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GRAVITATIONAL FORCES AND MOMENTS ON SPACECRAFT

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16. ABSTRACT

The solution of many problems of attitude dynamics of spacecraft requires a thorough understanding of gravitational forces and moments. This understanding can be communicated most directly in terms of arguments based on Newton's law of gravitation, employing the methods of Newtonian (vectorial) mechanics, and with minimal recourse to the classical concepts of potential theory. In this report the necessary ideas are developed and relationships are established to permit the representation of gravitational forces and moments exerted on bodies in space by other bodies, both in terms involving the mass distribution properties of the bodies, and in terms of vector operations on those scalar functions classically described as gravitational potential functions.

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PREFACE

This volume is the second in a series of reports being prepared by Professors Kane and Likins on various aspects of spacecraft attitude dynamics. The first volume, "Kinematics of Rigid Bodies in Spaceflight," was presented as Technical Report No. 204 of the Stanford University Department of Applied Mechanics, published in May of 1971. Subsequent volumes will deal with non-gravitational forces and moments on spacecraft, with the attitude dynamics of rigid spacecraft, and with the attitude dynamics of nonrigid spacecraft. In time, the series will be assembled as a book intended both for classroom use and as a reference for practicing engineers.

Most of the development herein is the unsponsored enterprise of the authors, forming the basis for the spacecraft attitude dynamics studies of Contract NAS8-28358.

The present volume, "Gravitational Forces and Moments on Spacecraft," furnishes a unique perspective on a subject that was developed in its classical form long before the era of modern space exploration. The development provided here differs from the traditional presentation of the subject by focusing attention on applications to problems of attitude dynamics of the spacecraft. In the opinion of the authors, understanding of gravitational forces and moments required for the solution of problems of attitude dynamics can be communicated most directly in terms of arguments based on Newton's law of gravitation, presented by employing the methods of Newtonian (vectorial) mechanics, with minimal recourse to the classical concepts of potential theory. Accordingly, in this report, the necessary ideas are developed and relationships are established to permit the representation of gravitational forces and moments exerted on bodies in space by other bodies, both in terms involving the mass distribution properties of the bodies, and in terms of vector operations on scalar functions classically described as gravitational potential functions.

CONTENTS

2.1 Gravitational interaction of two particles

A particle P of mass m experiences in the presence of a particle $\bar{P}$ of mass $\bar{m}$ a force $\underline{F}$ acting along the line joining P and $\bar{P}$, directed from P toward $\bar{P}$, and having a magnitude proportional to the product of m and $\bar{m}$ and inversely proportional to the square of the distance between P and $\bar{P}$. Hence, if $\underline{p}$ is the position vector of P relative to $\bar{P}$ (see Fig. 2.1.1), the force $\underline{F}$ can be expressed as*

$$\underline{F} = -G\bar{m}m\underline{p}(\underline{p}^2)^{-3/2} \tag{1}$$

where G is the universal gravitational constant, given numerically by† $G \cong 6.6732 \times 10^{-11} Nm^2kg^{-2}$.

The force $\bar{\underline{F}}$ experienced by $\bar{P}$ in the presence of P is

$$\bar{\underline{F}} = -\underline{F} \tag{2}$$

in conformity with Newton's third law.

Example: If $\bar{m}$ is the mass of the earth, m the mass of the moon, and $\underline{p}$ the vector from the mass center of the earth to the mass center of the moon, and if the numerical values of $\bar{m}$, m and $|\underline{p}|$ are given approximately by $\bar{m} \cong 5.97 \times 10^{24}$kg, $m \cong 7.34 \times 10^{22}$kg and $|\underline{p}| = 3.844 \times 10^8$m, what is the magnitude of the force $\underline{F}$ exerted on the moon by the earth?

As Eq. (1) applies to particles, it can be used for the purpose at hand only with the supposition that the earth and moon can be replaced by

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*The superscript 2 on a vector indicates scalar multiplication of the vector with itself, i.e., squaring the magnitude of the vector.

†E.A. Mechtly, "The International System of Units: Physical Constants and Conversion Factors," NASA SP-7012, Revised, 1969.

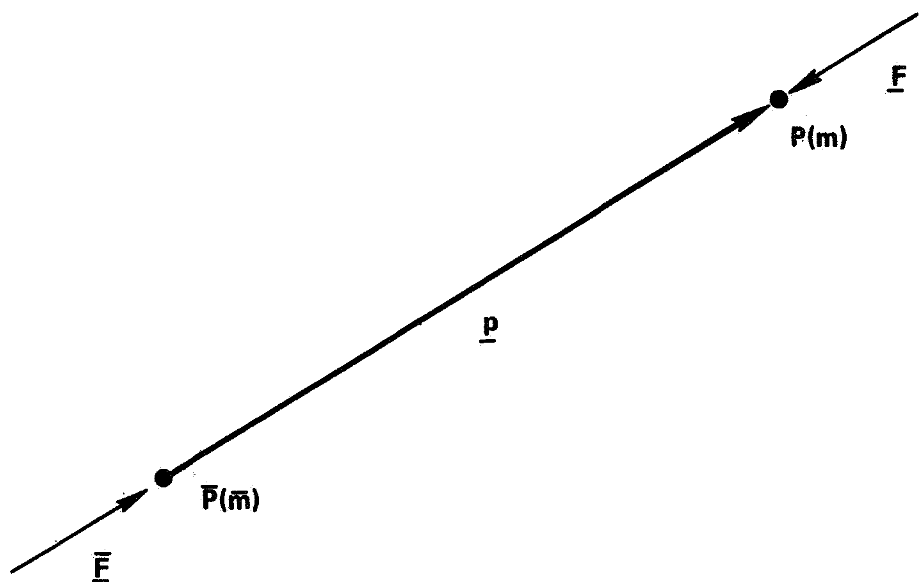


Figure 2.1.1

particles situated at their mass centers and having masses $\bar{m}$ and m, respectively. In that case,

$$
\begin{aligned}
|\underline{F}| &\underset{(1)}{=} G\bar{m}m|\underline{p}|(\underline{p}^2)^{-3/2} \\
&\cong (3.98602 \times 10^{14}\,\mathrm{Nm^2kg^{-1}})(7.34 \times 10^{22}\,\mathrm{kg})(3.844 \times 10^{8}\,\mathrm{m})^{-2} \\
&\cong 1.98 \times 10^{20}\ \mathrm{N} \qquad (3)
\end{aligned}
$$

The validity of this supposition is examined in Sec. 2.4.

2.2 Force exerted on a body by a particle

The system of gravitational forces exerted by a particle $\overline{P}$ of mass $\overline{m}$ on the particles of a (not necessarily rigid) body B is equivalent to a single force $\underline{F}$ whose line of action passes through $\overline{P}$, but not necessarily through the mass center B^* of B. If B consists of particles $P_1, \ldots, P_N$ of masses $m_1, \ldots, m_N$, and if $\underline{p}_1, \ldots, \underline{p}_N$ are the position vectors of $P_1, \ldots, P_N$ relative to $\overline{P}$ (see Fig. 2.2.1), then $\underline{F}$ is given by

$$\underline{F} = -G\overline{m} \sum_{i=1}^{N} m_i \underline{p}_i (\underline{p}_i^2)^{-3/2} \tag{1}$$

whereas, if B is a continuous distribution of matter, and ρ is the mass density of B at a generic point P of B, $\underline{p}$ is the position vector of P relative to $\overline{P}$ (see Fig. 2.2.2), and $d\tau$ is the length, area, or volume of a differential element of the figure (curve, surface, or solid) occupied by B, then $\underline{F}$ can be expressed as

$$\underline{F} = -G\overline{m} \int \underline{p}(\underline{p}^2)^{-3/2} \rho \, d\tau \quad . \tag{2}$$

