A NUMERICAL METHOD FOR THE PREDICTION OF HIGH-SPEED BOUNDARY-LAYER TRANSITION USING LINEAR THEORY*

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SUMMARY

This paper describes a method of estimating the location of transition in an arbitrary laminar boundary layer on the basis of linear stability theory. After an examination of experimental evidence for the relation between linear stability theory and transition, a discussion is given of the three essential elements of a transition calculation: (1) the interaction of the external disturbances with the boundary layer; (2) the growth of the disturbances in the boundary layer; and (3) a transition criterion. A brief discussion is given of the computer program which carries out these three calculations. The program is first tested by calculating the effect of free-stream turbulence on the transition of the Blasius boundary layer, and is then applied to the problem of transition in a supersonic wind tunnel. The effects of unit Reynolds number and Mach number on the transition of an insulated flat-plate boundary layer are calculated on the basis of experimental data on the intensity and spectrum of free-stream disturbances. Reasonable agreement with experiment is obtained in the Mach number range from 2 to 4.5.

INTRODUCTION

One of the most difficult problems in theoretical aerodynamics is the prediction of transition from laminar to turbulent flow, a problem that is especially severe for supersonic and hypersonic boundary layers. Examples ranging from high-velocity reentry vehicles to the wind-tunnel testing of transonic airfoil sections can be put forward to illustrate the dramatic effects on flow characteristics which result from differences in the location of transition. The search for some method of estimating whether the boundary layer will be laminar or turbulent for a particular external flow has mostly focussed on empirical correlations of some type. These methods are limited in scope and should be replaced by a more fundamental approach which involves the calculation of the development of the perturbed boundary layer as it responds to its disturbance environment. The direct solution of the three-dimensional time-dependent Navier-Stokes equation of compressible flow, which could be of great benefit, still lies in the future. The use of turbulence model equations is a promising approach although it remains to be demonstrated if enough of the complexity of the transition

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process is retained in these time-averaged equations, which were primarily intended for fully developed turbulent flow, to make them useful as a prediction technique. A third possibility is the use of linear stability theory. This approach might at first appear to be of little value because of the evident nonlinearity of the final breakdown of laminar flow. However, it does have the considerable advantage that within the restrictions of linearity and locally parallel flow one is dealing with solutions of the unsteady Navier-Stokes equations. Furthermore, in many disturbance environments most of the region preceding transition will involve a disturbance of small amplitude. In these cases, the process by which the dominant external disturbances form an organized wave structure in the boundary layer, and the subsequent growth of the internal boundary-layer disturbances both lie within the scope of a linear theory. Nonlinearity occurs only in a small region immediately preceding transition. Consequently, it should follow that at least the change in the transition Reynolds number as the mean boundary layer or the disturbance environment changes can be calculated from linear theory.

In reference 1, a detailed investigation was carried out to determine whether in a supersonic wind tunnel the change in the transition Reynolds number of a flat-plate boundary layer with Mach number and surface cooling can indeed be accounted for by linear theory. The results reported there are sufficiently promising to encourage taking the next step, which is to use linear theory to make a quantitative estimate of the transition Reynolds number. It is the purpose of this paper to describe a numerical method and computer program which combine information about the external disturbances with stability theory and a transition criterion to provide estimates of transition location in a wide variety of cases. As examples, the effect on transition of free-stream turbulence in low-speed flow, and of Mach number and unit Reynolds number in a supersonic wind tunnel are given.

SYMBOLS

A	disturbance amplitude
A_1	free-stream disturbance amplitude
A_o	initial disturbance amplitude
A_t	amplitude transition criterion
c_p	phase velocity, ω/α_r
$E(\omega)$	energy density of normalized power spectrum
f	frequency

F	dimensionless frequency, $\omega^* \nu_1^* / U_1^{*2}$
L^*	length scale
L_x^*	integral length scale of turbulence
M	Mach number
p'	pressure fluctuation
Re	free-stream x-Reynolds number
R_L	Reynolds number based on L_x^*
R_{δ^*}	displacement-thickness Reynolds number
u', v'	velocity fluctuations
U	mean longitudinal velocity
U_s	average source velocity
x,y,z	longitudinal, transverse and lateral coordinates
α	complex wave number in x-direction, $\alpha_r + i\alpha_i$
β	complex wave number in z-direction, $\beta_r + i\beta_i$
γ	ratio of specific heats
μ	viscosity coefficient
ν	kinematic viscosity coefficient
τ_R	Reynolds stress ratio
$(\tau_R)_t$	Reynolds stress transition criterion
ψ	wave angle, $\tan^{-1}(\beta_r/\alpha_r)$
ω	circular frequency

Superscripts:

$(\ )^*$	dimensional quantity
$(\ )'$	fluctuation quantity

$(\overline{\ \ })$	average quantity

Subscripts:

t	transition
1	free-stream condition
o	neutral-stability condition

LINEAR STABILITY THEORY AND TRANSITION

The idea of calculating boundary-layer transition by means of linear stability theory would be on much more solid ground if it were possible to point to experimental evidence that there is indeed a direct quantitative relation between linear instability and transition. Schubauer and Skramstad (ref. 2) demonstrated the correctness of the theory of Tollmien and Schlichting as a description of the behavior of small disturbances in the laminar boundary layer preceding transition. They also showed that the location of transition could be changed by varying either the frequency or amplitude of an artificial disturbance, but no quantitative results were given.

Apparently the only published experiment that does offer quantitative results in this regard is the one described in reference 3 by Jackson and Heckl. An axisymmetric model of circular cross section on which a Blasius boundary layer formed was mounted in a wind tunnel of moderate turbulence level (0.2 - 0.4%). A loudspeaker was placed inside the model and the sound introduced into the boundary layer through a circumferential slit located 12 in. from the effective start of the boundary layer. Transition was fixed at a point 15 in. downstream of the slit for a range of frequencies by adjusting the amplitude of the loudspeaker. The amplitude of the disturbance created in the boundary layer was monitored with a hot-wire anemometer located in the boundary layer above the slit. Transition was measured by a hot wire located at the downstream (15 in.) station and was judged to have occurred when the disturbance spectrum changed to a turbulent form. Thus the end rather than the start of transition was being measured.

