N76-10169

ON THE FORMULATION OF THE GRAVITATIONAL POTENTIAL IN TERMS OF EQUINOCTIAL VARIABLES*

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One year ago I presented a paper at this symposium that described some work we had been doing with the method of averaging. One of the aspects of that paper had to do with a particular choice of nonsingular variables that were called equinoctial variables. In particular, we investigated analytical averaging techniques in the equinoctial coordinate frame. Some very good results were obtained, and we developed some expressions for fairly low-order perturbations. We developed explicit expressions for the disturbing potential for the zonal harmonics up to J_4 and the third-body harmonics up to $(a/R_3)^6$, where a is the semimajor axis of the orbit and R_3 is the distance to the third body.

After that we reconsidered the problem and decided that one of the shortcomings of that work was the lack of general expressions for the disturbing potential. This lack was a problem for several reasons: First, serious problems were encountered in extending the results to higher degree terms. The need to extend the results occurs, for example, in mission analysis, where the problem of including the effect of one higher degree third-body term or including the effect of an additional zonal harmonic occurs frequently. Second, having explicit expressions for each term made software development complicated because, for each new term, additional software was required, and there was the possibility of errors at each step.

These are some of the reasons why it was desirable to develop general results for the disturbing potential in this coordinate frame, and that is what I propose to discuss now. We are going to consider the gravitational potential and will use several special functions–the Legendre and associated Legendre polynomials and the Q_{nm}, which are called the derived Legendre functions. Recently there has been a fair amount of interest in these functions in the context of the gravitational potential. (For example, Pines, 1973.) The various disturbing potential expansions are listed below.

* A significantly expanded version of this paper was presented at the AIAA Aerospace Sciences meeting in January 1975. Copies of the preprint are available from the author.

Disturbing Function Expansions

Perturbation	*nth* Term
Third-body harmonics	$\frac{\mu_3}{R_3}\left(\frac{a}{R_3}\right)^n\left(\frac{r}{a}\right)^n \quad P_n(\cos\psi)$
Zonal harmonics	$-J_n\frac{\mu}{a}\left(\frac{R_e}{a}\right)^n\left(\frac{a}{r}\right)^{n+1} \quad P_n(\cos\psi)$
Arbitrary geopotential term (n, m)	$\frac{\mu}{a}\left(\frac{R_e}{a}\right)^n\left(\frac{a}{r}\right)^{n+1} \quad P_{nm}(\cos\psi)$ $(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)$ $\lambda = \alpha - \theta$

Here we are considering the third-body harmonics and the zonal harmonics, in particular. (It was originally planned to have some expressions for the tesseral case, but they are still in the process of being checked.) We have introduced the semimajor axis into these expressions, even though it is not required, because we are going to average these expressions and want to use existing results in two-body mechanics. For the third-body harmonics, the ψ angle will be the angle between the vector to the satellite and the vector to the third body. In the case of zonal harmonics, ψ will be the colatitude.

In averaging these potentials, a two-step process is used. The purpose of step 1 is to obtain expressions for the Legendre functions in terms of equinoctial orbital elements. In step 2, the averaging is performed.

One of the important questions in deriving potentials in terms of classical or equinoctial elements is how to get the fast variable motion into the potential. There are several different options, and they have a direct effect upon the amount of manipulation and on the complexity of the final results. Also related to these options is the level of accuracy that can be achieved. For third-body harmonics, there are really two options: We can use the mean anomaly for both the satellite and for the third body and end up with a very general potential. This was done by Kaula in 1966. Or we can use direction cosines (relative to an orbital reference frame) to get the location of the third body. If direction cosines are used, there is not quite the flexibility in modeling that is had with the Kaula approach, but the expressions are a lot more compact. In our work with the method of averaging, the use of the direction cosines is quite appropriate.

For zonal harmonics, we bring in the motion of the fast variable by using the orbital true longitude. For the arbitrary geopotential term, by which is meant tesserals, there are two options that are somewhat related to the two options that exist for third-body harmonics: We can use the mean variable for the satellite and Greenwich sidereal time, which is particularly appropriate for the study of resonance cases; or we can use the true longitude of the satellite and Greenwich sidereal time, which appears to be most appropriate if we want to assume that the central body rotational angle is fixed during the averaging. Such a potential might be particularly appropriate for lunar satellites.