Once the line of action of the force $\underline{F}$ in Eq. (1) or Eq. (2) has been established, it is always possible to locate a point B' on this line such that the force exerted by $\overline{P}$ on a particle whose mass is equal to that of B and which is placed at B' is equal to the force $\underline{F}$ exerted on B by $\overline{P}$. The point B', called the <u>center of gravity</u> of B for the attracting particle $\overline{P}$, does not in general coincide with the mass center of B; and if R' is the distance from $\overline{P}$ to the center of gravity B', then

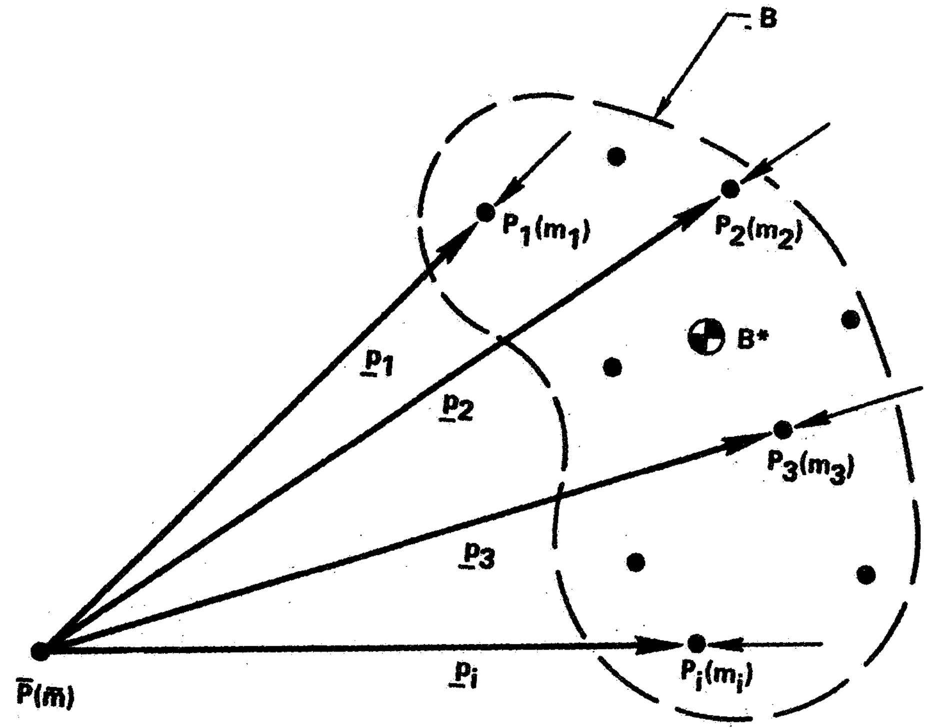

B
$P_1(m_1)$
$P_2(m_2)$
B*
$\underline{p}_1$
$\underline{p}_2$
$P_3(m_3)$
$\underline{p}_3$
$\bar{P}(\bar{m})$
$\underline{p}_i$
$P_i(m_i)$

Figure 2.2.1

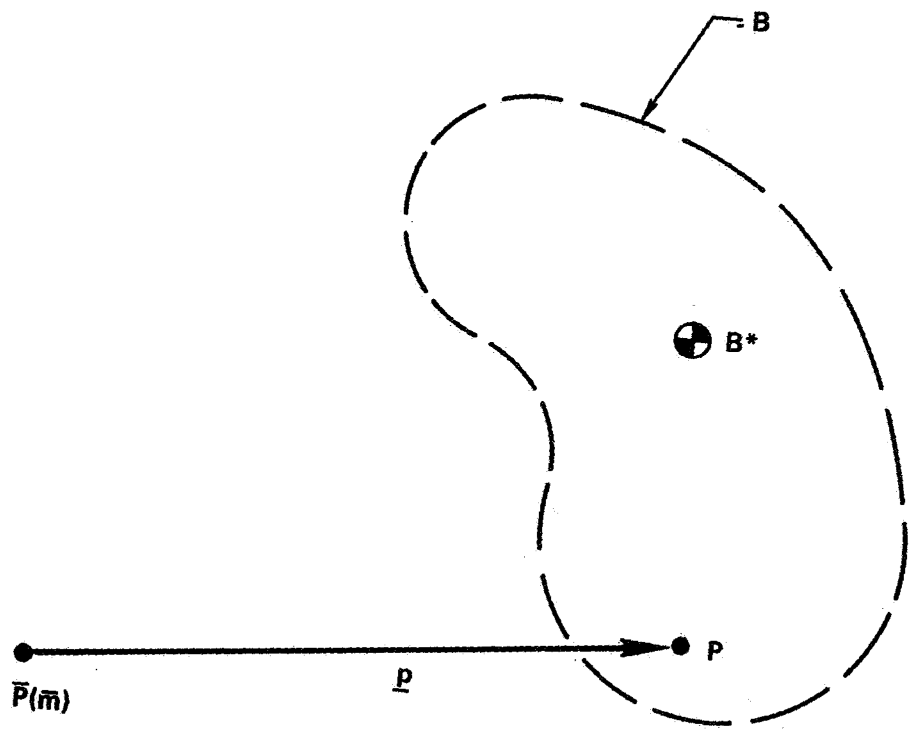


Figure 2.2.2

$$R' = \left(\frac{G\bar{m}m}{|\underline{F}|}\right)^{1/2} \tag{3}$$

where $\underline{F}$ is obtained from Eq. (1) or Eq. (2), and m is the mass of B.

<u>Derivations:</u> The force $d\underline{F}$ exerted by $\bar{P}$ on a differential element of B at point P acts along the line joining P to $\bar{P}$ and is given by

$$d\underline{F} \underset{(2.1.1)}{=} -G\bar{m}\underline{p}(\underline{p}^2)^{-3/2}\rho d\tau$$

By definition, two systems of forces are equivalent if they have equal resultants and equal moments about one point. Now, on the one hand, the resultant of the system of forces exerted by $\bar{P}$ on all the differential elements of B is given by

$$\int d\underline{F} = -G\bar{m}\int \underline{p}(\underline{p}^2)^{-3/2}\rho d\tau$$

and the moment of the system of forces about point $\bar{P}$ is equal to zero, because $d\underline{F}$, acting along the line joining P to $\bar{P}$, has zero moment about $\bar{P}$. On the other hand, the resultant of a system of forces containing but one force $\underline{F}$ is $\underline{F}$, itself, and the moment of this system about point $\bar{P}$ is zero if the line of action of $\underline{F}$ passes through $\bar{P}$. Hence, if $\underline{F}$ is given by Eq. (2) and acts along a line of action passing through $\bar{P}$, then $\underline{F}$ is equivalent to the system of forces exerted by $\bar{P}$ on all differential elements of B.

A parallel proof may be constructed for Eq. (1), which replaces Eq. (2) when B consists of a finite number of particles.

The truth of the assertion that the line of action of the force $\underline{F}$ does not necessarily pass through B^* is most easily demonstrated by an example;

and Eq. (3) follows directly from the definition of R' and Eq. (2.1.1).

Example: A uniform thin rod B of length L and mass m is subjected to the gravitational attraction of a particle $\overline{P}$ of mass $\overline{m}$ as shown in Fig. 2.2.3. The system of forces exerted by $\overline{P}$ on the particles comprising B is to be replaced with a force $\underline{F}$ whose line of action passes through $\overline{P}$, and an expression is to be found for the distance R' between $\overline{P}$ and the center of gravity B' of B for $\overline{P}$.

