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A STUDY OF ATTITUDE CONTROL CONCEPTS FOR PRECISION-POINTING NONRIGID SPACECRAFT

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16. ABSTRACT

The investigation reported upon in this report was undertaken to study attitude control concepts for use onboard structurally nonrigid spacecraft that must be pointed with great precision. It is an extension of work that has been carried on previously at UCLA with the same objective. The investigation was conducted in four separate areas, each of which is identified as a separate chapter.

In Chapter 1, the task of determining the eigenproperties of a system of linear time-invariant equations (in terms of hybrid coordinates) representing the attitude motion of a flexible spacecraft is attached. Literal characterizations are developed for the associated eigenvalues and eigenvectors of the system. In Chapter 2, a method is presented for determining the poles and zeros of the transfer function describing the attitude dynamics of a flexible spacecraft characterized by hybrid coordinate equations. The investigation reported on in Chapter 3 is motivated by the need for a control design procedure which is insensitive to modeling errors. Alterations are made to linear regulator and observer theory to accommodate modeling errors. The results (some of which are yet unproven) show that a "model error vector," which evolves from an "error system," can be added to a reduced system model, estimated by an observer, and used by the control law to render the system less sensitive to uncertain magnitudes and phase relations of truncated modes and external disturbance effects. Sometimes mode shapes are provided the designer from an outside source. This chapter provides a hybrid coordinate formulation using the provided assumed mode shapes, rather than incorporating the usual finite element approach.

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PREFACE

This report consists of four chapters, as noted in the Table of Contents following. Chapters 1 and 2 were the primary responsibility of Dr. Yoshiaki Ohkami, who was a NASA International University Fellow and Postgraduate Research Engineer at UCLA for two years, on leave from the National Aerospace Laboratory of Japan. Chapter 3 was the responsibility of Robert Skelton; this material represents a preliminary statement of the topic of his doctoral dissertation in the Dynamic Systems Control Field. Chapter 4 is the work of Joseph Canavin, who is beginning his doctoral research in the field of Dynamics. This chapter includes a development which is required for a projected digital computer simulation of the Large Space Telescope Satellite.

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CONTENTS

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Chapter 1. Eigenvalues and Eigenvectors for Hybrid Coordinate Equations of Motion for Flexible Spacecraft

ABSTRACT. Literal characterizations are developed for the eigenvalues and eigenvectors of a system of linear time-invariant equations which describes the attitude motion of flexible spacecraft in terms of hybrid coordinates. The eigenproblem is shown to reduce to that of a symmetric and positive definite matrix of lower dimension. Both analytical and minimax characterization methods prove to be useful in localizing the eigenvalues for the zero damping case. A perturbation method is employed to investigate the effects of modal damping. The resulting eigenvectors generate a canonical form, based on which the controllability is examined.

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1. INTRODUCTION

One of the most commonly accepted procedures for the description of the attitude dynamics and control of flexible spacecraft is a hybrid coordinate formulation, in which the attitude variables of a primary body are discrete coordinates while the displacements of elastic appendages relative to the primary body are characterized by distributed or modal coordinates. Since the formulation has been established and the linearized variational equations are at hand,[1] it concerns us greatly to examine the system behavior based on this representation. In this examination the eigenproblems play a key role for many reasons. Firstly, the eigenvalues or eigenvectors represent essential features of the system such as stability, steady state response and so on. Moreover, in application of existing control theory, whether it be modern control theory or classical, eigenvalues and eigenvectors are useful in canonical transformation, pole location and so on. Secondly, knowledge of eigenproperties of the system affords a sound basis for the truncation procedure which is essential to system simulation or control system design.

However, such systems are usually multivariable systems with an extremely high dimension, so that numerical calculation of the eigenvalues or eigenvectors is not an easy task. Hence, it is desirable to characterize them in some closed form.

The purpose of this paper is to characterize the eigenvalues and eigenvectors of the system in literal expressions, by utilizing peculiar properties of the system parameter matrices as they have been previously established.[1],[2] The eigenvalues are localized in terms of inertial matrices and modal parameters, and a procedure for calculation of the eigenvectors is proposed in some special cases of practical interest. The effects of truncation and damping are also examined.

A canonical form of the system state equations is derived, and controllability criteria are discussed and compared with the previous results.[3]

In the hybrid coordinate representation, the vehicle translational equations take the form[1]

$$I^*\ddot{\theta} - \delta^T\ddot{\eta} = T \tag{1a}$$

and the appendage deformation equations may be written[1],[3]

$$\ddot{\eta} + 2\zeta\sigma\dot{\eta} + \sigma^2\eta - \delta\ddot{\theta} = \phi^T\mathscr{L}_c T \tag{1b}$$

where T is the 3 × 1 matrix representing the external torque vector about the vehicle mass center c, for an orthonormal vector basis in the primary body b, I* is the 3 × 3 inertia matrix of the total vehicle for c, θ is the 3 × 1 matrix of 1-2-3 inertial attitude angles of b, and $\mathscr{L}_c$ establishes the location and type of the attitude control actuators.[3] Here η is the N × 1 matrix representing the modal coordinates, where N represents the number of appendage modes; ϕ is an N × 6n matrix whose columns ϕ^j represent the mode shapes of those appendage vibrations with nonzero natural frequencies σ_j, which could occur independently of each other if the primary body were translationally free but constrained against rotation; σ is an N × N diagonal matrix whose elements are σ_j arranged in non-decreasing order; and ζ is also an N × N diagonal matrix whose elements ζ_j represent the modal damping associated with modal frequencies σ_j, for j = 1, ..., N. The N × 3 matrix δ is established by ϕ and the geometry and mass distribution characteristics of the appendages, as in Eq. (278) of [1]. It should be noted that ϕ is normalized with respect to the generalized inertia matrix, as in Eq. (213) of [1], and that with this normalization δ has the dimension of square root of inertia.

For convenience in later discussions, we define the following matrices:

$$\gamma \triangleq (I^*)^{\frac{1}{2}}\theta \tag{2a}$$ †

$$\Delta \triangleq \delta\,(I^*)^{-\frac{1}{2}} \tag{2b}$$

$$u \triangleq (I^*)^{-\frac{1}{2}}T \tag{2c}$$

and

$$D \triangleq 2\zeta\sigma \tag{2d}$$

With the definitions of Eqs. (2), Eqs. (1) become

$$\ddot{\gamma} - \Delta^T\ddot{\eta} = u \tag{3a}$$

$$\ddot{\eta} + D\dot{\eta} + \sigma^2\eta - \Delta\ddot{\gamma} = \phi^T\mathscr{L}_c\,(I^*)^{\frac{1}{2}}u \tag{3b}$$

where

$$\sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{bmatrix} \tag{4a}$$

with $\sigma_1 \le \sigma_2 \le \ldots \le \sigma_N$.

Furthermore, it is convenient to decompose Δ into submatrices

$$\Delta = \begin{bmatrix} \Delta^1 \\ \Delta^2 \\ \vdots \\ \Delta^N \end{bmatrix} \tag{4b}$$

where the 1 x 3 matrix Δ^j is defined by

$$\Delta^j \triangleq [\ \Delta^j{}_1,\ \Delta^j{}_2,\ \Delta^j{}_3\ ]\ ,\ j = 1,\ 2,\ \ldots,\ N$$

and the scalar quantities $\Delta^j{}_k$ ($j = 1,2,\ldots, N$; $k = 1,2,3$) are dimensionless and [1]

$$0 \le \Delta^j{}_k < 1 \tag{4c}$$

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† Since I^* is symmetric and positive definite by nature, $(I^*)^{\frac{1}{2}}$ is always feasible with the following interpretation: there exists an orthogonal matrix R such that $RI^*R^T = {}^DI$ where ${}^DI$ is a diagonal matrix with positive elements, and $({}^DI)^{\frac{1}{2}}$ is defined. This permits $I^* = R^T\,{}^DIR = R^T({}^DI)^{\frac{1}{2}}\,RR^T({}^DI)^{\frac{1}{2}}R$, so that $(I^*)^{\frac{1}{2}} = R^T({}^DI)^{\frac{1}{2}}R$

In what follows, the 3×3 matrix $\Delta^T\Delta$ plays an important role so that some of its properties are stated: if we interpret δ in terms of primitive definitions, then we obtain the physical interpretation[2]

$$\delta^T\delta = I^* - I^0$$

where I^0 is the inertia matrix of the primary body referred to its own mass center. From Eq. (2b),

$$\Delta^T\Delta = (I^*)^{-\frac{1}{2}}(I^*-I^0)\ (I^*)^{-\frac{1}{2}} \tag{4d}$$

Obviously, the matrix (I^*-I^0) is symmetric and positive definite and hence $\Delta^T\Delta$ is positive definite. Furthermore, if we consider the matrix

$$U_3 - \Delta^T\Delta = (I^*)^{-\frac{1}{2}}I^0(I^*)^{-\frac{1}{2}} \tag{4e}$$

it is also positive definite. Therefore, the eigenvalues of $\Delta^T\Delta$ are greater than zero and less than unity, provided that there exist at least three independent rows in Δ (see Appendix A).

In terms of the $(2N + 6) \times 1$ state variable

$$X \triangleq \begin{bmatrix} \gamma \\ \dot{\gamma} \\ \eta \\ \dot{\eta} \end{bmatrix}$$

Eqs. (3) may be written as

$$\dot{X} = AX + Bu \tag{5}$$

where

$$A = \begin{bmatrix} 0 & U_3 & 0 & 0 \\ 0 & 0 & -\Delta^T M_1 \sigma^2 & -\Delta^T M_1 D \\ 0 & 0 & 0 & U_N \\ 0 & 0 & -M_1\sigma^2 & -M_1 D \end{bmatrix} \tag{6a}$$

and

$$B = \begin{bmatrix} 0 \\ B_2 \\ 0 \\ B_4 \end{bmatrix} \tag{6b}$$

with $$B_2 \triangleq M_2 + \Delta^T M_1 \phi^T \mathscr{L}_c(I^*)^{\frac{1}{2}} \tag{6c}$$

$$B_4 \triangleq \Delta M_2 + M_1 \phi^T \mathscr{L}_c(I^*)^{\frac{1}{2}} \tag{6d}$$

$$M_1 \triangleq (U_N - \Delta\Delta^T)^{-1} \tag{6e}$$

and

$$M_2 \triangleq (U_3 - \Delta^T\Delta)^{-1} \tag{6f}$$

The noted properties of $\Delta^T\Delta$ guarantee the nonsingularity of $(U_3 - \Delta^T\Delta)$, so that M_2 is always feasible. Since $|U_N - \Delta\Delta^T| = |U_3 - \Delta^T\Delta|$, M_1 is also feasible. Moreover, M_1 and M_2 are both symmetric and positive definite, as shown in Appendix A.

2. REDUCTION OF EIGENPROBLEM

Let x be a generic eigenvector of A corresponding to a generic eigenvalue λ; then the problem is to find λ and x which satisfy

$$[\lambda U_{2N+6} - A]\, x = 0 \tag{7}$$

with the characteristic equation

$$|\lambda U_{2N+6} - A| = 0 \tag{8}$$

By the determinant partitioning formula [6], Eq. (8) may be written from Eq. (6a) as

$$\begin{vmatrix} \lambda U_3 & -U_3 \\ 0 & \lambda U_3 \end{vmatrix} \cdot \begin{vmatrix} \lambda U_N & -U_N \\ M_1\sigma^2 & \lambda U_N + M_1 D \end{vmatrix} = 0 \tag{9}$$

From the first determinant of Eq. (9)

$$\lambda^6 = 0 \tag{10}$$

and we have $\lambda = 0$ with multiplicity 6.

The second determinant of Eq. (9) is further decomposed into

$$|\lambda U_N|\cdot|\lambda U_N + M_1 D - M_1\sigma^2(\lambda U_N)^{-1}(-U_N)| = 0$$

or

$$|\lambda^2 U_N + \lambda M_1 D + M_1\sigma^2| = 0 \tag{11}$$

If we also decompose the eigenvector x as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{12}$$

where x_1, x_2, x_3 and x_4 are 3×1, 3×1, N×1, and N×1 matrices associated respectively with $\gamma, \dot{\gamma}, \eta$ and $\dot{\eta}$, then Eq. (7) becomes

$$\begin{bmatrix} \lambda U_3 & -U_3 \\ 0 & \lambda U_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \Delta^T M_1 \sigma^2 & \Delta^T M_1 D \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = 0 \tag{13}$$

and

$$\begin{bmatrix} \lambda U_N & -U_N \\ M_1\sigma^2 & \lambda U_N + M_1 D \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = 0 \tag{14}$$

This decomposition indicates a procedure for obtaining the eigenvectors: since Eq. (14) does not include x_1 or x_2, we can first solve Eq. (14) for x_3 and x_4, and then substitute the solution into Eq. (13) to produce x_1 and x_2.

For the solution $\lambda = 0$ of Eq. (10), the determinant of Eq. (14) becomes

$$\begin{vmatrix} 0 & -U_N \\ M_1\sigma^2 & M_1 D \end{vmatrix} = (-1)^N \, |M_1\sigma^2| \neq 0$$

and hence there exists only the trivial solution for x_3, x_4, i.e.,

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = 0 \tag{15}$$

Substituting Eqs. (10) and (15) into Eq. (13) yields

$$\begin{bmatrix} 0 & -U_3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

implying $\quad x_1 = \text{arbitrary}, \quad x_2 = 0.$

We may introduce three independent and orthogonal 6×1 matrices $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$ defined by

$$C^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16a}$$

The six eigenvectors corresponding to $\lambda = 0$ are null except for the upper 6×1 partitions, which are $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, and three linear combinations of these matrices. We see that the number of independent eigenvectors corresponding to $\lambda=0$ is less than the multiplicity 6 of $\lambda = 0$, so the matrix A is not similar to a diagonal matrix but to a nondiagonal Jordan form. We can find such a transformation matrix if we construct the generalized eigenvectors, which are null except for upper 6×1 partitions given by (among many)

$$g^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad g^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad g^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{16b}$$

For the <u>nonzero</u> eigenvalues determined by Eq. (11), we have from Eqs. (13) and (14)

$$(M_1\sigma^2 + \lambda M_1 D + \lambda^2 U_N)\; x_3 = 0 \tag{17a}$$

$$x_4 = \lambda x_3 \tag{17b}$$

$$x_2 = -\frac{1}{\lambda}\,(\Delta^T M_1 \sigma^2 x_3 + \Delta^T M_1 D\, x_4) \tag{17c}$$

$$x_1 = \frac{1}{\lambda}\, x_2 \tag{17d}$$

Equations (17) indicate that x_3 is the only vector to be solved as the eigenvector, and that other vectors are all determined by matrix or scalar multiplications and additions.

Thus, the eigenvalue and eigenvector problem of A reduces to Eq. (17a), in which λ is in the second order form. Eq. (14) is also a reduced form which may be written in a standard form of eigenvalue problems as

$$(\lambda U_{2N} - \mathscr{A})\; Y = 0 \tag{18}$$

where

$$\mathscr{A} \triangleq \left[\begin{array}{c|c} 0 & U_N \\ \hline -M_1\sigma^2 & -M_1 D \end{array}\right] \tag{19a}$$

and

$$Y \triangleq \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \tag{19b}$$

In order to localize the eigenvalues and characterize the eigenvectors, we will take the following procedures:

We first assume that $\zeta = 0$ (so $D = 0$) and in the second order form of Eq. (17a) we define

$$\mu \triangleq -\lambda^2 \tag{20}$$

to which corresponds the eigenvector ψ. Then Eq. (17a) becomes

$$(\mu U_N - M_1\sigma^2)\psi = 0 \tag{21}$$

for which the noted properties of M_1 will be fully utilized.

After obtaining the results for D = 0, we will return to the first order form of Eq. (14) and treat D as a small perturbation.

3. CHARACTERIZATION OF EIGENVALUES

3.1 Analytical Method

Since M_1 is nonsingular, we may define the characteristic polynomial, $f(\mu)$, for Eq. (21) by

$$f(\mu) \triangleq |M_1^{-1}||\mu U_N - M_1\sigma^2| \tag{22a}$$

or, in view of Eq. (6e),

$$f(\mu) = |\mu U_N - \sigma^2 - \mu\Delta\Delta^T| \tag{22b}$$

If we rewrite $f(\mu)$ as

$$\begin{aligned} f(\mu) &= |M_1^{-1}|\ |\sigma|\ |\mu U_N - M_1\sigma^2|\ |\sigma^{-1}| \\ &= |M_1^{-1}|\ |\mu U_N - \sigma M_1\sigma| \end{aligned}$$

then we recognize that μ is the eigenvalue of an N×N real, symmetric, and positive definite matrix $\sigma M_1\sigma$. Thus, we have[4] the following fact.

Fact 1. All the roots of $f(\mu) = 0$ are real, and positive, i.e., all the eigenvalues of $M_1\sigma^2$ are real and positive, and hence, by Eqs. (10) and (20), all the nonzero eigenvalues of $\mathscr{A}$ are imaginary. In addition, the eigenvalue problem is well-conditioned (a small change in the elements of $\sigma M_1\sigma$ does not cause any abrupt change in the eigenvalues).

In what follows we attempt to characterize the eigenvalues further mainly based upon a particular structure of $f(\mu)$ (Eq. (22b)), in which $\mu U_N - \sigma^2$ is diagonal and $\Delta\Delta^T$ is of rank 3.

In order to do so, we first tentatively assume that for the root of $f(\mu) = 0$, $(\mu U_N - \sigma^2)$ is nonsingular, i.e.,

$$|\mu U_N - \sigma^2| \neq 0 \tag{23}$$

Under this assumption, we may rewrite Eq. (22) using the determinant identity[6] as

$$f(\mu) = |\mu U_N - \sigma^2| \cdot g(\mu) \tag{24a}$$

where

$$g(\mu) \triangleq |U_3 - \mu\Delta^T (\mu U_N - \sigma^2)^{-1}\Delta| \tag{24b}$$

Since in this case the roots of $f(\mu) = 0$ are identical to those of $g(\mu)=0$, we will treat $g(\mu)$ rather than $f(\mu)$ because the former is related to a 3×3 matrix while the latter is to an N×N matrix.

The cases in which Eq. (23) is violated will be discussed in such a way as to establish under what conditions the system eigenvalue becomes identical to σ_ℓ^2 for some ℓ. A sufficient condition is stated below for this to hold.

<u>Fact 2</u>. If $\Delta^\ell = 0$, then $f(\sigma_\ell^2) = 0$. This means that the natural frequency, σ_ℓ, of an appendage vibration mode with $\Delta^\ell = 0$ is also that of a system vibration mode.

Proof. If we rewrite Eq. (22b) as

$$f(\mu) = \begin{vmatrix} \mu - \sigma_1^2 & -\mu\Delta^1\Delta^{1T} & \cdots & \cdots & -\mu\Delta^1\Delta^{NT} \\ \vdots & \ddots & & & \vdots \\ -\mu\Delta^\ell\Delta^{1T} & \cdots & \mu - \sigma_\ell^2 & -\mu\Delta^\ell\Delta^{\ell T} \cdots & -\mu\Delta^\ell\Delta^{NT} \\ \vdots & & & \ddots & \vdots \\ -\mu\Delta^N\Delta^1 & \cdots & \cdots & \mu - \sigma_N^2 & -\mu\Delta^N\Delta^{NT} \end{vmatrix}$$

and expand about the ℓ-th row with $\Delta^\ell = 0$, we have

$$f(\mu) = (\mu - \sigma_\ell^2) \begin{vmatrix} \mu - \sigma_1^2 & -\mu\Delta^1\Delta^{1T} & \cdots & -\mu\Delta^1\Delta^{NT} \\ \vdots & \ddots & & \vdots \\ -\mu\Delta^N\Delta^{1T} & \cdots & \mu - \sigma_N^2 & -\mu\Delta^N\Delta^{NT} \end{vmatrix}$$

so that

$$f(\sigma_\ell^2) = 0 \quad .$$

It is noted that $f(\sigma_\ell^2) = 0$ does not necessarily imply $\Delta^\ell = 0$. In fact, if σ_j's and Δ^j's satisfy a certain equality, then it happens that $f(\sigma_\ell^2) = 0$ for nonzero Δ^ℓ as will be shown later.

In order to evaluate the roots of Eq. (24b), we define

$$d_j(\mu) \triangleq \frac{\mu}{\mu - \sigma_j^2} \qquad j = 1,2,\ldots,N \tag{25}$$

and, with this definition, we have

$$\mu(\mu U_N - \sigma^2)^{-1} = \begin{bmatrix} d_1(\mu) & & & \\ & d_2(\mu) & & \\ & & \ddots & \\ & & & d_N(\mu) \end{bmatrix} \tag{26}$$

and

$$\begin{aligned} g(\mu) &= \left| U_3 - \begin{bmatrix} \Delta^1 \\ \Delta^2 \\ \vdots \\ \Delta^N \end{bmatrix}^T \begin{bmatrix} d_1(\mu) & & \\ & \ddots & \\ & & d_N(\mu) \end{bmatrix} \begin{bmatrix} \Delta^1 \\ \Delta^2 \\ \vdots \\ \Delta^N \end{bmatrix} \right| \\ &= \left| U_3 - \sum_{j=1}^{N} d_j(\mu) \Delta^{jT} \Delta^j \right| \\ &= \left| \left[\delta^\alpha_\beta - \sum_{j=1}^{N} \Delta^j_\alpha \;\; \Delta^j_\beta \; d_j(\mu) \right] \right| \end{aligned} \tag{27}$$

where δ^α_β is the Kronecker delta defined as

$$\delta^\alpha_\beta = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

and the quantity in the brackets indicates the (α,β) element of $[U_3 - \mu\, \Delta^T(\mu\, U_N - \sigma^2)^{-1}\Delta]$. Some properties of $d_j(\mu)$ are given in Appendix B.

The function $g(\mu)$ is easily evaluated for some particular values of interest. Since $d_j(\mu) = 0$ for $\mu = 0$,

$$g(0) = |U_3| = 1, \tag{28}$$

and since

$$\begin{aligned} \lim_{\mu\to\pm\infty} d_j(\mu) &= 1, \\ \lim_{\mu\to\pm\infty} g(\mu) &= |U_3 - \Delta^T\Delta| \end{aligned} \tag{29}$$

Consequently

$$0 < \lim_{\mu\to\pm\infty} g(\mu) < 1 \tag{30}$$

in view of the noted property of Eq. (4e).

Considering the properties of $g(\mu)$ of Eqs. (28) and (30), we will further characterize the roots of $g(\mu) = 0$ based on the fact that it grows without bound when μ approaches to σ_j^2 for any $j = 1,2,\ldots,N$.

As proven in Appendix C, $g(\mu)$ is expressed by

$$g(\mu) = 1 - \sum_{\ell=1}^{N} G_\ell(\mu)\; d_\ell(\mu) \tag{31}$$

where

$$\begin{aligned} G_\ell(\mu) = \Delta^\ell\Delta^{\ell T} &- \sum_{j=1}^{\ell-1} d_j(\mu)\; K_j^\ell \\ &+ \sum_{1\le j<k\le \ell-1} d_j(\mu)\, d_k(\mu)\, L_{jk}^\ell \end{aligned} \tag{32}$$

with

$$K_j^\ell \overset{\Delta}{=} (\Delta^\ell \tilde{\Delta}^j)(\Delta^\ell \tilde{\Delta}^j)^T \tag{33}$$

and

$$L_{jk}^\ell \overset{\Delta}{=} \left\{ \Delta^\ell (\Delta^j \tilde{\Delta}^k)^T \right\}^2 \tag{34}$$

where

$$\tilde{\Delta}^j \overset{\Delta}{=} \begin{bmatrix} 0 & -\Delta_3^j & \Delta_2^j \\ \Delta_3^j & 0 & -\Delta_1^j \\ -\Delta_2^j & \Delta_1^j & 0 \end{bmatrix} . \tag{35}$$

We recognize the following properties of K_j^ℓ and L_{jk}^ℓ, noting that by the assumption in Eq. (23) and Fact 2 following Eq. (24b) we have excluded the case $\Delta^\ell = 0$ for all ℓ.

(i) $K_j^\ell \geq 0$ (36)

with the equality when $\Delta^\ell \tilde{\Delta}^j = 0$, implying that two vectors have the same direction.

(ii) $K_j^\ell = K_\ell^j$ (37)

(iii) $L_{jk}^\ell \geq 0$

with the equality when $\Delta^\ell (\Delta^j \tilde{\Delta}^k)^T = 0$. This takes place if any two of Δ^j, Δ^k and Δ^ℓ have the same direction or if one of these three is perpendicular to the cross-product of the others. Hence

(iv) If $K_j^\ell = 0$, then $L_{jk}^\ell = 0$. (38)

(v) $L_{jk}^\ell = L_{kj}^\ell = L_{\ell k}^j$ (39)

From the expression of Eq. (31) with the definition of Eq. (32) and the noted properties of Eqs. (36)-(39) we have the following fact:

Fact 3. All the roots of $g(\mu) = 0$ are greater than σ_1^2 .

Proof. Consider the value of $g(\mu)$ for $0 < \mu < \sigma_1^2$. Then, by the definition of Eq. (25)

$$d_\ell(\mu) = \frac{\mu}{\mu - \sigma_\ell^2} < 0 \qquad \text{for } \ell = 1,2,\ldots,N \tag{40a}$$

and hence

$$G_\ell(\mu) \geq 0 \qquad \text{for } \ell = 1,2,\ldots,N \tag{40b}$$

Equations (40) guarantee that

$$g(\mu) \geq 1 \qquad \text{for } 0 < \mu < \sigma_1^2$$

so that there exist no roots of $g(\mu) = 0$ in the range $0 < \mu < \sigma_1^2$. On the other hand, from Fact 1, there exists no negative root of $g(\mu) = 0$. Therefore, all the roots of $g(\mu) = 0$ are greater than σ_1^2.

One can interpret the Facts 1 and 3 in terms of the eigenvalues, λ, of A as follows. If the roots of $g(\mu) = 0$ are denoted by μ_j, $j = 1,2,\ldots,N$ and arranged in non-decreasing order, then the Fact 3 indicates that

$$\sigma_1^2 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_N .$$

In view of Eq. (20), this implies that for any eigenvalue λ of A,

$$\sigma_1^2 < -\lambda^2 \qquad \text{or} \qquad \lambda^2 < -\sigma_1^2$$

meaning λ is purely imaginary and $|\lambda| > \sigma_1$. Thus, if we denote the eigenvalues of A by λ_j, $j = 1,2,\ldots,2N$, then we may arrange them as

$$\left.\begin{aligned} &\lambda_j = i\sqrt{\mu_j} && j = 1,2,\ldots,N \\ \text{and} \quad &\lambda_j = -i\sqrt{\mu_{j-N}} = \lambda_{j-N}^* && j = N+1,\ldots,2N \end{aligned}\right\} \tag{41}$$

Although Eq. (31) was useful in proving the Fact 3, it is not appropriate for further localization of the eigenvalues utilizing the noted properties of

$d_j(\mu)$, because the $G_\ell(\mu)$ in Eq. (32) contains $d_j(\mu)$ $(j < \ell)$. To avoid this difficulty, we may expand the products $d_j(\mu)\, d_k(\mu)$ in $G_\ell(\mu)$ and collect all the terms including $d_\ell(\mu)$ in Eq. (31) to obtain (see Appendix C)

$$g(\mu) = 1 - \sum_{\ell=1}^{N} P_\ell \, d_\ell(\mu) \tag{42}$$

where

$$P_\ell \stackrel{\Delta}{=} \Delta^\ell \Delta^{\ell T} - \sum_{\substack{j=1 \\ j\neq\ell}}^{N} d_{j\ell}\, K_j^\ell + \sum_{\substack{1 \leq j < k \leq N \\ j,k \neq \ell}} d_{j\ell}\, d_{k\ell}\, L_{jk}^\ell \tag{43}$$

and

$$d_{j\ell} \stackrel{\Delta}{=} d_j(\sigma_\ell^2) = \frac{\sigma_\ell^2}{\sigma_\ell^2 - \sigma_j^2} \tag{44}$$

with

$$d_{j\ell} > 0 \qquad \text{if} \qquad j < \ell \tag{45a}$$

$$d_{j\ell} < 0 \qquad \text{if} \qquad j > \ell \tag{45b}$$

Consider the case when μ approaches to σ_ℓ^2. Then, $d_\ell(\mu)$ grows without bound, so we have

$$\lim_{\mu\to\sigma_\ell^2\pm0} g(\mu) = -\lim_{\mu\to\sigma_\ell^2\pm0} P_\ell d_\ell(\mu) = \begin{cases} \mp\infty & \text{if} \quad P_\ell > 0 \\ \pm\infty & \text{if} \quad P_\ell < 0 \end{cases} \tag{46}$$

Immediately from the properties of $g(\mu)$ of Eqs. (28), (30) and (46) follows Fact 4, as illustrated in Fig. 1.

<u>Fact 4</u>. If $P_\ell > 0$ for $\ell = 2,3,\ldots,N$, then the roots of $g(\mu) = 0$ are separated by the σ_j^2's as follows

$$\sigma_1^2 < \mu_1 < \sigma_2^2 < \mu_2 < \ldots < \sigma_N^2 < \mu_N \tag{47}$$

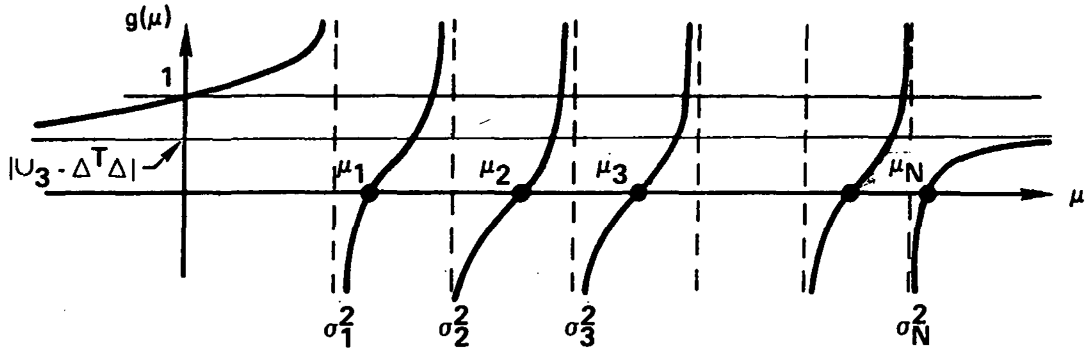

Figure 1. $g(\mu)$ Versus μ When $P_\ell > 0$, $\ell = 2,3\ldots, N$.

It is evident that $P_1 > 0$ because $d_{j1} < 0$ for $j = 2,3,\ldots,N$. However, P_ℓ ($\ell \geq 2$) is not necessarily of a definite sign; the sign of P_ℓ depends on the relations between the σ_j^2's and the Δ^j's. Loosely speaking, P_ℓ tends to be negative if Δ^j and Δ^ℓ are uncoupled or moderately coupled (meaning K_j^ℓ and possibly L_{jk}^ℓ are nonzero) and if the two adjacent frequencies σ_j and σ_ℓ are very close (meaning $d_{j\ell}$ is very large). In such a situation it is unlikely that there exists a root of $g(\mu) = 0$ between those frequencies. This is shown by the following example.

If we rewrite P_ℓ by

$$P_\ell = P_\ell^+ - P_\ell^- \tag{48}$$

where

$$P_\ell^+ \triangleq \Delta^\ell \Delta^{\ell T} - \sum_{j=\ell+1}^{N} d_{j\ell} K_j^\ell$$

$$+ \sum_{1\le j<k\le \ell} d_{j\ell} d_{k\ell} L_{jk}^\ell + \sum_{\ell<j<k\le N} d_{j\ell} d_{k\ell} L_{jk}^\ell \tag{49a}$$

and

$$P_\ell^- \triangleq \sum_{j=1}^{\ell-1} d_{j\ell} K_j^\ell - \sum_{1\le j<\ell<k\le N} d_{j\ell} d_{k\ell} L_{jk}^\ell , \tag{49b}$$

then for $\Delta^\ell \neq 0$

$$P_\ell^+ > 0 \quad \text{and} \quad P_\ell^- \geq 0$$

in view of Eqs. (45). (See Fig. C1 of Appendix C.)

Assume that σ_r is so close to σ_{r+1} as to satisfy

$$M \triangleq d_{r,r+1} = \frac{\sigma_{r+1}^2}{\sigma_{r+1}^2 - \sigma_r^2} >> 1$$

and $d_{r,r+1} >> |d_{jk}|$ for any $j \neq k$. Then, we may approximate $d_{r+1,r}$ by $-M$ and, neglecting terms not containing M, we have

$$P_r^+ \approx \left(K_{r+1}^r - \sum_{j=r+2}^{N} L_{r+1,j}^r \, d_{jr} \right) M \tag{50a}$$

$$P_r^- \approx \left(+ \sum_{j=1}^{r-1} L_{j,r+1}^r \, d_{jr} \right) M \tag{50b}$$

and

$$P_{r+1}^+ \approx \left(\sum_{j=1}^{r-1} L_{j,r}^{r+1} \, d_{j,r+1} \right) M \tag{50c}$$

$$P_{r+1}^- \approx \left(K_r^{r+1} - \sum_{j=r+2}^{N} L_{r,j}^{r+1} \, d_{j,r+1} \right) M \tag{50d}$$

From Eqs. (37) and (39) and Eqs. (50),

$$P^+_{r+1} \approx P^-_r$$

$$P^-_{r+1} \approx P^+_r$$

implying that

$$P_{r+1} \approx -P_r \quad .$$

Hence, for $\sigma_r^2 < \mu < \sigma_{r+1}^2$,

$$g(\mu) \approx 1 - \left(\frac{\mu P_r}{\mu-\sigma_r^2} + \frac{\mu P_{r+1}}{\mu-\sigma_{r+1}^2}\right) \approx 1 - \left(\frac{\mu}{\mu-\sigma_r^2} - \frac{\mu}{\mu-\sigma_{r+1}^2}\right) P_r$$

and

$$g'(\mu) \overset{\Delta}{=} \frac{d}{d\mu} g(\mu) \approx \left\{\frac{\sigma_r^2}{(\mu-\sigma_r^2)^2} - \frac{\sigma_{r+1}^2}{(\mu-\sigma_{r+1}^2)^2}\right\} P_r$$

If we denote the root of $g'(\mu) = 0$ by μ_m, then

$$\mu_m = \sigma_r \sigma_{r+1} \; ,$$

and the local extremal of $g(\mu)$ is given by

$$g(\mu_m) \approx 1 - \frac{2\sigma_r}{\sigma_{r+1} - \sigma_r} P_r \quad .$$

If $P_r > 0$, then $P_{r+1} < 0$ and

$$g(\mu_m) < 0 \quad \text{(see Fig 2a)}.$$

If $P_r < 0$, then $P_{r+1} > 0$ and

$$g(\mu_m) > 0 \text{ (see Fig 2b)}$$

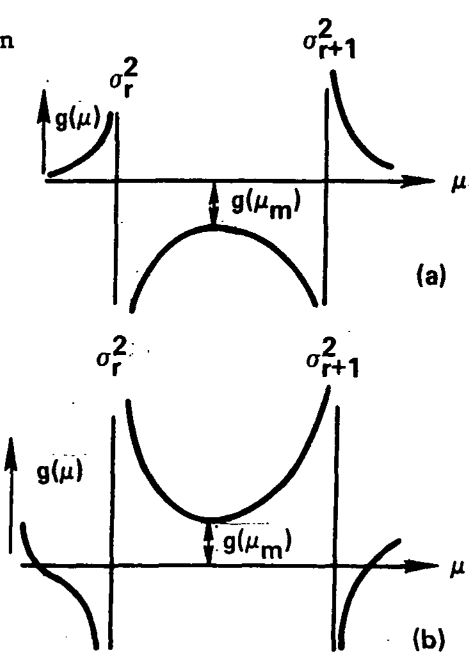

Figure 2. Roots Not Separated.