For a given free-stream velocity U_1^* (asterisks refer to dimensional quantities) the frequency and amplitude were varied to find the frequency which resulted in transition at the downstream station with the smallest initial amplitude. These frequencies are called the critical frequencies and are shown on a typical stability diagram in figure 1 where the dimensionless frequency $F = \omega^* \nu_1^* / U_1^{*2}$ is plotted against Re, the free-stream x-Reynolds number, and R_{δ^*}, the displacement-thickness Reynolds number. In terms of linear stability theory, the experimental procedure was equivalent to finding the frequency with maximum total amplification at a given Reynolds number. Consequently, if the location of transition is determined by the linear instability of the undisturbed laminar boundary layer, the critical-frequency data points should lie along the theoretical line of maximum amplification. Figure 1 shows that this condition is satisfied. Furthermore, this close relation between transition and linear instability is not restricted to what are commonly thought of as small disturbances. The transition Reynolds numbers of figure 1 are between 0.3×10^6 and 1.2×10^6 which would correspond to free-stream turbulence levels of 0.4 to 1.6% if transition were caused solely by free-stream turbulence.

Finally, it must be remarked that stability and transition experiments with artificially produced sound as the disturbance source are notoriously difficult to carry out, a situation already noted and discussed at some length in reference 2. It would be highly desirable to repeat the same type of experiment as in reference 3 with a different method of producing the artificial disturbances.

With some experimental support available for the idea of using linear theory for transition prediction, it only remains to decide on how to apply the theory. Any naturally occurring disturbance will have its energy distributed over a range of frequencies, and in the most general case its development in a boundary layer can only be calculated by considering the separate development of all frequencies with a significant portion of the total energy. However, amplification in a boundary layer is selective, and for small initial disturbance levels the selectivity, or tuning, is sufficiently sharp so that by the time transition is approached most of the disturbance energy is concentrated in a narrow band about the most-amplified frequency. This phenomenon suggests simplifying the application of stability theory by considering only disturbances of a single frequency. Such a procedure is possible because the amplification is linear and there is no transfer of energy from one frequency to another. Therefore, transition will be predicted in this paper on the basis of the single-frequency disturbance of maximum amplitude at each Reynolds number. Since the amplitude of a disturbance at any Reynolds number depends on its initial amplitude as well as on the amplification it has undergone, it is necessary to consider the initial energy spectrum of the complete wide-band disturbance to arrive at the single-frequency disturbance of maximum amplitude.

A comparison of the growth of the theoretical single-frequency disturbance of maximum amplitude at transition with the measured growth of the same frequency component is shown in figure 2 for an insulated flat-plate boundary layer in a supersonic wind tunnel at $M_1 = 4.5$ and Re/in. = 1.8×10^5. The dimensionless frequency of the two growth curves, $F = 0.3 \times 10^{-4}$, is the theoretical frequency of the disturbance of maximum amplitude only if the initial energy distribution of all single-frequency disturbances is identical to the power spectrum of the free-stream disturbances as measured by Laufer (ref. 4). The experimental narrow-band disturbance growth is taken from Kendall's measurements (ref. 5) in the JPL 20-in. wind tunnel, the same tunnel as used by Laufer. Because of the interaction of the irradiated sound from the turbulent boundary layers on the tunnel walls with the laminar boundary layer near the leading edge of the flat plate, there is no experimentally discernible neutral-stability point. The theoretical and experimental disturbance amplitudes are matched at the theoretical neutral point.

The theoretical disturbance of maximum amplitude has a wave angle ψ equal to 60°, while the experimental narrow-band disturbance includes all wave angles. Unfortunately, the distribution of energy with respect to wave angle in the experiment is unknown. Even so, the growths of the two disturbances are seen to be closely related. By coincidence, the two growth curves cross at almost exactly the start-of-transition Reynolds number measured by Coles (ref. 6), also in the JPL 20-in. wind tunnel. The fact that transition occurs where the theoretical disturbance is growing rapidly means that a transition criterion based on amplitude has a good chance of predicting the start of transition provided only that the region of maximum growth does vary, as assumed, with the mean-flow parameters in the same way as the transition Reynolds number.

REQUIREMENTS OF TRANSITION CALCULATION

In addition to the calculation of the velocity and temperature profiles of the mean boundary layer, the transition calculation can be divided into three distinct parts: (1) the interaction of the external disturbances which lead to transition with the boundary layer to form the internal boundary-layer disturbances of Tollmien-Schlichting type; (2) the growth of the internal disturbances; and (3) a transition criterion based on some property of the growing disturbances. In this section each of these three aspects will be discussed separately starting with the second for reasons of clarity in the exposition.

Spatial Stability Theory

The calculation of the disturbance growth in the boundary layer is the element which brings in the traditional linear stability theory. A detailed account of compressible stability theory may be found in reference 7. What is required of the theory are the eigenvalues of the stability equations for a spatially growing disturbance. For the parallel-flow form of the stability equations, a Fourier component of a typical three-dimensional fluctuation quantity is given by

$$q'(x,y,z,t) = Q(y) \exp \left[i(\alpha x + \beta y - \omega t) \right] \tag{1}$$

where q' is a small quantity; x,y,z are the longitudinal, transverse and lateral coordinates; $Q(y)$ is a complex amplitude function; ω is the real circular frequency; and α and β are the complex wave numbers

$$\alpha = \alpha_r + i\alpha_i \quad , \quad \beta = \beta_r + i\beta_i \tag{2}$$

All quantities have been made dimensionless with respect to a length scale L^* and a velocity scale V^*. The disturbance wave angle is

$$\psi = \tan^{-1}(\beta_r/\alpha_r) \tag{3}$$

and β_i/α_i is assumed equal to β_r/α_r. The phase velocity is

$$c_p = \omega/\alpha_r \tag{4}$$

and the imaginary part of α is the amplification rate

$$(1/A)(dA/dx) = -\alpha_i \tag{5}$$

The notation A has been introduced as the amplitude in equation (5) to emphasize that in the parallel-flow theory all flow variables grow at the same rate and independently of y. The amplitude is given as a function of Reynolds number by

$$A(Re)/A_0 = \exp\left(-\int_{Re_0}^{Re} \alpha_i \, dRe\right) \tag{6}$$

The subscript o refers to the lower-branch neutral point, i.e., where the disturbance has its minimum amplitude and first starts to amplify. The initial amplitude A_0 is obviously of as much importance in determining the amplitude A as the amplification, and is the quantity that must be obtained from the external disturbance and an interaction relation.

The eigenvalues α_i and c_p are obtained from repeated numerical integrations of the stability equations. For a three-dimensional compressible disturbance, the equations form an eighth-order system of complex linear ordinary differential equations (ref. 7). The four solutions which satisfy the boundary conditions as $y \to \infty$ provide the initial conditions for the numerical integration which proceeds from the free stream to the wall at $y = 0$. There are a total of 64 real equations to integrate. At $y = 0$, a linear combination of the four solutions satisfies three of the four homogeneous boundary conditions. With Re and the dimensionless frequency F fixed, an iterative linear search procedure finds the eigenvalues α_i and c_p which satisfy the remaining boundary condition.

