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4. STABILITY OF MULTIRATE SAMPLED-DATA SYSTEMS (10 JUN - 16 AUG 74)

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16. Abstract

Current adaptive sampling schemes that can be used in sampled-data systems with variable rate sampling does not guarantee the stability of the resulting closed-loop system. Therefore, this study has been undertaken with the objective of finding a stable control law for multirate sampled-data systems. The problem is formulated such that the sampling interval is selected from a set of fixed number of sampling intervals. A necessary and sufficient condition under which these types of systems can be stabilized is given. For a certain subclass of these types of systems, a sampling selection algorithm is given which results in a stable closed-loop system.

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I. Introduction

The advent of high speed digital computers has made a striking impact in aircraft flight control systems. In the flight control actuation systems alone, digital technology has brought about the possibility of quite a number of new design approaches [1]. For instance, the digital actuator is one of these new possibilities. Basically, a digital actuator is a device that can produce a desired quantized rate or a quantized position. The question that immediately arises is how often the on-board control computer must output digitized rate or position commands to the control actuators. If a fixed high output rate is selected, the control system analysis is greatly simplified enabling the designer to use the standard discrete control theory. On the other hand, a fixed high output rate increases the burden on the flight computer and the wear and tear of the associated mechanical equipment. Intuitively, a multirate output sampling is needed to improve the sampling efficiency. In a multirate sampled digital environment, the control computer outputs per unit time can be optimized to achieve a desired system performance with the least number of output commands.

The desire to achieve a high level of sampling efficiency in sampled-data control systems has been a challenging goal in control engineering. Many different methods have been used to obtain efficient adaptive sampling laws in the last decade [2] - [10]. In most of these methods, an integral difference criterion is used to make the difference of any two consecutive sampled errors equal. In general, these methods have been relatively simple formulations and yield implementable laws with low computational requirements. The major drawback in all of these approaches has been the lack of guarantee of stability in the resulting closed-loop system. One reason for this stability problem is that, in all these systems, the sampling interval is selected from a continuum of sampling intervals which makes the problem nonlinear. In this study, the sampling interval is selected from a finite number of sampling intervals. The use of a finite number of sampling intervals results in an analytically tractable formulation and also fits better to physical reality since the discretization of the continuum of sampling intervals is necessary in order to be implemented in a digital environment.

The outline of the report is as follows: In the next section, a mathematical statement of the problem is given. Basic notions of stability for time-varying systems is reviewed in Section III. In Section IV, it is shown that the stability problem as modelled in this study essentially reduces to one of finding a convergent sequence of matrices from a finite collection of matrices. The notions of convergent, contractive, and pre-contractive set of a finite number of matrices is introduced. It is shown that pre-contractiveness is a necessary and sufficient condition for finding a convergent sequence from a finite set of matrices. In Section V, computable criteria are found to test the contractiveness of a set of matrices. Section VI contains the conclusions and directions for further research.

II. Statement of the Problem

The objective of this investigation is to find mathematical conditions which will ensure stability in linear multirate-sampled-data control systems. Specifically, we take as given an $m \times m$ constant matrix A, and an $m \times n$ constant matrix B, and consider the system

$$(1) \cdots\cdots \quad x = Ax + Bu$$

where x is an m-component "state vector" depending on time, and u is an n-component "control vector" depending on time. Given an initial state $x_0 = x(0)$, we wish to construct a control u which will produce stability of the system at zero. (Various types of stability will be discussed in the next section.)

All numbers, vectors, and matrices considered in this report are assumed real. Complex numbers enter only in the brief allusions to the spectral radius of a matrix in section 4.

We must first discuss a solution of (1) satisfying an arbitrary condition of the form $x(t_1) = z$, where $t_1 > 0$ and $z \varepsilon R^m$. The following solution to this problem is standard.

Consider the equation $\dot{x} = Ax$. Any $m \times m$ matrix Φ whose columns are linearly independent solutions of $\dot{x} = Ax$ is called a <u>fundamental matrix</u> for the equation. One such matrix is $\Phi(t) = e^{tA}$. Furthermore, if $t_1 > 0$ and $z \varepsilon R^m$, a solution ϕ of $\dot{x} = Ax$ satisfying $\phi(t_1) = z$ is given by

$$\phi(t) = e^{(t-t_1)A}z$$

Also, a solution θ of the original equation (1) satisfying $\theta(t_1) = 0$ is given by

$$\theta(t) = \Phi(t)\int_{t_1}^{t} \Phi^{-1}(s)Bu(s)ds = e^{tA}\int_{t_1}^{t} e^{-sA}Bu(s)ds .$$

Now $(\phi + \theta)'(t) = A\phi(t) + A\theta(t) + Bu(t) = A(\phi + \theta)(t) + Bu(t)$ and $(\phi + \theta)(t_1) = \phi(t_1) + \theta(t_1) = z$. Thus $x = \phi + \theta$ is a solution of (1) satisfying $x(t_1) = z$.

Now letting $T = t-t_1$, we may write $x(t) = e^{(t-t_1)A}z + e^{tA}\int_{t_1}^{t} e^{-sA}Bu(s)ds = e^{TA}z + e^{(T+t_1)A}\int_0^T e^{-(s+t_1)A}Bu(s+t_1)ds = e^{TA}z + \int_0^T e^{(T-s)A}Bu(s+t_1)ds.$

Substituting $v = T-s$, we obtain

$$(2) \cdot \cdot \cdot \cdot \cdot \quad x(T) = e^{TA}z + \int_0^T e^{vA}Bu(t_1+T-v)dv .$$

We now think of constructing a "stable" solution of (1) by the method of sampling; that is, at particular instants in time we examine the value of x and, on the basis of that value, change the value of u so as to make x stable. The value of u is to be held constant until the next sampling instant. Thus if $\{t_k\}_{k=1}^{\infty}$ is an increasing sequence of sampling instants, $T_k = t_{k+1} - t_k$ is the k^{th} sampling interval, $x_k = x(t_k)$ is the state at time t_k, and u_k is the constant value of $u(t)$ for $t_k < t < t_{k+1}$, formula (2) can be rewritten as

$$x_{k+1} = e^{T_kA}\ x_k + \int_0^{T_k} e^{vA}Bu_k dv \quad , \text{ or}$$

$$(3) \cdot \cdot \cdot \cdot \cdot \quad x_{k+1} = e^{T_kA}\ x_k + \int_0^{T_k} e^{vA}dvBu_k \quad .$$

A variable rate sampling problem is a problem of the above type in which a collection $S = \{S_1, S_2, \cdot \cdot \cdot, S_N\}$ of positive numbers (sampling interval lengths) is given. The sequence $\{t_k\}$ of sampling instants is generated in time along with the sequence $\{u_k\}$ of controls, in that each time a sample x_k is observed, we are allowed to use its value in determining the next sampling interval, T_k, selected from the set S.