In any case, no root exists between $\sigma_r^2 < \mu < \sigma_{r+1}^2$.

In what follows, we will investigate some special cases in which some identifiable sets of the P_ℓ are positive. Suppose that $\Delta^1, \Delta^2, \ldots, \Delta^{N_1}$ have the same direction and they are orthogonal to the others, which necessarily lie in the plane normal to any Δ^j, $(j = 1,2,\ldots,N_1)$. Then, the system eigenvalues associated with $\Delta^1,\ldots,\Delta^{N_1}$ are independent of the others and in addition

$$K_j^\ell = 0, \qquad L_{jk}^\ell = 0 \qquad \text{for } j,k,\ell=1,\ldots,N_1$$

implying

$$P_\ell = \Delta^\ell \Delta^{\ell T} > 0 \qquad \text{for } \ell=1,2,\ldots,N_1 \ .$$

Therefore, Fact 4 always applies to this group of modes. For the remaining $(N-N_1)$ modes $L_{jk}^\ell = 0$ but in general $K_j^\ell \neq 0$ for $\ell,j \in \{N_1+1,\ N_1+2,\ldots,N\}$.

Furthermore, if the remaining modes are classified into two other groups, i.e., $\Delta^{N_1+1} \ldots \Delta^{N_1+N_2}$ and $\Delta^{N_1+N_2+1} \ldots \Delta^N$, which groups are also orthogonal, then the problem is decomposed into three single axis problems, each of which can be treated independently with $P_\ell = \Delta^\ell \Delta^{\ell T} > 0$.

3.2 Eigenvalue Localization by the Minimax Theorem

Taking advantage of the noted properties of the matrices M_1 and σ^2, another localization of the eigenvalues of Eq. (21) will be accomplished by utilizing a result based on the minimax method [4] as stated below.

Theorem: If A,B,C are symmetric matrices with the eigenvalues α_j, β_j and γ_j, respectively, which are arranged in non-increasing order, and if A,B and C are related by

$$C = A + B,$$

then for any s = 1,2,...,N,

$$\alpha_s + \beta_N \leq \gamma_s \leq \alpha_s + \beta_1 \ .$$

This means that if we add B to A, all of the eigenvalues of A are changed by an amount which lies between the smallest and greatest of the eigenvalues of B.

In order to apply this theorem, the characteristic equation of Eq. (21) is rewritten without changing the eigenvalues as follows.

Since $|M_1^{-1}| \neq 0$ and $|\sigma| \neq 0$,

$$|\mu U_N - M_1\sigma^2| = 0$$

is equivalent to

$$|\sigma^{-1} M_1^{-1}| \cdot |\mu U_N - M_1\sigma^2| \cdot |\sigma^{-1}| = 0$$

or

$$|\mu\sigma^{-1}M_1^{-1}\sigma^{-1} - U_N| = 0 \tag{51}$$

If we define ν by

$$\nu \overset{\Delta}{=} \frac{1}{\mu} \tag{52}$$

then we may rewrite Eq. (51) in terms of ν as

$$|\nu U_N - \sigma^{-1}M_1^{-1}\sigma^{-1}| = 0 \tag{53}$$

in which the matrix $\sigma^{-1}M_1^{-1}\sigma^{-1}$ is also symmetric and positive definite.

If we partition the matrices Δ and σ into submatrices by writing

$$\sigma = \left[\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & \hat{\sigma} \end{array}\right] \tag{54}$$

and

$$\Delta = \left[\begin{array}{c} \Delta^1 \\ \hline \hat{\Delta} \end{array}\right] \tag{55}$$

where

$$\hat{\sigma} = \begin{bmatrix} \sigma_2 & & \\ & \ddots & \\ & & \sigma_N \end{bmatrix} \quad \text{and} \quad \hat{\Delta} = \begin{bmatrix} \Delta^2 \\ \vdots \\ \Delta^N \end{bmatrix}, \tag{56}$$

then

$$\Delta\Delta^T = \left[\begin{array}{c|c} \Delta^1\Delta^{1T} & \Delta^1\hat{\Delta}^T \\ \hline \hat{\Delta}\,\Delta^{1T} & \hat{\Delta}\,\hat{\Delta}^T \end{array}\right]$$

so

$$\begin{aligned} M_1^{-1} &= U - \Delta\Delta^T \\ &= \left[\begin{array}{c|c} 1 - \Delta^1\Delta^{1T} & -\Delta^1\hat{\Delta}^T \\ \hline -\hat{\Delta}\,\Delta^{1T} & U_{N-1} - \hat{\Delta}\hat{\Delta}^T \end{array}\right] \end{aligned}$$

and

$$\sigma^{-1}M_1^{-1}\sigma^{-1} = \left[\begin{array}{c|c} \sigma_1^{-1}(1 - \Delta^1\Delta^{1T})\sigma_1^{-1} & -\sigma_1^{-1}\,\Delta^1\hat{\Delta}^T\,\hat{\sigma}^{-1} \\ \hline -\hat{\sigma}^{-1}\hat{\Delta}\Delta^{1T}\sigma_1^{-1} & \hat{\sigma}^{-1}(U_{N-1} - \hat{\Delta}\hat{\Delta}^T) \end{array}\right] \tag{57}$$

Writing

$$\sigma^{-1}M_1^{-1}\sigma^{-1} = \mathscr{D}^1 + \mathscr{F}^1 \tag{58}$$

where

$$\mathscr{D}^1 \triangleq \left[\begin{array}{c|c} \sigma_1^{-1}(1 - \Delta^1\Delta^{1T})\sigma_1^{-1} & 0 \\ \hline 0 & \hat{\sigma}^{-1}(U_{N-1} - \hat{\Delta}\hat{\Delta}^T)\hat{\sigma}^{-1} \end{array}\right] \tag{59}$$

and

$$\mathscr{F}^1 \triangleq \left[\begin{array}{c|c} 0 & -\sigma_1^{-1}\Delta^1\hat{\Delta}^T\,\hat{\sigma}^{-1} \\ \hline -\hat{\sigma}^{-1}\hat{\Delta}\Delta^{1T}\sigma_1^{-1} & 0 \end{array}\right], \tag{60}$$

we denote the eigenvalues of $\sigma^{-1}M_1^{-1}\sigma^{-1}$, $\mathscr{D}^1$ and $\mathscr{F}^1$ by ν_j, ν_j' and ω_j, respectively. Note that the matrices $\mathscr{D}^1$ and $\mathscr{F}^1$ are also symmetric.

The eigenvalues, ν_j', of $\mathscr{D}^1$ are the roots of

$$|\nu_j' U_N - \mathscr{D}^1| = 0$$

i.e.,

$$\left|\begin{array}{c|c} \nu' - \sigma_1^{-1}(1 - \Delta^1\Delta^{1T})\sigma_1^{-1} & 0 \\ \hline 0 & \nu' U_{N-1} - \hat{\sigma}^{-1}(U_{N-1} - \hat{\Delta}\hat{\Delta}^T)\hat{\sigma}^{-1} \end{array}\right| = 0$$

or

$$|\nu' - \sigma_1^{-1}(1 - \Delta^1\Delta^{1T})\sigma_1^{-1}| \cdot |\nu' U_{N-1} - \hat{\sigma}^{-1}(U_{N-1} - \hat{\Delta}\hat{\Delta}^T)\hat{\sigma}^{-1}| = 0$$

Obviously, one of the eigenvalues of $\mathscr{D}^1$, to be called ν_1', is given by

$$\nu_1' = \sigma_1^{-1}(1 - \Delta^1\Delta^{1T})\sigma_1^{-1} = \frac{1 - \Delta^1\Delta^{1T}}{\sigma_1^2} \tag{61}$$

The eigenvalues of $\mathscr{F}^1$ are the roots of

$$|\omega U_N - \mathscr{F}^1| = 0 \tag{62}$$

or

$$\left|\begin{array}{c|c} \omega & -\sigma_1^{-1}\Delta^1\hat{\Delta}^T\hat{\sigma}^{-1} \\ \hline -\hat{\sigma}^{-1}\hat{\Delta}\Delta^{1T}\sigma_1^{-1} & \omega U_{N-1} \end{array}\right| = 0$$

For the nonzero ω the determinant identity [6] applies to yield

$$|\omega U_{N-1}| \; |\omega - \sigma_1^{-1}\Delta^1\hat{\Delta}^T\hat{\sigma}^{-1}(\omega U_{N-1})^{-1}\hat{\sigma}^{-1}\hat{\Delta}\Delta^{1T}\sigma_1^{-1}| = 0$$

which reduces to

$$\left|\omega^2 - \sigma_1^{-1}\Delta^1\hat{\Delta}^T\hat{\sigma}^{-2}\hat{\Delta}\Delta^{1T}\sigma_1^{-1}\right| = 0 \tag{63}$$

It turns out that there exist only two nonzero roots of Eq. (62), and these are equal in magnitude and opposite in sign. If we denote the positive root of Eq. (63) by ω_1, then the eigenvalues of $\mathcal{F}^1$ are arranged in non-increasing order as

$$\omega_1, \underbrace{0, \ldots, 0,}_{(N-2)\text{ zeros}} -\omega_1$$

where

$$\omega_1 = \frac{1}{\sigma_1} \sqrt{\Delta^1 \hat{\Delta}^T \hat{\sigma}^{-2} \hat{\Delta} \Delta^{1T}} \tag{64}$$

As stated previously, the matrices $\mathcal{D}^1$ and $\mathcal{F}^1$ are both symmetric so that we may apply the minimax theorem to the matrices of Eq. (58) to find

$$\nu_1' - \omega_1 \leq \nu_1 \leq \nu_1' + \omega_1 \tag{65}$$

From Eqs. (61) and (64),

$$\nu_1' \pm \omega_1 = \frac{1 - \Delta^1 \Delta^{1T}}{\sigma_1^2} \pm \frac{\sqrt{\Delta^1 \hat{\Delta}^T \hat{\sigma}^{-2} \hat{\Delta} \Delta^{1T}}}{\sigma_1}$$

$$= \frac{1 - \Delta^1 \Delta^{1T} \pm \sqrt{\Delta^1 \hat{\Delta}^T (\sigma_1 \hat{\sigma}^{-1})^2 \hat{\Delta} \Delta^{1T}}}{\sigma_1^2}$$

On the other hand, if the definition of ν (Eq. (52)) is substituted into Eq. (65), we have

$$\frac{\mu_1'}{1 - \mu_1' \omega_1} \leq \mu_1 \leq \frac{\mu_1'}{1 - \mu_1' \omega_1}$$

where

$$\mu_1' \stackrel{\Delta}{=} \frac{1}{\nu_1'}$$

or

$$\frac{\sigma_1^2}{1 - \Delta^1 \Delta^{1T} + \sqrt{\Delta^1 \hat{\Delta}^T (\sigma_1 \hat{\sigma}^{-1})^2 \hat{\Delta} \Delta^{1T}}} \leq \mu_1 \leq \frac{\sigma_1^2}{1 - \Delta^1 \Delta^{1T} - \sqrt{\Delta^1 \hat{\Delta}^T (\sigma_1 \hat{\sigma}^{-1})^2 \hat{\Delta} \Delta^{1T}}} \tag{66}$$

We could apply the minimax theorem to the other roots of $\sigma^{-1}M_1^{-1}\sigma^{-1}$ to produce a similar result to Eq. (65). However, the eigenvalues of $\mathscr{D}^1$ other than ν_1' are not immediately feasible in view of the equations immediately preceding Eq. (61). For those roots of $\mathscr{D}^1$ which are expected to give the bounds of ν_s (s=2,...,N), we can reformulate the eigenvalue problem of Eq. (53) as follows.

Let V_k^j be an elementary matrix [6] which interchanges the j-th row (column) of an arbitrary square matrix A with the k-th row (column) if A is pre-multiplied (post-multiplied) by V_k^j. Specifically, V_k^j takes the form

$$V_k^j \triangleq \begin{array}{c@{}l} \left[\begin{array}{c|c|c|c|c} U_{j-1} & & & & \\ \hline & 0 & & 1 & \\ \hline & & U_{k-j-1} & & \\ \hline & 1 & & 0 & \\ \hline & & & & U_{N-k} \end{array}\right] & \begin{array}{l} \\ \leftarrow \text{j-th} \\ \\ \leftarrow \text{k-th} \\ \\ \end{array} \\ \begin{array}{ccccc} \quad & \uparrow & \qquad\quad & \uparrow & \quad \\ & \text{j-th} & & \text{k-th} & \end{array} & \end{array} \tag{67}$$

Note that

$$(V_k^j)^T = V_k^j \tag{68a}$$

$$(V_k^j)^2 = (V_k^j)(V_k^j)^T = (V_k^j)^T(V_k^j) = U_N$$

$$|V_k^j|^2 = 1 \tag{68b}$$

If we define V^ℓ by

$$V^\ell \triangleq V_2^1\, V_3^2\, V_4^3 \ldots V_\ell^{\ell-1} \quad , \tag{69}$$

then from Eqs. (68),

$$V^\ell V^{\ell T} = (V_2^1\ V_3^2\ \ldots\ V_\ell^{\ell-1})(V_2^1\ V_3^2\ \ldots\ V_\ell^{\ell-1})^T$$

$$= V_2^1\ V_3^2\ \ldots\ V_\ell^{\ell-1}(V_\ell^{\ell-1})^T\ \ldots\ V_3^2\ V_2^1$$

$$= U_N \tag{70a}$$

and similarly

$$V^{\ell T}\ V^\ell = U_N \tag{70b}$$

Premultiplying Eq. (53) by V^ℓ and postmultiplying by $V^{\ell T}$ yields

$$|\nu V^\ell V^{\ell T} - V^\ell \sigma^{-1}\ M_1^{-1}\ \sigma^{-1}\ V^{\ell T}| = 0$$

From Eqs. (70), we may write

$$|\nu U_N - (V^\ell \sigma^{-1} V^{\ell T})(V^\ell M_1^{-1} V^{\ell T})(V^\ell \sigma^{-1} V^{\ell T})| = 0$$

By the noted properties of V_k^j composing V^ℓ, it follows that

$$V^\ell \sigma^{-1}\ V^{\ell T} = \left[\begin{array}{c|c} \sigma_\ell^{-1} & 0 \\ \hline 0 & (\hat{\sigma}^\ell)^{-1} \end{array}\right]$$

and

$$V^\ell M_1^{-1} V^{\ell T} = \left[\begin{array}{c|c} 1 - \Delta^\ell \Delta^{\ell T} & 0 \\ \hline 0 & U_{N-1} - \hat{\Delta}^\ell \hat{\Delta}^{\ell T} \end{array}\right]$$

where

$$\hat{\sigma}^\ell \triangleq \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_{\ell-1} & & \\ & & & & \sigma_{\ell+1} & \\ & & & & & \ddots \\ & & & & & & \sigma_N \end{bmatrix} \tag{71}$$

and

$$\hat{\Delta}^{\ell} \overset{\Delta}{=} \begin{bmatrix} \Delta^{1} \\ \vdots \\ \Delta^{\ell-1} \\ \Delta^{\ell+1} \\ \vdots \\ \Delta^{N} \end{bmatrix} \tag{72}$$

By these definitions, we may reformulate the eigenvalue problem of Eq. (53) as

$$|\nu U_N - (\mathscr{D}^{\ell} + \mathscr{F}^{\ell})| = 0 \tag{73}$$

where

$$\mathscr{D}^{\ell} \overset{\Delta}{=} \left[\begin{array}{c|c} \sigma_{\ell}^{-1}(1-\Delta^{\ell}\Delta^{\ell T})\sigma_{\ell}^{-1} & 0 \\ \hline 0 & (\hat{\sigma}^{\ell})^{-1}(U_{N-1} - \hat{\Delta}^{\ell}\hat{\Delta}^{\ell T})(\hat{\sigma}^{\ell})^{-1} \end{array}\right] \tag{74}$$

$$\mathscr{F}^{\ell} \overset{\Delta}{=} \left[\begin{array}{c|c} 0 & -\sigma_{\ell}^{-1}\Delta^{\ell}\Delta^{\ell T}(\hat{\sigma}^{\ell})^{-1} \\ \hline -(\hat{\sigma}^{\ell})^{-1}\,\hat{\Delta}^{\ell}\hat{\Delta}^{\ell T}(\sigma_{\ell})^{-1} & 0 \end{array}\right] \tag{75}$$

for $\ell = 2,3,\ldots,N$.

Following the procedures taken for ν_1, we have similar results for $\nu_{\ell}(s = 2,3,\ldots,N)$:

$$\nu_{\ell}' - \omega_{\ell} \le \nu_{\ell} \le \nu_{\ell}' + \omega_{\ell} \tag{76}$$

where

$$\nu_{\ell}' \overset{\Delta}{=} \sigma_{\ell}^{-1}(1 - \Delta^{\ell}\Delta^{\ell T})\sigma_{\ell}^{-1} = \frac{1 - \Delta^{\ell}\Delta^{\ell T}}{\sigma_{\ell}^{2}} \tag{77}$$

and

$$\omega_\ell \triangleq \frac{1}{\sigma_1}\sqrt{\Delta^\ell \hat{\Delta}^{\ell T}(\hat{\sigma}^\ell)^{-2}\ \hat{\Delta}^\ell \Delta^{\ell T}} \tag{78}$$

and

$$\frac{\sigma_\ell^2}{1 - \Delta^\ell \Delta^{\ell T} + \sqrt{\Delta^\ell \hat{\Delta}^{\ell T}(\hat{\sigma}^\ell/\sigma_\ell)^{-2}\ \hat{\Delta}^\ell \Delta^{\ell T}}} \le \mu_\ell$$

$$\le \frac{\sigma_\ell^2}{1 - \Delta^\ell \Delta^{\ell T} - \sqrt{\Delta^\ell \hat{\Delta}^{\ell T}(\hat{\sigma}^\ell/\sigma_\ell)^{-2}\hat{\Delta}^\ell \Delta^{\ell T}}} \tag{79}$$

for $\ell = 2,3,\ldots,N$.

Thus, we have obtained a range in which $\mu_\ell(\ell=2,3,\ldots,N)$ exists, and the range is easily evaluated by computing the scalar quantity consisting of Δ^ℓ, $\hat{\Delta}^\ell$ and $\hat{\sigma}^\ell$ as defined by Eqs. (71) and (72). The ranges given by Eq. (79) may be very useful for localization of the μ_s if the quantity in the square root is so small that we may write

$$\mu_\ell \approx \frac{\sigma_\ell^2}{1 - \Delta^\ell \Delta^{\ell T}} \quad . \tag{80}$$

Strict equality holds if Δ^ℓ is orthogonal to any $\Delta^j (j \ne \ell)$ so that

$$\Delta^\ell\ \hat{\Delta}^{\ell T} = 0 \quad .$$

In general, however, the quantity is not necessarily small because some elements of the matrix $(\hat{\sigma}^\ell/\sigma_\ell)^{-2}$ can be very large. In fact, $(\hat{\sigma}^\ell/\sigma_\ell)^{-2}$ takes the form

$$(\hat{\sigma}^{\ell}/\sigma_{\ell})^{-2} = \left[\begin{array}{ccc|ccc} \sigma_{\ell}^2/\sigma_1^2 & & & & & \\ & \ddots & & & & \\ & & \sigma_{\ell}^2/\sigma_{\ell-1}^2 & & & \\ \hline & & & \sigma_{\ell}^2/\sigma_{\ell+1}^2 & & \\ & & & & \ddots & \\ & & & & & \sigma_{\ell}^2/\sigma_N^2 \end{array}\right]$$

in which the first (ℓ-1) elements are equal to or greater than unity in view of the ordering of Eq. (4a).

3.3 Multiplicity of Eigenvalues

We will first examine the case when σ_j's are all distinct. From Eqs. (24a) and (42),

$$f(\mu) = \prod_{m=1}^{N} (\mu-\sigma_m^2) - \mu \sum_{\ell=1}^{N} P_{\ell} \prod_{\substack{m=1 \\ m\neq j}}^{N} (\mu-\sigma_m^2) \tag{81}$$

because $|\mu U_N - \sigma^2| = \prod_{m=1}^{N} (\mu-\sigma_m^2)$. Although Eq. (42) is derived under the assumption (see Eq. (23)) that $|\mu U_N - \sigma^2| \neq 0$, Eq. (81) is not so restricted (the restriction can be removed by the continuity argument), and holds for any real value of μ for the distinct σ_j case.

We may rewrite $f(\mu)$ as

$$f(\mu) = (\mu-\sigma_s^2) \left\{ \prod_{\substack{m=1 \\ m\neq s}}^{N} (\mu-\sigma_m^2) - \mu \sum_{\substack{\ell=1 \\ \ell\neq s}}^{N} P_{\ell} \prod_{\substack{m=1 \\ m\neq \ell,s}}^{N} (\mu-\sigma_m^2) \right\}$$
$$-\sigma_s^2 P_s \prod_{\substack{m=1 \\ m\neq s}}^{N} (\sigma_s^2 - \sigma_m^2) \tag{82}$$

which indicates that

$$f(\sigma_s^2) = 0$$

if and only if $P_s = 0$. We have noted that $P_1 > 0$ if $\Delta^1 \neq 0$, so that σ_1^2 cannot be the eigenvalue of $M_1\sigma^2$ unless $\Delta^1 = 0$. However, even if $\Delta^\ell \neq 0$, P_ℓ can be zero for $\ell=2,\ldots,N$, so that $\sigma_\ell^2(\ell=2,\ldots,N)$ can be the eigenvalue of $M_1\sigma^2$. Moreover, if $P_s = 0$, we may write

$$f(\mu) = (\mu-\sigma_s^2)\left\{\prod_{\substack{m=1\\m\neq s}}^{N}(\mu-\sigma_m^2) - \mu\sum_{\substack{\ell=1\\\ell\neq s}}^{N} P_\ell \prod_{\substack{m=1\\m\neq\ell,s}}^{N}(\mu-\sigma_m^2)\right\}$$

$$= (\mu-\sigma_s^2)\left\{\prod_{\substack{m=1\\m\neq s}}^{N}(\mu-\sigma_m^2)\right\}\left\{1 - \mu\sum_{\substack{\ell=1\\\ell\neq s}}^{N}\frac{P_\ell}{\mu-\sigma_\ell^2}\right\}$$

in which σ_s^2 is one of the roots of

$$1 - \mu\sum_{\substack{\ell=1\\\ell\neq s}}^{N}\frac{P_\ell}{\mu-\sigma_\ell^2} = 0$$

if

$$1 - \sigma_s^2\sum_{\substack{\ell=1\\\ell\neq s}}^{N}\frac{P_\ell}{\sigma_s^2-\sigma_\ell^2} = 0. \tag{83}$$

This means that if $P_s = 0$ and if Eq. (83) holds, $\mu = \sigma_s^2$ is a repeated eigenvalue of $M_1\sigma^2$ with multiplicity 2 (at least).

Consider next the case when σ_s is repeated with multiplicity 2, i.e., $\sigma_s = \sigma_{s+1}$, assuming that all others are distinct. Then, Eq. (31) may be written as

$$g(\mu) = 1 - \sum_{\ell=1}^{N} d_\ell(\mu)\left\{\Delta^\ell\Delta^{\ell T} - \sum_{j=1}^{\ell-1} d_j(\mu)K_j^\ell + \sum_{1\leq j<k\leq\ell-1} d_j(\mu)\,d_k(\mu)\,L_{jk}^\ell\right\},$$

$$g(\mu) = + K_s^{s+1} \{d_s(\mu)\}^2 - \sum_{\ell=s+2}^{N} \{d_s(\mu)\}^2 L_{s,s+1}^{\ell} d_\ell(\mu) + \hat{g}(\mu)$$

and subsequently

$$f(\mu) = \prod_{m=1}^{N} (\mu-\sigma_m^2) \left[\left\{d_s(\mu)\right\}^2 K_s^{s+1} - \sum_{\ell=s+2}^{N} L_{s,s+1}^{\ell} \left\{d_s(\mu)\right\}^2 d_\ell(\mu) \right] + \hat{f}(\mu)$$

where $\hat{f}(\sigma_s^2) = 0$, or

$$f(\mu) = \prod_{\substack{m=1 \\ m\neq s,s+1}}^{N} (\mu-\sigma_m^2) \left\{ K_s^{s+1} - \sum_{\ell=s+2}^{N} L_{s,s+1}^{\ell} \; d_\ell(\mu) \right\} + \hat{f}(\mu) \tag{84}$$

It follows that

$$f(\sigma_s^2) = 0$$

if and only if

$$K_s^{s+1} - \sum_{\ell=s+2}^{N} L_{s,s+1}^{\ell} \; d_\ell(\mu) = 0 \tag{85}$$

This means that if σ_s^2 is repeated with multiplicity 2 and if Eq. (85) holds, then σ_s^2 is one of the eigenvalues with multiplicity at least unity.

If the multiplicity of σ_s^2 is three, i.e., $\sigma_s = \sigma_{s+1} = \sigma_{s+2}$ and others are all distinct, then as previously

$$f(\mu) = - \prod_{\substack{m=1 \\ \sigma_m\neq \sigma_s}}^{N} (\mu-\sigma_m^2) \cdot L_{s,s+1}^{s+2} + \hat{f}(\mu) \tag{86}$$

where

$\hat{f}(\mu) = 0$ for $\mu = \sigma_s^2$, and hence

$f(\sigma_s^2) = 0$

if and only if $L_{s,s+1}^{s+1} = 0$.

If the multiplicity, r_s, of σ_s^2 is greater than three, then σ_s^2 is one of the eigenvalues of $M_1\sigma^2$, with multiplicity r_s-3 (at least).

The results obtained in Section 3.3 are summarized as follows:

If σ_ℓ's are all distinct and P_ℓ's of Eq. (43) are all positive for $\ell=1,2,\ldots,N$, then the system eigenvalues of $M_1\sigma^2$ are all distinct as stated in Fact 4.

If σ_ℓ's are all distinct but $P_s = 0$ for some s, then σ_s^2 is the eigenvalue of $M_1\sigma^2$ with multiplicity at least unity.

If σ_s's are repeated with multiplicity 2, then σ_s^2 is an eigenvalue of $M_1\sigma^2$ if Eq. (85) holds. Otherwise, it is not so.

If σ_s's are repeated with multiplicity 3, then σ_s^2 is an eigenvalue of $M_1\sigma^2$ if Eq. (86) holds. Otherwise, it is not so.

If σ_s's are repeated with multiplicity r_s for $r_s > 3$, then σ_s^2 is an eigenvalue of $M_1\sigma^2$ with multiplicity at least (r_s-3).

4. EIGENVECTORS

Consider the eigenvector problem of Eq. (21) premultiplied by M_1^{-1}

$$(\mu M_1^{-1} - \sigma^2)\psi = 0 \ . \tag{87}$$

Since $M_1^{-1}(= U_N - \Delta\Delta^T)$ is symmetric and positive definite and so is σ^2, the results in [1] and [4] apply directly to this problem to yield:

<u>Fact 5</u>. There exist N independent eigenvectors to be called $\psi^1, \psi^2, \ldots, \psi^N$ regardless of multiplicity of the eigenvalues. Furthermore, we may choose a set of ψ^j's such that ψ^j and ψ^k are orthogonal with respect to M_1^{-1}, i.e.,

$$\psi^{jT} M_1^{-1} \psi^k = 0 \qquad \text{for } j \neq k \ .$$

whereby ψ^j and ψ^k are also orthogonal with respect to σ^2, i.e.

$$\psi^{jT} \sigma^2 \psi^k = 0 \qquad \text{for } j \neq k$$

If we normalize ψ^j by

$$\psi^{jT} M_1^{-1} \psi^j = 1, \qquad j = 1,2,\ldots,N, \tag{88a}$$

then

$$\psi^{jT} \sigma^2 \psi^j = \mu_j \qquad j = 1,2,\ldots,N \tag{88b}$$

These relations are conveniently expressed in matrix form: if we define N×N matrices Ψ and Λ^2 by

$$\Psi \triangleq [\psi^1, \psi^2, \ldots, \psi^N] \tag{89a}$$

and

$$\Lambda^2 \triangleq \begin{bmatrix} -\mu_1 & & & \\ & -\mu_2 & & \\ & & \ddots & \\ & & & -\mu_N \end{bmatrix} \tag{89b}$$

then

$$\Psi^T M_1^{-1} \Psi = U_N \tag{90a}$$

and

$$\Psi^T \sigma^2 \Psi = -\Lambda^2 \tag{90b}$$

and

$$M_1 \sigma^2 \Psi = -\Psi \Lambda^2 \tag{90c}$$

In what follows, it will be shown that the particular structure of M_1^{-1} simplifies the eigenvector calculations.

Assume again, as in Eq. (23), that $|\mu_j U_N - \sigma^2| \neq 0$ for μ_j. Then, from Eq. (87) with Eq. (6e),

$$\mu_j (U_N - \Delta\Delta^T)\psi^j - \sigma^2 \psi^j = 0$$

so that

$$(\mu_j U_N - \sigma^2)\psi^j - \mu_j \; \Delta\Delta^T \psi^j = 0$$

or

$$\psi^j = \mu_j (\mu_j U_N - \sigma^2)^{-1} \Delta\Delta^T \psi^j \tag{91}$$

It is noted that $\Delta^T \psi^j$ is a 3×1 matrix so that if we define χ^j by

$$\chi^j \triangleq \mu_j \Delta^T \psi^j \quad , \tag{92a}$$

then we may write ψ^j as

$$\psi^j = (\mu_j U_N - \sigma^2)^{-1} \Delta \chi^j \tag{92b}$$

Subsequently, the eigenproblem of Eq. (87) becomes

$$\begin{aligned} 0 &= (\mu_j U_N - \sigma^2)\psi^j - \mu_j \Delta\Delta^T \psi^j \\ &= \mu_j \Delta \chi^j - \mu_j^2 \Delta\Delta^T \; (\mu_j U_N - \sigma^2)^{-1} \Delta \chi^j \\ &= \mu_j \Delta \{U_3 - \mu_j \Delta^T \; (\mu_j U_N - \sigma^2)^{-1} \Delta\} \chi^j \end{aligned}$$

On the other hand, the characteristic equation of reduced form (Eq. (24b)) assures that the determinant of the matrix in the braces is zero for μ_j, and hence there exists a non-trivial solution for χ^j determined by

$$\{U_3 - \mu_j\Delta^T(\mu_j U_N - \sigma^2)^{-1}\Delta\}\chi^j = 0 \tag{93}$$

in which only two elements of χ^j are to be solved. The normalization condition of Eq. (88a) with Eq. (92b) substituted yields

$$\chi^{jT}\Delta^T(\mu_j U_N - \sigma^2)^{-1}(U_N - \Delta\Delta^T)(\mu_j U_N - \sigma^2)^{-1}\Delta\chi^j = 1$$

or in view of Eq. (93),

$$\chi^{jT}\left\{\Delta^T(\mu_j U_N - \sigma^2)^{-2}\Delta - \frac{1}{\mu_j^2}U_3\right\}\chi^j = 1 \tag{94}$$

Thus, we can determine $\chi^j (j = 1,2,\ldots,N)$ corresponding to μ_j by Eqs. (93) and (94) provided that $|\mu_j U_N - \sigma^2| \neq 0$, and then the ψ^j's can be calculated from Eq. (92b).

Next, we will show that even if $|\mu_j U_N - \sigma^2| = 0$, the ψ^j's can be determined in a similar manner.