If the rod is regarded as matter distributed along a straight line segment, then the position vector $\underline{p}$ of a generic point P of the rod relative to $\overline{P}$ can be expressed as

$$\underline{p} = y\underline{b}_1 - a\underline{b}_2 \tag{4}$$

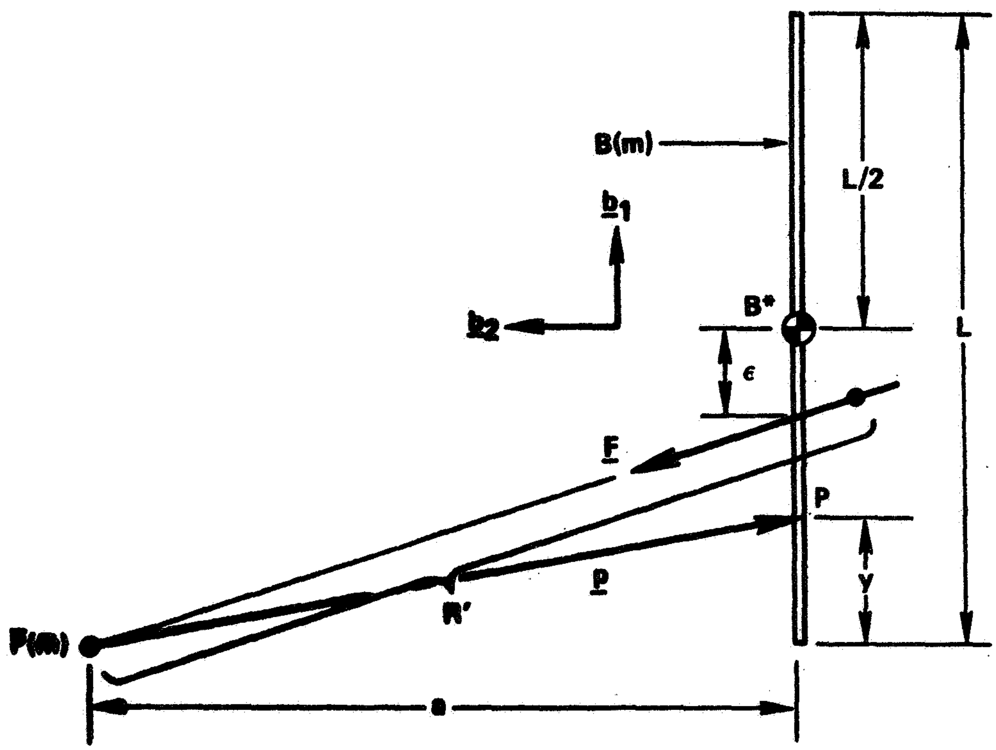
where $\underline{b}_1$ and $\underline{b}_2$ are unit vectors directed as shown in Fig. 2.2.3 and where y varies from 0 to L. The mass density ρ of B at P is then equal to m/L, and a differential element of B has a length dy. Hence the force $\underline{F}$, resolved into components parallel to $\underline{b}_1$ and $\underline{b}_2$, can be written

$$\underline{F} = F_1\underline{b}_1 + F_2\underline{b}_2 \underset{(2)}{=} -G\overline{m}\int_0^L (y\underline{b}_1 - a\underline{b}_2)(a^2 + y^2)^{-3/2}\frac{m}{L}\,dy \tag{5}$$

and integration yields

$$F_1 = -Gm\overline{m}\left[\left(1 + \frac{a^2}{L^2}\right)^{1/2} - \frac{a}{L}\right]\Big/ a(a^2 + L^2)^{1/2} \tag{6}$$

$$F_2 = Gm\overline{m}/a(a^2 + L^2)^{1/2} \tag{7}$$



B(m)
$\underline{b}_1$
$\underline{b}_2$
L/2
B*
L
ϵ
$\underline{F}$
P
$\underline{P}$
y
R'
F(m)
a

Figure 2.2.3

From Fig. 2.2.3 it can be seen that the distance ε between the mass center B^* of B and the interscetion of B with the line of action of $\underline{F}$ is related to F_1 and F_2 by

$$-\frac{F_1}{F_2} = \frac{\left(\frac{L}{2}-\varepsilon\right)}{a} \tag{8}$$

or

$$\frac{\varepsilon}{L} \underset{(8)}{=} \frac{1}{2} + \frac{aF_1}{LF_2} \underset{(6,7)}{=} \frac{1}{2} - \frac{a}{L}\left[\left(1 + \frac{a^2}{L^2}\right)^{1/2} - \frac{a}{L}\right] \tag{9}$$

The ratio ε/L from Eq. (9) is plotted vs. a/L in Fig. 2.2.4. Since two bodies cannot occupy the same point in space, the limiting case a/L = 0 must be excluded from consideration. However, as this limit is approached, the ratio ε/L approaches the value 1/2 and the line of action of $\underline{F}$ approaches coincidence with B. For any finite value of a/L, Fig. 2.2.4 indicates that the line of action $\underline{F}$ cannot pass through both $\bar{P}$ and B^*.

The distance R' from $\bar{P}$ to the center of gravity B' of B for $\bar{P}$ is given by

$$R' \underset{(3)}{=} \left[\frac{G\bar{m}m}{(F_1^2+F_2^2)^{1/2}}\right]^{1/2} \underset{(6,7)}{=} \frac{(aL)^{1/2}}{\{2[1-\frac{a}{L}(1+\frac{a^2}{L^2})^{-1/2}]\}^{1/4}}$$

It is clear from this example that the location of the center of gravity B' is not in general a property of body B alone, but depends also on the position of the attracting particle $\bar{P}$.

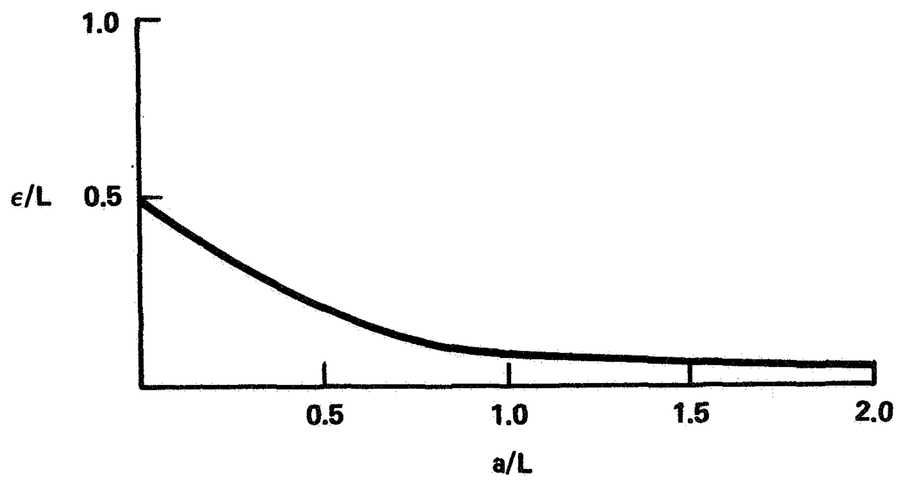


Figure 2.2.4

2.3 Force exerted on a small body by a particle

When a body B is subjected to the gravitational attraction of a particle $\bar{P}$ removed so far from the mass center B^* of B that the largest distance from B^* to any point P of B is considerably smaller than the distance R between $\bar{P}$ and B^*, a useful form of the expression for the force $\underline{F}$ given in Eq. (2.2.2) can be found as follows: Replace $\underline{p}$ with the sum of $\underline{R}$, the position vector of B^* relative to $\bar{P}$, and $\underline{r}$, the position vector of P relative to B^* (see Fig. 2.3.1), and then expand the integrand in ascending powers of $|\underline{r}|/R$ to obtain

$$\underline{F} = -\frac{Gm\bar{m}}{R^2}\left(\underline{a}_1 + \sum_{i=2}^{\infty} \underline{f}^{(i)}\right) \tag{1}$$

