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NUMERICAL IMPLICATIONS OF STABILIZATION BY THE USE OF INTEGRALS

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The subject matter of this paper is some numerical experiences that we have had involving some of the celebrated notions of dynamic stabilization. Figure 1 demonstrates Ljapunov stabilization, which is an analytic notion, and the example given is one of circular motion under an arbitrary attractive central force field. There is some particle moving around in a circle at some distance from an attractive center. The attractive force is designated by -f(r), the velocity of the spacecraft is v, and the centripetal acceleration is v^2/r. By equating the centripetal acceleration to the attractive force, we can get an expression for the angular frequency that is the square root of the attractive force over $r(\sqrt{-f/r})$.

Notions of stabilization involve questions associated with what happens to this motion under slight perturbation of initial conditions. Let us start out with some satellite or particle moving in a circular orbit at a radius of α; there will be a certain angular rate associated with that motion. If we cause a slight change to occur in the initial conditions so that the radius of the orbit is no longer α but is $\alpha + \Delta\alpha$, then it may happen that the angular rate may differ from that of the original orbit. These initial conditions should be selected in such a way that the motion will still be circular in this example.

As a result of the possible different rates in the angular frequency, the mean anomalies (θ's) between both satellites will be different and their difference will increase linearly in time. Such motion is dynamically unstable because the motion of the initial problem and that of the perturbed problem (that is, the problem with slightly perturbed initial conditions) will deviate arbitrarily; the deviation in the mean anomaly will be as great as desired if a sufficient length of time is allowed to elapse:

$$\omega = \frac{2\pi}{T} = \frac{v}{r} = \sqrt{\frac{-f(r)}{r}}$$

for any

$$\Delta\alpha > \epsilon \quad \theta_1(t) - \theta_2(t) \geq \delta$$

for sufficiently large t.

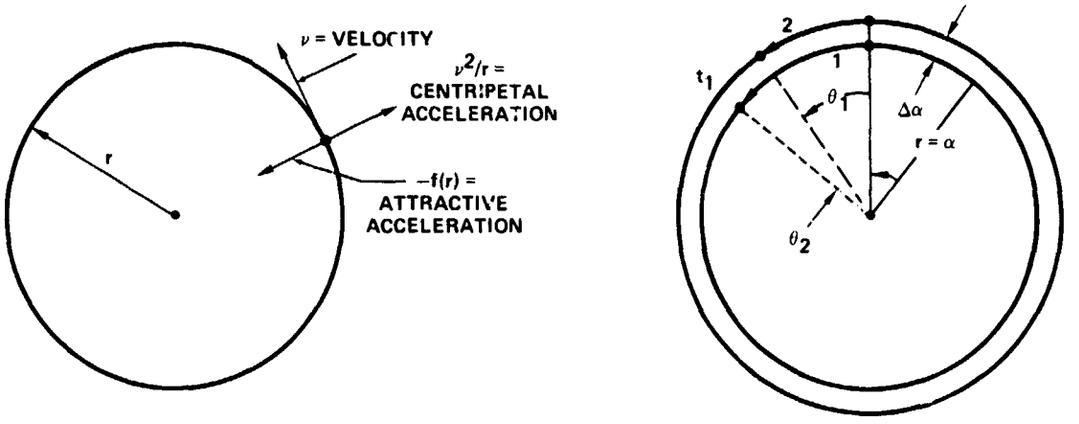

Figure 1. Ljapunov stability, circular motion under arbitrary attractive central force.

The single exception to this rule is the case where the angular frequencies are in fact the same for all possible radii. This implies that the force law should be given by $-\omega^2 r$, which is the simple harmonic oscillator:

$$\omega_1 = \omega_2 = \omega \rightarrow f(r) = -\omega^2 r.$$

Since the orbit (Keplerian) problem involves the force law, which is inversely proportional to the square of the field, this implies that the Keplerian problem is Ljapunov unstable:

$$f(r) = \frac{K^2}{r^2}$$

A dynamic problem is going to be either Ljapunov stable or unstable This is a physical concept and depends on the particular problem. The notion of stabilizing an unstable problem must then involve a change in the problem itself. We must look for another problem that happens to have the same solution as the original problem, and how this can be done will be explained later in the presentation.