Interaction of External Disturbances

Quantities computed directly from the linear stability theory such as the minimum critical Reynolds number and amplification rate are inherent properties of the mean boundary layer on the same basis as the displacement thickness or skin-friction coefficient. On the other hand, the transition Reynolds number is not at all an inherent property as it depends not only on the instability of the boundary layer, but also on external disturbances which interact with the boundary layer to form the internal disturbances which lead ultimately to transition. With no disturbances, transition can not occur no matter how unstable the boundary layer is. The external disturbances can arise from any one of several sources of unsteadiness such as free-stream turbulence, sound or vibration. Ideally, one would like to have a theory to give the initial amplitude of each Fourier component of the internal disturbance from the known external disturbance, but no such theory exists. A forced response of the boundary layer can be computed in certain instances, and the initial amplitude of the free internal disturbance assumed to be related in some way to the forced internal disturbance. An example of such a procedure is given in reference 1, where the effect of irradiated sound on the stable region of a laminar boundary layer is calculated from a simple forcing theory.

Yet a third procedure for determining the initial amplitude is to adopt an empirical relation. The simplest of these assumes that the square of the amplitude of each frequency component of the internal disturbance is directly proportional to the energy density of the same frequency of the external disturbance, and that the constant of proportionality is the same for all frequencies. That is, the initial amplitude A_0 of the single-frequency internal disturbance of frequency ω is related to A_1, the amplitude of the external disturbance, by

$$A_0(\omega) = A_z E^{\frac{1}{2}}(\omega) A_1 \qquad (7)$$

where $E(\omega)$ is the normalized (unit area) energy density of the one-dimensional power spectrum of A_1. The constant A_z can be regarded as an interaction or coupling coefficient which "couples" the external to the internal disturbance. It is determined by adjusting the calculated transition Reynolds number to a measured value. Once A_z is determined in conjunction with a specific transition criterion and for a specific disturbance source, there are no more free constants in the entire calculation. More generally, A_z is a function of ω and is so given when calculated from a forcing theory.

Equation (7) is in accord with the stated procedure of applying stability theory in the form of single-frequency disturbances. Otherwise, A_0^2 in equation (7) would be an energy density, and the internal disturbance amplitude A_d would be given by

$$A_d^2 = \int_0^\infty (A/A_0)^2 \, A_0^2(\omega) \, d\omega \tag{8}$$

where A/A_0 is the frequency-dependent amplification ratio given by equation (6). There is one circumstance under which A_d differs from the A of the most amplified frequency only by a constant, and that is when A/A_0 has the character of a delta function. For transition in a low-disturbance environment where large amplifications take place, A/A_0 does resemble a delta function near transition, but in many cases it does not. It must be kept in mind that the development of a disturbance composed of a whole spectrum of frequencies is being represented by a fictitious disturbance of only a single frequency. Such a representation can not always be adequate, and it is most likely to be seriously in error when the amplification is small.

A potentially serious problem for which there is no solution at the present time is that the available disturbance spectra both in the free stream and the boundary layer are one-dimensional. As can be seen from equation (1) the elementary disturbance of stability theory is an oblique wave in the x-z plane. For supersonic flow, the most unstable first-mode disturbance is oblique with a wave angle ψ of between 50° and 60° over a wide range of Mach numbers (ref. 7). What is needed, therefore, is the energy distribution with respect to ψ as well as frequency. In the absence of any measurements, it will be assumed that the frequency power spectra are the same for all wave angles.

Transition Criterion

The final step in the transition calculation is to apply a transition crit^rion, the simplest of which is an amplitude criterion based on a value of A, say A_t. The theoretical disturbance growth curve of figure 2 shows that the choice of A_t is not critical, as a rather large change in A_t makes only a small difference in the corresponding Reynolds number Re_t which is to be identified with the transition Reynolds number. The use of A itself as the transition criterion avoids the troublesome problem in the application of the parallel-flow theory of having to identify A with a particular fluctuation quantity. In a growing boundary layer, the eigenfunctions are functions of Re and as a result the different flow variables do not all grow in the same manner. Even within the scope of an amplitude criterion, one could identify A with, say, the mass-flow fluctuation and use the pressure fluctuation as the transition criterion with somewhat different results than if the mass-flow fluctuation were the transition criterion.

Some 30 years ago Liepmann (ref. 8), in an exceptionally clear presentation of the requirements of a transition calculation based on linear theory, proposed that transition starts when the Reynolds stress equals the mean viscous stress, i.e., when

$$\tau_R = \rho \overline{u'v'}/\mu \; \partial U/\partial y = 1 \tag{9}$$

The basis of this idea is that when the Reynolds stress reaches such a value the mean velocity profile must change in an important way. Liepmann's criterion can be modified somewhat by selecting a value $(\tau_R)_t$ different from unity as the criterion.

In order to calculate τ_R, it is necessary to first calculate the eigenfunctions. Since the amplitude in linear stability theory is arbitrary, A must be identified with the peak value of a particular fluctuation to set the amplitude, and only then can τ_R be calculated. Thus, as mentioned above, the transition Reynolds number obtained will depend to some extent on which fluctuation is chosen.

At present it is not clear how to use Liepmann's criterion in compressible flow. There are other momentum transfer terms besides $\bar{\rho}\,\overline{u'v'}$, and even if this single term can properly represent the distortion of the mean velocity profile there are still fluctuation heat-flux terms which perhaps should be included as a measure of the distortion of the mean temperature profile. For these reasons, only the amplitude criterion will be used in this paper for compressible flow.

A third criterion which also involves the Reynolds stress has recently been proposed by R. Kaplan of the University of Southern California. This criterion is based on an argument concerning the total stress tensor. Transition is considered to start when the transverse principal stress vanishes, a condition that is satisfied when

$$\rho \overline{u'v'} + \mu \; \partial U/\partial y = (\overline{u'^2} \; \overline{v'^2})^{\frac{1}{2}} \tag{10}$$

Again this criterion will only be used for incompressible flow.