The next control, u_k, is to be determined in the following manner. Corresponding to each S_i in S we determine an $n \times m$ matrix F_{S_i}, called the $\underline{i^{th}}$ feedback matrix. Whenever the sampling interval T_k is selected to be the member S_i of S, then u_k is determined by $u_k = F_{S_i} x_k$. Further, we wish to allow the choice of T_k to depend not only on the observed state x_k, but also on the previous sampling interval T_{k-1}.

Thus, given A, B, and S, we wish to define the feedback matrices $F_{S_1}, F_{S_2}, \cdots, F_{S_N}$ and a "decision function" $d(x_k, T_{k-1})$ whose value is a member of S which we will choose as T_k. In this way, starting from any initial state x, and an arbitrary member T_o of S, a sequence of state values is defined, using (3) by:

$$(4) \cdots\cdots \begin{cases} T_k = d(x_k, T_{k-1}) \\ x_{k+1} = \left[e^{T_k A} + \int_o^{T_k} e^{vA} dv B F_{T_k} \right] x_k \end{cases}$$

We want to choose the F_i's and the d in such a way that the resulting system is stable at 0.

III. Stability

We restrict our attention to notions of stability for sequences of vectors in R^m generated by some discrete system [11]; that is,

$$x_{k+1} = P(x_k) \quad , \quad x_1 \text{ given.}$$

In general, P may involve any number of parameters such as, in our problem, T_{k-1}. The notions of stability require a norm on R^m. It is the case that all norms on R^m are equivalent in the sense that, if $||\cdot||_1$ and $||\cdot||_2$ are any two norms, there are numbers $\alpha > 0$ and $\beta > 0$ such that

$$||x||_1 \le \alpha ||x||_2 \text{ and } ||x||_2 \le \beta ||x||_1$$

for all $x \varepsilon k^m$. As a result of this, it is immaterial which norm is used in the following definitions, and we denote a general norm by $||\cdot||$.

Several definitions for various types of stability have been defined. The ones given here essentially follow Ortega [12].

The discrete system P is <u>convergent of x*</u> if there is a $\delta > 0$ such that

$$\lim_{k\to\infty} ||x_k - x^*|| = 0 \text{ whenever } ||x_1 - x^*|| < \delta.$$

P is <u>globally convergent at x^*</u> if

$$\lim_{k\to\infty} ||x_k - x^*|| = 0 \text{ for all } x_1 \varepsilon R^m.$$

P is <u>Lyapunov stable at x^*</u> if, given $\epsilon > 0$, there is a $\delta > 0$ such that $||x_k - x^*|| < \epsilon$ for all $k \ge 1$ whenever $||x_1 - x^*|| < \delta$.

Global convergence does not imply Lyapunov stability, for consider in R^1, with $x^* = 0$,

$$x_{k+1} = P(k, x_k) = \frac{1}{k}, \ k \geq 1, \ x_1 \text{ arbitrary}.$$

Also, Lyapunov stability does not imply convergence, for consider in R^1 with $x^* = 0$,

$$x_{k+1} = P(k, x_k) = \begin{cases} 2x_1, & k=1 \\ x_k, & k>1 \end{cases} .$$

P is <u>asymptotically stable at x^*</u> if P is both Lyapunov stable and convergent at x^*.

P is <u>expontially convergent at x^*</u> if there is a $\Gamma > 0$, a positive $\beta < 1$, and a $\delta > 0$, such that

$$||x_k - x^*|| \leq \Gamma\beta^k||x_1 - x^*|| \text{ for all } k \geq 1, \text{ whenever}$$

$$||x_1 - x^*|| < \delta.$$

<u>Global asymptotic stability at x^*</u> is Lyapunov stability together with global convergence at x^*, and <u>global exponential convergence</u> is defined by requiring

$$||x_k - x^*|| \leq \Gamma\beta^k||x_1 - x^*||, \text{ all } k \geq 1 \text{ and all } x.$$

Clearly (global) exponential converge implies (global) asymptotic stability.

IV. Convergent, Contractive, and Pre-contractive Sets.

In order to attack the problem as stated at the end of section 2, we proceed as follows. Assuming the $m \times m$ matrix A, the $m \times n$ matrix B, and the set $S = \{S_1, S_2, \cdot \cdot \cdot, S_N\}$ of positive sampling interval lengths are given, set

$$C_i = e^{S_i A}, \text{ and } D_i = \int_0^{S_i} e^{vA} dvB \; .$$

If F_{S_i} is to be the feedback matrix corresponding to S_i, equations (4) at the end of section 2 can be written:

$$T_k = d(x_k, T_{k-1})$$

$$x_{k+1} = \left(C_{T_k} + D_{T_k} F_{T_k}\right) x_k \; .$$

If suitable feedback matrices and a suitable decision function are to be found, then surely the collection

$$\left\{H_i \;\middle|\; H_i = C_i + D_i F_{s_i}, \quad 1 \le i \le N\right\}$$

has the property that, given any vector $x \in R^m$, there is an order in which we can apply the H_i's to x to produce a sequence convergent to, or at least remaining near, zero. We will find conditions which assure that a set of matrices has this property, in the hope that then the F_{s_i} can be found to produce these conditions.

In our setting it will be clear that convergence, exponential converge, and asymptotic convergence are equivalent to the corresponding global types of stability. In the remainder of this report, the adjective "global" will be dropped. Also, all types of stability will be considered at $x^* = 0$, so the phrase "at x^*" will be dropped. Throughout the report, theorems are numbered consecutively, regardless of the section in which they appear, as are definitions, lemmas, corollaries, and numbered formulas.

