AVERAGED INITIAL CARTESIAN COORDINATES
FOR
LONG LIFETIME SATELLITE STUDIES

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SUMMARY

A set of initial Cartesian coordinates, which are free of ambiguities and resonance singularities, is developed to study satellite mission requirements and dispersions over long lifetimes.

The method outlined herein possesses two distinct advantages over most other averaging procedures. First, the averaging is carried out numerically using Gaussian quadratures, thus avoiding tedious expansions and the resulting resonances for critical inclinations, etc. Secondly, by using the initial rectangular Cartesian coordinates, conventional, existing acceleration perturbation routines can be absorbed into the program without further modifications, thus making the method easily adaptable to the addition of new perturbation effects.

The averaged nonlinear differential equations are integrated by means of a Runge Kutta method. A typical step size of several orbits permits rapid integration of long lifetime orbits in a short computing time.

INTRODUCTION

Several sets of averaged elements (Lorell, 1970; Broucke and Cefola, 1972; Uphoff, 1973) are in use for satellite lifetime studies. These usually suffer from ambiguities and resonance singularities for low inclinations, near circular orbit, near polar orbits, critical inclination resonances, and such. Moreover, it is necessary to develop the perturbation representations in these element coordinate systems, which often requires ingenuity and is difficult in application. The Cartesian coordinates have been extensively utilized and routines are available to numerically generate most of the significant perturbations. The initial conditions of the Cartesian solution of the classical two-body problem have been developed for variation of parameters (Pines, 1961; Christensen, 1970; Godal and Johansen, 1968), but has not found wide application. With the advent of averaging as a tool for eliminating long tedious numerical integrations in computing solutions, this study was undertaken to reestablish the initial Cartesian coordinates as a useful set of parameters for orbital analysis.

THE INITIAL CONDITION CARTESIAN ELEMENTS

The equations of motion of the satellite in the planetary reference frame are given by

$$\ddot{R} = -\mu \frac{R}{r^3} + F \tag{1}$$

where F represents the perturbation forces other than the central attraction of the principal body.

The initial Cartesian coordinate parameters which describe the motion are given in terms of the position and velocity vectors in Cartesian coordinates by

$$\dot{R}_0 = -\dot{f}R + f\dot{R} \tag{2}$$

where f, g, $\dot{g}$, $\dot{f}$ are given as functions of the difference in eccentric anomaly, θ, as

$$f = 1 - \frac{a(1 - \cos\theta)}{r_0}$$

$$g = \frac{1}{\mu}(r\sqrt{\mu a}\,\sin\theta - d\,a(1 - \cos\theta))$$

$$\dot{f} = -\frac{\sqrt{\mu a}\,\sin\theta}{r_0 r}$$

$$\dot{g} = 1 - \frac{a(1 - \cos\theta)}{r} \tag{3}$$

$$d = R \cdot \dot{R}$$

$$d_0 = R_0 \cdot \dot{R}_0 = -\sqrt{\mu a}\,\sin\theta\left(1 - \frac{r}{a}\right) + d\cos\theta$$

$$r_0 = a(1 - \cos\theta) + r\cos\theta - \frac{d}{\sqrt{\mu}}\sqrt{a}\,\sin\theta$$

$$\frac{1}{a} = \frac{2}{r_0} - \frac{R_0 \cdot \dot{R}_0}{\mu} = \frac{2}{r} - \frac{\dot{R} \cdot \dot{R}}{\mu}$$

The relationship of the time increment to θ, in the absence of perturbations, is given by a form of Kepler's equation

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$$t - t_0 = \frac{a^{3/2}}{\sqrt{\mu}} \left\{ \theta - \sin\theta + \frac{r}{a}\sin\theta - \frac{d}{\sqrt{\mu a}} (1 - \cos\theta) \right\} \tag{4}$$