COMPUTER PROGRAM

The computer program developed for the transition calculation is based on the author's stability program (ref. 9) which has been used for several years to work on a variety of incompressible and compressible boundary-layer stability problems. The stability program was first simplified and put in single-precision arithmetic except for the independent variable of the differential equations. The first new feature to be added was the automation of the eigenvalue computations so that a large number of eigenvalues can be obtained in a single computer run. Up to 12 dimensionless frequencies

F may be calculated at a given initial Reynolds number at either equal increments in F or at unequally spaced specified values of F. Then for each F in turn, up to 14 eigenvalues are computed over a range of Reynolds numbers which may also be unequally spaced.

The eigenvalue search procedure is set up to do a minimum of two iterations. Only one perturbation integration is required per iteration because with F and Re fixed, the secular determinant is an analytic function of the complex variable α for a spatial disturbance. Thus two iterations require four integrations of the 64 equations (16 for incompressible flow). Convergence is usually achieved after the two mandatory iterations, but if not, and the search has started to converge, up to two more iterations are allowed. If there is still no convergence, or the search did not give adequate signs of converging after the first two iterations, the increment in F or Re is halved, and if necessary, halved a second time. If nonsimilar boundary-layer profiles are being used, the Re increment can not be halved as the program is set up to use precomputed profiles which are read in from mass storage as needed.

After the eigenvalues have been obtained over a sufficient Reynolds number range for a given frequency, the next step is to compute the Reynolds numbers of the neutral-stability points. Up to four neutral points can be computed to allow for the possibility of two separate unstable regions. The neutral points are found by interpolation, and if desired the interpolated neutral points can be further refined by applying an eigenvalue search procedure which requires a minimum of six additional integrations per neutral point. When the lower-branch neutral point Re_o has been found, A/A_o is calculated from equation (6).

The next step is the calculation of the initial amplitude A_o from equation (7). For an empirical interaction relation, A_1 and A_z are both input quantities and $E(\omega)$ is calculated from one or several formulas which are specific to a particular problem. If the sound-forcing theory is used to calculate A_z, then two integrations of 80 equations each are needed for this purpose. With both A/A_o and A_o known, A(Re) can be calculated and the amplitude transition criterion applied. When A exceeds A_t, the corresponding Re_t is computed by inverse interpolation. When this series of calculations has been carried out for all of the frequencies of importance, the minimum Re_t is the predicted transition Reynolds number.

The evaluation of the Reynolds stress criteria requires the computation of the eigenfunctions at each Reynolds number. Two integrations are needed for this purpose. The peak value of the mass-flow fluctuation is identified with A to assign a magnitude to the eigenfunctions and thus to the Reynolds stress. The Liepmann and Kaplan criteria are evaluated, and when either criteria is exceeded the equivalent transition Reynolds number is found by inverse interpolation.

Storage and Time Requirements

The program requires a total of 49,000 single-precision words (36 bits) of storage, or 31,200 words with segmentation (overlays). On the UNIVAC 1108 time-sharing system, the basic integration time is 7.4×10^{-4} sec to integrate one equation across one step. For incompressible flow, where 80 steps are adequate and there are 16 equations for a two-dimensional disturbance, it requires 0.95 sec for each integration, and a total of 3.8 sec for each eigenvalue provided there is convergence in two iterations. For compressible flow with 100 steps and 64 equations, the respective times are 4.7 and 19 sec.

A minimum requirement for a transition calculation is about four frequencies with eight Reynolds numbers for each frequency. Consequently, to find the eigenvalues requires 122 sec for incompressible flow and 608 sec for compressible flow if all eigenvalue searches converge in two iterations. The time required to obtain the neutral-stability points by interpolation, evaluate the integral of α_i, calculate A_0 and determine Re_t on the basis of the amplitude criterion is negligible. For example, the results to be presented at $M_1 = 4.5$ were obtained with five frequencies and a total of 45 Reynolds numbers. The time to compute the eigenvalues was 855 sec, but to do all of the other calculations took only 1.5 sec.

The Reynolds stress transition criteria require two integrations per Reynolds number, and thus 50% as much time as the computation of the eigenvalues if the transition criteria are to be evaluated at all Reynolds numbers. However, if an Re_t is obtained first from the amplitude criterion, then the eigenfunction calculation need not start until one or two stations before this Reynolds number. In practice, the two Reynolds stress criteria required about 25% more time than for the eigenvalue calculation alone.

Since the transition Reynolds number is often computed as a function of some mean-flow parameter such as Mach number or altitude, a great many different boundary layers have to be evaluated. At 10-15 minutes per boundary layer, a large amount of computer time can be involved and a faster computer than the UNIVAC 1108 would be an advantage. It is estimated that the program would run about ten times faster on a CDC 7600. On a parallel-processor computer such as the ILLIAC, a further time advantage could be realized by integrating the independent solutions simultaneously instead of consecutively as at present, and by calculating the eigenvalues and eigenfunctions of different frequencies simultaneously.

EFFECT OF FREE-STREAM TURBULENCE

In order to debug the final program as economically as possible, but still work on an important problem, a calculation was made of the effect of free-stream turbulence on the transition of the Blasius boundary layer.

Figure 3 shows the published measurements as taken from reference 10. The ordinate is the start-of-transition Reynolds number, and the abscissa is the rms intensity of free-stream turbulence which is identified here with the amplitude A_1. The curve is from the present calculation and is discussed below. Unfortunately, none of the experimenters measured either the turbulence spectrum or the scale. In the absence of this essential information, the spectrum was assumed to be the Dryden spectrum (ref. 11) of isotropic turbulence,

$$U_1{}^*E^*(\omega^*)/L_x{}^* = 4/[1 + (\omega^*L_x{}^*/U_1{}^*)^2] \tag{11}$$

where $L_x{}^*$ is the integral length scale of turbulence. Although it is not necessary to set individual values of U_1^* and L_x^* in the present calculation at a single turbulent Reynolds number ($R_L = U_1{}^*L_x{}^*/\nu^*$), the spectrum is entered in the program as given by equation (11) in order to be able to calculate the separate effects of $U_1{}^*$ and $L_x{}^*$. Therefore, $U_1{}^*$ and $L_x{}^*$ had to be assigned and the values chosen were

$$U_1{}^* = 44 \text{ ft/sec} \quad , \quad L_x{}^* = 2.18 \text{ in.} \tag{12}$$

With $E^*(\omega^*)$ known, the next step is to set the interaction constant A_z. For this purpose, the start-of-transition Reynolds number $Re_t = 2.8 \times 10^6$ measured by Schubauer and Skramstad (ref. 2) for $A_1 = 0.1\%$ was used. Although the lowest measured disturbance level in their tunnel (0.02 - 0.03%) was mostly sound, particularly for the unstable frequencies, it will be assumed that disturbances of 0.1% and greater are primarily turbulence. Since the Kaplan transition criterion is the only one that does not require a numerical value to be chosen, it was used to calculate A_z. With A identified as the peak value of u', the rms longitudinal velocity fluctuation referenced to the free-stream mean velocity, it was found that $A_z = 0.086$ gives $Re_t = 2.8 \times 10^6$. With this A_z, the same Re_t is obtained with the amplitude criterion set at $A_t = 0.04$, or with the Liepmann criterion set at $(\tau_R)_t = 0.14$ instead of Liepmann's suggested value of 1.0.

