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SOLUTIONS OF THE MOTION OF SYNCHRONOUS SATELLITES WITH ARBITRARY ECCENTRICITY AND INCLINATION

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ABSTRACT

A first-order, semianalytical theory for the long-term motion of resonant satellites is presented. The theory is valid for all eccentricities and inclinations and for all commensurability ratios. The method allows the inclusion of all the zonal and tesseral harmonics as well as luni-solar perturbations and radiation pressure.

The method is applied to a synchronous satellite including only the J_2 and J_{22} harmonics. Global, long-term solutions for this problem, eccentricity, argument of perigee, and inclination are obtained.

INTRODUCTION

The method of solution presented here is based on a modified Von Zeipel method applied to resonant satellite systems with three degrees of freedom. The method allows any inclination and eccentricity and is a first-order semianalytical theory, yielding only the long-term perturbations.

The procedure first eliminates the short periodic terms numerically by a classical Von Zeipel averaging process. Since the averaging is performed numerically, developments in the eccentricity and inclination are not necessary. For the same reason, it is straightforward to add luni-solar perturbations and radiation pressure.

After the first Von Zeipel transformation, we have an averaged Hamiltonian that is a tabulated function of the variables: argument of perigee, longitude of the node, eccentricity, and inclination. The averaged Hamiltonian is obtained as a numerical table, not an analytical function.

The next procedure is then to eliminate the angular variable corresponding to the resonance argument, or the critical argument, by a modified Von Zeipel transformation. The principle here is that the averaged Hamiltonian at resonance is a minimum with respect to the critical argument. This principle was introduced by Hori in 1960 for his satellite critical inclination theory. It was used in a slightly different form by Musen and Bailie in their satellite tesseral resonance theory in 1962. During the transformation, the new Hamiltonian is set equal to the value of the old Hamiltonian at its minimum.

After these two canonical transformations, a one-degree-of-freedom system remains, and in that system the Hamiltonian is a constant. We are now able to analyze the new one-degree-of-freedom Hamiltonian and obtain all long-period and secular perturbations, without having to develop a series in eccentricity or inclination.

The procedure and algorithm of the modified Von Zeipel method that we are using was developed by G. Giacaglia in 1965 (and described in detail in 1969). He applied the method to resonant asteroids in a circular restricted three-body problem in 1968. It was then applied to the elliptic restricted problems in 1970 by Giacaglia and Nacozy. The results of the elliptic restricted problem are in agreement with the solutions of asteroidal motion performed by Schubart (1968) using methods of numerical averaging.

The method was applied to the Pluto-Neptune system by Nacozy and Diehl (1974) and the results agree very closely with the numerical integration performed by Williams and Benson (1971), who used methods of numerical averaging over a four and one-half million year period.

Recently, we have adapted the method to satellite systems in resonance with the tesseral harmonics and have applied it to the J_2-J_{22} problem. The adaptation and application are presented below.

DESCRIPTION OF THE METHOD

The method uses the Delaunay equations of motion. The angular variables are modified slightly so that one variable is the mean anomaly, Y_0. A second variable is the critical argument for the synchronous satellite, Y_1, and a third is the argument of perigee, Y_2. Letting the X variables refer to the conjugate momentums, and where F is the Hamiltonian, we have

$$
\begin{aligned}
X_0 &= L - H & \qquad Y_0 &= \ell \\
X_1 &= H & \qquad Y_2 &= \ell + \omega + \Omega + \theta \\
X_2 &= G - H & \qquad Y_3 &= \omega \\
F &= 1/2 \cdot L^2 + H + \text{geopotential (+luni-solar forces)}
\end{aligned}
$$

The first canonical transformation is the classical Von Zeipel transformation where the new Hamiltonian, F*, is defined as the average of the old Hamiltonian over the mean anomaly from 0 to 2π:

$$F^*(X_0^*, X_1^*, X_2^*; -, Y_1^*, Y_2^*) \equiv \frac{1}{2\pi} \int_0^{2\pi} F(X_0, X_1, X_2; Y_0, Y_1, Y_2)\, dy_0.$$

After the averaging, we replace the X and Y variables by X* and Y* and then a new Hamiltonian is obtained, free from short periodic terms. This quadrature is performed numerically, not analytically. The quadrature is performed for various values of X_1*, X_2*, Y_1*, and Y_2*, and the Hamiltonian is obtained as a tabulated function of these variables.