The differential equations for the variation of R_0 and $\dot{R}_0$ under the action of F are generated in Pines (1961) from the conditions that

$$\frac{d}{d\tau} R = 0$$

and $\tag{5}$

$$\frac{d}{d\tau} \dot{R} = F$$

Thus, we have

$$\frac{d}{d\tau} R_0 = \dot{g}_\tau R - g_\tau \dot{R} - gF$$

$$\tag{6}$$

$$\frac{d}{d\tau} \dot{R}_0 = - \dot{f}_\tau R + f_\tau \dot{R} + fF$$

Following Godel and Johansen (1968), we choose for the perturbation equation of the difference in eccentric anomaly, θ

$$\theta_\tau = 0 \tag{7}$$

This serves to eliminate mixed secular terms (see Christensen, 1970) from the perturbation derivatives of f, g, $\dot{f}$, $\dot{g}$, that arise in Pines (1961) where the time from the initial time was assumed unperturbed,

$$\frac{d}{d\tau} (t - t_0) = 0 \tag{8}$$

We can replace the vectors R, $\dot{R}$ from (6) using the inverse of (2)

$$R = fR_0 + g\dot{R}_0$$

$$\tag{9}$$

$$\dot{R} = \dot{f}R_0 + \dot{g}\,\dot{R}_0$$

$$\frac{d}{d\tau} R_0 = (\dot{g}_\tau f - \dot{f}g_\tau) R_0 + (g\dot{g}_\tau - g_\tau \dot{g}) \dot{R}_0 - gF \tag{9a}$$

$$\frac{d}{d\tau} \dot{R}_0 = (f_\tau \dot{f} - f\dot{f}_\tau) R_0 + (f_\tau \dot{g} - \dot{f}_\tau g) \dot{R}_0 + fF$$

An important stability condition on (9a) is derived. We have

$$f\dot{g} - g\dot{f} = 1 \tag{10}$$

It follows from the perturbation derivative of (10) that

$$f_\tau \dot{g} - \dot{f}_\tau g = -\dot{g}_\tau f + \dot{f} g_\tau \tag{11}$$

Thus the coefficient of R_0 in the $d/d\tau\, R_0$ expression is the negative of the coefficient of $\dot{R}_0$ in the $d/d\tau\, \dot{R}_0$ expression in Equations (9a). This will reduce the computing work in the averaging process.

Since the perturbation of the transit time is not zero, we require a differential equation for the time. Using (3), (4), and (5), we obtain the perturbation differential equations for the time,

$$\frac{d}{d\tau}(t - t_0) = \frac{1}{\sqrt{\mu}} \left\{ \left(\frac{3}{2} a^{1/2} (\theta - \sin\theta) + \frac{r}{2\sqrt{a}} \sin\theta - \frac{d(1 - \cos\theta)}{\sqrt{\mu}} \right) a_\tau - \frac{a(1 - \cos\theta)}{\sqrt{\mu}} \, d_\tau \right\} + 1 \tag{12}$$

where

$$a_\tau = 2\, a^2\, \dot{R} \cdot F$$

$$d_\tau = R \cdot F \tag{12a}$$

The perturbation derivatives of f, g, $\dot{f}$, and $\dot{g}$ are given below:

$$f_\tau = \frac{a}{r_0}\, (1 - \cos\theta) \left(\frac{r_{0_\tau}}{r_0} - \frac{a_\tau}{a} \right)$$

$$g_\tau = \left(\frac{r \sin\theta}{2\sqrt{a\mu}} - \frac{d(1 - \cos\theta)}{\mu} \right) a_\tau - \frac{a(1 - \cos\theta)}{\mu}\, d_\tau \tag{13}$$

$$\dot{f}_\tau = \dot{f} \left(\frac{a_\tau}{2a} - \frac{r_{0_\tau}}{r_0} \right)$$

$$\dot{g}_\tau = - \frac{(1 - \cos\theta)}{r}\, a_\tau$$

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where

$$(r_0)_\tau = (1 - \cos\theta)\, a_\tau - \frac{\sqrt{a}\,\sin\theta}{\sqrt{u}}\,(d_\tau + \frac{d}{2a}\,a_\tau)$$ (13a)

We now proceed to the averaged differential equations.

