3})(\tfrac{1}{4})\sqrt{\tfrac{5}{3}} = -\tfrac{1}{16}\sqrt{\tfrac{5}{2}}.$$

Then we have

$$C_4 = g(D,\tfrac{1}{2})f_{4\to9} = (-\tfrac{1}{2})(-\tfrac{1}{16})\sqrt{\tfrac{5}{2}},$$
$$C_3 = g(B,1)f(D,\tfrac{1}{2})f_{4\to9} = (-\tfrac{1}{3})(\tfrac{1}{2})(-\tfrac{1}{16})\sqrt{\tfrac{5}{2}},$$
$$C_2 = g(A,\tfrac{1}{2})f(B,1)f(D,\tfrac{1}{2})f_{4\to9} = (\tfrac{1}{2})(\tfrac{4}{3})(\tfrac{1}{2})(-\tfrac{1}{16})\sqrt{\tfrac{5}{2}},$$
$$C_1 = g(A,0)f(A,\tfrac{1}{2})f(B,1)f(D,\tfrac{1}{2})f_{4\to9}$$
$$= (1)(\tfrac{1}{2})(\tfrac{4}{3})(\tfrac{1}{2})(-\tfrac{1}{16})\sqrt{\tfrac{5}{2}}.$$

So the final result is

TABLE I. Factors in the recursion formulas for different types of sections, as defined in Fig. 2.

Types	$f(x,r)$	$g(x,r)$
A	$\frac{2r}{2r+1}$	$\frac{1}{2r+1}$
B	$\frac{2r+2}{2r+1}$	$-\frac{1}{2r+1}$
C	$-\frac{1}{[(2r+1)(2r-1)]^{1/2}}$	
D	$\frac{1}{2r+1}$	$-\frac{1}{2r+1}$
E	$\left(\frac{2r-1}{2r+1}\right)^{1/2}$	
F	$\left(\frac{2r+1}{2r-1}\right)^{1/2}$	

$$(\tfrac{3}{2};\tfrac{1}{2}\,1\tfrac{1}{2}\,1\tfrac{1}{2}\,1\tfrac{1}{2}\,2|R^{\frac{1}{2}}_{\frac{1}{2}}(9)|\tfrac{1}{2};\tfrac{1}{2}\,1\tfrac{1}{2}\,0\tfrac{1}{2}\,1\tfrac{1}{2}\,1)=-\tfrac{1}{10}\sqrt{\tfrac{3}{2}}\,(\tfrac{1}{3}e^{+i(\vec{r}-\vec{r}_1)\cdot\vec{x}_1}+\tfrac{1}{3}e^{+i(\vec{r}-\vec{r}_1)\cdot\vec{x}_2}-\tfrac{1}{6}e^{+i(\vec{r}-\vec{r}_1)\cdot\vec{x}_3}-\tfrac{1}{6}e^{+i(\vec{r}-\vec{r}_1)\cdot\vec{x}_4}).$$

IV. ASYMPTOTIC RESULTS FOR LARGE N

From the physical point of view, we are more interested in the asymptotic results for large N ($\sim 10^{20}$) than in the exact results for small N. When N is so large, we have $r \gg 1$ for an overwhelming majority of cases; so we can assume r to be of the same order as N without introducing any serious errors.

From Table I, we see that in the asymptotic limit the f factor is 1 for a section of type A, B, E, or F and of the order $1/N$ for a section of type C or D. Therefore, in the diagram representing a RME, whenever a section of type C or D occurs, the value of this RME will be reduced by a factor of the order of $1/N$. For the convenience of statements, we shall refer to a section of type D or C as branching or rejoining section, respectively.

A. Topological classification of reduced matrix elements

We can classify the RME according to the topological types of their diagrams, as shown in Table II. Obviously, type $C0$, representing a diagonal element, is the most important one among the r-conserving processes; and the same is true with type $NC1$ among the r-nonconserving processes. The value of each of an RME of type $C2$ or $NC3$ is smaller than the leading one by a factor of the order of $1/N^2$. Therefore, if the number of possible diagrams of these types corresponding to each of the leading one is much less than N^2, then we will be safe to discard them.