Suppose that $\mu_k = \sigma_s^2$ and $\mu_k \neq \sigma_j^2$ $(j \neq s)$; then $|\mu_k U_N - \sigma^2| = 0$. In this case, we may rewrite Eq. (87) (simply by changing the rows) as

$$\left[\begin{array}{ccc|c} \mu_k - \sigma_1^2 - \mu_k\Delta^1\Delta^{1T} & \cdots & -\mu_k\Delta^1\Delta^{NT} & -\mu_k\Delta^1\Delta^{ST} \\ \vdots & \ddots & \vdots & \vdots \\ -\mu_k\Delta^N\Delta^{1T} & \cdots & \mu_k - \sigma_N^2 - \mu_k\Delta^N\Delta^{NT} & -\mu_k\Delta^N\Delta^{ST} \\ \hline -\mu_k\Delta^S\Delta^{1T} & \cdots & -\mu_k\Delta^S\Delta^{NT} & -\mu_k\Delta^S\Delta^{ST} \end{array}\right] \left[\begin{array}{c} \psi_1^k \\ \vdots \\ \psi_N^k \\ \hline \psi_s^k \end{array}\right] = 0 .$$

With the definitions of Eq. (71) and (72), we may write

$$\left[\begin{array}{c|c} \mu_k U_{N-1} - (\hat{\sigma}^s)^2 - \mu_k \hat{\Delta}^S \hat{\Delta}^{ST} & -\mu_k \hat{\Delta}^S \Delta^{ST} \\ \hline -\mu_k \Delta^S \hat{\Delta}^{ST} & -\mu_k \Delta^S \Delta^{ST} \end{array}\right] \left[\begin{array}{c} \hat{\psi}^k \\ \hline \psi_s^k \end{array}\right] = 0$$

where $\hat{\psi}^k$ is the (N-1)×1 matrix. Hence

$$\{\mu_k U_{N-1} - (\hat{\sigma}^s)^2 - \mu_k \hat{\Delta}^S \hat{\Delta}^{ST}\} \hat{\psi}^k - \mu_k \hat{\Delta}^S \Delta^{ST} \psi_s^k = 0 \tag{95a}$$

and

$$\Delta^S \hat{\Delta}^{ST} \hat{\psi}^k + \Delta^S \Delta^{ST} \psi_s^k = 0 \tag{95b}$$

Assuming that $\Delta^S \neq 0$, we have from Eq. (95b),

$$\psi_s^k = - \frac{\Delta^S \hat{\Delta}^{ST}}{\Delta^S \Delta^{ST}} \hat{\psi}^k \quad . \tag{96a}$$

Substituting Eq. (96a) into Eq. (95a) provides

$$\left\{ \mu_k U_{N-1} - (\hat{\sigma}^s)^2 - \mu_k \hat{\Delta}^S \left(U_3 - \frac{\Delta^{ST} \Delta^S}{\Delta^S \Delta^{ST}} \right) \hat{\Delta}^T \right\} \hat{\psi}^k = 0$$

in which $|\mu_k U_{N-1} - (\hat{\sigma}^s)^2| \neq 0$, so that we can follow the procedures established above, simply changing the dimension of the matrices to (N-1). It immediately follows that $\hat{\psi}^k$ is generated by

$$\hat{\psi}^k = \left(\mu_k U_{N-1} - (\hat{\sigma}^s)^2 \right)^{-1} \hat{\Delta}^S \chi^k \tag{96b}$$

where χ^k is obtained by solving

$$\left\{ U_3 - \mu_k \left(U_3 - \frac{\Delta^{ST}\Delta^S}{\Delta^S\Delta^{ST}} \right) \hat{\Delta}^{ST} \left(\mu_k U_{N-1} - (\hat{\sigma}^s)^2 \right)^{-1} \hat{\Delta}^S \right\} \chi^k = 0 \quad (97a)$$

with the normalization condition

$$[\hat{\psi}^{kT} \mid \psi_s^k] \left[\begin{array}{c|c} U_{N-1} - \hat{\Delta}^S\hat{\Delta}^{ST} & 0 \\ \hline 0 & 1 - \Delta^S\Delta^{ST} \end{array} \right] \left[\begin{array}{c} \hat{\psi}^k \\ \hline \psi_s^k \end{array} \right] = 1 \quad (97b)$$

or

$$\chi^{kT}\hat{\Delta}^{ST}\left(\mu_k U_{N-1} - (\hat{\sigma}^s)^2\right)^{-1}\left(U_{N-1} - \hat{\Delta}^S\hat{\Delta}^{ST} + \right.$$
$$\left. + \frac{1 - \Delta^S\Delta^{ST}}{(\Delta^S\Delta^{ST})^2} \hat{\Delta}^S\Delta^{ST}\Delta^S\hat{\Delta}^{ST}\right) \left(\mu_k U_{N-1} - (\hat{\sigma}^s)^2\right)^{-1} \hat{\Delta}^S \chi^k = 1$$

(97c)

Thus, if $\sigma_s^2 = \mu_k$ and $|\mu_k U_{N-1} - (\hat{\sigma}^s)^2| \neq 0$, χ^k is determined by Eqs. (97a) and (97c), and then $\hat{\psi}^k$ is calculated by (96b) and ψ_s^k by (96a), so that ψ^k is obtained. Once again, the eigenvector calculation of an (N-1)×1 matrix has been reduced to simply that of 3×1 matrix.

5. TRUNCATION EFFECT

In this section, the eigenvalues are examined for a truncated system of equations represented by

$$\ddot{\gamma} - \bar{\Delta}^T \ddot{\bar{\eta}} = 0 \tag{98a}$$

and

$$\ddot{\bar{\eta}} + \bar{\sigma}^2\bar{\eta} - \bar{\Delta}\,\ddot{\gamma} = 0 \tag{98b}$$

where

$$\bar{\Delta} = \begin{bmatrix} \Delta^1 \\ \Delta^2 \\ \vdots \\ \Delta^{\bar{N}} \end{bmatrix} \tag{99}$$

and

$$\bar{\sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{\bar{N}} \end{bmatrix} \tag{100}$$

where $\bar{N}$ is the dimension of the truncated system and

$$\bar{N} < N$$

or usually

$$\bar{N} \ll N$$

When such a truncation is exercised, it is required (or sometimes presumed) that the nonzero eigenvalues of Eqs. (98), to be called $\lambda_1', \lambda_2' \ldots \lambda_{2\bar{N}}'$, should be a good approximation of the first $2\bar{N}$ smallest modulus eigenvalues of the original system, namely $\lambda_1, \lambda_2, \ldots, \lambda_{\bar{N}}$ and $\lambda_1^*, \lambda_2^* \ldots \lambda_{\bar{N}}^*$, as of Eqs. (3) or Eq. (8).

If we define μ_j' by

$$\mu_j' \stackrel{\Delta}{=} -(\lambda_j')^2 \tag{101}$$

then the problem is to compare μ_j and μ_j' where μ_j' is a root of

$$|\mu U_{\bar{N}} - \bar{M}_1 \bar{\sigma}^2| = 0 \tag{102}$$

and

$$\bar{M}_1 = (U_{\bar{N}} - \bar{\Delta}\bar{\Delta}^T)^{-1} \quad . \tag{103}$$

For convenience of later discussions, we also define $\bar{\bar{\Delta}}$, $\bar{\bar{\sigma}}$ and $\bar{\bar{M}}_1$ for the part of the system to be deleted by truncation:

$$\bar{\bar{\Delta}} \stackrel{\Delta}{=} \begin{bmatrix} \Delta^{\bar{N}+1} \\ \vdots \\ \Delta^N \end{bmatrix} \tag{104}$$

$$\bar{\bar{\sigma}} \stackrel{\Delta}{=} \begin{bmatrix} \sigma_{\bar{N}+1} & & \\ & \ddots & \\ & & \sigma_N \end{bmatrix} \tag{105}$$

and

$$\bar{\bar{M}}_1 \stackrel{\Delta}{=} \left(U_{\bar{\bar{N}}} - \bar{\bar{\Delta}}\bar{\bar{\Delta}}^T \right)^{-1} \tag{106}$$

where

$$\bar{\bar{N}} \stackrel{\Delta}{=} N - \bar{N} \tag{107}$$

The notation μ_j' will be used also for the roots of

$$|\mu U_{\bar{\bar{N}}} - \bar{\bar{M}}_1 \bar{\bar{\sigma}}^2| = 0 \tag{108}$$

with

$$j = \bar{N} + 1, \bar{N} + 2, \ldots, N.$$

With these definitions, we may write

$$\sigma = \left[\begin{array}{c|c} \bar{\sigma} & 0 \\ \hline 0 & \bar{\bar{\sigma}} \end{array}\right] \tag{109}$$

and

$$\Delta = \left[\begin{array}{c} \bar{\Delta} \\ \hline \bar{\bar{\Delta}} \end{array}\right] \tag{110}$$

In what follows, we will examine the reciprocals of the roots defined by

$$\nu_j' \stackrel{\Delta}{=} \frac{1}{\mu_j'} \tag{111}$$

as in Eq. (53), which satisfy

$$|\nu' U_{\bar{N}} - \bar{\sigma}^{-1} \bar{M}_1^{-1} \bar{\sigma}^{-1}| = 0 \tag{112}$$

for

$$\nu_j' = 1, 2, \ldots, \bar{N}$$

and

$$|\nu' U_{\bar{\bar{N}}} - \bar{\bar{\sigma}}^{-1} \bar{\bar{M}}_1^{-1} \bar{\bar{\sigma}}^{-1}| = 0 \tag{113}$$

for

$$\nu_j' = \bar{N} + 1,\ \bar{N} + 2, \ldots, N.$$

Since M_1^{-1} is partitioned as

$$M_1^{-1} = \left[\begin{array}{c|c} U_{\bar{N}} & 0 \\ \hline 0 & U_{\bar{\bar{N}}} \end{array}\right] - \left[\begin{array}{c} \bar{\Delta} \\ \hline \bar{\bar{\Delta}} \end{array}\right] \left[\begin{array}{c} \bar{\Delta} \\ \hline \bar{\bar{\Delta}} \end{array}\right]^T$$

$$= \left[\begin{array}{c|c} U_{\bar{N}} - \bar{\Delta}\bar{\Delta}^T & -\bar{\Delta}\,\bar{\bar{\Delta}}^T \\ \hline -\bar{\bar{\Delta}}\,\bar{\Delta}^T & U_{\bar{\bar{N}}} - \bar{\bar{\Delta}}\,\bar{\bar{\Delta}}^T \end{array}\right]$$

$$= \left[\begin{array}{c|c} \bar{M}_1^{-1} & 0 \\ \hline 0 & \bar{\bar{M}}_1^{-1} \end{array}\right] + \left[\begin{array}{c|c} 0 & -\bar{\Delta}\,\bar{\bar{\Delta}}^T \\ \hline -\bar{\bar{\Delta}}\,\bar{\Delta}^T & 0 \end{array}\right],$$

the matrix $\sigma^{-1}M_1^{-1}\sigma^{-1}$ is decomposed and rewritten as

$$\sigma^{-1}M_1^{-1}\sigma^{-1} = \mathscr{D} + \mathscr{F}$$

where

$$\mathscr{D} \triangleq \left[\begin{array}{c|c} \bar{\sigma}^{-1}\bar{M}_1^{-1}\bar{\sigma}^{-1} & 0 \\ \hline 0 & \bar{\bar{\sigma}}^{-1}\bar{\bar{M}}_1^{-1}\bar{\bar{\sigma}}^{-1} \end{array}\right] \tag{114}$$

and

$$\mathscr{F} \triangleq \left[\begin{array}{c|c} 0 & -\bar{\sigma}^{-1}\bar{\Delta}\bar{\bar{\Delta}}^{T}\bar{\bar{\sigma}}^{-1} \\ \hline -\bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\bar{\Delta}^{T}\bar{\sigma}^{-1} & 0 \end{array}\right] \tag{115}$$

It is recognized that the eigenvalues of $\mathscr{D}$ are ν_j', which are determined by Eqs. (112) and (113), because

$$|\nu U_N - \mathscr{D}| = |\nu U_{\bar{N}} - \bar{\sigma}^{-1}\bar{M}_1^{-1}\bar{\sigma}^{-1}| \cdot |\nu U_{\bar{\bar{N}}} - \bar{\bar{\sigma}}^{-1}\bar{\bar{M}}_1^{-1}\bar{\bar{\sigma}}^{-1}| \tag{116}$$

From Eq. (115), if $\bar{\Delta}\,\bar{\bar{\Delta}}^{T} = 0$ then $\mathscr{F} = 0$ and

$$\nu_j = \nu_j'$$

and hence

$$\mu_j = \mu_j' \quad .$$

In other words, if $\Delta^j\,\Delta^{kT} = 0$ for $j = 1,2,\ldots,\bar{N}$ and $k = \bar{N} + 1,\ldots,N$, then the eigenvalues of the truncated system together with those of the deleted system are identical to the eigenvalues of the original system.

If $\bar{\Delta}\,\bar{\bar{\Delta}}^{T} \neq 0$, then the matrix $\mathscr{F}$ usually causes a discrepancy between ν_j and ν_j'. In order to examine this influence, first consider the eigenvalues of $\mathscr{F}$, to be called ω_j. The ω_j are the roots of

$$|\omega U_N - \mathscr{F}| = 0$$

or from Eq. (115)

$$\left[\begin{array}{c|c} \omega U_{\bar{\bar{N}}} & -\bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\,\bar{\bar{\Delta}}\,\bar{\bar{\sigma}}^{-1} \\ \hline -\bar{\bar{\sigma}}^{-1}\,\bar{\bar{\Delta}}\,\bar{\bar{\Delta}}^T\,\bar{\bar{\sigma}}^{-1} & \omega U_{\bar{\bar{N}}} \end{array}\right] = 0 \tag{117}$$

For nonzero ω, the determinant identity [6] applies to yield

$$\left|\omega U_{\bar{\bar{N}}} - \bar{\bar{\sigma}}^{-1}\,\bar{\bar{\Delta}}\,\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-1}(\omega U_{\bar{\bar{N}}})^{-1}\bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\,\bar{\bar{\Delta}}^T\,\bar{\bar{\sigma}}^{-1}\right| = 0$$

or

$$\left|\omega^2 U_{\bar{\bar{N}}} - \bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}}\,\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-1}\right| = 0$$

or (118)

$$\left|\omega^2 U_3 - (\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})\right| = 0$$

It turns out from Eq. (118) that there exist at most six nonzero eigenvalues of $\mathscr{F}$. Moreover, since $\mathscr{F}$ is symmetric all the eigenvalues are real, so that if we call the roots of Eq. (118) $\pm\,\omega_1^o$, $\pm\,\omega_2^o$, $\pm\,\omega_3^o$ with $\omega_1^o \geq \omega_2^o \geq \omega_3^o \geq 0$, then the ω_j's are given in non-increasing order by

$$\omega_1^o,\ \omega_2^o,\ \omega_3^o,\ \underbrace{0,\ 0,\ldots,0}_{(N-6)\text{ zeros}}\ -\omega_3^o,\ -\omega_2^o,\ -\omega_1^o$$

From the fact that the matrix $\sigma^{-1}M_1^{-1}\sigma^{-1}$ is a sum of two matrices, $\mathscr{D}$ and $\mathscr{F}$, which are both symmetric, we may apply the Wielandt-Hoffman theorem [4], [8] stating that if ν_j and ν_j' and ω_j are arranged in non-increasing (or non-decreasing) order, then

$$\sum_{j=1}^{N}(\nu_j - \nu_j')^2 \leq \sum_{j=1}^{N}\omega_j^2 \tag{119}$$

Noting that

$$\sum_{j=1}^{N} \omega_j^2 = 2 \sum_{j=1}^{3} (\omega_j^o)^2$$

$$= 2\ \mathrm{tr}(\bar{\Delta}^T \bar{\bar{\sigma}}^{-2}\ \bar{\Delta})(\bar{\bar{\Delta}}^T\ \bar{\bar{\sigma}}^{-2} \bar{\bar{\Delta}}),$$

and recalling the definitions of Eqs. (52) and (111), we see that Eq. (119) immediately implies that

$$\sum_{j=1}^{N}\left(\frac{1}{\mu_j} - \frac{1}{\mu_j'}\right)^2 \leq 2\ \mathrm{tr}(\bar{\Delta}^T\ \bar{\sigma}^{-2}\ \bar{\Delta})(\bar{\bar{\Delta}}^T \bar{\bar{\sigma}}^{-2} \bar{\bar{\Delta}}) \qquad (120)$$

Eq. (120) gives a bound of errors which could take place if we approximated μ_j by μ_j'. If the relative errors are small, i.e.,

$$|\varepsilon_\ell/\mu_\ell'| << 1$$

where,

$$\varepsilon_\ell \stackrel{\Delta}{=} \mu_\ell' - \mu_\ell \quad ,$$

then

$$\frac{1}{\mu_\ell} - \frac{1}{\mu_\ell'} = \frac{1}{\mu_\ell' - \varepsilon_\ell} - \frac{1}{\mu_\ell'} = \frac{1}{\mu_\ell'} \cdot \frac{\varepsilon_\ell}{\mu_\ell'} + 0_\ell(\varepsilon_\ell^2)$$

Therefore,

$$\sum_{j=1}^{N}\left(\frac{1}{\mu_j} - \frac{1}{\mu_j'}\right)^2 = \sum_{j=1}^{N}\left(\frac{1}{\mu_\ell'}\right)^2\left(\frac{\varepsilon_\ell}{\mu_\ell'}\right)^2 + 0(\varepsilon_\ell^4)$$

and if we neglect $0(\varepsilon_\ell^4)$, then

$$\sum_{j=1}^{N}\left(\frac{1}{\mu_\ell'}\right)^2\left(\frac{\varepsilon_\ell}{\mu_\ell'}\right)^2 \leq 2\ \mathrm{tr.}(\bar{\Delta}^T \bar{\sigma}^{-2} \bar{\Delta})(\bar{\bar{\Delta}}^T\ \bar{\bar{\sigma}}^{-2}\ \bar{\bar{\Delta}}).$$

Each term in the summation of the left hand side is positive so that for any ℓ

$$\left(\frac{1}{\mu_\ell'}\right)^2\left(\frac{\varepsilon_\ell}{\mu_\ell'}\right)^2 \le \mathrm{tr.}(\bar{\Delta}^T\,\bar{\sigma}^{-2}\,\bar{\Delta})(\bar{\bar{\Delta}}^T\,\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})$$

or

$$\left|\frac{\varepsilon_\ell}{\mu_\ell'}\right| \le \mu_\ell'\sqrt{2\ \mathrm{tr.}(\bar{\Delta}^T\,\bar{\sigma}^{-2}\,\bar{\Delta})(\bar{\bar{\Delta}}^T\,\bar{\bar{\sigma}}^{-2}\,\bar{\bar{\Delta}})} \qquad \ell = 1,2,\ldots,N \tag{121}$$

Equation (121) gives a bound for the relative errors, and if the dimensionless quantity of the right-hand side of Eq. (121) is sufficiently small, judging from some practical point of view, then we may employ μ_j' as a satisfactory approximation of μ_j, permitting the truncation to be acceptable. This condition, however, is a sufficient one in that even if the quantity in question is not small enough this does not necessarily imply that the μ_j' are unacceptable. This is because the error limits of Eq. (121) are overestimated due to the neglect of the positive terms in the left-hand side of Eq. (121).

Equation (120) also implies that

$$\sum_{j=1}^{\bar{N}}\left(\frac{1}{\mu_j}-\frac{1}{\mu_j'}\right)^2 \le 2\ \mathrm{tr.}\ (\bar{\Delta}^T\,\bar{\sigma}^{-2}\,\bar{\Delta})(\bar{\bar{\Delta}}^T\,\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}}) \tag{122}$$

The requirement that the quantity of the right hand side of Eq. (122) is practically small is also a sufficient condition in the sense stated above.

The minimax theorem as applied to the matrices $\sigma^{-1}M_1^{-1}\sigma^{-1}$, $\mathcal{D}$ and $\mathcal{F}$ produces the following result. Since the eigenvalues of $\sigma^{-1}M_1^{-1}\sigma^{-1}$ and $\mathcal{D}$ are ν_j and ν_j', respectively, and the maximum and minimum eigenvalues of $\mathcal{F}$ are ω_1^o and $-\omega_1^o$, we have

$$\nu_j' - \omega_1^o \le \nu_j \le \nu_j' + \omega_1^o \tag{123}$$

or

$$\frac{\mu_j'}{1 + \mu_j' \omega_1^o} \leq \mu_j \leq \frac{\mu_j'}{1 - \mu_j' \omega_1^o} \tag{124}$$

Equation (124) affords an explicit error bound for μ_j' at the cost of solving the eigenvalue problem of $\mathscr{F}$, which is a 3×3 symmetric matrix.

6. DAMPING EFFECT

This section considers the effect of damping, represented by $D = 2\zeta\sigma$ in Eq. (14), on the system eigenvalues and eigenvectors. Rewriting Eq. (14) as

$$\left\{ \lambda\, U_{2N} - \left(\left[\begin{array}{c|c} 0 & U_N \\ \hline -M_1\sigma^2 & 0 \end{array} \right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -M_1 D \end{array} \right] \right) \right\} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = 0 \qquad (125)$$

we treat the second matrix including D as a small perturbation to the first matrix in the parentheses, whose eigenvalues and eigenvectors have been characterized to some extent in the previous sections. We employ the perturbation method established for diagonalizable matrices in [7]. Define

$$\mathscr{A} \triangleq \left[\begin{array}{c|c} 0 & U_N \\ \hline -M_1\sigma^2 & 0 \end{array} \right]$$

Let ε be the maximum damping ratio of the appendage vibration mode, i.e.,

$$\varepsilon = \max_j \zeta_j \qquad (126)$$

and define ρ_j by

$$\rho_j \triangleq \frac{\zeta_j}{\varepsilon} \; , \qquad j = 1,2,\ldots,N$$

then

$$0 < \rho_j \leq 1 \; .$$

By nature of the damping ratio, ζ_j is a small positive number (typically $0.001 < \zeta_j < 0.01$) and so is ε, i.e.,

$$0 < \varepsilon \ll 1 \; .$$

Defining

$$\rho \stackrel{\Delta}{=} \begin{bmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_N \end{bmatrix}, \tag{127}$$

we may rewrite the perturbation matrix as

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -M_1 D \end{array}\right] = \varepsilon P$$

where

$$P \stackrel{\Delta}{=} \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -2M_1 \rho\sigma \end{array}\right] \tag{128}$$

For $\varepsilon = 0$ (unperturbed system), the previous results indicate that the eigenvalues of $\mathscr{A}$ are given by

$$\lambda_j = \begin{cases} i\sqrt{\mu_j} = \lambda_j & j = 1,2,\ldots,N \\ -i\sqrt{\mu_{j-N}} = \lambda^*_{j-N} & j = N+1,\ldots,2N \end{cases}$$

to which correspond the eigenvectors

$$Y^j = \left[\begin{array}{c} \psi^j \\ \hline \lambda_j \psi^j \end{array}\right] \qquad j = 1,2,\ldots,2N$$

From the orthogonality of ψ^j together with the normalization condition of Eq. (87), the Y^j's are orthogonal with respect to K'_j defined by

$$K'_j \stackrel{\Delta}{=} \left[\begin{array}{c|c} \frac{1}{2} M_1^{-1} & 0 \\ \hline 0 & \frac{1}{2\lambda_j^2} M_1^{-1} \end{array}\right] \tag{129}$$

In fact,

$$Y^{kT}K'_j\, Y^j = \begin{bmatrix} \psi^k \\ \hline \lambda_k \psi^k \end{bmatrix} \left[\begin{array}{c|c} \frac{1}{2} M_1^{-1} & 0 \\ \hline 0 & \frac{1}{2\lambda_j^2} M_1^{-1} \end{array} \right] \begin{bmatrix} \psi^j \\ \hline \lambda_j \psi^j \end{bmatrix}$$

$$= \frac{1}{2} \left\{ \psi^{kT} M_1^{-1} \psi^j + \left(\frac{\lambda_k}{\lambda_j} \right) \psi^{kT} M_1^{-1} \psi^j \right\}$$

$$= \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases} \quad . \tag{130}$$

In the presence of nonzero ε, we write the eigenvector as $Y^j(\varepsilon)$ corresponding to the eigenvalue $\lambda_j(\varepsilon)$ and hence the eigenvalue problem becomes

$$\left\{ \lambda_j(\varepsilon)\ U_{2N} - \mathscr{A}(\varepsilon) \right\} Y^j(\varepsilon) = 0, \qquad j = 1,2,\ldots,2N \tag{131}$$

where

$$\mathscr{A}(\varepsilon) = \mathscr{A} + \varepsilon\, P \tag{132}$$

For a sufficiently small ε, we may assume that

$$\lambda_j(\varepsilon) = \lambda_j + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \ldots \tag{133}$$

$$Y^j(\varepsilon) = Y^j + \varepsilon\, \xi^{j1} + \varepsilon^2 \xi^{j2} + \ldots \tag{134}$$

Since $\{Y^1, Y^2, \ldots, Y^{2N}\}$ form a complete set of eigenvectors, any vector in the 2N-dimensional space can be expressed by a linear combination of Y^j's, so that we may write

$$\xi^{j\ell} = \sum_{k=1}^{2N} C^j_{\ell k}\, Y^k \qquad \ell = 1,2,\ldots \tag{135}$$

Substituting Eq. (135) into Eq. (134) yields

$$Y^j(\varepsilon) = Y^j + \varepsilon \sum_{k=1}^{2N} C^j_{1k} Y^k + \varepsilon^2 \sum_{k=1}^{2N} C^j_{2k} Y^k + \ldots$$

$$= (1 + \varepsilon C^j_{1j} + \varepsilon^2 C^j_{2j} + \ldots) Y^j$$

$$+ (\varepsilon C^j_{11} + \varepsilon^2 C^j_{21} + \ldots) Y^1 + \ldots$$

$$+ (\varepsilon C^j_{1N} + \varepsilon^2 C^j_{2N} + \ldots) Y^N$$

If we normalize the coefficient of Y^j to unity,

$$Y^j(\varepsilon) = Y^j + \left(\sum_{k=1}^{\infty} \varepsilon^k t_{k1}\right) Y^1 + \ldots + \left(\sum_{k=1}^{\infty} \varepsilon^k t_{kN}\right) Y^{2N}$$

$$= Y^j + \sum_{\substack{\ell=1 \\ \ell \neq j}}^{2N} \left(\sum_{k=1}^{\infty} \varepsilon^k t_{k\ell}\right) Y^\ell \tag{136}$$

or

$$Y^j(\varepsilon) = Y^j + \varepsilon \sum_{\substack{\ell=1 \\ \ell \neq j}}^{2N} t_{1\ell} Y^\ell + \varepsilon^2 \sum_{\substack{\ell=1 \\ \ell \neq j}}^{2N} t_{2\ell} Y^\ell + \ldots \tag{137}$$

Substituting Eqs. (133) and (137) into Eq. (131) yields

$$(\mathscr{A} + \varepsilon P) \left\{ Y^j + \varepsilon \sum_{\ell \neq j} t_{1\ell} Y^\ell + \varepsilon^2 \sum_{\ell \neq j} t_{2\ell} Y^\ell + \ldots \right\}$$

$$- (\lambda_j + \alpha_1 \varepsilon + \ldots) \left\{ Y^j + \varepsilon \sum_{\ell \neq j} t_{1\ell} Y^\ell + \varepsilon^2 \sum_{\ell \neq j} t_{2\ell} Y^\ell + \ldots \right\} = 0$$

Equating the coefficient of ε in the left hand side to zero produces

$$\mathscr{A} \sum_{\ell \neq j} t_{1\ell} Y^\ell + P Y^j = \lambda_j \sum_{\ell \neq j} t_{1\ell} Y^\ell + \alpha_1 Y^j \tag{138}$$

In view of the unperturbed relations

$$\mathscr{A}\, Y^{\ell} = \lambda_{\ell}\, Y^{\ell} \qquad (\ell = 1,2,\ldots,2N)$$

we have

$$\mathscr{A} \sum_{\ell \neq j} t_{1\ell} Y^{\ell} = \sum_{\ell \neq j} \lambda_{\ell}\, t_{1\ell} Y^{\ell}$$

and Eq. (138) becomes

$$PY^{j} + \sum_{\ell \neq j} (\lambda_{\ell} - \lambda_{j}) t_{1\ell} Y^{\ell} = \alpha_{1} Y^{j} \tag{139}$$

Determination of α_1

Premultiplying both sides of Eq. (139) by $Y^{jT}K_j'$ and considering the orthonormality of Eq. (130), we have

$$Y^{jT}\, K_j'\, P\, Y^{j} = \alpha_1 \ .$$

From Eqs. (128) and (129),

$$K_j'\, P = \left[\begin{array}{c|c} \frac{1}{2} M_1^{-1} & 0 \\ \hline 0 & \frac{1}{2\lambda_j^2} M_1^{-1} \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -2M_1\rho\sigma \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\frac{1}{\lambda_j^2}\rho\sigma \end{array}\right]$$

Hence

$$\alpha_1 = \left[\begin{array}{c} \psi \\ \hline \lambda_j \psi^j \end{array}\right]^T \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\frac{1}{\lambda_j^2}\rho\sigma \end{array}\right] \left[\begin{array}{c} \psi^j \\ \hline \lambda_j \psi^j \end{array}\right]$$

$$= -\psi^{jT}\rho\sigma\psi^{j} \tag{140}$$

and

$$\alpha_1 \varepsilon = (-\psi^{jT} \rho\sigma\psi^j) \cdot \varepsilon$$

$$= -\frac{1}{2} \psi^{jT}(2\zeta\sigma)\psi^j$$

$$= -\frac{1}{2} \psi^{jT} D\psi^j \tag{141}$$

Therefore,

$$\lambda_j(\varepsilon) \cong \lambda_j - \frac{1}{2} \psi^{jT} D\psi^j \tag{142}$$

Since $\psi^j D\psi^j$ is real and positive, the perturbed eigenvalue $\lambda_j(\varepsilon)$ has a negative real part. Moreover, the eigenfrequency $|\lambda_j|$ is not perturbed to within the first order of ε.

<u>Determination of t_{1k}</u>

Premultiplying both sides of Eq. (139) by $Y^{kT}K_k'$ with $k \neq j$, we have

$$Y^{kT}K_k' \, PY^j + (\lambda_k-\lambda_j)t_{1k} \, Y^{kT}K'_k \, Y^k = 0$$

in which

$$Y^{kT}K_k' \, Y^k = 1$$

and

$$Y^{kT}K_k' \, PY^j = \begin{bmatrix} \psi^k \\ \lambda_k\psi^k \end{bmatrix}^T \left[\begin{array}{c|c} 0 & \\ \hline 0 & -\frac{1}{\lambda_k^2}\rho\sigma \end{array}\right] \begin{bmatrix} \psi^j \\ \lambda_j\psi^j \end{bmatrix}$$

$$= -\left(\frac{\lambda_j}{\lambda_k}\right) \psi^{kT}\rho\sigma\psi^j$$

Subsequently,

$$(\lambda_k-\lambda_j)t_{1k} = \left(\frac{\lambda_j}{\lambda_k}\right)\psi^{kT}\rho\sigma\psi^j$$

$$k = 1,2,\ldots,2N; \quad k \neq j$$

or

$$t_{1k} = \frac{\lambda_j}{\lambda_k} \cdot \frac{1}{\lambda_k - \lambda_j} \psi^{kT} \rho\sigma\psi^j$$

$$k = 1,2,\ldots,2N$$

Therefore, the coefficient of ε in $Y^j(\varepsilon)$ of Eq. (147) becomes

$$\sum_{k \neq j} t_{1k} Y^k = \sum_{k \neq j} \left(\frac{\lambda_j}{\lambda_k}\right) \frac{1}{\lambda_k - \lambda_j} \psi^{kT} \rho\sigma\psi^j Y^k$$

and

$$Y^j(\varepsilon) \approx Y^j + \varepsilon \sum_{k \neq j} \left(\frac{\lambda_j}{\lambda_k}\right) \frac{1}{\lambda_k - \lambda_j} \psi^{kT} \rho\sigma\psi^j Y^k$$

$$= Y^j + \frac{1}{2} \sum_{k \neq j} \left(\frac{\lambda_j}{\lambda_k}\right) \frac{1}{\lambda_k - \lambda_j} \psi^{kT} D\psi^j Y^k \qquad (143)$$

It should be noted that Eq. (143) is valid only when λ_k and λ_j are apart for $k \neq j$ so that t_{1k} is sufficiently small.

7. CANONICAL TRANSFORMATION AND CONTROLLABILITY EVALUATION

This section discusses how to construct a transformation which carries the system equations (Eqs. (5)) into a canonical form, and provides interpretations of system controllability in terms of the transformed equations.

We consider the transformation

$$X = \begin{bmatrix} \gamma \\ \dot{\gamma} \\ \eta \\ \dot{\eta} \end{bmatrix} = T\, Z \tag{144}$$

where T is a (2N+6)×(2N+6) matrix and Z is a (2N+6)×1 matrix representing the system vibration modes. The matrix T is conveniently partitioned into the submatrices

$$T = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right] \tag{145}$$

where T_{11}, T_{12}, T_{21} and T_{22} are 6×6, 6×2N, 2N×6 and 2N×2N matrices, respectively, and the eigenvectors and generalized eigenvectors corresponding to $\lambda=0$ may be assigned to the first six columns, and the eigenvectors corresponding to the nonzero eigenvalues to the remaining 2N columns. Then, from Eqs. (15) and (16), we may write

$$T_{11} = [c^{(1)},\ g^{(1)},\ c^{(2)},\ g^{(2)},\ c^{(3)},\ g^{(3)}]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{146}$$

and

$$T_{21} = 0 \; . \tag{147}$$

In view of the identity (not the transformation)

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{bmatrix} = T_{11} \begin{bmatrix} \gamma_1 \\ \dot{\gamma}_1 \\ \gamma_2 \\ \dot{\gamma}_2 \\ \gamma_3 \\ \dot{\gamma}_3 \end{bmatrix} ,$$

we may well write the variables of Z corresponding to $\lambda = 0$ as the three 2×1 partitions

$$Z^j = \begin{bmatrix} z_j \\ \dot{z}_j \end{bmatrix} \qquad j = 1,2,3 \tag{148}$$

and the two $N\times1$ partitions Z^4 and Z^5, so that

$$Z = \begin{bmatrix} Z^1 \\ Z^2 \\ Z^3 \\ Z^4 \\ Z^5 \end{bmatrix} \tag{149}$$

where Z^4 and Z^5 represent the contribution of appendage vibration modes of number 2N with nonzero frequencies.

For the nonzero eigenvalues which are determined by $|M_1\sigma^2 + \lambda^2 U_N| = 0$, we have from Eq. (41)

$$\lambda_j = i\sqrt{\mu_j}$$

and $\qquad\qquad\qquad\qquad j = 1,2,\ldots,N$

$$\lambda_{j+N} = \lambda_j^* = -\sqrt{\mu_j}$$

We know that ψ^j satisfying Eq. (21) is the eigenvector (of x_3) corresponding to λ_j and λ_{j+N} for the zero damping case. In view of Eqs. (17) with D = 0, we may write the eigenvectors of the system corresponding to the λ_j in terms of λ_j, ψ^j and the parameter matrices as

$$x_1^j = -\frac{1}{\lambda_j^2}\Delta^T M_1 \sigma^2 \psi^j \tag{150a}$$

$$x_2^j = -\frac{1}{\lambda_j}\Delta^T M_1 \sigma^2 \psi^j \tag{150b}$$

$$x_3^j = \psi^j \tag{150c}$$

$$x_4^j = \lambda_j \psi^j \tag{150d}$$

for $\quad j = 1,2,\ldots,N$.