where $\underline{f}^{(i)}$ is a collection of terms of i^{th} degree in $|\underline{r}|/R$, m and $\bar{m}$ are the masses of B and $\bar{P}$, G is the universal gravitational constant, and $\underline{a}_1$ is a unit vector directed from $\bar{P}$ toward B^*, so that

$$\underline{R} = R\underline{a}_1 \tag{2}$$

In particular, $\underline{f}^{(2)}$ is given by

$$\underline{f}^{(2)} = \frac{1}{mR^2}\{\frac{3}{2}\,[\mathrm{tr}(\underline{I}) - 5\underline{a}_1\cdot\underline{I}\cdot\underline{a}_1]\underline{a}_1 + 3\underline{I}\cdot\underline{a}_1\} \tag{3}$$

where $\underline{I}$ is the inertia dyadic of B for B^*, and $\mathrm{tr}(\underline{I})$ denotes a scalar invariant of $\underline{I}$, called the trace of $\underline{I}$ and defined in terms of <u>any</u> mutually orthogonal unit vectors $\underline{n}_1$, $\underline{n}_2$, and $\underline{n}_3$ as

$$\mathrm{tr}(\underline{I}) \triangleq \underline{n}_1\cdot\underline{I}\cdot\underline{n}_1 + \underline{n}_2\cdot\underline{I}\cdot\underline{n}_2 + \underline{n}_3\cdot\underline{I}\cdot\underline{n}_3 \tag{4}$$

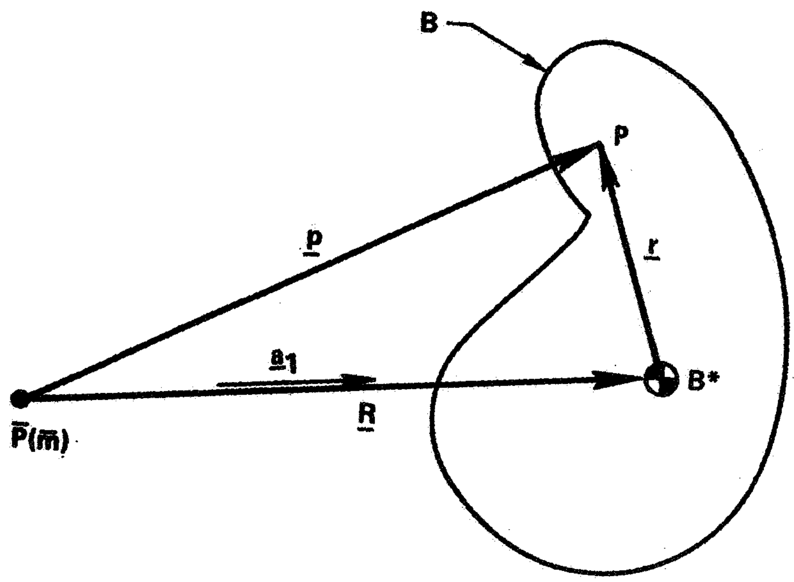


Figure 2.3.1

Eq. (1) suggests two approximations which we shall find useful in the sequel:

$$\underline{F} \cong \hat{\underline{F}} \triangleq -\frac{G\bar{m}m}{R^2}\underline{a}_1 \tag{5}$$

and

$$\underline{F} \cong \tilde{\underline{F}} \triangleq -\frac{G\bar{m}m}{R^2}\left(\underline{a}_1 + \underline{f}^{(2)}\right) \tag{6}$$

Two expressions for $\underline{f}^{(2)}$ which sometimes furnish convenient alternatives to the one given in Eq. (3) are obtained as follows: Introduce unit vectors $\underline{a}_2$ and $\underline{a}_3$ such that $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ form a dextral, orthogonal set, and let $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ be unit vectors respectively parallel to principal axes of B for B^* and also forming a dextral orthogonal set. Next, define I_{jk} and I_j as

$$I_{jk} \triangleq \underline{a}_j \cdot \underline{I} \cdot \underline{a}_k \qquad (j,k = 1,2,3) \tag{7}$$

and

$$I_j \triangleq \underline{b}_j \cdot \underline{I} \cdot \underline{b}_j \qquad (j = 1,2,3) \tag{8}$$

Finally, let

$$C_{ij} \triangleq \underline{a}_i \cdot \underline{b}_j \qquad (i,j = 1,2,3) \tag{9}$$

Then $\underline{f}^{(2)}$ may be written either as

$$\underline{f}^{(2)} = \frac{3}{mR^2}\left[\frac{1}{2}(I_{22} + I_{33} - 2I_{11})\underline{a}_1 + I_{21}\underline{a}_2 + I_{31}\underline{a}_3\right] \tag{10}$$

or as

$$\begin{aligned}\underline{f}^{(2)} = \frac{3}{mR^2}\Big\{&\frac{1}{2}[I_1(1-3C_{11}^2) + I_2(1-3C_{12}^2) + I_3(1-3C_{13}^2)]\underline{a}_1\\ &+ [I_1C_{21}C_{11} + I_2C_{22}C_{12} + I_3C_{23}C_{13}]\underline{a}_2\\ &+ [I_1C_{31}C_{11} + I_2C_{32}C_{12} + I_3C_{33}C_{13}]\underline{a}_3\Big\}\end{aligned} \tag{11}$$

The relative simplicity of Eq. (10) is a result of the use of time-dependent moments and products of inertia [see Eq. (7)]. By way of contrast, the principal moments of inertia appearing in Eq. (11) are constants, and the orientation of B relative to $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ now comes into evidence through the direction cosines relating the two sets of unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$.

Derivations: Replacing $\underline{p}$ in Eq. (2.2.2) by $\underline{R} + \underline{r}$ (see Fig. 2.3.1) provides

$$\underline{F} = -G\bar{m}\int(\underline{R} + \underline{r})\cdot(\underline{R}^2 + 2\underline{R}\cdot\underline{r} + \underline{r}^2)^{-3/2}\rho d\tau \tag{12}$$

In terms of the normalized position vector $\underline{q}$ defined by

$$\underline{q} \triangleq \underline{r}/R \tag{13}$$

and the unit vector $\underline{a}_1$ satisfying

$$\underline{a}_1 \underset{(2)}{=} \underline{R}/R \tag{14}$$

the force becomes

$$\underline{F} \underset{(12,13,14)}{=} -\frac{G\bar{m}}{R^2}\int(\underline{a}_1 + \underline{q})\ (1 + 2\underline{a}_1\cdot\underline{q} + \underline{q}^2)^{-3/2}\rho d\tau \tag{15}$$

Application of the binomial series expansion

$$(1+x)^n = 1 + nx + \frac{1}{2!}\,n(n-1)x^2 + \frac{1}{3!}\,n(n-1)(n-2)x^3 + \ldots \tag{16}$$

(valid for $x < 1$) to the exponentiated quantity then yields, for $2\underline{a}_1\cdot\underline{q} + \underline{q}^2 < 1$, the expression

$$\underline{F} \underset{(15,16)}{=} -\frac{G\bar{m}}{R^2}\int\{\underline{a}_1[1-\frac{3}{2}(2\underline{a}_1\cdot\underline{q} + \underline{q}^2) + \frac{15}{2}(\underline{a}_1\cdot\underline{q})^2 + \ldots]$$

$$+ \underline{q}[1 - 3\underline{a}_1\cdot\underline{q} + \ldots]\}\rho d\tau \tag{17}$$

where dots represent terms of degree three and higher in $|\underline{q}|$. This expression can be simplified by taking advantage of the fact that B^* is the mass center of B, for this means that

$$\int\underline{q}\rho d\tau = 0 \tag{18}$$

so that

$$\underline{F} \underset{(17,18)}{=} -\frac{G\bar{m}}{R^2}\int[\underline{a}_1 - \frac{3}{2}\,\underline{a}_1\underline{q}^2 + \frac{15}{2}\,\underline{a}_1(\underline{a}_1\cdot\underline{q})^2 - 3\underline{q}\underline{a}_1\cdot\underline{q}]\rho d\tau + \ldots \tag{19}$$