Definition 1: The set $K = \{H_1, H_2, \cdots, H_N\}$ of $m \times m$ matrices is convergent provided that, if $x \in R^m$, there is a sequence $\{p(x)_i\}_{i=1}^{\infty}$, with $p(x)_i \in \{1, 2, \cdots, N\}$ for all i, such that

$$\lim_{k\to\infty}\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x = 0 .$$

(The product is taken in reverse order so that $H_{p(x)_1}$, is applied first, $H_{p(x)_2}$ second, and so on.)

In the familiar case where $N = 1$, the only possible sequence is $\left\{H_1{}^k x\right\}_{k=1}^{\infty}$. It is well-known that this sequence converges to zero for all x if and only if the spectral redius of H_1 is less than 1, where the spectral radius is defined as the largest number which is the absolute value of an eigenvalue (real or complex). That this does not generalize in any foreseeable way to $N > 1$ is seen by the following example in which $N = 4$:

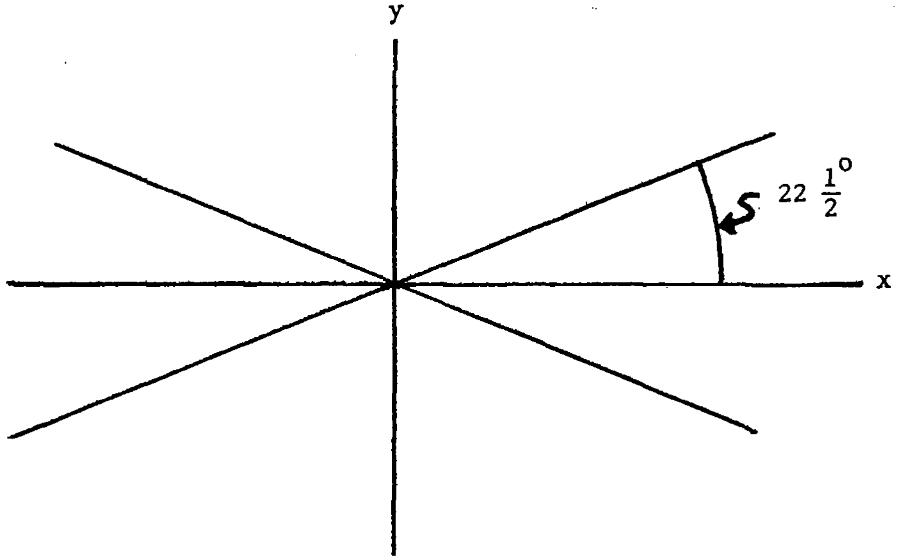
Example 1:

$$H_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1.25 & -.75 \\ -.75 & 1.25 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1.25 & .75 \\ .75 & 1.25 \end{bmatrix}$$

In this example, each H_i has spectral radius 2, and yet the set $K = \{H_1, H_2, H_3, H_4\}$ is convergent. This can be seen by observing that:

If $||\cdot||$ denotes the Euclidean norm, $||H_1x|| \le \beta||x||$ for all $x \ne 0$

in C_1, where $\beta = \sqrt{1-\frac{48-3\pi^2}{64+\pi^2}} < 1$ and C_1 is the closed cone pictured here:



To prove this, let $\alpha = \frac{48-3\pi^2}{64}$ and consider the function $f(t) = (1+t^2) - (1/4 + 4t^2) = 3/4 - 3t^2$. $f(0) = 3/4$, $f(1/2) = 0$, f decreases on $[0, 1/2]$, and f is even. Since $0 < \pi/8 < 1/2$, $f(\pi/8) > 0$. $f(\pi/8) = 3/4 - 3\pi^2/64 = \alpha$. Assume $|t| \le \pi/8$. Then $f(t) \ge f(\pi/8) = \alpha$. Thus $(1+t^2) - (1/4 + 4t^2) \ge \alpha$

$$1 - \frac{1/4 + 4t^2}{1 + t^2} \ge \frac{\alpha}{1+t^2}$$

$$\frac{1/4 + 4t^2}{1 + t^2} \le 1 - \frac{\alpha}{1+t^2} \le 1 - \frac{\alpha}{1+\pi^2/64} = 1 - \frac{48-3\pi^2}{64+\pi^2} = \beta^2.$$

$$(5) \quad \cdots \quad \frac{\sqrt{1/4 + 4t^2}}{\sqrt{1 + t^2}} \le \beta \text{ for } |t| \le \frac{\pi}{8}$$

Now suppose $(x,y) \in R^2$ and $\left|\frac{y}{x}\right| \le \frac{\pi}{8}$.

$$H_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2x \\ 2y \end{bmatrix}$$

$$||H_1\begin{bmatrix}x\\y\end{bmatrix}|| = \sqrt{\frac{x^2}{4} + 4y^2} = |x|\sqrt{\frac{1}{4} + 4\left(\frac{y}{x}\right)^2}$$

$$||\begin{bmatrix}x\\y\end{bmatrix}|| = \sqrt{x^2+y^2} = |x|\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Thus by (5),

$$\frac{||H_1\begin{bmatrix}x\\y\end{bmatrix}||}{||\begin{bmatrix}x\\y\end{bmatrix}||} = \frac{\sqrt{\frac{1}{4} + 4\left(\frac{y}{x}\right)^2}}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} \leq \beta, \text{ and}$$

$$||H_1\begin{bmatrix}x\\y\end{bmatrix}|| \leq \beta||\begin{bmatrix}x\\y\end{bmatrix}|| .$$

Since $C_1 = \left\{(x,y)\right\} \Big| \ |\frac{y}{x}| \leq \frac{\pi}{8} \cup \left\{0\right\}\Big\}$, the statement is justified.

Furthermore H_2, H_3, and H_4 contract the lengths of vectors in C_2, C_3, and C_4 in the same way, where C_2 is the counterclockwise rotation of C_1 through 45°, C_3 is the counterclockwise rotation of C_1 through 90°, and C_4 is the counterclockwise rotation of C_1 through 135°.