At this point everything should be in readiness to calculate the effect of A_1 on Re_t. However, the change of A_0 with A_1 as given by equation (7) and the Dryden spectrum, together with the frequency dependence of A/A_0 for a two-dimensional instability wave in the Blasius boundary layer, does not begin to give a large enough effect on Re_t to account for figure 3. There is no experimental information on the relation between the turbulence intensity in the free stream and the amplitude of the instability wave, so only conjectures are possible. One possibility is that the interaction is not linear as assumed by equation (7). A second possibility is that there is indeed a linear interaction, but that the initial disturbance forms in the damped region rather than at the neutral point. As a result, A_θ would vary with F along the neutral curve just from the different damping ratios between the initial point and the neutral point. Since the frequency of the disturbance of maximum growth also changes with A_1, the second possibility is in a certain sense equivalent to the first. Consequently, it will be assumed without further inquiry into its meaning that

$$A_z = 0.043 \left[1 + (A_1/0.001)^{2.3}\right] \tag{13}$$

where the multiplicative constant is just one-half the previous value of A_z in order to give the same Re_t as before when $A_1 = 0.1\%$. The exponent 2.3 was chosen to fit the experimental curve at $A_1 = 0.4\%$. As a note of caution, equation (13) is not intended to relate the entire free-stream spectrum to the internal spectrum, but only to give the A_θ, and hence the amplification ratio, that is required to account for the initial rapid decrease of Re_t with increasing A_1.

Re_t was computed with equations (7), (13) and the amplitude criterion of 4% up to $A_1 = 2\%$, and the result is the curve shown in figure 3. It is surprising that agreement with the experimental results is obtained all the way to $A_1 = 2\%$ where transition is not far from the minimum critical Reynolds number.

The curve shown in figure 3 is not much more than an empirical fit to the data. Unfortunately, until the amplitudes of Tollmien-Schlichting waves can be related in a fundamental way, either theoretically or experimentally, to the free-stream turbulence, nothing better can be done at the present time. The advantage of the present procedure over a direct curve fit of Re_t to the data is that the effects of turbulence scale, spectrum and free-stream velocity on Re_t, as well as the influence of turbulence on the transition of arbitrary boundary layers, can all be calculated with no further assumptions. Furthermore, it is possible to use the method to compare results obtained with the three transition criteria, and some of these calculations have been carried out. Simply stated, the computed transition Reynolds numbers appear to depend very little on the particular criterion used. For $A_1 < 0.5\%$, there is virtually no difference between the criteria; for $A_1 = 1\%$, the results for the amplitude, Liepmann and Kaplan criteria are, respectively, 0.567×10^6, 0.600×10^6 and 0.609×10^6, a maximum difference of 7%.

TRANSITION IN SUPERSONIC WIND TUNNELS

Determination of Input Quantities

Transition in a supersonic wind tunnel above $M_1 = 2$-3 is dominated by the sound radiated from the turbulent boundary layers on the tunnel walls.

In reference 1, linear stability theory was used together with an amplitude criterion to calculate the variation of Re_t with M_1 for an insulated flat-plate boundary layer. The external disturbances were included by the simple expedient of taking $A_0 \sim M_1^2$, as suggested by Laufer's finding in reference 4 that the free-stream rms pressure fluctuation p_1' varies essentially as M_1^2. The calculated variation of Re_t with M_1 bore a striking resemblance to the experimental measurements, although the unit Reynolds number dependence of p_1' was not sufficient to explain the measured dependence of Re_t on unit Reynolds number. The spectrum of p_1' plays an essential role in this dependence and must be included in the calculation.

With the interaction in the form of equation (7), $E^*(\omega^*)$ was obtained from the measurements of Laufer (ref. 4). The faired experimental spectra are shown in figure 4. The spectrum at $M_1 = 4.5$, $Re/in. = 3.4 \times 10^5$ is approximated in the computer program by curve fits accurate to about 5%. A unit Reynolds number correction as given by the spectrum at $M_1 = 4.5$, $Re/in. = 1.8 \times 10^5$, and a Mach number correction as given by the spectrum at $M_1 = 2.0$, $Re/in. = 3.4 \times 10^5$ are both included in the program. The energy density and frequency $f^*(= \omega^*/2\pi)$ were made dimensionless by Laufer with L_x^*, the integral length scale of the wall pressure fluctuations, and U_s^*, the average velocity of the sound sources. Both of these quantities are entered in the program as curve fits to the measured values.

Laufer measured p_1' at two Re/in. from $M_1 = 1.6$ to 5.0. Other measurements (ref. 12) have shown that p_1' varies with Re/in. as $(Re/in.)^n$. The power that agrees best with Laufer's two values over the Mach number range is $n = -0.2$. Consequently, the value of A_1 entered in the program is

$$A_1 = 0.00045\ \gamma M_1^2\ [(Re/in.)/(3.4 \times 10^5)]^{-0.2} \tag{14}$$

where $\gamma = 1.4$.

The remaining quantity in equation (7) is the coupling coefficient A_z. The program provides for A_z to be computed by the sound-forcing theory presented in reference 1. This theory requires a value of the sound source velocity which in turn defines ψ_c, the cut-off value of the wave angle ψ beyond which there is no sound radiation. The angle ψ_c is usually less than the angle of the disturbance of maximum amplification and can result in a marked reduction in the amplification ratio A/A_0. In addition, the source velocity is a strong function of frequency and this dependence has been measured only at $M_1 = 4.5$ (ref. 13). Even though with an average value of the source velocity the theory gives the result that A_z is inversely proportional to F and increases slowly with M_1 in agreement with the measurements of Kendall (ref. 5), it was decided that the uncertainties involved in using the sound-forcing theory in the present calculations are greater than just assuming A_z to be constant.