THE AVERAGED EQUATIONS OF MOTION

The principle here is to replace the nonlinear differential equations for $R_0(\tau)$ and $\dot{R}_0(\tau)$ by their average value over a single period of the reference orbit as defined by $R_0(\tau_1)$ and $\dot{R}_0(\tau_1)$.

Let the period of reference orbit be $T(\tau_1)$, then for

$$-\frac{T}{2}(\tau_1) \leqslant \tau \leqslant \frac{T}{2}(\tau_1)$$

$$R_0(\tau) = R_0(\tau_1)$$ (14)

$$\dot{R}_0(\tau) = \dot{R}_0(\tau_1)$$

$$t - t_1 = \tau - \tau_1 = \frac{1}{\sqrt{\mu}} \left\{ a^{3/2}(\theta - \sin\theta) + r_0 a^{1/2}\sin\theta + \frac{d_0 a(1 - \cos\theta)}{\sqrt{\mu}} \right\}$$

Thus,

$$\frac{d}{d\tau}\,\dot{\bar{R}}_0 = \bar{R}_0\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(f\dot{g}_\tau - \dot{f}g_\tau)\,d\theta + \dot{\bar{R}}_0\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(g\dot{g}_\tau - g_\tau\dot{g})\,d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\,gF\,d\theta$$

$$\frac{d}{d\tau}\,\bar{R}_0 = \bar{R}_0\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(f_\tau\dot{f} - f\dot{f}_\tau)\,d\theta + \dot{\bar{R}}_0\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(f_\tau\dot{g} - f\dot{g}_\tau)\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\,fF\,d\theta$$ (15)

$$\frac{d}{d\tau}\,(\overline{t - t_0}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(t - t_0)_\tau\,d\theta$$

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where

$$\frac{r}{a} = 1 - \cos\theta + \frac{r_0}{a}\cos\theta + \frac{d_0}{\sqrt{\mu a}}\sin\theta$$

(15.

Equation (15) could be integrated analytically, using Fourier series in θ. However, this would require a representation of F in θ and greatly burden the introduction of additional perturbations. Moreover, mathematical resonances would appear and require special techniques for each resonant term. In this, we follow the lead of Uphoff (1973) and adapt Gaussian quadratures s the technique for evaluating the integrals.

We note that F is a vector in three space, and recalling (11), we need to evaluate only 10 integrals. The average equations become:

$$\frac{d}{d\tau}\bar{R}_0 = a_1\bar{R}_0 + a_2\dot{\bar{R}}_0 + \bar{G}$$

$$\frac{d}{d\tau}\dot{\bar{R}}_0 = a_3\bar{R}_0 - a_1\dot{\bar{R}}_0 + \bar{F}$$

(16)

$$\frac{d}{d\tau}(t - t_0) = 1 + a_4$$

where

$$a_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(f\dot{g}_r - \dot{f}g_r)\,d\theta$$

$$a_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(g\dot{g}_r - g_r\dot{g})\,d\theta$$

$$a_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}(f_r\dot{f} - ff_r)\,d\theta$$

(16a)

$$\bar{G} = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\,gF\,d\theta = \begin{Bmatrix}\bar{g}_1\\ \bar{g}_2\\ \bar{g}_3\end{Bmatrix}$$

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$$\bar{F} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r}{a} f F \, d\theta = \begin{Bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{Bmatrix}$$

$$a_4 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r}{a} (t - t_0)_\tau \, d\theta$$

To obtain the averaged $\bar{F}, \bar{G}$ vectors, it is necessary to specify the perturbations. For the purpose of this study, we consider accelerations due to tesseral and zonal harmonics for a rotating planet, third body forces, such as the sun and moon, and drag. The detailed equations are given in Appendix A. It should be noted that where F is a function of R, $\dot{R}$ and t, for the purposes of averaging, these functions are given by the two body equations and Kepler's equation given in (17a) through (17d) referenced to the τ time at the midpoint.