Now $\bigcup_{i=1}^{4} C_i = R^2$. Thus, given $x \varepsilon R^2$, we select $p(x)_1 = j$ if $x \varepsilon C_j$ and, if $\left\{p(x)_i\right\}_{i=1}^{k}$ have been determined, we select $p(x)_{k+1} = j$ if $\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x \varepsilon C_j$ (If at some stage there are two possible choices for j, that is two cones overlap, we may choose either.)

Then $\left|\left| \left(\prod_{i=k}^{1} H_{p(x)_i}\right) x\right|\right| \leq \beta^k x \longrightarrow 0$ as $k \longrightarrow \infty$.

This example suggests the following definition.

<u>Definition 2:</u> A set $K = \left\{H_1, \cdot \cdot \cdot, H_N\right\}$ of $m \times m$ matrices is <u>contractive relative to the norm $||\cdot||_v$</u> provided that, if $x \varepsilon R^m$ and $x \neq 0$, there is an $i \varepsilon \left\{1, 2, \ldots, N\right\}$ such that $||H_i x||_v < ||x||_v$.

It is a theorem that if K is contractive relative to some norm, then K is convergent. This will be contained in the more general

Theorem 1, to be proved presently.

Clearly, contractiveness relative to a given norm is not a necessary condition for convergence of a set K of matrices. For example,

$$K:\ H_1 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix},\quad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \text{ is not contractive}$$

relative to the Euclidean norm, for neither matrix contracts $x = (1,1)$. Yet the matrix H_1H_2 is zero, so that, if

$$p(x)_i = \begin{cases} 2, & i=1 \\ 1, & i>1 \end{cases} \quad \text{for all } x,$$

then all x go to zero in a hurry. In fact, it is clear that, if $K = \{H_1, \cdots, H_N\}$ and some finite product of the H_i's has spectral radius less than 1, then the sequence p(x) can be chosen to produce that product repeatedly, and thus K will be convergent.

The authors have not proved that no finite product of the four matrices in Example 1 has spectral radius less than 1, but extensive experimentation suggests that this is the case. Yet the set K of Example 1 is convergent, and the algorithm for selecting p(x) does not deal with the spectral radii of products, but rather with the more near-sighted contraction of the length of the vector.

Consider another example with N = 2.

Example 2:

$$K:\ H_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix},\quad H_2 = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

H_1 is identical with the H_1 of Example 1, and H_2 effects rotation clockwise through 30°. Clearly K is convergent, for, given $x \in R^2$, we apply H_2 often enough to bring the vector into the cone C_1 described in Example 1, and then apply H_1. Since $||H_2 y|| = ||y||$ for all y, we have now contracted the vector. If the result of H_1 puts the vector outside of C_1, we again apply H_2 often enough to bring it back to C_1, then apply H_1, and so on. This example, together with the foregoing discussion, suggests the following definition.

Definition 3: A set $K = \{H_1, \cdots, H_N\}$ of $m \times m$ matrices is pre-contractive relative to the norm $||\cdot||_v$ provided that, if $x \in R^m$, $x \neq 0$, there is a finite sequence $\{q(x)_i\}_{i=1}^{n(x)}$, $q(x)_i \in \{1,2,\cdots,N\}$, such that

$$\left|\left|\left(\prod_{i=n(x)}^{1} H_{q(x)_i}\right)x\right|\right|_v < ||x||_v \,.$$

We now show that pre-contractiveness relative to some norm is equivalent to convergence. In the following discussion, $K = \{H_1, \cdots, H_N\}$ is a set of $m \times m$ matrices and $||\cdot||_v$ is a norm on R^m.

Lemma 1: If K is pre-contractive relative to $||\cdot||_v$, there is a positive integer M such that, if $x \in R^m$, $x \neq 0$, then there is a finite sequence $\{q(x)_i\}_{i=1}^{n(x)}$, $q(x)_i \in \{1,2,\cdots,N\}$, such that

$$\left|\left|\left(\prod_{i=n(x)}^{1} H_{q(x)_i}\right)x\right|\right|_v < ||x||_v$$

and $n(x) \leq M$.

Proof: Let $B = \left\{x \in R^m \,\middle|\, ||x||_v = 1\right\}$.

B is compact in the norm topology. For each positive integer n, let

$$S_n = \left\{x \in B \,\middle|\, \exists\, \{q(x)_i\}_{i=1}^{k} \ni \left|\left|\left(\prod_{i=k}^{1} H_{q(x)_i}\right)x\right|\right|_v < 1, \text{ and } k \leq n \right\}$$

S_n is open relative to B and $\bigcup_{n=1}^{\infty} S_n = B$, since K is pre-contractive relative to $||\cdot||_v$. Thus there are positive integers $n_1, n_2, \cdots, n_\ell$ such that

$$\bigcup_{i=1}^{\ell} S_{n_i} = B.$$

Let $M = \max\left\{n_1, n_2, \cdots, n_\ell\right\}$.

Given $0 \neq x \in R^m$, there is an j, $1 \leq j \leq \ell$ such that $\frac{x}{||x||_v} \in S_{n_j}$.

Then

$$\left|\left|\left(\prod_{i=n_j}^{1} H_{q\left(\frac{x}{||x||_v}\right)_i}\right) x\right|\right|_v = ||x||_v \left|\left|\left(\prod_{i=n_j}^{1} H_{q\left(\frac{x}{||x||_v}\right)_i}\right) \frac{x}{||x||_v}\right|\right|_v <$$

$$||x||_v \cdot 1 = ||x||_v \text{ and } n_j \leq M \quad .$$

QED

Lemma 2: If K is contractive relative to $||\cdot||_v$, there is a β, $0 < \beta < 1$, such that, if $x \in R^m$, $x \neq 0$, then there is an $i \in \left\{1,2,\cdots,N\right\}$ such that $||H_i x||_v \leq \beta ||x||_v$.