The reason for looking at analytic concepts of stability is to try to improve the accuracies associated with numerical integration. Two basic approaches have been examined: With the two-body motion, one possibility (A) is to find a different formulation of the equations of motion, which are dynamically stable (that is, Ljapunov stable). Such examples are given by Baumgarte or the simple harmonic oscillator Kustaanheimo Stiefel (KS) theory. For such formulations, the frequency of the motion–the energy–is an a priori constant. A

second procedure (B) is to add a constraint (for example, an energy constraint) in addition to the equations of motion.

$$H(X,\dot{X}) = -P_0 = \text{constant}$$

If there were such a constraint, then, when initial conditions were perturbed, those initial conditions would have to satisfy the constraints. Variations in the initial conditions that change the energy (in the case of frequency constraint) would not be permissible alterations in the initial conditions.

So it happens that, for the two-body problem, concepts A and B are analytically, but not numerically, equivalent, as will be shown later.

With perturbed motion, the Baumgarte and KS theories are not Ljapunov stable. In fact, they are unstable, but the degree to which this instability occurs is of the order of the perturbations and numerically this does not present any difficulty (concept A). On the other hand, employing an energy constraint (B), the negative energy P_0 may or may not be a constant. If we had a J_2 problem, the Keplerian energy would be a constant, yet the Baumgarte or the KS theories would still be unstable. So, for perturbed motions, concepts A and B are not equivalent, and the question arises as to whether it is better for numerical integration to look for Ljapunov stable or nearly Ljapunov stable equations of motion or to apply some kind of energy constraint. All of the evidence indicates that slightly better answers are obtained, at least for nearly circular motions, by using energy constraints.

We now take a look at some of the methods of dynamic stabilization, by which is meant either formulations that are Ljapunov stable or nearly Ljapunov stable and/or applying an energy constraint. The crux of all the stabilization procedures that exist are presently being investigated: The Baumgarte approach; a recent approach that was proposed by Stiefel, with which we have not yet had any numerical experiences; the Baumgarte-Stiefel stabilization procedure, which is an energy-constraint-type formulation; and a formulation of applying an energy constraint directly, which is due to Nacozy and will be discussed later.

The Baumgarte procedure involves a Ljapunov-stable system of equations for the two-body problem and is based on a method of Poincaré. If we start with a Hamiltonian, which is a function of the position and conjugate momentum (velocities), then the Keplerian equations of motion are given by the following canonical equations:

$$H(X,P) \rightarrow \dot{X} = \frac{\partial H}{\partial P} \quad \text{and} \quad \dot{P} = -\frac{\partial H}{\partial X}$$

In numerical integration, it is very often desirable, especially for eccentric motion, to make a transformation from time as an independent variable to some other independent variable, an S variable. This S variable could be the eccentric anomaly for certain cases or a mean anomaly in another case, but it does not matter. The type of equation which will relate time with the new independent variable is given by dt/ds as some function and, in general, that function may be a function of both position and momentum:

$$\frac{dt}{ds} = \mu(x,P)$$

Poincaré defined a new Hamiltonian, which is related to the old Hamiltonian by the function (dt/ds) times the old Hamiltonian plus a constant, and it turns out that, with this new Hamiltonian, the equations of motion for both the space variables and the time variable are canonical, as shown below:

$$X' = \frac{\partial \hat{H}}{\partial P} = \mu \frac{\partial H}{\partial P} + \frac{\partial \mu}{\partial P}(H + P_0)$$

$$P' = -\frac{\partial \hat{H}}{\partial X} = -\mu \frac{\partial H}{\partial X} - \frac{\partial \mu}{\partial X}(H + P_0)$$

$$t' = \frac{\partial \hat{H}}{\partial P_0} = \mu + \frac{\partial \mu}{\partial P}(H + P_0)$$

$$P_0' = -\frac{\partial}{\partial t} = 0$$

P_0 = constant such that $[H + P_0] \equiv 0$

There are three equations here that are the correct equations for dynamics, except for certain terms which are called control terms. The equation for P_0, which is the momentum conjugate to the time, defines P_0. The S-derivative of P_0 should vanish, and this implies that P_0 is a constant. If that constant is chosen to be equal to the negative of the energy or the value of the Hamiltonian, then the control terms would be numerically equal to zero, but dynamically they are not. Dynamically, they are some function of r and the constant energy. So this new canonical system of equations is really an entirely different problem that happens to have the same solution as the old problem.