Let us define

$$d \equiv \tfrac{1}{2}N - r. \tag{28}$$

Then for an eigenstate $|r, m; r_1 \cdots r_{N-1}\rangle$, we will have $r_{i+1} - r_i = -\frac{1}{2}$ exactly d times. In other words, the diagram representing this state will drop exactly d times.

For a fixed ket vector $|r, m; r_1 \cdots r_{N-1}\rangle$, there is only one bra vector to form a process of type $C0$; and there are less than $d(N-d-1)$ possible bra vectors to form processes of type $C2$. The latter can be seen as follows: In the branching diagram, one can see easily that one end of the "loop" in a diagram of type $C2$ must occur at one of the d sections with $r_{i+1} - r_i = -\frac{1}{2}$, while the other end can occur at any of the other $N-d-1$ sections with $r_{i+1} - r_i = +\frac{1}{2}$. So there are $d(N-d-1)$ possibilities. However, some of these possibilities may have to be dropped because of the restriction that r_i can never be negative. Hence, the real possibilities must be less than $d(N-d-1)$. Therefore, the condition that we can discard processes of type $C2$ is $d(N-d-1) \ll N^2$; which implies

$$d/N \ll 1. \tag{29}$$

On the other hand, for a fixed ket vector $|r, m; r_1 \cdots r_{N-1}\rangle$, there are d possible bra vectors $\langle r+1, m \pm 1; r'_1 \cdots r'_{N-1}|$ and $N-d-1$ possible bra vectors $\langle r-1, m \pm 1; r'_1 \cdots r'_{N-1}|$ to form processes of type $NC1$. So under condition (29), we can discard processes in which r increases.

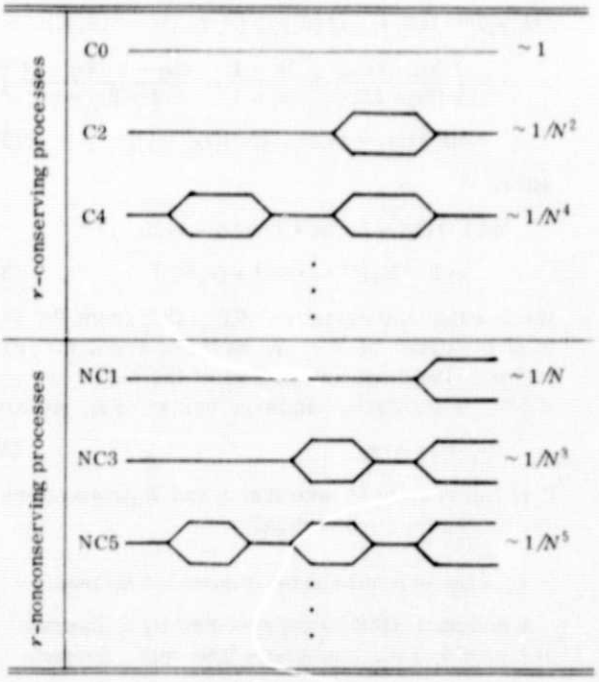
Consider a diagram of type $NC1$ with the branching point occurring at the N_1th section such that $\frac{1}{2}N_1 - r_{N_1} = d_1$. There will be $d_1(N_1 - d_1 - 1)$ possible ways to form the "loop" of a diagram of type $NC3$ corresponding to this fixed diagram of type $NC1$. Therefore, the condition that we can discard processes of type $NC3$ is $d_1(N_1 - d_1 - 1) \ll N_1^2$, which implies

$$d_1/N_1 \ll 1. \tag{29'}$$

For an overwhelming majority of cases, we can say that condition (29) implies condition (29').

Fortunately, from the physical point of view, we are most interested in situations where condition (29) is satisfied.

TABLE II. Classification of reduced matrix elements according to the topological types of their diagrams and the order of magnitude of each type.



B. Propagators

As can be seen from Table II, a diagram of RME consists of single lines and double lines alternatively connected by branching and rejoining sections. We shall refer to a single line as an S "propagator" and a double line as a T "propagator." However, the first single line must be treated differently and shall be called the "generator" because, while others contribute only a common factor to all the nonvanishing coefficients $C^{\alpha\alpha'}_{irr'}$, this single line determines the structure or variation of the coefficients. We shall denote the contribution to the RME by an S propagator, a T propagator, and the generator by S, T, and Q, respectively.

