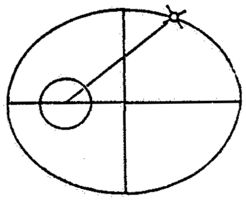
SPACE MATHEMATICS

A RESOURCE FOR TEACHERS

outlining supplementary space-related problems in mathematics



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outlining supplementary space-related problems in mathematics

Developed at Duke University
under the auspices of the
Department of Mathematics

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
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introduction

PURPOSE

No longer is a single textbook sufficient as a source of teaching material for high school mathematics classes. Although the majority of the recent curriculum-revision projects have tended to emphasize primarily the development of pure mathematics, there is a growing awareness that students should also know something about the applications of mathematics. The National Aeronautics and Space Administration has recognized the appeal of aerospace activities, and has initiated and supported the development of curriculum supplements for several high school courses. It is hoped that they will fill a need felt by many teachers.

Because the present attainments in aerospace would not be possible without mathematics, it is most appropriate that supplementary publications dealing with space activities be made available to teachers of mathematics. It is our hope that students will become more interested in mathematics as the result of seeing some of its significant current space-related applications. Working problems such as those in this book should enhance both the mathematical knowledge and skill of the student and his appreciation and understanding of space technology.

CONTENT AND ORGANIZATION

SPACE MATHEMATICS, A Resource for Teachers consists of a collection of mathematical problems related to space science. Because the emphasis is on the mathematics, the problems have been grouped according to mathematical topics. A minimum amount of attention has been given to the development of theory. In general, the new formulas that are necessary for understanding the text have been quoted but not derived. In some cases, as in Chapter 10, formulas have been derived from more basic equations. The theory that has been presented is explained only to the extent needed to make the problem understandable. A rigorous discussion of the principles of astronautics is beyond the scope of this book, and would in fact be inconsistent with the purpose of the book. It is largely for these reasons that no calculus problems appear in the text. The development of the theoretical basis of spaceflight depends heavily upon calculus, but the level of sophistication required is far above the high school level. The reader who is interested in this type of material will find it in other publications, several of which are listed in the Bibliography.

The problems range in difficulty from very easy ninth grade level ones to very challenging twelfth grade level applications. Within the chapters the problems are arranged roughly in order of increasing difficulty. Solutions are provided for all problems. A list of topics in mathematics presented in the text can be found in the Table of Contents, and the types of problems in each topic are listed in the introductions to the individual chapters.

The problems were written by various writers. During the process of compiling and editing the material, an attempt was made to retain, whenever possible, the style of the individual writer. Thus styles and arrangement vary somewhat from problem to problem and from chapter to chapter. Although some sets of problems are sequential, the authors have tried for the most part to make each problem self-explanatory. The lack of continuity in content throughout most of the book should not be disturbing. Actually it enables a teacher to select problems at will without preliminary study.

NOTATION AND COMPUTATION

For ease of reading, the most conventional notations and language have been used. The variety of writers, however, introduces certain inconsistencies that might bother a reader who is not prepared for them. First, notation is only locally consistent. In problems dealing with rocket propulsion, for instance, the letter c is used consistently to denote the exhaust velocity of a rocket. In other problems, however, such as those concerning relativity, c is used to denote the speed of light. In both cases, standard notation is followed. Similarly, e is used to denote both the eccentricity of a conic section and the base of natural logarithms. As a subscript, it may refer to Earth (as in r_e, the radius of Earth) or escape (as in v_e, the escape velocity). The reader should be able to interpret such symbols correctly from context.

Second, the way in which units are handled is not consistent. In many problems, especially those in the early part of the book, the units are carried throughout the entire course of a computation. Such a practice should be encouraged initially, and it can be very helpful when the data involve an assortment of units. When, however, it is clear what the appropriate units are, they are often withheld until the final answer.

Third, the matter of accuracy can become a thorny matter. Naturally it is not intended that the equation $10^3 \times \sqrt{13} = 3{,}600$ be interpreted literally. The irrational number on the left-hand side cannot possibly be equal to the integer which stands on the right-hand side. In this case, 3,600 is the value of $10^3 \times \sqrt{13}$ rounded in such a way as to be consistent with the accuracy of the data given in the statement of the problem. Unfortunately there are no simple, sure-fire rules for the rounding of answers — which is not to say that correct rounding is unimportant. In careful scientific work, great attention often must be paid to error analysis. It is usually not enough to determine a numerical value for a quantity, but one must also determine its degree of accuracy. A distance, for instance, might be quoted as 3.71 ± 0.02 centimeters, rather than 3.71 centimeters. The estimation of errors is frequently a complicated and tedious task. The authors have deliberately shied away from such tasks, partly to make the computations less burdensome, but mainly because such considerations could detract from the real point of a problem. In summary, some equal signs must be taken with a grain of salt. Perhaps the only general rule which we can state is that one should not expect greater accuracy in the answer than he has in the data.

As those interested in the teaching of mathematics, whether they be classroom teachers, supervisors, curriculum specialists, or textbook writers, may have noted, this publication is essentially a supplement to the several courses in mathematics, grades 9 through 14. It is neither a text nor a syllabus; it is a rich resource of real problems through which it is hoped that students, because of their interest in aerospace, may be motivated toward a better understanding of mathematics as well as of the space program in general.

ACKNOWLEDGMENTS

The writers are grateful to the many NASA scientists who discussed their research with them and made helpful suggestions. In particular, S. Walter Hixon, Jr., former Head of Training and Educational Services, NASA Langley Research Center, deserves thanks for his cooperation in serving as host to the writers during various visits to the Center and for introducing them to scientists there.

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chapter 1

chapter 1

CONVERSION FACTORS, NOTATION, AND UNITS OF MEASUREMENT

The conversion factors presented in this chapter will be new to many teachers of mathematics. As will be apparent from the problems presented, conversion from one set of units to another is often made easier by the use of conversion factors. The notation involved in using conversion factors and in some problems in other parts of the book will be new and perhaps controversial. The procedure of writing the units into the computation and then dividing, multiplying, adding, and subtracting units as if they were numbers is not often used in mathematics.

Some physics and engineering textbooks do use the "factor label" technique. Sometimes the use of this technique offers the best way for one to know what units are involved in the final answer. If the engineer does not write the units into the equation, he goes through a similar process mentally or on scratch paper. The evidence is that as the engineer gains experience in a given field, he finds it less and less necessary to make the units part of the computation. It should be understood that this labeling technique is not new, and it has no connection with the space program itself, except as the individual engineer or scientist finds it useful.

The chapter also introduces a few of the units of measure used in space technology and incidentally provides some information on temperatures, distances, velocities, and the like that are characteristic of space exploration.

Measurements expressed in one set of units can be converted to another set by using conversion factors. A conversion factor expresses the relationship between two units as a ratio equal to 1. Therefore, multiplication or division by this factor does not alter the size of the original expression.

To obtain conversion factors we can begin with an equation which expresses the relationship between two units. Division of both members of the equation by either the left or right member results in a conversion factor. Thus, beginning with

$$1 \text{ yd} = 3 \text{ ft},$$

we may obtain either

$$\frac{1 \text{ yd}}{3 \text{ ft}} = 1$$

or

$$1 = \frac{3 \text{ ft}}{1 \text{ yd}}.$$

Some uses of these conversion factors are shown in the problems.

PROBLEMS

1. When a spacecraft returns from the Moon, lunar gravity will slow it down until it enters the sphere of Earth's gravitational influence. Then Earth's gravity will cause it to accelerate until it reaches a speed of nearly 25,000 miles per hour. Convert this speed to feet per second, using the relationship

$$60 \text{ mi/hr} = 88 \text{ ft/sec}.$$

Solution. We are given

$$v = 25{,}000 \text{ mi/hr}.$$

Using a conversion factor from the given relationship, we get

$$\begin{aligned} v &= 25{,}000 \text{ mi/hr} \times \frac{88 \text{ ft/sec}}{60 \text{ mi/hr}} \\ &= \frac{2{,}200{,}000}{60} \text{ ft/sec} \\ &= 36{,}700 \text{ or } 37{,}000 \text{ ft/sec}. \end{aligned}$$

2. The speed of light is about 186,300 miles per second.

a. Calculate its speed in miles per hour.

Solution. Using two conversion factors, we obtain

$$\begin{aligned} v &= 186{,}300 \frac{\text{mi}}{\text{sec}} \times \frac{60 \text{ sec}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hr}} \\ &= (1.863 \times 10^5)(6.0 \times 10)(6.0 \times 10) \text{ mi/hr} \\ &= 6.707 \times 10^8 \text{ mi/hr}. \end{aligned}$$

b. Calculate the number of miles in 1 light year, the distance light can travel in 1 year. Use 365 days = 1 year.

Solution. Using two conversion factors, we obtain

$$\begin{aligned} 1 \text{ light yr} &= 6.707 \times 10^8 \text{ mi/hr} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{365 \text{ days}}{1 \text{ yr}} \\ &= 5.875 \times 10^{12} \text{ mi/yr} = 5.88 \times 10^{12} \text{ mi}. \end{aligned}$$

3. A typical altitude for manned spacecraft about Earth is 100 miles because this is the lowest altitude at which air resistance becomes small enough to make a stable orbit possible. Because the speed in a circular orbit at this altitude is about 17,500 miles per hour, this speed is sometimes quoted as a typical one for space travel. How many years would it take for a spaceship to travel 1 light year if its rate is 17,500 miles per hour?

Solution. Solving the distance-time-rate equation, $d = vt$, for t, we obtain

$$t = \frac{d}{v}$$

$$= \frac{5.88 \times 10^{12}\text{ mi}}{1.75 \times 10^{4}\text{ mi/hr}} = 3.36 = 10^{8}\text{ hr.}$$

Converting 3.36×10^8 hours to years yields

$$3.36 \times 10^{8}\text{ hr} = 3.36 \times 10^{8}\text{ hr} \times \frac{1\text{ day}}{24\text{ hr}} \times \frac{1\text{ yr}}{365\text{ days}}$$

$$= 3.36 \times 10^{8}\text{ hr} \times \frac{1\text{ yr}}{8{,}760\text{ hr}}$$

$$= 3.84 \times 10^{4}\text{ yr}$$

$$= 38{,}400\text{ yr.}$$

4. If a spacecraft were to escape from our solar system, it would need, if departing at a distance equal to Earth's distance from the Sun, a speed of 94,200 miles per hour or more. Because Earth is moving about the Sun at the rate of 66,600 miles per hour, the spacecraft could be given the required speed if launched from Earth in the direction of the Earth's motion about the Sun with a speed of 27,600 miles per hour relative to Earth. Suppose that a spacecraft of sufficient size can be given this initial speed, and that in addition a source of propulsion on board will enable it to maintain 94,200 miles per hour as an average speed. How long would it take the spacecraft to reach the nearest star, Alpha Centauri, which is 4.3 light years away?

NOTE: Average speeds have little meaning in the operation of spacecraft. Speed is constantly changing as a result of propulsion and gravity forces. Only if a spacecraft were located out in space, far from any significant gravity field, could it coast with a nearly constant speed.

Solution.

$$t = \frac{d}{v}$$

$$= \frac{4.3 \times 5.88 \times 10^{12}\text{ mi}}{94{,}200\text{ mi/hr}}$$

$$= \frac{25.28 \times 10^{12}}{9.42 \times 10^{4}}\text{ hr}$$

$$= 2.68 \times 10^{8}\text{ hr.}$$

Converting 2.68×10^8 hours to years yields, using the computation from the previous problem,

$$\begin{aligned} 2.68 \times 10^8 \text{ hr} &= 2.68 \times 10^8 \text{ hr} \times \frac{1 \text{ yr}}{8{,}760 \text{ hr}} \\ &= 3.06 \times 10^4 \text{ yr} \\ &= 3.1 \times 10^4 \text{ yr.} \end{aligned}$$

NOTE: In this problem, the value 4.3 for the number of light years has only two significant digits. Therefore, for consistency in our computation, we must round the final answer to two significant digits.

5. The average radii of Earth and the Moon are approximately 6,371 and 1,738 kilometers, respectively.

a. What is the ratio of the volume of Earth to the volume of the Moon?

Solution. Using the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, we get

$$\frac{V_e}{V_m} = \frac{\frac{4}{3}\pi(6{,}371 \text{ km})^3}{\frac{4}{3}\pi(1{,}738 \text{ km})^3} = \frac{6{,}371^3}{1{,}738^3} = 49.3.$$

Thus the volume of Earth is 49.3 times as large as the volume of the Moon.

b. If the volume of Earth is 1.082×10^{21} cubic meters, what is the volume of the Moon?

Solution. Because the volume of Earth is 49.3 times as large as that of the Moon, the volume of the Moon is

$$\frac{1.082 \times 10^{21} \text{ m}^3}{49.3} = 2.19 \times 10^{19} \text{ m}^3.$$

6. The temperature on the surface of the Moon is thought to vary from a low of 120° K to a high of 383° K. (The Kelvin temperature scale has the same size degree as the Celsius, or centigrade, scale but is measured from absolute zero as the starting point. Students who are not acquainted with the different temperature scales may get information by reading appropriate reference books.) What are the extremes of temperature on the Moon, expressed in degrees Celsius and Fahrenheit?

Solution. Changing 120° K to degrees Celsius, we find

$$120° - 273° = -153°.$$

Thus the Celsius temperature is −153° C. Converting this temperature to degrees Fahrenheit, we have

$$\left(\frac{180^\circ \text{ F}}{100^\circ \text{ C}}\right)(-153^\circ \text{ C}) + 32^\circ \text{ F} = -243^\circ \text{ F}.$$

Changing 383° K to degrees Celsius, we find

$$383^\circ - 273^\circ = 110^\circ.$$

Thus the Celsius temperature is 110° C. Converting this temperature to degrees Fahrenheit, we have

$$\left(\frac{180^\circ \text{ F}}{100^\circ \text{ C}}\right)(110^\circ \text{ C}) + 32^\circ \text{ F} = 230^\circ \text{ F}.$$

Hence the temperature on the Moon varies from –153° to 110° C or from –243° to 230° F.

7. The temperature of liquid hydrogen, the propellant used in the second and third stages of the Saturn V launch vehicle, is about –253° C. What would this temperature read on the Fahrenheit scale?

Solution. Converting –253° C to degrees Fahrenheit, we find

$$\left(\frac{180^\circ \text{ F}}{100^\circ \text{ C}}\right)(-253^\circ \text{ C}) + 32^\circ \text{ F} = -423^\circ \text{ F}.$$

8. The temperature of the surface of the Sun has been computed to be 5,800° K. What temperature is this on the Celsius and Fahrenheit scales?

Solution. Changing 5,800° K to degrees Celsius, we get

$$5{,}800^\circ - 273^\circ = 5{,}527^\circ.$$

Thus the Celsius temperature is 5,527° C. Converting 5,527° C to degrees Fahrenheit, we have

$$\left(\frac{180^\circ \text{ F}}{100^\circ \text{ C}}\right)(5{,}527^\circ \text{ C}) + 32^\circ \text{ F} = 9{,}981^\circ \text{ F}.$$

9. Sounding rockets have reported the lowest temperature ever measured for Earth's atmosphere. U.S.-Swedish cooperative sounding rocket studies conducted from Swedish Lapland found temperatures as low as –225° F in the upper atmosphere. What is this temperature in degrees Celsius?

Solution. Using the conversion factor for changing Fahrenheit to Celsius, we get

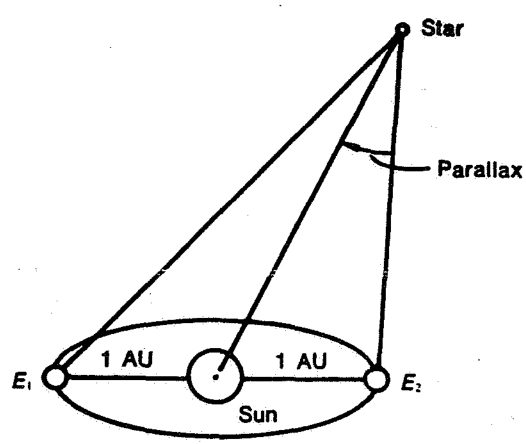
$$(-225^\circ\ \mathrm{F} - 32^\circ\ \mathrm{F})\left(\frac{100^\circ\ \mathrm{C}}{180^\circ\ \mathrm{F}}\right) = (-257)\left(\frac{5}{9}\right)^\circ\ \mathrm{C}$$
$$= -143^\circ\ \mathrm{C}.$$

10. The astronomical unit (AU) is the average distance of Earth from the Sun. One AU equals approximately 92,960,000 miles. How many astronomical units are there in a light year?

Solution. Using the conversion factor derived in problem 2b, we find

$$5.88 \times 10^{12}\ \mathrm{mi} = 5.88 \times 10^{12}\ \mathrm{mi} \times \frac{1\ \mathrm{AU}}{9.296 \times 10^{7}\ \mathrm{mi}}$$
$$= 6.33 \times 10^{4}\ \mathrm{AU}.$$

11. The parsec is a unit of distance used to measure the great distances to stars. Two observations of a distant star with respect to a fixed, more distant star field are made at 6-month intervals (see figure) when Earth is on opposite sides of its orbit around the Sun. The distance between the observation points E_1 and E_2 is 2 AU. The star in question is 1 parsec distant from Earth if the parallax angle, one-half the angle subtending an arc of 2 AU, is 1 second. The arc length and the chord of the arc are close to being equal, and are considered to be the same. This is why the parallax angle is the angle Sun-star-Earth at E_2. The farther away the star is from Earth, the smaller the parallax angle will be.



How many astronomical units are there in 1 parsec? Use the approximation 3.14159 for π.

Solution. In a complete circle, there are 2π radians, which equal $360 \times 60 \times 60$ seconds. Thus, if the parallax angle θ equals 1 second, then,

$$\theta = \frac{2\pi}{360 \times 60 \times 60} \text{ radian} = \frac{\pi}{648{,}000} \text{ radian.}$$

$$\begin{aligned} r = \frac{s}{\theta} &= \frac{1 \text{ AU}}{\pi/648{,}000} \\ &= \frac{648{,}000}{\pi} \text{ AU} \\ &= 206{,}265 \text{ AU.} \end{aligned}$$

In actual use, the length of a parsec is often rounded to 206,300 AU.

12. How many light years are there in a parsec?

Solution. By the preceding two problems, there are 206,300 AU in 1 parsec, and 63,300 AU in 1 light year. Therefore,

$$\begin{aligned} 1 \text{ parsec} &= \frac{206{,}300}{63{,}300} \text{ light yr} \\ &= 3.26 \text{ light yr.} \end{aligned}$$

13. Among the planets of the solar system, Pluto is the most distant from the Sun. Its maximum distance from the Sun is about 4.60 billion miles.

a. How long does it take the light of the Sun to reach Pluto at this distance?

Solution. Using the distance-rate-time equation, we have

$$\begin{aligned} t &= \frac{4.60 \times 10^9 \text{ mi}}{1.863 \times 10^5 \text{ mi/sec}} \\ &= 2.47 \times 10^4 \text{ sec} \\ &= 6 \text{ hr } 52 \text{ min.} \end{aligned}$$

b. What is the maximum distance from the Sun to Pluto in terms of astronomical units? Use the conversion factors previously derived.

Solution.

$$\begin{aligned} 4.60 \times 10^9 \text{ mi} &= 4.60 \times 10^9 \text{ mi} \times \frac{1 \text{ AU}}{9.296 \times 10^7 \text{ mi}} \\ &= 49.5 \text{ AU.} \end{aligned}$$

c. Find the distance in terms of light years.

Solution.

$$
\begin{aligned}
4.60 \times 10^{9} \text{ mi} &= 4.60 \times 10^{9} \text{ mi} \times \frac{1 \text{ light yr}}{5.88 \times 10^{12} \text{ mi}} \\
&= 0.782 \times 10^{-3} \text{ light yr} \\
&= 0.000782 \text{ light yr} \left(\text{less than } \frac{1}{1000}\right).
\end{aligned}
$$

14. The chances of penetration of space vehicles by meteoroids has recently been shown to be several thousand times lower than estimated several years ago. Except for travel in the asteroid belt, it would appear that the meteoroid problem would rank low as a criterion in the selection of space-cabin materials. A recent estimate is that the shortest average interval of time between perforations of an aluminum skin 10^{-1} centimeter thick is 1.0×10^{8} seconds. Compute the number of years between perforations.

Solution. Converting 1.0×10^{8} seconds to years, we have

$$
\begin{aligned}
1.0 \times 10^{8} \text{ sec} &= 1.0 \times 10^{8} \text{ sec} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{1 \text{ hr}}{60 \text{ min}} \times \frac{1 \text{ day}}{24 \text{ hr}} \times \frac{1 \text{ yr}}{365 \text{ days}} \\
&= \frac{1.0 \times 10^{8} \text{ sec}}{3.15 \times 10^{7} \text{ sec/yr}} \\
&= 3.17 \text{ or } 3.2 \text{ yr}.
\end{aligned}
$$

NOTE: This estimate is pessimistic because it gives the minimum number of years expected between perforations. A more optimistic estimate is one perforation every 100 years.

chapter 2

chapter 2

ELEMENTARY ALGEBRA

This chapter contains problems which in general require only algebra, and is limited largely to equations of the first degree. Algebraic problems of a more advanced nature are provided in Chapter 3, "Ratio, Proportion, and Variation," and Chapter 4, "Quadratic Equations." In other chapters, algebra is used to solve problems involving probability, exponential and logarithmic functions, geometry, and trigonometry.

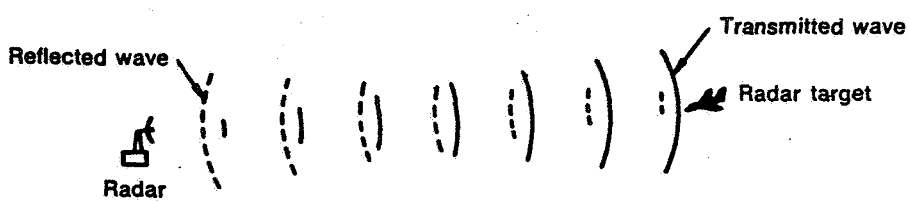
The chapter presents problems related to radio transmission, Mach number, launch and reentry velocities and accelerations, pumping rate of an astronaut's heart during launch, barycenter, periods of certain planets, and sidereal and synodic period of a satellite.

These problems are designed to demonstrate that even very elementary algebraic formulas are useful in the space and technological age.

PROBLEMS

1. A radar transmits pulses of electromagnetic waves, which travel at the speed of light, approximately 186,300 miles per second. Directional antennas radiate the energy in narrow beams. If the radiated waves strike an object such as a plane, ship, or rocket, some of the energy may be reflected back to the radar. The indicator on the radar usually is calibrated to convert the time between transmission and reception into units of distance.

Given that t is the length of time for a pulse of energy to be both transmitted by and reflected back to the radar, c is the speed of light, and d is the distance between the radar and the object, devise a formula for the distance d.



Solution. The total time of travel for the pulse will be the time it takes

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to get to the target plus the time it takes to rebound to the radar station; that is,

$$t = t_1 + t_2.$$

Because t_1 and t_2 are equal, we can say

$$t_1 = \frac{1}{2}t.$$

By the distance-rate-time equation, we have

$$t_1 = \frac{d}{c} = \frac{1}{2}t.$$

Hence the required formula is

$$d = \frac{1}{2}ct.$$

2. In 0.01 second after transmission, a Texas radar station receives a reflection from a Saturn rocket.

a. How many seconds did it take for the pulse to reach the rocket?

Solution. Since the pulse must travel a certain time before hitting the rocket and then must return along the same path in the same amount of time, it must take half the total time for the pulse to reach the rocket, or

$$\frac{1}{2}(0.01 \text{ sec}) = 0.005 \text{ sec}.$$

b. How far away is the rocket from the radar station?

Solution. Substituting the time calculated in part a and the speed of light into the distance-rate-time equation, we have

$$\begin{aligned} d &= ct \\ &= (186{,}300 \text{ mi/sec})(0.005 \text{ sec}) \\ &= 932 \text{ mi}. \end{aligned}$$

3. A 10- by 10-foot-square supersonic wind tunnel is operated at Mach 3. Find the volume of airflow per second through the wind tunnel.

NOTE: The speed of sound is dependent upon temperature. In the wind tunnel where the temperature is approximately 212° F, the speed of sound is 1,200 feet per second.

Solution. Under the given conditions, the speed of sound, Mach 1, is 1,200 feet per second; therefore

$$\text{Mach } 3 = 3{,}600 \text{ ft/sec}.$$

The cross-sectional area of the wind tunnel is 10 by 10 feet or 100 square feet, and the volume of airflow per second is

$$\begin{aligned} \text{Volume/sec} &= (\text{distance air travels/sec})(\text{cross-sectional area}) \\ &= (3{,}600 \text{ ft/sec})(100 \text{ ft}^2) \\ &= 360{,}000 \text{ ft}^3/\text{sec} \\ &= 3.6 \times 10^5 \text{ ft}^3/\text{sec}. \end{aligned}$$

4. A meteorite crashed to Earth in Siberia on February 12, 1947. It was found to contain a number of elements. There were 70 pounds of iron, 20 pounds of calcium, and 30 pounds of unknown material. What was the percentage by weight of unknown material?

Solution. In calculating the total weight of the meteorite, we get

70 lb iron
20 lb calcium
30 lb unknown
120 lb total

To obtain the percentage by weight of the unknown material, the fraction $\frac{30 \text{ lb}}{120 \text{ lb}}$ is multiplied by $\frac{100}{100}$, which gives the following result:

$$\frac{30 \text{ lb}}{120 \text{ lb}} \times \frac{100}{100} = \frac{1/4 \times 100}{100} = 25 \text{ percent of unknown material.}$$

5. A two-stage rocket is fired vertically and has a speed v_0 when the second-stage motor ignites, providing an average acceleration a. Two seconds after the second-stage ignition, the speed of the rocket is 1,700 feet per second, and after 5 seconds it is 2,900 feet per second. (Note that we are concerned only with the time that elapses after second-stage ignition.) Find a and v_0, given that the final speed is equal to the initial speed plus the product of acceleration and time.

Solution. Applying the given equation yields

$$2{,}900 = v_0 + a(5)$$

and

$$1{,}700 = v_0 + a(2).$$

Subtracting yields

$$1{,}200 = a(5 - 2)$$

$$a = 400 \text{ ft/sec}^2.$$

Substituting this value of a into either of the preceding equations yields $v_0 = 900$ feet per second.

NOTE: See the comment in Chapter 1, problem 4, regarding the use of averages.

6. A scientific capsule was carried aloft and released at the peak of the trajectory by a rocket that had an average vertical speed of 570 miles per hour. The capsule made a controlled descent with an average vertical speed of 240 miles per hour and landed 67.5 minutes after the rocket was launched. Find the maximum height reached by the rocket.

Solution. First the time of ascent in hours t is found by equating expressions for the distance of ascent and descent:

$$\begin{aligned} 570t &= 240\left(\frac{67.5}{60} - t\right) \\ &= 270 - 240t \\ t &= \frac{1}{3}. \end{aligned}$$

Thus the rocket reached the peak of its trajectory 1/3 hour after launch. The maximum height reached is equal to the product of the speed and time; i.e., (570 miles per hour) (1/3 hour) or 190 miles.