Baumgarte noticed that for the function $\mu = r$, the equations of motion with respect to the new, independent variable S (that is, the spatial equation of motion with respect to the new, independent variable S) are dynamically stable in the Ljapunov sense and, in fact, can be transformed to the KS simple harmonic oscillator equations. The time equation, $t' = r$, is not dynamically stable, but there are ways of getting around that problem as will be shown later in the paper.

Even though the control term, $H + P_0$, is analytically equal to zero from the analytic solution, when these equations of motion are numerically integrated, the control term may develop some error. We will then want to know how this error is going to behave dynamically.

To examine this formulation in the time domain, we use the following equation:

$$\ddot{\vec{X}} = -\frac{K^2}{r^3}\vec{X} + \underbrace{\frac{\vec{X}}{r^2}[H + P_0]}_{\text{CONTROL TERM}}$$

We can derive an equation of motion for the control term by multiplying (dot product) this equation by the velocity and doing some manipulation. The general solution of this equation has the control term as a constant multiplied by r. For circular motions this would basically be a constant. This solution tells us that if we develop some numerical (nonzero) errors in the control terms, then those errors are going to persist. Hence, from a numerical point of view, the orbital state will not be on the correct energy surface.

This is different from our analytical approach, in which we saw that, for the two-body problem, Baumgarte stabilization (or Ljapunov stabilization) and applying an energy constraint are, in a way, equivalent. From the numerical point of view we see that this is not true, because in numerical integration we create errors and those errors persist in the Baumgarte approach, so the state is not forced back onto the correct energy surface. Thus, applying the Ljapunov stable system of equations does not ensure that the state is going to be on the correct energy surface.

There are other methods of stabilization. The Stiefel approach involves multiplying the forcing terms of the differential equations by the a priori energy constant divided by the Hamiltonian:

Given

$$\ddot{\vec{X}} = f(\vec{x},t)$$

use

$$\ddot{\vec{X}} = \left(\frac{-P_0}{H}\right)^{3/2} f(\vec{x},t).$$

Now this parentheses is ideally equal to 1 for the analytic solution. If we raise the exponent to the 3/2 power, numerically it does nothing, but from a stability point of view, it does. This Stiefel system of equations is dynamically stable in the Ljapunov sense.

The Baumgarte-Stiefel control term is similar to the Baumgarte control term, except that the multiplier is proportional to the velocity vector over the square of the velocity. The multiplier is sometimes called a dissipative term, because, by looking at the equation of motion that governs the control term, it can be seen that it has a solution which decays in time:

$$\ddot{\vec{X}} = \frac{K^2\vec{X}}{r} - \frac{\gamma\dot{\vec{X}}}{\dot{X}^2}\,[H + P_0]$$

leads to

$$\frac{d}{dt}\,[H + P_0] = -\gamma\,[H + P_0]$$

$$[H + P_0] = \text{constant} * e^{-\gamma t}.$$

Hence, the Baumgarte-Stiefel control term forces the state back onto the correct energy surface, because the control term is forced to decay to zero.

In the Nacozy approach, at each step of the integration, the control term is forced to zero by a virtual displacement made in the state, such that the new energy after such a displacement should be the correct energy.

$$\delta X = -\frac{[H + P_0]\,\vec{F}}{\vec{F}^2} \qquad \vec{F} = \text{acceleration.}$$

It happens that the Nacozy procedure is not compatible with multistep integration processes because of the discontinuity that occurs in the state variables. The numerical results using single-step numerical integration methods apparently work; with the multistep methods, there are problems, as will be shown later.