With an amplitude transition criterion of 1%, the constant A_z was adjusted to give $Re_t = 1.45 \times 10^6$ at $M_1 = 4.5$, Re/in. = 3×10^5, the start-of-transition Reynolds number measured by Coles (ref. 6). It must be pointed out that the amplitude criterion is here completely arbitrary. A different value of A_t would merely change A_z in proportion. What is really being set is the amplification A/A_o needed for transition at the specified Reynolds number. The coefficient A_z would acquire a physical meaning only if the entire spectrum were being used to compute the amplitude rather than a single frequency. However, it is helpful to use constants whose magnitudes make physical sense, and 1% was chosen on the idea that it represents the pressure fluctuation. The mass-flow and pressure fluctuation both become large in the free stream as M_1 increases. In the boundary layer, the mass-flow fluctuation, which is mostly a density fluctuation at high Mach numbers, is larger than in the free stream. On the other hand, p' is smaller than in the free stream and declines relative to the mass-flow fluctuation as M_1 increases. Since it is known that a boundary layer at hypersonic Mach numbers can support large mass-flow fluctuations without becoming turbulent, it may be that p' is the more suitable quantity to relate to transition.

Results of Calculations

With the constant A_z set once and for all, a series of calculations were carried out for M_1 = 2.2, 3.0 and 4.5, and $1 < \text{Re/in.} \times 10^{-5} < 4$. These Mach numbers were chosen because most of the eigenvalues needed were already available. The results are shown in figures 5 and 6 where they are compared with Coles' measurements at four Mach numbers, only one of which is the same as the Mach numbers of the calculations. Figure 6 is cross plotted from figure 5, and there is one additional point shown at M_1 = 1.6 that does not appear in figure 5.

There is seen to be reasonable agreement between the calculated and experimental values, with a maximum difference of about 15%. The unit Reynolds number effect has been a particularly difficult one to account for in anything resembling a fundamental manner (ref. 14), and it is encouraging to see some features of the measured effect appear in the calculated results. Many measurements of the unit Reynolds number effect can be fitted by the relation

$$Re_t \sim (\text{Re/in.})^m \tag{15}$$

A common value of m is 0.4, although a wide range of values have been encountered, and there are measurements which do not fit this relation at all. The measurements for M_1 = 2.57 in figure 5 fall into this latter category when the entire Re/in. range is included. However, for Re/in. > 1×10^5 a power law is a reasonable fit to the data with m = 0.28, 0.36, 0.63, 0.47 at M_1 = 2.0, 2.57, 3.7, 4.5, respectively. The calculated slope at $M_1 = 2.2$ is $m = 0.35$ which is close to the experimental value at $M_1 = 2.57$. Although the calculated curves at the other two Mach numbers are not straight lines, they are in agreement with experiment to the extent that their slopes increase with Mach numbers.

The increased slope calculated at the lower Re/in. may possibly be a reflection of this same tendency in the experimental results for $M_1 = 2.57$, but it more likely comes from an inadequacy in the method. The best agreement is found at $M_1 = 2.2$, and this agreement would be even better as to the magnitude of Re_t if the actual measured value of $p_1'/\gamma M_1^2$ at $M_1 = 2.2$, 0.00055, were used instead of the average value 0.00045. At $M_1 = 2.2$, the total amplification at Re/in. = 1.5×10^5 is 11.1, and the representation of the disturbance growth by a single frequency should be valid. In contrast, at $M_1 = 4.5$ the amplification at the same Re/in. is only 3.5, and it is possible that the single-frequency approximation at this and lower Re/in. is not valid because of the small overall amplifications involved.

In support of this conjecture, a calculation made by the author a number of years ago (ref. 7, fig. 13-46) is helpful. In this calculation, growth curves were obtained at $M_1 = 4.5$ with the complete frequency spectrum taken into account, but with still only a single wave angle of 60° (there is a similar calculation with energy distributed uniformly with respect to ψ). In this calculation the spectrum and p_1' were assumed to be independent of Re/in., and α_i was computed approximately from the temporal stability theory. Of these simplifications, the most serious is believed to be the one concerning p_1'. A unit Reynolds number effect smaller than in the present calculation was found. With an amplitude criterion set to yield $Re_t = 1.45 \times 10^6$ at Re/in. = 3×10^5 as here, the Re_t at Re/in. = 1×10^5 can be determined from reference 7 to be 1.15×10^6. This value can be compared with $Re_t = 0.66 \times 10^6$ of the present calculation with $n = -0.20$ in equation (14). If n is set equal to zero, then Re_t increases to 0.83×10^6. If the influence of n on the result with the complete spectrum is in the same ratio as with a single frequency, then with $n = -0.20$, Re_t would decrease from 1.15×10^6 to 0.91×10^6. It can be seen from figure 5 that this value fits the measurements quite well, and the value of m in equation (15) is 0.43 as compared to the experimental value of 0.47.

Figure 6 requires little comment except to point out that computations are needed at more Mach numbers to better define the curves drawn in the figure. In order to extend the calculations to higher Mach numbers, additional p_1' and spectrum measurements are needed. For $M_1 < 2$, there is a different sort of problem. With decreasing Mach number, the influence of the irradiated sound decreases and that of free-stream vorticity increases. Consequently, the nature of the interaction changes and A_z can not be expected to remain constant. The present indications are that the sound is more effective in creating instability waves than is vorticity, so that A_z should decrease below $M_1 = 2$ with resulting larger values of Re_t. In support of this reasoning, $Re_t = 3.4 \times 10^6$ at $M_1 = 1.6$, Re/in. = 3×10^5 in figure 6, while an experiment by Kendall in the JPL 20-in. tunnel showed no transition on a flat plate at Re = 4.3×10^6 with Re/in. = 3.4×10^5.

CONCLUDING REMARKS

The origin of transition has been viewed here as the result of specific external disturbances with well-defined characteristics interacting with the boundary layer and being amplified according to linear stability theory until a critical state is reached. The example of the effect of free-stream turbulence on transition could not be carried to a conclusion because not enough is known of the all important interaction process. It is in the supersonic wind tunnel that the most complete information exists. The free-stream disturbances have been measured in the necessary detail, transition data are available, and the interaction appears to be linear and of such a nature that it can be represented by a simple relation. In these favorable circumstances, linear stability theory has been shown capable of providing reasonable estimates of the start of transition as a function of unit Reynolds number and Mach number for the simplest possible boundary layer, the boundary layer on a smooth, insulated, flat plate. However, there appears no reason to doubt that the method, perhaps modified to include the complete spectrum, will work for more complicated boundary layers and in different disturbance environments if the interaction can be properly accounted for.