Proof: Let B be defined as in Lemma 1. For each positive integer n, let

$$S_n = \left\{x \in B \,\middle|\, \exists i \in \left\{1,2,\cdots,N\right\} \ni ||H_i x||_v < \frac{n-1}{n}\right\} .$$

Each S_n is open relative to B, and $\bigcup_{n=1}^{\infty} S_n = B$. Thus there are positive integers $n_1, n_2, \cdots, n_\ell$ such that

$$\bigcup_{i=1}^{\ell} S_{n_i} = B.$$

Let $n_0 = \max\left\{n_1, n_2, \cdots, n_\ell\right\}$, and let $\beta = \frac{n_0 - 1}{n_0}$. Then $0 < \beta < 1$, and, given $x \in R^m$, $x \neq 0$, choose j so that $\frac{x}{||x||_v} \in S_{n_j}$, i.e.

$$\left|\left|H_j \frac{x}{||x||_v}\right|\right| < \frac{n_j - 1}{n_j}$$

Then $||H_j x||_v = ||x||_v ||H_j \frac{x}{||x||_v}|| < \frac{n_j-1}{n_j} ||x||_v \le \beta ||x||_v$

QED.

Theorem 1: K is convergent if and only if K is pre-contractive relative to $||.||_v$.

Proof: Assume K is convergent. Choose $x \in R^m$, $x \ne 0$, and let $p(x)$ be a sequence such that

$$\lim_{k\to\infty}\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x = 0 \; .$$

There is an $n(x)$ such that $||\left(\prod_{i=n(x)}^{1} H_{p(x)_i}\right)x||_v < ||x||_v$, and we may choose $q(x)_i = p(x)_i$ for $1 \le i \le n(x)$. Thus K is pre-contractive relative to $||\cdot||_v$.

Now assume K pre-contractive relative to $||\cdot||_v$. Choose a positive integer M as given in Lemma 1. Then the set

$$\hat{K} = \left\{\prod_{i=k}^{1} H_{p_i} \;\middle|\; k \le M,\ p_i \in \{1,2,\cdots,N\}\right\}$$

is contractive relative to $||\cdot||_v$. Choose β, $0 < \beta < 1$, for $\hat{K}$ as given in Lemma 2. Thus we have:

If $x \in R^m$, $x \ne 0$, $\exists \{q(x)_i\}_{i=1}^{n(x)} \ni$

$$||\left(\prod_{i=n(x)}^{1} H_{q(x)_i}\right)x||_v \le \beta ||x||_v \quad \text{and } n(x) \le M.$$

Let $||\cdot||_0$ be the operator norm on $m \times m$ matrices generated by $||\cdot||_v$; i.e. $||H||_0 = \sup\{||Hx||_v \mid ||x||_v = 1\}$. Then $||Hy||_v \le ||H||_0 ||y||_v$ for all $y \in R^m$, and $||H_i H_j||_0 \le ||H_i||_0 ||H_j||_0$.

Let $B > \max\{||H_1||_0, ||H_2||_0, \cdots, ||H_N||_0\}$

Fix an $x \in R^m$. We must produce a $\{p(x)_i\}_{i=1}^{\infty}$ so that $\lim_{k\to\infty}\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x = 0$.

If $x = 0$, any $p(x)$ will do, so assume $x \neq 0$.

Define $y^0 = x$ and, for $k > 0$,

$$y^k = \left(\prod_{i=n(y^{k-1})}^{1} H_{q(y^{k-1})_i}\right) y^{k-1} ,$$

unless $y^{k-1} = 0$, in which case let $y^k = 0$. Then

$$||y^k||_v \le \beta||y^{k-1}||_v \le \beta^2||y^{k-2}||_v \le \cdots \le \beta^k||y^0||_v = \beta^k||x||_v .$$

For each positive integer k, let m_k be the unique non-negative integer satisfying:

$$s(k) = \sum_{j=0}^{m_k-1} n(y^j) < k \text{ and } t(k) = \sum_{j=0}^{m_k} n(y^j) \ge k,$$

(where we may take $n(y^j) = 1$ if $y^j = 0$). Then $m_{k+1} \ge m_k$ for $k \ge 0$, and $\lim_{k\to\infty} m_k = \infty$.

Let $p(x)_k = q(y^{m_k})_{k-s(k)}$. Given $\epsilon > 0$, choose a positive integer ℓ so that $\beta^{m_\ell} < \dfrac{\epsilon}{||x||_v \cdot B^M}$.

Suppose $k \ge \ell$. Then $k-s(k) \le t(k) - s(k) = n(y^{m_k}) \le M$. Thus

$$\left|\left|\left(\prod_{i=k}^{1} H_{p(x)_i}\right) x\right|\right|_v = \left|\left|\left(\prod_{i=k-s(k)}^{1} H_{q(y^{m_k})_i}\right) y^{m_k}\right|\right|_v \le$$

$$\left(\prod_{i=k-s(k)}^{1} \left|\left|H_{q(y^{m_k})_i}\right|\right|_0\right) \cdot ||y^{m_k}||_v \le$$

$$\left(\prod_{i=M}^{1} B\right) \beta^{m_k}||x||_v = \beta^{m_k}||x||_v B^M < \epsilon$$

Therefore $\lim_{k\to\infty}\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x = 0$ and K is convergent.

QED.

We list three corollaries to this theorem. The first involves the concept of exponential convergence, which, in this setting, is defined as follows:

Definition 4: A set $K=\{H_1,H_2,\cdots,H_N\}$ of $m \times m$ matrices is exponentially convergent provided that there is a norm $||\cdot||_v$ on R^m, a $\Gamma > 0$ and a β, $0 < \beta < 1$, such that, if $x \in R^m$, there is a sequence $\{p(x)_i\}_{i=1}^{\infty}$, $p(x)_i \in \{1,2,\cdots,N\}$ such that

$$||\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x||_v \leq \Gamma\beta^k ||x||_v \text{ for each positive integer } k.$$

Corollary 1: If K is contractive relative to $||\cdot||_v$, then K is exponentially convergent.

Proof: Since K is contractive relative to $||\cdot||_v$, the M in the proof of Theorem 1 can be taken as 1. Thus each n(x) is 1 and it follows that $m_k = k-1$ for $k > 0$. Therefore the last inequality in the proof becomes:

$$||\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x||_v \leq \beta^{k-1}||x||_v B^M = \Gamma\beta^k ||x||_v, \text{ where } \Gamma = \frac{B^M}{\beta}.$$

QED.