7. During a spacecraft launching, an astronaut's heart pumps blood at a rate of 10 pints per minute greater than when he is sitting in normal conditions. Under launching conditions his blood makes two times as many complete circulations in 8 minutes as when normally sitting. The astronaut's body contains 10 pints of blood. Find the rate his heart pumps during launching.

Solution. Letting $L =$ rate during launch and $S =$ rate at normal sitting, we have

$$S = (1/2)L$$

and

$$\begin{aligned} L &= S + 10 \\ &= (1/2)L + 10 \\ &= 20. \end{aligned}$$

The rate during launch is then 20 pints per minute.

8. The planets Earth, Jupiter, Saturn, and Uranus revolve around the Sun approximately once each 1, 12, 30, and 84 years, respectively.

a. How often will Jupiter and Saturn appear in the same direction in the night sky, as seen from Earth?

Solution. Let J and S represent the periods of revolution of Jupiter and Saturn, respectively. In 1 Earth year, Jupiter revolves $(1/12)J$ and Saturn revolves $(1/30)S$. Equating these times yields $(1/12)J = (1/30)S$, or $5J = 2S$. Hence Jupiter makes five revolutions, while Saturn completes two revolutions. Thus the planets will appear in the same direction, as seen from Earth in the night sky, once every 5(12 years) = 2(30 years) = 60 years.

Alternate solution. It is clear that the time required is a multiple of 30 years because the period of revolution of Saturn is 30 years. Hence, the result is the least common multiple of the two periods, which is 60 years.

b. About how often will the planets Jupiter, Saturn, and Uranus all appear in the same direction in the night sky as seen from Earth?

Solution. Utilizing the method of part a, we have

$$(1/12)J = (1/30)S = (1/84)U,$$

or

$$35J = 14S = 5U.$$

Substituting for J, S, and U, we have

$$35(12 \text{ yr}) = 14(30 \text{ yr}) = 5(84 \text{ yr}) = 420 \text{ yr}.$$

Alternate solution. The least common multiple of 12, 30, and 84 years is 420 years. Thus the three planets will appear as described once every 420 years.

9. The huge 10-story-high Echo satellite was designed to reflect radio waves back to Earth. To be a good reflector, the spherical satellite required a diameter that was at least as large as the wavelength λ of the wave reflected, that is

$$\frac{D}{\lambda} \geq 1.$$

Determine the minimum diameter D in meters needed for the satellite to be a good reflector of waves with frequency of 10^7 hertz. (The hertz is the new unit recently adopted for indicating the frequency in cycles per second. One hertz is 1 cycle per second.) The length of a wave is the distance traveled by a series of waves during a given time divided by the frequency or number of waves propagated during that time. That is,

$$\lambda = \frac{c}{f}$$

where c is the speed of light, 3×10^8 meters per second, and f is the frequency. Also find the surface area and volume of the Echo satellite.

Solution. The wavelength is

$$\begin{aligned}\lambda &= \frac{c}{f} \\ &= \frac{3 \times 10^8 \text{ m/sec}}{10^7 \text{ Hz}} \\ &= 30 \text{ m}.\end{aligned}$$

Substituting this value into the diameter relation gives

$$\frac{D}{30\ \text{m}} \geq 1$$
$$D \geq 30\ \text{m}.$$

Hence the Echo satellite must be at least 30 meters in diameter. Assuming this value to be the approximate diameter, the surface area is

$$\begin{aligned} S &= 4\pi r^2 \\ &= 4\pi(15\ \text{m})^2 \\ &= 2{,}828 \text{ or } 2{,}800\ \text{m}^2. \end{aligned}$$

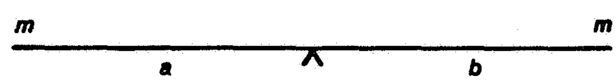
The volume is

$$\begin{aligned} v &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(15\ \text{m})^3 \\ &= 14{,}137 \text{ or } 14{,}000\ \text{m}^3. \end{aligned}$$

10. The moment of a mass is the product of the mass and the distance of the mass from the center of rotation. The point at which the sum of the moments is zero is called the center of mass. This point is usually called the barycenter. An example is the fulcrum at which a teeter-totter is balanced.

a. Determine the center of mass of two equal point masses.

Solution. Let mass m be located the distance a from the center of mass, while another equal mass m is located at distance b from the center of mass, as indicated in the drawing.



Then, if the moments are in balance,

$$\begin{aligned} ma &= mb \\ a &= b \end{aligned}$$

Hence the center of mass, or barycenter, is midway between two equal masses.

b. If M is the mass of Earth, the mass of the Moon is about $M/81$. The distance between the centers of Earth and the Moon is about 239,000 miles. Find the location of the barycenter of the Earth-Moon system.

Solution. Let x be the number of miles between the barycenter and the center of Earth. Then

$$Mx = (239{,}000 - x)\frac{M}{81}$$
$$81x = 239{,}000 - x$$
$$x = \frac{239{,}000}{82}$$
$$= 2{,}915 \text{ or } 2{,}900 \text{ mi.}$$

Thus the barycenter is below the surface of Earth, about 2,900 miles from its center.

c. During the rotation of Earth about the Sun, the Earth-Moon barycenter follows a path about the Earth-Moon-Sun barycenter. The Sun is about 332,500 times more massive than Earth. The distance between the center of the Sun and the Earth-Moon barycenter is about 93 million miles. Find the location of the Earth-Moon-Sun barycenter.

Solution. Let M' be the mass of the Earth-Moon system. By simple arithmetic, $M = (81/82)M'$, then the mass of the Sun will be

$$332{,}500M = \frac{332{,}500 \times 81 \times M'}{82} = 328{,}450M'.$$

If x is the number of miles between the center of the Sun and the Earth-Moon barycenter,

$$(328{,}450M')x = M'(93{,}000{,}000 - x)$$
$$328{,}450x = 93{,}000{,}000 - x$$
$$x = \frac{93{,}000{,}000}{328{,}451}$$
$$= 283 \text{ or } 300 \text{ mi.}$$

Thus the barycenter of the Earth-Moon-Sun system is inside the Sun about 300 miles from its center. (This solution assumes that the mass of each body is concentrated at its center. Thus the figure is not precise, and gives us only a rough idea of the location of the barycenter.)

11. The time required for an orbiting satellite to make one complete revolution of Earth is called its period. The length of the period depends upon the location of the observer making the measurement.

Suppose the observer is located far out in space and views the satellite against the background of fixed stars. The period measured in this manner is called the *sidereal* period of revolution or "the period in relation to the stars." Note that the rotation of Earth does not affect the sidereal period.

Now suppose the observer is on Earth standing on the Equator. A satellite in low Earth orbit moving directly eastward is overhead. When the satellite has made one complete transit of its orbit, it will be behind the observer because the rotation of Earth will have carried him a distance eastward. The satellite must travel an additional distance to again be directly over the head of the observer. The observer measures the period of the satellite as the time elapsing between successive passes directly overhead. This is referred to as the *synodic* period of revolution or "the period between successive conjunctions." The synodic period takes into account the rotation of Earth. It is greater than the sidereal period if the satellite travels in an easterly direction.

The Gemini 7 spacecraft with astronauts Borman and Lovell aboard completed 206 synodic periods with respect to Cape Kennedy and 220 sidereal periods with respect to a fixed point in space during its 14-day mission.

The sidereal period in seconds can be computed by the formula $P = 2\pi\sqrt{a^3/GM}$ where a is the average radius of orbit measured in miles from the center of the body about which the satellite is in orbit. (The derivation of this formula is given in Chapter 10.) G is the constant of universal gravitation and M is the mass of the body about which the satellite orbits. The average radius of Earth is 3,960 miles, and for the Moon it is 1,080 miles. When the units of measurement are miles and seconds, the product GM is 9.56×10^4 for Earth and 1,176 for the Moon.

Usually spacecraft orbit in the same easterly direction as Earth's rotation and are said to be in a *posigrade* orbit. All U.S. manned spaceflights have been launched in posigrade orbits to take advantage of the additional velocity imparted to the spacecraft by Earth's rotation.

If the direction of orbiting is westerly, or opposite to Earth's rotation, the orbit is said to be *retrograde*. In this case an Earth observer would meet the satellite before it made one complete revolution around Earth. Accordingly, the synodic period would be less than the sidereal period.

a. Find the sidereal period of a satellite with an average altitude above Earth of 100 miles.

Solution. The radius of orbit is equal to the radius of Earth plus the average altitude of the satellite, or

$$\begin{aligned} a &= 3{,}960 \text{ mi} + 100 \text{ mi} \\ &= 4{,}060 \text{ mi}. \end{aligned}$$

Hence the sidereal period in seconds is

$$\begin{aligned} P &= 2(3.14)\sqrt{\frac{(4{,}060)^3}{9.56 \times 10^4}} \\ &= (6.28)(4{,}060)\sqrt{\frac{4{,}060}{9.56 \times 10^4}} \\ &= (25{,}500)(10^{-2})\sqrt{425} \\ &= 5{,}258. \end{aligned}$$

Thus the sidereal period is 5,258 seconds = 87.6 minutes = 1.46 hours.

b. Find the sidereal period of Lunar Orbiter 3, which traveled an orbit of 89 by 196 miles above the Moon's surface.

Solution. The average radius of the orbit is the average radius of the Moon plus the average altitude of the satellite, or

$$\begin{aligned} a &= 1{,}080 \text{ mi} + \frac{1}{2}(89 + 196) \text{ mi} \\ &= 1{,}223 \text{ mi}. \end{aligned}$$

Therefore the sidereal period is

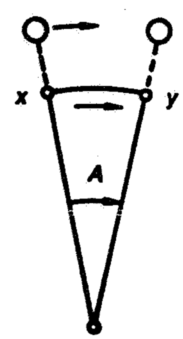
$$\begin{aligned} P &= 2(3.14)\sqrt{\frac{(1{,}223)^3}{1{,}176}} \\ &= (6.28)(1{,}223)\sqrt{\frac{1{,}223}{1{,}176}} \\ &= 7{,}680\sqrt{1.040} \\ &= 7{,}834. \end{aligned}$$

Thus the sidereal period is 7,834 seconds = 131 minutes = 2 hours 11 minutes.

12. For the satellite in problem 11a:

a. Compute the synodic period of the satellite, assuming it is in a *posigrade* equatorial orbit.

Solution. Let x be a position on the Equator at which the satellite is directly over the observer. During one synodic period the rotation of Earth carries the observer to position y, where the satellite "overtakes" him again. The basic problem is to find the angular distance A.



In one synodic period the observer traveled an angular distance A, and the satellite traveled an angular distance $360° + A$. The observer travels 360° in 24 hours, or 1° in 24/360 hours. Thus, during one synodic period the observer travels $(24/360)A$ hours.

From problem 11a the sidereal period is 1.46 hours. Thus the satellite travels 1° in 1.46/360 hours. During one synodic period the satellite travels (1.46/360) $(360 + A)$ hours. The synodic period for the satellite and observer are, of course, equal. Therefore, we have

$$\left(\frac{1.46 \text{ hr}}{360°}\right)(360° + A) = \left(\frac{24 \text{ hr}}{360°}\right)A$$

$$(1.46)(360° + A) = 24A$$

$$525.6° + 1.46A = 24A$$

$$22.54A = 525.6°$$

$$A = 23.3°.$$

Hence, the synodic period is

$$\left(\frac{1.46 \text{ hr}}{360°}\right)(360° + 23.3°) = 1.555 \text{ hr}$$
$$= 93.3 \text{ min.}$$

Note that the synodic period is 5.7 minutes greater than the sidereal period.

b. Compute the synodic period of the satellite, assuming it is in a *retrograde* equatorial orbit.

Solution. The observer would travel an angular distance A, but the satellite would travel only $360° - A$ during the synodic period. By using the same approach as in part a and equating times, we have

$$\left(\frac{1.46 \text{ hr}}{360°}\right)(360° - A) = \left(\frac{24 \text{ hr}}{360°}\right)A$$

$$A = 20.6°.$$

Therefore, the synodic period is

$$\left(\frac{1.46 \text{ hr}}{360°}\right)(360° - 20.6°) = 1.376 \text{ hr}$$
$$= 82.6 \text{ min.}$$

Note that the posigrade synodic period is 10.7 minutes greater than the retrograde synodic period.

chapter 3

chapter 3

RATIO, PROPORTION, AND VARIATION

This chapter contains problems in algebra that illustrate the concepts of ratio, proportion, and variation. Although some of these problems could be presented in another context and be solved by other means, placing them in this chapter provides a selected list of problems illustrating these concepts. In a number of instances, these problems are related to and augment problems in other chapters. When this situation arises, cross-references are used.

The rich variety of space-related topics discussed in the chapter includes thrust-to-weight ratio, mass ratio, specific impulse and exhaust velocity, derivation of Newton's law of universal gravitation, problems involving force and acceleration of gravity on the Moon and on an asteroid, "g forces" on an astronaut, artificial gravity, a number of interesting consequences of Einstein's theory of relativity, variation of weight with distance from the center of gravity, strength of a reflected radio signal, temperature equilibrium of a satellite, and the roles of the Sun and Moon in producing tides on Earth.

PROBLEMS

1. A fundamental concept in the design and operation of launch vehicles is the thrust-to-weight ratio. Because most launches begin vertically, it is apparent that the thrust, the force that lifts the vehicle, must be greater than the weight. That is, the thrust-to-weight ratio must be greater than 1. (According to Newton's second law of motion, $F = ma$, the thrusting force F will give the vehicle an acceleration. The thrust remains constant or tends to increase a little as the propellant is burned. Meanwhile, the mass m is rapidly reduced as the propellant is burned. The result is an increasing acceleration. From this acceleration must be subtracted, of course, the acceleration of gravity, which acts as a retarding force. For additional information about launch vehicle behavior, see Chapter 6.)

Find the thrust-to-weight ratio of the following launch vehicles.

Vehicle	*Thrust*	*Weight*
Delta	170,000	114,200
Atlas-Centaur	368,000	300,000
Gemini-Titan II	430,000	300,000
Saturn IB	1,600,000	1,294,000
Saturn V	7,700,000	6,400,000

Solution. The answers, found by simple division, are respectively 1.49, 1.23, 1.43, 1.24, and 1.20.

2. The mass ratio of a launch vehicle is defined as $R = \frac{\text{takeoff weight}}{\text{burnout weight}}$. The weight of a rocket or launch vehicle can be divided into three parts: the weight of the structure S; the weight of the propellant or fuel F; and the weight of the payload P. The part of the weight that disappears between liftoff and burnout is F. Burnout occurs, of course, when all the fuel has been burned. Thus the mass ratio is usually defined as

$$R = \frac{S + F + P}{S + P}$$

(Further information about the relationship of the mass ratio to launch vehicle operation will be found in Chapter 6.)

If the mass ratio of a launch vehicle is 7, the weight of the structure is 2 tons, and the weight of the payload is 1 ton, find the weight of the fuel.

Solution. Applying the given equation yields

$$R = \frac{S + F + P}{S + P}$$

$$7 = \frac{2 \text{ tons} + F + 1 \text{ ton}}{2 \text{ tons} + 1 \text{ ton}}$$

$$F = 18 \text{ tons.}$$

3. The Mach number M is a measure of speed and is defined as the ratio of the vehicle's speed v to the speed of sound at that altitude v_a. (The Mach number varies with temperature, and the temperature varies with altitude.) What is the Mach number of an aircraft flying at 845 feet per second at an altitude of 30,000 feet? (Assume that the speed of sound at this altitude is 995 feet per second.)

Solution. By definition,

$$\begin{aligned} M &= \frac{v}{v_a} \\ &= \frac{845 \text{ ft/sec}}{995 \text{ ft/sec}} \\ &= 0.85. \end{aligned}$$

4. The specific impulse I_{sp} of a propellant-engine combination is the thrust produced when 1 pound of propellant is burned in 1 second. That is, $I_{sp} = \frac{F}{w/t}$, where F is the thrust or force measured in pounds, w is the weight in pounds of the propellant burned, and t is the time in seconds. Rearranging the equation to read $I_{sp} = \frac{Ft}{w}$, we note that the numerator is expressed in

pounds of force and seconds of time, whereas the denominator is expressed in pounds of weight. It is common practice to divide out the pounds, leaving the answer in seconds. Finally the ratio $\frac{w}{t}$, which represents pounds of propellant used per second and is called the weight flowrate, is commonly written as $\dot{w}$, leaving us with the equation

$$I_{sp} = \frac{F}{\dot{w}}.$$

a. Find the specific impulse of a propellant when the burning of 1 pound per second produces a thrust of 400 pounds.

Solution. Evidently $\dot{w} = 1$ pound per second, and

$$I_{sp} = \frac{400}{1} = 400 \text{ sec.}$$

b. When 4,735,000 pounds of propellant are burned in 161 seconds, the thrust produced at sea level is 7,700,000 pounds. Find the specific impulse at sea level.

Solution.

$$\dot{w} = \frac{4{,}735{,}000}{161} = 29{,}410 \text{ lb/sec}$$

$$I_{sp} = \frac{7{,}700{,}000}{29{,}410} = 262 \text{ sec.}$$

(These data represent the performance of the Saturn V launch vehicle at sea level. Although $\dot{w}$ remains essentially constant, the thrust F increases with altitude; as a result the specific impulse at burnout of the S-IC stage is higher.)

c. If a propellant can be found that delivers 50 percent more thrust with the same weight flowrate, how does this affect the specific impulse?

Solution. If the weight flowrate is constant, the specific impulse is directly proportional to the thrust. In this case, therefore, the specific impulse would be 50 percent higher than for the first propellant.

5. The exhaust velocity c produced by a rocket engine is directly proportional to the specific impulse of the fuel; that is, $c = gI_{sp}$, where g is the acceleration of gravity at the surface of Earth. We may derive the formula as follows. In the equation $F = ma$, $m = \frac{w}{g}$, obtaining $F = \frac{w}{g}a$. This form of the equation merely enables us to work with units of weight rather than mass. Acceleration is change in velocity per unit of time, or $a = \frac{c}{t}$.

Thus

$$F = \frac{w}{g} \times \frac{c}{t}.$$

Rearranging,

$$\frac{Ft}{w} = \frac{c}{g}.$$

But as noted in the previous problem,

$$I_{sp} = \frac{Ft}{w} = \frac{c}{g},$$

which yields

$$c = gI_{sp}.$$

(The relationship of exhaust velocity to launch vehicle operation is discussed in Chapter 6.)

a. What exhaust velocity will be produced by a propellant with a specific impulse of 360 seconds?

Solution. The value of g is 32.2 feet per second per second.

$$\begin{aligned} c &= \frac{32.2 \text{ ft}}{\text{sec/sec}} \times 360 \text{ sec} \\ &= 11{,}592 \text{ ft/sec.} \end{aligned}$$

b. If an exhaust velocity of 14,000 feet per second is needed, what must be the specific impulse of the fuel?

Solution.

$$I_{sp} = \frac{c}{g} = \frac{14{,}000 \text{ ft/sec}}{32.2 \text{ ft/sec/sec}} = 435 \text{ sec.}$$

(The maximum specific impulse available from present chemical propellants is 450 to 460 seconds.)

6. The statement has been made that Newton's derivation of his inverse-square law of gravity from Kepler's third law is among the most important calculations ever performed in the history of science. Kepler's third law, based upon observation rather than theory, states that the squares of the periods of any two planets are to each other as the cubes of their average distances from the Sun. Derive Newton's law from Kepler's law.

Solution. If we represent the periods of any two planets by t and T and their distances from the Sun by r and R, respectively, then

$$\frac{T^2}{t^2} = \frac{R^3}{r^3}$$

or

$$T^2 = \frac{t^2R^3}{r^3}.$$

Assuming that we know the values of t and r, and substituting for them a constant C, the equation can be reduced to

$$T^2 = CR^3.$$

Thus if we know either T or R for the second planet, we can solve for the unknown quantity. In this problem, however, we wish to use this equation to discover a new relationship, Newton's law of gravitation. For a body moving in a circular path, the acceleration toward the center is

$$a = \frac{v^2}{r}.$$

Substituting in $F = ma$,

$$F = \frac{mv^2}{r}.$$

The velocity of the body in the circular orbit is

$$v = \frac{2\pi r}{T}.$$

Thus,

$$F = \frac{mv^2}{r} = \frac{m4\pi^2R^2}{RT^2}.$$

Because $T^2 = CR^3$, we find by substitution in the previous equation that

$$F = \frac{m4\pi^2}{C} \times \frac{1}{R^2} = \frac{K}{R^2}.$$

That is, the force holding a planet in orbit falls off as the square of the distance R to the Sun. Newton expressed the value of K and obtained his law of universal gravitation

$$F = \frac{GMm}{r^2}.$$

This law applies not only to the attraction between a planet and the Sun, but to the attraction between any two bodies. G is the constant of universal gravitation, M and m are the masses of the two bodies, and r is the distance between their centers of mass.

7. If M is the mass of Earth, then the mass of the Moon is $0.012M$. The radii of Earth and the Moon are 3,960 and 1,080 miles, respectively. Use these facts with Newton's law of universal gravitation to find the ratio of surface gravity on the Moon to surface gravity on Earth.

Solution. If we place a mass m at the surface of Earth, then the gravitational attraction between the mass and Earth is

$$F_e = \frac{GMm}{3{,}960^2}.$$

Similarly, the attraction between the Moon and an equal mass m placed on its surface is

$$F_m = \frac{G(0.012M)m}{1{,}080^2}.$$

The ratio of F_m to F_e is

$$\frac{F_m}{F_e} = \frac{0.012}{1{,}080^2} \times \frac{3{,}960^2}{1} = \frac{188{,}179}{1{,}166{,}400}$$

$$= \frac{1}{6}$$

That is, gravity at the surface of the Moon is 1/6 as great as gravity at the surface of Earth.

8. Several scientists have suggested that manned landings eventually be made on asteroids. With the equations that we now have available we can investigate many phenomena related to landing on and exploring an asteroid. Asteroids exist in many shapes and sizes, with diameters ranging from less than 1 mile to several hundred miles. It has been estimated that the density of asteroids is about three-fifths that of Earth. In the following problems we consider an asteroid with a diameter of 14 miles. We assume that it is spherical. Let us name it A–14. (A–14 has about the same diameter as Eros, but its mass is greater because Eros is believed to be brick shaped rather than spherical.) Find the ratio of the surface gravity on A–14 to the surface gravity on Earth.

Solution. Because A–14 is spherical, we know that its volume is $\left(\frac{14}{7{,}920}\right)^3$ times the volume of Earth. Then if M is the mass of Earth, the mass of A–14 is

$$\left(\frac{14}{7{,}920}\right)^3 \times \frac{3}{5}M = 3{,}314 \times 10^{-12}\, M.$$

Expressing the force of gravity at the surface of each body,

$$F_e = \frac{GMm}{3{,}960^2}$$

and

$$F_A = \frac{G(3{,}314 \times 10^{-12})\, Mm}{7^2}.$$

Therefore

$$\frac{F_A}{F_e} = \frac{3{,}314 \times 10^{-12} \times 3{,}960^2}{49}$$
$$= 106 \times 10^{-5}$$
$$= 0.00106.$$

Thus a person on A–14 would weigh just a trifle more than one-thousandth his weight on Earth.

9. If a man weighs 180 pounds on Earth, what would he weigh on the Moon and on A–14?

Solution. Weight on the Moon would be

$$\frac{1}{6} \times 180 \text{ lb} = 30 \text{ lb}.$$

Weight on A–14 would be

$$0.00106 \times 180 \text{ lb} = 0.191 \text{ lb},$$

or just over 3 ounces.

10. Compute the acceleration of gravity at the surfaces of the Moon and A–14.

Solution. The equation $F = ma$ tells us that the acceleration is directly proportional to the force that produces it. The force that causes a body to fall is its weight. The acceleration of a freely falling body near the surface of Earth is 32.2 feet per second per second. In the case of the Moon, the weight is 1/6 of Earth weight, and therefore the acceleration near the surface is

$$\frac{1}{6} \times 32.2 = 5.4 \text{ ft/sec}^2.$$

Similarly for A–14, the acceleration near the surface is

$$0.00106 \times 32.2 = 0.034 \text{ ft/sec}^2.$$

11. Galileo found that when a body falls from rest, the distance s traveled is directly proportional to the square of the time t of travel, or

$$s = kt^2.$$

Experiment shows that $k = \frac{1}{2}a$, where a is the local acceleration caused by gravity. Thus we obtain the familiar equation found in physics,

$$s = \frac{1}{2}at^2.$$

Find the distance that a body will fall in 10 seconds on each of the following bodies.

a. Earth.

Solution. Because

$$a = 32.2 \text{ ft/sec}^2,$$

$$s = \frac{1}{2} \times 32.2 \times 10^2$$

$$= 16.1 \times 100$$

$$= 1{,}610 \text{ ft.}$$

b. Moon.

Solution. From problem 10,

$$a = 5.4 \text{ ft/sec}^2,$$

$$s = \frac{1}{2} \times 5.4 \times 10^2$$

$$= 2.7 \times 100$$

$$= 270 \text{ ft.}$$

c. A–14.

Solution. From problem 10,

$$a = 0.034 \text{ ft/sec}^2,$$

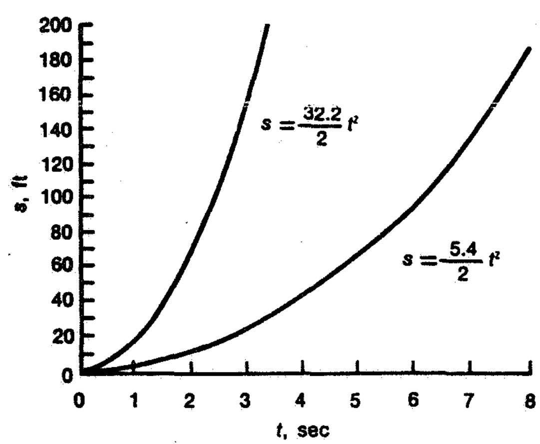
$$s = \frac{1}{2} \times 0.034 \times 10^2$$

$$= 0.017 \times 100$$

$$= 1.7 \text{ ft.}$$

12. Graph the equation $s = \frac{1}{2}at^2$ for Earth and the Moon on the same set axes for $t \leqslant 8$ and $s \leqslant 200$.

Solution.



13. Using the data found in the preceding problem, find the time required for an object to fall 100 feet on Earth, the Moon, and A–14. (Ignore air resistance for a body falling to Earth.) By definition, acceleration is change in velocity per unit of time, or $a = \frac{v}{t}$. From this equation we may write $v = at$. Use this equation to find the velocity at impact on the same bodies.

a. Earth.

Solution.

$$100 \text{ ft} = \frac{1}{2}(32.2 \text{ ft/sec}^2)(t^2)$$

$$t^2 = 6.21 \text{ sec}^2$$

$$t = 2.49 \text{ sec.}$$

$$v = (32.2 \text{ ft/sec}^2)(2.49 \text{ sec})$$

$$= 80.2 \text{ ft/sec.}$$

b. Moon.

Solution.

$$100 \text{ ft} = \frac{1}{2}(5.4 \text{ ft/sec}^2)(t^2)$$

$$t^2 = 37 \text{ sec}^2$$

$$t = 6.1 \text{ sec.}$$

$$v = (5.4 \text{ ft/sec}^2)(6.1 \text{ sec})$$

$$= 33 \text{ ft/sec.}$$

c. A–14.