Furthermore, replacing $\underline{q}$ with $\underline{r}/R$ and observing that

$$\int \rho d\tau = m \tag{20}$$

one arrives at

$$\underline{F} \underset{(19,20)}{=} -\frac{G\bar{m}m}{R^2}\,\underline{a}_1 + \frac{3G\bar{m}}{2R^4}\Big[\underline{a}_1\int \underline{r}^2\rho d\tau - 5\underline{a}_1\underline{a}_1\cdot\int \underline{r}\underline{r}\rho d\tau\cdot\underline{a}_1$$

$$+ 2\int \underline{r}\underline{r}\rho d\tau\cdot\underline{a}_1\Big] + \ldots \tag{21}$$

The integrals appearing in Eq. (2) can be related to inertia properties of B by introducing two quantities: the inertia dyadic $\underline{I}$ of B for B*, defined as

$$\underline{\underline{I}} \triangleq \int (\underline{\underline{U}}\underline{r}^2 - \underline{r}\underline{r})\rho d\tau \tag{22}$$

where $\underline{U}$ denotes the unit dyadic, that is, a dyadic such that, for any vector $\underline{v}$, $\underline{v}\cdot\underline{U} = \underline{U}\cdot\underline{v} = \underline{v}$; and the trace of $\underline{I}$, defined in Eq. (4). For it follows from Eqs. (22) and (4) that

$$\mathrm{tr}(\underline{I}) = \int [3\underline{r}^2 - (\underline{n}_1\cdot\underline{r}\underline{r}\cdot\underline{n}_1 + \underline{n}_2\cdot\underline{r}\underline{r}\cdot\underline{n}_2 + \underline{n}_3\cdot\underline{r}\underline{r}\cdot\underline{n}_3)]\rho d\tau$$

$$= \int (3\underline{r}^2 - \underline{r}^2)\rho d\tau = 2\int \underline{r}^2\rho d\tau \tag{23}$$

so that

$$\int \underline{r}^2\rho d\tau \underset{(23)}{=} \frac{1}{2}\,\mathrm{tr}(\underline{I}) \tag{24}$$

and

$$\int \underline{r}\,\underline{r}\rho d\tau \underset{(22)}{=} \underline{U}\int \underline{r}^2 \rho d\tau - \underline{I} \underset{(24)}{=} \underline{U}\,\frac{tr(\underline{I})}{2} - \underline{I} \tag{25}$$

Consequently

$$\underline{F} \underset{(21,24,25)}{=} -\frac{G\bar{m}m}{R^2}\underline{a}_1 - \frac{3G\bar{m}}{2R^4}[tr(\underline{I}) - 5\underline{a}_1\cdot\underline{I}\cdot\underline{a}_1]\underline{a}_1 - \frac{3G\bar{m}}{R^4}\underline{I}\cdot\underline{a}_1 + \ldots \tag{26}$$

and the equivalence of Eqs. (1) and (26) becomes apparent after using Eq. (3) and recognizing that the three dots in Eq. (26) stand for terms of degree three and higher in $|\underline{r}|/R$, these terms being represented in Eq. (1) by $\underline{f}^{(i)}$ for $i \geq 3$.

Referring to Eq. (7), one can express $\underline{I}$ as

$$\begin{aligned}\underline{I} = {} & I_{11}\underline{a}_1\underline{a}_1 + I_{12}\underline{a}_1\underline{a}_2 + I_{13}\underline{a}_1\underline{a}_3 \\ & + I_{21}\underline{a}_2\underline{a}_1 + I_{22}\underline{a}_2\underline{a}_2 + I_{23}\underline{a}_2\underline{a}_3 \\ & + I_{31}\underline{a}_3\underline{a}_1 + I_{32}\underline{a}_3\underline{a}_2 + I_{33}\underline{a}_3\underline{a}_3\end{aligned}$$

from which it follows that

$$\underline{I}\cdot\underline{a}_1 = I_{11}\underline{a}_1 + I_{21}\underline{a}_2 + I_{31}\underline{a}_3$$

Substituting from this equation into Eq. (3) and noting that

$$tr(\underline{I}) \underset{(4,7)}{=} I_{11} + I_{22} + I_{33}$$

while

$$\underline{a}_1 \cdot \underline{I} \cdot \underline{a}_1 \underset{(7)}{=} I_{11}$$

one arrives at Eq. (10).

Finally, Eq. (11) may be obtained from Eq. (10) after observing that, in view of Eq. (8) and of the assumption that $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are parallel to principal axes of inertia of B for B^*, $\underline{I}$ can be expressed as

$$\underline{I} = I_1\underline{b}_1\underline{b}_1 + I_2\underline{b}_2\underline{b}_2 + I_3\underline{b}_3\underline{b}_3$$

so that

$$I_{11} \underset{(7)}{=} I_1\underline{a}_1 \cdot \underline{b}_1\underline{b}_1 \cdot \underline{a}_1 + I_2\underline{a}_1 \cdot \underline{b}_2\underline{b}_2 \cdot \underline{a}_1 + I_3\underline{a}_1 \cdot \underline{b}_3\underline{b}_3 \cdot \underline{a}_1$$

$$\underset{(9)}{=} I_1C_{11}^2 + I_2C_{12}^2 + I_3C_{13}^2$$

and, similarly,

$$I_{21} = I_1C_{21}C_{11} + I_2C_{22}C_{12} + I_3C_{23}C_{13}$$

and so forth. Substitution into Eq. (10) then yields Eq. (11) if one makes use of the fact that $C_{i1}^2 + C_{i2}^2 + C_{i3}^2 = 1$ for $i = 1, 2, 3$.

<u>Example</u>: An approximate expression is required for the force $\underline{F}$ exerted on a uniform thin rod B of mass m and length L by a particle $\overline{P}$ of mass $\overline{m}$ located relative to B as shown in Fig. 2.3.2, with R >> L. Body B is to be