The new Hamiltonian, F*, has a minimum at Y_1* near 90°. This minimum corresponds to the libration center for Y_1 (also 270°). The exact value of Y_1* depends on the value of Y_2* (the perigee). The partial derivative of the Hamiltonian with respect to Y_1*, at Y_1* near 90°, is zero.

The next canonical transformation defines a new Hamiltonian, F**, to be the value of F* at its minimum value:

$$F^{**}(X_0^{**}, X_1^{**}, X_2^{**}; -, -, Y_2^{**})$$

$$= F^* \qquad Y_1^* \cong 90° \text{ (or } 270°).$$

This canonical transformation is that used in 1960 by Hori for his critical inclination solution, and we are using the same principle here. Since F** does not depend on Y_1**, we have:

$$X_1^{**} = \frac{\partial F^{**}}{\partial Y_1^{**}} = 0.$$

This gives a quasi-integral for X_1**:

$$X_1^{**} = H^{**} = \text{constant} = \sqrt{1 - e^{**2}} \cos i^{**}.$$

This relation between e** and i** is the component of the angular momentum in the Z direction. Since F** is a constant of the motion (conservative system), we now have two relations between three variables–eccentricity, inclination, and perigee–and we can rewrite the two relations as:

$$F^{**}(e, i, \omega) = \text{constant}$$

$$H^{**}(e, i) = \text{constant}$$

These two functions define curves in an e versus ω plane with H** as a parameter or curves in an i versus ω plane, also with H** as a parameter. This then gives us the long-term solution for e, i, and ω.

DETAILS OF AN APPLICATION

The first part of the numerical results for the $J_2 - J_{22}$ problem are shown in figure 1 for F* versus the critical argument, Y_1*. Only one plot is shown here for Y_2* (or ω*) 90°, but many other plots were obtained for different values of Y_2* (ω*). We have values of eccentricity ranging from 0.0 to 0.9, but only six values are shown in figure 1. Also, only one value for the parameter H** is given, corresponding to H** to 0.3. It should be noted that there is a minimum of F* at Y_1* = 90°. This corresponds to the libration center for the synchronous satellite at Y_1* = 90°.

We define the new Hamiltonian to have the minimum value of the old Hamiltonian, and then we plot the new Hamiltonian. This is shown in figure 2, for F** versus Y_2**, where Y_2** goes from 0° to 180°. The problem is periodic in π, so this will repeat from 180° to 360°.

The plot of figure 2 may be considered as a three-dimensional plot by visualizing an eccentricity axis as perpendicular to and coming out of the plane of the paper. This will then produce a two-dimensional surface and it can be seen that there will be a valley in the surface centered at Y_2** = 90° and e** = 0.44.

Since F** is a constant for a given trajectory, we can construct planes parallel to the e versus ω plane (in the three-dimensional plot), one plane for each value of the constant Hamiltonian. In other words, we take several contour levels and each of the levels corresponds to a constant Hamiltonian and hence to a certain trajectory. The contour curves give the long-term global solutions for eccentricity versus perigee and are shown in figure 3. Contour levels near the bottom of the valley produce libration. At the bottom of the valley is the libration center, and at the top of the valley there is circulation.

Consider an eccentricity of 0.1. We see in figure 3 that, for this eccentricity, there is circulation of the perigee by following the curve, beginning at e = 0.1 and Y_2** = 0. In other words, the e** versus ω** on that curve corresponds to an actual trajectory, the long-term solution to the problem.

If we have an eccentricity of about 0.4, we have libration of the perigee about the value of perigee equal to 90°. At nearly e = 0.44, there is a stationary solution, and this corresponds to a periodic solution. Apparently, these periodic solutions have not been found prior to this work for the J_2 - J_{22} problem.

Using the quasi-integral relation, we can obtain corresponding plots for the inclination versus the perigee, and this is shown in figure 4. The libration center that was shown in figure 3 corresponding to e = 0.44 shows up at about i** = 70° and ω** = 90°.

The stationary solution (libration center) that we have found and presented here has an eccentricity of 0.44, an inclination of 70°, and a perigee of 90°. We have found that as H** is varied, these stationary solutions form a family of periodic solutions.