Solution.

$$100 \text{ ft} = \frac{1}{2}(0.034 \text{ ft/sec}^2)(t^2)$$

$$t^2 = 5{,}882 \text{ sec}^2$$

$$t = 76.7 \text{ sec} = 1.28 \text{ min}$$

$$v = (0.034 \text{ ft/sec}^2)(76.7 \text{ sec})$$

$$= 2.6 \text{ ft/sec.}$$

14. The centripetal acceleration a on a body in circular motion varies directly as the square of its rotational speed v (feet per second) and inversely as the radius r (feet). Astronauts are sometimes conditioned and tested in giant centrifuges that follow this law.

a. Find the acceleration on an astronaut in a centrifuge with a diameter of 100 feet and a speed of 80 feet per second.

Solution. From the given statement we find

$$
\begin{aligned}
a &= \frac{v^2}{r} \\
&= \frac{(80\ \text{ft/sec})^2}{50\ \text{ft}} \\
&= 128\ \text{ft/sec}^2.
\end{aligned}
$$

b. The acceleration due to gravity g is about 32 feet per second per second. Find the number of g's on the astronaut in part a.

Solution. Applying the relation $1g = 32$ ft per second per second gives

$$
\begin{aligned}
128\ \text{ft/sec}^2 &= 128\ \text{ft/sec}^2 \times \frac{1g}{32\ \text{ft/sec}^2} \\
&= 4g.
\end{aligned}
$$

c. If the astronaut's normal weight is 170 pounds, find the force that the side of the centrifuge exerts on him.

NOTE: Force is equal to mass times acceleration.

Solution.

$$
\begin{aligned}
F &= ma = \frac{w}{g}a = \frac{170\ \text{lb}}{32\ \text{ft/sec}^2}(128\ \text{ft/sec}^2) \\
&= 680\ \text{lb}.
\end{aligned}
$$

15. It is expected that in some future space stations, artificial gravity will be created by rotation of all or part of the station. Gas jets or other propulsion devices can be used to control the rate of rotation of the station. As in the case of the centrifuge, the rotation will produce a force against the astronaut that cannot be distinguished from gravity. If r is the distance of a point in the station from the center of rotation, then the velocity of the point for N rotations per second is

$$v = 2\pi rN.$$

As noted above,

$$a = \frac{v^2}{r} \qquad \text{or} \qquad v = \sqrt{ar}.$$

Setting the two velocities equal,

$$
\begin{aligned}
2\pi rN &= \sqrt{ar} \\
N^2 &= \frac{ar}{(2\pi)^2 r^2} \\
N &= \frac{1}{2\pi}\sqrt{\frac{a}{r}}
\end{aligned}
$$

If r is given in feet, then a is the acceleration in feet per second per second. By controlling the values of r and N, any desired artificial gravity can be produced.

a. Compute the rotational rate needed if the radius of the station is 100 feet and a gravity equal to one-half the gravity of Earth is desired. (Use $g = 32$ feet per second per second.)

Solution.

$$N = \frac{1}{2\pi}\sqrt{\frac{16}{100}}$$

$$= \frac{1}{6.283} \times \frac{4}{10} = \frac{4}{62.83}$$

$$= 0.064$$

The rate of rotation must be 0.064 rotation per second or $60 \times 0.064 = 3.8$ rotations per minute.

b. Compute the needed rotational rate if the radius of the station is 500 feet and Earth surface gravity is desired.

Solution.

$$N = \frac{1}{2\pi}\sqrt{\frac{32}{500}}$$

$$= \frac{1}{6.283}\sqrt{0.064} = \frac{0.253}{6.283}$$

$$= 0.04.$$

The rate of rotation must be 0.04 rotation per second or 2.4 rotations per minute.

16. A jet pilot coming out of a dive flying at 600 feet per second experiences a centrifugal force of 1,800 pounds. If the centrifugal force F is proportional to the square of the velocity v, find the force on a pilot flying the same path at 800 feet per second.

Solution. From the given information we have

$$\frac{F_1}{v_1^2} = \frac{F_2}{v_2^2}$$

where $F_1 = 1{,}800$ pounds, $v_1 = 600$ feet per second, and $v_2 = 800$ feet per second. Thus the force on the pilot flying at 800 feet per second is

$$\frac{1{,}800 \text{ lb}}{(600 \text{ ft/sec})^2} = \frac{F_2}{(800 \text{ ft/sec})^2}$$

$$F_2 = 3{,}200 \text{ lb}.$$

17. At speeds close to that of light, the theory of relativity plays an important role. One of the relativistic effects of high speeds is an increase in mass. If an object has a mass m_0 when at rest, then its relativistic mass m_v when moving with velocity v is

$$m_v = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

where c is the velocity of light, approximately 186,300 miles per second.

a. Find the percent increase in mass of a unit particle when its speed is equal to 60 percent of the speed of light.

Solution. Applying the given equation yields

$$\begin{aligned} m_v &= \frac{1}{\sqrt{1-\left(\frac{0.6c}{c}\right)^2}} \\ &= \frac{1}{\sqrt{1-0.36}} \\ &= \frac{1}{0.8} = 1.25. \end{aligned}$$

The difference between the relativistic mass and rest mass gives the increase in mass, 0.25 or 25 percent.

b. Find the percent change in the mass of an electron when its speed is equal to 80 percent of the speed of light.

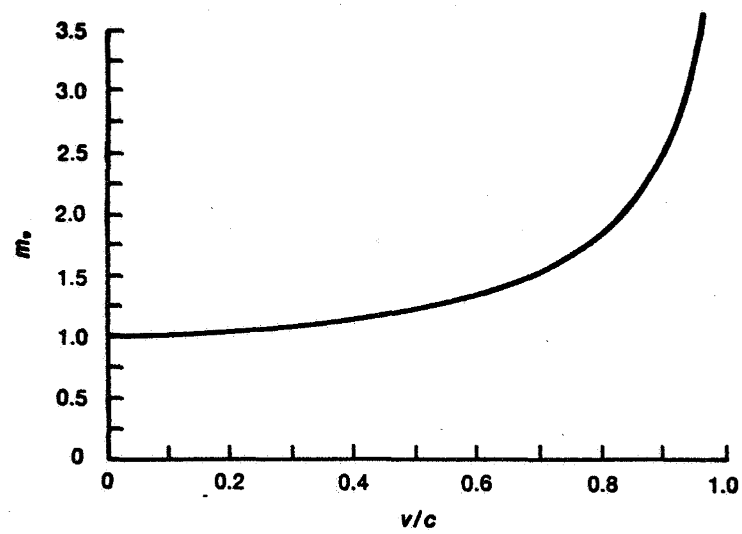
Solution. The relativistic mass of the electron is

$$\begin{aligned} m_v &= \frac{m_0}{\sqrt{1-\left(\frac{0.8c}{c}\right)^2}} \\ &= \frac{m_0}{\sqrt{1-0.64}} \\ &= \frac{m_0}{0.6} = 1.67m_0 \end{aligned}$$

Thus the change in mass is $1.67m_0 - m_0 = 0.67m_0$, or 67 percent increase.

c. Plot a graph of the relativistic mass as a function of the ratio of particle speed v to the speed of light c. Assume a unit rest mass. How does your graph indicate that the speed of light is an unreachable speed?

Solution.



As the speed of the particle approaches the speed of light, the mass becomes increasingly large without bound, which means that the speed of light is a limiting speed that can never be achieved.

18. This problem uses an interesting application of a binomial expansion to investigate the relationship between Newton's and Einstein's formulas for kinetic energy. The exploration of space outside our solar system will be feasible only if we can produce spacecraft that will travel nearly as fast as light. (Even if we could travel at the speed of light, it would require a little more than 4 years to reach Alpha Centauri, the closest star outside the solar system.) At speeds close to that of light, the theory of relativity changes the formula for kinetic energy E_K. Whereas in Newtonian mechanics we have

$$E_K = \frac{1}{2}m_0v^2,$$

the relativistic formula is

$$E_K = (m_v - m_0)c^2.$$

At first sight these formulas look quite different. We shall see, however, that the Newtonian formula can be regarded as an approximation to the relativistic one.

a. Verify that the binomial expansion of

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

is

$$1 + \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{3}{4}x^4 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6 + \cdots.$$

Solution. Using the binomial expansion for $(a+b)^n$, namely

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2\cdot 1}a^{n-2}b^2 + \cdots$$

with $a=1$, $b=-x^2$, and $n=-\frac{1}{2}$, we find

$$\begin{aligned}(1-x^2)^{-1/2} &= (1)^{-1/2} + \frac{\left(-\frac{1}{2}\right)}{1}(1)^{-3/2}(-x^2)\\ &\quad + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2}(1)^{-5/2}(-x^2)^2\\ &\quad + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\cdot 2\cdot 3}(1)^{-7/2}(-x^2)^3 + \cdots\\ &= 1 + \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{3}{4}x^4 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6 + \cdots .\end{aligned}$$

b. Use the first two terms of the expansion as an approximation to $(1-x^2)^{-1/2}$ and set $x = v/c$. Show that the relativistic kinetic energy formula reduces to the Newtonian one.

Solution. We are given that

$$m_v = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

Hence

$$\frac{m_v}{m_0} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

Using $1+\frac{1}{2}\left(\frac{v}{c}\right)^2$ as an approximation to $\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, we have

$$\begin{aligned}\frac{m_v}{m_0} &\doteq 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\\ m_v &\doteq m_0 + \frac{1}{2}m_0\left(\frac{v}{c}\right)^2\\ m_v - m_0 &\doteq \frac{1}{2}m_0\left(\frac{v}{c}\right)^2\\ (m_v - m_0)c^2 &\doteq \frac{1}{2}m_0v^2.\end{aligned}$$

(We are using $x \doteq y$ to mean "x is approximately equal to y.") Thus we have shown that the relativistic kinetic energy formula reduces approximately to the Newtonian one when v is small compared with c.

19. According to Einstein's theory of relativity, if one system is moving rapidly with respect to another system, time passes more slowly in the moving system. Suppose that an astronaut is in a futuristic spacecraft that travels at the speed $v = 0.5c$, where c is the speed of light. The astronaut has a brother on Earth who was 1 year younger at the time of launch. The aging rate R_a of the astronaut is related to the aging rate R_b of his brother by the equation

$$R_a = R_b\sqrt{1 - \left(\frac{v}{c}\right)^2}.$$

How long must the astronaut travel so that upon his return to Earth, he is exactly as old as his brother?

Solution. From the given equation, we get

$$\begin{aligned} R_a &= R_b\sqrt{1 - \left(\frac{0.5\,c}{c}\right)^2} \\ &= R_b\frac{\sqrt{3}}{2}. \end{aligned}$$

Hence, for each year the brother ages, the astronaut ages only $\frac{\sqrt{3}}{2}$ years. Expressed in another way, the astronaut's clock moves more slowly than the same clock would move on Earth. While the clock advances 1 year on Earth, the same clock would advance only $\frac{\sqrt{3}}{2}$ years while moving at half the speed of light. Let x be the time needed. Then

$$\begin{aligned} x &= \frac{\sqrt{3}}{2}x + 1 \text{ yr} \\ &= (4 + 2\sqrt{3}) \text{ yr}. \end{aligned}$$

As a check, if when the astronaut leaves Earth, the age of the brother is N, the age of the astronaut is $N + 1$. When the astronaut returns after $(4 + 2\sqrt{3})$ years of travel, the ages are

$$\begin{aligned} A_b &= N + 4 + 2\sqrt{3} \text{ yr}. \\ A_a &= N + 1 + \frac{\sqrt{3}}{2}(4 + 2\sqrt{3}) \\ &= N + 1 + 2\sqrt{3} + 3 \\ &= N + 4 + 2\sqrt{3} \text{ yr}. \end{aligned}$$

20. Find the general equations required to solve problem 19 if the age of the brother is N and the astronaut is d years older. Apply the equations to the case in which the brother is 20 years old and the astronaut is 5 years older.

Solution. From problem 19, the number of Earth years needed is

$$x = \frac{\sqrt{3}}{2}x + d$$

$$x\left(1 - \frac{\sqrt{3}}{2}\right) = d$$

$$x = \frac{2d}{2 - \sqrt{3}} = \frac{2d(2 + \sqrt{3})}{4 - 3}$$

$$= (4 + 2\sqrt{3})d \text{ yr.}$$

The ages of the brothers are

$$A_b = N + (4 + 2\sqrt{3})d$$

$$= N + 4d + 2\sqrt{3}d \text{ yr}$$

and

$$A_a = N + d + \frac{\sqrt{3}}{2}(4 + 2\sqrt{3})d$$

$$= N + 4d + 2\sqrt{3}d \text{ yr.}$$

Because $d = 5$ and $N = 20$,

$$x = (4 + 2\sqrt{3})5 = 20 + 17.3 = 37.3 \text{ yr}$$

and

$$A_b = A_a = 20 + 4(5) + 10\sqrt{3}$$

$$= 57.3 \text{ yr.}$$

Thus the brothers will be the same age when the astronaut returns after traveling for 37.3 years at half the speed of light.

21. At what velocity must the astronaut travel in order that he may age one-third as rapidly as his Earth-bound brother?

Solution. From the equation in problem 19,

$$\frac{R_a}{R_b} = \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

$$\frac{1}{3} = \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

$$\frac{1}{9} = 1 - \left(\frac{v}{c}\right)^2$$

$$v^2 = \frac{8}{9}c^2$$

$$v = \frac{\sqrt{8}}{3}c = \frac{2.83}{3}c = 0.94c.$$

Thus the astronaut must travel at 94 percent of the speed of light.

22. How would the rates of aging compare if the astronaut were able to travel at the speed of light?

Solution. When $v = c$,

$$\frac{R_a}{R_b} = \sqrt{1 - 1} = 0$$

which implies that $R_a = 0$. This would mean that those aboard a spacecraft traveling at the speed of light would not age at all.

23. The world's champion weight lifter in 1968 lifted 1,280 pounds. If he were on the Moon and able to exert the same lifting force, what would be the Earth weight of the greatest mass that he could lift?

Solution. As noted in previous problems, a mass on the Moon will weigh only one-sixth of its weight on Earth because gravity on the Moon is 1/6 that of Earth. The Earth weight that the champion could lift would be $6 \times 1{,}280$, or 7,680 pounds.

24. According to Newton's law of universal gravitation, the acceleration of gravity at a point in space varies inversely with the square of the distance from the center of gravity of the primary body. Because the weight of a given mass varies directly with the local acceleration of gravity, as indicated in the equation $w = mg$, the weight of a body in space also varies inversely with the square of the distance from the center of the primary body. We can investigate this matter with the following computation.

Let r = the distance from the center of gravity; g_r = the acceleration of gravity at distance r; R = the radius of the primary body; g_R = the acceleration of gravity at the surface of the primary body; w_r = the weight of a mass at distance r; and w_R = the weight of the mass at the surface of the primary body. Then,

$$F = \frac{GMm}{d^2} = ma$$

from which we obtain

$$a = \frac{GM}{d^2}.$$

At distance r,

$$g_r = \frac{GM}{r^2}.$$

At distance R,

$$g_R = \frac{GM}{R^2}.$$

Then, by division,

$$\frac{g_r}{g_R} = \left(\frac{R}{r}\right)^2$$

$$g_r = \left(\frac{R}{r}\right)^2 g_R.$$

Because w varies directly as g, we may also derive in a similar manner

$$w_r = \left(\frac{R}{r}\right)^2 w_R.$$

The acceleration of gravity at Earth's surface is about 32.2 feet per second per second.

a. Find the acceleration of gravity at an altitude of 100 miles above the surface of Earth.

Solution. Because $g_R = 32.2$, $R = 3{,}960$ miles, and $r = 4{,}060$ miles, we obtain

$$\begin{aligned} g_r &= \left(\frac{3{,}960}{4{,}060}\right)^2 \times 32.2 \\ &= (0.975)^2 \times 32.2 \\ &= 0.95 \times 32.2 = 30.6 \text{ ft/sec}^2. \end{aligned}$$

b. Find the weight of a 100-pound object at the altitude given in part **a.**

Solution. Evidently $w_R = 100$ pounds. Using the previous computation,

$$w_r = 0.95 \times 100 \text{ lb} = 95 \text{ lb}.$$

NOTE: This problem should illustrate the fact that although a body in orbit acts as though it were "weightless," this lack of weight is apparent rather than real. Under what circumstances would a body be physically weightless?

25. a. Find the acceleration of gravity at an altitude of 70 miles above the surface of the Moon.

Solution. In a previous problem we computed that on the Moon $g_R = 5.4$ feet per second per second. Also $R = 1{,}080$ miles and $r = 1{,}150$ miles.

$$\begin{aligned} g_r &= \left(\frac{1{,}080}{1{,}150}\right)^2 \times 5.4 \\ &= (0.939)^2 \times 5.4 \\ &= 0.882 \times 5.4 = 4.8 \text{ ft/sec}^2. \end{aligned}$$

b. At the same altitude as part a, find the weight of a mass with an Earth weight of 120 pounds.

Solution. If the Earth weight of the mass is 120 pounds,

$$w_R = \frac{120}{6} = 20 \text{ lb}$$

$$w_r = 0.882 \times 20 \text{ lb} = 17.6 \text{ lb}.$$

26. Find the frequency of a simple pendulum on the Moon in terms of its frequency f on Earth if the frequency is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}},$$

where g is the acceleration of gravity and L is the pendulum's length.

Solution. Because g on the Moon is only 1/6 as great as on Earth, we have

$$\begin{aligned} f_m &= \frac{1}{2\pi}\sqrt{\frac{(1/6)g}{L}} \\ &= \frac{1}{2\pi}\sqrt{\frac{g}{6L}} \\ &= \frac{1}{2\pi}\sqrt{\frac{g}{L}}\sqrt{\frac{1}{6}} \\ &= \frac{f}{\sqrt{6}}. \end{aligned}$$

27. In a game of skip rope a minimum speed of 60 revolutions per minute is required to keep the rope rotating. Find the minimum speed for "Moon children" using an identical rope. Assume that the "centrifugal" force necessary to keep the rope rotating is proportional to the square of the speed.

Solution. Equating the ratios of force F to the square of velocity v for the rope on Earth and on the Moon yields

$$\begin{aligned} \frac{F_e}{v_e^2} &= \frac{F_m}{v_m^2} \\ \frac{F_e}{(60 \text{ rpm})^2} &= \frac{\frac{1}{6}F_e}{v_m^2} \\ v_m^2 &= 600 \text{ rpm}^2 \\ v_m &= 24.5 \text{ rpm}. \end{aligned}$$

28. The strength of a radio signal is inversely proportional to the square of the distance from the source of the signal. Consider a radio signal that is reflected by a spacecraft and picked up by a receiver on the ground. How does the strength of the signal at the receiving station vary with the altitude of the satellite?

Solution. Let h be the altitude of the satellite. The intensity of the signal when it reaches the satellite may be written

$$I_s = \frac{cI_0}{h^2},$$

where I_0 is the strength of the source and c is a constant. The strength of the reflected signal when it returns to the Earth's surface can be written

$$I_e = \frac{kI_s}{h^2},$$

where k is another constant. Hence

$$I_e = \frac{ckI_0}{h^4}.$$

Thus the strength of the signal at the receiving station is inversely proportional to the fourth power of the altitude.

29. How does the equilibrium temperature of a satellite vary with its distance from the Sun? Base your answer upon the following assumptions: (*a*) The solar energy received from the Sun is inversely proportional to the square of the distance from the Sun, (*b*) the energy radiated by the satellite is directly proportional to the fourth power of its absolute temperature, and (*c*) temperature equilibrium is achieved when the energy received from the Sun is equal to the energy radiated from the satellite.

Solution. If r denotes the distance from the Sun, then the energy received from the Sun can be written c/r^2, when c is a positive constant. If T denotes the absolute temperature of the satellite, then the energy radiated from the satellite is given by σT^4, where σ is another positive constant. In the case of temperature equilibrium we have

$$\frac{c}{r^2} = \sigma T^4.$$

Thus $T = k/\sqrt{r}$, where the constant k is equal to the fourth root of c/σ. Hence the absolute temperature of the satellite is inversely proportional to the square root of the distance from the Sun.

30. The force of gravitation with which one body attracts another is inversely proportional to the square of the distance between them. Consequently, the pull of the Moon on the oceans is greater on one side of Earth than on the other. This gravitational imbalance produces tides. The Sun affects the tides similarly. Because the Sun exerts an enormously greater pull on Earth than the Moon, one might think that the Sun would influence the tides more than the Moon. Just the opposite is true. How can this be?

Solution. Let N be the point on Earth nearest the Moon and let F be the point on Earth farthest from the Moon. We shall assume that the tide-raising force of the Moon is in some sense measured by the difference in the pull of the Moon on unit masses located at N and F. If r is the distance from the center of the Moon to N and if D_e is the diameter of Earth, then the forces acting at N and F are, respectively, $\frac{GM}{r^2}$ and $\frac{GM}{(r + D_e)^2}$, M being the mass of the Moon and G the universal gravitational constant. The difference between these two forces is the tide-raising force, which we shall call F_t. Then,

$$F_t = GM\left[\frac{1}{r^2} - \frac{1}{(r + D_e)^2}\right]$$

$$= \frac{2GMD_e\left(1 + \frac{D_e}{2r}\right)}{r^3\left(1 + \frac{D_e}{r}\right)^2}.$$

Because $\frac{D_e}{r}$ is very small, this expression is approximately

$$F_t = 2GM\frac{D_e}{r^3}.$$

Thus we would expect the tidal effect to be inversely proportional to the *cube* of the distance, whereas gravity is inversely proportional to the *square* of the distance. Because the distance from Earth to the Sun is enormously greater than the distance to the Moon, it is not surprising that the Moon provides the dominant tide-raising force. Local horizontal components of this force cause the tides to roll in and roll out; i.e., the horizontal movement of the water.

chapter 4

chapter 4

QUADRATIC EQUATIONS

Quadratic equations are used in this chapter to analyze in detail the behavior of sounding rockets both when launched and when returning to Earth. The lift generated by a wing is analyzed, and flying and landing speeds of a jet transport plane are computed. Other second-degree equations are found in Chapter 3.

PROBLEMS

1. The height s of an object t seconds after being given an upward velocity of v feet per second from an altitude h is given by the formula

$$s = vt - 16t^2 + h$$

Determine when a toy rocket fired with an upward velocity of 80 feet per second from a 624-foot cliff will be 224 feet below the cliff.

Solution. When the rocket is 224 feet below the cliff it will have an altitude of $624 - 224$, or 400 feet. Applying the given formula yields

$$\begin{aligned} 400 &= 80t - 16t^2 + 624 \\ 0 &= 16t^2 - 80t - 224 \\ &= 16(t^2 - 5t - 14) \\ &= 16(t + 2)(t - 7) \\ t &= -2, 7. \end{aligned}$$

The -2 seconds is extraneous; hence the toy rocket will be 224 feet below the cliff 7 seconds after firing.

NOTE: The solution of -2 seconds can be given a meaning. If another rocket had been fired from an altitude of 400 feet, 224 feet lower, a firing 2 seconds earlier would have been necessary to make it follow the above trajectory. That is, it would have been at 400 feet twice, both at the beginning and end of the flight. One can verify that the initial velocity required in this instance would have been 144 feet per second.

2. The acceleration of one sounding rocket is two-thirds that of a second rocket. Both are launched vertically at the same time. After 4 seconds the second rocket is 96 feet higher than the first. Given that distance = ½ (acceleration) (time)^2, find the acceleration of both rockets.

Solution. Let $s + 96$ and a be the height and acceleration, respectively, of the higher rocket, while s and $(2/3)a$ represent the same quantities for the other rocket. Substituting into the given formula yields

$$s + 96 = \frac{1}{2}(a)(4)^2$$

and

$$s = \frac{1}{2}\left(\frac{2}{3}a\right)(4)^2.$$

Subtracting equations yields

$$\begin{aligned} 96 &= \frac{1}{2}(4)^2 a\left(1 - \frac{2}{3}\right) \\ &= \frac{8}{3}a \\ a &= 36. \end{aligned}$$

Thus the second rocket has an acceleration of 36 feet per second per second, whereas the first rocket's acceleration is $(2/3)36 = 24$ feet per second per second.

3. A sounding rocket is thrust vertically upward with an initial velocity v_0. The height h of the rocket at time t is equal to the height it would attain in the absence of gravity $v_0 t$ minus the free-fall distance due to gravity $gt^2/2$. Thus

$$h = v_0 t - \frac{gt^2}{2}.$$

We are neglecting air resistance and the variation of g with altitude. Show that the rocket attains a maximum height of $v_0^2/2g$ and that this height is attained at time v_0/g.

Solution. We complete the square.

$$\begin{aligned} h &= -\frac{g}{2}\left[t^2 - 2\frac{v_0}{g}t + \left(\frac{v_0}{g}\right)^2\right] + \frac{v_0^2}{2g} \\ &= \frac{v_0^2}{2g} - \frac{g}{2}\left(t - \frac{v_0}{g}\right)^2. \end{aligned}$$

If $t \neq v_0/g$, then the second term is strictly negative and consequently $h < v_0^2/2g$. If, on the other hand, $t = v_0/g$, then $h = v_0^2/2g$.

4. Suppose a model rocket weighs one-fourth of a pound; its engine propels it vertically to a height of 52 feet and a speed of 120 feet per second at burnout. If the parachute fails to open, what will be the approximate time to fall to Earth, according to the following equation for free fall in vacuum.

The free-fall equation is

$$h(t) = h_0 + v_0 t - \frac{1}{2}gt^2,$$

where $h(t)$ is the height at time t, h_0 and v_0 are the height and velocity at the time selected at $t = 0$, and g is approximately 32 feet per second per second. Note that v_0 should be assigned a positive (negative) value if the object is moving upward (downward) at $t = 0$.

Solution. The altitude at time t is $h(t) = 52 + 120t - 16t^2$; ground is reached when $h(t) = 0$. Hence,

$$0 = 16t^2 - 120t - 52$$

or

$$0 = 4t^2 - 30t - 13.$$

Thus

$$t = \frac{30 \pm \sqrt{30^2 + (16 \times 13)}}{8}.$$

Because we reject $t < 0$, we have

$$t = \frac{1}{8}(30 + \sqrt{900 + 208})$$

$$= \frac{1}{8}(30 + \sqrt{1{,}108})$$

$$= \frac{1}{8}(30 + 10\sqrt{11.08})$$

$$= 7.91 \text{ sec.}$$

5. The lift generated by a wing is given as

$$L = C_L \frac{\rho}{2} v^2 S$$

where

ρ is the density of air (0.002378 slugs/ft^3) (a mass of 1 slug weighs 32.2 lb at the surface of Earth),
v is the forward velocity in feet per second,
S is the wing area, and
C_L is an experimentally determined constant called the lift coefficient.

One model of the Boeing 727 jet transport weighs 142,500 pounds and has a wing area of 1,550 square feet. In the landing configuration, the maximum lift coefficient is 3.2. At what speed does the aircraft land? (Assume that landing speed is 1.1 times the minimum flying speed.)

Solution. The minimum flying speed would be

$$v_{\min} = \sqrt{\frac{2L}{C_{L_{\max}} \rho S}}$$

$$= \sqrt{\frac{2(142{,}500)}{3.2(0.002378)(1{,}550)}}$$

$$= 155.5$$

The landing speed is (1.1) (155.5 feet per second) = 171 feet per second = 117 miles per hour.

chapter 5

chapter 5

PROBABILITY

Some of the principles of elementary probability theory and simple combinatorials are applied in this chapter. The problems involve primarily the number of combinations of n objects taken r at a time, and independent events. Of special interest is the application of probabilities in determining the reliability of spacecraft systems.