If we define Λ and Λ^* by

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \quad \text{and} \quad \Lambda^* = \begin{bmatrix} \lambda_1^* & & & \\ & \lambda_2^* & & \\ & & \ddots & \\ & & & \lambda_N^* \end{bmatrix} \tag{151}$$

then Eqs. (150) are expressed in matrix form ((2N+6)×2N)

$$\left[\begin{array}{cccc|cccc} x_1^1 & x_1^2 & \cdots & x_1^N & x_1^{N+1} & \cdots & & x_1^{2N} \\ x_2^1 & x_2^2 & \cdots & x_2^N & x_2^{N+1} & \cdots & & x_2^{2N} \\ \hline x_3^1 & x_3^2 & \cdots & x_3^N & x_3^{N+1} & \cdots & & x_3^{2N} \\ x_4^1 & x_4^2 & \cdots & x_4^N & x_4^{N+1} & \cdots & & x_4^{2N} \end{array}\right] = \left[\begin{array}{c|c} -\Delta^T M_1 \sigma^2 \Psi \Lambda^{-2} & -\Delta^T M_1 \sigma^2 \Psi \Lambda^{*-2} \\ -\Delta^T M_1 \sigma^2 \Psi \Lambda^{-1} & -\Delta^T M_1 \sigma^2 \Psi \Lambda^{*-1} \\ \hline \Psi & \Psi \\ \Psi\Lambda & \Psi\Lambda^* \end{array}\right] \tag{152}$$

The upper and lower submatrices of Eq. (152) are identified as T_{12} and T_{22} of Eq. (145), respectively, i.e.,

$$T_{12} = \left[\begin{array}{c|c} -\Delta^T M_1 \sigma^2 \Psi \Lambda^{-2} & -\Delta^T M_1 \sigma^2 \Psi \Lambda^{*-2} \\ \hline -\Delta^T M_1 \sigma^2 \Psi \Lambda^{-1} & -\Delta^T M_1 \sigma^2 \Psi \Lambda^{*-1} \end{array}\right] \tag{153}$$

and

$$T_{22} = \left[\begin{array}{c|c} \Psi & \Psi \\ \hline \Psi\Lambda & \Psi\Lambda^* \end{array}\right] \tag{154}$$

Thus, we have established the transformation matrix, T, whose submatrices are given by Eqs. (146), (147), (153) and (154).

Since $|T_{11}| = 1 \neq 0$, and $|T_{22}| \neq 0$ by Fact 5, and since $T_{21} = 0$,

$$|T| = |T_{11}| \cdot |T_{22}| . \tag{155}$$

T is also nonsingular and

$$T^{-1} = \left[\begin{array}{c|c} T_{11}^{-1} & -T_{11}^{-1} T_{12} T_{22}^{-1} \\ \hline 0 & T_{22}^{-1} \end{array}\right] \tag{156}$$

where

$$T_{11}^{-1} = T_{11}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{157}$$

and

$$T_{22}^{-1} = \frac{1}{2} \left[\begin{array}{c|c} \Psi^{-1} & \Lambda^{-1}\Psi^{-1} \\ \hline \Psi^{-1} & \Lambda^{*-1}\Psi^{-1} \end{array}\right] \tag{158}$$

Next, we will show that the transformation of Eq. (144) transforms Eq. (5) into a canonical form

$$\dot{Z} = T^{-1}AT\,Z + T^{-1}Bu$$

where

$$T^{-1}AT = \left[\begin{array}{ccc|cc} \begin{array}{c|c|c} J & 0 & 0 \\ \hline 0 & J & 0 \\ \hline 0 & 0 & J \end{array} & & & \multicolumn{2}{c}{0} \\ \hline \multicolumn{3}{c|}{0} & \begin{array}{c|c} \Lambda & 0 \\ \hline 0 & \Lambda^{*} \end{array} & \end{array}\right] \tag{159}$$

with

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{160}$$

Equation (159) is derived as follows: Partitioning A into submatrices as

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

where

$$A_{11} = \left[\begin{array}{c|c} 0 & U_3 \\ \hline 0 & 0 \end{array}\right] \qquad A_{12} = \left[\begin{array}{c|c} 0 & 0 \\ \hline -\Delta^T M_1 \sigma^2 & 0 \end{array}\right]$$

and

$$A_{22} = \left[\begin{array}{c|c} 0 & U_3 \\ \hline -M_1 \sigma^2 & 0 \end{array}\right]$$

yields

$$\begin{aligned} T^{-1}AT &= \left[\begin{array}{c|c} T_{11}^{-1} & -T_{11}^{-1}T_{12}T_{22}^{-1} \\ \hline 0 & T_{22}^{-1} \end{array}\right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline 0 & T_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} T_{11}^{-1}A_{11} & T_{11}^{-1}A_{12} - T_{11}^{-1}T_{12}T_{22}^{-1}A_{22} \\ \hline 0 & T_{22}^{-1}A_{22} \end{array}\right] \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline 0 & T_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} T_{11}^{-1}A_{11}T_{11} & T_{11}^{-1}A_{11}T_{12} + T_{11}^{-1}A_{12}T_{22} - T_{11}^{-1}T_{12}T_{22}^{-1}A_{22}T_{22} \\ \hline 0 & T_{22}^{-1}A_{22}T_{22} \end{array}\right] \end{aligned}$$

$$
T_{11}^{-1}A_{11}T_{11} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|ccc} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ \hline & & & \\ 0 & & 0 & \\ & & & \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
= \left[\begin{array}{c|ccc} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ \hline & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
= \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} J & 0 & 0 \\ \hline 0 & J & 0 \\ \hline 0 & 0 & J \end{array}\right]
$$

$$
T_{22}^{-1}A_{22}T_{22} = \frac{1}{2}\left[\begin{array}{c|c} \Psi^{-1} & \Lambda^{-1}\Psi^{-1} \\ \hline \Psi^{-1} & \Lambda^{*-1}\Psi^{-1} \end{array}\right] \left[\begin{array}{c|c} 0 & U_N \\ \hline -M_1\sigma^2 & 0 \end{array}\right] \left[\begin{array}{c|c} \Psi & \Psi \\ \hline \Psi\Lambda & \Psi\Lambda^* \end{array}\right]
$$

$$
= \frac{1}{2}\left[\begin{array}{c|c} -\Lambda^{-1}\Psi^{-1}M_1\sigma^2\Psi + \Lambda & -\Lambda^{-1}\Psi^{-1}M_1\sigma^2\Psi + \Lambda^* \\ \hline -\Lambda^{*-1}\Psi^{-1}M_1\sigma^2\Psi + \Lambda & -\Lambda^{*-1}\Psi^{-1}M_1\sigma^2\Psi + \Lambda^* \end{array}\right]
$$

But $M_1\sigma^2\Psi = -\Psi\Lambda^2$ and $\Lambda + \Lambda^* = 0$, so

$$
T_{22}^{-1}A_{22}T_{22} = \left[\begin{array}{c|c} \Lambda & 0 \\ \hline 0 & \Lambda^* \end{array}\right]
$$

$$
A_{11}T_{12} = \left[\begin{array}{c|c} 0 & U_3 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} -\Delta^T M_1\sigma^2\Psi\Lambda^{-2} & -\Delta^T M_1\sigma^2\Psi\Lambda^{*-2} \\ \hline -\Delta^T M_1\sigma^2\Psi\Lambda^{-1} & -\Delta^T M_1\sigma^2\Psi\Lambda^{*-1} \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} -\Delta^T M_1\sigma^2\Psi\Lambda^{-1} & -\Delta^T M_1\sigma^2\Psi\Lambda^{*-1} \\ \hline 0 & 0 \end{array}\right]
$$

$$A_{12}T_{22} = \left[\begin{array}{c|c} 0 & 0 \\ \hline -\Delta^T M_1\sigma^2 & 0 \end{array}\right] \left[\begin{array}{c|c} \Psi & \Psi \\ \hline \Psi\Lambda & \Psi\Lambda^* \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline -\Delta^T M_1\sigma^2\Psi & -\Delta^T M_1\sigma^2\Psi \end{array}\right]$$

$$T_{12}T_{22}^{-1}A_{22}T_{22} = \left[\begin{array}{c|c} -\Delta^T M_1\sigma^2\Psi\Lambda^{-2} & -\Delta^T M_1\sigma^2\Psi\Lambda^{*-2} \\ \hline -\Delta^T M_1\sigma^2\Psi\Lambda^{-1} & -\Delta^T M_1\sigma^2\Psi\Lambda^{*-1} \end{array}\right] \left[\begin{array}{c|c} \Lambda & 0 \\ \hline 0 & \Lambda^* \end{array}\right]$$

$$= \left[\begin{array}{c|c} -\Delta^T M_1\sigma^2\Psi\Lambda^{-1} & -\Delta^T M_1\sigma^2\Psi\Lambda^{*-1} \\ \hline -\Delta^T M_1\sigma^2\Psi & -\Delta^T M_1\sigma^2\Psi \end{array}\right]$$

Hence

$$A_{11}T_{12} + A_{12}T_{22} - T_{12}T_{22}^{-1}A_{22}T_{22} = 0$$

and Eq. (159) is proven.

Next, we will express $T^{-1}B$ of Eq. (158) in an explicit form.

$$T^{-1}B = \left[\begin{array}{c|c} T_{11}^{-1} & -T_{11}^{-1}T_{12}T_{22}^{-1} \\ \hline 0 & T_{22}^{-1} \end{array}\right] \left[\begin{array}{c} 0 \\ B_2 \\ \hline 0 \\ B_4 \end{array}\right]$$

$$= \left[\begin{array}{c} T_{11}^{-1}\left[\begin{array}{c} 0 \\ \hline B_2 \end{array}\right] - T_{11}^{-1}T_{12}T_{22}^{-1}\left[\begin{array}{c} 0 \\ \hline B_4 \end{array}\right] \\ \hline T_{22}^{-1}\left[\begin{array}{c} 0 \\ \hline B_4 \end{array}\right] \end{array}\right]$$

$$T_{22}^{-1}\left[\begin{array}{c} 0 \\ \hline B_4 \end{array}\right] = \frac{1}{2}\left[\begin{array}{c|c} \Psi^{-1} & \Lambda^{-1}\Psi^{-1} \\ \hline \Psi^{-1} & \Lambda^{*-1}\Psi^{-1} \end{array}\right] \left[\begin{array}{c} 0 \\ \hline M_1\Delta_c \end{array}\right]$$

$$= \frac{1}{2}\left[\begin{array}{c} \Lambda^{-1} \\ \hline \Lambda^{*-1} \end{array}\right] \Psi^{-1}M_1\Delta_c$$

where $\Delta_c \overset{\Delta}{=} \Delta + \phi^T \mathscr{L}_c(I^*)^{1/2}$

$$T_{12}T_{22}^{-1}\left[\frac{0}{B_4}\right] = \frac{1}{2}\left[\begin{array}{c|c} -\Delta^T M_1 \sigma^2 \Psi \Lambda^{-2} & -\Delta^T M_1 \sigma^2 \Psi \Lambda^{*-2} \\ \hline -\Delta^T M_1 \sigma^2 \Psi \Lambda^{-1} & -\Delta^T M_1 \sigma^2 \Psi \Lambda^{*-1} \end{array}\right]\left[\frac{-\Lambda^{-1}}{\Lambda^{*-1}}\right] \Psi^{-1} M_1 \Delta_c$$

$$= -\frac{1}{2}\begin{bmatrix} -\Delta^T M_1 \sigma^2 \Psi(\Lambda^{-3} + \Lambda^{*-3}) \\ \Delta^T M_1 \sigma^2 \Psi(\Lambda^{-2} + \Lambda^{*-2}) \end{bmatrix} \Psi^{-1} M_1 \Delta_c$$

But $\Lambda^{-3} + \Lambda^{*-3} = 0$ and $\Lambda^{-2} + \Lambda^{*-2} = 2\Lambda^{-2}$

$$\therefore \quad T_{12}T_{22}^{-1}\left[-\frac{0}{B_4}-\right] = -\begin{bmatrix} 0 \\ \Delta^T M_1 \sigma^2 \Psi \Lambda^{-2} \end{bmatrix} \Psi^{-1} M_1 \Delta_c$$

and with $M_1 \sigma^2 \Psi = -\Psi \Lambda^2$, this becomes

$$T_{12}T_{22}^{-1}\left[-\frac{0}{B_4}-\right] = \begin{bmatrix} 0 \\ \Delta^T(\Psi\Lambda^2)\Lambda^{-2} \end{bmatrix} \quad \Psi^{-1} M_1 \Delta_c = \begin{bmatrix} 0 \\ \Delta^T M_1 \Delta_c \end{bmatrix}$$

Hence

$$\begin{bmatrix} 0 \\ B_2 \end{bmatrix} - T_{12}T_{22}^{-1}\begin{bmatrix} 0 \\ B_4 \end{bmatrix} = \begin{bmatrix} 0 \\ B_2 - \Delta^T M_1 \Delta_c \end{bmatrix}$$

But

$$\begin{aligned} B_2 - \Delta^T M_1 \Delta_c &= M_2 + \Delta^T M_1 \phi^T \mathscr{L}_c^* - \Delta^T M_1 (\Delta + \phi^T \mathscr{L}_c^*) \\ &= M_2 - \Delta^T M_1 \Delta \\ &= M_2 - \Delta^T \Delta M_2 \\ &= (U_3 - \Delta^T \Delta) M_2 = U_3 \end{aligned}$$

Hence

$$T^{-1}B = \left[\begin{array}{c} T_{11}^{-1}\left[\frac{0}{U_3}\right] \\ \hline \frac{1}{2}\left[\frac{\Lambda^{-1}}{\Lambda^{*-1}}\right]\Psi^{-1} M_1 \Delta_c \end{array}\right] \qquad \text{with } T_{11}^{-1} = T_{11}^T$$

Thus, we have obtained a canonical form

$$\begin{bmatrix} \dot{Z}^1 \\ \dot{Z}^2 \\ \dot{Z}^3 \\ \hline \dot{Z}^4 \\ \dot{Z}^5 \end{bmatrix} = \left[\begin{array}{ccc|cc} J & 0 & 0 & & \\ 0 & J & 0 & & 0 \\ 0 & 0 & J & & \\ \hline & & & \Lambda & 0 \\ & 0 & & 0 & \Lambda^* \end{array}\right] \begin{bmatrix} Z^1 \\ Z^2 \\ Z^3 \\ \hline Z^4 \\ Z^5 \end{bmatrix} + \left[\begin{array}{c} T_{11}^T \begin{bmatrix} 0 \\ \hline U_3 \end{bmatrix} \\ \hline \frac{1}{2} \begin{bmatrix} \Lambda^{-1} \\ \hline \Lambda^{*-1} \end{bmatrix} \Psi^{-1} M_1 \Delta_c \end{array}\right] u \tag{161}$$

We can examine the controllability of the system described by Eq. (5), based upon the canonical transformation of Eq. (161).

It is obvious that the modes Z^1, Z^2 and Z^3 are all controllable by the input u in such a fashion that Z^j is controllable by u_j for j=1,2,3 independent of each other. In other words zero frequency modes are controllable by u independently, where u is "normalized" by the definition of Eq. (2c). (Note that the physical implementation of torquers T does not necessarily apply independently to each of three attitude directions.)

For nonzero frequency modes, it will suffice to examine the controllability of Z^4, because it implies that of Z^5 and vice versa as seen from Eq. (161). From Eq. (161),

$$\dot{Z}^4 = \Lambda Z^4 + \frac{1}{2} \Lambda^{-1} \Psi^{-1} M_1 \Delta_c u \tag{162}$$

With the normalization of Eq. (91),

$$\Psi^{-1} M_1 = \Psi^T$$

so that Eq. (162) becomes

$$\dot{Z}^4 = \Lambda Z^4 + \frac{1}{2} \Lambda^{-1} \Psi^T \Delta_c u \tag{163}$$

Since Λ^{-1} is diagonal, we may evaluate the controllability of each mode based on the presence or absence of nonzero values in the corresponding row of the matrix $\Psi^T\Delta_c$, and thereby establish the system controllability conditions, after noting the multiplicity of the λ_j's comprising the nonzero elements of Λ.

It should be noted that the controllability arguments stated above are referred to Z^4 (or Z^5) but not to η. Subsequently, justification for truncation based upon the controllability properties is also referred to Z^4, e.g., if $||\Psi_s^T\Delta_c|| = 0$ then Z_s^4 is uncontrollable and "truncatable" when it is also unobservable. However, this does not necessarily imply that η_s is truncatable or uncontrollable in view of the transformation of Eq. (156).

At this point, Eq. (162) is to be compared with Eq. (57) of [3], which says

$$\dot{v} = \sigma^2 M_1 v + \Delta_c u \tag{164}$$

If we transform v into w by

$$v = M_1^{-1}\Psi w \tag{165}$$

then Eq. (164) becomes

$$M_1^{-1}\Psi\dot{w} = \sigma^2 M_1(M_1^{-1}\Psi w) + \Delta_c u$$

or

$$\dot{w} = \Psi^{-1}M_1\sigma^2\Psi w + \Psi^{-1}M_1\Delta_c u \quad .$$

But from Eq. (91),

$$\Psi^{-1}M_1 = \Psi^T$$

and from Eq. (93)

$$\Psi^{-1}M_1\sigma^2\Psi = -\Lambda^2$$

so

$$\dot{w} = -\Lambda^2 w + \Psi^T \Delta_c u \tag{166}$$

By comparing Eq. (165) with Eq. (163), we recognize that the mode controllability conditions are identical, both being determined by the corresponding row of $\Psi^T\Delta_c$. In addition, the system controllability conditions are identical, because the multiplicity of the elements in $-\Lambda^2$ is the same as that of Λ.

Although Eq. (164) was derived in [3] by rank calculations without utilizing any knowledge about the system eigenvalues or eigenvectors, it turns out that the implications of both approaches are identical as far as controllability is concerned.

In either way, the canonical form of Eq. (162) or (163) with $\Psi^T\Delta_c$ specified will be required for further arguments. It should be noted that uncontrollability of $\dot{w}_s$ (or Z_s^4) does not necessarily imply that of any particular element of v (e.g. v_s) because of the transform relation of Eq. (165).

However, in some special cases, the controllability condition for Z_s^4 (or w_s) does coincide with that of η_s (or v_s), and hence truncation of η_s is justified from the controllability point of view. For example, this is the case when $\Psi_j^s = 0$ for $j \neq s$, permitting

$$v_s = \Psi^{sT} w = \sum_{j=1}^{N} \psi_j^s w_j = \psi_s^s w_s$$

in which v_s is affected only by w_s and vice-versa.

8. CONCLUSION

The eigenvalue and eigenvector problem associated with the original system of dimension (2N+6) reduces to that of a symmetric and positive definite matrix of dimension N with the zero damping assumption (Eq. (21)). The results from the analytical method show that the lowest eigenfrequency of the system vibration modes is always equal to or greater than the lowest of the appendage vibration frequencies (Fact 3). In some special cases, including the case when the system is decomposable into three single axis subsystems, the system eigenvalues separate the appendage frequencies at least in a weak sense (Eq. (47) of Fact 4). The results from the minimax characterization localize the eigenvalues as given by Eq. (79). This procedure requires only simple calculations of modal matrices.

The multiplicity of the eigenvalues is dependent on the inertial properties and modal parameters in a somewhat complicated manner. It is suggested that careful examination should be made of the quantities involved, such as P_ℓ's, K_j^ℓ and L_{jk}^ℓ (Eqs. (83)-(86)).

If the appendage natural frequencies $\sigma_1,\ldots,\sigma_N$ are distinct, then expressions for reduced system eigenvalues $\mu_1,\ldots,\mu_N$ are always available as follows: (a) If $\Delta^\ell = 0$, then $\mu_\ell = \sigma_\ell^2$; (b) If $\Delta^\ell \neq 0$ but $P_\ell = 0$, then $\mu_\ell = \sigma_\ell^2$; (c) If $\Delta^\ell \neq 0$ and $P_\ell > 0$, then the bounds in Eq. (66) apply and in addition, from Eq. (47)

$$\sigma_1^2 < \mu_1 < \sigma_2^2 < \mu_2 < \cdots \quad \sigma_N^2 < \mu_N \quad ;$$

(d) If $\Delta^\ell \neq 0$ and $P_\ell < 0$, then the bounds in Eq. (66) apply. Finally, if the appendage natural frequencies are not distinct then, in addition to the bounds in Eq. (65), further restricted results are available in Sec. 3.3.

The orthogonal properties of the eigenvectors are presented with normalization conditions employed (Eqs. (90)). The ($N \times 1$) eigenvectors are generated by a 3×1 matrix (Eqs. (92b) and (93)), that reduces the calculations greatly.

Sufficient condition for acceptability of truncation is given by Eqs. (120)-(122) as a result of the Wielandt-Hoffman theorem. This evaluation is made by the calculation of the trace of a 3×3 matrix (Eq. (120)). An explicit error bound is derived in terms of the eigenvalues of the 3×3 matrix (Eq. (124)).

The effect of modal damping is examined by a perturbation method applied to the first order form of the eigenvalue problem. The result assures that the system eigenvalues have negative real parts and that it does not affect the eigenfrequency to within the first order of maximum modal damping (Eq. (42)). The eigenvectors can be significantly changed as shown in Eq. (143).

Based on the eigenvectors of the reduced system, a matrix is constructed that transforms the original system equation into the Jordan canonical form, which is useful for the system controllability evaluation. The results are compared with those of [3] and physical interpretations are given.

Although the exact eigenvalues and eigenvectors are available only by numerical calculation, the characterizations established in this paper will be useful in that they afford some insight into the eigenvalue localization and eigenvector properties. These procedures are recommended especially for preliminary analysis, because the requirements for calculations are not burdensome — nothing more than algebraic manipulations of matrices such as additions and multiplications.

Eigenvalue and eigenvector sensitivity analysis is left for further research. This is particularly important for the eigenvector characterization, because it may be greatly affected by a small change in the parameters (ill-conditioned), even though the eigenvalues of the system are always well conditioned. It is speculated that a perturbation method similar to that employed in the damping effect examination will be useful for sensitivity analysis, if we can define an appropriate perturbation matrix.

Since the quantities which give the bounds for the system eigenvalues or truncation error are considered to be the Euclidean norm of 3×1 matrices, further evaluation of these quantities may be possible by virtue of norm characterization (perhaps by linear programming).

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APPENDICES

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APPENDIX A. SOME PROPERTIES OF $\Delta^T\Delta$ AND M_1

1. All the eigenvalues of $\Delta^T\Delta$ lie between zero and unity.

<u>Proof</u>. From Eq. (4d),

$$\Delta^T\Delta = (I^*)^{-1/2}(I^*-I^o)(I^*)^{-1/2}$$

with

$$I^* - I^o = I_A^* - \mathcal{M}_B \tilde{q}\tilde{q} \qquad \text{(A1)}$$

where I_A^* is the inertia matrix of the appendage about the total vehicle mass center c, and q is a matrix representation of the vector from c to the primary body mass center, and $\mathcal{M}_B$ is the primary body mass.

In Eq. (A1), I_A^* and $-\mathcal{M}_B \tilde{q}\tilde{q}$ are both symmetric and positive definite and hence $I^* - I^o$ is symmetric and positive definite. Therefore, $\Delta^T\Delta$ is positive definite and the eigenvalues α_j of $\Delta^T\Delta$ are positive. In addition,

$$U_3 - \Delta^T\Delta = (I^*)^{-1/2} I^o(I^*)^{-1/2}$$

is also positive definite, so that the eigenvalues of $(U_3 - \Delta^T\Delta)$, $(1-\alpha_j)$'s, are positive. Thus,

$$0 < \alpha_j < 1 \quad . \qquad \text{(A2)}$$

2. The matrix M_1 is positive definite.

<u>Proof</u>. Let β_j be the eigenvalues of M_1^{-1}. Then, $\beta_1,\ldots,\beta_N$ are the roots of

$$|(\beta U_N - M_1)| = 0$$

or

$$|\beta U_N(U_N - \Delta\Delta^T) - U_N| = 0$$

or

$$\left| \frac{\beta-1}{\beta} \; U_N - \Delta\Delta^T \right| = 0$$

Let $\gamma \overset{\Delta}{=} \frac{\beta-1}{\beta}$, then

$$|\gamma \; U_N - \Delta\Delta^T| = 0$$

meaning γ is the eigenvalue of $\Delta\Delta^T$. But the eigenvalues of $\Delta\Delta^T$ are identical to those of $\Delta^T\Delta$ except for (N-3) zero eigenvalues [4], so that

$$\gamma_j = \alpha_j \quad (j=1,2,3)$$

and

$$\gamma_j = 0 \qquad (j=4,5,\ldots,N)$$

with

$$0 \leq \gamma_j < 1 \qquad (j=1,2,\ldots,N) \; .$$

It follows that

$$\beta_j = \frac{1}{1-\alpha_j} \qquad j=1,2,3$$

and

$$\beta_j = 1 \qquad j=4,5,\ldots,N$$

implying that β_j are all positive, $j = 1,\ldots,N$.

APPENDIX B. PROPERTIES OF $d_j(\mu) = \dfrac{\mu}{\mu - \sigma_i^2}$

Assume that the σ_i are all distinct, i.e.,

$$(\sigma_i^2 - \sigma_j^2)(\sigma_j^2 - \sigma_k^2)(\sigma_k^2 - \sigma_i^2) \neq 0$$

for

$$i \neq j \neq k \neq i$$

Then,

$$d_i(\mu)\, d_j(\mu) = d_{ji} d_i(\mu) + d_{ij} d_j(\mu) \tag{B1}$$

and

$$d_i(\mu)\, d_j(\mu)\, d_k(\mu) = d_{ji} d_{ki} d_i(\mu) + d_{kj} d_{ij} d_j(\mu) + d_{ik} d_{jk} d_k(\mu) \tag{B2}$$

where

$$d_{ij} \overset{\Delta}{=} d_i(\sigma_j^2) = \frac{\sigma_j^2}{\sigma_j^2 - \sigma_i^2} \tag{B3}$$

with

$$\left.\begin{array}{lll} d_{ij} > 0 & & \text{if } \; j > i \\ d_{ij} < 0 & & \text{if } \; j < i \end{array}\right\} \tag{B4}$$

<u>Proof</u>. Under the assumption, we can expand the product of $d_i(\mu)$ and $d_j(\mu)$ in the partial fraction as follows.

$$d_i(\mu)d_j(\mu) = \frac{\mu}{\mu-\sigma_i^2} \; \frac{\mu}{\mu-\sigma_j^2}$$

$$= \frac{1}{\sigma_i^2-\sigma_j^2}\left(\frac{\mu\,\sigma_i^2}{\mu-\sigma_i^2} \quad \frac{\mu\,\sigma_j^2}{\mu-\sigma_j^2}\right)$$

$$= \frac{\sigma_i^2}{\sigma_i^2-\sigma_j^2}\; d_i(\mu) + \frac{\sigma_j^2}{\sigma_j^2-\sigma_i^2}\; d_j(\mu)$$

$$= d_{ji}d_i(\mu) + d_{ij}d_j(\mu)$$

Using (B1),

$$d_i(\mu)d_j(\mu)d_k(\mu) = \{d_{ji}d_i(\mu) + d_{ij}d_j(\mu)\}\; d_k(\mu)$$

$$= d_{ji}d_i(\mu)d_k(\mu) + d_{ij}d_j(\mu)d_k(\mu)$$

$$= d_{ji}\;\{d_{ki}d_i(\mu) + d_{ik}d_k(\mu)\} + d_{ij}\{d_{kj}d_j(\mu) + d_{jk}d_k(\mu)\}$$

$$= d_{ji}d_{ki}d_i(\mu) + d_{kj}d_{ij}d_j(\mu) + \{d_{ji}d_{ik} + d_{ij}d_{jk}\}\; d_k(\mu)$$

But $d_{ji}d_{ik} + d_{ij}d_{jk} =$

$$= \frac{\sigma_i^2}{\sigma_i^2-\sigma_j^2}\,\frac{\sigma_k^2}{\sigma_k^2-\sigma_i^2} + \frac{\sigma_j^2}{\sigma_j^2-\sigma_i^2}\,\frac{\sigma_k^2}{\sigma_k^2-\sigma_j^2}$$

$$= \frac{\sigma_k^2}{\sigma_k^2-\sigma_i^2}\left[\frac{\sigma_i^2}{\sigma_i^2-\sigma_j^2} + \frac{\sigma_j^2}{\sigma_j^2-\sigma_i^2}\cdot\frac{\sigma_k^2}{\sigma_k^2-\sigma_j^2}\cdot\frac{\sigma_k^2-\sigma_i^2}{\sigma_k^2}\right]$$

$$= \frac{\sigma_k^2}{\sigma_k^2-\sigma_i^2}\cdot\frac{\sigma_i^2(\sigma_k^2-\sigma_j^2) - \sigma_j^2(\sigma_k^2-\sigma_j^2)}{(\sigma_i^2-\sigma_j^2)(\sigma_k^2-\sigma_j^2)}$$

$$= \frac{\sigma_k^2}{\sigma_k^2-\sigma_i^2}\cdot\frac{(\sigma_i^2-\sigma_j^2)\sigma_k^2}{(\sigma_i^2-\sigma_j^2)(\sigma_k^2-\sigma_j^2)}$$

$$= d_{ik}d_{jk}$$

Thus, we have (B2).

APPENDIX C. DERIVATION OF $g(\mu)$, $G_\ell(\mu)$ AND P_ℓ (Eqs. (31), (32) AND (43))

Associated with the definition of $g(\mu)$ (Eq. (27)), we define $g_\ell(\mu)$ by

$$g_\ell(\mu) \overset{\Delta}{=} \left| \left[\delta^\alpha_\beta - \sum_{j=1}^{\ell} \Delta^j_\alpha \Delta^j_\beta \, d_j(\mu) \right] \right| \tag{C1}$$

where the quantity in the square brackets indicates the (α,β) element of the 3×3 matrix with $\alpha,\beta = 1,2,3$. Note that

$$g_N(\mu) = g(\mu) \tag{C2}$$

and

$$g_0(\mu) = 1. \tag{C3}$$

If we define 3×3 matrices Q^ℓ and R^ℓ by

$$Q^\ell \overset{\Delta}{=} \left[\delta^\alpha_\beta - \sum_{j=1}^{\ell} \Delta^j_\alpha \Delta^j_\beta \, d_j(\mu) \right] \tag{C4}$$

and

$$R^\ell \overset{\Delta}{=} \left[\Delta^\ell_\alpha \Delta^\ell_\beta \, d_\ell(\mu) \right] , \tag{C5}$$

then

$$g_\ell(\mu) = |Q^\ell| \tag{C6}$$

and

$$Q^\ell = Q^{\ell-1} - R^\ell \tag{C7}$$

For the determinant of a sum (or difference) of two 3×3 matrices, the following identity holds.

$$|Q^{\ell-1} - R^\ell| = |Q^{\ell-1}| - |R^\ell| + \sum_{j,k=1}^{3} (-1)^{j+k} (q_{jk}|R_{jk}| - r_{jk}|Q_{jk}|) \tag{C8}$$

where q_{jk}, r_{jk} are the (j,k) elements of the matrices $Q^{\ell-1}$, R^ℓ, respectively, and Q_{jk} and R_{jk} are the (j,k) minor of $Q^{\ell-1}$ and R^{ℓ}, respectively. (The superscripts on $q_{jk}, r_{jk}, Q_{jk}, R_{jk}$ are deleted for simplicity of notation.)

Noting that $R^{\ell} = d_{\ell}(\mu)\Delta^{\ell T}\Delta^{\ell}$, whose rank is at most unity, we have

$$|R^{\ell}| = 0 \quad \text{and} \quad |R_{jk}| = 0 \tag{C9}$$

From Eqs. (C6)-(C9),

$$\begin{aligned} q_{\ell}(\mu) &= |Q^{\ell}| \\ & |Q^{\ell-1} - R^{\ell}| \\ &= |Q^{\ell-1}| - \sum_{j,k=1}^{3} (-1)^{j+k}\, r_{jk}|Q_{jk}| \end{aligned} \tag{C10}$$

But $\quad |Q^{\ell-1}| = g_{\ell-1}(\mu)$

and

$$r_{jk} = d_{\ell}(\mu)\Delta_j^{\ell}\Delta_k^{\ell}$$

so that

$$g_{\ell}(\mu) = g_{\ell-1}(\mu) - \left\{ \sum_{j,k=1}^{3} (-1)^{j+k}\Delta_j^{\ell}\Delta_k^{\ell}|Q_{jk}| \right\} d_{\ell}(\mu) \tag{C10'}$$

Defining (with the superscript on Q_{jk} restored)

$$G_{\ell}(\mu) \stackrel{\Delta}{=} \sum_{j,k=1}^{3} (-1)^{j+k}\Delta_j^{\ell}\Delta_k^{\ell}|Q_{jk}^{\ell-1}| \quad , \tag{C11}$$

we obtain a recursive formula for $g_{\ell}(\mu)$ as

$$g_{\ell}(\mu) = g_{\ell-1} - G_{\ell}(\mu)\ d_{\ell}(\mu) \tag{C12}$$

Note that $G_{\ell}(\mu)$ does not contain $d_j(\mu)(j \geq \ell)$. From Eq. (C12) together with Eqs. (C2) and (C3), we have

$$g(\mu) = 1 - \sum_{\ell=1}^{N} G_{\ell}(\mu)\ d_{\ell}(\mu) \ . \tag{C13}$$

In what follows, $G_{\ell}(\mu)$ of Eq. (C11) will be expressed more explicitly.

For simplicity of notation, define $\sum_{jk}$ by

$$\sum_{jk} \stackrel{\Delta}{=} \sum_{i=1}^{\ell-1} \Delta_j^i \Delta_k^i \, d_i(\mu) \tag{C14}$$

with

$$\sum_{jk} = \sum_{kj} \tag{C15}$$

then from Eq. (C4),

$$Q^{\ell-1} = \left[\delta_\beta^\alpha - \sum_{\alpha\beta}\right]$$

and

$$G_\ell(\mu) = \sum_{j,k=1}^{3} (-1)^{j+k} \Delta_j^\ell \Delta_k^\ell \left| \left[\delta_\beta^\alpha - \sum_{\alpha\beta}\right]_{jk} \right| .$$

Direct expansion of the determinants of the minors $[\ \cdot\ ]_{\alpha\beta}$ produces

$$\begin{aligned} G_\ell(\mu) = {} & (\Delta_1^\ell)^2 \left\{\left(1 - \sum_{22}\right)\left(1 - \sum_{33}\right) - \left(\sum_{23}\right)^2\right\} \\ & + (\Delta_2^\ell)^2 \left\{\left(1 - \sum_{33}\right)\left(1 - \sum_{11}\right) - \left(\sum_{31}\right)^2\right\} \\ & + (\Delta_3)^2 \left\{\left(1 - \sum_{11}\right)\left(1 - \sum_{22}\right) - \left(\sum_{12}\right)^2\right\} \\ & + 2\Delta_1^\ell \Delta_2^\ell \left\{\sum_{12}\left(1 - \sum_{33}\right) + \sum_{13} \sum_{23}\right\} \\ & + 2\Delta_2^\ell \Delta_3^\ell \left\{\sum_{23}\left(1 - \sum_{11}\right) + \sum_{21} \sum_{31}\right\} \\ & + 2\Delta_3^\ell \Delta_1^\ell \left\{\sum_{31}\left(1 - \sum_{22}\right) + \sum_{32} \sum_{12}\right\} \end{aligned} \tag{C16}$$

The terms of the right-hand side of Eq. (16) are conveniently classified as

$$G_\ell(\mu) = G_\ell^0 + G_\ell^1(\mu) + G_\ell^2(\mu) \tag{C17}$$

where G_ℓ^0, $G_\ell^1(\mu)$ and $G_\ell^2(\mu)$ are the collections of the zero-th, 1-st and 2-nd order terms in $d_j(\mu)$ (or $\sum_{\alpha\beta}$), respectively.