CONCLUSION

Our future studies will be to trace the family of periodic (stationary) solutions. We will also add more zonals and tesseral harmonics, the effects of the sun and the moon, and radiation pressure to determine what effect they will have on the family of stationary, librating, and circulating solutions.

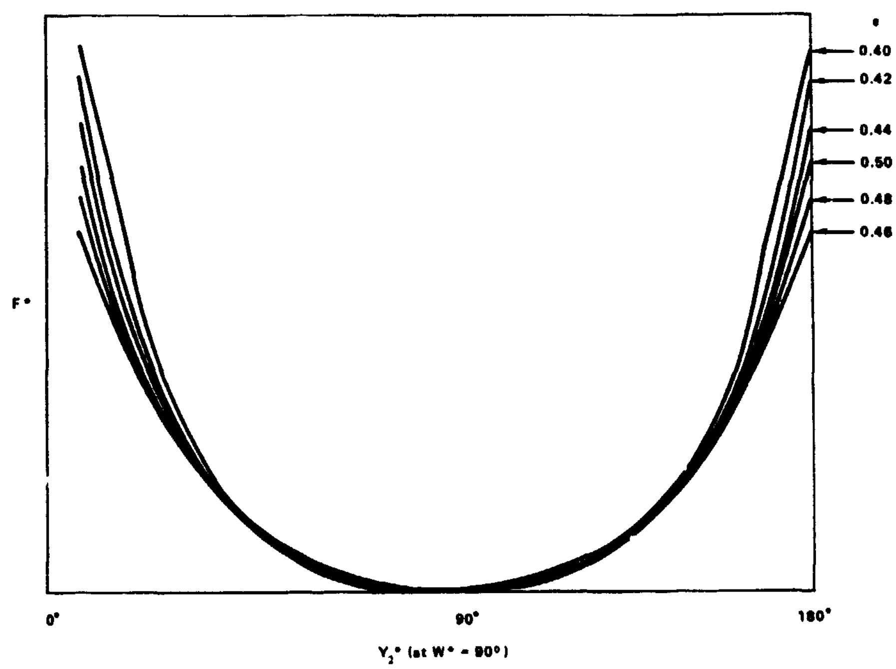

Figure 1. F^* versus ω^* for various values of e for the synchronous satellite with J_2 and J_{22}, F^* = average F and $H^{**} = 0.03$.

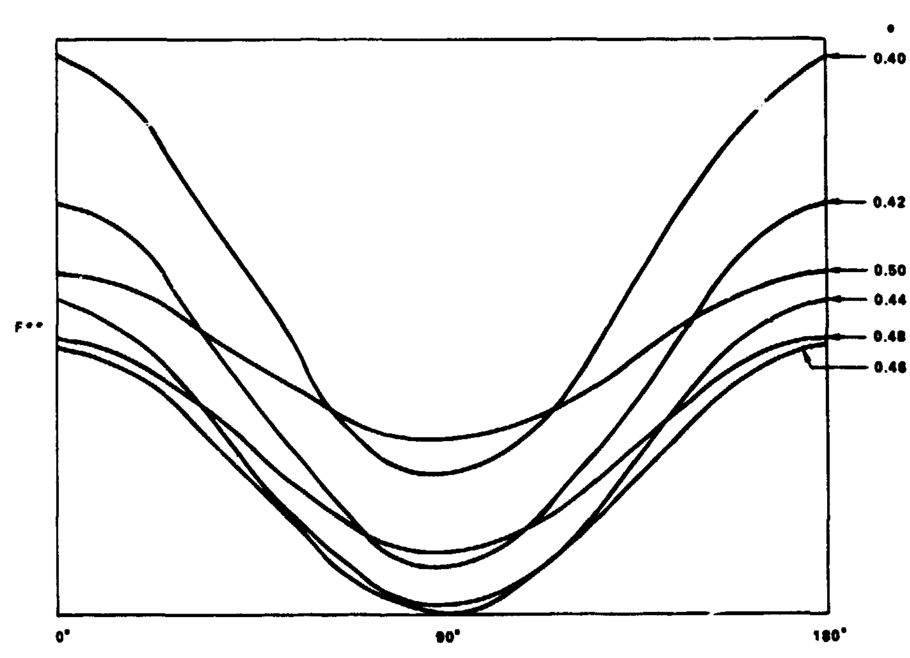

Figure 2. F^{**} versus ω^{**} for various values of e, F^{**} = minimum F^* with respect to $Y_1{}^*$.