1. S propagators

An S propagator is a consecutive product of a mixture of $f(A,r)$ and $f(B,r)$ factors. It turns out that the result of this product always gives the numerator of the first factor divided by the denominator of the last factor. This can be seen easily by considering the following simple example:

$$f(E,r_i)f(E,r_{i+1})f(E,r_{i+2})\cdots f(E,r_{j-2})f(E,r_{j-1})f(E,r_j)$$
$$=\left(\frac{2r_i-1}{2r_i+1}\times\frac{2r_i}{2r_i+2}\times\frac{2r_i+1}{2r_i+3}\cdots\frac{2r_j-3}{2r_j-1}\times\frac{2r_j-2}{2r_j}\times\frac{2r_j-1}{2r_j+1}\right)^{1/2}$$
$$=[(2r_i-1)(2r_i)/(2r_j)(2r_j+1)]^{1/2}\ . \tag{32}$$

Then consider the case when one of the factors $f(E,r_i)$ is replaced by $f(F,r_i)$. We have

$$f(E,r_i)\cdots f(E,r_{i-1})f(F,r_i)f(E,r_{i+1})\cdots f(E,r_j)$$
$$=\left(\frac{(2r_i-1)2r_i}{(2r_i-1)2r_i}\times\frac{2r_i+1}{2r_i-1}\times\frac{(2r_i-2)(2r_i-1)}{2r_j(2r_j+1)}\right)^{1/2}$$
$$=h(r_i)[(2r_i-1)(2r_i)/(2r_j)(2r_j+1)]^{1/2}, \tag{33}$$

where

$$h(r_i)=[(2r_i-2)(2r_i+1)/(2r_i-1)(2r_i)]^{1/2}$$
$$=1-(2r_i)^{-2}+\cdots=1-O(N^{-2})\ . \tag{34}$$

We see that the deviation of Eq. (33) from Eq. (32) is of the order of N^{-2}. When there are $d'f(F,r)$ factors, the deviation will be of the order of d'/N^2. Therefore, for large values of r_i, we have

$$T(r_i,r_j)=r_i/r_j\ . \tag{35}$$

It is interesting to note that S and T propagators are the same asymptotically.

C. Diagonal reduced matrix elements and the generator

A diagonal RME is represented by a diagram of type $C0$; i.e., one single line only. Suppose

$$f(A,r_i)f(A,r_{i+1})f(B,r_{i+2})f(A,r_{i+3})f(A,r_{i+4})$$
$$=\frac{2r_i}{2r_i+1}\,\frac{2r_i+1}{2r_i+2}\,\frac{2r_i+2}{2r_i+1}\,\frac{2r_i+1}{2r_i+2}\,\frac{2r_i+2}{2r_i+3}$$
$$=\frac{2r_i}{2r_i+3}=\frac{2r_i}{2r_{i+4}+1}\ .$$

Therefore, an S propagator is a function of the types and r values of the beginning and ending sections only. And we have

$$S(A,r_i;A,r_j)=S(A,r_i;B,r_j)=2r_i/(2r_j+1),$$
$$S(B,r_i;A,r_j)=S(B,r_i;B,r_j)=(2r_i+2)/(2r_j+1)\ . \tag{30}$$

For large values of r_i, these two expressions are almost the same and can be summarized as

$$S(r_i,r_j)=r_i/r_j\ . \tag{31}$$

2. T propagators

A T propagator is a consecutive product of a mixture of $f(E,r)$ and $f(F,r)$ factors. Let us first obtain the result of a consecutive product of pure $f(E,r)$ factors. We have

that sections of type A occur $N-d-1$ times and those of type B occur d times at the (i_s-1)th section with $s=1,2,\ldots,d$. Assuming that the last section is of type A, we can calculate the coefficients $C^{\alpha\alpha}_{irr}$ for four different cases according to the types of the $(i-1)$th and the ith sections, namely AA, AB, BA, or BB, as follows:

$$C^{AA}_i=g(A,r_{i-1})S(A,r_i;A,r-\tfrac{1}{2})$$
$$=(1/2r_i)(2r_i/2r)=1/2r\ , \tag{36a}$$
$$C^{AB}_{i_s-1}=g(A,r_{i_s-2})S(B,r_{i_s-1};A,r-\tfrac{1}{2})$$
$$=[1/(2r_{i_s}+1)][(2r_{i_s}+3)/2r]$$
$$=(1/2r)[(2r_{i_s}+3)/(2r_{i_s}+1)]\ , \tag{36b}$$
$$C^{BA}_{i_s}=g(B,r_{i_s-1})S(A,r_{i_s};A,r-\tfrac{1}{2})$$
$$=[-1/(2r_{i_s}+2)][2r_{i_s}/2r]$$
$$=-(1/2r)[(2r_{i_s})/(2r_{i_s}+2)]\ , \tag{36c}$$
$$C^{BB}_{i_s}=g(B,r_{i_s-1})S(B,r_{i_s};A,r-\tfrac{1}{2})$$
$$=[-1/(2r_{i_s}+2)][(2r_{i_s}+2)/2r]$$
$$=-1/2r\ , \tag{36d}$$

where we have dropped the regular subscripts and

superscripts of the coefficients.

The probability that AA will occur is about $(N-d)^2/N^2$; that of AB or BA is about $d(N-d)N^2$; and that of BB is about d^2/N^2. Therefore, under condition (29), cases of AA are an overwhelming majority and those of BB are negligible.

Let us consider a consecutive pair of AB and BA. This pair will contribute the following to the RME:

$$P_s = C^{AB}_{i_s-1} e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_{i_s-1}} + C^{BA}_{i_s} e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_{i_s}}$$
$$= (1/2r)e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_{i_s-1}}\{[1-e^{\pm i(\vec{k}-\vec{k}_1)\cdot(\vec{X}_{i_s}-\vec{X}_{i_s-1})}] + (1/r_{i_s})[1+e^{\pm i(\vec{k}-\vec{k}_1)\cdot(\vec{X}_{i_s}-\vec{X}_{i_s-1})}]+\cdots\}. \quad (37)$$

The value of P_s as given by Eq. (37) depends on the index $(\vec{k}-\vec{k}_1)\cdot(\vec{X}_{i_s}-\vec{X}_{i_s-1})$; which, in turn, depends very critically on how we order the atoms in the sample. Assuming that the atoms are randomly distributed, we can order them in such a way that two consecutive atoms are very close neighbors and $(\vec{X}_{i_s}-\vec{X}_{i_s-1})$ is almost perpendicular to $(\vec{k}-\vec{k}_1)$, as illustrated in Fig. 3. Suppose in this way we can manage to reduce the index to a value of the order

$$(\vec{k}-\vec{k}_1)\cdot(\vec{X}_{i_s}-\vec{X}_{i_s-1}) = O(\epsilon); \quad (38)$$

then we have

$$P_s = O(\epsilon/N) + O(1/N^2) = O(\epsilon/N), \quad (39)$$

where the last step is taken because we believe that the order of ϵ is higher than that of $1/N$.

Now we can write the diagonal RME as

$$(r;\alpha|R^{\pm}_{k}|r;\alpha) = \frac{1}{2r}\sum_{i=1}^{N}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_i} + \sum_{s=1}^{d} P_s + O(d^2/N^3)$$
$$= \frac{1}{2r}\sum_{i=1}^{N}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_i} + O(d\epsilon/N), \quad (40)$$

where a possible error of the order of d^2/N^3 is introduced because we have ignored the existence of BB. The prime on the summation sign indicates that i_s-1 and i_s are to be skipped; this means that the number of terms to be summed is exactly $2r$. This would be true even if we had taken cases of BB into consideration because for every term of BB, a term of AA is canceled almost exactly.

If the last section of the diagram is of type B, then the last factors in Eqs. (36) should be $f(B, r+\frac{1}{2})$ instead of $f(A, r-\frac{1}{2})$, and $2r$ in all expressions in this subsection should be replaced by $2r+2$. But this is of no significance at all.