Obviously,

$$
\begin{aligned}
G_\ell^0 &= (\Delta_1^\ell)^2 + (\Delta_2^\ell)^2 + (\Delta_3^\ell)^2 \\
&= \Delta^\ell \Delta^{\ell T}
\end{aligned}
\tag{C18}
$$

$$
\begin{aligned}
G^1(\mu) = &-(\Delta_1^\ell)^2\left(\Sigma_{22} + \Sigma_{33}\right) - (\Delta_2^\ell)^2\left(\Sigma_{33} + \Sigma_{11}\right) \\
&-(\Delta_3^\ell)^2\left(\Sigma_{11} + \Sigma_{22}\right) + 2\Delta_1^\ell\Delta_2^\ell\,\Sigma_{12} \\
&+ 2\Delta_2^\ell\Delta_3\,\Sigma_{23} + 2\Delta_3^\ell\Delta_1^\ell\,\Sigma_{31}
\end{aligned}
$$

From Eq. (C1)

$$
\begin{aligned}
G_\ell^1(\mu) = &-(\Delta_1^\ell)^2\left\{\sum_{j=1}^{\ell-1}(\Delta_2^j)^2\,d_j(\mu) + \sum_{j=1}^{\ell-1}(\Delta_3^j)^2 d_j(\mu)\right\} \\
&-(\Delta_2^\ell)^2\left\{\sum_{j=1}^{\ell-1}(\Delta_3^j)^2\,d_j(\mu) + \sum_{j=1}^{\ell-1}(\Delta_1^j)^2 d_j(\mu)\right\} \\
&-(\Delta_3^\ell)^2\left\{\sum_{j=1}^{\ell-1}(\Delta_1^j)^2\,d_j(\mu) + \sum_{j=1}^{\ell-1}(\Delta_2^j)^2 d_j(\mu)\right\} \\
&+ 2\Delta_1^\ell\Delta_2^\ell\sum_{j=1}^{\ell-1}\Delta_1^j\Delta_2^j\,d_j(\mu) \\
&+ 2\Delta_2^\ell\Delta_3^\ell\sum_{j=1}^{\ell-1}\Delta_2^j\Delta_3^j\,d_j(\mu) \\
&+ 2\Delta_3^\ell\Delta_1^\ell\sum_{j=1}^{\ell-1}\Delta_3^j\Delta_1^j\,d_j(\mu) \\
= &-\sum_{j=1}^{\ell-1} d_j(\mu)\left\{(\Delta_1^\ell\Delta_2^j)^2 + (\Delta_1^\ell\Delta_3^j)^2 + (\Delta_2^\ell\Delta_3^j)^2\right. \\
&+ (\Delta_2^\ell\Delta_1^j)^2 + (\Delta_3^\ell\Delta_1^j)^2 + (\Delta_3^\ell\Delta_2^j)^2 \\
&- 2\Delta_1^\ell\Delta_2^\ell\Delta_1^j\Delta_2^j - 2\Delta_2^\ell\Delta_3^\ell\Delta_2^j\Delta_3^j \\
&\left.- 2\Delta_3^\ell\Delta_1^\ell\Delta_3^j\Delta_1^j\right\} \\
= &-\sum_{j=1}^{\ell-1} d_j(\mu)\left\{(\Delta_1^\ell\Delta_2^\ell - \Delta_2^\ell\Delta_1^j)^2 + (\Delta_2^\ell\Delta_3^j - \Delta_3^\ell\Delta_2^j)^2 + (\Delta_3^\ell\Delta_1^j - \Delta_1^\ell\Delta_3^j)^2\right\}
\end{aligned}
$$

If we define K_j^ℓ by

$$K_j^\ell \triangleq (\Delta_1^\ell \Delta_2^j - \Delta_2^\ell \Delta_1^j)^2 + (\Delta_2^\ell \Delta_3^j - \Delta_3^\ell \Delta_1^j)^2 + (\Delta_3^\ell \Delta_1^j - \Delta_1^\ell \Delta_3^j)^2 \tag{C19}$$

then it is also written as

$$K_j^\ell = (\Delta^\ell \tilde{\Delta}^j)(\Delta^\ell \tilde{\Delta}^j)^T \tag{C20}$$

where

$$\tilde{\Delta}^j = \begin{bmatrix} 0 & -\Delta_3^j & \Delta_2^j \\ \Delta_3^j & 0 & -\Delta_1^j \\ -\Delta_2^j & \Delta_1^j & 0 \end{bmatrix} \tag{C21}$$

It is recognized that

$$K_j^\ell \geq 0 \tag{C22}$$

with the equality when $\Delta^\ell \tilde{\Delta}^j = 0$, implying that the two row matrices have the same direction, i.e., with some scalar α_j,

$$\Delta^j = \alpha_j \Delta^\ell$$

We also recognize that

$$K_j^\ell = K_\ell^j \tag{C23}$$

With K_j^ℓ thus defined, we may write $G_\ell^1(\mu)$ as

$$G_\ell^1(\mu) = -\sum_{j=1}^{\ell-1} d_j(\mu) K_j^\ell \tag{C24}$$

$$
\begin{aligned}
G_\ell^2(\mu) = {} & (\Delta_1^\ell)^2 \left\{ \Sigma_{22}\Sigma_{33} - \left(\Sigma_{23}\right)^2 \right\} \\
& + (\Delta_2^\ell)^2 \left\{ \Sigma_{33}\Sigma_{11} - \left(\Sigma_{31}\right)^2 \right\} \\
& + (\Delta_3)^2 \left\{ \Sigma_{11}\Sigma_{22} - \left(\Sigma_{12}\right)^2 \right\} \\
& + 2\Delta_1\Delta_2 \left\{ \Sigma_{13}\Sigma_{23} - \Sigma_{12}\Sigma_{33} \right\} \\
& + 2\Delta_2\Delta_3 \left\{ \Sigma_{21}\Sigma_{31} - \Sigma_{13}\Sigma_{11} \right\} \\
& + 2\Delta_3\Delta_1 \left\{ \Sigma_{32}\Sigma_{12} - \Sigma_{32}\Sigma_{11} \right\}
\end{aligned}
\tag{C25}
$$

By the Lagrange's identity [8],

$$
\begin{aligned}
\Sigma_{22}\Sigma_{33} - \left(\Sigma_{23}\right)^2 &= \left\{ \sum_{j=1}^{\ell-1} (\Delta_2^j)^2 d_j(\mu) \right\} \left\{ \sum_{j=1}^{\ell-1} (\Delta_3^j)^2 d_j(\mu) \right\} - \left\{ \sum_{\ell=1}^{\ell-1} \Delta_2^j \Delta_3^j d_j(\mu) \right\}^2 \\
&= \sum_{1\ i<j\le\ell-1} (\Delta_2^i\Delta_3^j - \Delta_3^i\Delta_2^j)^2 d_i(\mu) d_j(\mu)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
& \Sigma_{13}\Sigma_{23} - \Sigma_{12}\Sigma_{33} \\
&= \left(\sum_{j=1}^{\ell-1} \Delta_1^j\Delta_3^j \, d_j(\mu) \right) \left(\sum_{j=1}^{\ell-1} \Delta_2^j\Delta_3^j \, d_j(\mu) \right) - \left(\sum_{j=1}^{\ell-1} \Delta_1^j\Delta_2^j \, d_j(\mu) \right) \left(\sum_{j=1}^{\ell-1} (\Delta_3^j)^2 d_j(\mu) \right) \\
&= \sum_{1\le i<j\le\ell-1} (\Delta_3^i\Delta_1^j - \Delta_1^i\Delta_3^j)(\Delta_2^i\Delta_3^j - \Delta_3^i\Delta_2^j) d_i(\mu) d_j(\mu)
\end{aligned}
$$

We have similar relations for the other terms in Eq. (C25) so that

$$G_\ell^2(\mu) = \sum_{1\le i<j\le \ell-1} d_i(\mu)d_j(\mu)\Big\{(\Delta_1^\ell)^2(\Delta_2^i\Delta_3^j - \Delta_3^i\Delta_2^j)^2$$

$$+ (\Delta_2^\ell)^2(\Delta_3^i\Delta_1^j - \Delta_1^i\Delta_3^j)^2 + (\Delta_3^\ell)^2(\Delta_1^i\Delta_2^j - \Delta_2^i\Delta_1^j)^2$$

$$+2\Delta_1^\ell(\Delta_2^i\Delta_3^j - \Delta_3^i\Delta_2^j)\cdot\Delta_2^\ell(\Delta_3^i\Delta_1^j - \Delta_1^i\Delta_3^j)$$

$$+2\Delta_2^\ell(\Delta_3^i\Delta_1^j - \Delta_1^i\Delta_3^j)\cdot\Delta_3^\ell(\Delta_1^i\Delta_2^j - \Delta_2^i\Delta_1^j)$$

$$+2\Delta_3^\ell(\Delta_1^i\Delta_2^j - \Delta_2^i\Delta_1^j)\cdot\Delta_1^\ell(\Delta_2^i\Delta_3^j - \Delta_3^i\Delta_1^j)\Big\}$$

$$= \sum_{1\le i<j\le \ell-1} d_i(\mu)d_j(\mu)\Big\{\Delta_1^\ell(\Delta_2^i\Delta_3^j - \Delta_3^i\Delta_2^j)$$

$$+ \Delta_2^\ell(\Delta_3^i\Delta_1^j - \Delta_1^i\Delta_3^j) + \Delta_3^\ell(\Delta_1^i\Delta_2^j - \Delta_2^i\Delta_1^j)\Big\}^2$$

If we designate the square of the quantity in the braces by L_{ij}^ℓ, then we may write

$$G_\ell^2(\mu) = \sum_{1\le i<j\le \ell-1} d_i(\mu)d_j(\mu)L_{ij}^\ell \tag{C26}$$

where

$$L_{ij}^\ell = (\Delta^\ell\tilde{\Delta}^i\Delta^{jT})^2 \tag{C27}$$

Since $\Delta^\ell(\tilde{\Delta}^i\Delta^{jT}) = -\Delta^\ell(\tilde{\Delta}^j\Delta^{jT}) = -\Delta^i(\tilde{\Delta}^\ell\Delta^{jT})$, we have

$$L_{ij}^\ell = L_{ji}^\ell = L_{\ell j}^i \tag{C28}$$

It is obvious that

$$L_{ij}^\ell \ge 0 \tag{C29}$$

with the equality when $\Delta^\ell(\tilde{\Delta}^i\Delta^{jT}) = 0$, which takes place if the two of Δ^ℓ,Δ^i and Δ^j have the same direction or if the third vector is perpendicular to the cross-product of two vectors. Hence, if $K_j^\ell = 0$ then $L_{ij}^\ell = 0$ (not vice versa).

Substituting Eqs. (C18), (C24) and (C26) into Eq. (C17), we have

$$G_\ell(\mu) = \Delta^\ell \Delta^{\ell T} - \sum_{j=1}^{\ell-1} K_j d_j(\mu) + \sum_{1 \le i < j \le \ell-1} L_{ij}^\ell \, d_i(\mu) d_j(\mu) \tag{C30}$$

with the definitions of K_j^ℓ in Eq. (C19) and of L_{ij}^ℓ in Eq. (C27).

In what follows an alternative form of $g(\mu)$ is derived for the distinct eigenvalue case. That is, it is shown that if the σ_j's are all distinct, then $g(\mu)$ can be expressed in the form of

$$g(\mu) = 1 - \sum_{\ell=1}^{N} P_\ell d_\ell(\mu) \tag{C31}$$

where P_ℓ is a constant expressed in terms of d_{ij}, L_{ij}^ℓ and K_j^ℓ previously defined.

We start with collecting terms containing $d_\ell(\mu)$ in Eq. (C30) rewritten as

$$\begin{aligned} 1 - g(\mu) = {} & G_1(\mu) d_1(\mu) + \ldots + G_{\ell-1}(\mu) d_{\ell-1}(\mu) \\ & + G_\ell(\mu) d_\ell(\mu) + \ldots + G_N(\mu) d_N(\mu) \end{aligned}$$

Since $G_1(\mu), \ldots$ and $G_{\ell-1}(\mu)$ do not contain $d_\ell(\mu)$ by definition, we are concerned only with the terms typified by $G_k(\mu) d_k(\mu)$ with $k \ge \ell$

By Eq. (C17) with ℓ replaced by k,

$$G_k(\mu) d_k(\mu) = G_k^0 d_k(\mu) + G_k^1(\mu) d_k(\mu) + G_k^2(\mu) d_k(\mu)$$

where

$$G_k^0 \, d_k(\mu) = \Delta^k \Delta^{kT} d_k(\mu)$$

and in view of Eq. (B1),

$$
\begin{aligned}
G_k^1(\mu)\, d_k(\mu) &= -\sum_{j=1}^{k-1} d_k(\mu)\, d_j(\mu) K_j^k \\
&= -\sum_{j=1}^{k-1} \left\{ d_{jk} d_k(\mu) + d_{kj} d_j(\mu) \right\} K_j^k \\
&= -\, d_k(\mu) \sum_{j=1}^{k-1} d_{jk} K_j^k - \sum_{j=1}^{k-1} d_{kj} d_j(\mu) K_j^k
\end{aligned}
$$

and in view of Eq. (B2)

$$
\begin{aligned}
G_k^2(\mu)\, d_k(\mu) &= \sum_{1\le i<j\le k-1} d_i(\mu)\, d_j(\mu)\, d_k(\mu)\, L_{ij}^k \\
&= \sum_{1\le i<j<k-1} \left\{ d_{ji} d_{ki} d_i(\mu) + d_{kj} d_{ij} d_j(\mu) + d_{ik} d_{jk} d_k(\mu) \right\} L_{ij}^k \\
&= d_k(\mu) \sum_{1\le i<j\le k-1} d_{ik} d_{jk} L_{ij}^k \\
&\quad + \sum_{1\le i<j<k-1} \left\{ d_{ji} d_{ki} d_i(\mu) + d_{ij} d_{kj} d_j(\mu) \right\} L_{ij}^k
\end{aligned}
$$

For $k=\ell$, the coefficient of $d_\ell(\mu)$ is given by

$$
\Delta^\ell \Delta^{\ell T} - \sum_{j=1}^{\ell-1} d_{j\ell} K_j^\ell + \sum_{1\le i<j\le \ell-1} d_{i\ell} d_{j\ell} L_{ij}^\ell \quad , \quad k=\ell
$$

For $k > \ell$, the coefficient of $d_\ell(\mu)$ is

$$
\begin{aligned}
-d_{k\ell} K_\ell^k &+ \sum_{j=\ell+1}^{k-1} d_{j\ell} d_{k\ell} L_{\ell j}^k \\
&+ \sum_{i=1}^{\ell-1} d_{i\ell} d_{k\ell} L_{i\ell}^k \quad , \quad k = \ell+1,\ldots,N
\end{aligned}
$$

Summing the coefficients of all terms for $k \ge \ell$,

$$P_\ell = \Delta^\ell \Delta^{\ell T} - \sum_{j=1}^{\ell-1} d_{j\ell} K_j^\ell + \sum_{1 \le i < j \le \ell-1} d_{i\ell} d_{j\ell} L_{ij}^\ell$$

$$+ \sum_{k=\ell+1}^{N} \left\{ - d_{k\ell} K_\ell^k + \sum_{j=\ell+1}^{k-1} d_{j\ell} d_{k\ell} L_{\ell j}^k + \sum_{i=1}^{\ell-1} d_{i\ell} d_{k\ell} L_{i\ell}^k \right\} \tag{C32}$$

Changing the dummy indices for summation into i,j with $i < j$ is possible, and utilizing the identities (Eqs. (C23) and (C28))

$$L_{ij}^\ell = L_{ji}^\ell = L_{\ell j}^i \quad \text{and} \quad K_j^\ell = K_\ell^j$$

we may rewrite the double summations in Eq. (C32) as follows:

$$\sum_{k=\ell+1}^{N} (-d_{k\ell} K_\ell^i) = - \sum_{j=\ell+1}^{N} d_{j\ell} K_\ell^j = - \sum_{j=\ell+1}^{N} d_{j\ell} K_j^\ell$$

and

$$\sum_{k=\ell+1}^{N} \sum_{j=\ell+1}^{k-1} d_{j\ell} d_{k\ell} L_{\ell j}^k$$

$$= \sum_{\ell < j < k \le N} d_{j\ell} d_{k\ell} L_{\ell j}^k$$

$$= \sum_{\ell < i < j \le N} d_{i\ell} d_{j\ell} L_{\ell i}^j$$

$$= \sum_{\ell < i < j \le N} d_{i\ell} d_{j\ell} L_{ij}^\ell$$

which corresponds to the region 2 in Fig. C1 implying that in view of Eq. (B4) each term is positive and so is the summation.

Moreover

$$\sum_{k=\ell+1}^{N} \sum_{i=1}^{\ell-1} d_{i\ell} d_{k\ell} L_{i\ell}^{k}$$

$$= \sum_{1\le i<\ell<k\le N} d_{i\ell} d_{k\ell} L_{i\ell}^{k}$$

$$= \sum_{1<i<\ell<j\le N} d_{i\ell} d_{j\ell} L_{ij}^{\ell}$$

which corresponds to the region 3 in Fig. C1, implying that this summation is negative. Finally

$$P_\ell = \Delta^\ell \Delta^{\ell T} - \sum_{j=1}^{\ell-1} d_{j\ell} K_j^\ell + \sum_{1\ i<j\le\ell} d_{i\ell} d_{j\ell} L_{ij}^\ell$$

$$- \sum_{j=\ell+1}^{N} d_{j\ell} K_j^\ell + \sum_{\ell<i<j\le N} d_{i\ell} d_{j\ell} L_{ij}^\ell$$

$$+ \sum_{1\le i<\ell<j\le N} d_{i\ell} d_{j\ell} L_{ij}^\ell \tag{C33}$$

We can now identify which term is positive or negative, so if we decompose P_ℓ into

$$P_\ell = P_\ell^+ - P_\ell^- \tag{C34}$$

where P_ℓ^+ and P_ℓ^- are the collection of terms which are positive and negative, respectively, then

$$P_\ell^+ = \Delta^\ell \Delta^{\ell T} - \sum_{j=1}^{\ell-1} d_{j\ell} K_j^\ell + \sum_{1\le i<j<\ell} d_{i\ell} d_{j\ell} L_{ij}^\ell$$

$$+ \sum_{\ell<i<j<N} d_{i\ell} d_{j\ell} L_{ij}^\ell \tag{C35a}$$

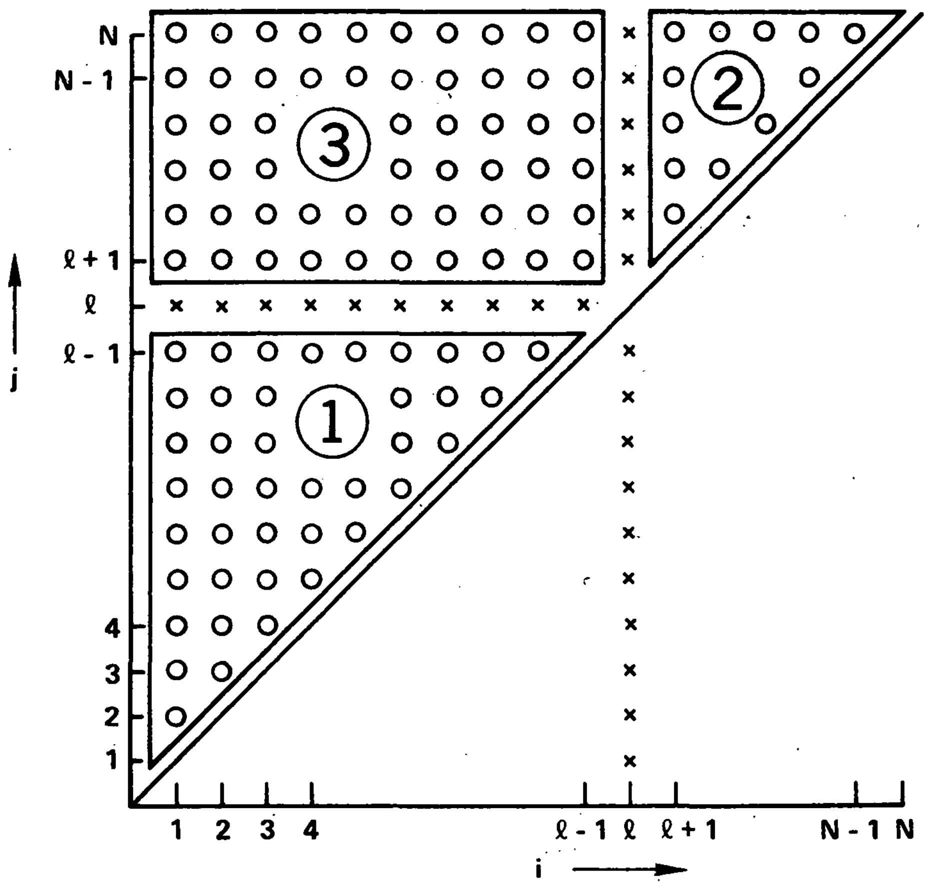
and

$$P_\ell^- = + \sum_{j=\ell+1}^{N} d_{j\ell} K_j^\ell - \sum_{1 \le i < \ell < j \le N} d_{i\ell} d_{j\ell} L_{ij}^\ell \tag{C35b}$$

In a more compact but less informative form, P_ℓ may be rewritten as

$$P_\ell = \Delta^\ell \Delta^{\ell T} - \sum_{\substack{j=1 \\ j \ne \ell}}^{N} d_{j\ell} K_j^\ell + \sum_{\substack{1 \le i < j \le N \\ i,j \ne \ell}} d_{i\ell} d_{j\ell} L_{ij}^\ell \tag{C36}$$

Fig. C1 illustrates the implication of the summation in Eq. (C36).



REGION	SUMMATION LIMITS	$d_i(\sigma_\ell^2)$	$d_j(\sigma_\ell^2)$	$d_i(\sigma_\ell^2)\,d_j(\sigma_\ell^2)$
1	$1 \leqslant i < j < \ell$	+	+	+
2	$\ell < i < j \leqslant N$	–	–	+
3	$1 \leqslant i < \ell < j \leqslant N$	+	–	–

Figure C1. Implication of $\displaystyle\sum_{\substack{1 \leqslant i < j \leqslant N \\ i,j \neq \ell}}$

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Chapter 2. Determination of Poles and Zeros of Transfer Functions for Flexible Spacecraft Attitude Control

ABSTRACT. A method is presented for determining the poles and zeros of the transfer function describing the attitude dynamics of a flexible spacecraft characterized by hybrid coordinate equations. It is shown that the problem reduces to that of finding the eigenvalues of matrices which are constructed by simple manipulations of the inertia and modal parameter matrices. Particular emphasis is put on the determination of the zeros, which depend also on the sensor and/or actuator location. The established procedure will be useful for numerical determination on the digital computer.

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INTRODUCTION

The system discussed in Chapter 1 is a multi-variable linear constant coefficient system with three inputs and three outputs and a large number of states. Such a system is sometimes conveniently described by a "transfer function matrix," in which the determination of the poles and zeros is of primary importance in designing a conventional compensator by the classical approach or an optimal controller by modern techniques.

The determination of the poles of the system is relatively simple from a computational point of view, because it is just a calculation of the system eigenvalues, for which computer algorithms and/or programs are well established (see also Chapter 1 for eigenvalue characterization).

The calculation of the zeros, however, is not an easy task even numerically, and even the definition of the "zeros" is not unique. We will first survey some of the recent papers dealing with this problem, and thereby demonstrate the difficulty of the zero determination problem.

For the purpose of this survey, we start with a more general description of the system (after Laplace transformation* of Eq. (5) in Chapter 1 with $X(0) = 0$)

$$sX(s) = AX(s) + B\,u(s)$$

with the observation equation (a)

$$y(s) = C\,X(s)$$

whose transfer function matrix (sometimes called "frequency response" [1]) is given by [1],[2]

*The Laplace transform of a variable, say $X(t)$, is written as $X(s)$, and when the distinction between $X(t)$ and $X(s)$ is clear from the context the arguments may be omitted.

so that
$$G(s) = C\,(sU_N - A)^{-1}B$$
$$y(s) = G(s)\,u(s) \tag{b}$$

Definition 1: The zeros of the transfer function are those of the individual entries in G(s) which give the relationship between the r-th input u_r and the j-th output y_j.

This form of G(s) (sometimes called the first form) and the definition that follows seem to be standard. Rosenbrock, [3],[4] however, insists that the following form (called the second form) is more general.

Consider the system

$$\begin{aligned} J(s)\,X(s) &= K(s)\,u(s) \\ y(s) &= L(s)\,X(s) + M(s)\,u(s) \end{aligned} \tag{c}$$

where in particular we may have, as in Eq. (a),

$$J(s) = s\,U_n - A,\ K(s) = B,\ L(s) = C,\ \text{and}\ \ M(s) = 0$$

and define the system matrix in polynomial form by

$$P(s) = \left[\begin{array}{c|c} J(s) & K(s) \\ \hline -L(s) & M(s) \end{array}\right]$$

which has Smith normal form [4] (uniquely determined)

$$N(s) = \begin{bmatrix} \mathrm{diag}(\varepsilon_j(s)) & 0 \\ 0 & 0 \end{bmatrix}$$

Definition 2: The zeros of the system are the zeros of the polynomial $\varepsilon_j(s)$ taken all together.

The formulation by Rosenbrock (Eq. (c)) is certainly more general than Eq. (a) since the former permits the case when $|T(s)| \equiv 0$ while the latter does not. Usually, the zeros are not identical for the two definitions, except for the single-input, single-output case.

In the problem under consideration, $|T(s)| \not\equiv 0$ so that it is not necessary to take the form of Eq. (c).

In the case where the number of inputs equals the number of outputs, as in the present problem, we may consider the determinant of G(s), since it is square. In this case, the zeros of this determinant are identical to the zeros of the definition 2, because the elementary operations used in forming the Smith normal form do not produce excess zeros and the determinant remains unchanged except for a scalar multiplier.

Brockett [2] has presented a way of zero determination by deriving the "inverse equation" whose poles include the zeros of the original system; hence the problem reduces to the eigenvalue problem. However, his results are limited to the single-input, single-output case, and, as pointed out by Davison [5], the formulation of the matrix representing the inverse system *is computationally difficult, because the prescription for forming the matrix* relies on logic statements such as $\alpha = 0$ if $d = 0$ and $\alpha = 1$ if $d \neq 0$, and because of the finite word length of the digital computer, a subjective judgment must always be made to determine if $d = 10^{-8}$ or $d = 10^{-20}$ may be taken to equal zero. Also the system must be controllable and observable. In addition this approach supplies excess roots at the origin which must be sorted out by judgment.

Davison [5] has proposed a computational method and computer algorithm for zero calculation, which also reduces it to eigenvalue calculation of a matrix formed by Cramer's method. Although this method also supplies excess roots, it is always possible by repeating some steps of the calculation to determine which roots should be rejected. The advantage of this method lies in the fact that this approach can reduce the inaccuracy due to word length limitation and round-off error of the computer. On the other hand, this method has a disadvantage for large-scale systems because it requires the solution of an eigenproblem of higher dimension than truly necessary, and hence the number of excess zeros can be very large. In addition repetition of some calculation steps is not desirable. Kropholler and Neale [6] have discussed a similar method.

The method proposed by Guidorzi and Terragni [7] operates only on a subsystem of minimal dimension completely describing the dynamical behavior between the input and output of interest. The zeros are finally determined as the roots of a polynomial of the minimal dimension (no excess zeros are produced). Although some of the difficulties in Davison's method are overcome, the Guidorzi-Terragni approach requires the number of independent vectors, which may be dependent on subjective judgment, and the calculation of the zeros of a polynomial can be less accurate than that of eigenvalue calculation for some problems.

Yokoyama [8] has discussed a method of obtaining transfer functions by transformation to a phase variable canonical form. He has overcome the difficulty in the inversion of (sU - A) in Eq. (b), but the calculation of the canonical form is another burdensome task, so that his method does not seem to be appropriate for the present problem.

In this paper, we will show that the zero determination problem reduces to eigenvalue calculations of a matrix. The construction of the matrix is discussed in detail for the cases when the sensors and/or the actuators are mounted on the primary body. The method presented here does not require any repetition of calculation steps or any subjective judgment, unlike most of the alternative methods noted.

We will consider primarily the zero determination problem in the sense of Definition 1. We will also discuss the zeros of the determinant of the G(s) which is related to Definition 2.

DERIVATION OF TRANSFER FUNCTION

As in Eqs. (1) of Chapter 1, we start with describing the system in terms of hybrid coordinates as

$$I^{*}\ddot{\theta} - \delta^{T}\ddot{\eta} = T \tag{1a}$$

$$\ddot{\eta} + D\dot{\eta} + \sigma^{2}\eta - \delta\ddot{\theta} = \mathscr{L}_c' T \tag{1b}$$

with the observation equation [9]

$$y = \theta + \mathscr{L}_o'^{T}\eta \tag{1c}$$

where

$$\mathscr{L}_c' \overset{\Delta}{=} \phi^{T} \mathscr{L}_c \tag{2a}$$

and

$$\mathscr{L}_o' \overset{\Delta}{=} \phi^{T} \mathscr{L}_o \tag{2b}$$

establish the actuator and sensor locations, repsectively, and all other matrices are defined as previously.

For convenience of later discussion, we partition the matrices δ, $\mathscr{L}_c'$ and $\mathscr{L}_o'$ as

$$\delta = [a^1 \quad a^2 \quad a^3] \tag{3a}$$

$$\mathscr{L}_c' = [b^1 \quad b^2 \quad b^3] \tag{3b}$$

and

$$\mathscr{L}_o' = [c^1 \quad c^2 \quad c^3] \tag{3c}$$

where a^j, b^j and c^j (j=1,2,3) are N×1 matrices.

Laplace transformation of Eqs. (1) with the zero initial state (i.e. $\theta=0$, $\dot{\theta}=0$, $\eta=0$ and $\dot{\eta}=0$) yields

$$s^2 I^*\theta - s^2\delta^T\eta = T \tag{4a}$$

$$(s^2U_N + sD + \sigma^2)\eta - s^2\delta\theta = \mathscr{L}_c'T \tag{4b}$$

$$y = \theta + \mathscr{L}_o'^T\eta \tag{4c}$$

Define the N×N matrix Q(s) by

$$Q(s) = s^2(s^2U_N + sD + \sigma^2)^{-1} , \tag{5}$$

then from Eq. (4b)

$$\eta = Q(s)\delta\theta + \frac{1}{s^2} Q(s) \; \mathscr{L}_c' \; T \tag{6a}$$

and from Eq. (4a) with Eq. (6a) substituted

$$s^2I^*\theta - s^2\delta^TQ(s)\delta\theta = T + \delta^T \; Q(s) \; \mathscr{L}_c' \; T$$

or

$$s^2(I^* - \delta^TQ(s)\delta)\theta = \left\{ U_3 + \delta^TQ(s)\mathscr{L}_c' \right\} T$$

or

$$\theta = \frac{1}{s^2} \left\{ I^* - \delta^TQ(s)\delta \right\}^{-1} \left\{ U_3 + \delta^TQ(s) \; \mathscr{L}_c' \right\} T \tag{6b}$$

From Eqs. (6a) and (6b),

$$\eta = \frac{1}{s^2} \left[Q(s)\delta \left\{ I^* - \delta^TQ(s)\delta \right\}^{-1} \left\{ U_3 + \delta^TQ(s) \; \mathscr{L}_c' \right\} + Q(s)\mathscr{L}_c' \right] T \tag{6c}$$

Substituting Eqs. (6b) and (6c) into Eq. (4c) yields

$$y = \frac{1}{s^2} \left[\left\{ U_3 + \mathscr{L}_o'^TQ(s)\delta \right\} \left\{ I^* - \delta^TQ(s)\delta \right\}^{-1} \cdot \left\{ U_3 + \delta^TQ(s) \; \mathscr{L}_c' \right\} + \mathscr{L}_o'^TQ(s) \; \mathscr{L}_c' \right] T \tag{7}$$

Thus, we may define the transfer function G(s) from the control input T to the sensor output, excluding the poles at the origin, by

$$G(s) \triangleq \left\{U_3 + \mathscr{L}_o'^T Q(s)\delta\right\}\left\{I^* - \delta^T Q(s)\delta\right\}^{-1}\left\{U_3 + \delta^T Q(s)\,\mathscr{L}_c'\right\} + \mathscr{L}_o'^T Q(s)\,\mathscr{L}_c' \tag{8a}$$

and denote the α-β element of G(s) by $G_{\alpha\beta}(s)$, i.e.,

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & G_{13}(s) \\ G_{21}(s) & G_{22}(s) & G_{23}(s) \\ G_{31}(s) & G_{32}(s) & G_{33}(s) \end{bmatrix} \tag{8b}$$

Hence, the purpose is to establish a procedure to determine the poles and zeros of $G_{\alpha\beta}(s)$.

We will examine the following cases:

(1) $\mathscr{L}_c' = \mathscr{L}_o' = 0$

meaning that the actuators and the sensors are attached to the primary body and

(2) $\mathscr{L}_c' = 0$ and $\mathscr{L}_o' \neq 0$

meaning that the actuators are on the primary body and the sensors on the sub-body characterized by $\mathscr{L}_o'$ and

(3) $\mathscr{L}_c' \neq 0$ and $\mathscr{L}_o = 0$

meaning that the actuators are on the sub-body characterized by $\mathscr{L}_c$ and the sensors on the primary body.

The reason for omitting the general case when $\mathscr{L}_c' \neq 0$ and $\mathscr{L}_o' \neq 0$ may may be justified by the possibility that we may consider either of these sub-bodies as the primary body.