We now turn to the perturbed problem, where we have nonconservative perturbation, such as drag or solar radiation pressures, conservative forces, which are derived from the grading of some potential, and control terms:

$$\ddot{\vec{X}} = -\frac{K^2}{r^3}\,\vec{X} + \vec{P} - \bar{\nabla}V + [\text{control term}]$$

There are two types of integral constraints to discuss: The Keplerian energy is a near integral of the motion and has the equation:

$$\frac{d}{dt}\left[\frac{\dot{\vec{X}}^2}{r} + V - \frac{K^2}{r}\right] = \dot{\vec{X}} \cdot \vec{P}\,.$$

There is also a term not included here, which is the partial of the potential with respect to time. In the absence of perturbation, this is zero, so it is a near constant of the motion. Instead of having a P_0 that is a constant, we have to numerically integrate it. The following is a comparison of that integration of P_0 with the computation of the Hamiltonian in a

conservative system. The sum of H and P_0 is ideally zero and will be used as a control term:

$$S_0 \left| \begin{array}{l} \dfrac{d}{dt} P_0 = -\dot{\vec{X}} \cdot \vec{P} \\ \hline H = \dfrac{\dot{\vec{X}}^2}{2} - \dfrac{K^2}{r} + V \end{array} \right.$$

The other type of integral constraint is a Jacobi integral, which is what the energy looks like in the rotated coordinate system of the earth. It is an exact constant of the motion for the full J_2 potential including tesseral harmonics; it is given by the Hamiltonian for the inertial energy plus the angular velocity of the earth dotted into the angular momentum. When V is time-dependent due to the earth's rotation (tesseral harmonics):

$$\frac{d}{dt} [H + \vec{\omega} \cdot (\vec{R} \times \dot{\vec{R}})] = + (\ddot{\vec{X}} - \dot{\vec{\omega}} \times \vec{R}) \cdot \vec{P} = 0 \text{ IF } \vec{P} = 0.$$

A control term based on this integral of motion can be applied, designated in later equations by R, indicating the energy in the rotating coordinate system:

$$[H + \vec{\omega} \cdot (\vec{R} \times \dot{\vec{R}}) + P_0]$$

where

$$\frac{dP_0}{dt} = -(\ddot{\vec{X}} - \omega \times \dot{\vec{R}}) \cdot \vec{P}$$

None of these equations are integrated in the time domain. They are integrated usi g the eccentric anomaly as an independent variable and, in addition, the time equation must be integrated.

Baumgarte dynamic stabilization also requires a time equation that is dynamically stable. There are five options for the time equation. The normal one would be $t' = r$, the defining relationship between t and s.

If we differentiate this equation with respect to s, we get $t'' = r'/K$. The reason we consider this is that second-order systems of equations are often easier to integrate using class 2 methods.

The next option is the ordinary time element option:

$$t = \tau - \frac{Kr'}{2P_0}$$

The equations of motion are essentially a constant term plus perturbation:

$$\tau' = \frac{K}{2P_0}\left\{1 + \frac{r}{K^2}\left[\vec{R}\cdot(\vec{P} - \bar{V}V) - 2V\right] - \frac{r'P_0'}{P_0}\right\}$$

This is a constant when the energy P_0 is a constant. If the energy P_0 has to be integrated, then some errors in P_0 might be expected to crop up; if those errors are significant, then there can be secular error type terms that will grow. There will then be timing errors associated with the integration of t'.

This analysis leads to a third system of equations, in which a new eccentric anomaly is related to the old eccentric anomaly by the following combination of variables:

$$\frac{d\eta}{ds} = \frac{K}{2P_0} \qquad \frac{d}{ds} = \frac{K}{2P_0}\frac{d}{d\eta}$$

Here the energy P_0 is in the denominator. We can transform all of the equations of motion by using this relationship.

The time is then related to a new time element, t*, by adding to it a modified eccentric anomaly; as the equations of motion for t* involve only perturbations, in the absence of such perturbations, the anomaly is zero:

$$t = \tau^* + \eta - \frac{Kr'}{2P_0}$$

$$\frac{d\tau^*}{d\eta} = \frac{r}{K}\left[\vec{R}\cdot(\vec{P} - \bar{V}V) - 2V\right] - \frac{\gamma'P_0'}{P_0}$$

The fifth possibility for deriving the time as a function of s is to use a third-order differential equation as shown here:

$$t''' + 2P_0 t' = K$$

We have not had any numerical experiences doing this, but this equation is dynamically stable, as are the time element equations.