The generator of an RME is represented by the first single line of its diagram. Suppose this single line consists of N_1-1 sections with N_1-d_1-1 of them being of type A and d_1 of type B. The expression for the generator can then be obtained from that for the diagonal RME in Eq. (40) by replacing N and r by N_1 and r_{N_1}, respectively, as follows:

$$Q^{\pm}_{k}(N_1) = \frac{1}{2r_{N_1}}\sum_{i=1}^{N_1}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{X}_i} + O(d_1\epsilon/N_1). \quad (41)$$

D. Off-diagonal reduced matrix elements

With the expressions for propagators and the generator available, we can very easily write down the RME for all types of diagrams. For off-diagonal RME, the generator is always followed by a section of type D. It is convenient to combine the contribution of these two to the RME once for all as follows:

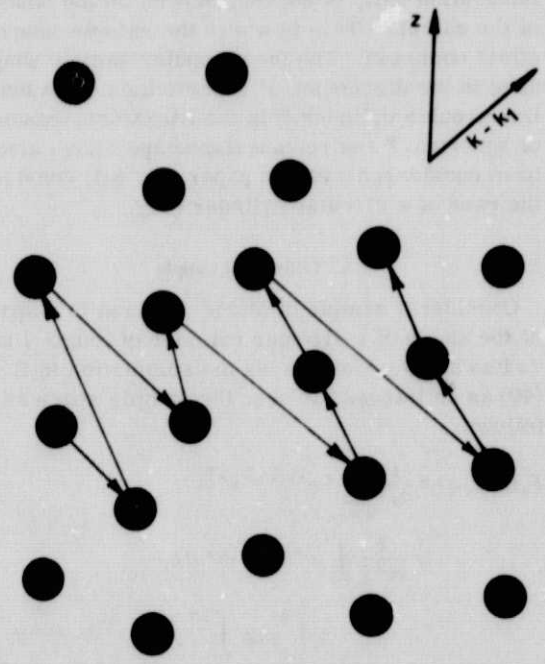


FIG. 3. An illustration of the way to choose the sequence of atoms in order to reduce the possible error introduced in simplifying the expression for a diagonal reduced matrix element or a "generator."

$$Q_k^{\omega}(N_1) = Q_k^{\frac{1}{2}}(N_1) f(D, r_{N_1}) + g(D, r_{N_1}) e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_{N_1+1}}$$

$$= \frac{1}{2r_{N_1}} \left(\frac{1}{2r_{N_1}} \sum_{i=1}^{N_1}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_i} - e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_{N_1+1}} \right) + O(d_1 \epsilon / N_1^2) . \tag{42}$$

(a) Type NC1.

$$(r \pm 1; \alpha' | R_k^{\frac{1}{2}} | r; \alpha) = Q_k^{\frac{1}{2}}(N_1) T(r_{N_1+1}, r_{N-1}) \simeq \frac{1}{2r} \left(\frac{1}{2r_{N_1}} \sum_{i=1}^{N_1}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_i} - e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_{N_1+1}} \right) . \tag{43}$$

(b) Type C(2n). A diagram of type $C(2r)$ consists of the generator, n T propagators, and n S propagators connected by n sections of type D and n sections of type C. We can write the RME as

$$(r; \alpha' | R_k^{\frac{1}{2}} | r; \alpha) = Q_k^{\omega}(N_1) T(r_{N_1+1}, r_{N_2-1}) f(C, r_{N_2}) S(r_{N_2+1}, r_{N_3-1}) f(D, r_{N_3}) \cdots S(r_{N_{2n}+1}, r_{N-1})$$

$$\simeq \frac{(-1)^n}{2r \prod_{i=2}^{2n} (2r_{N_i})} \left(\frac{1}{2r_{N_1}} \sum_{i=1}^{N_1}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_i} - e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_{N_1+1}} \right) . \tag{44}$$

(c) Type NC(2n + 1). A diagram of type $NC(2n+1)$ consists of the generator, $(n+1)$ T propagators, and n S propagators connected by $(n+1)$ sections of type D and n sections of type C. We can write the RME as

$$(r \pm 1; \alpha' | R_k^{\frac{1}{2}} | r; \alpha) \simeq \frac{(-1)^n}{2r \prod_{i=2}^{2n+1} (2r_{N_i})} \left(\frac{1}{2r_{N_1}} \sum_{i=1}^{N_1}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_i} - e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_{N_1+1}} \right) . \tag{45}$$

V. CYLINDRICAL SAMPLE AND TRANSITION PROBABILITIES

To obtain numerical values of RME by using Eqs. (40), (43), (44), and (45), we replace the summation by integration. The result of such integration will, of course, depend on the boundary of the sample. This is where the sample-shape effect comes in. The most popular sample shape used in the discussion of superradiance has been the circular cylinder.[9] In the literature, cases of spherical[10] and rectangular shape[11] have also been considered. In this paper, we will consider the case of a circular cylinder only.