PROBLEMS

1. Suppose 21 astronauts are available for the lunar landing program and 12 have had orbital experience.

a. How many crews of three can be made up?

Solution. Because in this problem the order of arrangement of the men in the crews is immaterial, it is necessary to use a combination rather than a permutation. Using $\binom{n}{r}$ to denote the number of combinations of n things taken r at a time, we have

$$\binom{21}{3} = \frac{21!}{3!(21-3)!} = 1{,}330.$$

Thus 1,330 crews of three men each can be made up.

b. How many crews of three can be made with at least one experienced and one inexperienced man on each crew?

Solution. There are two cases to be considered here: We can have two experienced and one inexperienced or one experienced and two inexperienced astronauts make up the crew of three.

To get the number of crews of two experienced men and one inexperienced man, we count $\binom{12}{2} = 66$ ways of choosing two men from the 12 with experience, and for each such choice, there are $\binom{9}{1} =$ nine ways of choosing one man from the nine without experience. Hence, there are $\binom{12}{2}\binom{9}{1}$ $= (66)(9) = 594$ different three-men crews having two experienced and one inexperienced astronaut.

Similarly, there are $\binom{12}{1}\binom{9}{2} = 12(36) = 432$ different three-man crews with one experienced and two inexperienced astronauts. Hence the total number of possible crews is $594 + 432 = 1{,}026$, 304 less than the number of possible unrestricted crews. (See problem 1a.)

NOTE: The reader should not infer from this problem that astronaut teams are selected by chance. Many other factors enter into the making of the selection.

2. The electronic telemetry system aboard a spacecraft transmits data of spacecraft motion in the x, y, and z directions. The system consists of three motion sensors, a signal conditioner, and a transmitter. The probability of failure for each motion sensor and for the signal conditioner is 0.0001. The probability of failure for the transmitter is 0.001. Assuming that component failures are independent events and that the failure of any component will render the telemetry system inoperative, compute the probability of a spacecraft telemetry success.

Solution. The probability of success is equal to one minus the probability of failure. Therefore, the probability of success for each sensor and the signal conditioner is

$$\begin{aligned} P &= 1 - 0.0001 \\ &= 0.9999. \end{aligned}$$

Similarly, the probability of success for the transmitter is

$$\begin{aligned} P &= 1 - 0.001 \\ &= 0.999. \end{aligned}$$

The probability of success for the telemetry system is the product of probabilities of success for each component; that is,

$$\begin{aligned} P &= (0.9999)^4(0.999) \\ &= 0.9986. \end{aligned}$$

3. The Service Module engine, whose thrust provided the velocity changes needed to control the Apollo 8 spacecraft in lunar orbit on Christmas Day 1968, has been described as an extremely reliable engine with a failure to start occurring about "once in a million times." During the Apollo 8 mission, the 20,500-pound-thrust engine was started seven times. Write an expression for the probability of success of the engine on the mission. Note that each start of the engine is an independent event.

Solution. Because the probability of failure is "one in a million times," or 0.000001, the probability of success for each start is $1 - 0.000001$, or 0.999999. The probability of success on seven consecutive starts is

$$\begin{aligned} P &= (0.999999)^7 \\ &= (1 - 0.000001)^7 \\ &= 1 - 7(0.000001) + \frac{7 \cdot 6}{2 \cdot 1}(0.000001)^2 - \cdots \\ &= 1 - 0.000007 = 0.999993. \end{aligned}$$

(We have used the binomial theorem to do this evaluation.)

4. A pilot who was forced to land because of an electrical malfunction in his radar equipment is told that an improvised repair was made and that there is a 25 percent chance that the radar will fail before he reaches his home base. The weather report for his home base is as follows: 90 percent chance of complete overcast, 50 percent chance of foggy conditions, and 20 percent chance of rain. Consider each condition independent of the others.

a. The pilot is willing to risk a 10 percent chance of landing in the rain with the radar inoperative. Should he proceed or should he wait for a more favorable weather report?

Solution. The probability of the pilot's having to land in the rain with the radar out is the product of the probabilities of the two events, which is

$$\begin{aligned} P &= P_{\text{rain}} P_{\text{radar out}} \\ &= (0.20)(0.25) \\ &= 0.05. \end{aligned}$$

The risk of 5 percent is less than that which the pilot is willing to take. Thus the pilot would proceed to his home base.

b. Determine the probability that the pilot will land in foggy and overcast conditions with the radar operating.

Solution. The probability of three independent events occurring is the product of the probabilities of the events; that is,

$$\begin{aligned} P &= P_{\text{fog}} P_{\text{overcast}} P_{\text{radar operating}} \\ &= (0.50)(0.90)(1 - 0.25) \\ &= 0.3375 \text{ or } 0.34. \end{aligned}$$

c. What is the probability that the pilot will land in the rain with foggy and overcast conditions while the radar is inoperative?

Solution. The probability of these four independent events occurring is

$$\begin{aligned} P &= P_{\text{rain}} P_{\text{fog}} P_{\text{overcast}} P_{\text{radar out}} \\ &= (0.20)(0.50)(0.90)(0.25) \\ &= 0.0225 \text{ or } 0.02. \end{aligned}$$

d. Determine the probability that the pilot will land with clear, sunny conditions and with the radar operating.

Solution. The desired probability is

$$\begin{aligned} P &= P_{\text{no overcast}} P_{\text{no rain}} P_{\text{no fog}} P_{\text{radar operating}} \\ &= (1-0.90)(1-0.20)(1-0.50)(1-0.25) \\ &= 0.037 \text{ or } 0.04. \end{aligned}$$

5. An aerospace consulting company is working on the design of a spacecraft system composed of three main subsystems, A, B, and C. The reliability, or probability of success, of each subsystem after three periods of operation is displayed in the following table:

	1 day	*3⅓ months*	*8½ months*
A	0.9997	0.8985	0.6910
B	1.0000	.9386	.7265
C	.9961	.9960	.9959

These reliabilities have been rounded to four significant digits. For example, subsystem B could fail during the first day of operation, but the likelihood of failure is so remote that more than four significant digits are needed to indicate it. If P_S is the total probability of success of the system, find P_S for each of the three time periods.

Solution. For the first 24 hours,

$$\begin{aligned} P_S &= P_A P_B P_C \\ &= (0.9997)(1.0000)(0.9961) \\ &= 0.9958 \end{aligned}$$

For a period of 3⅓ months,

$$\begin{aligned} P_S &= P_A P_B P_C \\ &= (0.8985)(0.9386)(0.9960) \\ &= 0.8399 \end{aligned}$$

For a period of 8½ months,

$$\begin{aligned} P_S &= P_A P_B P_C \\ &= (0.6910)(0.7265)(0.9959) \\ &= 0.4999 \end{aligned}$$

6. In problem 5 we saw that the total reliability of the system deteriorates rather rapidly in its present stage of design, with less than a 50 percent chance that it will operate after 8½ months. The reliability of subsystem C remains nearly constant, whereas the greatest decline in reliability takes place in subsystem A, which contains one particular part that is expected to wear out rapidly. The consulting firm is asked whether enough improvement could be made in subsystem A to provide a reliability after 8½ months of 0.7500. Compute the improvement needed in subsystem A.

Solution. Let x be the factor by which the reliability of subsystem A must be multiplied. Then, as before

$$P_S = P_A P_B P_C$$

$$0.7500 = (0.6910x)(0.7265)(0.9959) = 0.4999x$$

$$x = \frac{0.7500}{0.4999} = 1.500$$

The reliability of subsystem A must be $1.500 \times 0.6910 = 1.037$. The increase in reliability cannot be obtained by improving subsystem A alone, because the reliability cannot be greater than 1.

chapter 6

chapter 6

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The 12 problems in this chapter range from simple topics such as the half-life of a radioisotope power supply to the more challenging applications of logarithmic and exponential functions found in multistage rocket design. To work successfully through the set of problems, it is necessary to derive, solve, and write exponential and logarithmic equations.

Other topics upon which problems are based are sound intensity and the decibel unit of measure, atmospheric pressure at varying altitudes, radioactive materials, and electron beam intensity. A number of problems investigate the relationship of mass and mass ratios of a rocket, the impossibility of orbiting a payload with a single-stage rocket, the characteristics associated with multistage rockets, and the actual design of a two-stage launch vehicle.

PROBLEMS

1. The difference in intensity (energy) level of two sounds with intensities I and I_0 is defined to be 10 log (I/I_0) decibels, where I_0 is the minimum intensity detectable by the human ear. When two sounds differ in intensity by a factor of 10, they differ in loudness by 1 bel; a difference of 100 means a loudness difference of 2 bels. In practice the unit used is the decibel, one-tenth of a bel. Find the intensity level in decibels of the sound produced by an electric motor which is 189 times greater than I_0.

Solution. Substituting $189I_0$ for I, we have

$$\begin{aligned} 10 \log \frac{I}{I_0} &= 10 \log \frac{189I_0}{I_0} \\ &= 10 \log 189 \\ &= 10(2.28) \\ &= 22.8 \end{aligned}$$

Thus the intensity level is 22.8 decibels.

2. Testing a rocket engine for a certain spacecraft on the launch pad, the noise level is found to be 100 decibels outside the spacecraft and 45 decibels inside. How many times greater is the noise intensity outside the spacecraft than inside?

Solution.

$$\text{Let } x = \text{intensity level outside}$$

and

$$y = \text{intensity level inside.}$$

Then

$$100 = 10 \log x$$

and

$$45 = 10 \log y.$$

Subtracting the equations and solving for the ratio x/y gives

$$55 = 10(\log x - \log y)$$

$$5.5 = \log \frac{x}{y}$$

$$\frac{x}{y} = \text{antilog of } 5.5 = 10^{5.5}$$

$$= 316{,}230.$$

Therefore, the noise intensity on the outside is approximately 316,000 times greater than that on the inside of the spacecraft.

3. An approximate rule for atmospheric pressure at altitudes less than 50 miles is the following: Standard atmospheric pressure, 14.7 pounds per square inch, is halved for each 3.25 miles of vertical ascent.

a. Write a simple exponential equation to express this rule.

Solution. Letting P denote the atmospheric pressure at altitudes less than 50 miles and h the altitude, we have

$$P = 14.7 \text{ lb/in}^2 \left(\frac{1}{2}\right)^{h/3.25 \text{ mi}}.$$

b. Compute the atmospheric pressure at an altitude of 19.5 miles.

Solution. Using the equation derived in part a,

$$P = (14.7 \text{ lb/in}^2)\left(\frac{1}{2}\right)^{19.5 \text{ mi}/3.25 \text{ mi}}$$

$$= (14.7 \text{ lb/in}^2)\left(\frac{1}{2}\right)^6$$

$$= (14.7 \text{ lb/in}^2)\frac{1}{64}$$

$$= 0.23 \text{ lb/in}^2.$$

c. Find the altitude at which the pressure is 20 percent of standard atmospheric pressure.

Solution. Solving the derived equation for h, we have

$$\begin{aligned}
\frac{1}{5}(14.7 \text{ lb/in}^2) &= (14.7 \text{ lb/in}^2)\left(\frac{1}{2}\right)^{h/3.25 \text{ mi}} \\
\frac{1}{5} &= \left(\frac{1}{2}\right)^{h/3.25 \text{ mi}} \\
\log \frac{1}{5} &= \left(\frac{h}{3.25 \text{ mi}}\right) \log \frac{1}{2} \\
h &= \frac{(3.25 \text{ mi}) \log \frac{1}{5}}{\log \frac{1}{2}} \\
&= 7.54 \text{ mi.}
\end{aligned}$$

d. What altitude is just above 99 percent of the atmosphere?

Solution. Because pressure and density are proportional, the desired altitude is the point at which the pressure is 1 percent of standard atmospheric pressure. Hence

$$\begin{aligned}
(0.01)(14.7 \text{ lb/in}^2) &= (14.7 \text{ lb/in}^2)\left(\frac{1}{2}\right)^{h/3.25 \text{ mi}} \\
0.01 &= \left(\frac{1}{2}\right)^{h/3.25 \text{ mi}} \\
\log 0.01 &= \left(\frac{h}{3.25 \text{ mi}}\right) \log \frac{1}{2} \\
h &= \frac{(3.25 \text{ mi}) \log 0.01}{\log \frac{1}{2}} \\
&= \frac{(3.25 \text{ mi})(-2)}{-0.301} \\
&= 21.6 \text{ mi.}
\end{aligned}$$

4. A certain radioactive material decays at a rate given by the equation

$$A = A_0 \, 10^{-kt}$$

where A is in grams and t is in years. If A_0 is 500 grams, find k if A is 450 grams when t is 1,000 years.

Solution. Applying the given equation and solving for k yields

$$\begin{aligned}
450 \text{ g} &= 500 \text{ g} \times 10^{k(-1{,}000 \text{ yr})} \\
\log 450 &= \log 500 - k(1{,}000 \text{ yr})(\log 10) \\
k(1{,}000 \text{ yr}) &= \log 500 - \log 450 \\
&= 2.6990 - 2.6532 \\
&= 0.0458 \\
k &= 0.0000458/\text{yr.}
\end{aligned}$$

5. The intensity of a beam of radiation after passing through a material is given by the equation

$$I = I_0\, 10^{-kt},$$

where I_0 is the original intensity, t the thickness in centimeters, and k an absorption coefficient. If a beam of gamma radiation is reduced from 1 million electron volts to 100,000 electron volts while passing through a sheet of material with $k = 0.08$, find the thickness of the material.

Solution. Using the given values and solving for t gives

$$\begin{aligned} 10^5 &= (10^6)10^{-0.08t} \\ \frac{1}{10} &= 10^{-0.08t} \\ \log\frac{1}{10} &= -0.08t \log 10 \\ t &= \frac{-1}{-0.08} \\ &= 12.5. \end{aligned}$$

Thus the thickness is 12.5 centimeters.

6. A satellite has a radioisotope power supply. The power output in watts is given by the equation

$$P = 50e^{-t/250}$$

where t is the time in days and e is the base of natural logarithms.

a. How much power will be available at the end of 1 year?

Solution. Applying the given equation, we have

$$\begin{aligned} P &= 50e^{-365/250} \\ &= 50e^{-1.46} \\ &= 50 \times 0.232236 \\ &= 11.6. \end{aligned}$$

Thus approximately 11.6 watts will be available at the end of 1 year.

b. What is the half-life of the power supply? In other words, how long will it take for the power to drop to half its original strength?

Solution. To find the half-life, we solve the equation

$$25 = 50e^{-t/250}$$

for t and obtain

$$\begin{aligned} \frac{-t}{250} &= \ln 0.5 \\ &= -0.69315 \\ t &= 250 \times 0.69315 \\ &= 173. \end{aligned}$$

Thus the half-life of the power supply is approximately 173 days. (Note that $\ln x$ is a shorter expression for $\log_e x$.)

c. The equipment aboard the satellite requires 10 watts of power to operate properly. What is the operational life of the satellite?

Solution. Solving the equation

$$10 = 50e^{-t/250}$$

for t gives

$$\begin{aligned}\frac{-t}{250} &= \ln\frac{10}{50}\\ &= \ln 0.2\\ &= -1.60944\\ t &= 250 \times 1.60944\\ &= 402.\end{aligned}$$

Hence the operational life of the satellite is 402 days.

7. The velocity gained by a launch vehicle when its propellant is burned to depletion is expressed by the equation

$$v = c \log_e R.$$

The velocity gained during the burn is v, the exhaust velocity is c, e is the base of natural logarithms, and R is the mass ratio. Because some high school students may not be acquainted with natural logarithms (base e), it may be convenient to use the rule for changing the base to express the given equation in base 10, the base of common logarithms. The conversion of natural logarithms to logarithms on the base 10 simply involves multiplication by a fixed number, because

$$\log_e R = (\log_e 10)(\log_{10} R).$$

The conversion factor $\log_e 10$ is, like π and e, a transcendental number. To two decimal places, $\log_e 10 = 2.30$. Thus our equation can be replaced by the approximate equation

$$v = c(2.30) \log_{10} R.$$

The mass ratio R is defined by $R = \frac{\text{takeoff weight}}{\text{burnout weight}}$. This definition applies whether we are considering the entire launch vehicle or just a single stage. The takeoff weight consists of propellant or fuel F, structure S, and payload P. Thus the mass ratio may be written as

$$R = \frac{F + S + P}{S + P}.$$

At burnout all of the fuel has been used and $F = 0$. It has been found that the weight of fuel cannot be more than about 10 times the weight of the structure, because if the structure is too weak for the weight of fuel carried, the vehicle may not stand the stresses of operation. Thus the largest possible value for R is

$$\frac{10S + S + P}{S + P} = \frac{11S + P}{S + P}.$$

Because

$$\frac{11S + P}{S + P} = \frac{11S + 11P - 10P}{S + P}$$

$$= 11 - \frac{10P}{S + P} \leq 11,$$

we see that the largest possible value for R is 11. Unfortunately, for R to be equal to 11, P must be zero; i.e., a vehicle designed with $R = 11$ has no room for a payload.

The minimum altitude for a stable orbit about Earth is about 100 miles. At lower altitudes, air resistance slows the spacecraft and causes rapid deterioration of the orbit. As will be noted in Chapter 10, the orbital velocity at 100 miles is nearly 17,500 miles per hour or about 25,600 feet per second. The rocket equation gives the ideal velocity, and ignores losses resulting from the pull of Earth's gravity and the resistance of the heavy atmosphere surrounding Earth at low altitudes. The total drag losses are of the order of 4,000 feet per second for a launch to a 100-mile orbit, so that the total velocity imparted by the launch vehicle must be $25{,}600 + 4{,}000 = 29{,}600$ feet per second, which we shall round for convenience to 30,000 feet per second. If the highest energy propellant available for takeoff from the surface of Earth has an average exhaust velocity of 9,600 feet per second, compute the performance of a launch vehicle with $R = 11$.

Solution. Substituting $c = 9{,}600$ feet per second and $R = 11$ in the rocket equation, we obtain

$$v = (9{,}600)(2.30) \log 11$$

$$= (22{,}080)(1.04)$$

$$= 22{,}960 \text{ or } 23{,}000 \text{ ft/sec.}$$

Thus the launch vehicle cannot fly itself, much less a payload, into Earth orbit. An additional velocity of 7,000 feet per second is needed.

8. What exhaust velocity must the propellant supply to place the launch vehicle from the previous problem in orbit?

Solution. Solving the equation

$$30{,}000 \text{ ft/sec} = c(2.30) \log 11,$$

we get

$$c = \frac{30{,}000}{2.30 \log 11} \text{ ft/sec}$$

$$= 12{,}500 \text{ ft/sec.}$$

Exhaust velocities of 12,500 feet per second and more are available from a mixture of liquid hydrogen and liquid oxygen. However, large engines suitable for using this propellant mixture for launches from the surface of Earth have not yet been perfected.

9. It is apparent from the rocket equation that the burnout velocity increases when the mass ratio increases. We can get a higher mass ratio by using a solid propellant because the stiff rubberlike propellant mass serves as part of the structure. If no payload, or a very small payload, is included, a solid-propellant rocket could have a mass ratio of about 19. A typical average exhaust velocity for a solid propellant might be about 8,000 feet per second. Could this launch vehicle achieve a 100-mile Earth orbit?

Solution. Using the rocket equation,

$$\begin{aligned} v &= (8{,}000)(2.30)\log 19 \\ &= (18{,}400)(1.28) \\ &= 23{,}550 \text{ or } 23{,}600 \text{ ft/sec.} \end{aligned}$$

The speed achieved is much less than that needed for orbit.

10. The solution to the problem pointed out in the previous examples is to use staging. That is, the launch vehicle is divided into two or more parts or stages. As soon as the propellant has been all burned in the first stage, there is a brief coast during which the heavy motors and structure in the first stage are jettisoned and permitted to fall into the ocean. Freed from this deadweight, the second-stage motors are much more effective; the same procedure is repeated for the remaining stages.

Let us design a two-stage vehicle to place a payload into Earth orbit. We shall make three assumptions: (1) that the structure weight of each stage is 10 percent of the fuel weight, the remaining weight being payload; (2) that the gain in velocity is divided equally among the stages, each contributing 15,000 feet per second to the required final velocity of 30,000 feet per second; and (3) that all stages use the same propellant with an exhaust velocity of 12,000 feet per second. This third assumption in particular is unrealistic because no first-stage propellant in use today produces an exhaust velocity this high, whereas second- and third-stage propellants produce higher exhaust velocities than this. However, an assumed exhaust velocity of 12,000 feet per second is satisfactory as an overall average. The total weight at liftoff is to be 100,000 pounds.

Solution. First stage:

$$\begin{aligned} v &= c(2.30)\log R_1 \\ 15{,}000 &= (12{,}000)(2.30)\log R_1 \end{aligned}$$

$$\log R_1 = \frac{15{,}000}{27{,}600} = 0.543$$

$$R_1 = 3.492 = 3.5$$

$$R_1 = \frac{F_1 + S_1 + P_1}{S_1 + P_1} = 3.5$$

$$= \frac{100{,}000}{S_1 + P_1} = 3.5$$

$$S_1 + P_1 = \frac{100{,}000}{3.5} = 28{,}600$$

$$F_1 = 100{,}000 - 28{,}600 = 71{,}400$$

By assumption (1),

$$S_1 = (0.10)(71{,}400) = 7{,}140 \text{ lb}$$

$$P_1 = 28{,}600 - 7{,}140 = 21{,}460 \text{ lb.}$$

NOTE: The payload of 21,460 pounds for the first stage includes all of the remaining weight, including the entire second stage and orbital payload.

Second stage:

$$\log R_2 = \frac{15{,}000}{27{,}600} = 0.543$$

$$R_2 = \frac{F_2 + S_2 + P_2}{S_2 + P_2} = 3.5$$

$$S_2 + P_2 = \frac{21{,}460}{3.5} = 6{,}130$$

$$F_2 = 21{,}460 - 6{,}130 = 15{,}330$$

$$S_2 = (0.10)(15{,}300) = 1{,}533$$

$$P_2 = 6{,}130 - 1{,}533 = 4{,}597 \text{ or } 4{,}600 \text{ lb.}$$

Our design for the two-stage launch vehicle may be checked as follows:

	Pounds
Weight of fuel:	
F_1	71,400
F_2	15,330
Total	86,730
Weight of structure:	
S_1	7,140
S_2	1,533
Total	8,673
Weight of orbital payload	4,597
Total weight of vehicle	100,000

Thus, although the single-stage launch vehicle discussed in problem 7 could not place any payload into orbit, this two-stage vehicle can place nearly 5 percent of its weight into Earth orbit.

11. Show that when all stages use the same propellant, the total mass ratio of a multiple-stage launch vehicle is equal to the product of the individual mass ratios.

Solution. Indicate the burnout velocities and mass ratios of the first, second, third stages, etc., by the subscripts 1, 2, 3, etc. Then, using a three-stage vehicle as an example,

$$v_1 + v_2 + v_3 = 2.30c \log R_1 + 2.30c \log R_2 + 2.30c \log R_3$$
$$v = (2.30c)(\log R_1 + \log R_2 + \log R_3)$$
$$v = (2.30c)(\log R_1R_2R_3)$$

NOTE: Making the structure stronger so that it can support large payloads reduces the mass ratios. However, if we have several stages, the total mass ratio can become very high, producing much greater performance.

12. Using the equation derived in problem 11, show that the launch vehicle constructed in problem 10 can indeed orbit its payload.

Solution. Given $R_1R_2 = (3.5)\ (3.5) = 12.25$

$$v = 2.30c \log 12.25$$
$$= (2.30)(12{,}000)(1.09)$$
$$= (27{,}600)(1.09)$$
$$= 30{,}084 \text{ or } 30{,}000 \text{ ft/sec.}$$

The launch vehicle will impart sufficient velocity to overcome drag losses and insert the payload into a 100-mile Earth orbit. Note that dividing the launch vehicle into stages increases the overall mass ratio to 12.25.

chapter 7

chapter 7

GEOMETRY AND RELATED CONCEPTS

The analysis of many mathematical problems involves geometrical concepts that are not always apparent. The 13 problems contained in this chapter range from the pure and obviously geometrical problems to some which seem, at first reading, only vaguely related to geometry.

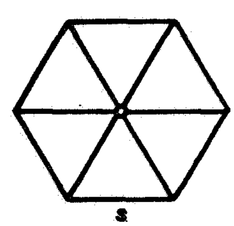
In the first category are problems concerning solar cells, area and load on the feet of a Moon landing craft, and the distance to the horizon from a given altitude above Earth or the Moon.

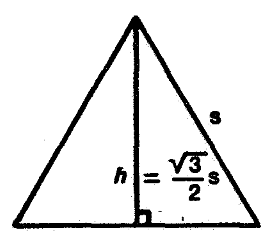
Problems based on geometry but more algebraic in nature include transforming a rectangular map into an isosceles trapezoidal map, relationships of volumes and areas in spacecraft pressure and storage tanks, measuring the diameter of the Moon and Sun, determining the period of a planet, and measuring the distance between Earth and Mars.

PROBLEMS

1. Solar cells convert the energy of sunlight directly into electrical energy. For each square centimeter of solar cell in direct overhead sunlight, about 0.01 watt of electrical power is available. A solar cell in the shape of a regular hexagon is required to deliver 10.4 watts. Find the minimum length of a side.

Solution. The total area required is 10.4 watts/0.01 watt per square centimeter, or 1,040 square centimeters. The regular hexagon can be partitioned into six congruent equilateral triangles, each with an area of $1{,}040/6 = 173$ square centimeters.





The area A of any equilateral triangle with side s may be expressed

$$A = \frac{1}{2}(\text{base})(\text{altitude})$$

$$= \frac{s}{2} \cdot \frac{\sqrt{3}}{2}s$$

$$= \frac{\sqrt{3}}{4}s^2.$$

Solving for s, we have

$$s = \sqrt{\frac{4A}{\sqrt{3}}}$$

$$= \sqrt{\frac{4(173)\text{ cm}^2}{1.73}}$$

$$= \sqrt{400\text{ cm}^2}$$

$$= 20\text{ cm}.$$

2. Solar cells are made in various shapes to utilize most of the lateral area of satellites. A certain circular solar cell with radius r will produce 5 watts. Two equivalent solar cells are made, one being a square with side s and the other an equilateral triangle with side p. Find r in terms of p and also in terms of s.

Solution. For the solar cells to have equivalent outputs, their areas must be equal. Thus for the circle and square, we have

$$A_{\text{circle}} = A_{\text{square}}$$

$$\pi r^2 = s^2$$

$$r = \frac{s}{\sqrt{\pi}}$$

$$= 0.564s.$$

For the circle and equilateral triangle, we have

$$A_{\text{circle}} = A_{\text{equilateral triangle}}$$

$$\pi r^2 = \frac{\sqrt{3}\,p^2}{4}$$

$$r = p\sqrt{\frac{\sqrt{3}}{4\pi}}$$

$$= 0.371p.$$

3. A spacecraft is to be soft landed on the Moon with a maximum impact force of 1,500 pounds. Three legs, each with a large circular foot, will support the spacecraft after landing. As a safety factor, it is assumed that the Moon's surface will support a maximum of 1.5 pounds per square inch.

Allowing for a possible 50-percent overload on any foot, determine the minimum radius of the foot.