Corollary 2: If K is pre-contractive relative to $||\cdot||_v$, then K is pre-contractive relative to any other norm $||\cdot||_w$ on R^m.

Proof: K pre-contractive relative to $||\cdot||_v$ implies K is convergent, which implies that K is pre-contractive relative to $||\cdot||_w$.

QED.

Thus pre-contractiveness is norm-independent and we have:

Definition 3': K is pre-contractive if there is a norm $||\cdot||_v$ on R^m such that K is pre-contractive relative to $||\cdot||_v$.

Theorem 1 can now be restated:

Theorem 1': K is convergent if and only if K is pre-contractive.

In the present setting, the notion of asymptotic stability is given by:

Definition 5: K is asymptotically stable provided that there is a norm $||\cdot||_v$ on R^m such that, if $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in R^m$ and $||x||_v < \delta$ then there is a sequence $\{p(x)_i\}_{i=1}^{\infty}$ such that

$$\lim_{k\to\infty} \left(\prod_{i=k}^{1} H_{p(x)_i}\right)x = 0 \text{ and } \left|\left|\left(\prod_{i=k}^{1} H_{p(x)_i}\right)x\right|\right|_v < \epsilon$$

for each positive integer k.

Corollary 3: K is convergent if and only if K is asymptotically stable.

Proof: Clearly, asymptotic stability implies convergence. Assume, then, that K is convergent. Let $||\cdot||_v$ be a norm. K is pre-contractive relative to $||\cdot||_v$. Given $\epsilon > 0$, choose M, β, and B as in the proof of Theorem 1. Let $\delta = \frac{\epsilon}{B^M}$, and suppose $x \in R^m$ with $||x||_v < \delta$. Choose $\{p(x)_i\}_{i=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ as in the proof and we have, for each positive integer k,

$$||\prod_{i=k}^{1} H_{p(x)_i} x||_v \leq \beta^{m_k}||x||_v B^M < B^M\delta = \epsilon.$$

Therefore K is asymptotically stable. QED.

V. Criteria for Contractiveness--Positive Definite Sets of Matrices.

We now seek criteria for contractiveness and pre-contractiveness of a finite set of $m \times m$ matrices. We have some information concerning contractiveness for which we must define the notion of a positive definite set of matrices. Recall that a symmetric $m \times m$ matrix A is said to be <u>positive definite</u> provided $x^T Ax > 0$ for all non-zero x in R^m. Equivalently, A is positive definite provided that all the eigenvalues of A, which must be real because of the symmetry, are positive.

<u>Definition 6</u>: A set $W = \{A_1, A_2, \cdots, A_N\}$ of symmetric $m \times m$ matrices is a <u>positive definite set</u> provided that, if $x \in R^m$, $x \neq 0$, then there is an $i \in \{1,2,\cdots,N\}$ such that $x^T A_i x > 0$.

We will show that a finite set of $m \times m$ matrices is contractive relative to a certain norm if and only if a certain related set of matrices is a positive definite set.

<u>Definition 7</u>: If B is a symmetric $m \times m$ positive definite matrix and $x \in R^m$, the <u>B-norm of x</u> is defined by

$$||x||_B = \sqrt{x^T Bx} \ .$$

It is well-known that $||\cdot||_B$ is a norm on R^m. Notice that $B = I$ gives the Euclidean norm.

<u>Lemma 3</u>: If H is an $m \times m$ matrix, B an $m \times m$ positive definite matrix, and $A = B - H^T BH$, then $||x||_B{}^2 - ||Hx||_B{}^2 = x^T Ax$ for all $x \in R^m$, and so $\left\{x \in R^m \,\middle|\, ||Hx||_B < ||x||_B\right\} = \left\{x \in R^m \,\middle|\, x^T Ax > 0\right\}$.

<u>Proof</u>: Let $x \in R^m$. Then $||x||_B{}^2 - ||Hx||_B{}^2 =$

$$x^T Bx - (Hx)^T BHx = x^T (B - H^T BH)x = x^T Ax.$$

QED.

Definition 8: If $K = \{H_1, H_2, \cdots, H_N\}$ is a set of $m \times m$ matrices and B is a positive definite matrix, the B-symmetric set of K is the set

$$\left\{B - H_i^T B H_i \,\middle|\, 1 \le i \le N\right\}$$

of symmetric $m \times m$ matrices.

In [13], Stein proves:

Theorem: If H is a real or complex square matrix, a necessary and sufficient condition that $\lim_{n\to\infty} H^n = 0$ is that there exist a positive definite Hermitian matrix B for which $B - H^* B H$ is positive definite. If H is real, B may be taken real and symmetric.

Since contractiveness relative to some norm implies convergence, the following theorem contains a generalization of one direction of Stein's theorem.

Theorem 2: If $K = \{H_1, H_2, \cdots, H_N\}$ and B is a positive definite matrix, then K is contractive relative to $||\cdot||_B$ if and only if the B-symmetric set of K is a positive definite set.

Proof: Let $A_i = B - H_i^T B H_i$, so that $W = \{A_1, A_2, \cdots, A_N\}$ is the B-symmetric set of K. The following statements are equivalent:

K is contractive relative to B.
$x \in R^m$, $x \ne 0$, implies $||H_i x||_B < ||x||_B$ for some i.
$x \in R^m$, $x \ne 0$, implies $x^T A_i x > 0$ for some i.
W is a positive definite set.

QED.

Theorem 2 together with Corollary 1 yields:

Corollary 4: If $K = \{H_1, H_2, \cdots, H_N\}$, B is a positive definite matrix, and the B-symmetric set of K is a positive definite set, then K is exponentially convergent.

Thus we would like to know conditions related to positive definiteness of a set of symmetric matrices.

Recall the following theorem from matrix theory:

Theorem: If A is a symmetric $m \times m$ matrix, there is a unique diagonal matrix D of the form $D = \text{diag}\left\{1,1,\cdot\cdot,1,-1,-1,\cdots,-1,0,0,\cdots,0\right\}$ such that there is a non-singular matrix P with $D = P^T AP$. The matrix D can be computed from by several standard techniques, one at which is to repeatedly apply similar row and column operations to A. The product of the corresponding elementary column matrices gives a matrix P such that $D = P^T AP$.