A. Cylindrical sample

Consider a sample of atoms confined in a space of the shape of a circular cylinder of length l and radius a. We can express the summation in Eq. (40) as an integration over the sample space as follows:

$$F_1(\vec{k}, l, a) \equiv \frac{1}{2r} \sum_{i=1}^{N}{}' e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_i}$$

$$= \frac{1}{\pi a^2 l} \int_0^l e^{\pm i(\vec{k}-\vec{k}_1)_z z} dz \times \int_0^a \rho d\rho \int_0^{2\pi} e^{\pm i(\vec{k}-\vec{k}_1)_\rho \rho \cos\theta} d\theta , \tag{46}$$

where $(\vec{k}-\vec{k}_1)_{z,\rho}$ are the transversal and axial components of $(\vec{k}-\vec{k}_1)$, and the z axis has been chosen in the $\vec{k}_1$ direction. Carrying out the integration in Eq. (46), we obtain

$$F_1(\vec{k}, l, a) = e^{\pm i(\vec{k}-\vec{k}_1)_z l/2} \times \frac{\sin[(\vec{k}-\vec{k}_1)_z l/2]}{[(\vec{k}-\vec{k}_1)_z l/2]} \frac{2J_1((\vec{k}-\vec{k}_1)_\rho a)}{[(\vec{k}-\vec{k}_1)_\rho a]} , \tag{47}$$

where J_1 is the Bessel function of order 1.

Now we can write Eq. (43) as

$$(r \pm 1, \alpha' | R_k^{\frac{1}{2}} | r, \alpha) = (1/2r)(F_1(\vec{k}, l_1, a) - e^{\pm i(\vec{k}-\vec{k}_1)\cdot\vec{x}_{N_1+1}}) , \tag{48}$$

where we have assumed that the order of the atoms is such that a disk of radius a and thickness dz is completely filled before we go to the next disk along the axis of the cylinder; we have also defined

$$l_1 \equiv l(N_1/N) . \tag{49}$$

B. Transition probabilities

With our knowledge of the matrix elements but without solving the master equation, we should be able to get some idea about the time evolution of the atomic system. We assume that at time $t=0$, the system consists of completely inverted atoms with $r = \frac{1}{2}N$ and a vacuum of the radiation field. The initial evolution when the value of r is still

close to $\frac{1}{2}N$ will be essentially determined by the collective operator $R_{\vec{k}}^{\dagger}$. Suppose at a certain instant the atomic system is represented by an eigenstate $|r, m; \alpha\rangle$; we will consider the transition probabilities to the single state $|r, m-1; \alpha\rangle$ and to the whole set of states $|r-1, m-1; \alpha'\rangle$, respectively. They will be proportional to the following quantities:

(a) r-conserving process.

$$|\langle r, m-1; \alpha | R_{\vec{k}}^{\dagger} | r, m; \alpha\rangle|^2 = (r+m)(r-m+1)|(r; \alpha | R_{\vec{k}}^{\dagger} | r; \alpha)|^2$$
$$\simeq (r+m)(r-m+1)\frac{\sin^2[(\vec{k}-\vec{k}_1)_z l/2]}{[(\vec{k}-\vec{k}_1)_z l/2]^2}\ \frac{4J_1^2((\vec{k}-\vec{k}_1)_\rho a)}{[(\vec{k}-\vec{k}_1)_\rho a]^2}\ . \tag{50}$$

(b) r-nonconserving processes.