Tables 1 through 7 give the numerical errors of the GEOS-B test orbit from computer simulations. The errors are given in units of 10^{-6} kilometers (millimeters) after integrating the GEOS-B orbit over 50 revolutions. The perturbations are a 15-by-15 geopotential field, and the results are given over a range of 60 to 100 integration steps per orbit. In the first column, an I indicates the integration of the inertial energy, and an R indicates the integration of the rotational or body-fixed energy. The second column provides the particular time equation used to get the time from the eccentric anomaly. A zero indicates the t''

time equation, 1 corresponds to the t′ equation, 2 is the ordinary time element equation, and 3 is the modified time element equation. The last four columns give the errors for the Baumgarte dynamic stabilization procedure (B); the Baumgarte-Stiefel control term, which forces the state back onto the correct energy surface (BS); the experiences using the Nacozy approach (N); and the ordinary time-regularized formulation, which does not have any control term and is dynamically unstable (TR), respectively.

Related to these tables are the experiences associated with integrating Cowell's equations of motion using time as an independent variable. Since the motion is nearly circular, equal step sizes in time will correspond very closely to equal step sizes in eccentric anomaly.

In the last four columns of the tables, the three numbers in each entry correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively. There are three errors here because we are integrating spatial equations and a time equation with respect to eccentric anomalies. The final time at the end of the run is one in which the eccentric anomaly has reached a final fixed value at the end of 50 revolutions. The results of these runs are as follows: The time element with the Baumgarte control terms works with the timing error and the intrack error is still somewhat large compared to the crosstrack error. This result might be expected, because the state is no longer on the exact energy surface but persists off the energy surface. When a modified time element is applied, the timing error almost completely disappears, but only at the expense of intrack errors in the spatial equation. So all the modified time element has done for us is to shift the timing errors to spatial errors. The Nacozy process has really done nothing for us either, and the reason for that is the discontinuities in the state. Taking finite differences just adds errors in a multistep integration process. If we had a single-step integration process, these problems would disappear, as other experiences have indicated. The time-regularized processes are unstable and about two-thirds of the intrack error is in the timing and about one-third is in the spatial equations. This is in agreement with analytic work that has been done by Baumgarte.

It is the Baumgarte-Stiefel control term that gives the best answers. If, for example, the Jacobi integral is used, we get very little timing error. The spatial errors are still a little bit larger, but this control term gives us the best overall results. An important note is that the Baumgarte-Stiefel control term does not require a time element. In fact, the best results were achieved when the time element was not used and when the T'' equations or the T' equation were integrated. This is very important. Basically, the timing error results from consistent errors in r; $T' = r$. If there are consistent errors in r, timing errors will develop in integrating that system, the double integral problem.

By applying the energy constraint, the radius r is adjusted so that consistent errors in the time do not occur, and this has led to the results presented here. It is recommended that an energy constraint can be applied without worrying about the timing problem at all.

Table 1
Numerical Errors of the GEOS-B Test Orbit
for the BDSP. 60 Steps Per Orbit

ENERGY	ITELEM	B	BS	N	TR
I	0	UNSTABLE	−127.07899 28.97938 561.96729	−5802.32261 84.70741 12330.44769	−5326.43758 13.81227 12123.72627
I	1	UNSTABLE	−127.07197 28.97945 282.55283	−5802.18569 84.70873 6753.60074	−5326.43611 13.81225 11831.85920
I	2	UNSTABLE	−127.11421 28.98998 244.29093	−5803.50299 84.74010 1405.50988	−5327.36544 13.81054 665.25084
I	3	UNSTABLE	−341.064 28.773919 42.056	5693.628 87.159853 1136.962	−5244.249 6.950228 438.872
R	0	UNSTABLE	− 22.85710 29.93991 331.36557	UNSTABLE	SAME AS I0
R	1	UNSTABLE	− 22.84957 29.93998 50.23874	UNSTABLE	SAME AS I1
R	2	UNSTABLE	− 22.84268 29.95027 12.51864	UNSTABLE	−5327.29464 13.81072 1799.09322
R	3	UNSTABLE	− 22.84268 29.95027 12.41565	UNSTABLE	−5327.29498 13.81074 1798.98859

NOTES:

In the last four columns, the three numbers in each entry (given in units of 10^{-6} km) correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively.

By integrating Cowell's equations of motion with time as an independent variable, the intrack spatial error was 3401.4027, and the crosstrack spatial error was 26.77772.