INVERSION OF $(I^* - \delta^T Q(s)\delta)$

Let $F(s) = [F_{\alpha\beta}(s)]$ be the adjoint matrix of $(I^* - \delta^T Q(s)\delta)$. Then

$$(I^* - \delta^T Q(s)\delta)^{-1} = \frac{F(s)}{|I^* - \delta^T Q(s)\delta|} \tag{9}$$

Since, with the definitions of Eqs. (3), $I^* - \delta^T Q(s)\delta$ takes the form

$$I^* - \delta^T Q(s)\delta = \begin{bmatrix} I_{11} - a^{1T}Qa^1 & I_{12} - a^{1T}Qa^2 & I_{13} - a^{1T}Qa^3 \\ I_{12} - a^{2T}Qa^1 & I_{22} - a^{2T}Qa^2 & I_{23} - a^{2T}Qa^3 \\ I_{13} - a^{3T}Qa^1 & I_{23} - a^{3T}Qa^2 & I_{33} - a^{3T}Qa^3 \end{bmatrix}$$

Then the α-β element, $F_{\alpha\beta}(s)$, of $F(s)$ is expressed by

$$F_{\alpha\beta}(s) = (-1)^{\alpha+\beta}|I^{\alpha\beta} - \bar{\delta}^{\beta T} Q\, \bar{\delta}^{\alpha}| \qquad \alpha,\beta = 1,2,3 \tag{10}$$

where

$$I^{11} \triangleq \begin{bmatrix} I_{22} & I_{23} \\ I_{23} & I_{33} \end{bmatrix}, \quad I^{22} \triangleq \begin{bmatrix} I_{11} & I_{13} \\ I_{13} & I_{33} \end{bmatrix}, \quad I^{33} \triangleq \begin{bmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{bmatrix}$$
$$I^{12} \triangleq \begin{bmatrix} I_{12} & I_{13} \\ I_{23} & I_{33} \end{bmatrix}, \quad I^{13} \triangleq \begin{bmatrix} I_{12} & I_{13} \\ I_{22} & I_{23} \end{bmatrix}, \quad I^{23} \triangleq \begin{bmatrix} I_{11} & I_{13} \\ I_{12} & I_{23} \end{bmatrix} \tag{11}$$

and

$$I^{21} = I^{12T}, \quad I^{31} = I^{13T}, \quad I^{32} = I^{23T}$$

and

$$\bar{\delta}^1 \triangleq [a^2\ a^3], \quad \bar{\delta}^2 \triangleq [a^1\ a^3], \quad \bar{\delta}^3 \triangleq [a^1\ a^2] \tag{12}$$

Therefore, we have obtained the inverse

$$(I^* - \delta^T Q(s)\delta)^{-1} = \left[(-1)^{\alpha+\beta} \frac{|I^{\alpha\beta} - \bar{\delta}^{\beta T} Q(s)\bar{\delta}^{\alpha}|}{|I^* - \delta^T Q(s)\delta|} \right] \tag{13}$$

It is noted that since $I^* - \delta^T Q(s)\delta$ is symmetric, the inverse is also symmetric, implying

$$F_{\alpha\beta}(s) = F_{\beta\alpha}(s)$$

This is confirmed for example by considering the case of $\alpha=1$, $\beta=2$ as follows

$$F_{12}(s) = - \begin{vmatrix} I_{12} - \delta^{2T}Q\delta^1 & I_{13} - \delta^{3T}Q\delta^1 \\ I_{23} - \delta^{3T}Q\delta^2 & I_{33} - \delta^{3T}Q\delta^3 \end{vmatrix}$$

$$= - \begin{vmatrix} I_{12} - \delta^{2T}Q\delta^1 & I_{23} - \delta^{2T}Q\delta^3 \\ I_{13} - \delta^{3T}Q\delta^1 & I_{33} - \delta^{3T}Q\delta^3 \end{vmatrix}$$

$$= - F_{21}$$

DETERMINATION OF POLES AND ZEROS

(1) In the case of $\mathscr{L}'_c = \mathscr{L}'_o = 0$, from Eq. (8a),

$$G(s) = (I^* - \delta^T Q(s)\delta)^{-1} \tag{14a}$$

so that, with Eq. (13), the transfer function, $G_{\alpha\beta}(s)$, from the β-th input to the α-th output becomes

$$G_{\alpha\beta}(s) = (-1)^{\alpha+\beta} \frac{|I^{\alpha\beta} - \bar{\delta}^{\beta T} Q(s)\bar{\delta}^{\alpha}|}{|I^* - \delta^T Q(s)\delta|} \tag{14b}$$

In what follows, it will be shown that the determinants in Eq. (14b) reduce to characteristic equations of (2N×2N) matrices. This reduction comes from the observation that the poles of the transfer function must be the eigenvalues of the matrix A in Eq. (1) or its equivalent form.

Consider the eigenvalues of the matrix of reduced form (Eq. (19a) of Chapter 1)

$$\mathscr{A} = \left[\begin{array}{c|c} 0 & U_N \\ \hline -M_1\sigma^2 & -M_1 D \end{array}\right] \tag{15}$$

where

$$M_1 = (U_N - \delta(I^*)^{-1}\delta^T)^{-1} \tag{16}$$

The eigenvalues, s, of $\mathscr{A}$ are the roots of

$$|sU_{2N} - \mathscr{A}| = 0 \tag{17}$$

But

$$\begin{aligned}
&|sU_{2N} - \mathscr{A}| \\
&= \begin{vmatrix} sU_N & -U_N \\ M_1\sigma^2 & sU_N + M_1 D \end{vmatrix} \\
&= |sU_N| \cdot |sU_N + M_1 D + M_1\sigma^2(sU_N)^{-1}U_N| \\
&= |M_1| \cdot |s^2 M_1^{-1} + sD + \sigma^2| \\
&= |M_1| \cdot |s^2(U_N - \delta(I^*)^{-1}\delta^T) + sD + \sigma^2| \\
&= |M_1| \cdot |s^2 U_N + sD + \sigma^2 - s^2\,\delta(I^*)^{-1}\delta^T| \\
&= |M_1| \cdot |s^2 U_N + sD + \sigma^2| \cdot |U_N - Q(s)\delta(I^*)^{-1}\delta^T| \\
&= |M_1| \cdot |s^2 U_N + sD + \sigma^2| \cdot |I^*|^{-1} \cdot |I^* - \delta^T Q(s)\delta|
\end{aligned}$$

Hence

$$|I^* - \delta^T Q(s)\delta| = |M_1|^{-1}|s^2 U_N + sD + \sigma^2|^{-1}|I^*|\,|sU_{2N} - \mathscr{A}| \tag{18}$$

Similarly, if we define the 2N×2N matrix $\mathscr{A}^{\alpha\beta}$ by

$$\mathscr{A}^{\alpha\beta} \triangleq \left[\begin{array}{c|c} 0 & U_N \\ \hline -M^{\alpha\beta}\sigma^2 & -M^{\alpha\beta}D \end{array}\right] \tag{19}$$

where

$$M^{\alpha\beta} \triangleq \left[U_N - \bar{\delta}^{\alpha}(I^{\alpha\beta})^{-1}\,\bar{\delta}^{\beta T}\right]^{-1} \tag{20}$$

then the numerator of Eq. (14b) is written

$$\begin{aligned} &|I^{\alpha\beta} - \bar{\delta}^{\beta T}Q(s)\,\bar{\delta}^{\alpha}| \\ &\quad = |M^{\alpha\beta}|^{-1}|s^2U_N + sD + \sigma^2|^{-1}|I^{\alpha\beta}|\;|sU_{2N} - \mathscr{A}^{\alpha\beta}| \end{aligned} \tag{21}$$

Substituting Eqs. (18) and (21) into Eq. (14), we obtain

$$G_{\alpha\beta}(s) = (-1)^{\alpha+\beta}\,\frac{|M^{\alpha\beta}|^{-1}|s^2U_N+sD+\sigma^2|\cdot|I^{\alpha\beta}|\cdot|sU_{2N} - \mathscr{A}^{\alpha\beta}|}{|M_1|^{-1}|s^2U_N+sD+\sigma^2|\cdot|I^*|\cdot|sU_{2N} - \mathscr{A}|} \tag{22}$$

Noting that

$$\begin{aligned} |M_1|^{-1}|I^*| &= |I^*|\;|U_N - \delta(I^*)^{-1}\delta^T| \\ &= |I^*|\cdot|U_3 - (I^*)^{-1}\delta^T\delta| \\ &= |I^* - \delta^T\delta| \end{aligned}$$

and similarly

$$|M^{\alpha\beta}|^{-1}|I^*| = |I^{\alpha\beta} - \bar{\delta}^{\beta T}\bar{\delta}^{\alpha}|$$

we have

$$G_{\alpha\beta}(s) = (-1)^{\alpha+\beta}\,\frac{|I^{\alpha\beta} - \bar{\delta}^{\beta T}\bar{\delta}^{\alpha}|}{|I^* - \delta^T\delta|}\;\frac{|sU_N - \mathscr{A}^{\alpha\beta}|}{|sU_N - \mathscr{A}|} \tag{23}$$

Thus, the poles and the zeros are the eigenvalues of the matrices $\mathscr{A}$ (Eq. (15)) and $\mathscr{A}^{\alpha\beta}$ (Eq. (19)), respectively. The matrix $\mathscr{A}^{\alpha\beta}$ is constructed by the definitions of Eqs. (11), (12) and (19), but since it contains the inversion of $(I^{\alpha\beta})$ and $(U_N - \bar{\delta}^{\alpha T}(I^{\alpha\beta})^{-1}\bar{\delta}^{\beta})$ Eq. (23) is restricted to the case when

$$|I^{\alpha\beta}| \neq 0 \tag{24}$$

and

$$|U_N - \bar{\delta}^{\alpha}(I^{\alpha\beta})^{-1}\ \bar{\delta}^{\beta}| \neq 0 \tag{25}$$

However, Eqs. (24) and (25) always hold for the case of $\alpha = \beta$, implying that the poles and zeros are always calculated by Eq. (23) if we consider the transfer functions from the α-th input to the α-th output (α=1,2,3) which are usually of primary importance. This comes from the symmetry and positive definiteness of the matrices I^* and $U_N - \delta(I^*)^{-1}\delta$, which guarantee that every principal minor of these matrices is also symmetric and positive definite (by Sylvester's theorem) with the recognition that $I^{\alpha\beta}$ is the $\alpha\beta$ minor of I^* and $|U_N - \bar{\delta}^{\alpha}(I^{\alpha\beta})^{-1}\ \bar{\delta}^{\beta}|$ is related to the $\alpha\beta$ minor of $U_3 - (I^*)^{-1}\delta^T\delta$.

(2) In the case of $\mathscr{L}'_c = 0$ and $\mathscr{L}'_o \neq 0$, from Eq. (8a)

$$G(s) = \{U_3 + \mathscr{L}_o'^T Q(s)\delta\}\ \{I^* - \delta^T Q(s)\delta\}^{-1} \tag{26}$$

By partitioning of Eqs. (3a) and (3c),

$$\begin{aligned} & U_3 + \breve{\mathscr{L}}_o'^T Q(s)\delta \\ &= U_3 + \begin{bmatrix} c^{1T} \\ c^{2T} \\ c^{3T} \end{bmatrix} Q(s)\,[a^1\ a^2\ a^3] \\ &= \begin{bmatrix} 1 + c^{1T}Qa^1 & c^{1T}Qa^2 & c^{1T}Qa^3 \\ c^{2T}Qa^1 & 1 + c^{2T}Qa^2 & c^{2T}Qa^3 \\ c^{3T}Qa^1 & c^{2T}Qa^3 & 1 + c^{3T}Qa^3 \end{bmatrix} \end{aligned} \tag{27}$$

Carrying out the matrix multiplication in Eq. (26) with Eq. (9) substituted yields

$$G_{\alpha\beta}(s) = \frac{(-1)^{\alpha+\beta}}{|I^* - \delta^T Q\delta|} \sum_{\gamma=1}^{3} (\delta_{\alpha\gamma} + c^{\alpha T} Q a^{\gamma}) F_{\gamma\beta}(s) \tag{28}$$

where $\delta_{\alpha\gamma}$ denotes the Kronecker delta defined by

$$\delta_{\alpha\gamma} = \begin{cases} 1 & \text{if } \alpha = \gamma \\ 0 & \text{if } \alpha \neq \gamma \end{cases}$$

We record $G_{\alpha\beta}(s)$ for $\alpha,\beta = 1,2,3$ as follows, deleting the factors $|I^* - \delta^T Q\delta|$ and $(-1)^{\alpha+\beta}$

$$\begin{aligned}
G_{11}&: (1 + c^{1T}Qa^1)\ F_{11} + c^{1T}Qa^2\ F_{21} + c^{1T}Qa^3\ F_{31} \\
G_{12}&: (1 + c^{1T}Qa^1)\ F_{12} + c^{1T}Qa^2\ F_{22} + c^{1T}Qa^3\ F_{32} \\
G_{13}&: (1 + c^{1T}Qa^1)\ F_{13} + c^{1T}Qa^2\ F_{23} + c^{1T}Qa^3\ F_{33} \\
G_{21}&: (c^{2T}Qa^1F_{11} + (1 + c^{2T}Qa^2)\ F_{21} + c^{2T}Qa^3\ F_{31} \\
G_{22}&: c^{2T}Qa^1\ F_{12} + (1 + c^{2T}Qa^2)\ F_{22} + c^{2T}Qa^3\ F_{32} \\
G_{23}&: c^{2T}Qa^1\ F_{13} + (1 + c^{2T}Qa^2)\ F_{23} + c^{2T}Qa^3\ F_{33} \\
G_{31}&: c^{3T}Qa^1\ F_{11} + c^{3T}Qa^2\ F_{21} + (1 + c^{3T}Qa^3)\ F_{31} \\
G_{32}&: c^{3T}Qa^1\ F_{12} + c^{3T}Qa^2\ F_{22} + (1 + c^{3T}Qa^3)\ F_{32} \\
G_{33}&: c^{3T}Qa^1\ F_{13} + c^{3T}Qa^2\ F_{23} + (1 + c^{3T}Qa^3)\ F_{33}
\end{aligned}$$

Note: $F_{\alpha\beta} = F_{\beta\alpha}$

It is easy to verify that Eq. (28) may be rewritten

$$G_{\alpha\beta}(s) = (-1)^{\alpha+\beta} \frac{|\hat{I}^{\alpha\beta} - \delta^T Q\hat{\delta}^{\alpha\beta}|}{|I^* - \delta^T Q\delta|} \tag{29}$$

where $\hat{I}^{\alpha\beta}$ is the 3×3 matrix formed by replacing the β^{th} column of I^* with the 3×1 matrix

$$\begin{bmatrix} \delta_{\alpha 1} \\ \delta_{\alpha 2} \\ \delta_{\alpha 3} \end{bmatrix}$$

and $\hat{\delta}^{\alpha\beta}$ is the N×3 matrix formed by replacing the β-th column of δ with $(-c^{\alpha})$. We record all the $\hat{I}^{\alpha\beta}$ and $\hat{\delta}^{\alpha\beta}$ in the following.

$$\hat{I}^{11} = \begin{bmatrix} 1 & I_{12} & I_{13} \\ 0 & I_{22} & I_{23} \\ 0 & I_{23} & I_{33} \end{bmatrix}, \quad \hat{I}^{12} = \begin{bmatrix} I_{11} & 1 & I_{13} \\ I_{12} & 0 & I_{23} \\ I_{13} & 0 & I_{33} \end{bmatrix}, \quad \hat{I}^{13} = \begin{bmatrix} I_{11} & I_{12} & 1 \\ I_{12} & I_{22} & 0 \\ I_{13} & I_{23} & 0 \end{bmatrix}$$

$$\hat{I}^{21} = \begin{bmatrix} 0 & I_{12} & I_{13} \\ 1 & I_{22} & I_{23} \\ 0 & I_{23} & I_{33} \end{bmatrix}, \quad \hat{I}^{22} = \begin{bmatrix} I_{11} & 0 & I_{13} \\ I_{12} & 1 & I_{23} \\ I_{13} & 0 & I_{33} \end{bmatrix}, \quad \hat{I}^{23} = \begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 1 \\ I_{13} & I_{23} & 0 \end{bmatrix}$$

$$\hat{I}^{31} = \begin{bmatrix} 0 & I_{12} & I_{13} \\ 0 & I_{22} & I_{23} \\ 1 & I_{23} & I_{33} \end{bmatrix}, \quad \hat{I}^{32} = \begin{bmatrix} I_{11} & 0 & I_{13} \\ I_{12} & 0 & I_{23} \\ I_{13} & 1 & I_{33} \end{bmatrix}, \quad \hat{I}^{33} = \begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ I_{13} & I_{23} & 1 \end{bmatrix} \tag{30}$$

$$\hat{\delta}^{11} = [-c^1 \quad a^2 \quad a^3] \tag{31a}$$

$$\hat{\delta}^{12} = [a^1 \quad -c^1 \quad a^3] \tag{31b}$$

$$\hat{\delta}^{13} = [a^1 \quad a^2 \quad -c^1] \tag{31c}$$

$$\hat{\delta}^{21} = [-c^2 \quad a^2 \quad a^3] \tag{31d}$$

$$\hat{\delta}^{22} = [a^1 \quad -c^2 \quad a^3] \tag{31e}$$

$$\hat{\delta}^{23} = [a^1 \quad a^2 \quad -c^2] \tag{31f}$$

$$\hat{\delta}^{31} = [-c^3 \quad a^2 \quad a^3] \tag{31g}$$

$$\hat{\delta}^{32} = [a^1 \quad -c^3 \quad a^3] \tag{31h}$$

$$\hat{\delta}^{33} = [a^1 \quad a^2 \quad -c^3] \tag{31i}$$

Eq. (29) is confirmed for a specific combination of indices below:

If $\alpha = \beta = 1$, then

$$G_{11}(s)\cdot|*| = (-1)^2\left|\begin{bmatrix}1 & I_{12} & I_{13}\\ 0 & I_{22} & I_{23}\\ 0 & I_{23} & I_{33}\end{bmatrix} - \begin{bmatrix}a^{1T}\\ a^{2T}\\ a^{3T}\end{bmatrix} Q[-c^1,a^2,a^3]\right|$$

$$= \begin{vmatrix}1 + a^{1T}Qc^1 & I_{12} - a^{1T}Qa^2 & I_{13} - a^{1T}Qa^3\\ a^{2T}Qc^1 & I_{22} - a^{2T}Qa^2 & I_{23} - a^{2T}Qa^3\\ a^{3T}Qc^1 & I_{23} - a^{3T}Qa^2 & I_{33} - a^{3T}Qa^3\end{vmatrix}$$

$$= (1 + a^{1T}Qc^1)\ F_{11} + a^{2T}Qc^1\ F_{12} + a^{3T}Qc^1\ F_{13}$$

$$= (1 + c^{1T}Qa^{1T})\ F_{11} + c^{1T}Qa^2\ F_{21} + c^{1T}Qa^3\ F_{31}$$

$$= \sum_{\gamma=1}^{3}(\delta_{1\gamma} + c^{1T}Q\ a^{\gamma})F_{\gamma 1}$$

which is identical to Eq. (28) with $\alpha = \beta = 1$.

Recognizing the similarity of Eq. (29) to Eq. (14), we may rewrite (as in Eqs. (21) and (23))

$$|\hat{I}^{\alpha\beta} - \delta^T Q\ \hat{\delta}^{\alpha\beta}|$$
$$= |\hat{M}^{\alpha\beta}|^{-1}|s^2U_N + sD + \sigma^2|^{-1}\cdot|\hat{I}^{\alpha\beta}|\cdot|sU_{2N} - \hat{\mathscr{A}}^{\alpha\beta}|$$

so that

$$G_{\alpha\beta}(s) = (-1)^{\alpha+\beta}\frac{|\hat{I}^{\alpha\beta} - \delta^T\hat{\delta}^{\alpha\beta}|}{|I^* - \delta^T\delta|}\ \frac{|sU_{2N} - \hat{\mathscr{A}}^{\alpha\beta}|}{|sU_{2N} - \mathscr{A}|} \tag{32}$$

where

$$\hat{M}^{\alpha\beta} \triangleq (U_N - \hat{\delta}^{\alpha\beta}(\hat{I}^{\alpha\beta})^{-1}\delta^T)^{-1} \tag{33}$$

and

$$\hat{\mathscr{A}}^{\alpha\beta} \triangleq \left[\begin{array}{c|c} 0 & U_N\\ \hline -\hat{M}^{\alpha\beta}\sigma^2 & -\hat{M}^{\alpha\beta}D\end{array}\right] \tag{34}$$

Thus, we have reduced the zero determination problem to the eigenvalue problem of the matrix $\hat{\mathscr{A}}^{\alpha\beta}$ defined by Eq. (34). As previously, the construction of the matrices $\hat{I}^{\alpha\beta}$ and $\hat{\delta}^{\alpha\beta}$ is straightforward, but since $\hat{\mathscr{A}}^{\alpha\beta}$ contains the inversions of $\hat{I}^{\alpha\beta}$ and $(U_N - \hat{\delta}^{\alpha\beta}(\hat{I}^{\alpha\beta})^{-1}\delta^T)$, Eq. (32) is restricted to the case when

$$|\hat{I}^{\alpha\beta}| \neq 0 \tag{35}$$

and

$$|U_N - \hat{\delta}^{\alpha\beta}(\hat{I}^{\alpha\beta})^{-1}\delta^T| \neq 0 \tag{36}$$

In view of Eqs. (30),

$$|\hat{I}^{\alpha\beta}| = |I^{\alpha\beta}|$$

so that Eq. (35) always holds for $\alpha = \beta$. However, Eq. (36) is not necessarily guaranteed even for $\alpha = \beta$, as opposed to the previous case (Eq. (25)) because $\hat{\delta}^{\alpha\beta}$ contains the sensor location parameter $(-c^{\alpha})$ as seen from Eq. (31).

It is worth noting that $|U_N - \hat{\delta}^{\alpha\beta}(\hat{I}^{\alpha\beta})^{-1}\delta^T|$ in Eq. (36) gives the coefficient of the highest order term in s, namely s^{2N}, and $|\hat{I}^{\alpha\beta}|$ gives the constant term in the numerator polynomial. Therefore, if Eq. (36) is violated, then the number of the zeros is less than 2N. In addition if

$$|U_N - \hat{\delta}^{\alpha\beta}(\hat{I}^{\alpha\beta})^{-1}\delta^T| \cdot |\hat{I}^{\alpha\beta}| < 0$$

then there exists at least one zero in the right half plane.

(3) In the case $\mathscr{L}'_c \neq 0$ and $\mathscr{L}'_o = 0$, in parallel with the preceding case, Eq. (8a) becomes

$$G(s) = \{I^* - \delta^T Q(s)\delta\}^{-1} \{U_3 + \delta^T Q(s)\mathscr{L}'_c\} \tag{37}$$

If we transpose G(s), then

$$G^T(s) = \{U_3 + \mathscr{L}_c'^T Q(s)\delta\}\{I^* - \delta^T Q(s)\delta\}^{-1} \tag{38}$$

(Note that $Q^T(s) = Q(s)$ since it is diagonal.)

We recognize that Eq. (38) is identical to Eq. (26) if $\mathscr{L}_c'$ is replaced by $\mathscr{L}_o'$. Therefore, we may treat this case simply by changing c^α with b^α and noting that $G_{\alpha\beta}(s)$ of Eq. (37) is given by $G_{\beta\alpha}(s)$ of Eq. (26). All the results in case (2) immediately apply to this case.

ZEROS OF DETERMINANT OF G(s)

The zeros of $|G(s)|$ are related to those of definition 2 as mentioned previously, and they are easily determined for the cases (1), (2) and (3).

If $\mathscr{L}_c' = \mathscr{L}_o' = 0$, then from Eqs. (14a) and (18) with Eq. (16)

$$\begin{aligned} |G(s)| &= \frac{1}{|I^* - \delta^T Q(s)\delta|} \\ &= \frac{|s^2 U_N + sD + \sigma^2|}{|U_N - \delta(I^*)^{-1}\delta^T|\,|I^*|\,|sU_{2N} - \mathscr{A}|} \end{aligned} \tag{39}$$

so that the zeros of $|G(s)|$ satisfy

$$|s^2 U_N + sD + \sigma^2| = 0 \tag{40}$$

Therefore,

$$s_j = \sigma_j(-\zeta_j \pm i\sqrt{1 - \zeta_j^2}\,) \qquad j = 1,\ldots,N$$

If $\mathscr{L}_c' = 0$ and $\mathscr{L}_o' \neq 0$, then from Eq. (26)

$$|G(s)| = \frac{|U_3 + \mathscr{L}_o'^T Q(s)\delta|}{|I^* - \delta^T Q(s)\delta|} \tag{41}$$

In parallel with the derivation of Eq. (18)

$$
\begin{aligned}
&|U_3 + \mathscr{L}_o'^{T} Q(s)\delta| \\
&= |U_N + \delta\, \mathscr{L}_o'^{T}| \cdot |s^2 U_N + sD + \sigma^2| \cdot |sU_{2N} - \mathscr{A}^o| \\
&= |U_3 + \mathscr{L}_o'^{T}\, \delta|\ |s^2 U_N + sD + \sigma^2|\,|sU_{2N} - \mathscr{A}^o|
\end{aligned}
\tag{42}
$$

where

$$
\mathscr{A}^o \triangleq \left[\begin{array}{c|c} 0 & U_N \\ \hline -M^o\sigma^2 & -M^o D \end{array}\right] \tag{43}
$$

with

$$
M^o \triangleq (U_N - \delta\ \mathscr{L}_o'^{T})^{-1} \tag{44}
$$

Substituting Eqs. (18) and (42) into Eq. (41) yields

$$
|G(s)| = \frac{|U_3 + \mathscr{L}_o'^{T}\delta|}{|I^* - \delta^T\delta|} \cdot \frac{|sU_{2N} - \mathscr{A}^o|}{|sU_{2N} - \mathscr{A}\ |} \tag{45}
$$

Thus, the zeros are determined by

$$
|sU_N - \mathscr{A}^o| = 0 \tag{46}
$$

which are the eigenvalues of $\mathscr{A}^o$.

If $\mathscr{L}_o' = 0$ and $\mathscr{L}_c' \neq 0$, then from Eq. (37)

$$
|G(s)| = \frac{|U_3 + \delta^T Q(s)\, \mathscr{L}_c'|}{|I^* - \delta^T Q(s)\delta|} \tag{47}
$$

which reduces (as in the previous case) to

$$
|G(s)| = \frac{|U_3 + \delta^T\ \mathscr{L}_c'|}{|I^* - \delta^T\delta|} \cdot \frac{|sU_{2N} - \mathscr{A}^c|}{|sU_{2N} - \mathscr{A}\ |} \tag{48}
$$

where

$$\mathscr{A}^c \triangleq \left[\begin{array}{c|c} 0 & U_N \\ \hline -M^c\sigma^2 & -M^cD \end{array}\right] \tag{49}$$

with

$$M^c \triangleq (U_N - \mathscr{L}_c'\delta^T)^{-1} \tag{50}$$

Thus, the zeros of $|G(s)|$ are the eigenvalues of $\mathscr{A}^c$ determined by

$$|sU_{2N} - \mathscr{A}^c| = 0 \tag{51}$$

The zeros determined by Eq. (40), Eq. (46) or Eq. (51) do not correspond to the zeros of any particular transfer function, $G_{\alpha\beta}(s)$, which are determined by Eq. (23) or Eq. (32).

It is pointed out by Brockett [1] that the zeros of $|G(s)|$ play a fundamental role in least-square optimization theory and they are important in determining if a "plant" can be decoupled by state variable feedback.

CONCLUSION

First we discussed briefly the two different definitions of the zeros of the transfer function matrix for linear time-invariant systems, to which the attitude control system of flexible spacecraft under consideration belongs.

The transfer function matrix is obtained for the model of three inputs and three outputs (Eq.(8a)). Based on this representation, it is shown that the zero determination problem reduces to the eigenvalue problem of the matrices defined by Eqs. (19) or (34) for the cases when the sensors and/or the actuators are on the primary body. These matrices are formed by interchanging the columns or rows of the system matrix $\mathscr{A}$ (Eq. (15)) with those of the sensor or actuator location matrix. They are always feasible for the transfer function from the j-th input to the j-th output (j=1,2,3) in the primary body instrumentation case. However, for the other cases, the procedure is restricted by the assumption of Eqs. (35) and (36). The results above stated are related to the zeros of Definition 1, and they will be useful for designing a controller by means of classical techniques, such as the root locus method.

A method for determining the zeros of the determinant of the transfer function matrix is also presented (Eqs. (40), (46) and (51)), and the zeros thus determined are related to Definition 2. They are useful for designing an optimal controller by modern control theory.

Although no algorithm has been shown, the procedures may be implemented on the digital computer, without any repetition of calculation steps or any subjective judgment.

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APPENDIX A. EXAMPLE OF TRANSFER FUNCTION DETERMINATION

Consider the single axis rotational motion of a spacecraft with a single appendage mode as shown in Fig. A1.

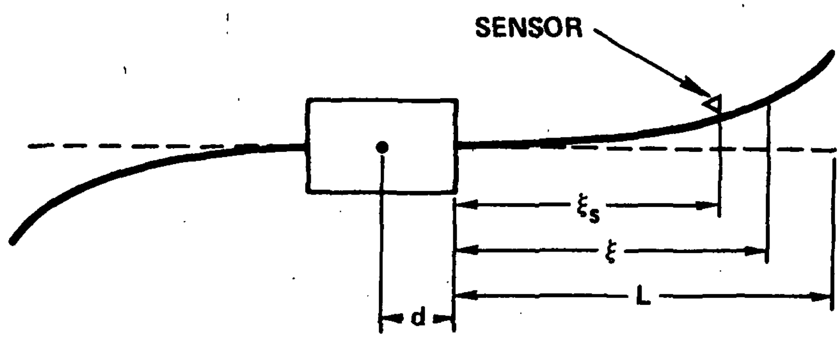

Fig. A1. Model of Example

The mode shape is assumed to be given by a function of ξ, i.e., $\phi(\xi)$. From Eqs. (1a), (1b) and (1c), with scalar variables and parameters,

$$\left\{\begin{array}{ll} I\ddot{\theta} - \delta^T\ddot{\eta} = T & \text{(A1)} \\ \ddot{\eta} + 2\zeta\sigma\dot{\eta} + \sigma^2\eta - \delta\ddot{\theta} = 0 & \text{(A2)} \\ y = \theta + \mathscr{L}_0'^T\eta & \text{(A3)} \end{array}\right.$$

where

$$\delta = -\frac{m}{L}\int_0^L \phi(\xi)(d+\xi)\,d\xi \tag{A4}$$

and we introduce the symbol c to represent $\mathscr{L}'^{T}_{o}$, as given by

$$c \triangleq \mathscr{L}'^{T}_{o} = \frac{\partial}{\partial \xi}\, \phi(\xi)\Big|_{\xi=\xi_s} \tag{A5}$$

Then, the transfer function G(s) from T to y (with the poles at the origin) is given by

$$G(s) = \frac{|1 - \delta c|}{|I - \delta^2|} \cdot \frac{|sU_2 - \hat{\mathscr{A}}^{11}|}{|sU_2 - \mathscr{A}|} \cdot \frac{1}{s^2} \tag{A6}$$

where

$$\mathscr{A} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -M_1\sigma^2 & -M_1 D \end{array}\right] \tag{A7}$$

$$\hat{\mathscr{A}}^{11} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\hat{M}^{11}\sigma^2 & -\hat{M}^{11} D \end{array}\right] \tag{A8}$$

with

$$M_1 = (1 - \delta^2/I)^{-1} \tag{A9}$$

and

$$\hat{M}^{11} = (1 + \delta c)^{-1} \tag{A10}$$

Since

$$|I - \delta^2| \cdot |sU_2 - \mathscr{A}| = I\{(1-\delta^2/I)s^2 + Ds + \sigma^2\}$$

and

$$|1 + \delta b| \cdot |sU_2 - \hat{\mathscr{A}}^{11}| = (1 + \delta c)s^2 + Ds + \sigma^2 \quad ,$$

then

$$G(s) = \frac{1}{Is^2} \cdot \frac{(1 + \delta c)s^2 + Ds + \sigma^2}{(1 - \delta^2/I)s^2 + Ds + \sigma^2} \tag{A11}$$

which is also derived by direct calculations.

If we further assume that $\phi(\xi)$ takes the form

$$\phi(\xi) = \frac{1}{2} \ell \xi^2$$

where ℓ is a scalar constant, then by Eq. (A5)

$$c = \frac{\partial}{\partial \xi} \left(\frac{1}{2} \ell \xi^2 \right) \Big|_{\xi = \xi_s} = \ell \xi_s$$

and

$$\delta = -mL^2 \phi_o \left[\frac{d}{3} + \frac{L}{4} \right]$$

where ϕ_o is determined by the normalization condition and becomes in this case

$$\phi_o = \sqrt{\frac{5}{mL^4}}$$

Hence, the highest order coefficient of the numerator of Eq. (A11) is given by

$$1 + \delta b = 1 - 5 \left(\frac{d}{3L} + \frac{1}{4} \right) \left(\frac{\xi_s}{L} \right)$$

Now, we confirm that the poles are not affected by the sensor location, while the zeros are significantly affected in view of the equation

$$\left(1 - \alpha(\xi_s)\right) s^2 + Ds + \sigma^2 = 0$$

with

$$\alpha(\xi_s) = 1 - 5 \left(\frac{d}{3L} + \frac{1}{4} \right) \frac{\xi_s}{L} .$$

We observe that as ξ_s increases from zero to the maximum attainable value, L, α changes continuously from unity to some negative value. Fig. A2 illustrates the dependency of the zeros of G(s) on the sensor location parameter α.