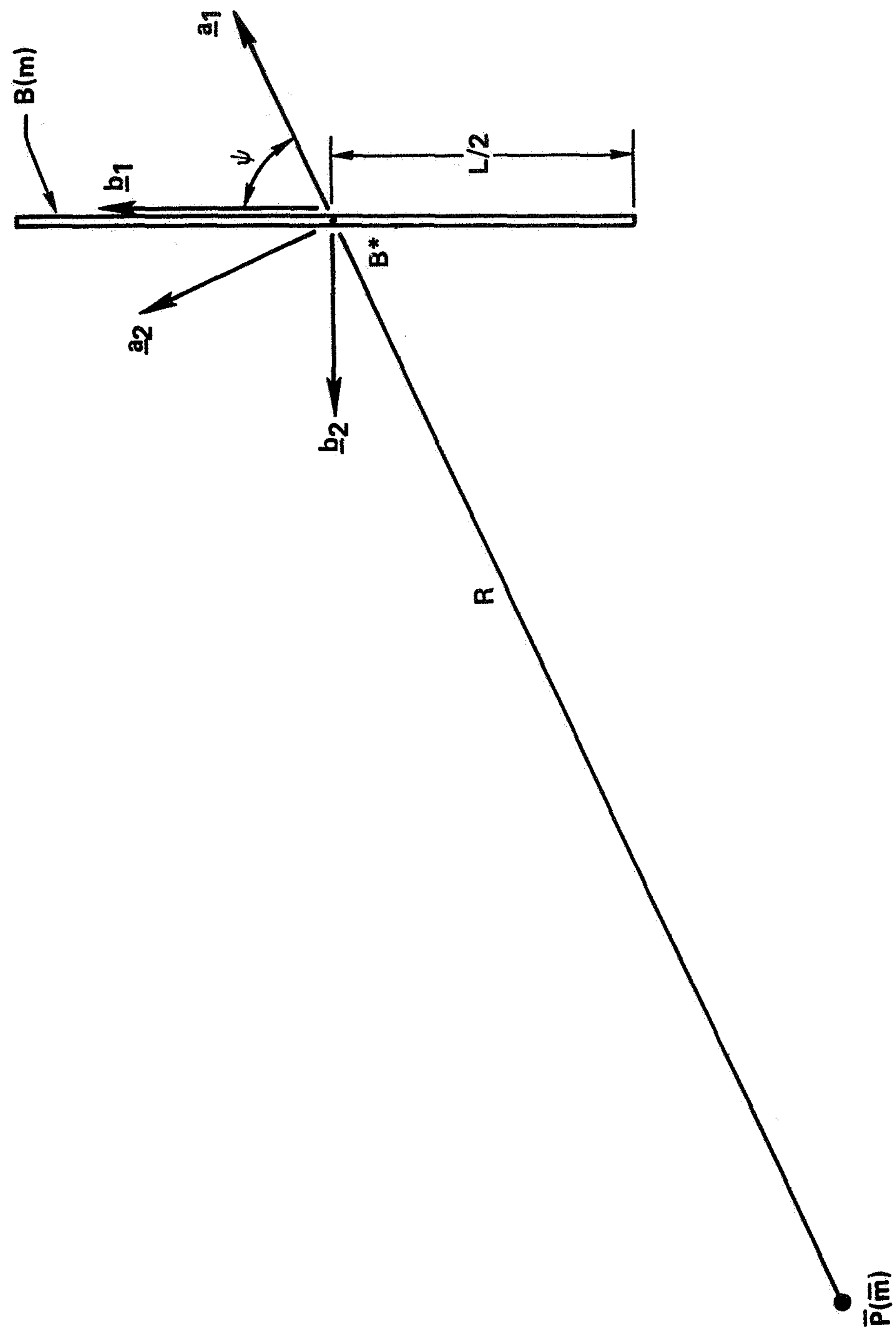

B(m)
$\underline{b}_1$
ψ
$\underline{a}_1$
L/2
B*
$\underline{a}_2$
$\underline{b}_2$
R
$\bar{P}(\bar{m})$

Figure 2.3.2

idealized as matter distributed uniformly along a straight line segment.

When $\underline{f}^{(3)}$, $\underline{f}^{(4)}$, etc. are omitted, Eq. (6) provides the required approximate force expression. If $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are introduced as a set of dextral, orthogonal unit vectors, with $\underline{b}_1$ parallel to the rod axis, then the inertia dyadic $\underline{I}$ of B for B^* is given by

$$\underline{I} = \frac{mL^2}{12} (\underline{b}_2\underline{b}_2 + \underline{b}_3\underline{b}_3) \tag{27}$$

which, when substituted into Eq. (3), introduces into the force expression the dot products $\underline{b}_2 \cdot \underline{a}_1$ and $\underline{b}_3 \cdot \underline{a}_1$. To evaluate these, let ψ be the angle between $\underline{b}_1$ and $\underline{a}_1$, and note that only these two unit vectors have physical significance inherent to the problem, so that $\underline{b}_2$, $\underline{b}_3$, $\underline{a}_2$ and $\underline{a}_3$ are defined arbitrarily, as shown in Fig. 2.3.2. It follows that

$$\underline{b}_2 \cdot \underline{a}_1 = -\sin \psi \tag{28}$$

$$\underline{b}_3 \cdot \underline{a}_1 = 0 \tag{29}$$

and

$$\underset{(3)}{\underline{f}^{(2)}} = \frac{1}{mR^2} \left[\frac{3}{2} \left(\frac{mL^2}{6} - \frac{5mL^2}{12} \sin^2 \psi \right) \underline{a}_1 - \frac{3mL^2}{12} \sin \psi \, \underline{b}_2 \right] \tag{30}$$

so that, after using

$$\underline{b}_2 = \underline{a}_2 \cos \psi - \underline{a}_1 \sin \psi \tag{31}$$

and omitting $\underline{f}^{(3)}$, $\underline{f}^{(4)}$, etc., one obtains

$$\underline{F} \underset{(6)}{\cong} \underset{(6,30,31)}{\tilde{\underline{F}}} = -\frac{G\bar{m}m}{R^2}\left\{\underline{a}_1\left[1 + \frac{L^2}{8R^2}(2-3\sin^2\psi)\right] - \underline{a}_2\frac{L^2}{8R^2}\sin 2\psi\right\} \quad (32)$$

This result could have been obtained as easily from Eqs. (1), (6), and (11), with the substitution of direction cosines available from Fig. 2.3.2 as

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and with $I_1 = 0$, $I_2 = I_3 = \frac{mL^2}{12}$.

When the rod is aligned with the line joining its mass center B^* to $\bar{P}$, so that $\psi = 0$, the force becomes

$$\tilde{\underline{F}} \underset{(32)}{=} -\frac{G\bar{m}m}{R^2}\left(1 + \frac{L^2}{4R^2}\right)\underline{a}_1 \quad (33)$$

and when the rod is perpendicular to this line, so that $\psi = \pi/2$ rad.,

$$\tilde{\underline{F}} \underset{(32)}{=} -\frac{G\bar{m}m}{R^2}\left(1 - \frac{L^2}{8R^2}\right)\underline{a}_1 \quad (34)$$

Since in these special cases lines passing through B^* and paralleling $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ are principal axes of B for B^*, Eqs. (33) and (34) could have been obtained most directly by using Eq. (10), with $I_{21} = I_{31} = 0$ and for $\psi = 0$,

$$I_{11} = 0, \quad I_{22} = I_{33} = \frac{mL^2}{12}$$

whereas, for $\psi = \pi/2$,

$$I_{11} = \frac{mL^2}{12}, \quad I_{22} = 0, \quad I_{33} = \frac{mL^2}{12}$$

Note that the term $(G\bar{m}mL^2/8R^4)\sin 2\psi \underline{a}_2$ in Eq. (32) represents a force directed normal to the line joining $\bar{P}$ to B^*. Such force components cause bodies to move in orbits differing from the Keplerian orbits associated with particles; but this effect is so small that it can generally be ignored even in high precision orbital calculations.

Certain differences between the Example in Sec. 2.2 and the present example should be noted carefully. For instance, Eq. (32) is valid only when $\frac{L}{R}\underline{a}_1 \cdot \underline{b}_1 + \frac{L^2}{4R^2} < 1$, as required for the underlying series expansion in Eq. (17), whereas Eqs. (2.2.5)-(2.2.7) in the Example in Sec. 2.2 are not subject to any such restriction; and the line through $\bar{P}$ and normal to B passes through the end of B in Fig. 2.2.3, but need not do so in Fig. 2.3.2. For the special case in which $\bar{P}$ in Fig. 2.3.2 lies on the normal to B passing through the end of B, as in Fig. 2.2.3, $\cos\psi$ and $\sin\psi$ are given by

$$\cos\psi = L/2R \tag{35}$$

and

$$\sin\psi = [1-(L/2R^2)]^{1/2} \tag{36}$$

but the form of Eq. (32) applicable to this case would not be obtained by simply substituting these expressions into Eq. (32); rather, it is necessary to expand sin ψ in ascending powers of L/R and then to drop terms of degree three or higher after substituting into Eq. (32), which gives

$$\underset{\sim}{\tilde{F}} = -\frac{G\bar{m}m}{R^2}\left(1 - \frac{L^2}{8R^2}\right)\underline{a}_1 \tag{37}$$