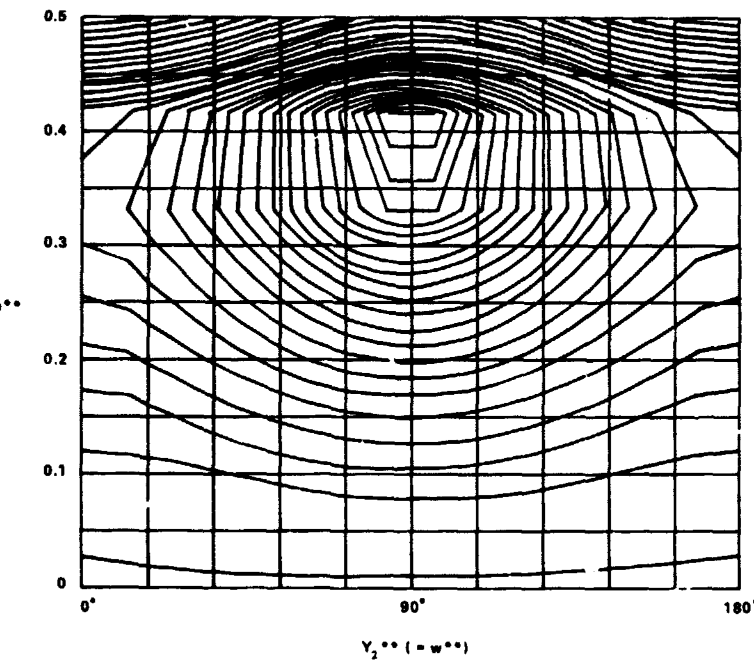

Figure 3. Phase-space diagram for e** versus ω** with H** = 0.3.

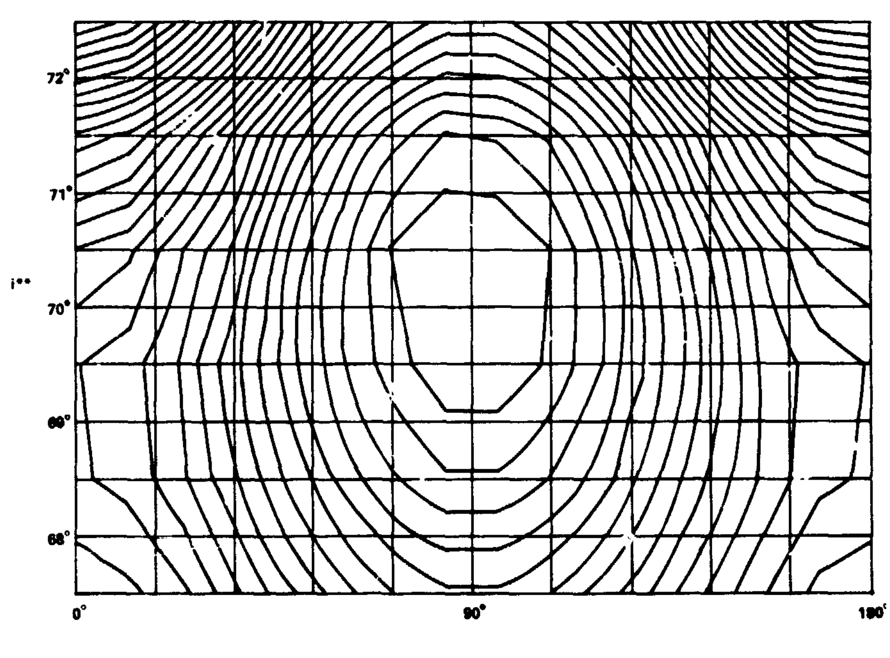

Figure 4. Phase-space diagram for i** versus ω** near libration center.

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DISCUSSION

VOICE: What do you mean by stable stationary solutions?

NACOZY: I mean stability in the sense that if you place a particle on any one of the contour curves (corresponding to a trajectory), any slight deviation from the curve will cause the particle merely to move to another adjacent curve.

VOICE: What do you mean by a quasi-integral?

NACOZY: The integrals that I have presented are accurate only to first order and hence are often referred to as quasi-integrals.