Solution. Each foot must be able to support 500 pounds plus a 250-pound overload. Hence, the minimum area A for each foot is

$$A = \frac{750 \text{ lb}}{1.5 \text{ lb/in}^2}$$

$$= 500 \text{ in}^2$$

and the radius of the foot is

$$500 \text{ in}^2 = \pi r^2$$

$$r = \sqrt{\frac{500 \text{ in}^2}{\pi}}$$

$$= 12.6 \text{ in.}$$

4. Because a sphere has the minimum surface area for a given volume and a spherical container has the maximum strength for a given thickness of metal, spherical tanks are often used on spacecraft to hold pressurized gases and propellants. It is decided for a certain application that the volume of a spherical tank must be doubled. What increase is required in the radius?

Solution. Let r and R be the radii of the smaller and larger tanks respectively. Then,

$$V = \frac{4}{3}\pi r^3$$

$$2V = \frac{4}{3}\pi R^3.$$

Dividing,

$$\frac{1}{2} = \frac{r^3}{R^3}$$

$$R^3 = 2r^3$$

$$R = \sqrt[3]{2}\, r = 1.26r.$$

5. Consider a spherical tank of radius r, and a cylindrical tank with radius R and altitude equal to the diameter $2R$.

a. Compute R in terms of r if the volumes of the two tanks are equal.

Solution. If R is the radius of the cylindrical tank, then

$$V = \pi R^2 \times 2R = 2\pi R^3.$$

Because the volumes are equal,

$$2\pi R^3 = \frac{4}{3}\pi r^3$$

$$R = \sqrt[3]{\frac{2}{3}}\, r = 0.874r.$$

b. How do the surface areas of the two tanks compare?

Solution. The area of the cylinder is

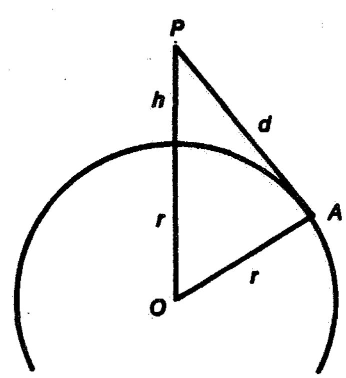
$$\begin{aligned} A &= 2\pi R^2 + (2\pi R \times 2R) \\ &= 6\pi R^2 \\ &= 6\pi\left(\sqrt[3]{\frac{2}{3}}\,r\right)^2 \\ &= 6\pi r^2(0.874)^2 \\ &= 14.4r^2. \end{aligned}$$

The area of the sphere with equal volume is

$$\begin{aligned} A &= 4\pi r^2 \\ &= 12.6r^2. \end{aligned}$$

Thus the surface of the sphere is about 87.5 percent of the area of the cylinder.

6. A spacecraft is at P, at an altitude h above Earth's surface, as pictured in the accompanying drawing. The distance to the horizon is d, and r is the radius of Earth.



a. Derive an equation for d in terms of r and h.

Solution. Because PA is tangent to the circle at A, angle PAO is a right angle. Then

$$\begin{aligned} r^2 + d^2 &= (r+h)^2 \\ d^2 &= (r+h)^2 - r^2 \\ &= 2rh + h^2 \\ d &= \sqrt{2rh + h^2}. \end{aligned}$$

b. Find the distance to the horizon if $h = 100$ miles. Use 3,960 miles for the radius of Earth.

Solution.

$$\begin{aligned} d &= \sqrt{2(3{,}960)(100) + (100)^2} \\ &= \sqrt{802{,}000} \\ &= 10^2\sqrt{80.2} \\ &= 896 \text{ mi.} \end{aligned}$$

c. It is apparent that for near-Earth orbits, h will be small in comparison with r, so that discarding the h^2 term introduces only a small error. The formula then simplifies to $d = \sqrt{2rh}$. Find d with the simplified formula, and compute the percent of error that results when the h^2 term is dropped.

Solution.

$$\begin{aligned} d &= \sqrt{2(3{,}960)(100)} \\ &= \sqrt{792{,}000} \\ &= 10^2\sqrt{79.2} \\ &= 890 \text{ mi.} \end{aligned}$$

The percent of error is $\frac{6}{896} = 0.0067 = 0.67$ percent.

7. Solve problem 6 with respect to the Moon's horizon for a spacecraft 70 miles above the surface of the Moon. Use 1,080 miles for the radius of the Moon.

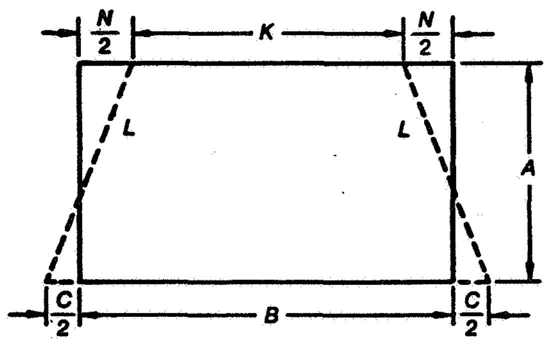
Solution.

$$\begin{aligned} d &= \sqrt{2(1{,}080)(70) + (70)^2} \\ &= \sqrt{156{,}100} \\ &= 10^2\sqrt{15.6} \\ &= 395 \text{ mi.} \end{aligned}$$

$$\begin{aligned} d &= \sqrt{2(1{,}080)(70)} \\ &= \sqrt{151{,}200} \\ &= 10^2\sqrt{15.1} \\ &= 389 \text{ mi.} \end{aligned}$$

The percent of error is $\frac{6}{395} = 0.015 = 1.5$ percent.

8. Some phases of instrumentation mapping on space shots require that a rectangular map be transformed into an isosceles trapezoid map with the same height and perimeter. Consider the following problem.



Transform the rectangle with sides A and B into an isosceles trapezoid with sides L and bases K and $B + C$. The perimeter P and height A must remain constant. Find the new length K in terms of A, B, and C. (Note that the lower base increases by an amount C, whereas the upper base decreases by an amount N, where $N > C$.)

Solution. Equating the perimeters, we have

$$\begin{aligned} P_{\text{rectangle}} &= P_{\text{trapezoid}} \\ 2A + 2B &= (B + C) + K + 2L \\ K &= 2A + 2B - (B + C) - 2L. \end{aligned}$$

Note that $B = K + N$, or $N = B - K$. Applying the Pythagorean theorem, we have

$$L = \sqrt{A^2 + \left(\frac{N}{2} + \frac{C}{2}\right)^2} = \sqrt{A^2 + \left(\frac{B - K + C}{2}\right)^2}.$$

Then

$$K = 2A + 2B - B - C - 2\sqrt{A^2 + \left(\frac{B - K + C}{2}\right)^2}$$

or

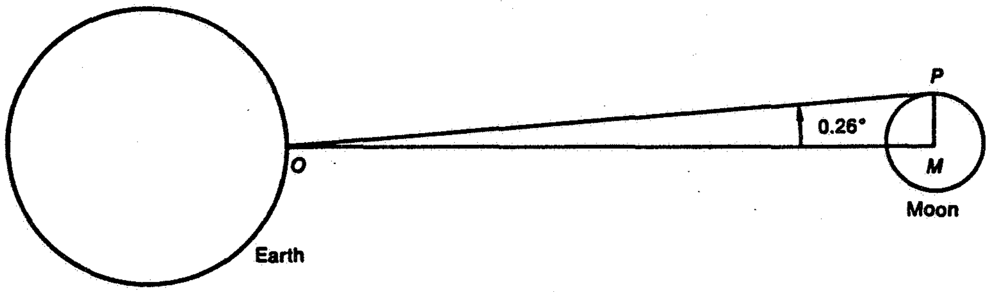
$$K - 2A - B + C = -2\sqrt{A^2 + \left(\frac{B - K + C}{2}\right)^2}.$$

Squaring both sides and simplifying, we have

$$\begin{aligned} [-2A - (B - K - C)]^2 &= 4A^2 + (B - K + C)^2 \\ 2BC - 2KC &= 4AB - 4AK - 4AC - 2BC + 2KC \\ -4KC + 4AK &= 4AB - 4AC - 4BC \\ 4K(A - C) &= 4(AB - AC - BC) \\ K &= \frac{AB - AC - BC}{A - C}. \end{aligned}$$

9. The average angle subtended by the Moon for an observer on Earth is 0.52° or 0.00907 radian. If the average distance from an observer on

Earth to the center of the Moon is known to be 384,400 kilometers, find the diameter of the Moon. Assume *POM* is a right triangle.



Solution. Using the tangent function, the radius of the Moon *MP* is found to be

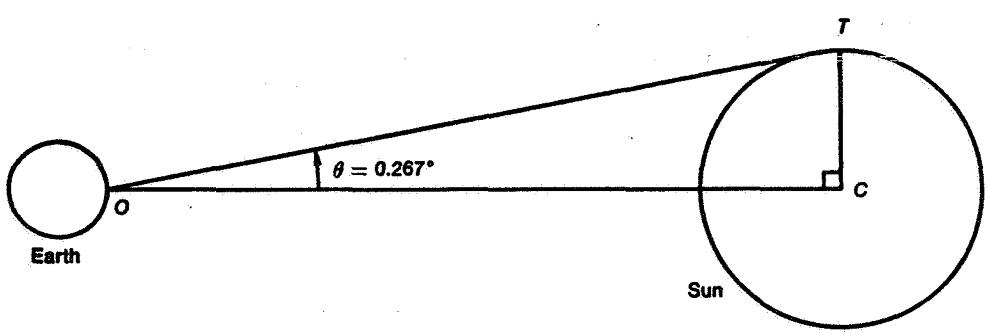
$$\begin{aligned} MP &= (\tan 0.26°)(384{,}400 \text{ km}) \\ &= (0.00454)(384{,}400 \text{ km}) \\ &= 1{,}745 \text{ km}, \end{aligned}$$

and the diameter of the Moon is $2MP =$ approximately 3,490 kilometers, or 2,168 miles. The accepted diameter is 2,160 miles.

Alternate solution. Because 0.00907 radian is such an extremely small angle, the length of the arc it subtends very closely approximates the radius of the Moon. Using the formula $S = r\theta$, where S is the length of a circular arc, r is the radius of the circle, and θ is the radian measure of the angle subtended by the arc, the diameter is found to be

$$\begin{aligned} D_m &= (384{,}400 \text{ km})(0.00907) \\ &= 3{,}487 \text{ km or } 2{,}167 \text{ mi}. \end{aligned}$$

10. The average angle subtended by the Sun for an observer on the surface of Earth is 0.533°. Assuming that the diameter of the Sun is 866,000 miles, find the distance from the surface of Earth to the center of the Sun. Assume *OCT* is a right triangle.



Solution. Consider the right triangle *OCT*. Because the total angle subtended by the Sun as viewed by an Earth observer is 0.533°, angle TOC is one-half the angle subtended or 0.267°.

The distance OC between Earth's surface and the center of the Sun may be calculated by using the tangent function:

$$\tan \theta = \frac{TC}{OC} \quad \text{or} \quad OC = \frac{TC}{\tan \theta}$$

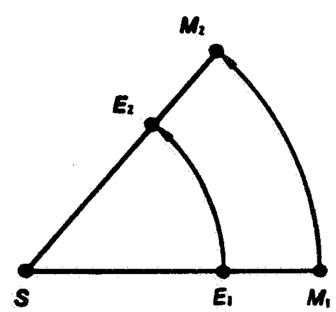
$$= \frac{433{,}000 \text{ mi}}{0.00466} = 92{,}900{,}000 \text{ mi.}$$

11. Determine the period of revolution of the planet Mars about the Sun. The period of Earth is 365 days (approximately), and Earth and Mars are in opposition (Earth is directly between the Sun and Mars), about every 780 days.

We know according to Kepler's laws that the period of Mars is greater than the period of Earth because the radius of orbit is greater for Mars.

NOTE: The period of Mars is less than 1,000 days.

Solution. Let S, E_1, M_1 and S, E_2, M_2 represent the positions at the first and second oppositions as indicated in the figure.



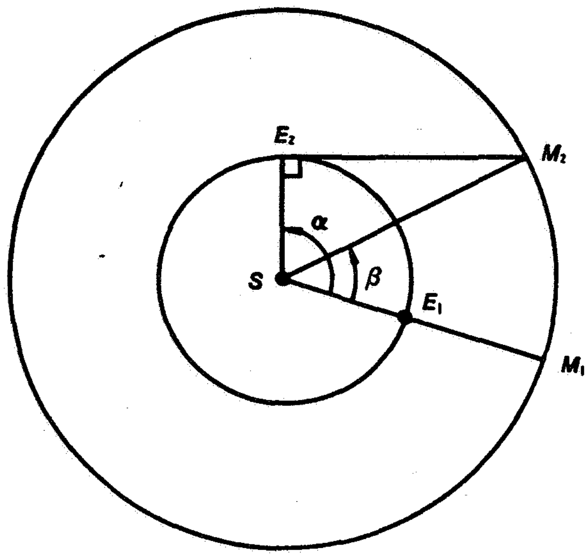
Each day, Earth moves an angular distance of (360°/365). Thus in the 780 days between oppositions, Earth moves an angular distance of 780 days $\times$ (360°/365 days) $= 769°$. Accordingly, angles E_1SE_2 and M_1SM_2 $= 769° - 720° = 49°$. Between oppositions Mars moves an angular distance of 360° plus 49°, or 409°. Therefore the period of Mars is

$$780 \text{ days } \frac{360°}{409°} = 686.5 \text{ days.}$$

Note that if we assumed Mars moves only 49° between oppositions, the period would be greater than 5,000 days. On the other hand, if we assumed Mars moves $2(360°) + 49°$, its period would be equal to that of Earth, which is impossible because the radius of orbit for Mars is greater than that of Earth.

12. Determine the distance between the Sun and Mars in astronomical units, AU. One AU is the mean distance from Earth to the Sun. We are given that 106 days after Mars is in opposition (see previous problem), the Sun, Earth, and Mars form a right triangle with the right angle at Earth.

Solution.



Angle E_2SM_2 is equal to angle $\alpha - \beta$, where α is the angle through which Earth has moved,

$$\begin{aligned}\alpha &= 106 \text{ days} \frac{360°}{365 \text{ days}} \\ &= 104.5°,\end{aligned}$$

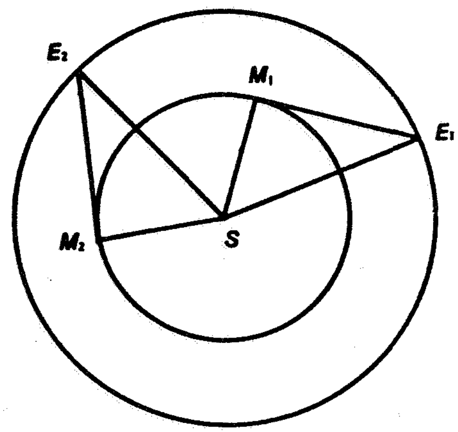
and β is the angle through which Mars has moved,

$$\begin{aligned}\beta &= 106 \text{ days} \frac{360°}{687 \text{ days}} \\ &= 55.5°.\end{aligned}$$

The distance SM_2 between the Sun and Mars in astronomical units is

$$\sec(\alpha - \beta) = \sec 49° = 1.52.$$

13. From Earth, the planet Mercury appears to oscillate about the Sun, appearing at elongation (its maximum angular distance from the Sun as seen from Earth) every 58 days. Earth and Mercury revolve in the same direction, counterclockwise as viewed from the north pole of the Sun. Determine the period of revolution of the planet Mercury.



Solution. An elongation occurs at SE_1M_1. Fifty-eight days later Mercury is at elongation on the other side of the Sun, and another 58 days later it is at the elongation SE_2M_2. During the 116 days Mercury has traversed one revolution plus the arc M_1M_2. Earth has traversed in 116 days the angular distance $116 \frac{360°}{365} = 114°$. Now the triangles SE_1M_1 and SE_2M_2 are congruent. Arc E_1E_2 is 114°, and therefore arc M_1M_2 is also 114°. Hence the period of Mercury is

$$360° \frac{116 \text{ days}}{360° + 114°} = 88.1 \text{ days}$$

chapter 8

chapter 8

TRIGONOMETRY

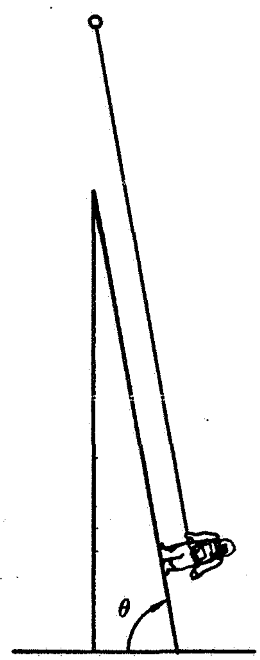
In space-related science, trigonometry has many applications ranging from solutions of right triangles to problems of a complex analytical nature. Seventeen problems from diversified areas are presented in this chapter.

Problems requiring basically the solution of right triangles involve finding lengths of parallels of latitude, angles between satellites, altitudes, climb rates, climb angles, and the tracking of model rockets. A series of navigation problems deals with both oblique and right triangles.

The power output of a solar cell is investigated in terms of the angle of the incident sunlight. The law of sines and the law of cosines are used in several problems concerning radar acquisition of satellites.

PROBLEMS

1. The weight of an astronaut on the Moon is one-sixth his weight on Earth. This fact has a marked effect on such simple acts as walking, running, jumping, and the like. To study these effects and to train astronauts for working under lunar gravity conditions, scientists at NASA Langley Research Center have designed an inclined plane apparatus to simulate reduced gravity.



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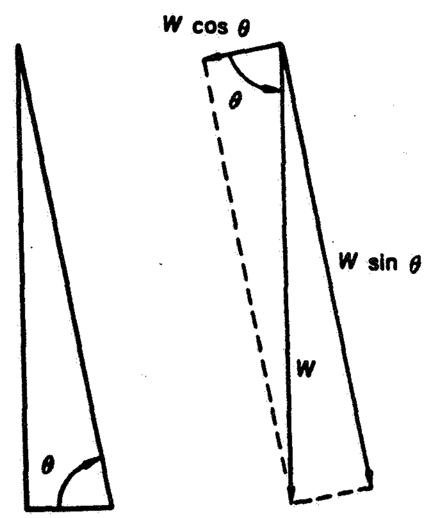
The apparatus consists of an inclined plane and a sling that holds the astronaut in a position perpendicular to the inclined plane. The sling is attached to one end of a long cable which runs parallel to the inclined plane. The other end of the cable is attached to a trolley that runs along a track high overhead. This device allows the astronaut to move freely in a plane perpendicular to the inclined plane.

a. Let W be the weight of the astronaut and θ the angle between the inclined plane and the ground. Make a vector diagram to show the tension in the cable and the force exerted by the inclined plane against the feet of the astronaut.

Solution. The weight of the astronaut is resolved into two components, one parallel to the inclined plane, the other perpendicular to it. These components are $W \sin \theta$ and $W \cos \theta$, respectively. To be in equilibrium, the component $W \sin \theta$ must be balanced by the tension in the cable, and the component $W \cos \theta$ must be balanced by the force exerted by the inclined plane.

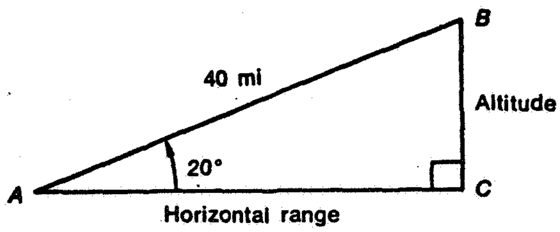


b. From the point of view of the astronaut in the sling, the inclined plane is the ground and his weight, that is, the downward force against the inclined plane, is $W \cos \theta$. What is the value of θ required to simulate lunar gravity? What is the tension in the cable?

Solution. To simulate lunar gravity we must have $W \cos \theta = W/6$. Thus $\cos \theta = 1/6 = 0.1667$, and $\theta = 80°24'$ to the nearest minute. The tension in the cable is $W \sin 80°24' = 0.986\ W$.

2. A radar station tracking an aircraft indicates the elevation angle to be 20° and the slant range to be 40 miles. Determine the altitude and horizontal range of the aircraft.

Solution.



The altitude is

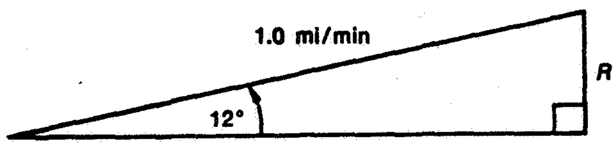
$$BC = (40 \text{ mi})(\sin 20°)$$
$$= 13.7 \text{ mi}$$

and the horizontal range is

$$AC = (40 \text{ mi})(\cos 20°)$$
$$= 37.6 \text{ mi}.$$

3. In 1 minute, an airplane climbing at a constant angle of 12° has flown a distance of 1.0 mile measured along its line of flight. Find the rate of climb of the airplane in miles per minute.

Solution.

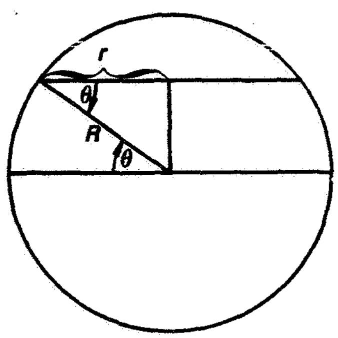


Computing the rate of climb R using the sine function yields

$$\sin 12° = \frac{R}{1.0 \text{ mi/min}}$$
$$R = 0.2079 = 2.1 \text{ mi/min}.$$

4. Show that the length of any parallel of latitude around Earth is equal to the equatorial distance around Earth times the cosine of the latitude angle.

Solution.



By the definition of the cosine function, $\cos \theta = r/R$ or $r = R \cos \theta$. The length of the parallel of latitude is C_p. If C_e denotes the average circumference of Earth, then

$$\begin{aligned} C_p &= 2\pi r \\ &= 2\pi R \cos\theta \\ &= C_e \cos\theta. \end{aligned}$$

5. Find the length of the 30° parallel, north or south latitude. Use $2\pi = 6.283$ and $R = 3{,}960$ miles.

Solution. Applying the formula for the length of a parallel of latitude derived in problem 4 gives

$$\begin{aligned} C_p &= (24{,}900 \text{ mi})(\cos 30°) \\ &= (24{,}900 \text{ mi})(0.866) \\ &= 21{,}560 \text{ or } 21{,}600 \text{ mi}. \end{aligned}$$

6. Determine the length of the Arctic Circle (66°33′ N).

Solution. Using the formula from problem 4, the length is

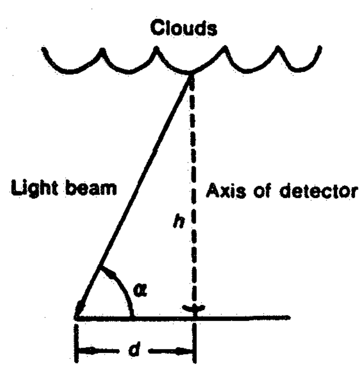
$$\begin{aligned} C_p &= (24{,}900 \text{ mi})(\cos 66°33') \\ &= (24{,}900 \text{ mi})(0.39795) \\ &= 9{,}910 \text{ mi}. \end{aligned}$$

7. How far is it "around the world" along the parallel of 80° N latitude?

Solution. Using the result of problem 4, the distance is

$$\begin{aligned} C_p &= (24{,}900 \text{ mi})(\cos 80°) \\ &= (24{,}900 \text{ mi})(0.17365) \\ &= 4{,}320 \text{ mi}. \end{aligned}$$

8. A sweeping light beam is used with a light-source detector to determine the height of clouds directly above the detector, as in the following diagram.



With the axis of the detector vertical, the light beam is allowed to sweep from the horizontal ($\alpha = 0$) to the vertical ($\alpha = 90°$). When the beam illuminates the base of the clouds directly above the detector, as in the figure, the angle α is read, and with d known, the height h can be computed.

a. Express h in terms of an appropriate trigonometric function of α and d.

Solution. Applying the definition of the tangent function gives

$$h = d(\tan \alpha).$$

b. If the light source is 1,000 feet from the detector and the angle is 45°, compute the height of the cloud.

Solution. Using the equation from part a gives

$$\begin{aligned} h &= (1{,}000 \text{ ft})(\tan 45°) \\ &= 1{,}000 \text{ ft}. \end{aligned}$$

c. If the height of the cloud is 2,050 feet and the distance d is 1,000 feet, compute the angle α.

Solution. Using the same equation, we find

$$\begin{aligned} 2{,}050 \text{ ft} &= (1{,}000 \text{ ft})(\tan \alpha) \\ 2.050 &= \tan \alpha \\ \alpha &= 64°. \end{aligned}$$

d. Find the angle α when clouds are 1,000 feet high and the light source is located 100 feet from the detector.

Solution. Applying the same equation again gives

$$\begin{aligned} 1{,}000 \text{ ft} &= (100 \text{ ft})(\tan \alpha) \\ 10 &= \tan \alpha \\ \alpha &= 84.29° \text{ or about } 84°. \end{aligned}$$

9. The light source in problem 8 must be reasonably close to the detector so that the illumination of the cloud above the detector is sufficiently strong to be detected. At many U.S. National Weather Service stations two beam sources are used, one 800 feet and the other 1,600 feet from the detector. To have reliable readings, α may not exceed 85°.

a. If $\alpha = 85°$ and $d = 1{,}600$ feet, compute the height of the cloud.

Solution. Using the equation from problem 8a, the height is

$$\begin{aligned} h &= (1{,}600 \text{ ft})(\tan 85°) \\ &= 18{,}288 \text{ or } 18{,}300 \text{ ft}. \end{aligned}$$

b. If $\alpha = 85°$ for the light source at 800 feet, find α for the light source at 1,600 feet (assuming the same cloud height).

Solution. Working with the closer light source, the cloud height is found to be 9,144 feet. Thus the angle for the other light source is

$$\begin{aligned} 9{,}144 \text{ ft} &= (1{,}600 \text{ ft})(\tan \alpha) \\ 5.715 &= \tan \alpha \\ \alpha &= 80.07° \text{ or } 80°. \end{aligned}$$

10. In problem 8, notice that as the beam rotates from $\alpha = 0°$ to $\alpha = 90°$ at a constant angular rate, the point of intersection of the beam with the axis accelerates upward.

a. Complete the sentence "The smaller the angle the ____ the speed of the point of intersection."

Solution. Because the point of intersection of the beam accelerates upward as α goes from 0° to 90°, "the smaller α, the *slower* the speed of the point of intersection."

b. If $d = 1{,}000$ feet, compute the difference of the height of intersection of beam and axis for $\alpha = 20°$ and $\alpha = 25°$.

Solution. The heights when α is 20° and 25° are, respectively

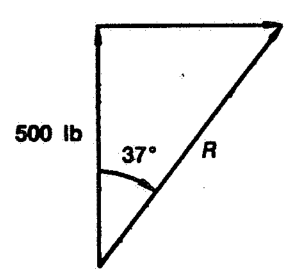
$$\begin{aligned} h &= (1{,}000 \text{ ft})(\tan 20°) \\ &= 364 \text{ ft} \end{aligned}$$

and

$$\begin{aligned} h &= (1{,}000 \text{ ft})(\tan 25°) \\ &= 466 \text{ ft.} \end{aligned}$$

Thus the difference in height is 466 − 364 feet or 102 feet.

11. A spacecraft designed to soft land on the Moon has three feet that form an equilateral triangle on level ground and each of the three legs makes an angle of 37° with the vertical. If the impact force of 1,500 pounds is evenly distributed, find the force in each leg.