Several expressions for cos ψ are listed below. These correspond to various attempts to develop the third-body potential. The first expression listed for cos ψ in terms of right ascension and declination of the satellite and the third body was used by Kaula. He then used the addition theorem and the definition of the inclination function to obtain expressions for the Legendre polynomials in terms of the classical elements. These results have great generality because we can impose all types of resonance constraints on them, yet they are also very complex.

Expressions for Cosine ψ

$$\cos\psi = \alpha \cos v + \beta \sin v$$
$$\alpha = \hat{R}_3 \cdot \hat{P}, \beta = \hat{R}_3 \cdot \hat{Q}$$

$$\cos\psi = \alpha \cos u + \beta \sin u$$
$$u = v + \omega, \alpha = \hat{R}_3 \cdot \hat{N}, \beta = \hat{R}_3 \cdot \hat{M}$$

$$\cos\psi = \alpha \cos L + \beta \sin L$$
$$L = v + \omega + \Omega, \alpha = \hat{R}_3 \cdot \hat{f}, \beta = \hat{R}_3 \cdot \hat{g}$$

More recent work in third-body perturbations uses the concept of expressing third-body motion by direction cosines relative to an orbital coordinate frame. Kaufman in 1970 and Lorell and Liu in 1971, in particular, worked with the apsidal coordinate frame. Subsequently, Kozai worked with a coordinate frame where the orbital x-axis was along the nodal crossing and the third axis was in the orbit plane. Essentially, he used the argument of latitude as his fast variable. At about the same time, we used the true longitude as our fast variable and also used the direction cosines of the third body relative to the equinoctial coordinate frame (FGW), in which the x-axis points at the origin of the latitudes.

One of the problems involved in working with Legendre polynomials and, in particular, with the direction cosines formulation, was that there did not appear to be an addition theorem for that case. In most of the work using that approach, the basic recursion relationships have been used to generate the Legendre functions. There have not been general formulas for the required Legendre functions with the argument $\alpha \cos L + \beta \sin L$. This is the same problem I reported on a year ago. Recently, we took another look at the addition theorem, looked at the expressions in terms of the direction cosines, and found that we could reformulate the problem in such a way that the addition theorem still applied; the steps in that derivation are shown in equations 1 and 2.

First, we introduce a phase angle, L', and define the trigonometric functions—$\cos L'$ and $\sin L'$—to fit the form of the addition theorem. Then α, δ, α_3, and δ_3 must be defined.

$$\alpha = L \qquad \cos\delta = 1$$
$$\alpha_3 = L' \qquad \sin\delta = 0 \tag{1}$$
$$\cos\delta_3 = \sqrt{\alpha^2 + \beta^2} \qquad \sin\delta_3 = \sqrt{1 - \alpha^2 - \beta^2} = \gamma$$

Here those angles are treated as arbitrary quantities. By substituting equation 1 into the addition theorem:

$$P_n(\cos\psi) = P_n(0)\,P_n(\gamma) + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_{nm}(0)\,P_{nm}(\gamma)\cos m\,(L - L'). \tag{2}$$

In equation 2, γ is the direction cosine of the third body relative to the vector normal to the orbit plane.

If we define some C_m and S_m polynomials in α and β similar to the w~ · Pines did in 1973, the addition theorem can be simplified. The final result is

$$P_n(\cos\psi) = P_n(0)\,P_n(\gamma) + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_{nm}(0)\,Q_{nm}(\gamma)\,[C_m(\alpha,\beta)\cos mL + S_m(\alpha,\beta)\sin mL] \tag{3}$$

where

$$C_m(\alpha,\beta) = \mathrm{Re}\,(\alpha + j\beta)^m \tag{4}$$

and

$$S_m(\alpha,\beta) = I_m\,(\alpha + j\beta)^m \tag{5}$$

Because $P_{nm} = 0$ for $n - m$ equal to an odd integer, Q_{nm} gives even powers of the γ function. It can be expressed quite well in terms of a polynomial in α and β. Again, it is finite, so we have a general form, a modified addition theorem, that is useful in this case.