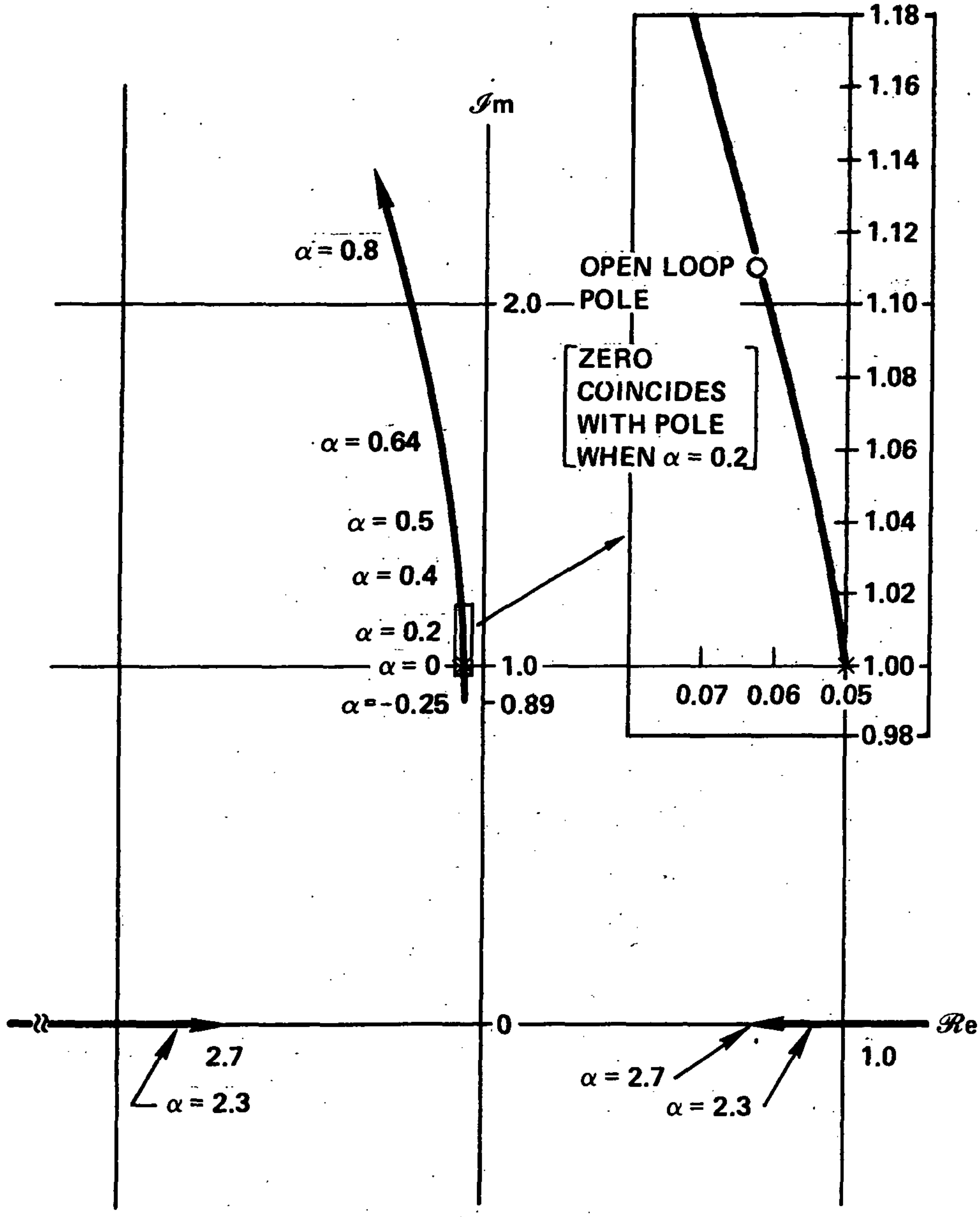

Figure A2. Zero Dependency on Sensor Location Parameter α, for σ = 1.0 and D = 0.1.

Chapter 3. Accommodating Model Error in Linear Systems Control

ABSTRACT. This study is motivated by the need for a control design procedure which gives to the controlled system three basic properties: a) insensitivity to modeling errors (such as from truncated modes and uncertain external disturbances); b) simplicity of controller (of small order); and c) guaranteed value of the performance measure of the actual system. This status report sketches some progress on the first two goals but does not touch on the third. (Actually, property a) is desirable only because it might not be possible to mathematically insure property c.)

Alterations are made to linear regulator and observer theory to accommodate modeling errors. The results (some of which are yet unproven) show that a "model error vector," which evolves from an "error system," can be added to a reduced system model, estimated by an observer, and used by the control law to render the system less sensitive to uncertain magnitudes and phase relations of truncated modes and external disturbance effects. A procedure is outlined to give the observer a "model learning" quality. Two parameters of the error system, an "observation window" τ, and the dimension of the error system, d, are related to parameters of the control problem. By choosing the optimal cost as a Liapunov function, the "observation window" (τ) of the observer is related to the minimum eigenvalue of the closed loop system. Necessary conditions are given for the solution of τ and d. In a rather novel turn of events we find that instead of the usual pattern of "given a model, apply the control theories" we are using given control theories to help construct a more appropriate model of the system.

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I. INTRODUCTION

The theory available for analyzing and controlling linear systems is quite extensive for LINEAR QUADRATIC problems provided the model (a set of ordinary differential equations) is perfectly accurate. Even if the model is in error, one can analyze the effects of these errors <u>post facto</u>. There is not, however, a well developed theory which allows one to consider the effects of model error during the control design.

The exact problem which we would like to solve is the following: Find a control policy $u^o(t)$ of minimal complexity which guarantees the performance, V^o. Mathematically, we say it this way:

<u>THE MINIMAL CONTROLLER PROBLEM:</u>

Find the control policy, $u^o = \mathscr{F}(z^o, A^j, B^j, C^j, M^j, t)$, which though based upon the model, $\mathscr{S}_j$, of the physical system, $\mathscr{S}_o$,

$$\mathscr{S}_j \left\{ \begin{array}{lll} \dot{x}^j - A^j\, x^j + B^j u^o\ , & x^j = n_j & \text{- vector (STATE)} \\ y^j = C^j\, x^j & y^j = k & \text{- vector (OUTPUT)} \\ z^j = M^j\, x^j & z^j = \ell & \text{- vector (MEASUREMENT)} \end{array} \right.$$

$$u^o = m \quad \text{- vector (CONTROL)}$$

$\mathscr{S}_o$ { Physical system, with actual inputs u^o, and measurements z^o, and output quantities,* y^o

satisfies the performance requirement

$$V^j \triangleq \int_0^\infty \left(||y^o||_Q + ||u^o(\mathscr{S}_j, z^o)|| \right) dt \le V^o \text{ (specified number)}$$

for minimum n_j, for all expected conditions (disturbance and control inputs and parameter variations).

*In this paper, the word "output" is defined to be the vector of variables which we wish to control, (y). The vector z represents those quantities actually measured.

Even this mathematical statement of the designer's goal has some vagueness that must be interpreted. The implication is that the selection of those variables which are to be controlled (identification of u^o); selection of parameters (A^j, B^j, C^j, M^j, t); and even the dimension of the model $\mathcal{S}_j$ must all be determined as part of this "minimal controller problem." Efforts to separate the problem into two separate problems (a "modeling problem" and a "control problem") usually yields controllers which either

(a) suffer from modeling errors, causing performance requirements to be violated

or,

(b) satisfy performance requirements at a cost of high controller complexity.

The conditions which cause (a) are sometimes not discovered before "flight." The condition (b) often prevents use of modern control theory in an application. Determination of "expected" disturbances and selection of the "requirement," V^o, can be difficult decisions in the evaluation of a control law.

Presently the "minimal controller problem" can only be solved by trial and error. Because we cannot solve this problem and guarantee the cost mathematically, we sometimes seek certain precautionary measures by

1. "worst case" designs (a deterministic approach of using conservative conditions in design to add reliability to "flights." This can cost us in controller complexity).
2. Sensitivity approaches (making the system less sensitive to some uncertain parameter or disturbances adds some confidence to the solution).

3. Stochastic approaches (offer good performance "on the average." A particular experiment might violate requirements).

In this paper the minimal controller problem (MCP) is not solved, but we are guided by the MCP objective to suggest a procedure which allows certain parameters and dimensions of the modeling problem (MP) to be related to parameters of the control problem (CP). The procedure offers to the controlled system (in lieu of a MCP solution) a certain type of model error "forgiveness." To date, however, the cost, V^j, is not guaranteed a priori.

In section II the modeling problem (MP) is divided into a two phase task: a part of the model is determined without regard to the control problem (CP); a second part of the model is presented in structure but the dimension and certain parameters are left to be determined in the CP. Section III presents an observer for estimating all the states of the finally selected model. Special cases of this observer are discussed which produce the Luenberger observer, the Disturbance Absorbing Controller and a Model Learning Observer. Finally, in section IV, necessary conditions are given for the selection of the control parameters by viewing the optimal cost as a Liapunov function. Some concluding remarks appear in section V.

II. THE MODELING PROBLEM

Suppose we are given some model of a physical system, $\mathcal{S}_o$. Let us label this model $\mathcal{S}_3$.

$$\mathcal{S}_3 \begin{cases} \dot{x}^3 = A^3 x^3 + B^3 u^o & x^3 = n_3 \text{ vector} \\ y^3 = C^3 x^3 & y^3 = k \text{ vector} \\ z^3 = M^3 x^3 & z^3 = \ell \text{ vector} \end{cases} \tag{2.1}$$

For the moment we disregard the means used to obtain $\mathcal{S}_3$. We consider $\mathcal{S}_3$ to be the actually realized model* and we wish to assess the errors which accompany $\mathcal{S}_3$ relative to a better model, $\mathcal{S}_1$.

$$\mathcal{S}_1 \begin{cases} \dot{x}^1 = A^1x^1 + B^1u^o + W^1, & x^1 = m_1 \text{ vector} \\ y^1 = C^1x^1 & y^1 = k \text{ vector} \\ z^1 = M^1x^1 & z^1 = \ell \text{ vector} \end{cases} \tag{2.2}$$

The model $\mathcal{S}_1$ is the most general description of $\mathcal{S}_0$ which we write ($\mathcal{S}_1$ might be used, for example to evaluate performance predictions and controller designs which are based upon simpler models). The model $\mathcal{S}_3$ may not have proceeded directly from a prior system description such as $\mathcal{S}_1$. However, there exists a transformation on $\mathcal{S}_1$ such that $\mathcal{S}_3$ is a truncation of the transformed $\mathcal{S}_1$. To illustrate this point of view imagine the transformation, $x^1 = T\chi$ which has the transformed system description $\mathcal{S}_1'$

$$\mathcal{S}_1' \begin{cases} \begin{pmatrix} \dot{x}^{3'} \\ \dot{x}_t \end{pmatrix} = \begin{bmatrix} A^3 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x^{3'} \\ x_t \end{pmatrix} + \begin{bmatrix} B^3 \\ \mathscr{B}_2 \end{bmatrix} u^o + \begin{bmatrix} \mathscr{E}_1 \\ \mathscr{E}_2 \end{bmatrix} W^1 \\ y^1 = C^3, C_t] \begin{pmatrix} x^{3'} \\ x_t \end{pmatrix} \\ z^1 = [M^3, M^t] \begin{pmatrix} x^{3'} \\ x_t \end{pmatrix} \end{cases} \tag{2.3}$$

where

$$\chi \triangleq \begin{pmatrix} x^{3'} \\ x_t \end{pmatrix}, \quad T^{-1}A^1T = \begin{bmatrix} A^3 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} \mathscr{E}_1 \\ \mathscr{E}_2 \end{bmatrix}$$

$$T^{-1}B^1 = \begin{bmatrix} B^3 \\ \mathscr{B}_2 \end{bmatrix}, \quad C^1T = [C^3, C_t], \quad M^1T = [M^3, M_t] \tag{2.4}$$

*Suppose, for instance, that the parameters actually used in a controller design or implementation are (A^3, B^3, C^3, M^3).

Thus $x^{3'}$ is the n_3-vector which satisfies

$$\dot{x}^{3'} = A^3 x^{3'} + B^3 u^o + e'(t) \tag{2.5}$$

where the vector $e'(t)$ can be considered a "model error vector" of the model $\mathcal{S}_3$ (compare (2.5) and (2.1)). The vector $e'(t)$ evolves from the "error system"

$$\begin{aligned} \dot{x}_t &= A_{22}x_t + B_2 u^o + A_{21}\, x^{3'} + \mathscr{E}_2\, W^1 \\ e'(t) &= A_{12}x_t + \mathscr{E}_1\, W^1 \end{aligned} \tag{2.6}$$

which is coupled with and is an integral part of the system description, $\mathcal{S}_1$, but which is neglected in the system description, $\mathcal{S}_3$. Equation (2.6) clearly focuses the fact that for any model of a physical system there is associated with this model an error vector that can be considered as a combination of external disturbances and truncated states of a more accurate model. Furthermore, such an $e'(t)$ exists which compensates for any parameter errors in (A^3,B^3,C^3,M^3) relative to some intermediate, n_3-dimensional, model which the designer may have intended to implement. The converse of this statement is not true, however. That is, there is no set of parameters (A^3,B^3,C^3,M^3) which can necessarily compensate for the effects of truncated states and external disturbances ($e'(t)$) which have been neglected in the model. The effects of such persistently acting "disturbances" can cause instability in the control problem or instability of Luenberger Observers in the observation problem. In stochastic descriptions of a system, $e'(t)$ represents correlated disturbances which have been ignored. It has been shown by Fitzgerald [1], Price [2] and others [3]-[5], that the Kalman filter can diverge when such correlated "disturbances" are either neglected or modeled as white. Moreover, even adaptive techniques (which update parameters of the assumed white noise statistics) may fail in such cases for reasons stated above. Thus, "structural" errors in modeling seem much more critical than so called "parameter errors."

It is the purpose of this section to propose a model error vector, e(t), which is an approximation to the vector e'(t), and which when added to model $\mathcal{S}_3$ can effectively compensate for modeling errors arising from,

- truncated states (regarded as "internal disturbances")
- external disturbances (which also includes effects of "weak" non-linearities)
- parameter changes

The vector e(t) is considered to evolve from a much simpler "error system" than described by the $(n_1 - n_3)^{th}$ order system (2.6). The error vector e which is added to $\mathcal{S}_3$ thusly,

$$\begin{cases} \dot{x}^3 = A^3x^3 + B^3u^o + e \\ y^3 = C^3x^3 \\ z^3 = M^3x^3 \end{cases} \tag{2.7}$$

is considered to evolve from the error system

$$\begin{aligned} \dot{\gamma} &= D\gamma \qquad & \gamma &= d\text{ -vector} \\ e &= P\gamma \qquad & P &= n_3 \times d \text{ matrix} \end{aligned} \tag{2.8}$$

Now $\gamma(t)$ is a small dimensional vector which represents a "compression" of information in x_t, W^1 and retains only important effects of large dimensional vectors $x_t(t)$ and $W^1(t)$. Thus, $\gamma(t)$ is an artificial vector whose space should not be presumed completely controllable because external disturbances W^1 are not controllable (by definition of the word "external," $W^1(t)$ is not causally related to $x^3(t)$ or $u^o(t)$). As an approximation, we write the error system

(2.8) unforced by u^o. The truncated states may well appear in the measurements since M_t in (2.3) would not be zero, in general. We therefore allow the artificial variables $\gamma(t)$ to influence the modeled measurement vector, $z^2(t)$ (via matrix M_γ), in the proposed model structure $\mathcal{S}_2$.

$$\mathcal{S}_2 \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\gamma} \end{pmatrix} = \begin{bmatrix} A & P \\ 0 & D \end{bmatrix} \begin{pmatrix} x \\ \gamma \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u^o \\ y^2 = [C, 0] \begin{pmatrix} x \\ \gamma \end{pmatrix} , \quad C \stackrel{\Delta}{=} C^3 \\ z^2 = [M, M_\gamma] \begin{pmatrix} x \\ \gamma \end{pmatrix} , \quad M \stackrel{\Delta}{=} M^3 \end{cases} \tag{2.9}$$

where

$$\begin{pmatrix} x \\ \gamma \end{pmatrix} \stackrel{\Delta}{=} \begin{pmatrix} x^3 \\ \gamma \end{pmatrix} = x^2 , \quad A \stackrel{\Delta}{=} A^3, \quad B = B^3 \tag{2.10}$$

and the superscript 3 is dropped now that discussion on the hierarchy of models is completed. For our purposes we assume in $\mathcal{S}_2$ (which has the error system augmented to model $\mathcal{S}_3$) that A,B,C,M are given and we must find the error system parameters P,D,M_γ which are appropriate for a given problem. The reader may well wonder at this point why we wouldn't simply incorporate the knowledge of important disturbance effects in the basic model reduction decisions which yield $\mathcal{S}_3$ so that $\mathcal{S}_3$ is a completely adequate model for control design. However, we do not usually know what disturbance effects (external and internal) are important to keep in the model <u>prior</u> to controller design. Performance evaluations tell the control designer, <u>in a trial and error fashion</u>, what disturbance effects must fall within the spectrum of control authority (i.e. what disturbance effects must be actively controlled). Also, parameter uncertainties (whether constant or due to in-flight changes) and variations in the disturbance environment cannot always be reliably predicted. This perplexity causes the designer to seek a conservative (i.e. "worst case") design with deterministic procedures or to seek statistical procedures which, in essence, can promise good performance only on the average (a particular experiment might behave quite poorly).

It is for these several reasons that e(t)* cannot be specified prior to control system design. We therefore set ourselves the task of constructing an analytical model $\mathcal{S}_2$ in two steps:

I. Controller independent model reduction decisions, $\mathcal{S}_1 \to \mathcal{S}_3$.

II. Controller dependent model decisions $\mathcal{S}_3 \to \mathcal{S}_2$

Figure 1 illustrates the sequence of events in the modeling problem.

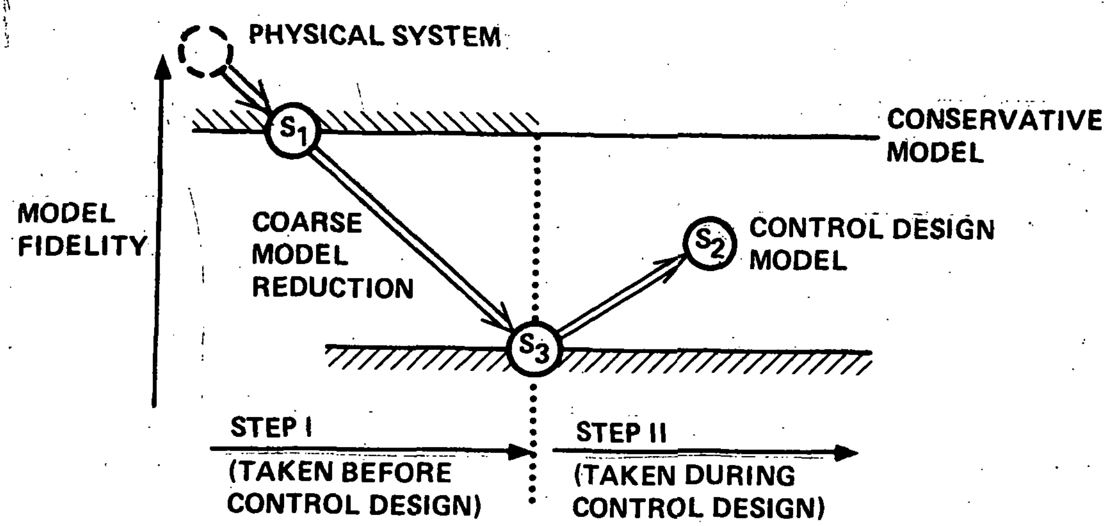

Figure 1. Modeling Process.

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*The vector e(t) has the nature of a "correlated" disturbance in the stochastic view of the problem and "persistently acting" disturbances in the deterministic view.

Specifying a Structure for P, D

Since $\gamma(t)$ is to be selected so that the approximate error system (2.8) represents the important effects of the actual error system (2.6), we could select the parameters of (2.8) so that

$$\int_0^{\tau} ||e'(t) - e(t)||_{g(t)} dt \tag{2.11}$$

is a minimum, where τ is the time interval over which good error system information is needed by the controller and g(t) is a weighting scalar. Now, if we view this intermediate task as a curve fitting problem in which the variables $\gamma_i(t)$ are a set of known functions, then the set of coefficients of those functions which will minimize (2.11) is

$$P = \int_0^{\tau} g(t)\ e'(t)\gamma^T(t)dt \left[\int_0^{\tau} g(t)\gamma(t)\gamma(t)^T dt\right]^{-1} . \tag{2.12}$$

One possible disadvantage of least squares methods in modeling problems is that the solution which minimizes (2.11), subject to $g(t) \equiv 1$, can permit large instantaneous deviations between e' and e. This is not a fault of least squares theory but can result from poor judgement in deciding what to take the least squares of. A wiser choice of the integrand of (2.11) might include rate terms $(\dot{e}'(t) - \dot{e}(t))$ as well. Another option, which we pursue here, is to minimize the maximum deviation between e' and e; a result which can be obtained simply by choosing an appropriate time weighting g(t) and interval τ. The result is that (2.12) takes on the form

$$\begin{aligned} P &= \frac{2}{\pi} \left[\int_{-1}^{1} \frac{e'(\sigma)\gamma^T(\sigma)}{\sqrt{1-\sigma^2}}\, d\sigma\right] \mathbb{J} \\ \mathbb{J} &\triangleq \begin{bmatrix} 1/2 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}_{d\times d} \end{aligned} \tag{2.13}$$

where a change of variable has been made for convenience

$$t = \frac{\tau}{2}(\sigma + 1)$$

$$g(\sigma) = (1 - \sigma^2)^{-1/2}$$

$$\gamma_i(\sigma) = \cos(i \cos^{-1}\sigma) \tag{2.14}$$

The functions $\gamma_i(\sigma)$ are called Chebyshev polynomials of the first kind of degree i. To obtain the matrix D, we may differentiate (2.14) and rearrange the form to yield

$$\dot{\gamma} = D\gamma \tag{2.15}$$

where

$$D = \frac{1}{\tau}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & \\ 2 & 0 & 0 & 0 & 0 & & \\ 0 & 8 & 0 & 0 & 0 & & \\ 6 & 0 & 12 & 0 & 0 & & \\ 0 & 16 & 0 & 16 & 0 & & \\ \vdots & & & & & \ddots & \\ \vdots & & & & & & 0 \end{bmatrix}_{d\times d} \tag{2.16}$$

Now the matrices P and D are determined to within two parameters; the "observation window," τ, and d, the dimension of the γ vector. A nontrivial task in the computation of P is the determination of e', the error vector we desire to approximate. It may be too difficult to compute e' exactly, by solving (2.6). Also the question of what conditions to impose upon $\mathcal{A}_1$ to obtain e' might be in doubt. Because an observer will be used in section III to continually update an estimate of $\gamma(t)$, we will find that the system will be forgiving of certain kinds of errors (uncertainties) in our selection of e'(t). Specifically, magnitudes and phase relationships of the various modes that comprise e' need not be known to guarantee proper convergence of the error system.

This notion is made more specific in section IV. For the present, we assume that some reasonable facsimile of e' is obtained for use in (2.13). We conclude this section by noting that (2.9) is the form of the "control design" model where the parameter matrices (A,B,C,M) are specified at this point and specification of the parameters of the error system $P(\tau,d)$, $D(\tau,d)$ and M_γ must await solution of the control problem in sections III and IV.

III. OBSERVER DESIGN

In this section we are committed to the design of an observer to continually estimate x(t) and $\gamma(t)$, as defined by (2.9). While constructing such an observer we will again, as in the modeling problem of the last section, refer to a better model $\mathcal{S}_1$ to interpret certain observation errors.

Suppose $x^2(t)$ is defined as the vector which satisfies model

$$\mathcal{S}_2 \begin{cases} \dot{x}^2 = A^2\, x^2 + B^2\, u^o & x = n \text{ vector} \\ y^2 = C^2\, x^2 & y = k \text{ vector} \\ z^2 = M^2\, x^2 & z = \ell \text{ vector} \end{cases} \tag{3.1}$$

where

$$A^2 = \begin{bmatrix} A & P \\ 0 & D \end{bmatrix}, \quad B^2 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C^2 = [C,0]$$
$$M^2 = [M, M_\gamma], \quad x^2 = \begin{pmatrix} x \\ \gamma \end{pmatrix}, \quad (n+d)\text{-vector} \tag{3.2}$$

Now $z^2 = M^2x^2$ represents the best combination of $x_i^2(t)$ which corresponds to the <u>actual</u> measurement record $z^o(t)$. The actual measurement $z^o(t)$ does not satisfy (3.1), however, because of model errors. In order to write an equality relating the state of the model, $x^2(t)$, and the actual measurement record, $z^o(t)$, we must include an error term, ε_z,

$$z^o(t) = M^2x^2 + \varepsilon_z \qquad \Rightarrow \qquad \varepsilon_z \overset{\Delta}{=} z^o - M^2x^2 \tag{3.3}$$

where ε_z is a result of the kind of error e'(t) shown in (2.5). Indeed, if $\mathcal{S}_1$ is considered to be a model of evaluation quality* then we can write from (2.3),

$$z^1(t) = Mx^{3'} + M_t\, x_t \approx z^o(t) \tag{3.4}$$

then from (3.3) and (3.4)

$$\varepsilon_z \approx M_t\, x_t - M_\gamma \gamma \tag{3.5}$$

bearing in mind that (3.5) is an approximation and (3.3) is the definition of ε_z. Furthermore, the approximation (3.5) is good only if e'(t) is small in (2.5) so that $x^{3'} \approx x^3$.

We define an $(m+d-\ell)$ vector, $\xi(t)$, by

$$\xi(t) \overset{\Delta}{=} \Gamma\, x^2 \tag{3.6}$$

where the $(m+d-\ell) \times (m+d)$ matrix Γ is to be defined momentarily. Now, if we augment (3.6) with the actual measurement relation (3.3) we have

$$\begin{pmatrix} \xi \\ z^o \end{pmatrix} = \begin{bmatrix} \Gamma \\ M^2 \end{bmatrix} x^2 + \begin{pmatrix} 0 \\ \varepsilon_z \end{pmatrix} \tag{3.7}$$

Assuming M^2 is of maximal rank we can choose Γ so that the inverse relation of (3.7) exists,

$$x^2 = L_1\xi + L_2(z^o - \varepsilon_z) \tag{3.8}$$

---

*If system tests are not economical, then performance evaluations are usually conducted with computer simulations. Here we assume that model $\mathcal{S}_1$ is of sufficient fidelity to warrant the confidence of performance decisions made from it. By this we mean that $z^1(t)$ is sufficiently close to $z^o(t)$ to write $z^1(t) = z^o(t)$.

where

$$[L_1, L_2] \triangleq \begin{bmatrix} \Gamma \\ M^2 \end{bmatrix}^{-1} \tag{3.9}$$

Thus, for later reference we have the relations, from (3.9),

$$\begin{aligned} &L_1\Gamma + L_2M^2 = I \\ &\Gamma L_1 = I \qquad M^2L_2 = I \\ &\Gamma L_2 = 0 \qquad M^2L_1 = 0 \end{aligned} \tag{3.10}$$

By differentiating (3.6) and using (3.1), (3.8) we can construct a differential equation from which ξ is considered to evolve.

$$\dot{\xi} = \Gamma\, A^2L_1\xi + \Gamma\, A^2L_2(z^o - \varepsilon_z) + \Gamma\, B^2u^o \tag{3.11}$$

Equation (3.11) together with (3.8) form an "observer"; a linear dynamical system whose output (relation (3.8)) yield x^2 (if estimation is perfect and $\hat{x}^2$ otherwise) and whose inputs are the real measurements $z^o(t)$ and the real controls $u^o(t)$, (see (3.11)). Of course $\varepsilon_z(t)$ must be available if the estimate of x^2 is to be perfect. Actually equations (3.11), (3.8) offer a structure from which a number of different "observers" can be discussed.

CASE I: <u>The perfect observer</u>

Equations (3.11), (3.8) and Figure 2 describe a perfect state observer, $\hat{x}^2 \equiv x^2$, if $x^2(t_o)$ and $\varepsilon_z(t)$ are known. (If $x^2(t_o)$ is known, then $\xi(t_o) = \Gamma x^2(t_o)$ and $\varepsilon_z = z^o - M^2x^2$ are available, as required.) Of course, $x^2(t_o)$ is usually not available, else we have no need for the observer at all.

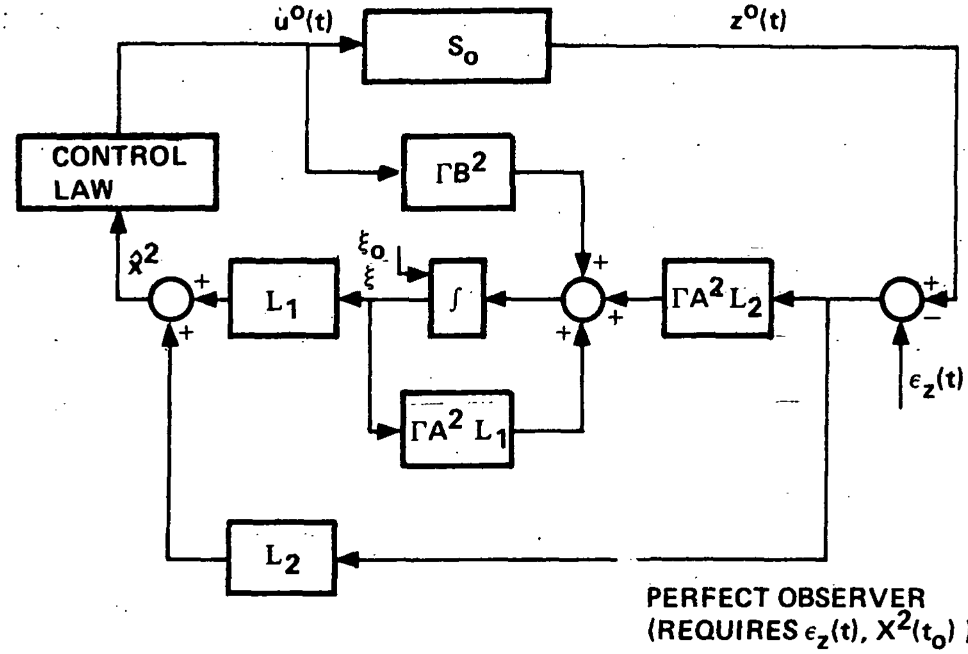

Figure 2. Perfect Observer. (Requires $\epsilon_z(t)$, $x^2(t_o)$).

CASE 2: <u>The Luenberger Observer</u>

If we presume that our model is perfect, then $\varepsilon_z = 0$ and (3.11), (3.8) reduce to the Luenberger observer for the system $\mathcal{S}_2$.

CASE 3: <u>A Model Learning Observer</u>

Given the model structure of (3.1), (3.2), the observer could "learn" the correct matrix M_γ which would make the "measurement residual," $\hat{\varepsilon}=z^o-M^2\hat{x}^2$, approach zero. In Figure 3, $\hat{\varepsilon}_z$ is used as an error signal to correct M_γ, which appears within the gains L_1,L_2. The physical interpretation of this scheme is the following: For a given character of the error system specified by (2.9), this observer will "learn" how the error variable $\gamma(t)$ propagates to the measurements, $z^2=Mx + M_\gamma\gamma$, by finding the matrix M_γ which will make the model predictions compatible with the measurements, i.e. $M^2\hat{x}^2 \rightarrow z^o$. If an algorithm can be found which will converge on the correct M_γ then the observer could be called a "model learning" observer, in the sense that it learns how the error vector $e = P\gamma$ influences the measurements. The matrices P, D characterize the the error vector e(t). For Chebyshev polynomial approximation the matrices P, D are given by (2.13), (2.16).

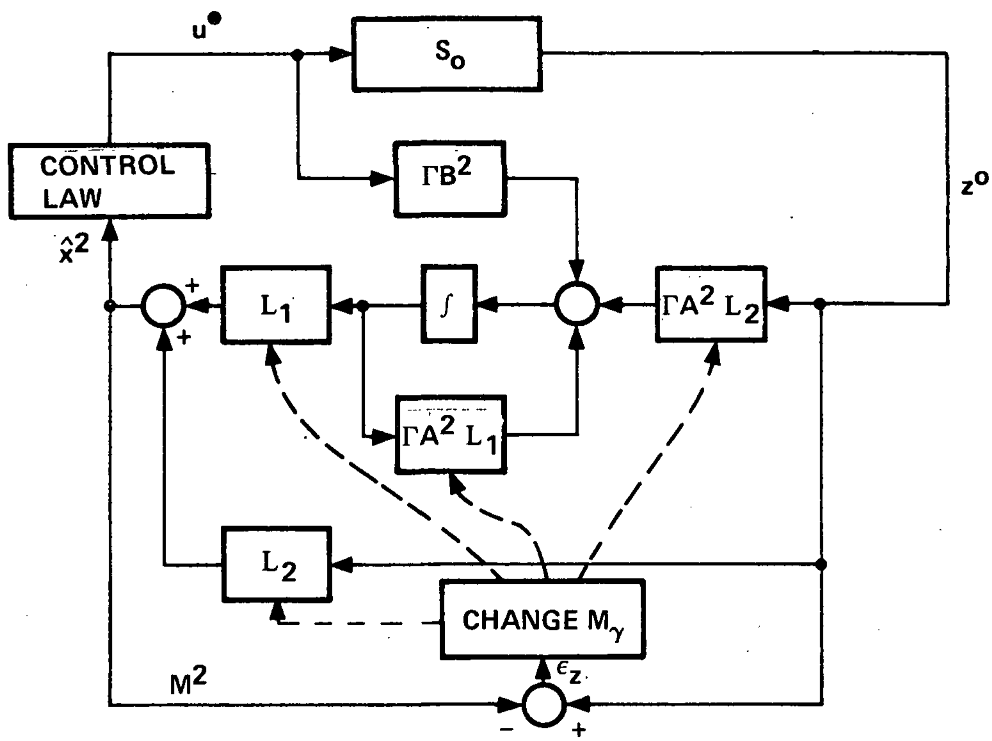

Figure 3. A Model Learning Observer.

CASE 4: <u>The Disturbance Absorbing Controller</u>

If we assume the model is perfect (that is, $e(t) \equiv e'(t)$ in (2.11), such that $\varepsilon_z(t) \equiv 0$) and that the error vector γ is totally composed of <u>external</u> disturbance effects which cannot be measured (that is $M_\gamma \equiv 0$), then the control structure of Figure 2 reduces to the so-called Disturbance Absorbing Controller whose structure was proposed in (6) and which is developed extensively in (7)-(9).

One final note is of interest on these observers. Since ε_z is defined to be $\varepsilon_z = z^o - M^2x^2$ it is perhaps tempting to construct the approximation to ε_z, namely $\hat{\varepsilon}_z = z^o - M^2\hat{x}^2$. If one implements $\hat{\varepsilon}_z(t)$ this way then the observer becomes completely disconnected from the measurements $z^o(t)$, (see Figure 2). A similar circumstance occurs in Kalman filtering when the plant noise is small or zero. Then the Kalman gain can become zero, essentially disconnecting the filter from the measurements. Of course, instability is assured in both cases since the model which the observer/filter has "learned" contains error and its predictions eventually diverge from the real system. Thus, the requirements stated in case 1 for the perfect observer can not be met and the attempt as outlined in case 1 will guarantee instability because it will not be causally related to the measurements.