We now consider the numerical integration of the averaged equations. We propose to use a Runge Kutta method with a τ step equal to several periods. It must be borne in mind that the time variable is τ and not time. For each evaluation of the seven derivatives of (16), at a specific τ time, τ_i, we must carry out the averaging procedure. Using $\bar{R}_0(\tau_i) \, \dot{\bar{R}}_0(\tau_i)$ as the reference orbit, we compute the 10 integrals given by (16a).

The Runge Kutta solution will produce $\bar{R}_0(\tau)$, $\dot{\bar{R}}_0(\tau)$ and $(t - t_0)(\tau)$. To compute an ephemeris of the state, R (t) and $\dot{R}$ (t), we proceed as follows:

Let the period of the $\bar{R}_0(\tau)$, $\dot{\bar{R}}_0(\tau)$ orbit be T (τ),

$$T(\tau) = \frac{2\pi}{\mu} (a(\tau))^{3/2}$$

where

(17a)

$$a(\tau) = \left(\frac{2}{\bar{r}_0(\tau)} - \frac{\dot{\bar{R}}_0(\tau) \cdot \dot{\bar{R}}_0(\tau)}{\mu} \right)^{-1}$$

Let N be the integer part of τ/T (τ), and θ is given by,

$$\theta = 2\pi N + \sigma$$

(17b)

Using Newton's method, solve Kepler's equation for σ

$$\Delta\tau = \tau - NT = \frac{1}{\sqrt{\mu}} \left\{ a^{3/2} (\sigma - \sin\sigma) + r_0 a^{1/2} \sin\sigma + \frac{d_0 a(1 - \cos\sigma)}{\sqrt{\mu}} \right\}$$

(17c)

For any position in the N + 1 revolution, where the incremental eccentric anomaly, θ, lies between 0 and 2π.

Let

$$t = t(\tau) + \frac{a^{3/2}}{\sqrt{\mu}} \left\{ \alpha - \sin \alpha + \frac{r(\tau)}{a} \sin \alpha + \frac{d(\tau)(1 - \cos \alpha)}{\sqrt{\mu a}} \right\} (1 + a_4(\tau))$$

$$R(t) = f(\alpha) R_0(\tau) + g(\alpha) \dot{R}_0(\tau)$$

$$\dot{R}(t) = \dot{f}(\alpha) R_0(\tau) + \dot{g}(\alpha) \dot{R}_0(\tau) \tag{17d}$$

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APPENDIX A

We consider several typical perturbations.

Tesseral and Zonal Harmonics.
Let the central body rotate about its polar axis, $\hat{k}$ (t) with a uniform angular rate ω. If the inertial Cartesian coordinates of the vehicle is R, then the unit vector to the vehicle in planet fixed coordinates is given by,

$$u = \hat{k}(t) \cdot \frac{R}{r}$$

$$s = \hat{i}(t) \cdot \frac{R}{r} \qquad (A.1)$$

$$w = \hat{j}(t) \cdot \frac{R}{r}$$

where $\hat{i}$ (t) and $\hat{j}$ (t) are orthogonal reference axes fixed in the rotating body equation, perpendicular to the polar axis $\hat{k}$ (t), expressed in the inertial Cartesian system of the vehicle.