Solution. Consider one leg. Five hundred pounds is the vertical component of force R acting at 37° from the vertical. Thus

$$\cos 37° = \frac{500 \text{ lb}}{R}$$

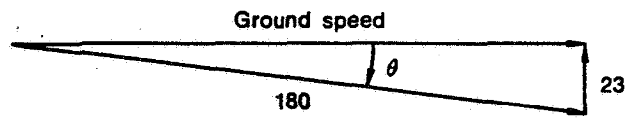
$$R = \frac{500 \text{ lb}}{\cos 37°}$$

$$= 626 \text{ lb}.$$

12. Consider a flight from Chicago to Boston to be along a west-to-east direction, with an airline distance of 870 statute miles. A light plane having an airspeed of 180 miles per hour makes the round trip.

a. How many flying hours does it take for the round trip with a constant southerly wind of 23 miles per hour? What are the headings for the two parts of the round trip? Disregard magnetic variation.

Solution. Let θ be the angle necessary to compensate for the wind.



Then

$$\sin \theta = \frac{23}{180} = 0.128$$

and

$$\theta = 7°21'.$$

Hence the ground speed of the plane is

$$(\cos 7°21')(180 \text{ mi/hr}) = 179 \text{ mi/hr}.$$

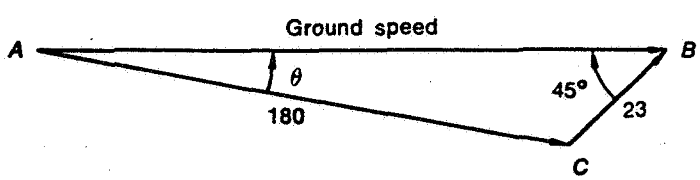
The round trip will take

$$\frac{1{,}740 \text{ mi}}{179 \text{ mi/hr}} = 9.72 \text{ hr or about 9 hr 43 min.}$$

The heading for the trip from west to east is $90° + 7°21'$, or $97°21'$, and the heading for the trip from east to west is $270° - 7°21'$, or $262°39'$.

b. How many flying hours will it take for the round trip with a constant southwest wind of 23 miles per hour? What headings will the pilot use for the two parts of the trip? Disregard magnetic variation.

Solution. For the eastbound trip the law of sines is applied to determine θ.



$$\sin \theta = \frac{23}{180} \sin 45° = 0.090$$

$$\theta = 5°10'.$$

Applying the law of sines again, the ground speed represented by AB is determined as follows:

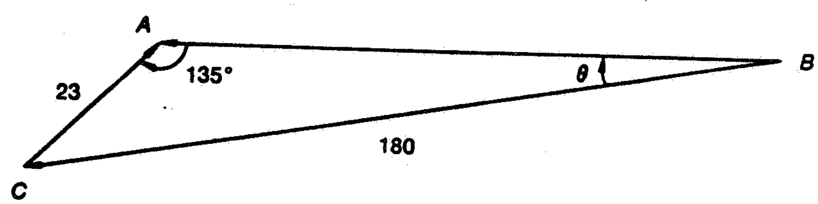
$$\frac{AB}{\sin C} = \frac{180}{\sin B} \qquad \text{or} \qquad AB = \frac{\sin 129°50'}{\sin 45°} \times 180 \text{ mi/hr}$$

$$= 197 \text{ mi/hr}.$$

Thus the time required for the eastbound trip is

$$\frac{870 \text{ mi}}{197 \text{ mi/hr}} = 4.42 \text{ hr}.$$

For the westbound trip, θ is again $5°10'$ and the ground speed is found by use of the law of sines.



$$AB = \frac{\sin 39°50'}{\sin 135°} \times 180 \text{ mi/hr} = 163 \text{ mi/hr},$$

and the time required is

$$\frac{870 \text{ mi}}{163 \text{ mi/hr}} = 5.34 \text{ hr}.$$

Thus the total time for the round trip is 9.76 hours, or 9 hours 46 minutes. The heading for the eastbound trip is $90° + 5°10'$ or $95°10'$ and the heading for the westbound trip is $270° - 5°10'$, or $264°50'$.

13. Magnetic variation is a correction or adjustment that has to be considered after you have computed the heading for an aircraft flight. Because of the fact that the magnetic north pole is not located at the geographic north pole (it is actually in northern Canada), the north-seeking compass will point *west* of north in the eastern part of the United States, and *east* of north in the western part. These are called west and east (magnetic) variation, respectively, and are indicated on some maps by lines called isogonic lines (lines of indicated value of magnetic declination) showing values of the magnetic variation for any point. If you flew on a magnetic course of 0° (north) from Boston, you would be flying about 15°30′ west of north. If you really want to fly north, your compass would read 15°30′. Thus you must add 15°30′ to your computed heading to allow for the magnetic variation, on any heading. Conversely, if you were flying in the vicinity of Seattle, you would have to subtract 22° from any computed heading because the magnetic deviation there is 22° E. (For example, if you want a heading of due east (90°) from Seattle, your compass would read 68°.)

For the trip and conditions of problem 12a, adjust the computed headings to take into account magnetic variation. Consider variation at Chicago to be 2° E and at Boston 15°30′ W.

a. Find initial and final headings for the eastbound trip.

Solution. Making the proper adjustments gives

$$\text{Initial heading (at Chicago)} = 97°21' - 2° = 95°21',$$

$$\text{Final heading (at Boston)} = 97°21' + 15°30' = 112°51'.$$

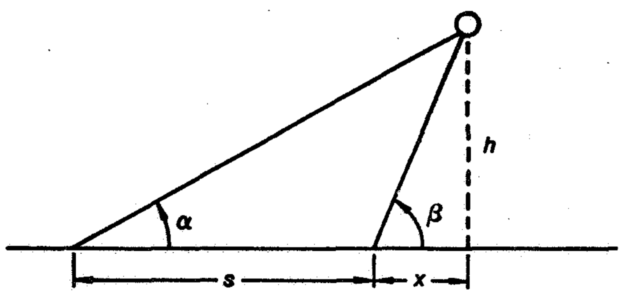
b. Find the initial and final headings for the westbound trip.

Solution. The desired headings are

$$\text{Initial heading (at Boston)} = 262°39' + 15°30' = 278°9',$$

$$\text{Final heading (at Chicago)} = 262°39' - 2° = 260°39'.$$

14. Two tracking stations s miles apart measure the elevation angle of a weather balloon to be α and β, respectively. Derive a formula for the altitude h of the balloon in terms of the angles α and β. Ignore the Earth's curvature.



Solution. Writing an equation for the cotangent of each angle and solving for x gives

$$\cot \alpha = \frac{s + x}{h}$$

$$x = h \cot \alpha - s$$

and

$$\cot \beta = \frac{x}{h}$$

$$x = h \cot \beta.$$

Now the two expressions for x are equated:

$$h \cot \alpha - s = h \cot \beta.$$

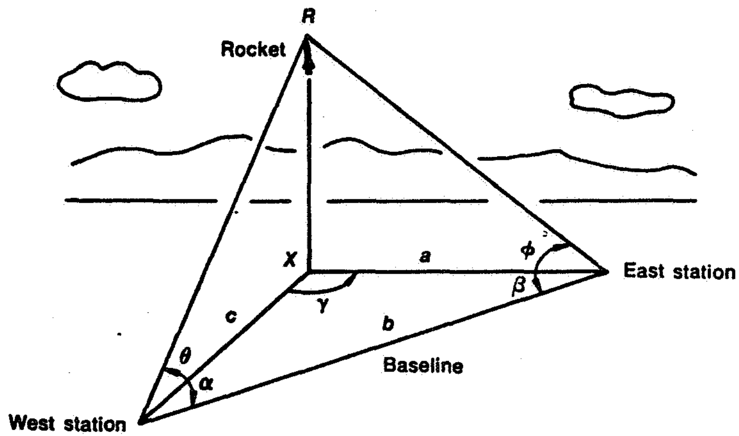
Thus

$$h(\cot \alpha - \cot \beta) = s$$

or

$$h = \frac{s}{\cot \alpha - \cot \beta}.$$

15. A typical setup for tracking model rockets is shown in the following sketch.



Theodolites, which are instruments used for measuring horizontal and vertical angles, are set up and leveled so that their azimuth dials are horizontal. They are zeroed in by sighting at each other along the baseline. While zeroed in, their azimuth and elevation dials are set at zero.

When a model rocket is launched, both station operators track the rocket until it reaches maximum altitude. Tracking then ceases and the scopes are locked in final position. Azimuth and elevation angles on each theodolite area are read. On some ranges, these data are communicated to the launch area by means of a telephone system. On other ranges, data are recorded at each tracking station and later taken to the launch area for final reduction.

a. Assume that you are given distance b and angles α, β, θ, and ϕ. Derive an equation for RX, the altitude of the model rocket, in terms of the given data.

Solution. Point X is directly beneath the model R, and the distance RX is the altitude of the model. We find an expression for the distance and solve the triangle R-X-West in the vertical plane to find RX.

Using the law of sines in trigonometry gives

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \gamma} = \frac{c}{\sin \beta}$$

or

$$c = \frac{b \sin \beta}{\sin \gamma} = \frac{b \sin \beta}{\sin [180° - (\alpha + \beta)]}.$$

Because R is directly above X by definition, the angle R-X-West is a right angle. We can therefore compute the western triangle as follows:

$$\tan \theta = \frac{RX}{c} \qquad \text{or} \qquad RX = c \tan \theta.$$

Substituting for c, we find

$$RX = \frac{b \sin \beta \tan \theta}{\sin [180° - (\alpha + \beta)]}.$$

In a similar manner the other right vertical triangle may be solved to give

$$RX = \frac{b \sin \alpha \tan \phi}{\sin [180° - (\alpha + \beta)]}.$$

The two values of RX may be compared; and if they differ by more than about 10 percent, an error is apparent. Otherwise, the track is good. The average of the two values of RX gives a more accurate value of the altitude achieved by the model.

A general rule for accuracy in the tracking of model rockets is that the angles are rounded to the nearest degree and the altitude is rounded to the nearest 10 feet. If the digit to be rounded is a 5 and the preceding digit is an odd digit, then the 5 is dropped and the preceding digit is increased by 1. If the digit preceding the 5 is an even digit, the 5 is simply dropped. Accordingly, a correctly rounded altitude will always be an even number.

b. Given a 1,000-foot baseline, tracking East azimuth $\beta = 23°$, tracking East elevation $\theta = 36°$, tracking West azimuth $\alpha = 45°$, and tracking West elevation $\phi = 53°$, find RX and determine whether the track is good.

Solution. Applying the derived equation gives

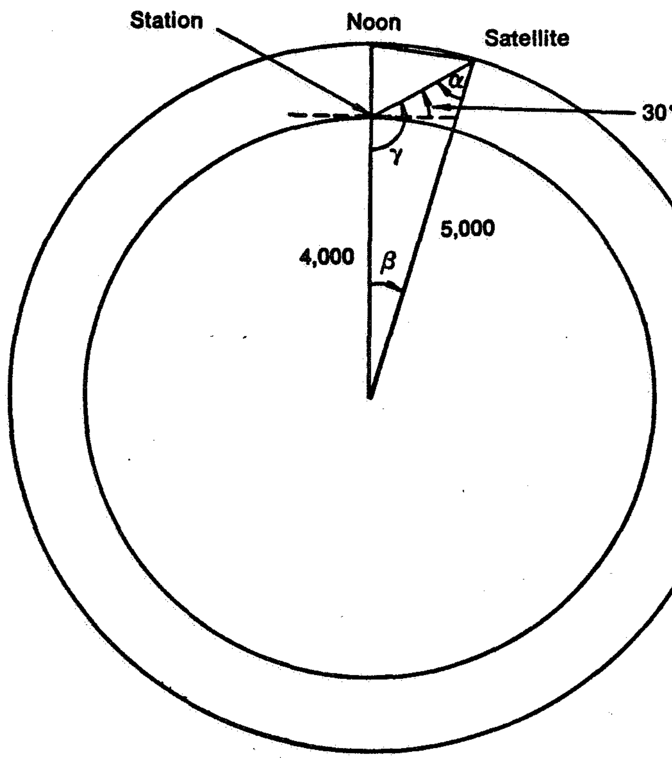
$$\begin{aligned} RX &= \frac{b \sin \beta \tan \theta}{\sin [180° - (\alpha + \beta)]} \\ &= \frac{1{,}000 \text{ ft } (\sin 23°)(\tan 53°)}{\sin [180° - (45° + 23°)]} \\ &= \frac{1{,}000 \text{ ft } (0.391)(1.327)}{\sin 112°} \\ &= (0.391)(1.327)(1{,}079 \text{ ft}) \\ &= 560 \text{ ft}. \end{aligned}$$

Similarly, solving the other triangle gives $RX = 554$ feet.

The average altitude is 557 feet, but rounding makes it 560 feet. Both 560 and 554 are within 10 percent of the average, so the track is good.

16. A satellite traveling in a circular orbit 1,000 miles above Earth is due to pass directly over a tracking station at noon. Assume that the satellite takes 2 hours to make an orbit and that the radius of Earth is 4,000 miles.

a. If the tracking antenna is aimed 30° above the horizon, at what time will the satellite pass through the beam of the antenna?



Solution. From the law of sines,

$$\frac{\sin \alpha}{4{,}000} = \frac{\sin \gamma}{5{,}000}$$

$$\sin \alpha = \frac{4{,}000 \sin 120^\circ}{5{,}000} = 0.693.$$

Hence

$$\alpha = 43.9^\circ$$

and

$$\beta = 180^\circ - (120^\circ + 43.9^\circ) = 16.1^\circ.$$

Time between $\beta = 16.1^\circ$ and $\beta = 0.0^\circ$ is $\frac{16.1^\circ}{360^\circ}$ (120 min) $= 5.4$ min. Thus the satellite will pass through the beam of the antenna at 12:00 − 5.4 minutes or 11:54.6 a.m.

b. Find the distance between the satellite and tracking station at 12:03 p.m.

Solution. Computing angle β gives

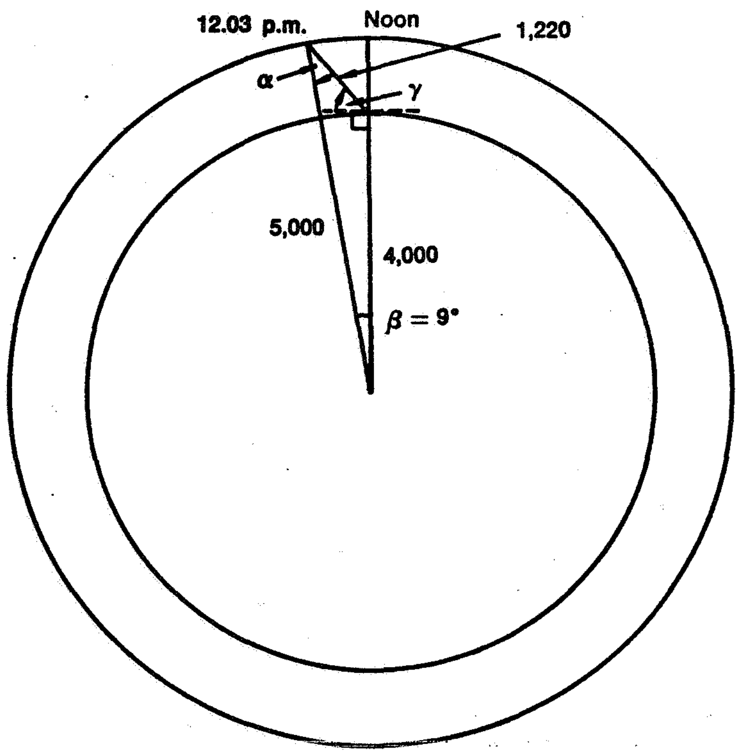
$$\beta = \frac{3\text{ min}}{120\text{ min}} 360° = 9°.$$

By the law of cosines,

$$\begin{aligned} x^2 &= (4{,}000)^2 + (5{,}000)^2 - 2(4{,}000)(5{,}000)\cos 9° \\ &= (16 + 25 - 39.51) \times 10^6 \\ &= 1.49 \times 10^6 \\ x &= 1.22 \times 10^3 = 1{,}220. \end{aligned}$$

Thus the distance between the satellite and tracking station is 1,220 miles.

c. At what angle above the horizon should the antenna be pointed so that its beam will intercept the satellite at 12:03 p.m.?



Solution. Again, applying the law of sines,

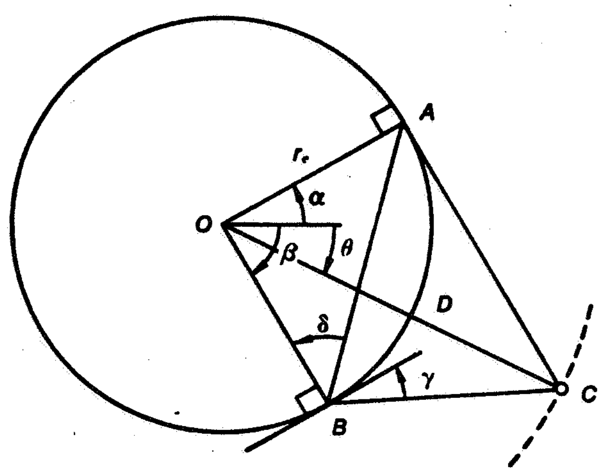
$$\frac{\sin 9°}{1{,}220} = \frac{\sin(\gamma + 90°)}{5{,}000}$$

$$\sin(\gamma + 90°) = \frac{5{,}000}{1{,}220}\sin 9° = 0.641$$

$$\cos\gamma = 0.641$$

$$\gamma = 50°8' \text{ or } 50°$$

17. A satellite C in an equatorial orbit is being tracked by two stations A and B both located on the Equator.



Given $r_e = 4{,}000$ miles, $\alpha = 30°$, $\beta = 50°$, and $\gamma = 30°$, compute the height h of the satellite C above the Equator, by following these steps:

a. Notice that triangle AOB is isosceles. Compute the length AB using the law of sines and the fact that $2\delta = 180° - (\alpha + \beta) = 100°$.

b. Compute the length AC. The angle ABC at B is $90° - \delta + \gamma = 70°$, and angle OAC is a right angle.

c. Now use the Pythagorean theorem in triangle ACO to compute the length OC.

d. Compute the altitude h.

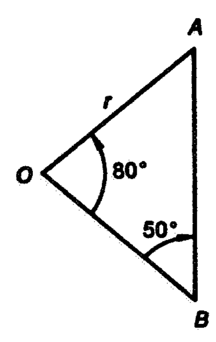
Solution.

a.

$$2\delta = 180° - (\alpha + \beta) = 180° - 80° = 100°$$

$$\delta = 50°.$$

From the given data it is known that angle $AOB = \alpha + \beta = 80°$.



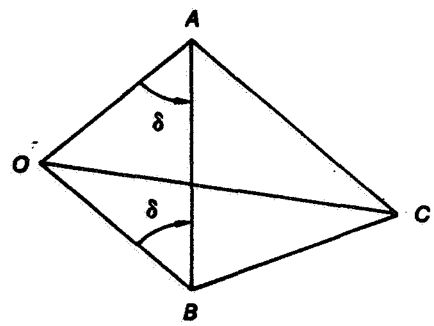
Now by the law of sines

$$\frac{AB}{\sin 80°} = \frac{4{,}000\text{ mi}}{\sin 50°}$$

$$AB = \frac{(4{,}000\text{ mi})(0.985)}{0.766}$$

$$= 5{,}140\text{ mi.}$$

b. We know that $\angle BAC = 90° - \angle OAB$, but $\angle OAB = \delta = 50°$. Thus $\angle BAC = 40°$ and $\angle ACB = 180° - (70° + 40°) = 70°$.



We see that triangle ABC is isosceles, therefore $AB = AC = 5{,}140$ miles.

c. By the Pythagorean theorem

$$(OC)^2 = (AO)^2 + (AC)^2$$
$$= (4{,}000\text{ mi})^2 + (5{,}140\text{ mi})^2$$
$$= 16{,}000{,}000\text{ mi}^2 + 26{,}420{,}000\text{ mi}^2$$
$$= 42{,}420{,}000\text{ mi}^2$$
$$OC = 6{,}513\text{ mi.}$$

d. The height h of the satellite above the Equator is represented by CD, which is

$$OC - OD = 6{,}513\text{ mi} - 4{,}000\text{ mi}$$
$$= 2{,}513\text{ mi.}$$

chapter 9

chapter 9

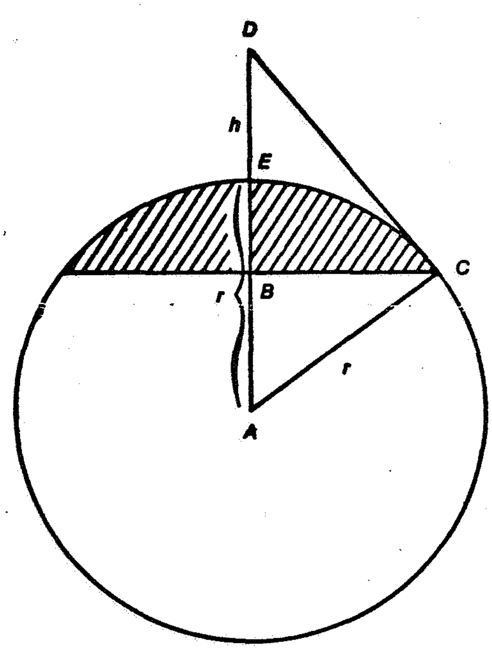
GEOMETRY AND TRIGONOMETRY RELATED TO THE SPHERE

This chapter deals only with mathematics related to the sphere. Some problems involve spherical geometry, some use plane trigonometry to analyze plane figures related to the sphere, and a few use spherical trigonometry to study figures on the surface of the sphere.

A series of problems deals with the percent of the surface that is visible from a given altitude above a spherical body. Others are concerned with distances and angles of lines of sight involved in the tracking of satellites by tracking stations. One problem considers the rotation of the "line of apsides" of an orbit caused by the equatorial bulge, and gives the formula for the angle of inclination that yields zero rotation. Other problems are concerned with the launch azimuth needed to achieve a given angle of inclination and with the location of the highest and lowest latitudes of an orbit.

PROBLEMS

The following figure applies to problems 1 through 8. The radius of Earth AE is taken to be 3,960 miles.



1. Derive a formula for finding what fraction of the surface of a sphere of radius r can be seen from an altitude h above the surface of the sphere.

Solution. In the preceding drawing, we note that triangles ABC and ACD are similar.

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$AB = \frac{(AC)^2}{AD} = \frac{r^2}{r+h}$$

$$\begin{aligned} BE &= r - AB \\ &= r - \frac{r^2}{r+h} = \frac{r^2 + rh - r^2}{r+h} \\ &= \frac{rh}{r+h}. \end{aligned}$$

Let A_z be the area of the zone with altitude BE. Then

$$A_z = 2\pi r(BE) = 2\pi r\left(\frac{rh}{r+h}\right).$$

Let A_e be the area of Earth. Then

$$A_e = 4\pi r^2$$

$$\begin{aligned} \frac{A_z}{A_e} &= \frac{2\pi r}{4\pi r^2}\left(\frac{rh}{r+h}\right) \\ &= \frac{h}{2(r+h)}. \end{aligned}$$

2. Gemini 10, with astronauts Collins and Young aboard, flew in an orbit with perigee of 100 miles and apogee of 168 miles. What percent of Earth's surface was visible from each of these two altitudes? Assume that Earth is a sphere with radius of 3,960 miles.

Solution. Substituting $h = 100$ and $r = 3{,}960$ in the derived formula,

$$\frac{A_z}{A_e} = \frac{100}{2(3{,}960 + 100)} = \frac{50}{4{,}060} = 0.012.$$

Thus the astronauts were able to observe 1.2 percent of Earth's surface from the perigee altitude of 100 miles. The problem from the apogee is solved in a similar manner. In this case $h = 168$, and

$$\frac{A_z}{A_e} = \frac{168}{2(3{,}960 + 168)} = \frac{84}{4{,}128} = 0.02.$$

Therefore 2.0 percent of Earth's surface was visible from the apogee altitude of 168 miles.

3. Gemini 11 achieved an orbit with an apogee of 853 miles, a new altitude record for manned flight at that time. What percent of Earth's surface was visible to astronauts Conrad and Gordon aboard Gemini 11 from apogee altitude?

Solution. Because $h = 853$,

$$\frac{A_z}{A_e} = \frac{853}{2(3{,}960 + 853)} = \frac{853}{9{,}626} = 0.0886 = 0.089.$$

The astronauts were able to observe 8.9 percent of Earth's surface from an altitude of 853 miles.

4. Discuss the manner in which the fraction $\frac{A_z}{A_e}$ varies with the altitude h.

Solution. Intuition suggests that as h increases, the value of $\frac{A_z}{A_e}$ should vary from zero to ½. On the surface of Earth, the fraction is zero. As h increases, so does the fraction, and yet it must always be less than ½; i.e., one cannot hope to view more than a hemisphere at any one time. A little algebra bears this out.

$$\frac{A_z}{A_e} = \frac{h}{2(r + h)}$$

is certainly zero when $h = 0$. Observe that

$$\frac{A_z}{A_e} = \frac{1}{2\left(\frac{r}{h} + 1\right)}.$$

As h increases, the denominator of the right-hand side decreases, which forces the entire fraction $\frac{A_z}{A_e}$ to increase. Furthermore, as $h \to \infty, \frac{r}{h} \to 0$, and consequently $\frac{A_z}{A_e}$ approaches $\frac{1}{2(1 + 0)} = ½$.

5. Find what altitude from Earth the astronaut must be to see one-quarter of Earth's surface at one time.

Solution. Substituting in the equation,

$$\begin{aligned} \frac{1}{4} &= \frac{h}{2(3{,}960 + h)} \\ 4h &= 2(3{,}960 + h) \\ 2h &= 7{,}920 \\ h &= 3{,}960. \end{aligned}$$

Therefore the astronaut would have to be 3,960 miles above Earth. The first astronauts to travel that far from Earth were the three American astronauts, Anders, Borman, and Lovell, on board Apollo 8, which orbited the Moon on Christmas Day, 1968.

6. What percent of Earth's surface were the Apollo 8 astronauts able to see as they passed the Moon, a distance of about 235,000 miles from Earth?

Solution. Because $h = 235{,}000$ miles,

$$\frac{A_s}{A_e} = \frac{235{,}000}{2(3{,}960 + 235{,}000)} = \frac{117{,}500}{238{,}960}$$
$$= 0.4917 \text{ or } 49.2 \text{ percent.}$$

7. The lunar altitude of the Command Module on several Apollo flights has been 69 miles. What fraction of the surface of the Moon can be seen from this altitude?

Solution. The formula previously derived applies as well to the Moon as to Earth. Evidently $h = 69$ and $r = 1{,}080$.

$$\frac{A_s}{A_e} = \frac{69}{2(1{,}080 + 69)} = \frac{69}{2{,}298} = 0.030 = 3.0 \text{ percent.}$$

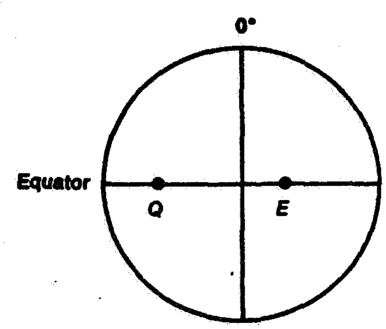
8. What percent of Earth's surface can be "seen" from a synchronous satellite, whose altitude is 22,300 miles above Earth?