Table 2
Numerical Errors of the GEOS-B Test Orbit
for the BDSP, 65 Steps Per Orbit

ENERGY	ITELEM	B	BS	N	TR
I	0	−114.19500 2.53846 1828.84252	−115.66616 14.78646 339.48653	−2006.31579 45.74424 4266.29205	−1814.58633 1.25166 4120.60271
I	1	−114.19606 2.53844 1867.57285	−115.66429 14.78647 261.06752	−2006.27240 45.74470 2416.86411	−1814.58677 1.25165 4031.98843
I	2	−114.18013 2.53900 204.35943	−115.69274 14.79099 259.28441	−2007.21942 45.76519 648.56484	−1815.31460 1.25176 399.31962
I	3	−337.19 3.09228 11.296	−338.474 14.558205 43.667	2117.405 46.343831 409.824	1936.262 3.169659 175.349
R	0	−101.88979 2.52544 1825.45528	− 29.81388 15.44724 148.96701	UNSTABLE	SAME AS I0
R	1	−101.89076 2.52543 1858.42729	− 29.81174 15.44727 7.00168	UNSTABLE	SAME AS I1
R	2	−101.87554 2.52593 307.78874	− 29.80779 15.45175 65.45256	UNSTABLE	−1815.25454 1.25175 667.86354
R	3	−101.87549 2.52593 307.55044	29.80778 15.45176 65.31316	UNSTABLE	−1815.25555 1.25173 667.72535

NOTES:

In the last four columns, the three numbers in each entry (given in units of 10^{-6} km) correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively.

By integrating Cowell's equations of motion with time as an independent variable, the intrack spatial error was 3401.4027, and the crosstrack spatial error was 26.77772.

Table 3
Numerical Errors of the GEOS-B Test Orbit
for the BDSP, 70 Steps Per Orbit

ENERGY	ITELEM	B	BS	N	TR
I	0	− 56.39602 3.11021 − 83.00906	− 55.74016 3.72737 110.25181	89.17622 9.39301 − 190.00997	59.21339 3.54422 − 144.20532
I	1	− 56.39740 3.11020 − 32.41277	− 55.74068 3.72737 126.90710	89.17136 9.39302 − 39.18658	59.21225 3.54421 − 130.56524
I	2	− 56.41604 3.10982 108.19045	− 55.75580 3.72860 136.67825	88.64037 9.40287 109.40972	58.77161 3.54320 129.18683
I	3	−172.115 2.960583 2.72	−171.421 3.569666 25.41	1.232 8.767146 7.63	31.457 3.383486 15.855
R	0	− 18.50051 3.14627 −127.93949	− 22.49304 3.90129 36.38436	UNSTABLE	SAME AS I0
R	1	− 18.50192 3.14625 − 77.63998	− 22.49356 3.90129 52.98888	UNSTABLE	SAME AS I1
R	2	− 18.518117 3.14591 15.28445	− 22.49164 3.90259 60.43499	UNSTABLE	58.81103 3.54332 35.63308
R	3	− 18.51815 3.14591 15.35058	− 22.49178 3.90258 60.50244	UNSTABLE	58.80972 3.54331 35.70119

NOTES:

In the last four columns, the three numbers in each entry (given in units of 10^{-6} km) correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively.

By integrating Cowell's equations of motion with time as an independent variable, the intrack spatial error was 3401.4027, and the crosstrack spatial error was 26.77772.

Table 4
Numerical Errors of the GEOS-B Test Orbit
for the BDSP, 75 Steps Per Orbit

ENERGY	ITELEM	B	BS	N	TR
I	0	− 19.62149 1.84296 −545.21889	− 19.12725 .99247 10.04084	631.43636 5.75062 −1342.56385	534.84907 2.43962 −1222.06456
I	1	− 19.62227 1.84295 −515.99802	− 19.12819 .99247 44.14811	631.41960 5.75071 − 698.77660	534.84874 2.43962 −1187.46233
I	2	− 19.64372 1.84251 41.23325	− 19.13503 .99227 53.53911	631.16381 5.74633 − 77.89860	534.62285 2.43881 9.64151
I	3	64.153 1.592328 .078	63.651 .5224b 12.097	588.847 5.790954 − 119.748	492.282 1.808203 31.944
R	0	9.20689 1.86947 −581.840	− 13.39047 1.023638 − 2.75868	UNSTABLE	SAME AS I0
R	1	9.20615 1.86948 −551.67891	− 13.39146 1.02364 31.42207	UNSTABLE	SAME AS I1
R	2	9.18710 1.86908 − 64.66871	− 13.39082 1.02341 39.45957	UNSTABLE	534.64250 2.43889 − 142.20691
R	3	9.18710 1.86908 − 64.61750	− 13.39086 1.02342 39.51002	UNSTABLE	534.64266 2.43889 − 142.15558

NOTES:

In the last four columns, the three numbers in each entry (given in units of 10^{-6} km) correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively.