2.4 Force exerted on a small body by a small body

When the distance R between the mass centers B^* and $\bar{B}^*$ of two (not necessarily rigid) bodies B and $\bar{B}$ substantially exceeds the greatest distance in either body from the mass center to any point of the body, the system of gravitational forces exerted on B by $\bar{B}$ has a resultant $\underline{F}$ which can be expressed as

$$\underline{F} = -\frac{G\bar{m}m}{R^2}\left(\underline{a}_1 + \sum_{i=2}^{\infty} \underline{f}^{(i)} + \sum_{i=2}^{\infty}\underline{\bar{f}}^{(i)} + \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} \underline{f}^{(ij)}\right) \tag{1}$$

where $\underline{a}_1$ is a unit vector directed from $\bar{B}^*$ toward B^*, G is the universal gravitational constant, and m and $\bar{m}$ are the masses of B and $\bar{B}$, respectively, and where $\underline{f}^{(i)}$ is a collection of terms of i^{th} degree in $|\underline{r}|/R$, $\underline{\bar{f}}^{(i)}$ is a collection of terms of i^{th} degree in $|\underline{\bar{r}}|/R$, and $\underline{f}^{(ij)}$ is a collection of terms in the product $(|\underline{r}|/R)^i(|\underline{\bar{r}}|/R)^j$, with $\underline{r}$ and $\underline{\bar{r}}$ respectively position vectors of generic points of B and $\bar{B}$ relative to B^* and $\bar{B}^*$. In particular,

$$\underline{f}^{(2)} \triangleq \frac{1}{mR^2}\left\{\frac{3}{2}\,[tr(\underline{I})-5\underline{a}_1\cdot\underline{I}\cdot\underline{a}_1]\underline{a}_1 + 3\underline{I}\cdot\underline{a}_1\right\} \tag{2}$$

and

$$\underline{\bar{f}}^{(2)} \triangleq \frac{1}{\bar{m}R^2}\left\{\frac{3}{2}\,[tr(\underline{\bar{I}})-5\underline{a}_1\cdot\underline{\bar{I}}\cdot\underline{a}_1]\underline{a}_1 + 3\underline{\bar{I}}\cdot\underline{a}_1\right\} \tag{3}$$

where $\underline{I}$ and $\underline{\bar{I}}$ are the inertia dyadics of B for B^* and of $\bar{B}$ for $\bar{B}^*$, respectively.

If $\underline{a}_2$ and $\underline{a}_3$ are defined so as to establish a dextral, orthogonal set of unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, then $\underline{f}^{(2)}$ and $\underline{\bar{f}}^{(2)}$ can be expressed in terms of these unit vectors and the moments and products of inertia of B and $\bar{B}$

for axes paralleling $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ and passing through the mass centers of the individual bodies. To this end, I_{jk} and $\overline{I}_{jk}$ are defined as

$$I_{jk} \triangleq \underline{a}_j \cdot \underline{I} \cdot \underline{a}_k \qquad (j,k = 1,2,3) \tag{4}$$

and

$$\overline{I}_{jk} \triangleq \underline{a}_j \cdot \underline{\overline{I}} \cdot \underline{a}_k \qquad (j,k = 1,2,3) \tag{5}$$

after which $\underline{f}^{(2)}$ and $\underline{\overline{f}}^{(2)}$ may be written

$$\underline{f}^{(2)} = \frac{3}{mR^2}\left[\frac{1}{2}\left(I_{22} + I_{33} - 2I_{11}\right)\underline{a}_1 + I_{21}\underline{a}_2 + I_{31}\underline{a}_3\right] \tag{6}$$

and

$$\underline{\overline{f}}^{(2)} = \frac{3}{\overline{m}R^2}\left[\frac{1}{2}\left(\overline{I}_{22} + \overline{I}_{33} - 2\overline{I}_{11}\right)\underline{a}_1 + \overline{I}_{21}\underline{a}_2 + \overline{I}_{31}\underline{a}_3\right] \tag{7}$$

Alternatively, $\underline{f}^{(2)}$ and $\underline{\overline{f}}^{(2)}$ can be expressed in terms of principal moments of inertia of B for B^* and of $\overline{B}$ for $\overline{B}^*$. To accomplish this, two sets of dextral, orthogonal unit vectors, $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ and $\underline{\overline{b}}_1$, $\underline{\overline{b}}_2$, $\underline{\overline{b}}_3$, paralleling principal axes of B for B^* and of $\overline{B}$ for $\overline{B}^*$, respectively, are introduced, and I_j, $\overline{I}_j$, C_{ij}, and $\overline{C}_{ij}$ are defined as

$$I_j = \underline{b}_j \cdot \underline{I} \cdot \underline{b}_j \qquad (j = 1,2,3) \tag{8}$$

$$\overline{I}_j = \underline{\overline{b}}_j \cdot \underline{\overline{I}} \cdot \underline{\overline{b}}_j \qquad (j = 1,2,3) \tag{9}$$

and

$$C_{ij} \triangleq \underline{a}_i \cdot \underline{b}_j \qquad (i,j = 1,2,3) \tag{10}$$

$$\bar{C}_{ij} \triangleq \underline{a}_i \cdot \underline{\bar{b}}_j \qquad (i,j = 1,2,3) \tag{11}$$

Thus one obtains

$$\begin{aligned} \underline{f}^{(2)} = \frac{3}{mR^2} \Big\{ & \frac{1}{2} [I_1(1-3C_{11}^2) + I_2(1-3C_{12}^2) + I_3(1-3C_{13}^2)]\underline{a}_1 \\ & + [I_1C_{21}C_{11} + I_2C_{22}C_{12} + I_3C_{23}C_{13}]\underline{a}_2 \\ & + [I_1C_{31}C_{11} + I_2C_{32}C_{12} + I_3C_{33}C_{13}]\underline{a}_3 \Big\} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \underline{\bar{f}}^{(2)} = \frac{3}{\bar{m}R^2} \Big\{ & \frac{1}{2} [\bar{I}_1(1-3\bar{C}_{11}^2) + \bar{I}_2(1-3\bar{C}_{12}^2) + \bar{I}_3(1-3\bar{C}_{13}^2)] \\ & + [\bar{I}_1\bar{C}_{11}\bar{C}_{22} + \bar{I}_2\bar{C}_{22}\bar{C}_{12} + \bar{I}_3\bar{C}_{23}\bar{C}_{13}]\underline{a}_2 \\ & + [\bar{I}_1\bar{C}_{31}\bar{C}_{11} + \bar{I}_2\bar{C}_{32}\bar{C}_{12} + \bar{I}_3\bar{C}_{33}\bar{C}_{13}]\underline{a}_3 \Big\} \end{aligned} \tag{13}$$

A useful approximation of $\underline{F}$ in Eq. (1) may be obtained by defining $\underline{\tilde{F}}$ such that

$$\underline{F} \cong \underline{\tilde{F}} \triangleq - \frac{G\bar{m}m}{R^2} (\underline{a}_1 + \underline{f}^{(2)} + \underline{\bar{f}}^{(2)}) \tag{14}$$

with $\underline{f}^{(2)}$ and $\underline{\bar{f}}^{(2)}$ substituted from either Eqs. (2) and (3), or Eqs. (6) and (7), or Eqs. (12) and (13).