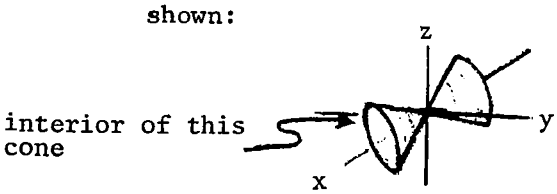
Definition 9: In the preceding theorem, D will be called the canonical form of A, any non-singular matrix P satisfying $D = P^T AP$ will be called a canonizing matrix for A, the number of 1's and -1's on the diagonal of D will be called the index and co-index of A, respectively.

The positive definiteness of a set of matrices will be studied by considering images under linear transformations of certain fixed sets in R^m. We now define these sets.

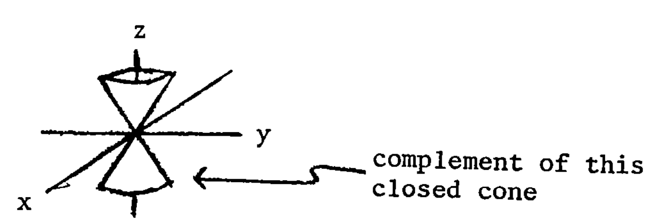
Definition 10: If $0 \le p$, $0 \le n$, and $p + n \le m$, then

$$S(p,n) = \left\{ y \in R^m \;\middle|\; \sum_{j=1}^{p} y_j^{\,2} > \sum_{j=p+1}^{p+n} y_j^{\,2} \right\}.$$

Notice that $0 \notin S(p,n)$, and $S(p,n)$ is closed under multiplication by non-zero scalars. Examples of some $S(p,n)$ for $m = 3$ are shown:



S(1,2)



S(2,1)

Theorem 3: If A is symmetric, P is a canonizing matrix for A, and p and n are the index and co-index, respectively, of A, then

$$\left\{x \in R^m \,\middle|\, x^T A x > 0\right\} = P\big(S(p,n)\big).$$

Proof: Let D be the canonical form of A, so that $P^T A P = D = \operatorname{diag}\left\{1,\cdots,1,-1,\cdots,-1,0,\cdots,0\right\}$ with p 1's and n -1's. Let $x \in R^m$ and let $y = P^{-1}x$. Then the following statements are equivalent:

$$x^T A x > 0.$$

$$x^T (P^{-1})^T D P^{-1} x > 0.$$

$$(P^{-1}x)^T D (P^{-1}x) > 0.$$

$$y^T D y > 0.$$

$$\sum_{j=1}^{p} y_j^{\,2} - \sum_{j=p+1}^{p+n} y_j^{\,2} > 0.$$

$$y \in S(p,n).$$

$$x \in P(S(p,n)).$$

QED.

Corollary 5: If $W = \left\{A_1, A_2, \cdots, A_N\right\}$ is a set of symmetric matrices, P_i is a canonizing matrix for A_i, p_i and n_i are the index and co-index of A_i, respectively, for $1 \le i \le N$, then W is a positive definite set if and only if

$$\bigcup_{i=1}^{N} P_i\big(S(p_i,n_i)\big) = R^m - \left\{0\right\}.$$

Proof: This follows immediately from Definition 6 and the theorem.

It will be recalled that a matrix A is positive definite if and only if every principal submatrix of A is positive definite. This fact generalizes to positive definite sets as follows.

Definition 11: If $1 \le t \le m$, G_t denotes the collection of all increasing t-termed sequences selected from $\{1,2,\cdots,m\}$. If $\alpha = \{\alpha_i\}_{i=1}^{t}$ is in G_t and C is an $m \times m$ matrix, $C[\alpha]$ denotes the principal $t \times t$ submatrix of C lying in the $\alpha_1,\alpha_2,\cdots,\alpha_t$ rows and $\alpha_1,\alpha_2,\cdots,\alpha_t$ columns of C. That is,

$$C[\alpha]_{ij} = C_{\alpha_i,\alpha_j} \text{ for } 1 \le i \le t,\ 1 \le j \le t.$$

Theorem 4: Let $W = \{A_1,A_2,\cdots,A_N\}$ be a set of symmetric $m \times m$ matrices. Then W is a positive definite set if and only if, for each t, $1 \le t \le m$, and each $\alpha \in G_t$, $W_\alpha = \{A_1[\alpha],\ A_2[\alpha],\ \cdots,\ A_N[\alpha]\}$ is a positive definite set.

Proof: Suppose W is a positive definite set. Let $1 \le t \le m$ and $\alpha \in G_t$, and let $x \in R^t$, $x \ne 0$. Define $y \in R^m$ by

$$y_k = \begin{cases} x_j \text{ if } k = \alpha_j,\ j = 1,2,\cdots,t \\ 0 \text{ if } k \notin \{\alpha_1,\alpha_2,\cdots,\alpha_t\}. \end{cases}$$

There is an $i \in \{1,2,\cdots,N\}$ such that $y^T A_i y > 0$. But then

$$x^T A_i[\alpha]x = \sum_{p=1}^{t} \sum_{q=1}^{t} A_i[\alpha]_{pq} x_p x_q =$$

$$\sum_{p=1}^{t} \sum_{q=1}^{t} (A_i)_{\alpha_p \alpha_q} y_{\alpha_p} y_{\alpha_q} = \sum_{j=1}^{m} \sum_{k=1}^{m} (A_i)_{jk} y_j y_k =$$

$y^T A_i y > 0$. Thus W_α is a positive definite set.

The reverse implication is seen by choosing $\alpha \in G_m$, $\alpha_i = i$ for $1 \le i \le m$, so that $W_\alpha = W$.

QED.

Finally, we relate positive definiteness of a set to the number of positive eigenvalues of the matrices in the set.