Further progress would seem to require more study of each of the three parts of the transition calculation. The stability theory must be extended beyond flat-plate boundary layers; the spectral characteristics of the disturbances which occur in different flow environments must be measured; and the interaction of these disturbances with the boundary layer to create instability waves must be understood. Some factors which influence transition and are commonly thought of as causes of transition, such as surface roughness and waviness, are not true sources of instability waves in the absence of an unsteady local separation, but act to influence existing disturbances which have arisen from other sources. This influence is exerted through a modification of the mean boundary layer which sharply increases the instability amplification (ref. 15). The requirement in these instances is the capability of computing the modified mean boundary layer.

To the objection that it is very difficult to obtain the necessary information about the external disturbances, it can be replied that otherwise the prospects for real progress in the ability to predict transition are indeed bleak. Repeated experimentation in similar disturbance and flight environments can result in some definite conclusions, but once the environment is changed the whole procedure must start all over again. Even when it becomes possible, as it will one day, to replace the linear stability theory with the three-dimensional time-dependent Navier-Stokes equations, this part of the problem will not change. The interaction can then be solved directly, but the necessity of defining the external disturbances will remain exactly what it is today. Without quantitative knowledge of the disturbances, transition prediction, difficult enough in any case, is likely to remain forever just out of reach.

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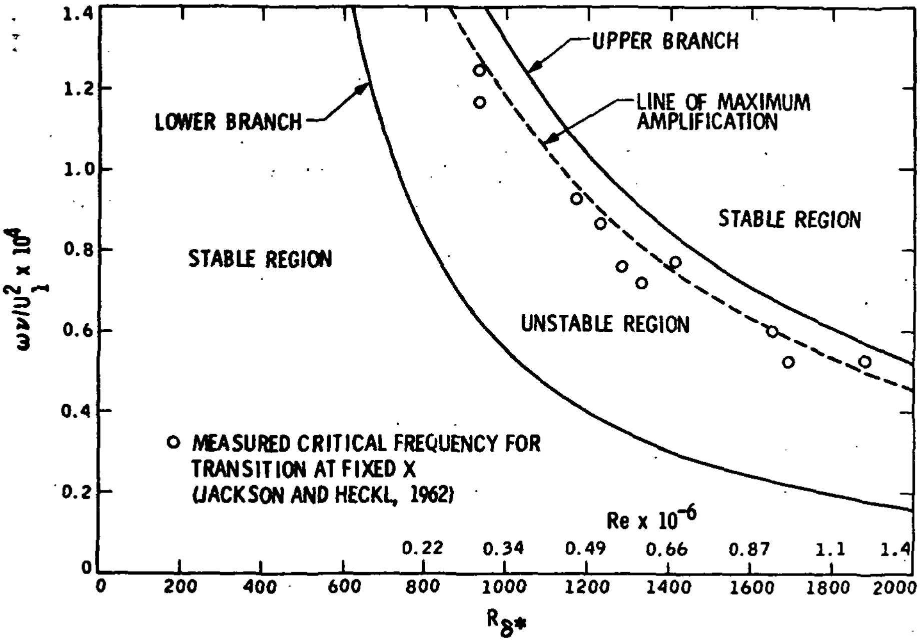

Figure 1. Experimental evidence for the relation between transition and linear stability theory.

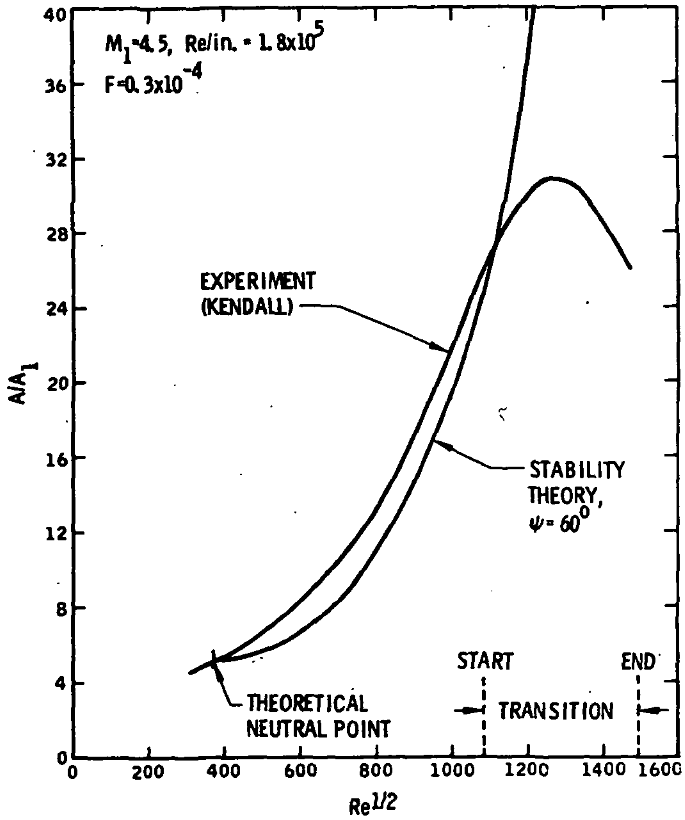

Figure 2. Comparison of the theoretical and experimental growth of a single-frequency disturbance.