Solution. Because $h = 22{,}300$ miles,

$$\frac{A_s}{A_e} = \frac{22{,}300}{2(3{,}960 + 22{,}300)} = 0.425 = 42.5 \text{ percent.}$$

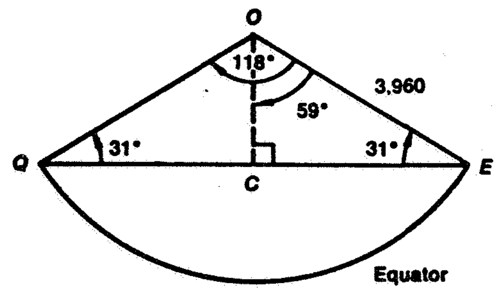
A synchronous satellite can relay messages to about 42.5 percent of Earth's surface. Thus three such satellites evenly spaced around the Earth over the Equator could form the basis of a communications network covering the entire Earth.

9. NASA tracking stations are located near the Equator; one in Ethiopia at 40° E longitude, the other near Quito, Ecuador, at 78° W longitude. Assume both stations, represented by E and Q in the figure, are on the Equator and that the radius of Earth is 3,960 miles.



a. Find the distance between the two stations on a straight line through the Earth. The angular distance between the two meridians of longitude is $78° + 40° = 118°$.

Solution.



To find the distance, it suffices to consider right triangle OCE, for $2CE = QE$. Hence

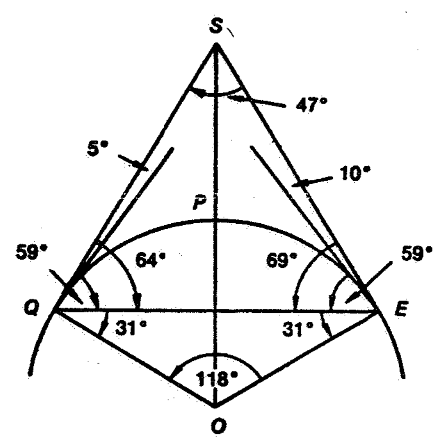
$$\begin{aligned} QE &= 2CE = 2(3{,}960 \text{ mi})(\sin 59°) \\ &= (7{,}920)(0.85717) \\ &= 6{,}790 \text{ mi}. \end{aligned}$$

b. Given that the circumference of Earth is 24,900 miles, find the distance along the surface of Earth between the two tracking stations.

Solution. To find the circumference, 360° or the whole circle was considered. In this case, however, only 118° are to be considered. Thus the distance is

$$\frac{118°}{360°}(24{,}900 \text{ mi}) = 8{,}160 \text{ mi}.$$

10. A satellite in equatorial orbit is observed at the same instant from the tracking stations in Quito and Ethiopia. The angle of elevation from Quito, above horizontal, is 5° and from Ethiopia, 10°.



Find the distance of the satellite from Earth at the instant of observation.

Solution. From the preceding problem, $QE = 6{,}790$ miles. Angles OQE and OEQ each measure 31°. Therefore we know that the angles between QE and the local horizontals at Q and E each equal 59°. Because the angle of elevation at Q is 5°, angle $EQS = 64°$. Similarly, angle $QES = 69°$. We apply the law of sines to triangle QSE. (Note that angle OES contains 100°.)

$$\frac{SE}{\sin 64°} = \frac{6{,}790 \text{ mi}}{\sin 47°}$$

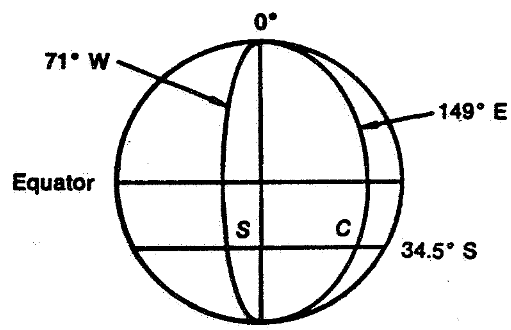
$$SE = 8{,}340 \text{ mi.}$$

Applying the law of cosines in triangle OSE gives

$$OS = \sqrt{(3{,}960 \text{ mi})^2 + (8{,}340 \text{ mi})^2 - 2(3{,}960 \text{ mi})(8{,}340 \text{ mi})(\cos 100°)}$$
$$= 9{,}834 \text{ mi.}$$

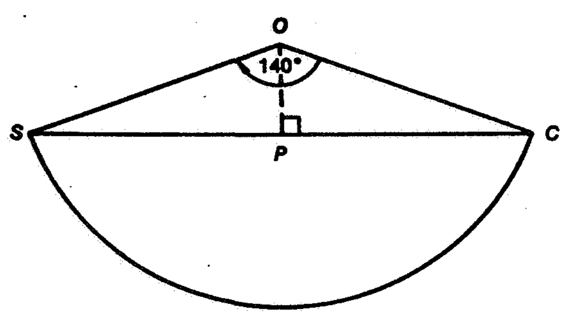
Thus the distance from Earth $PS = 9{,}834$ mi $- 3{,}960$ mi $= 5{,}874$ miles, or about 5,870 miles.

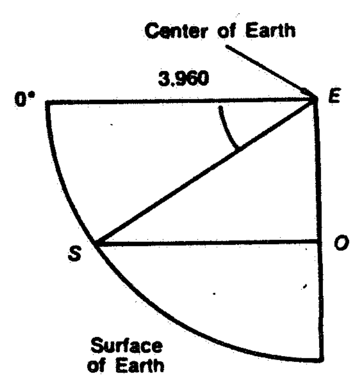
11. Two NASA tracking stations are located near the 34.5° parallel of south latitude, one near Santiago, Chile, at 71° W longitude; the other near Canberra, Australia, at 149° E longitude. Assume that both stations, represented by C and S in the figure, are at 34.5° S latitude and that the radius of Earth is 3,960 miles.



a. What is the distance between the two stations in a straight line through Earth?

Solution.





In the first drawing, angle SOC is $360° - (71° + 149°) = 140°$. It is necessary to use the right triangle SOE in the second drawing to find OS. We note that angle ESO contains 34.5°.

Now

$$\begin{aligned} OS &= (\cos 34.5°)(3{,}960 \text{ mi}) \\ &= (0.82413)(3{,}960 \text{ mi}) \\ &= 3{,}264 \text{ mi}. \end{aligned}$$

The required distance is SC in the first drawing, but $SC = 2SP$. Hence

$$\begin{aligned} SC = 2SP &= 2(3{,}264 \text{ mi})(\sin 70°) \\ &= (6{,}528 \text{ mi})(0.93969) \\ &= 6{,}134 \text{ or } 6{,}130 \text{ mi}. \end{aligned}$$

b. Given that the circumference of Earth at the Equator is 24,900 miles, find the distance between the two stations along the surface of Earth on the 34.5° S parallel.

Solution. We use the formula derived in Chapter 8, problem 4, $C_p = C_e \cos \theta$, where C_p and C_e are the lengths of the parallel of latitude and of the Equator, respectively.

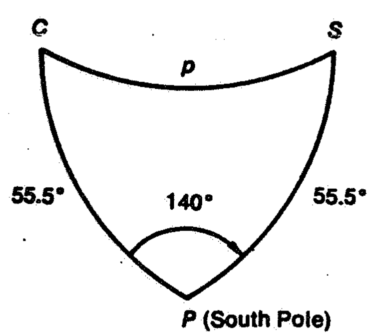
Then

$$\begin{aligned} C_p &= (24{,}900 \text{ mi})(\cos 34.5°) \\ &= (24{,}900 \text{ mi})(0.82413) = 20{,}520 \text{ mi}. \end{aligned}$$

Because only 140° of the total 360° are to be considered, the required distance is

$$\frac{140°}{360°}\, 20{,}520 \text{ mi} = 7{,}980 \text{ mi}.$$

c. Find the distance between the two stations along the surface of the Earth on the great circle passing through the two stations. Note that arcs CP and SP each contain $90° - 34.5° = 55.5°$.



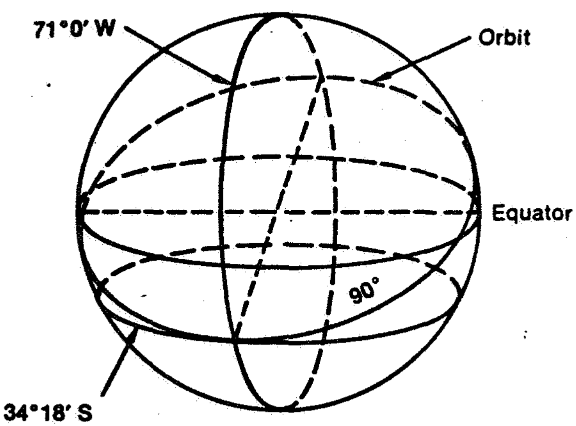
Solution. Using the law of cosines from spherical trigonometry gives

$$\begin{aligned}\cos p &= \cos c \cos s + \sin c \sin s \cos P \\ &= (\cos 55.5°)^2 + (\sin 55.5°)^2(\cos 140°) \\ &= 0.32082 - 0.52029 \\ &= -0.19947 \\ p &= 101°30' = 6{,}090'\end{aligned}$$

Converting p to nautical and statute miles, we have

$$p = 6{,}090 \text{ n mi} = 7{,}004 \text{ s mi} = 7{,}000 \text{ mi.}$$

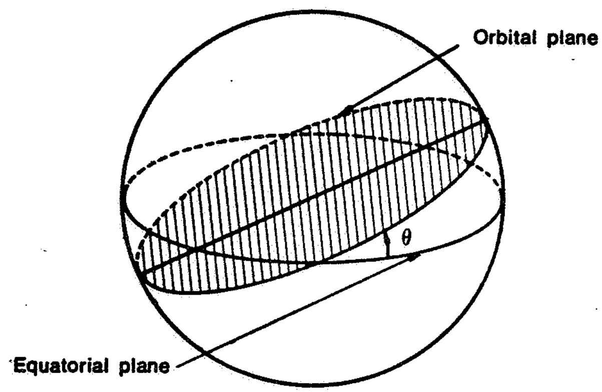
12. A satellite passes directly over Santiago, Chile (34°18′ S, 71°0′ W), at a 150-mile altitude on a circular orbit heading due east at 17,350 miles per hour. How long after passing over Santiago and at what longitude will it next cross Earth's Equator?



Solution. Because the satellite is observed heading due east, it is known to be at its apex, or point of greatest latitude (in this case south latitude). Halfway around its orbit it will be at greatest north latitude. Midway between these apexes it crosses the Equator. Thus the satellite will cross the Equator after an angular distance of 90°, or one-quarter of a sidereal period later. Because an angular distance of 90° at the satellite's altitude is equal to $\frac{2\pi(3{,}960 + 150)}{4} = 6{,}456$ statute miles, the time to travel this distance is $\frac{6{,}456 \text{ miles}}{17{,}350 \text{ miles per hour}}$ (60 minutes per hour) $= 22.33$ minutes. In 22.33 minutes the Earth rotates through 5.57°. Therefore the satellite crosses Earth's Equator 90° east of Santiago minus 5.57° for Earth's rotation, or 84.43° east of Santiago at 13.43° = 13°26′ E longitude.

13. If Earth were a perfect sphere, a satellite in orbit about Earth would travel in a perfect ellipse with the center of Earth at one focus. Actually, there are deviations in Earth satellite orbits because of the equatorial bulge. The bulge causes a rotation in the orbital plane of the major axis (often called "line of apsides") of the orbit. (More information on orbits and the technical language used will be found in the next chapter.) If θ denotes the angle of inclination (angle between Earth's equatorial plane and the orbital plane of the satellite), then the approximate rate of rotation, in degrees per day, of the major axis is given by

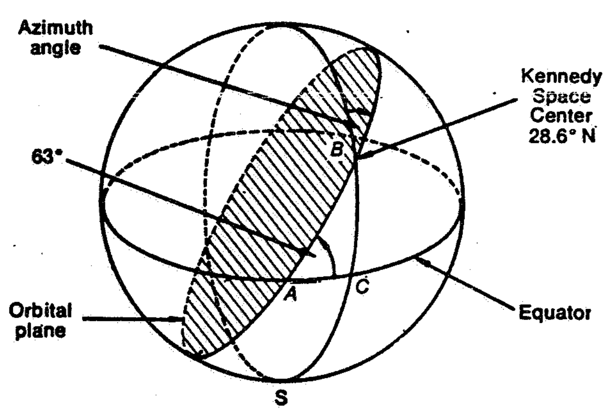
$$\omega = 4(5\cos^2\theta - 1).$$



a. Show that there is no rotation effect ($\omega = 0$) if θ is roughly 63°. For this reason the early Soviet satellites were launched in such a way that their angles of inclination were about 63°. The result was that the perigee point of such an orbit remained over the U.S.S.R., making data transmission optimal.

Solution. Solving the equation $4(5\cos^2\theta - 1) = 0$ gives $\cos\theta = \sqrt{5}/5 = 0.4472$ or $\theta = 63°26'$.

b. At what azimuth angle should a satellite at Kennedy Space Center (latitude 28.6° N) be launched so that its angle of inclination is about 63°? (The orbital plane intersects the surface of Earth in a great circle. By the azimuth angle we mean the angle between orbital plane and the plane of the great circle determined by Kennedy Space Center, the North Pole, and the center of Earth.)



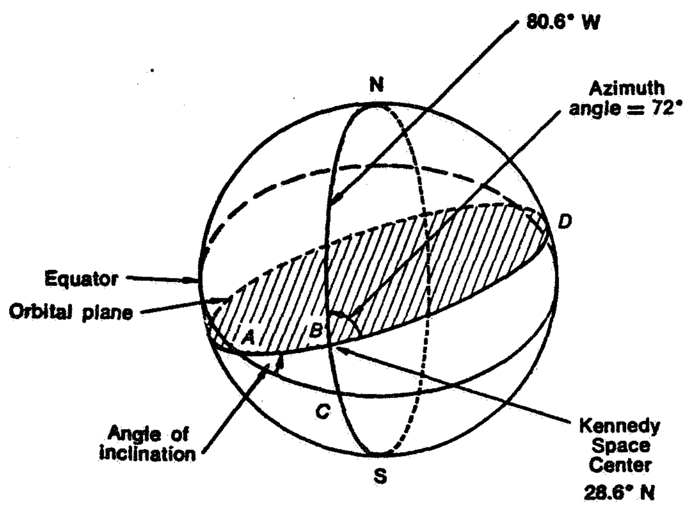
Solution. Consider the right spherical triangle ABC shown in the figure. By Napier's rules for a right spherical triangle

$$\cos 63° = \cos 28.6° \sin B.$$

Now $\log \sin B = \log \cos 63° - \log \cos 28.6° = 9.71356 - 10$, and $B = 31°$ to the nearest degree. Hence the azimuth angle is 31°.

14. On July 16, 1969, Apollo 11, the first flight for a lunar landing, was launched from Kennedy Space Center into a temporary parking orbit, prior to translunar injection. The launch was at an azimuth of 72°. The Kennedy Space Center is located at 28.6° N latitude and 80.6° W longitude.

a. Compute the inclination of the orbital plane to Earth's equatorial plane.



Solution. To find the angle of inclination to the orbital plane, apply Napier's rules for a right spherical triangle to triangle ABC in the drawing.

$$\begin{aligned} \cos A &= \sin 72° \cos 28.6° \\ \log \cos A &= \log \sin 72° + \log \cos 28.6° \\ &= 9.92170 - 10 \\ A &= 33°23' \text{ or } 33°. \end{aligned}$$

b. Compute the highest and lowest latitudes over which the orbit passed.

Solution. This part can be solved by plane geometry. Because the orbital plane is inclined 33° to the plane of the Equator, it will intersect Earth's surface at 33° N latitude. Similarly, the low point of the orbit with reference to the plane of the Equator will occur at 33° S latitude.

chapter 10

chapter 10

CONIC SECTIONS

The mathematics of orbits is one of the most rewarding areas that the teacher or student interested in space technology can study. The theory of orbits grows, of course, out of mathematical properties of the conic sections. The purely mathematical characteristics of the conics have long been of interest to mathematicians. But when one realizes that the conic sections describe the paths along which all bodies in the universe have moved since the beginning of time, these "celestial highways" take on added interest.

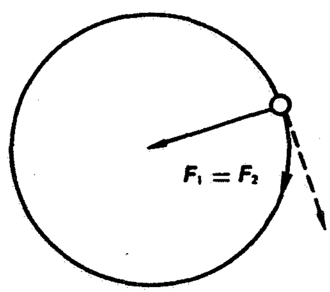
This chapter is organized somewhat differently from the others in the book. Instead of listing individual problems with only occasionally a sequential development of a topic, this chapter attempts to build a logical basis for understanding orbits. Therefore the textual material is longer and the number of individual problems somewhat smaller than in other chapters. In some cases, the text develops a concept and then presents as a problem a similar development or proof that the reader should be able to do. The material is far from complete, and the interested reader may wish to study further the laws of Newton and Kepler and the additional light they throw upon orbit theory. The chapter also deals only with the mathematics of ideal or simple situations, and does not consider the interaction of three or more bodies, nor the effect of perturbing forces. Some individual problems on perturbing forces are found in other chapters.

In contrast to the other chapters, however, this one is open ended. After the basic formulas of orbital mechanics are understood, the teacher or student can find an unlimited number of numerical examples to which these formulas are applicable. News stories of launches of satellites usually give the orbital parameters. It is interesting to check the report mathematically to see whether our mathematical prediction of the behavior of the satellite agrees with that given in the news report. One can investigate many kinds of orbital situations involving not only spacecraft but the celestial bodies in the universe. In fact, the study of the information in this chapter may give many readers their first real glimpse of why bodies throughout the universe move as they do.

PROBLEMS

An understanding of the conic sections is of vital importance to any individual who wishes to understand the basic facts of orbital mechanics. Every gravitational orbit of a satellite, planet, comet, meteor, star, galaxy, or other celestial body is a conic section, with the center of mass of the primary body located at one focus of the conic. Because the simplest nontrivial conic section is the circle, we shall begin with a consideration of circular orbits. Most of us understand from experience Newton's first law of motion, which states that an object in motion continues in a straight line unless it is acted upon by some force. If we wish to make an object move in a circular path rather than in a straight line, we must give it a constant push toward the center. Thus a central, or centripetal, force is required. For example, when we tie a string to an object and whirl it in a circle, the pull of the string is the force which keeps the object in the circular path. If we represent the centripetal force by F_1, then $F_1 = \frac{mv^2}{r}$, where m is the mass of the object, v is its speed or velocity, and r is the radius of the circle.

When a spacecraft is moving in a circular orbit about any primary body, the force toward the center is supplied by the force of gravity F_2. According to Newton's law of universal gravitation, $F_2 = \frac{GMm}{r^2}$. In this equation, G is the constant of universal gravitation, assumed to be constant for all bodies in the universe; M and m are the masses of any two bodies; and r is the distance between their centers of gravity. The physical situation, if these two forces are equal, is represented in the following drawing.



The arrow toward the center represents the force of gravity, the dashed arrow represents the speed, or tangential velocity, of the spacecraft, and the curved arrow indicates the circular path. (In rigorous use, velocity is a vector quantity, because it has both magnitude and direction; whereas speed, having magnitude only, is a scalar quantity. The two terms are often used interchangeably in space literature, and there will be no need for us to differentiate between them here.) Thus the force of gravity holds the body in the circular orbit.

If we set $F_1 = F_2$, we obtain $\frac{mv^2}{r} = \frac{GMm}{r^2}$. Solving for v gives us

$$v = \sqrt{\frac{GM}{r}}.$$

This simple equation enables us to find circular orbital velocities about any primary body, if M is the mass of the body and r is the radius of the orbit measured from the center of mass of the body. Because the value of GM is constant for any primary body, it is convenient to substitute its numerical value rather than to compute the value of the product for each individual problem. If the primary body is Earth, then $GM = 1.24 \times 10^{12}$ cubic miles per hour per hour. Thus for bodies in circular orbits around Earth,

$$v_{\text{Earth}} = \sqrt{\frac{1.24 \times 10^{12}}{r}} \text{ mi/hr},$$

where, of course, the distance r is expressed in miles. (Note that the value $GM = 9.56 \times 10^4$ cubic miles per second per second was used in Chapter 2.)

1. Most manned spacecraft in Earth orbit have been placed at altitudes of about 100 miles or more because atmospheric drag at altitudes below 100 miles causes a rather rapid deterioration of the orbit. Find the velocity needed for a body to stay in Earth orbit at an altitude of 100 miles.

Solution. Using the given equation,

$$\begin{aligned} v_{\text{Earth}} &= \sqrt{\frac{1.24 \times 10^{12}}{3{,}960 + 100}} = \sqrt{\frac{1.24 \times 10^{12}}{4{,}060}} \\ &= 10^3\sqrt{305} = 10^3 \times 17.464 \\ &= 17{,}500 \text{ mi/hr}. \end{aligned}$$

2. The formula for circular orbital velocity is perfectly general and can be applied to orbits about any primary body. G is a universal constant. We need only to change the value of M when we are concerned with another primary of different mass.

a. The mass of the Moon is approximately 0.012 times the mass M of Earth. Write a formula for finding circular orbital velocities about the Moon.

Solution. Multiplying the numerator in the previous equation by 0.012,

$$\begin{aligned} v_{\text{Moon}} &= \sqrt{\frac{1.24 \times 0.012 \times 10^{12}}{r}} \\ &= \sqrt{\frac{1.49 \times 10^{10}}{r}} \text{ mi/hr}. \end{aligned}$$

b. During the Apollo flights the parking orbit for the Command Module about the Moon has an altitude of 69 statute miles. The radius of the Moon is about 1,080 miles. Find the velocity in this orbit.

Solution.

$$v_{\text{Moon}} = \sqrt{\frac{1.49 \times 10^{10}}{1{,}080 + 69}} = \sqrt{\frac{1.49 \times 10^{10}}{1{,}149}}$$
$$= 10^3\sqrt{13} = 3{,}600 \text{ mi/hr.}$$

3. A synchronous Earth satellite is one which is placed in a west-to-east orbit over the Equator at such an altitude that its period of revolution about Earth is 24 hours, the time for one rotation of Earth on its axis. Thus the orbital motion of the satellite is synchronized with Earth's rotation, and the satellite appears from Earth to remain stationary over a point on Earth's surface below. Such communication satellites as Syncom, Early Bird, Intelsat, and ATS are in synchronous orbits. Find the altitude for a synchronous Earth satellite.

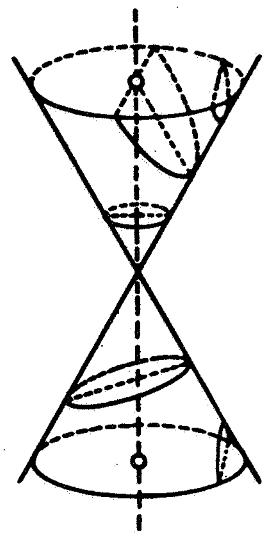
Solution. The velocity can be found from the equation for circular orbital velocity. It can also be found by dividing the distance around the orbit by the time required; that is, $v = \frac{2\pi r}{t}$. Because the two velocities are equal,

$$\frac{2\pi r}{t} = \sqrt{\frac{GM}{r}}$$
$$\left(\frac{2\pi r}{t}\right)^2 r = GM$$
$$r^3 = \frac{GMt^2}{4\pi^2}$$
$$r = \sqrt[3]{\frac{GMt^2}{4\pi^2}}.$$

It is apparent that $t = 24$ hours. Substituting the other values yields

$$r = \sqrt[3]{\frac{1.24 \times 10^{12} \times 24^2}{4 \times 3.14^2}} = 10^4\sqrt[3]{\frac{178.56}{9.86}}$$
$$= 10^4\sqrt[3]{18.1} = 26{,}260 \text{ mi.}$$
$$\text{Altitude} = 26{,}260 - 3{,}960 = 22{,}300 \text{ mi.}$$
$$v = \frac{2 \times 3.14 \times 26{,}260}{24} = 6{,}870 \text{ mi/hr.}$$

To understand orbits, we must know something of the nature and properties of the conic sections. They get their name, of course, from the fact that they may be formed by cutting or sectioning a complete right circular cone (of two nappes) with a plane. Any plane perpendicular to the axis of the cone cuts a section that is a circle. Incline the plane a bit, and the section formed is an ellipse. Tilt the plane still more until it is parallel to a ruling of the cone and the section is a parabola. Let the plane cut both nappes, and the section is a hyperbola, a curve with two branches. It is apparent that closed orbits are circles or ellipses. Open or escape orbits are parabolas or hyperbolas.

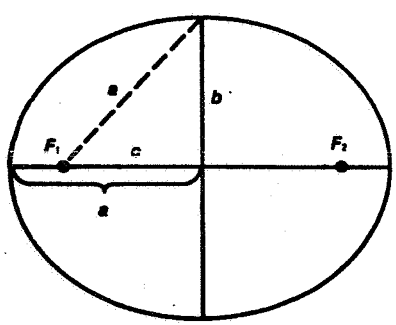


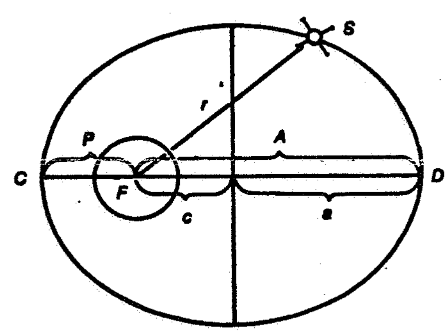
Another way of classifying the conic sections is by means of their eccentricity. If we represent the eccentricity by e, then a conic section is

A circle if $e = 0$,
An ellipse if $0 < e < 1$,
A parabola if $e = 1$,
A hyperbola if $e > 1$.

In actual practice, orbits that are exactly circular or parabolic do not exist because the eccentricity is never exactly equal to 0 or 1.

We shall now derive a group of formulas that are needed in working with orbits. The reader should carefully study and frequently refer to the following drawings. The formulas derived will be numbered for easy reference in solving the problems that follow the discussion.





The first drawing shows the ellipse as it is presented in the literature on analytical geometry. F_1 and F_2 are the two foci, a is the semimajor axis, b is the semiminor axis, c is the distance from a focus to the center, and the eccentricity $e = \frac{c}{a}$. It is apparent that if $c = 0$, F_1 coincides with the center, and $e = 0$. Thus a circle is an ellipse with eccentricity equal to zero. If we move the foci farther from the center, the ellipse becomes stretched out horizontally and narrower vertically, and the eccentricity increases.