$$\sum_{\alpha'}|\langle r-1, m-1; \alpha' | R_{\vec{k}}^{\dagger} | r, m; \alpha\rangle|^2 = (r+m)(r+m-1)\sum_{N_1=2}^{N}{}'|(r-1; \alpha' | R_{\vec{k}}^{\dagger} | r; \alpha)|^2$$
$$\simeq (r+m)(r+m-1)\sum_{N_1=2}^{N}{}'|F_-(\vec{k}, l_1, a) - e^{-i(\vec{k}-\vec{k}_1)\cdot\vec{x}_{N_1+1}}|^2/(2r)^2$$
$$\simeq \frac{(r+m)(r+m-1)}{2r}\left(\frac{1}{l}\int_0^l |F_-(\vec{k}, l_1, a)|^2 dl_1 - \frac{2}{\pi a^2 l}\ \frac{2J_1((\vec{k}-\vec{k}_1)_\rho a)}{[(\vec{k}-\vec{k}_1)_\rho a]}\right.$$
$$\times\int_0^l \frac{\sin[(\vec{k}-\vec{k}_1)_z l_1]}{[(\vec{k}-\vec{k}_1)_z l_1]}\ dl_1$$
$$\left.\times\int_0^a \rho d\rho\int_0^{2\pi} e^{i(\vec{k}-\vec{k}_1)_\rho \rho\cos\theta} d\theta + 1\right)$$
$$= \frac{(r+m)(r+m-1)}{2r}\left(1 - \frac{\sin^2[(\vec{k}-\vec{k}_1)_z l/2]}{[(\vec{k}-\vec{k}_1)_z l/2]^2}\ \frac{4J_1^2((\vec{k}-\vec{k}_1)_\rho a)}{[(\vec{k}-\vec{k}_1)_\rho a]^2}\right). \tag{51}$$

From Eqs. (50) and (51) we can see the general behavior of the evolution of the atomic system. At the beginning, when m is close to r, the factors $(r+m)(r-m+1)$ of Eq. (50) and $(r+m)(r+m-1)/2r$ of Eq. (51) are of the same order of magnitude. So the relative importance of the two kinds of processes is determined by the other factors; namely, $|F(\vec{k}, l, a)|^2$ of Eq. (50) and $1-|F(\vec{k}, l, a)|^2$ of Eq. (51). Rehler and Eberly[2] have calculated numerically the function $|F(\vec{k}, l, a)|^2$, which shows a high peak along the direction of $\vec{k}_1$. Therefore, for radiation around the direction of $\vec{k}_1$, it is predominantly r conserving; and for radiation in other directions, it is predominantly r nonconserving. Later on, as m approaches zero, we have $(r+m)(r-m+1) \gg (r+m)(r+m-1)/2r$, if $r \gg 1$. Then the r-conserving process dominates, and the radiation will be confined within a small solid angle around the direction of $\vec{k}_1$. We can see then the condition for superradiance is that the state of low m must be reached before r drops too low due to r-nonconserving processes.

VI. SUMMARY

In the Dicke-Schwendimann approach to the problem of N two-level atoms spread over large regions, interacting with multimode-radiation field, the major difficulty has been lack of explicit expression for the reduced matrix elements (RME's) of collective operators which determine the static aspect of the problem. The aim of this paper has been to clear up this difficulty. The genealogical construction was adopted to obtain Dicke states corresponding to a specified mode. Then a set of recursion formulas of the RME's of collective operators corresponding to arbitrary modes was obtained. Using the branching diagram usually used to determine the dimensions of binary irreducible representations of the permutation group S_N, a simple and systematic method was developed for calculating RME's for any number of atoms in a system. However, from the physical point of view, one is most interested in the asymptotic results for large N. In this asymptotic limit, it is found that the order of magnitude of an RME is determined by the topological structure of its diagram. Explicit expressions have been obtained for RME's with diagrams of all types of topological structures. They are presented in Eqs. (40), (43), (44), and (45). Equation (40) gives the diagonal r-conserving RME with r close to $\frac{1}{2}N$, which is not much different, especially when the summation is replaced by an integration, from

the case with $r = \frac{1}{2}N$ which is well known. However, because this expression has been obtained with the order of magnitude of possible error clearly in mind, the result certainly enhances our confidence in extending the usage of this expression from the case of $r = \frac{1}{2}N$ to cases of r close to $\frac{1}{2}N$. Equation (43) gives the most important off-diagonal and r-nonconserving RME. This is, perhaps, a new result, although Schwendimann has speculated about a similar expression (Ref. 4, p. 276, footnote). Anyway, in my judgement, the most important contribution of this paper is in gaining insight into r-nonconserving processes.

ACKNOWLEDGMENTS

The author is grateful to Dr. C. M. Bowden, Dr. R. L. Kurtz, and Dr. R. Gilmore for valuable discussions.

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*Work supported by NASA under Grant No. NSG-8011.

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