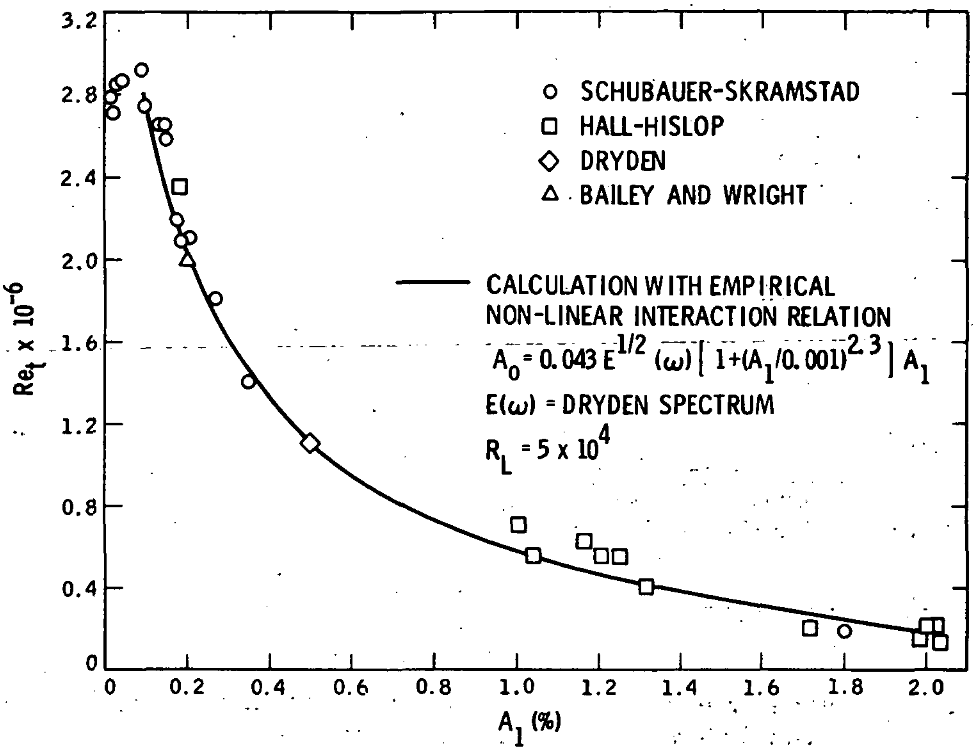

Figure 3. Effect of free-stream turbulence on transition.

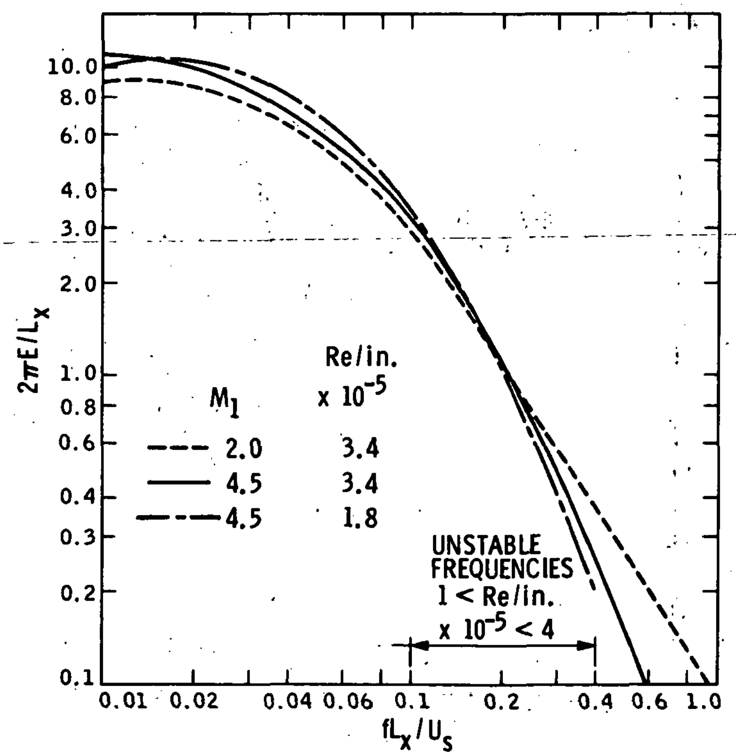

Figure 4. Experimental free-stream disturbance spectra (ref. 4).

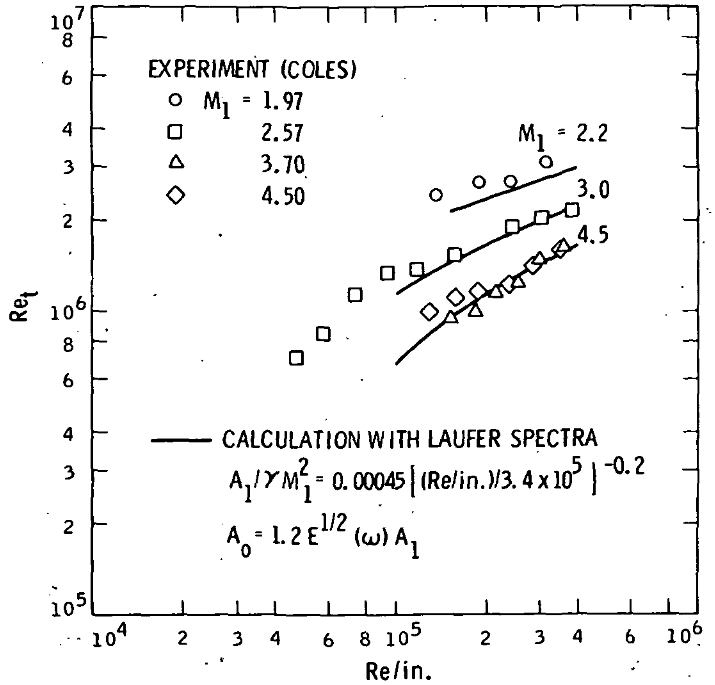

Figure 5. Calculated effect of unit Reynolds number on the transition of insulated flat-plate boundary layers at M_1 = 2.2, 3.0, 4.5 and comparison with experiment.

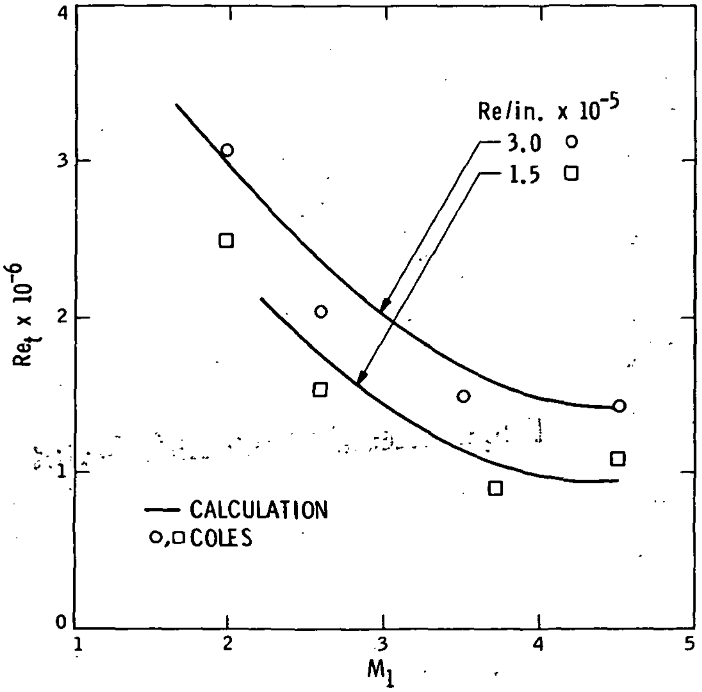

Figure 6. Calculated effect of Mach number on the transition of insulated flat-plate boun-ary layers at Re/in. x 10^{-5} = 1.5 and 3.0 and comparison with experiment.