Lemma 4: If $V_1, V_2, \cdots, V_N$ are subspaces of R^m, then

$$\dim \bigcap_{i=1}^{N} V_i \geq \left(\sum_{i=1}^{N} \dim V_i\right) - (N-1)m.$$

Proof: Let $d_i = \dim V_i$ and $d_0 = \dim \left(\bigcap_{i=1}^{N} V_i\right)$. For $1 \leq i \leq N$, there is an $(m - d_i) \times m$ matrix A_i such that V_i is the null space of A_i. Let A be the block matrix given by

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix}$$

A is $\left(\sum_{i=1}^{N}(m-d_i)\right) \times m$; that is A is $\left(Nm - \sum_{i=1}^{N} d_i\right) \times m$. For $x \varepsilon R^m$, $Ax = 0 \Leftrightarrow A_i x = 0$ for $1 \leq i \leq N \Leftrightarrow x \varepsilon \bigcap_{i=1}^{N} V_i$. Thus d_0 is the nullity of A. Now $Nm - \sum_{i=1}^{N} d_i \geq$ rank of $A = m-d_0$, so

$$d_0 \geq \sum_{i=1}^{N} d_i + m - Nm .$$

That is, $\dim \left(\bigcap_{i=1}^{N} V_i\right) \geq \left(\sum_{i=1}^{N} \dim V_i\right) - (N-1)m.$

QED.

Theorem 5: If $W = \left\{A_1, A_2, \cdots, A_N\right\}$ is a positive definite set of $m \times m$ symmetric matrices and p_i = index of A_i for $1 \leq i \leq N$, then

$$\sum_{i=1}^{N} p_i \geq m .$$

Proof: For $1 \leq i \leq N$, let P_i be a canonizing matrix for A_i and let n_i be the co-index of A_i. Since W is a positive definite set, Corollary 5 gives

$$\bigcup_{i=1}^{N} P_i\left(S(p_i n_i)\right) = R^m - \left\{0\right\}.$$

Thus $\bigcap_{i=1}^{N} P_i\left(S\ (p_i, n_i)\right)^c = \left\{0\right\}$, where Q^c denotes the complement of Q in R^m.

Now, for each i, $S(p_i,n_i) \subset S(p_i,0)$, so $P_i\Big(S(p_i,n_i)\Big) \subset P_i\Big(S(p_i,0)\Big)$ and $P_i\Big(S(p_i,0)\Big)^c \subset P_i\Big(S(p_i,n_i)\Big)^c$.

$$(6) \quad \cdots \text{ Thus } \bigcap_{i=1}^{N} P_i\Big(S(p_i,0)\Big)^c \subset \bigcap_{i=1}^{N} P_i\Big(S(p_i,n_i)\Big)^c = 0 \ .$$

But $S(p_i,0)^c = \left\{ y \in R^m \,\middle|\, y_1 = y_2 = \cdots = y_{p_i} \right\} = 0$ is a subspace of R^m of dimension $m-p_i$. Since P_i is non-singular, $P_i\Big(S(p_i,0)\Big)^c = P_i\Big(S(p_i,0)^c\Big)$ is a subspace of R^m of dimension $m-p_i$.

Thus by (6) and Lemma 4 we have

$$0 = \dim\left(\bigcap_{i=1}^{N} P_i\Big(S(p_i,0)\Big)^c \right) \geq$$

$$\left(\sum_{i=1}^{N}(m-p_i)\right) - (N-1)m = m - \sum_{i=1}^{N} p_i \ , \text{ or } \sum_{i=1}^{N} p_i \geq m$$

QED.

Corollary 6: If $W = \left\{A_1, A_2, \cdots, A_N\right\}$ is a positive definite set of symmetric $m \times m$ matrices, then the total number of positive roots of A_1, A_2, $\cdots$, A_N, counting multiplicities, is not less than m.

Proof: This follows from the theorem and the fact that the index of A_i is the number of positive eigenvalues of A_i, counting multiplicities.

QED.

VI. Conclusions.

In an attempt to secure sufficient conditions on the feedback matrices F_{s_i} and the decision function d that will ensure stability, we have come only this far:

If the F_{s_i} are selected so that the set

$$K = \left\{ C_i + D_i F_{s_i} \,\middle|\, 1 \le i \le N \right\} \quad \text{(see section 3)}$$

has a positive definite Q-symmetric set for some positive definite matrix Q, then a decision function exists which will produce exponential convergence (see Corollary 1). One such decision function would be constructed along the lines of the algorithm given in example 1, section 4.

We have also shown that if the feedback matrices are selected so that the resulting set is convergent, then the resulting set produces asymptotic stability, and so Lyapunov stability. For convergence implies that the set $\left\{ C_i + D_i F_{s_i} \,\middle|\, 1 \le i \le N \right\}$ is pre-contractive (Theorem 1'), which implies asymptotic stability (Corollary 3).

Some of the questions which immediately suggest themselves are: (in the notation of the preceding sections)

1. Under what conditions on A, B, and S can the F be selected to achieve for some positive definite matrix Q, a positive definite Q-symmetric set, and how should the F_{s_i} be determined?
2. *Can we find more and better characterizations of positive definite* sets of matrices than those given in section 5? It would seem

that there might be some conditions on the canonizing matrices P_i of section 5 which would produce positive definiteness, using Corollary 5.

3. Can we find some useful conditions relating to pre-contractiveness? If K is pre-contractive, then for some positive integer M, the set of all products of M or fewer members of K is contractive, so perhaps the material in section 5 will be useful here, too.

4. Related to 3 is the problem of taking a pre-contractive set K and forming a contractive set with finite products of members of K. Such a process might lead to the development of decision functions producing stability.

5. In the formation of a contractive set from a pre-contractive one, as mentioned in 4, we may want to minimize the number of matrices in the resulting contractive set. Alternatively, we may want to minimize the number of matrices from the pre-contractive set which are used in forming the contractive one. It would be useful to have conditions on contractive and pre-contractive sets which guarantee that they are "minimal"; that is, that if any matrix is removed from the set, the set no longer has the desired property.

6. We have observed that if the set $K = \left\{ C_i + D_i F_{s_i} \,\middle|\, 1 \le i \le N \right\}$ is contractive, the construction of a decision function depending only on x_k is possible. In general, the decision function should depend only on x_k and T_{k-1}. If K is pre-contractive, we know that any initial x_1 can be steered to zero, but we do not yet know

whether there necessarily exists a decision function depending only on x_k and T_{k-1} that can do it. If not, are there conditions on a pre-contractive K which guarantee the existence of the decision function without implying contractiveness?

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