To give an idea of what these polynomials look like, the first one that appears in the third-body perturbation is evaluated as follows:

$$P_2(\alpha\cos L + \beta\sin L) = -\frac{1}{4}(3\gamma^2 - 1) + \frac{3}{4}[(\alpha^2 - \beta^2)\cos 2L + 2\alpha\beta\sin 2L]$$
$$= \frac{1}{2}\left[\frac{3}{2}(\alpha^2 + \beta^2) + \frac{3}{2}(\alpha^2 - \beta^2)\cos 2L + 3\alpha\beta\sin 2L - 1\right]. \tag{6}$$

This checks against the results that were presented in our paper a year ago. It has been checked for several cases and gives exactly the results obtained previously through very long and arduous manipulations; in fact, some errors were found in those previous results through comparison with the general formula.

Another major point of this paper is to emphasize the computation of the potential by using the recursion formula, and below are listed the recursion formulas (equations 7 to 10) that are appropriate to the computation of P_m ($\alpha \cos L + \beta \sin L$).

$$(n+1)\,P_{n+1}(\gamma) = (2n+1)\,\gamma\,P_n(\gamma) - n\,P_{n-1}(\gamma) \quad (7)$$

$$(n-m)\,Q_{nm}(\gamma) = (2n-1)\,\gamma\,Q_{n-1,m}(\gamma) - (m+n-1)\,Q_{n-2,m}(\gamma) \quad (8)$$

$$C_{m+1}(\alpha,\beta) = \alpha\,C_m(\alpha,\beta) - \beta\,S_m(\alpha,\beta) \quad (9)$$

$$S_{m+1}(\alpha,\beta) = \beta\,C_m(\alpha,\beta) + \alpha\,S_m(\alpha,\beta) \quad (10)$$

Equation 7 is the standard recursion for the Legendre polynomials, and equation 8 is the recursion for the derived Legendre functions (this is given in Ananda and Broucke, 1973, and Pines, 1973). Equations 9 and 10 are the recursions for the C_m and S_m polynomials, respectively.

The final formula for the averaged potential also has the same C_m and S_m polynomials, with k and h as the argument, and the same recursions will still apply. We are interested in obtaining averaged potentials, and we want to average with respect to a time-oriented variable, so we average with respect to the mean longitude. In equation 11, we take the standard formula for the average of $(r/a)^n \cos m v$ and recast it in terms of the true longitude:

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^n e^{imL}\,d\lambda$$

$$= \left(-\frac{1}{2}\right)^m \binom{n+m+1}{m} (k+jh)^m F\left(\frac{m-n-1}{2}, \frac{m-n}{2}, m+1, h^2+k^2\right). \quad (11)$$

It should be noted that the formulas are a hypergeometric series and that the polynomial previously referred to as appearing in these integrals has the argument k + jh. The symbol j is the square root of -1.

We wanted to obtain a recursion formula for computing the right-hand sides of these integrals, and it happens that one has already been derived and is presented in Cook's paper on the PROD program (1973). The formula for these integrals is as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^n e^{imL}\,d\lambda = \left(-\frac{1}{2}\right)^m \binom{n+m+1}{m} (k+ih)^m B^m_{m+2}(h^2+k^2) \quad (12)$$

where

$$B^{\ell}_{\ell} = 0,\; B^{\ell-1} = 1,\; B^{\bar{k}-1}_{\ell} = B^{\bar{k}}_{\ell} + \frac{(\ell+\bar{k})(\ell-\bar{k}-1)}{4\bar{k}(\bar{k}+1)}(h^2+k^2)\,B^{\bar{k}+1}_{\ell}$$

So along with P_n ($\alpha \cos L + \beta \sin L$), the hypogeometric series and the eccentricity polynomials can be reached recursively.