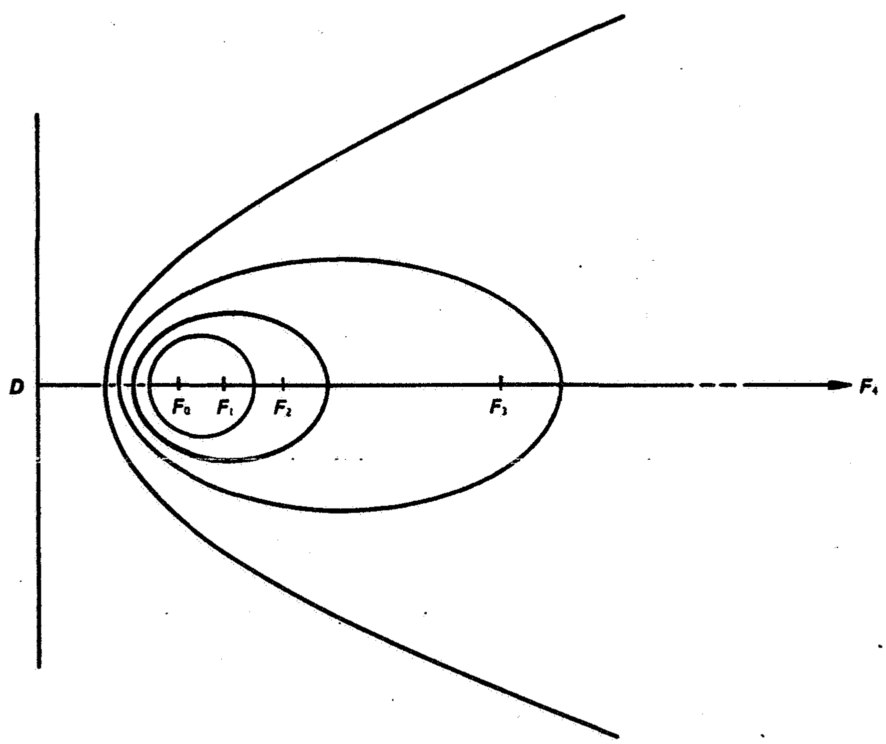
What such changes mean in an orbit can be explained with the second drawing and the two-body formula, or "vis-viva integral,"

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}. \tag{1}$$

(Deriving this equation is beyond the scope of this book.) This drawing shows the primary body located at a focus, while r is the "radial" distance of the satellite S from the center of its primary. If r has the constant value a, the ellipse is a circle and the formula reduces to the familiar one for circular orbital velocity,

$$v_c = \sqrt{\frac{GM}{r}}. \tag{2}$$

The formula for the velocity of a satellite in a parabolic escape orbit can be obtained as a limiting case of equation (1). The following illustration was obtained by drawing graphs of a conic, expressed in polar coordinates, $r = \frac{ep}{1 - e\cos\theta}$. $\left(\text{In this case, the particular conic used was } r = \frac{4e}{1 - e\cos\theta}.\right)$



The directrix and the prime focus F_0 are fixed. The values of e used are, from right to left (toward the directrix), ⅓, ½, ¾, and 1. For the three ellipses obtained, the empty foci are at F_1, F_2, and F_3, respectively. We note that as the eccentricity increases, the empty focus moves to the right, and the vertex moves toward the directrix. As we allow the eccentricity to approach unity, the semimajor axis tends to infinity. When the eccentricity is 1, the ellipse opens to a parabola, and the empty focus F_4 is at infinity. Thus in the case of a parabolic escape orbit,

$$v = \lim_{a\to\infty} \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{\frac{2GM}{r}}.$$

Noting the similarity to equation (2) of the expression for minimum escape velocity, we write

$$v_e = \sqrt{2}\sqrt{\frac{GM}{r}} = \sqrt{2}\, v_c. \tag{3}$$

Thus the minimum, or parabolic, escape velocity can be obtained readily by multiplying the circular orbital velocity at that radius by $\sqrt{2}$. If the velocity imparted to the satellite is greater than this, the satellite simply follows a hyperbolic path, and the eccentricity is greater than 1.

Before we discuss elliptical orbits, it will be necessary for us to avoid ambiguity by clarifying our terminology and mathematical notation. Most of us know from our reading of space events that in NASA news reports the point in an orbit nearest the surface of Earth is called *perigee*, whereas the farthest point from the surface is called *apogee*. These points are indicated by C and D, respectively, in the second ellipse drawn on page 122. In common usage the word is used to refer to either the position of the point or the distance to the point.

However, usage is not uniform and some references state that the distances are measured, not from the surface, but from the center of Earth. In this book, we shall use distances measured from the center. The distances from the center to perigee and apogee will be indicated by P and A, respectively. In most discussions, the context will make this clear. If in any situation confusion could result, then distances from the surface, if used, will be called *perigee altitude* or *apogee altitude*, whereas distances from the center will be called *perigee radius* or *apogee radius*. Incidentally, the mathematics is simpler when distances are measured from the center.

4. Derive a formula for the eccentricity of an elliptical orbit in terms of A and P.

Solution. The following relationships are apparent from the aforementioned drawing,

$$a = \tfrac{1}{2}(A + P),$$
$$c = a - P = \tfrac{1}{2}(A + P) - P = \tfrac{1}{2}(A - P),$$

and

$$e = \frac{c}{a} = \frac{\frac{1}{2}(A - P)}{\frac{1}{2}(A + P)}$$

$$e = \frac{A - P}{A + P}. \qquad (4)$$

This formula provides a quick and easy way of finding the eccentricity of an elliptical orbit. As a check, we note by inspection that $e = 0$ when $A = P$, which is the condition for a circular orbit.

Formulas for velocities at apogee and perigee can be obtained easily from equation (1), the two-body formula. Because $e = \frac{c}{a}$, $c = ea$. But, as we note in the drawing on page 122,

$$A = c + a = ea + a = a(1 + e).$$

Rearranging this equation,

$$\frac{1}{a} = \frac{1 + e}{A}.$$

Obviously at apogee $r = A$. Let $v_A =$ the velocity at apogee. Substituting in equation (1),

$$v_A = \sqrt{GM\left(\frac{2}{A} - \frac{1 + e}{A}\right)},$$

which simplifies to

$$v_A = \sqrt{\frac{GM}{A}(1 - e)}. \qquad (5)$$

5. Derive a formula for v_P, the velocity at perigee.

Solution. Prcceeding as for the velocity at apogee,

$$P = a - c = a - ae = a(1 - e)$$

and

$$\frac{1}{a} = \frac{1 - e}{P}.$$

Substituting in (1),

$$v_P = \sqrt{GM\left(\frac{2}{P} - \frac{1 - e}{P}\right)}$$

which simplifies to

$$v_P = \sqrt{\frac{GM}{P}(1 + e)} \qquad (6)$$

These equations can be written in various other ways, because there are numerous possible ways of expressing relationships among e, c, a, A, and P. The particular form for the formulas reflects personal preference.

6. Show that the velocities at apogee and perigee are inversely proportional to the distances from the center.

Solution. If we divide equation (5) by equation (6), we obtain

$$\frac{v_A}{v_P} = \sqrt{\frac{(1-e)P}{(1+e)A}} = \sqrt{\frac{a(1-e)P}{a(1+e)A}}$$
$$= \sqrt{\frac{P^2}{A^2}} = \frac{P}{A}.$$

Thus the velocity at perigee is inversely proportional to P, etc. That is, when the orbital distance from the center of the primary body is small, the velocity at that point is large; and when the distance is large, the orbital velocity is small. This result agrees with Kepler's second law of planetary motion, which states that a planet moves about the Sun in such a way that the radius vector from Sun to planet sweeps out equal areas in equal times.

7. Derive a formula for the period of an elliptical orbit, given that the period of an elliptical orbit with semimajor axis a is the same as that for a circle with radius $r = a$.

Solution. Following the method used in problem 3, we express the velocity in terms of the distance around the orbit and the time p required to make one transit of the orbit,

$$v = \frac{2\pi r}{p}.$$

Also

$$v = \sqrt{\frac{GM}{r}}.$$

Then

$$\frac{2\pi r}{p} = \sqrt{\frac{GM}{r}}$$
$$\frac{(2\pi r)^2}{p^2} = \frac{GM}{r}$$
$$p^2 = \frac{(2\pi r)^2 r}{GM}$$
$$p = 2\pi\sqrt{\frac{r^3}{GM}}.$$

Because the period is the same when $r = a$, we may write

$$p = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{7}$$

8. An Earth satellite is placed in an elliptical orbit with perigee altitude of 100 miles and apogee altitude of 10,000 miles. Use 3,960 for the radius of Earth.

a. *If injection is at perigee, what must be the injection velocity?*

Solution. We first find the eccentricity as follows:

$$P = 3{,}960 + 100 = 4{,}060.$$

$$A = 3{,}960 + 10{,}000 = 13{,}960.$$

By equation (4),

$$e = \frac{13{,}960 - 4{,}060}{13{,}960 + 4{,}060} = \frac{9{,}900}{18{,}020} = 0.55.$$

By equation (6),

$$v_P = \sqrt{\frac{1.24 \times 10^{12}}{4{,}060}(1.55)} = 10^3\sqrt{473} = 10^3 \times 21.749$$

$$= 21{,}800 \text{ mi/hr.}$$

b. Find the speed at apogee.

Solution. Using equation (5),

$$v_A = \sqrt{\frac{1.24 \times 10^{12}}{13{,}960}(1 - 0.55)} = \sqrt{\frac{1.24 \times 10^{12}}{13{,}960}(0.45)}$$

$$= 10^3\sqrt{40} = 10^3 \times 6.32 = 6{,}320 \text{ mi/hr.}$$

c. Find the period in this orbit.

Solution. Using equation (7),

$$a = \frac{13{,}960 + 4{,}060}{2} = 9{,}010$$

and,

$$p = 2\pi\sqrt{\frac{9{,}010^3}{1.24 \times 10^{12}}} = 6.283\sqrt{\frac{73.1 \times 10^{10}}{1.24 \times 10^{12}}}$$

$$= 6.283\sqrt{59 \times 10^{-2}} = 6.283 \times 0.768$$

$$= 4.825 = 4.83 \text{ hr.}$$

9. During the Apollo flights, the Apollo spacecraft and the third stage (SIVB) of the Saturn V launch vehicle are placed in a parking orbit 117 miles above Earth. Find the velocity and period in this orbit.

Solution. Because $r = 3{,}960 + 117 = 4{,}077$ miles, we find from equation (2),

$$v_c = \sqrt{\frac{1.24 \times 10^{12}}{4{,}077}} = 10^3\sqrt{304} = 17{,}400 \text{ mi/hr.}$$

From equation (7),

$$p = 6.283\sqrt{\frac{4{,}077^3}{1.24 \times 10^{12}}} = 6.283 \times 0.234 = 1.47 \text{ hr.}$$

10. During the flight of Apollo 11, the SIVB stage was reignited and burned long enough to place the Apollo spacecraft on a trajectory to the Moon. At the end of the burn, the spacecraft had a velocity of about 24,230 miles per hour at an altitude of 209 miles. Was the Apollo spacecraft given escape velocity?

Solution. Using equations (2) and (3),

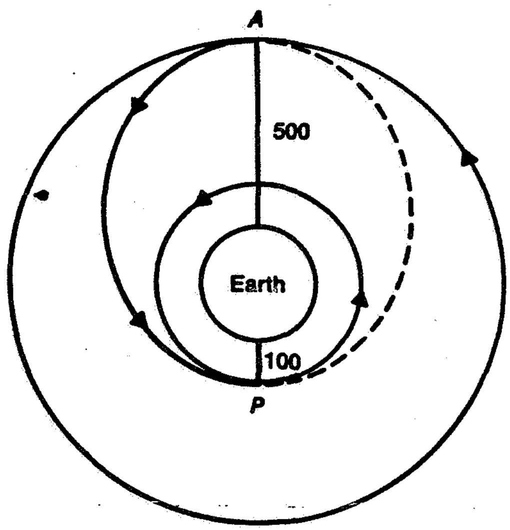
$$v_c = \sqrt{\frac{1.24 \times 10^{12}}{4{,}169}} = 10^3\sqrt{297.4} = 17{,}250 \text{ mi/hr}$$

and

$$v_e = \sqrt{2} \times 17{,}250 = 1.414 \times 17{,}250 = 24{,}390 \text{ mi/hr.}$$

Thus the velocity imparted was about 160 miles per hour less than escape velocity, thereby assuring a free return trajectory. That is, if the major propulsion systems failed, the spacecraft would be going slowly enough to be pulled around and oriented back toward Earth by lunar gravity, the attitude-control system being adequate to make needed course corrections.

11. A spacecraft, as illustrated in the following drawing, is in a circular orbit 500 miles above Earth. It is desired to transfer the spacecraft to a lower circular orbit 100 miles above Earth. Compute the velocity changes needed at A and P to achieve this transfer.



Solution. We first find the eccentricity of the transfer orbit, which is, of course, an ellipse, with $A = 4{,}460$ miles and $P = 4{,}060$ miles.

$$e = \frac{4{,}460 - 4{,}060}{4{,}460 + 4{,}060} = \frac{400}{8{,}520} = 0.047.$$

We then compare the velocities at A in the circular orbit and the elliptical orbit to find what change must be made. Using equations (2) and (5),

$$v_c = \sqrt{\frac{1.24 \times 10^{12}}{4{,}460}} = 16{,}700 \text{ mi/hr}$$

and

$$v_A = \sqrt{\frac{1.24 \times 10^{12}}{4{,}460}(0.953)} = 16{,}300 \text{ mi/hr.}$$

Therefore a propulsion engine on board the spacecraft must be fired long enough so that a retrothrust (opposite to the direction of motion) will slow down the spacecraft by 400 miles per hour. The spacecraft will then leave the 500-mile circular orbit and will follow the elliptical transfer orbit, remaining in it indefinitely unless additional changes in velocity are made.

When the spacecraft reaches the point P, however, we want it to move from the elliptical orbit into the 100-mile circular orbit. Therefore we must use equations (2) and (6) to investigate velocity changes at P.

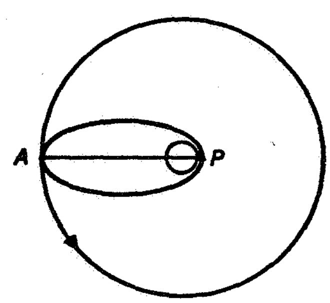
$$v_c = \sqrt{\frac{1.24 \times 10^{12}}{4{,}060}} = 17{,}500 \text{ mi/hr.}$$

$$v_P = \sqrt{\frac{1.24 \times 10^{12}}{4{,}060}(1.047)} = 17{,}900 \text{ mi/hr.}$$

That is, a retrothrust must reduce velocity again, this time also by about 400 mi/hr.

This method of transferring a spacecraft from one orbit to another is known as a Hohmann transfer, named after Walter Hohmann, city engineer of Essen, Germany, who published the method in 1925. There are many paths that could be used to move the spacecraft from the 500-mile to the 100-mile orbit. But the Hohmann-transfer ellipse, requiring only two short burns, is the most economical, taking the minimum amount of energy. Therefore, this method is called a minimum-energy transfer. It has many applications.

12. A satellite is placed into a synchronous orbit by a technique involving a Hohmann-transfer ellipse. We computed in problem 3 that the altitude of such a satellite is about 22,300 miles, and its orbital speed is about 6,870 miles per hour. The following drawing suggests the details.



We shall assume that injection is at the perigee point, which we shall place 100 miles above Earth. Then obviously $P = 3{,}960 + 100 = 4{,}060$, and $A = 3{,}960 + 22{,}300 = 26{,}260$. We wish to find the velocity change needed at A.

Solution.

$$e = \frac{26{,}260 - 4{,}060}{26{,}260 + 4{,}060} = \frac{22{,}200}{30{,}320} = 0.732.$$

$$v_P = \sqrt{\frac{1.24 \times 10^{12}}{4{,}060}(1.732)} = 10^3\sqrt{529} = 23{,}000 \text{ mi/hr.}$$

$$v_A = \sqrt{\frac{1.24 \times 10^{12}}{26{,}260}(0.268)} = 10^3\sqrt{12.7} = 3{,}560 \text{ mi/hr.}$$

But the tangential velocity needed at point A is 6,870 miles per hour. Therefore the velocity of the satellite must be increased in the direction of Earth's rotation by $6{,}870 - 3{,}560 = 3{,}310$ miles per hour. This extra push or kick would be provided by the firing of a motor on board the satellite, and the thrust and firing time must be such as to give the desired increment in velocity. Such a motor to be fired at apogee is called an *apogee motor,* and the thrust it provides is called an *apogee kick.*

The relative efficiency of using this method is easy to understand. The placing of a heavy final stage of the launch vehicle at the synchronous altitude and then having a burn to give the entire assembly circular orbital velocity would take much fuel. Instead we send up to the synchronous altitude only a relatively light satellite and a small apogee motor. The numerical values used in this problem are merely illustrative. If the perigee altitude is higher or lower than the one we have assumed, all of the other numbers are changed.

One more maneuver is needed to make the satellite synchronous. It now has a period equal to the time of rotation of Earth. However, the satellite will appear to be stationary over a given point only if it is in equatorial orbit. Unless corrections were made during launch, the plane of the orbit will be inclined to the plane of the Equator. One method of solving this problem is to fire a motor at the precise instant when the satellite crosses the Equator, adjusting the burning time and direction of thrust so that the vector sum of the burn velocity and the orbital velocity make the angle of inclination equal to zero.

13. The first step in lunar orbit injection in the Apollo 11 flight was to place the spacecraft in an elliptical orbit of 69 by 196 miles, the low point or perilune (corresponding to perigee for Earth) being on the back side of the Moon.

a. Compute the velocity needed at perilune to inject the Apollo spacecraft into this orbit.

Solution. Using the data developed for lunar orbits in problem 2,

$$P = 1{,}080 + 69 = 1{,}149.$$

$$A = 1{,}080 + 196 = 1{,}276.$$

$$e = \frac{1{,}276 - 1{,}149}{1{,}276 + 1{,}149} = \frac{127}{2{,}425} = 0.052.$$

$$v_p = \sqrt{\frac{1.49 \times 10^{10}}{1{,}149}(1.052)} = 10^3\sqrt{13.6} = 3{,}690 \text{ mi/hr}.$$

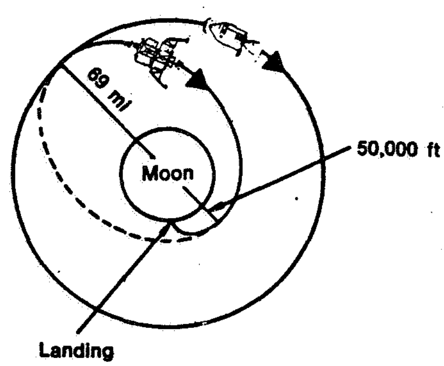
b. Find the period in this orbit.

Solution. Evidently $a = \frac{1}{2}(1{,}276 + 1{,}149) = 1{,}212$ and

$$p = 6.283\sqrt{\frac{1{,}212^3}{1.49 \times 10^{10}}} = 6.283 \times 0.3456$$

$$= 2.17 \text{ hr} = 130 \text{ min}.$$

14. The Lunar Module descent orbit insertion during the Apollo 11 mission began with a Hohmann transfer. The Command and Lunar Modules were in a circular orbit 69 miles above the Moon. The Lunar Module was detached and its descent engine was fired to reduce velocity so that it would enter a 69- by 9-mile lunar orbit. (The perilune altitude of 9 miles was usually given in news reports as 50,000 feet.) Find the reduction in velocity needed to achieve this orbit. The Command Module remained in the 69-mile parking orbit.



Solution. In this case, the change to the elliptical transfer orbit was made at apolune (corresponding to apogee for Earth).

$$A = 1{,}080 + 69 = 1{,}149.$$

$$P = 1{,}080 + 9 = 1{,}089.$$

$$e = \frac{1{,}149 - 1{,}089}{1{,}149 + 1{,}089} = \frac{60}{2{,}238} = 0.027.$$

$$v_A = \sqrt{\frac{1.49 \times 10^{10} \times 0.973}{1{,}149}} = 10^3\sqrt{12.6} = 3{,}550 \text{ mi/hr}.$$

We found in problem 2 that the circular velocity in the 69-mile orbit was 3,600 miles per hour. Thus the reduction in velocity needed, achieved by a retroburn of the Lunar Module descent engine, was 50 miles per hour. At perilune altitude of 9 miles, several retroburns and attitude changes were made, both automatically and manually by the pilot, causing the spacecraft to descend to the surface. If for any reason the descent from the 9-mile (50,000-foot) perilune could not be made, the Lunar Module could have remained indefinitely in the elliptical transfer orbit until a rendezvous and docking with the Command Module could be made. Thus this maneuver, which seemed so tricky and dangerous as we watched before our television sets, was actually a routine Hohmann transfer. The tricky maneuver, requiring some manual control, came when the powered descent to the lunar surface was made from the 50,000-foot altitude.

15. In Chapter 3, problem 8, we discussed some problems related to manned exploration of an asteroid. It has been suggested that if an asteroid were sufficiently small, a source of propulsion could be placed on it to move it into Earth orbit, where it might be used as an object of study or even as a space station. We assumed a spherical asteroid with a diameter of 14 miles. For convenience, we named it A–14. Compute circular and escape velocity at the surface of A–14. Could an astronaut run fast enough on A–14 to put himself into a circular or escape orbit?

Solution. We found in the previous asteroid problem that if M is the mass of Earth, the mass of A–14 is $3{,}314 \times 10^{-12}\,M$. Inserting this multiplier into equation (2),

$$v_c = \sqrt{\frac{1.24 \times 10^{12} \times 3{,}314 \times 10^{-12}}{7}}$$

$$= \sqrt{\frac{4{,}109}{7}} = \sqrt{587} = 24.2 \text{ mi/hr}$$

and

$$v_e = \sqrt{2} \times 24.2 = 1.414 \times 24.2 = 34.2 \text{ mi/hr.}$$

A study of track and field records set on Earth will show that a world's champion sprinter cannot run, even for a short distance, at the rate of 24.2 miles per hour, the circular orbital speed at the surface of A–14. Newton's second law of motion, $F = ma$, indicates that a man cannot run faster on the Moon or on an asteroid than on Earth. The forward force F exerted by his muscles is the same, and his mass m is unchanged by the reduced gravity. Thus each forward push should give him the same acceleration as on Earth. Furthermore, his cumbersome spacesuit would interfere with motion. Thus an astronaut on A–14 could not by his own physical activity put himself into orbit. He would not need a tether to keep himself from floating away. For an investigation of other factors related to physical activity on A–14, see Chapter 3.

16. We have noted that the velocity of a spacecraft in a circular orbit decreases when its distance from the center of the primary body increases. Therefore it requires less kinetic energy to orbit a spacecraft at a higher altitude. The question is then often asked why spacecraft are not orbited at higher altitudes to conserve energy. Given that the gravitational potential energy is $\frac{GMm}{r}$, show that the saving in kinetic energy is more than offset by the work required to give the satellite greater height.

Solution. Let p and q be the radii of the orbits, where $q > p$. Then the change in potential energy is

$$\Delta E_P = \frac{GMm}{q} - \frac{GMm}{p}$$
$$= GMm\left(\frac{1}{q} - \frac{1}{p}\right).$$

We have noted that the required velocity for a circular orbit of radius r is given by

$$v^2 = \frac{GM}{r}.$$

Because kinetic energy $E_K = \frac{mv^2}{2}$, we multiply the given equation by $\frac{m}{2}$ to obtain

$$E_K = \frac{mv^2}{2} = \frac{GMm}{2}.$$

Then

$$\Delta E_K = \frac{GMm}{2q} - \frac{GMm}{2p}$$
$$= \frac{1}{2}GMm\left(\frac{1}{q} - \frac{1}{p}\right)$$
$$= \frac{1}{2}\Delta E_P$$

That is,

$$\Delta E_P = 2\Delta E_K.$$

Thus the change in potential energy is twice the change in kinetic energy; and as a result, more energy is needed to launch at high altitudes than at lower altitudes.

bibliography

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Ahrendt, Myrl H. "The Flight of Apollo 11 Analyzed With High School Mathematics," *School Science and Mathematics*. June 1970. Pages 549–562.

Uses high school mathematics to analyze the orbits and velocity changes needed in the flight of Apollo 11. Includes an introduction to orbit theory.

——— *The Mathematics of Space Exploration.* Holt, Rinehart & Winston, Inc., New York. 1965. 160 pp.

Discusses motion, mass, weight, gravity, measurements, launch vehicle theory, orbits, and flights in space. Includes problems with answers.

Glasstone, Samuel. *Sourcebook on the Space Sciences.* D. Van Nostrand Co., Inc., Princeton, N.J. 1965. 937 pp.

Contains mathematical information on orbits and trajectories, propulsion, and astronomical measurements. Although the amount of mathematics used in the remainder of the book is minimal, this is an excellent resource for background information on nearly all phases of space exploration.

Hunter, Maxwell W., II. *Thrust Into Space.* Holt, Rinehart & Winston, Inc., New York. 1966. 224 pp.

Emphasis is on propulsion, but mathematical information is also included on orbits and spaceflight mechanics.

Massachusetts Department of Education and National Aeronautics and Space Administration. *Aerospace Curriculum Resource Guide.* Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1968. Pages 140–164.

Gives learning activities, with problems, on applications of mathematics in spaceflight and space technology. Materials range in difficulty from simple exercises for kindergarten to challenging problems for grade 12.

National Aeronautics and Space Administration. *Aerospace Bibliography.* Fifth ed., NASA EP–48. Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1970. 102 pp.

Lists a variety of publications and teaching aids on many phases of aerospace education.

——— *Exploring in Aerospace Rocketry.* NASA EP–88. Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1971. 354 pp.

Gives an easily read introduction to numerous problems and phenomena of spaceflight. Develops or quotes numerous equations. Useful primarily for background information. Contains only a few problems to be worked, and these are incidental to the discussion.

——— *From Here, Where?* Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1965. 192 pp.

A space mathematics supplement for secondary school. Brief discussions on getting into space, space and weather, space navigation, matter and motion, and the measure of space. Contains exercises with answers.

——— *Orbits and Revolutions.* NASA S–7. Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1968. 4 pp.

A simple, clear mathematical explanation of synodic and sidereal periods, with a group of related activities.

——— *The Shapes of Tomorrow.* Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1957. 204 pp.

A supplement in space-oriented geometry for secondary school. Includes a range of mathematical materials related to astronomy, motion in space, and space mechanics. Contains exercises with answers.

——— *Space Flight With Electric Propulsion.* NASA Lewis Research Center, Cleveland, Ohio. 1969. 49 pp.

Brief discussion of the theory and application of propulsion systems, including derivation of the rocket equation.

——— *Space Resources for Teachers: Space Science.* NASA EP–64. Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1969. 144 pp.

Introduces briefly the use of mathematics in astronomical measurements, measurement of radiation, atomic reactions, and radioactive decay. Includes an introduction to the mathematics of motion, rockets, and gravity.

Ordway, Frederick I., III; Gardner, James Patrick; Sharpe, Mitchell R., Jr.; and Wakeford, Ronald C. *Applied Astronautics: An Introduction to Space Flight.* Prentice-Hall, Inc., Englewood Cliffs, N.J. 1963. Pages 174–238.

A readable, yet rigorous, discussion of fundamental spaceflight mechanics. Includes derivation of the rocket equation and the vis-viva equation. Gives a mathematical discussion of orbits and trajectories, energy relationships, and the like. Advanced college level.

Seifert, Howard S.; and Seifert, Mary Harris. *Orbital Space Flight.* Holt, Rinehart & Winston, Inc., New York. 1964. 139 pp.

Gives an easy-to-read mathematical introduction to the principles of orbital spaceflight.

Van de Kamp, Peter. *Elements of Astromechanics.* W. H. Freeman & Co. Publishers, San Francisco. 1964. 140 pp.

Presents a rather rigorous discussion of astromechanics, including conic sections, Kepler's laws, Newton's law of gravitation, orbital motion, energy relations in orbits, the vis-viva equation, and similar topics. For the advanced reader.

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