By integrating Cowell's equations of motion with time as an independent variable, the intrack spatial error was 3401.4027, and the crosstrack spatial error was 26.77772.

Table 5
Numerical Errors of the GEOS-B Test Orbit
for the BDSP, 80 Steps Per Orbit

ENERGY	ITELEM	B	BS	N	TR
I	0	− 3.88340 .85629 −459.15305	− 3.74365 1.48006 − 15.88953	547.01168 7.57810 −1161.99962	462.17194 1.23206 −1052.62631
I	1	− 3.88368 .85629 −448.19016	− 3.74432 1.48006 9.26784	546.99724 7.57919 − 619.64184	462.17242 1.23206 −1026.10445
I	2	− 3.89834 .85601 10.98153	− 3.74692 1.48023 15.50786	546.89183 7.57760 − 96.19190	462.07088 1.23162 − 21.79383
I	3	− 16.867 .699993 .639	− 16.724 1.182493 4.999	530.505 7.653788 − 105.966	446.289 .752743 32.042
R	0	12.32661 .86888 −480.49666	− 6.75389 1.54807 − 9.24393	UNSTABLE	SAME AS I0
R	1	12.32631 .86888 −468.66261	− 6.75457 1.54807 15.97804	UNSTABLE	SAME AS I1
R	2	12.31317 .86863 − 59.98543	− 6.75434 1.54821 21.53818	UNSTABLE	462.07985 1.23164 − 134.05535
R	3	12.31314 .86863 − 60.08864	− 6.75426 1.54821 21.43552	UNSTABLE	462.079696 1.23164 − 134.15899

NOTES:

In the last four columns, the three numbers in each entry (given in units of 10^{-6} km) correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively.

By integrating Cowell's equations of motion with time as an independent variable, the intrack spatial error was 3401.4027, and the crosstrack spatial error was 26.77772.

Table 6
Numerical Errors of the GEOS-B Test Orbit
for the BDSP, 90 Steps Per Orbit

ENERGY	ITELEM	B	BS	N	TR
I	0	1.94060 .17462 −135.07205	1.88804 .65661 − 9.08666	172.74506 3.25877 − 365.23276	144.39606 .21146 − 326.48960
I	1	1.94069 .17463 −137.79058	1.88791 .65661 − 3.40434	172.74175 3.25882 − 198.53333	144.39652 .21147 − 319.98778
I	2	1.93719 .77468 − 1.66455	1.88796 .65678 − 1.60664	172.73643 3.25888 − 36.33632	144.38806 .21146 − 13.10179
I	3	1.632 .212515 .340	1.581 .342734 .362	170.214 3.062124 − 33.904	142.24 .192607 − 10.976
R	0	4.84812 .17485 −139.27516	− .91961 .68519 − 2.84861	UNSTABLE	SAME AS I0
R	1	4.8481 .17485 −141.66259	− .91983 .68519 2.84094	UNSTABLE	SAME AS I1
R	2	4.84502 .17489 − 18.61224	− .91976 .68534 4.51218	UNSTABLE	144.38850 .21146 − 43.66740
R	3	4.845096 .17489 − 18.82819	− .91971 .68534 − 4.29666	UNSTABLE	144.38868 .21146 − 43.88346

NOTES:

In the last four columns, the three numbers in each entry (given in units of 10^{-6} km) correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively.

By integrating Cowell's equations of motion with time as an independent variable, the intrack spatial error was 3401.4027, and the crosstrack spatial error was 26.77772.