Following Pines, (1973), the acceleration in the inertial system is given by

$$F_1 = \alpha_1 \hat{i}(t) + \alpha_2 \hat{j}(t) + \alpha_3 \hat{k}(t) + \alpha_4 \frac{R(t)}{r} \qquad (A.2)$$

For the purpose of completeness, we list the α_i coefficients for J_2, J_3, J_4, C_{22}, and S_{22}.

$$\alpha_1(J_2) = \alpha_2(J_2) = \alpha_1(J_3) = \alpha_2(J_3) = \alpha_1(J_4) = \alpha_2(J_4) = 0$$

$$\alpha_4(J_2) = - \frac{\mu a_E^2 \, 3J_2 \, (1 - 5 u^2)}{2r^4}$$

$$\alpha_4(J_3) = - \frac{\mu a_E^2 \, 5J_3 \, (3u - 7 u^3)}{2r^5} \qquad (A.3)$$

$$\alpha_4(J_4) = - \frac{\mu a_E^4 \, 15(-1 + 14u^2 - 21 u^4)}{8 \, r^6}$$

$$\alpha_3(J_2) = - \frac{\mu a_E^2 \, 3J_2 u}{r^4}$$

$$\alpha_3(J_3) = -\frac{\mu a_E^3 \, 3J_3 \, (5u^2 - 1)}{2\, r^5}$$

$$\alpha_3(J_4) = -\frac{\mu a_E^4 \, 5J_4 \, (7\,u^3 - 3\,u)}{2\, r^6}$$

$$\alpha_1(C_{22}\cdot S_{22}) = \frac{\mu a_E^2 6}{r^4}\,(wS_{22} + sC_{22})$$

$$\alpha_2(C_{22}\cdot S_{22}) = \frac{\mu a_E^2 6}{r^4}\,(sS_{22} - wC_{22})$$

$$\alpha_3(C_{22}\cdot S_{22}) = 0$$

$$\alpha_4(C_{22}\cdot S_{22}) = -\frac{\mu a_E^2 15}{r^4}\,(C_{22}(s^2 - w^2) + 2s\,w\,S_{22})$$

Third Body Acceleration.

Let the gravitational mass coefficient of third body be u_i. Let the ephemeris of the third body, $R_{ci}(t)$, be given with respect to the central planet, about which the vehicle is orbiting, then the indirect acceleration is given by

$$F_2 = -\sum_{i=1}^{M}\frac{\mu_i}{r_{vi}^3}\,R - \sum_{i=1}^{M}\frac{\mu_i}{r_{ci}^3}\left(\frac{r_{ci}^3}{r_{vi}^3} - 1\right) R_{ci} \tag{A.4}$$

where

$$r_{ci} = |R_{ci}|$$

$$r_{vi} = |R - R_{ci}| \tag{A.5}$$

Atmospheric Drag.

Let ρ be the air density, given as a function of the vehicle position, R, and the time, t. The drag acceleration is given by

$$F_3 = -\frac{1}{2}\rho(R, t)\,\frac{v\,C_D S}{m}\,\dot{R} \tag{A.6}$$

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where

$$v = |R|$$

$$C_D = \text{drag coefficient}$$

$$m = \text{vehicle mass}$$

$$s = \text{effective vehicle drag area.}$$

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NONLINEAR COUNTER EXAMPLE FOR BATCH
AND EXTENDED SEQUENTIAL ESTIMATION ALGORITHMS

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A simple nonlinear example is presented which shows the well-known iterated batch least-squares and extended sequential estimation algorithms may converge to different estimates. For this example one may even say the extended sequential algorithm converges to the "wrong" value.

ORBIT DETERMINATION IN THE PRESENCE OF
UNCERTAIN ATMOSPHERIC DRAG

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Uncertainties in the knowledge of the atmospheric density in the associated drag parameter constitute one of the primary limitations on the accuracy on which the orbits of near earth satellites can be determined and predicted. In most orbit determination programs, the effect of uncertainties in atmospheric drag are determined by adopting a standard atmosphere and estimating the drag parameters, β C_D A ÷ m.

However, for most missions, C_D and A vary and the standard atmosphere will contain errors. Each of these factors will lead to errors in the orbit determination and prediction operation. In this presentation, an approach for estimating the drag parameter, the effective satellite cross sectional area in the atmospheric density simultaneously with the satellite state, is described.