The following is a result for the *nth* term in the third-body potential expressed in terms of equinoctial elements and the direction cosines:

$$\frac{\mu_3}{R_3}\left(\frac{a}{R_3}\right)^n P_n(0) P_n(\gamma) B_{n+2}^0 (h^2 + k^2)$$

$$+ 2 \frac{\mu_3}{R_3}\left(\frac{a}{R_3}\right)^n \sum_{m=1}^{n} V_{n,m} B_{n+2}^m (h^2 + k^2) Q_{n,m}(\gamma) [C_m(\alpha,\beta) C_m(k,h) + S_m(\alpha,\beta) S_m(k,h)]$$

where

$$V_{n,m} = \left(-\frac{1}{2}\right)^m P_{n,m}(0) \frac{(n-m)!}{(n+m)!} \left(\frac{n+m+1}{m}\right). \qquad (13)$$

All of the terms in equation 13 can be obtained recursively. In writing a program, we would probably initially compute the coefficients $V_{n,m}$ and then store them as data at the beginning of the program. We do not obtain all of the terms indicated in the equation. We get approximately (n/2) terms in that summation because some of the associated Legendre functions with zero argument are zero, depending on whether n – m is an even or odd number. We have a finite, closed-form expression, and we verified several of the results presented a year ago with this formula. The next step is to show how the same analysis is done for zonal harmonics. The potential due to the *nth* zonal harmonic is

$$-J_n \frac{\mu}{a}\left(\frac{R_e}{a}\right)^n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\psi) \qquad (14)$$

where ψ is the colatitude. Cos ψ can be expressed in classical elements by

$$\cos\psi = \sin i \sin(v + \omega). \qquad (15)$$

In equation 15, i is the inclination and v is the true anomaly. We can express cos ψ quite well in terms of the equinoctial elements as is shown in the next equation.

$$\cos\psi = \frac{-2p}{1 + p^2 + q^2} \cos L + \frac{2q}{1 + p^2 + q^2} \sin L \qquad (16)$$

In equation 16, p and q are equinoctial elements. If we assume auxiliary variables defined by

$$\alpha \equiv \frac{-2p}{1 + p^2 + q^2}$$

(17)

and

$$\beta \equiv \frac{2q}{1 + p^2 + q^2},$$

the argument of the Legendre function in equation 14 has the same form as it did in the third-body case, and we can use the addition theorem we derived previously for P_n ($\alpha \cos L + \beta \sin L$).

For averaging over the mean longitude, we have to somewhat modify our averaging formula. We could use the previous averaging formula with negative values of n, but that would result in an infinite series in the eccentricity, which can be seen by examining the hypogeometric series involved. Essentially, we made a quadratic transformation in the hypogeometric series, changed the arguments a little, introduced a variable x defined as $1/\sqrt{1 - h^2 - k^2}$, and found that we could get a finite series in the eccentricity and x for the required integrals:

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^{n+1} e^{imL}\, d\lambda = \left(\frac{1}{2}\right)^m \binom{n-1}{m} x^{2n-1} (k + ih)^m B_n^m (h^2 + k^2). \quad (18)$$

We also use the same polynomials, the $P_{m,n}$ polynomials in this case, which can be used in computing the potential for both third-body and zonal effects. There will be another saving in computation if both effects are included.

Next we give a general expression for the potential due to a zonal harmonic, that is, the averaged potential in terms of the same eccentricity functions, derived Legendre functions, and the same α–β polynomials:

$$-J_n\left(\frac{\mu}{a}\right)\left(\frac{R_e}{a}\right)^n P_n(0) P_n(\gamma) x^{2n-1} B_m^0 (h^2 + k^2)$$

$$-2J_n\left(\frac{\mu}{a}\right)\left(\frac{R_e}{a}\right)^n \sum_{m=1}^{n-1} W_{n,m}\, x^{2n-1} B_n^m (h^2 + k^2)\, Q_{n,m}(\gamma) \left[C_m(\alpha,\beta)\, C_m(k,h) + S_m(\alpha,\beta)\, S_m(k,h)\right] \quad (19)$$

where

$$W_{n,m} = \left(\frac{1}{2}\right)^m P_{n,m}(0) \frac{(n-m)!}{(n+m)!} \binom{n-1}{m}$$

Again, everything can be derived recursively; terms that are useful in the the third-body case are useful here, and terms that are useful for the lower order zonal harmonics can be used to compute the higher order ones. Therefore, I suspect that a very efficient computation program can be built using this formula, with the added advantage of having only one formula rather than many to verify.

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