Table 7
Numerical Errors of the GEOS-B Test Orbit
for the BDSP, 100 Steps Per Orbit

ENERGY	ITELEM	B	BS	N	TR
I	0	1.00589 .078112 – 10.73311	.99013 .15110 – .96997	19.86950 .65629 – 40.52781	16.67995 .072534 – 36.09153
I	1	1.00598 .078109 – 13.79768	.99015 .15109 – 1.27553	19.86833 .65628 – 22.83707	16.68023 .07253 – 36.10359
I	2	1.0 .97 .078140 – .59675	.99046 .15116 – .92698	– 19.87749 .65649 – 4.78582	16.68774 .072546 – 2.22072
I	3	.775 .241308 .155	.790 .171367 .180	17.38 .348605 – 3.891	14.315 .236638 – 1.423
R	0	.83340 .078159 – 10.70273	.24629 .15553 .68240	UNSTABLE	SAME AS I0
R	1	.83356 .078162 – 13.68274	.24624 .15553 .38025	UNSTABLE	SAME AS I1
R	2	.83348 .07819 – 1.26500	.24621 .15559 .71750	UNSTABLE	16.68663 .072566 – 4.77124
R	3	.83348 .07819 – 1.31470	.24632 .15559 .66760	UNSTABLE	16.68570 .072560 4.82004

NOTES:

In the last four columns, the three numbers in each entry (given in units of 10^{-6} km) correspond to intrack spatial errors, crosstrack spatial errors, and a timing error (which is also an intrack error), respectively.

By integrating Cowell's equations of motion with time as an independent variable, the intrack spatial error was 3401.4027, and the crosstrack spatial error was 26.77772.

SOURCES

Baumgarte, J., "Numerical Stabilization of the Differential Equations of Keplerian Motion," *Celestial Mechanics*, **5**, pp. 490-501, 1972.

Baumgarte, J., "Stabilization of Constraints and Integrals of Motion in Dynamical Systems," *Computer Methods in Applied Mechanics and Engineering*, **1**, pp. 1-16, 1972.

Baumgarte, J. and E. Stiefel, "Examples of Transformations Improving the Numerical Accuracy of the Integration of Differential Equations," (presented at the Conference on Numerical Solution of Ordinary Differential Equations, Austin, Texas, October 1972).

LaSalle, J. and S. Lefschetz, *Stability by Ljapunov's Direct Method*, Academic Press, 1961.

DISCUSSION

VOICE: In the simulation of the Baumgarte-Stiefel stabilization procedure, corrections are applied in only one direction of a six-dimensional manifold. Wouldn't you expect more than errors in time? Is all the instability controlled by just the energy constraints?

BEAUDET: Yes.

VOICE: It seems that you ought to do more than just force your solution back into the correct energy surface.

BEAUDET: That is in fact correct, but when you look at it, you have to ask yourself the question: "What direction in this six-dimensional state manifold corresponds to the unstable direction, the direction in which errors are going to consistently grow?"

It's like that first diagram I gave you (figure 1). It is the fact that the frequency is in error that gave rise to an ever-increasing mean anomaly. It is an intrack error. In other words, when we integrate, we don't worry at all about crosstrack radial errors. It's intrack errors that are going to grow in time.

It turns out that the uncertainties in the frequency of the motion cause these errors and, since the frequency is somewhat related to the energy, at least in the two-body problem, applying an energy constraint solves the stability problem.

VOICE: But there is no constraint that will give a stable manifold, except for very special problems, which indicates that most problems should be unstable.

BEAUDET: In the perturbed problem that is the case and, if we apply J_2 perturbation, we're going to have an instability associated with the rotating plane of the orbit. In that direction, the manifold space is going to be unstable, even though we might apply an energy constraint. We're simply fixing up the worst part of the things associated with two-body-type motion.

VOICE: What have you done to take care of the high drag case?

BEAUDET: I haven't added anything into the energy to take care of drag. We have integrated orbits where drag was a perturbation and in which I knew that drag was not too severe a perturbation. We still get stabilization as a result. Of course, we have to integrate the energy equation, and drag appears on the right-hand side. The question is always associated with how accurately we can integrate this energy equation.

VOICE: You mean you can stabilize the problem in the case of drag?

BEAUDET: If the drag is not too severe. If the drag becomes very severe, we've come across another problem, that the eccentric anomaly is not the right independent variable to use. If we start hitting a big wall associated with drag, we would like to use a different independent variable than eccentric anomaly while integrating through that region. It is the difficulty associated with finding such an independent variable or using it that has given rise to our inconclusive results in high drag cases.

C.3