The observers described in this section, in conjunction with the model given in (3.1), (3.2), can be interpreted as performing the following service for the control system: The observer has within it a model of the uncorrected system $\mathcal{S}_3$. When measurement records do not correspond to this model the difference is curve fitted with a set of Chebyshev polynomials. It should be mentioned that the estimates of $\gamma_i(t)$ are not themselves Chebyshev polynomials. But over any time interval, $(t-\tau,t)$, the observer provides γ_o, (or equivalently, $\gamma(t)$, since $\gamma = e^{Dt}\gamma_o$), those initial conditions which correspond to a best fit of a set of Chebyshev polynomials of degree d over the interval $(t-\tau,t)$, to the actual error vector (MAX$[e'-Pe^{Dt}\gamma_o]$ is minimized over $(t-\tau,t)$ by the choice of γ_o, produced by the observer). For these observers to be stable (convergent on x^2) the matrix ΓAL_1 must be a stability matrix and (3.10) must be satisfied. The error analysis is incomplete, but the first observation is that if $\xi(t)$ is the (perfect) observer state which produces

$x^2(t)$ (via (3.8) and (3.11) with $\xi_o = \Gamma x_o^2$) and if $\hat{\xi}(t)$ is the state of the physically realizable observer which produces $\hat{x}^2(t)$ (via (3.8) with $\varepsilon_z = 0$ in (3.11) and ξ_o arbitrary), then the observer error

$$\Delta \overset{\Delta}{=} \xi(t) - \hat{\xi}(t) \tag{3.12}$$

obeys the differential equation

$$\dot{\Delta} = \Gamma A^2 L_1 \Delta - \Gamma A^2 L_2 \varepsilon_z \tag{3.13}$$

It is the purpose of the model learning observer to keep ε_z small by an appropriate choice (perhaps iterative) of M_γ. Noting that ε_z is not zero for any finite dimensional linear model of a physical dynamical system this is an important step in an attempt for better controllers. Luenberger observer and Kalman filter techniques assume $\varepsilon_z = 0$ (or white) and for that reason stability of observer or estimator cannot be assured in general. It is the purpose of Γ to make (3.13) stable and to make (3.7) invertible. This is possible under certain observability conditions that will be detailed in further work.

IV. THE CONTROL PROBLEM

The model and observer are characterized by the parameters $(A, B, C, M, M_\gamma, \Gamma, P(\tau,d), D(\tau,d))$. The control law to be utilized is the linear form, $u^o = G\hat{x}^2$. Therefore, the parameters that remain to be determined in this section are τ, d, Γ and G, assuming that M_γ is determined in section III.

Selection of τ, d:

Suppose $V(x^2,t)$ is a Liapunov function for $\mathcal{J}_2$, Equation (3.1), with the control

$$u^o = G\,\hat{x}^2 \tag{4.1}$$

Then, by our previous interpretation of τ as an "information window" over which good information is needed by the controller, we define τ to be a minimum time constant of the system in the following sense

$$\tau \overset{\Delta}{=} \underset{x^2(t)}{\mathrm{MIN}} \left[\frac{V(x^2,t)}{-\dot{V}(x^2,t)} \right] \tag{4.2}$$

Now if $V(x^2,t)$ is a quadratic form

$$V(x^2,t) = x^2(t)^T \mathscr{P} x^2(t) \tag{4.3}$$

$$\dot{V}(x^2,t) = -x^2(t)^T \mathscr{N} x^2(t) \tag{4.4}$$

Then we utilize the fact that the ratio $= x^{2^T} \mathscr{P} x^2 / x^{2^T} \mathscr{N} x^2$ is bounded above and below by

$$\lambda_{\mathrm{MIN}} \left[\mathscr{N}^{-1} \mathscr{P} \right] \le \frac{x^{2^T} \mathscr{P} x^2}{x^{2^T} \mathscr{N} x^2} \;\tau\; \le \lambda_{\mathrm{MAX}} \left[\mathscr{N}^{-1} \mathscr{P} \right] \tag{4.5}$$

to obtain

$$\tau = \lambda_{\mathrm{MIN}} \left[\mathscr{N}^{-1} \mathscr{P} \right] \tag{4.6}$$

Since the matrices $\mathscr{N}$ and $\mathscr{P}$ are functions of $P(\tau,d)$ and $D(\tau,d)$, we could find the best τ by minimizing (4.6). That is

$$\underset{\tau}{\mathrm{MIN}} \left[\underset{x^2(t)}{\mathrm{MIN}} \left\{ \frac{V}{-\dot{V}} \right\} \right] \tag{4.7}$$

yields τ. Rather than solving this problem directly, we will establish an upper bound on

$$\underset{x^2(t)}{\mathrm{MIN}} \left(\frac{V}{-\dot{V}} \right)$$

and minimize the upper bound. From matrix algebra (10) we can write

$$\lambda_{\mathrm{MIN}}[\mathscr{N}^{-1}\mathscr{P}] \le \lambda_{\mathrm{MIN}}[\mathscr{N}^{-1}] \cdot \lambda_{\mathrm{MIN}}[\mathscr{P}] \le ||\mathscr{N}^{-1}||^{+1/2} \cdot ||\mathscr{P}||^{1/2} \tag{4.8}$$

where

$$
\begin{aligned}
||\mathscr{P}|| &= \left[\sum_{i,j=1}^{n+d} \mathscr{P}_{ij}^2\right]^{1/2} \\
||\mathscr{N}^{-1}|| &= \left[\sum_{i,j=1}^{n+d} (\mathscr{N}^{-1})_{ij}^2\right]^{1/2}
\end{aligned}
\tag{4.9}
$$

If we choose $V(x^2,t)$ to be the optimal cost functional

$$
V(x^2,t) \triangleq \int_t^{\infty} (y^{2^T} Q\, y^2 + u^{o^T} R\, u^o)\, dt \tag{4.10}
$$

and u^o to be the corresponding optimal control

$$
u^o(t) = -R^{-1} B^{2^T} \bar{K}\, \hat{x}^2(t) \tag{4.11}
$$

where $\bar{K}$ satisfies the degenerate Riccati equation

$$
0 = -\bar{K} A^2 - A^{2^T} \bar{K} + \bar{K} B^2 R^{-1} B^{2^T} \bar{K} - C^{2^T} Q\, C^2 \tag{4.12}
$$

then $\mathscr{P} = \bar{K}$, $G = -R^{-1}B^{2^T}\bar{K}$, $\mathscr{N} = C^{2^T} Q C^2 + \bar{K}B^2 R^{-1} B^{2^T}\bar{K}$, (the observer is considered perfect for the present purpose, $\hat{x}^2 \equiv x^2$). If we minimize the upper bound as specified by (4.8) we have the following necessary conditions for τ.

$$
\begin{aligned}
\frac{\partial}{\partial \tau}\left[||\mathscr{N}^{-1}||^{1/2} \cdot ||\mathscr{P}||^{1/2}\right] &= \frac{\partial ||\mathscr{N}^{-1}||^{1/2}}{\partial \tau} ||\mathscr{P}||^{1/2} \\
&\quad + ||\mathscr{N}^{-1}||^{1/2} \frac{\partial ||\mathscr{P}||^{1/2}}{\partial \tau} = 0
\end{aligned}
\tag{4.13}
$$

Equation (4.13) is quite difficult to solve and we must show that $\mathscr{N}^{-1}$ exists. To obtain $||\mathscr{P}||^{1/2}$, $||\mathscr{N}^{-1}||^{1/2}$ in a form explicit in τ requires a Kronecker product (11) solution of

$$
-L(\tau,d)D(\tau,d) + [KBR^{-1}B^T - A^T]\, L(\tau,d) = K\, P(\tau,d) \tag{4.14}
$$

for L and of

$$-F(\tau,d)D(\tau,d) - D^T(\tau,d)F = L^T BR^{-1}B^T L + P^T(\tau,d)L + L^T P(\tau,d) \tag{4.15}$$

for F, where L and F are partitioned parts of $\bar{K}$ in (4.12),

$$\bar{K} = \begin{bmatrix} K & L(\tau,d) \\ L^T(\tau,d) & F(\tau,d) \end{bmatrix} = \mathscr{P} \tag{4.16}$$

Therefore

$$||\mathscr{P}|| = \sum_{i=1}^{d} \ell_i^T \ell_i \tag{4.17}$$

where ℓ_i are columns of the matrix

$$\begin{bmatrix} L \\ F \end{bmatrix} = [\ell_1, \ell_2, \ldots \ell_d] \tag{4.18}$$

Further work is required to determine if τ can be solved from (4.13) and whether it is unique.

Assuming that the norm of $\mathscr{P}$ is a convex function of d, we will select d to satisfy

$$\ell_d^T \ell_d = \alpha \sum_{i=1}^{d-1} \ell_i^T \ell_i \tag{4.19}$$

for small $\alpha \approx .01$. This in effect forces a minimum degree of observability of "error system" modes (i.e. so that the d^{th} (and final) mode of the error system is say, 1% as observable in the output as all other modes of the error system). Mathematical justification of this interpretation will be offered later. Choosing α to be as <u>large</u> as, say .01, serves to constrain the order of the error system. Otherwise d might be quite large (to obtain arbitrarily small errors in fitting e(t) to e'(t)).

V. CONCLUSIONS

An "error system" is augmented to a system model which is intended for controller design. The parameters of the error system are determined in such a way that the observer, which is used to estimate the states of the model and error system, serves to fit actual model errors (the difference between real measurements and those predicted by the model) with Chebyshev polynomials so that the maximum modeling error is minimized over an "observation window," τ. In this view of the operation of the observer, the system can perform, in essence, "adaptive curve fitting" of internal and external disturbances without recourse to "adaptive" techniques. Alternately, in the mode of operation in which the observer is adaptive, the observer could be called a "model learning observer" in the sense that it learns how the model errors influence the measurements. To accomplish this an algorithm must be found for changing a certain matrix, M_γ, so that the measurement residual, $\hat{\varepsilon}_z = z^o - z^2$, is driven toward zero. This adaptive feature has not been completed, however.

Certain parameters of the error system model must be determined simultaneous with control policy design. Specifically, the "observation window," τ, associated with the error system is shown to be related to the minimum eigenvalue of the closed loop system. This relation is accomplished by viewing the optimal cost as a Liapunov function. The dimension of the error system, d, is determined from a required degree observability of the modes of the error system, although this feature has not been completed and only necessary conditions are shown for τ and d.

As a special case of the observer derived herein, the Luenberger observer is obtained. As another special case the method reduces to the Disturbance Absorbing Controller reported in [9], [8] and [7]. The present method has several advantages over the Disturbance Absorbing Controllers. Most significantly, the effects of "internal disturbances" (truncated modes) are not ignored (i.e. they are not treated simply as external, unmeasurable disturbances).

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Chapter 4. Hybrid Coordinate Method Using Assumed Mode Shapes for Elastic Continua

ABSTRACT: The hybrid coordinate method provides equations of motion of minimum dimension for a spacecraft with flexible appendages. Instead of the usual finite element approach, in which mode shapes are calculated from equations of vibration of the finite element assembly, this chapter provides an alternative formulation using assumed mode shapes. This proves useful for a class of simply modeled appendages for which mode shapes are provided by an outside agency, or are otherwise known. The results are shown to be compatible with the finite element formulation, as previously described.

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I. INTRODUCTION

The hybrid-coordinate method provides equations of motion of a spacecraft with elastic (flexible) appendages. The appendages are modeled as an interconnected set of small rigid bodies interconnected by massless or massive elastic bodies (finite elements). From the Newton-Euler approach, the equations of motion for each finite element and the rigid body portion of the spacecraft are formulated. The introduction of an appropriate coordinate transformation allows the finite element equations to be represented as decoupled vibration equations, which involve mode shapes and modal coordinates. Since the vibration equations have been decoupled from each other, significant truncation of the higher order mode shapes can be accomplished. This leads to a set of equations where rotation of the rigid body portion of the spacecraft is coupled to the vibration of the flexible appendages. These equations are of great practical use because the truncation procedure has significantly reduced the number of degrees of freedom of the system without substantially sacrificing the fidelity of the results.

The purpose of this chapter is to provide an alternative formulation for the hybrid coordinate method using assumed mode shapes. This approach will prove useful for simply modeled appendages. For these the mode shapes can be determined from a continuum analysis using partial differential equation methods. The truncation procedure is accomplished at the outset by eliminating the higher order modes of vibration. The equations of motion are formulated using a Lagrangian approach and the coordinate transformation is accomplished using the assumed mode shapes. The resulting equations of motion are then seen to be compatible with those arising from the finite element method.

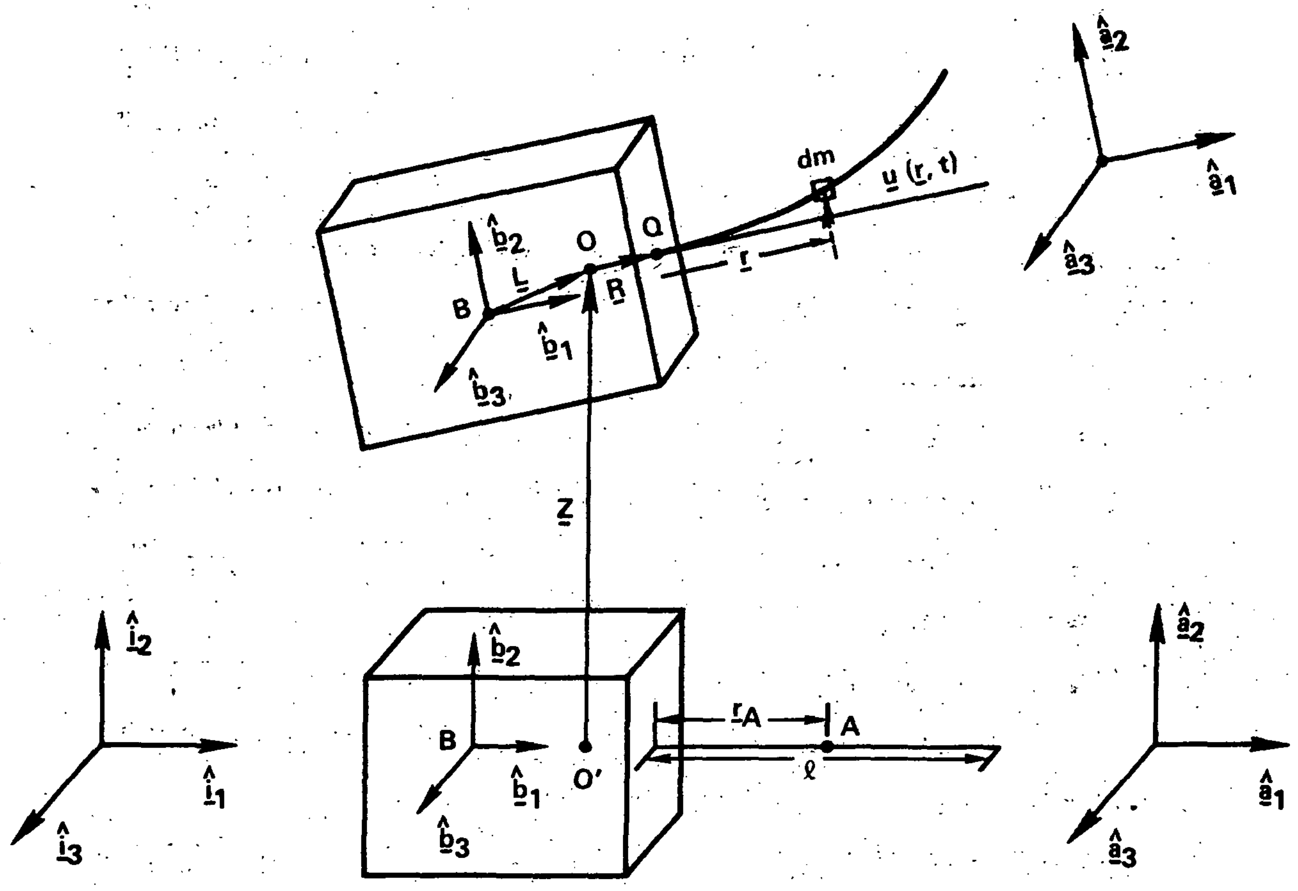

Figure 1. System Diagram.

where

O : Center of mass of undeformed system (body fixed)

O' : Position of O at rest (inertially fixed)

B : Center of mass of rigid body

A : Center of mass of undeformed appendage

$\mathcal{M}$: Total System mass

M : Appendage mass

m : Appendage mass/length

Q : Connection point of appendage

II. MODEL

The following derivation of equations of motion uses a model comprised of a central rigid body and a flexible cantilevered beam. Extensions to several appendages of arbitrary configurations may be made from the results of this simple model. The undeformed position of the appendage is taken to be constant relative to the rigid body. The transformation between the two is included in the derivation to facilitate the extension of the equations to cover a driven appendage. The angular rotations are assumed to be small as are the translational displacements. A diagram of the model is shown in Figure 1. To summarize, the assumptions used in the following derivation are:

- rigid body with cantilevered beam,
- beam rest position constant relative to base,
- small translations and rotations.

No orthogonality requirements have been placed on the assumed mode shapes. The vibration equations are therefore coupled. Further coordinate transformations may be employed to decouple the vibration equations or to achieve vehicle normal modes, but the truncation procedure does not require this as it does with the finite element procedure.

The vector bases employed in the derivation are:

$\{\underline{i}\}$: Inertially fixed basis

$\{\underline{b}\}$: Basis fixed in the rigid body

$\{\underline{a}\}$: Basis fixed in appendage prior to deformation

where

$$\{\underline{b}\} = [\theta]\{\underline{i}\}$$

$$[\theta] = (E - \tilde{\theta}) \text{ for small rotations } \tilde{\theta} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}$$

$$\{a\} = [c]\{b\}$$

$[c]$ = constant for an undriven appendage.

For the undeformed system, the location of the center of mass is defined by

$$\int_{SYS} \underline{\rho}\, dm = 0$$

where $\underline{\rho}$ is the generic position vector from the center of mass to the differential mass element. Evaluating this expression leads to

$$-(\mathcal{M}-M)\,\underline{L} + M\left(\underline{R} + \underline{r}_A\right) = 0$$

where the quantities are shown in Figure 1 with $\underline{L}$ being the location of the rigid body center of mass and $\underline{r}_A$ the appendage center of mass.

The dyadic of the undeformed system is defined by

$$\square^{*} = \square^{0}_{RB} + \square^{0}_{AP-u}$$

$$\square^{0}_{RB} = \square^{B}_{RB} + (\mathcal{M}-M)(\underline{L}\cdot\underline{L}\ \underline{U}-\underline{L}\underline{L})$$

$$\square^{0}_{AP-u} = \square^{A}_{AP-u} + M\left[\left(\underline{R}+\underline{r}_A\right)\cdot\left(\underline{R}+\underline{r}_A\right)\underline{U} - \left(\underline{R}+\underline{r}_A\right)\left(\underline{R}+\underline{r}_A\right)\right]$$

III. THE LAGRANGIAN OF THE SYSTEM

The kinetic energy of the system is

$$T_{SYS} = \frac{1}{2}\int_{RB} \underline{V}\cdot\underline{V}\, dm + \frac{1}{2}\int_{APP} \underline{V}\cdot\underline{V}\, dm$$

where $\underline{V}$ is the inertial velocity of a generic mass element. The kinetic energy for the rigid body yields

$$\frac{1}{2}\int_{RB} \underline{V}\cdot\underline{V}\, dm = \frac{1}{2}\left(\mathcal{M}-M\right)\dot{\underline{R}}^{B}\cdot\dot{\underline{R}}^{B} + \frac{1}{2}\,\underline{\omega}\cdot\square^{B}_{RB}\cdot\underline{\omega}$$

where $\mathcal{M}$ is the system mass, M the appendage mass, and B is the center of mass of the rigid body with

$$\underline{R}_B = \underline{Z} - \underline{L}$$

$$\dot{\underline{R}}_B = \dot{\underline{Z}} - \underline{\omega} \times \underline{L}.$$

Expanding this expression and switching the reference point of the inertia dyadic to the system mass center gives

$$\frac{1}{2}\int_{RB} \underline{V} \cdot \underline{V}\, dm = \frac{1}{2}\left(\mathcal{M} - M\right) \dot{\underline{Z}} \cdot \dot{\underline{Z}} + \frac{1}{2}\, \omega \cdot \square^{0}_{RB} \cdot \omega - \left(\mathcal{M} - M\right)\left[\dot{\underline{Z}} \cdot (\omega \times \underline{L})\right]$$

The kinetic energy of the appendage is

$$\frac{1}{2}\int_{APP} \underline{V} \cdot \underline{V}\, dm = \frac{1}{2}\int \dot{\underline{R}}_m \cdot \dot{\underline{R}}_m\, dm$$

where

$$\underline{R}_m = \underline{Z} + \underline{R} + \underline{r} + \underline{u}$$

$$\dot{\underline{R}}_m = \dot{\underline{Z}} + \dot{\underline{u}} + \overset{\circ}{\underline{R}} + \overset{\circ}{\underline{r}} + \underline{\omega} \times \left(\underline{R} + \underline{r}\right).$$

Vector differentiation with respect to the rotating reference frame is denoted by the "circle" above the vector. Here $\overset{\circ}{\underline{R}}$ and $\overset{\circ}{\underline{r}}$ are zero since they are fixed in the frame. The "dot" denotes differentiation relative to an inertial reference frame. Expanding the expression and making use of the dyadic of the undeformed appendage about the system mass center $\left(\square^{0}_{AP-u}\right)$ yields.

$$\begin{aligned}\frac{1}{2}\int_{APP} \underline{V} \cdot \underline{V}\, dm = {} & \frac{1}{2} M\dot{\underline{Z}} \cdot \dot{\underline{Z}} + \frac{1}{2}\, \underline{\omega} \cdot \square^{0}_{AP-u} \cdot \underline{\omega} + \frac{1}{2}\int \dot{\underline{u}} \cdot \dot{\underline{u}}\, dm \\ & + \dot{\underline{Z}} \cdot \int_{APP} \dot{\underline{u}}\, dm + \dot{\underline{Z}} \cdot \underline{\omega} \times \left[\left(M\underline{R} + \underline{r}_A\right)\right] \\ & + \int_{APP} \left[\dot{\underline{u}} \cdot \underline{\omega} \times \left(\underline{R} + \underline{r}\right)\right] dm\end{aligned}$$

Combining the terms for kinetic energy and eliminating terms produces

$$T_{SYS} = \frac{1}{2}\mathcal{M}\dot{Z}\cdot\dot{Z} + \frac{1}{2}\omega \cdot \square^{*} \cdot \omega$$

$$+ \frac{1}{2}\int_{APP} \underline{\dot{u}}\cdot\underline{\dot{u}}\ dm + \underline{Z}\cdot\int_{APP} \underline{\dot{u}}\ dm$$

$$+ \underline{\omega}\cdot\underline{R}\times\int_{APP} \underline{\dot{u}}\ dm + \underline{\omega}\cdot\int_{APP} (\underline{r}\times\underline{\dot{u}})\ dm.$$

The inertia dyadic of the undeformed system about the system center of mass is $\square^{*}$. The center of mass expression eliminated the term in the kinetic energy containing $\underline{Z}$ and $\underline{\omega}$.

From beam theory, the strain energy of the appendage is

$$u = \frac{1}{2}\int_{APP} EI\left(\frac{\partial^2 \underline{u}}{\partial r^2}\right)\cdot\left(\frac{\partial^2 \underline{u}}{\partial r^2}\right) dr$$

The Lagrangian for the system is then

$$\mathcal{L} = \frac{1}{2}\mathcal{M}\,\underline{\dot{Z}}\cdot\underline{\dot{Z}} + \frac{1}{2}\omega\cdot\square^{*}\cdot\underline{\omega} + \frac{1}{2}\int_{APP} \underline{\dot{u}}\cdot\underline{\dot{u}}\ dm$$

$$+ \underline{\dot{Z}}\cdot\int_{APP} \underline{\dot{u}}\ dm + \underline{\omega}\cdot\underline{R}\times\int_{APP} \underline{\dot{u}}\ dm$$

$$+ \underline{\omega}\cdot\int_{APP} (\underline{r}\times\underline{\dot{u}})\ dm - \frac{1}{2}\int_{APP} EI\left(\frac{\partial^2 \underline{u}}{\partial r^2}\right)\cdot\left(\frac{\partial^2 \underline{u}}{\partial r^2}\right) dr.$$

The formation of the above has assumed that the undeformed appendage is fixed relative to the base $\left(\underline{\omega}^{Bi} = \underline{\omega}^{Ai}\right)$. The next step to be taken is to assume small angle rotations and represent the Lagrangian in matrix form. The following matrices are used:

$$\underline{\dot{Z}} = \{\underline{i}\}^T\{\dot{Z}\}$$

$$\underline{\omega} = \{\underline{b}\}^T\{\dot{\theta}\} \qquad \text{(small rotations)}$$

$$\underline{u} = \{\underline{a}\}^T \{u\}$$

$$\square^{*} = \{\underline{b}\}^T I^{*} \{\underline{b}\}$$

$$\underline{R} = \{\underline{b}\}^T \{R\}$$

$$\underline{r} = \{\underline{b}\}^T \{r\}$$

$$\tilde{R} = \begin{bmatrix} 0 & -R_3 & R_2 \\ R_3 & 0 & -R_1 \\ -R_2 & R_1 & 0 \end{bmatrix} \quad ; \quad \tilde{r} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

$$\tilde{\dot{\theta}} = \begin{bmatrix} 0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ \dot{\theta}_3 & 0 & -\dot{\theta}_1 \\ -\dot{\theta}_2 & \dot{\theta}_1 & 0 \end{bmatrix}$$

$$[\theta] = [E - \tilde{\theta}]$$

$$\frac{\partial^2 u}{\partial r^2} = \{\underline{a}\}^T \begin{Bmatrix} \dfrac{\partial^2 u_1}{\partial r^2} \\ \dfrac{\partial^2 u_2}{\partial r^2} \\ \dfrac{\partial^2 u_3}{\partial r^2} \end{Bmatrix} = \{\underline{a}\}^T \{u''\}$$

Retaining second order terms in the Lagrangian produces

$$\begin{aligned} \mathscr{L} \doteq {} & \frac{1}{2}\mathscr{M}\{\dot{z}\}^T \{\dot{z}\} + \frac{1}{2}\{\dot{\theta}\}^T I^{*} \{\dot{\theta}\} \\ & + \frac{1}{2}\int_{APP} \{\dot{u}\}^T \{\dot{u}\}\, dm + \{\dot{z}\}[C]\int_{APP} \{\dot{u}\}\, dm \\ & + \{\dot{\theta}\}^T C\, \tilde{R}\int_{APP} \{\dot{u}\}\, dm + \{\dot{\theta}\}^T C\int_{APP} \tilde{r}\, \{\dot{u}\}\, dm \\ & - \frac{1}{2}\int_{APP} EI\, \{u''\}^T \{u''\}\, dr. \end{aligned}$$

Distributed coordinates are introduced by the coordinate transformation

$$\underline{u}(r,t) = \sum_{i=1}^{n} \underline{\phi}^{i}(r)\ \eta^{i}\ (t)$$

where n is the number of modes used to represent the displacement. In matrix form, the transformation is

$$\begin{aligned}\{u\} &= \sum_{i=1}^{n} \{\phi^{i}\}\ \eta^{i} \\ &= \left[\phi^{1} \mid \phi^{2} \mid \ldots \mid \phi^{\eta}\right]\{\eta\} \\ &= [\phi]\{\eta\}\end{aligned}$$

where $[\phi]$ is a $3 \times n$ matrix with each column corresponding to a mode shape and $\{\eta\}$ contains n modal coordinates.

This coordinate transformation yields the Lagrangian

$$\begin{aligned}\mathscr{L} = \frac{1}{2} \mathscr{M}\ \{\dot{z}\}^{T}\ \{\dot{z}\} &+ \frac{1}{2}\ \{\dot{\theta}\}^{T}\ I^{*}\ \{\dot{\theta}\} \\ &+ \frac{1}{2}\ \{\dot{\eta}\}^{T}\ X_{2}\ \{\dot{\eta}\} + \{\dot{z}\}^{T}\ c\ X_{1}\ \{\dot{\eta}\} \\ &+ \{\dot{\theta}\}^{T}\ c(\widetilde{R}\ X_{1}+X_{3})\ \{\dot{\eta}\} - \frac{1}{2}\ \{\eta\}^{T}\ X_{4}\ \{\eta\}\end{aligned}$$

where

$$X_{1} = \int_{APP} [\phi]\ dm$$

$$X_{2} = \int_{APP} [\phi]^{T}\ [\phi]\ dm$$

$$X_{3} = \int_{APP} \widetilde{r}\ [\phi]\ dm$$

$$X_{4} = \int_{APP} EI\ [\phi'']^{T}\ [\phi'']\ dr$$

The matrices X_1 and X_3 are of dimension $3 \times n$ while the matrices X_2 and X_4 are symmetric and of dimension $n \times n$. The Lagrangian depends on n+6 generalized coordinates. Six coordinates describe the translation and rotation of the undeformed system and n modal coordinates describe the displacement from rest of the flexible appendage relative to the rigid base.

IV. EQUATIONS OF MOTION

The equations of motion for the system may now be derived from the Lagrangian in the traditional manner. The resulting n+6 equations may be represented in matrix form as

$$\mathcal{M}\{\ddot{Z}\} + X_1 \{\ddot{\eta}\} = 0$$

$$I^* \{\ddot{\theta}\} + \left(\tilde{R} X_1 + X_3\right)\{\ddot{\eta}\} = \{T\}$$

$$X_2 \{\ddot{\eta}\} + X_4 \{\eta\} = -X_1^T \{\ddot{Z}\} + \left(X_1^T \tilde{R} - X_3^T\right)\{\ddot{\theta}\}$$

where $\{T\}$ is the externally applied torque. The first matrix equation may be used to eliminate the translation from the vibration equations. This produces n + 3 equations of the form

$$I^* \{\ddot{\theta}\} + \left(\tilde{R} X_1 + X_3\right)\{\ddot{\eta}\} = \{T\}$$

$$\left(X_2 - \frac{1}{\mathcal{M}} X_1^T X_1\right)\{\ddot{\eta}\} + X_4 \{\eta\} = \left(X_1^T \tilde{R} - X_3^T\right) \{\ddot{\theta}\}$$

The matrices that provide coupling between the rotation and vibration in each equation may be seen to be transposes of each other. The equations can be written as

$$I^* \{\ddot{\theta}\} - \delta^T \{\ddot{\eta}\} = \{T\}$$

$$\left(X_2 - \frac{1}{\mathcal{M}} X_1^T X_1\right) \{\ddot{\eta}\} + X_4 \{\eta\} = \delta\{\ddot{\theta}\}$$

where

$$\delta = X_1^T \tilde{R} - X_3^T.$$

V. COMPATIBILITY WITH FINITE ELEMENT EQUATIONS OF MOTION

The equations of motion derived from the continuum analysis are similar in structure to those derived from the finite element analysis (Ref. 1). The differences between the two appear in the assumptions made in the continuum analysis:

- No orthogonality properties
- No differential rotation of appendage mass elements due to deformation.

Orthogonality properties can be applied to the continuum analysis vibration equations by a suitable coordinate transformation. The orthogonality properties are not needed to permit truncation as is the case in the finite element analysis.

The equations of motion from a finite element analysis are shown by equations (287) to (289) of Reference 1.

$$I^* \ddot{\theta} - \bar{\delta}^T \ddot{\bar{\eta}} = T$$

$$\ddot{\bar{\eta}} + 2\bar{\delta}\bar{\sigma}\bar{\eta} + \bar{\sigma}^2\bar{\eta} = \bar{\delta}\, \ddot{\theta}$$

$$\bar{\delta} = -\phi^T M\left(\Sigma_{OE} - \Sigma_{EO}\tilde{R} - \tilde{r}\,\Sigma_{EO}\right)$$

The overbar indicates truncation. If the damping is eliminated and the orthogonality condition relaxed (after truncation), the equations become

$$I^* \ddot{\theta} - \bar{\delta}^T \ddot{\bar{\eta}} = T$$

$$\phi^T M' \phi\, \ddot{\bar{\eta}} + \phi^T K' \phi\, \bar{\eta} = \bar{\delta}\, \ddot{\theta}$$

where

$$M' = M\left(E - \Sigma_{EO}\Sigma_{EO}^T M/\mathcal{M}\right)$$

K' = Stiffnes- matrix.

Normally, the coordinate transformation $\bar{\phi}$ includes mode shapes with translation and rotation of the finite elements. To agree with the continuum analysis, no rotations of the finite elements will be allowed. The coordinate transformation $\bar{\phi}$ will then be a $6n \times N$ matrix represented by

$$\bar{\phi} = \left[\{\phi^1\}\ \{\phi^2\}\ \ldots\{\phi^N\}\right]$$

where

$$\{\phi^i\} = \left\{\begin{bmatrix} \{\phi^i_{D1}\} \\ 0 \\ \cdot \\ \vdots \\ \cdot \\ \{\phi^i_{D_n}\} \\ 0 \end{bmatrix}\right\}$$

With the above limitations, the matrix multiplication can be performed in the finite element equations and the terms may be compared with those from the continuum analysis.

For the augmented mass matrix, the finite element analysis results in

$$\left(\bar{\phi}^T M \bar{\phi}\right)_{ij} = \sum_{k=1}^{n} m_k \left\{\phi^i_{D_k}\right\}^T \left\{\phi^j_{D_k}\right\}$$

$$+ \sum_{\ell=1}^{n} \sum_{k=1}^{n} \left(\frac{m^k m^\ell}{\mathcal{M}} \left\{\phi^i_{D_\ell}\right\}^T \left\{\phi^j_{D_k}\right\}\right).$$

This is compatible with the result from the continuum analysis

$$\left(X_2 - \frac{1}{\mathcal{M}} X_1^T X_1\right)_{ij} = \int_{APP} m(r) \left[\phi^i\right]^T \left[\phi^j\right] dr$$

$$+ \iint_{APP} \frac{m(r)m(r)}{\mathcal{M}} \left[\phi^i\right]^T \left[\phi^j\right] dr\ dr.$$

For the δ matrix, the finite element analysis results in

$$[\bar{\delta}]_j = \sum_{i=1} m_i \left\{ \phi_{D_i}^j \right\}^T \left(\tilde{R} + \tilde{r}_i \right)$$

This is compatible with the result from the continuum analysis

$$[\delta]_j = \int m(r) \left[\phi_D^j \right]^T \left(\tilde{R} + \tilde{r} \right) dr$$

Thus, if the number of finite elements were increased without limit, the finite element equations would be identical to the continuum analysis equations.

VI. CONCLUSION

With the foregoing results it becomes possible to accomplish a hybrid coordinate dynamic analysis for a system with appendages defined only in terms of modal data based on a continuum analysis. This is a necessary step before we can simulate an LST vehicle with FRUSCA solar panels defined by the modal data provided by HAC.

REFERENCE

1. P. W. Likins, "Dynamics and Control of Flexible Space Vehicles," JPL Technical Report 32-1329, Rev. 1, January 15, 1970.

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