Derivations: To establish the validity of Eqs. (1), (2), (3), (6), (7), (12), and (13), expressions given in Sec. 2.3 may be used to represent the force exerted on B by a differential element of $\bar{B}$ located at a generic point $\bar{P}$ of $\bar{B}$ (see Fig. 2.4.1), and the total force $\underline{F}$ applied to B by $\bar{B}$ can then be obtained by integrating over the figure of $\bar{B}$. Specifically, if $\bar{R}$ is the distance between $\bar{P}$ and B^*, $\underline{\bar{a}}_1$ is a unit vector directed from $\bar{P}$ toward B^*, $\bar{\rho}$ is the mass density of $\bar{B}$ at $\bar{P}$, and $d\bar{\tau}$ is the volume of a differential element of $\bar{B}$ at $\bar{P}$, then, from Eqs. (2.3.1)-(2.3.3) with R, $\underline{a}_1$, and $\bar{m}$ replaced by $\bar{R}$, $\underline{\bar{a}}_1$, and $\bar{\rho}d\bar{\tau}$, respectively,

$$\underline{F} = -\int \frac{Gm}{\bar{R}^2}\left\langle \underline{\bar{a}}_1 + \frac{1}{m\bar{R}^2}\left\{\frac{3}{2}\,[\mathrm{tr}(\underline{I}) - 5\underline{\bar{a}}_1\cdot\underline{I}\cdot\underline{\bar{a}}_1]\underline{\bar{a}}_1 + 3\underline{I}\cdot\underline{\bar{a}}_1\right\} + \ldots\right\rangle \bar{\rho}d\bar{\tau} \tag{15}$$

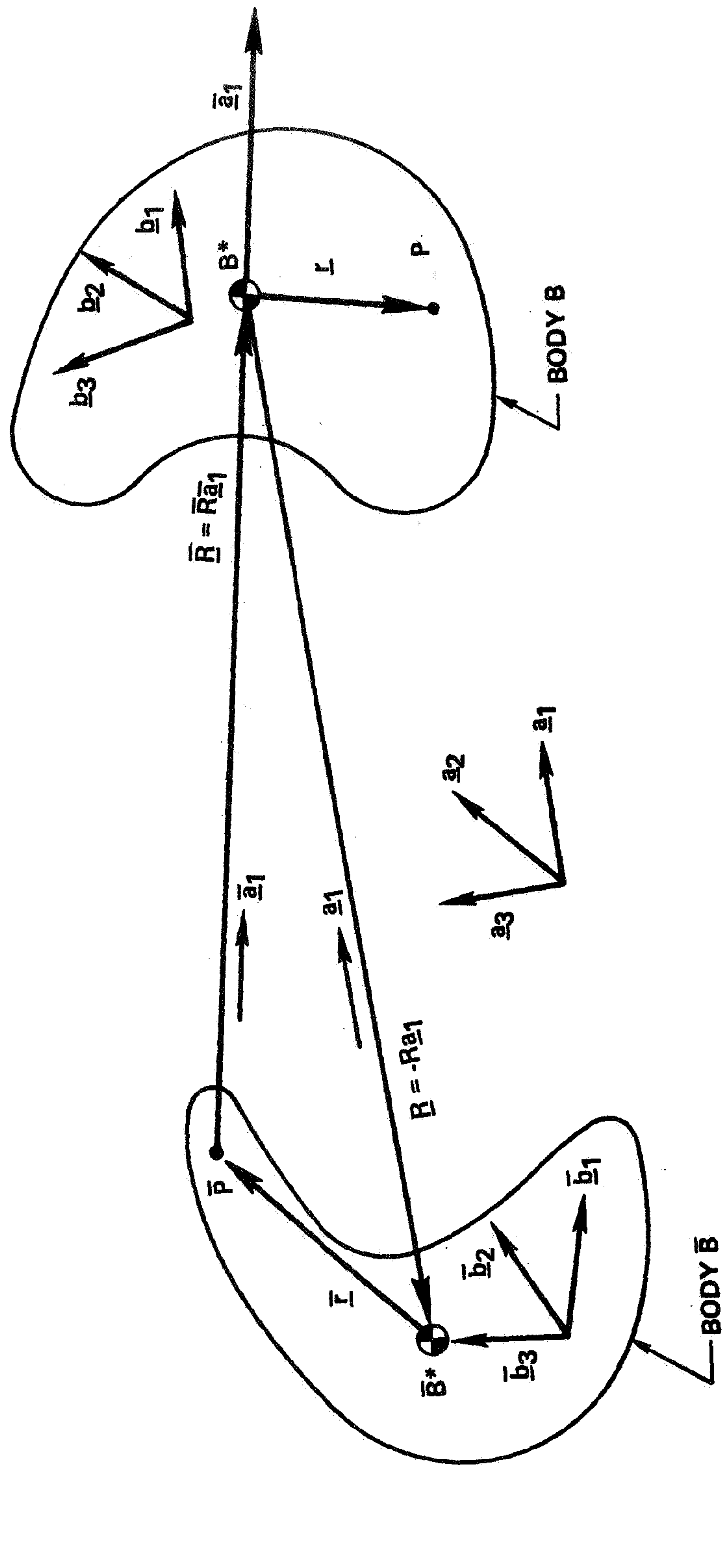
where the three dots represent terms of third or higher degree in $|\underline{r}|/R$. Now, if $\underline{\bar{R}}$ denotes the position vector of B^* relative to $\bar{P}$, and $\underline{R}$ the position vector of $\bar{B}^*$ relative to B^*, then (see Fig. 2.4.1)

$$\frac{\underline{\bar{a}}_1}{\bar{R}^2} = \underline{\bar{R}}(\underline{\bar{R}}^2)^{-3/2} = -\,(\underline{R} + \underline{\bar{r}})(\underline{R}^2 + 2\underline{R}\cdot\underline{\bar{r}} + \underline{\bar{r}}^2)^{-3/2} \tag{16}$$

Hence

$$\underline{F} \underset{(15,16)}{=} Gm\int(\underline{R} + \underline{\bar{r}})(\underline{R}^2 + 2\underline{R}\cdot\underline{\bar{r}} + \underline{\bar{r}}^2)^{-3/2}\,\bar{\rho}d\bar{\tau}$$

$$- G\int\left\langle\frac{1}{\bar{R}^4}\left\{\frac{3}{2}\,[\mathrm{tr}(\underline{I}) - 5\underline{\bar{a}}_1\cdot\underline{I}\cdot\underline{\bar{a}}_1]\underline{\bar{a}}_1 + 3\underline{I}\cdot\underline{\bar{a}}_1\right\} + \ldots\right\rangle\bar{\rho}d\bar{\tau} \tag{17}$$



$\bar{a}_1$
b_1
b_2
b_3
B^*
r
P
BODY B
$\bar{R} = \bar{R}\bar{a}_1$
$\bar{a}_1$
a_1
a_2
a_1
a_3
$R = -Ra_1$
$\bar{P}$
$\bar{r}$
$\bar{B}^*$
$\bar{b}_1$
$\bar{b}_2$
$\bar{b}_3$
BODY $\bar{B}$

Figure 2.4.1

The first integral in this equation has precisely the same form as the integral in Eq. (2.3.12). Consequently, proceeding as in that derivation, one obtains results analogous to Eqs. (2.3.26), namely

$$G m \int (\underline{R} + \underline{\bar{r}})(\underline{R}^2 + 2\underline{R}\cdot\underline{\bar{r}} + \underline{\bar{r}}^2)^{-3/2}\, \bar{\rho} d\bar{\tau}$$

$$= -\frac{G\bar{m}m}{R^2}\left\langle \underline{a}_1 + \frac{1}{\bar{m}R^2}\left\{\frac{3}{2}\,[\mathrm{tr}(\underline{\bar{I}}) - 5\underline{a}_1\cdot\underline{\bar{I}}\cdot\underline{a}_1]\underline{a}_1 + 3\underline{\bar{I}}\cdot\underline{a}_1 + \ldots\right\}\right\rangle \tag{18}$$

where the three dots represent terms of third or higher degree in $|\underline{\bar{r}}|/R$. In the second integral in Eq. (17), $\bar{R}$ and $\underline{\bar{a}}_1$ may be replaced with R and $\underline{a}_1$, respectively, because every term in the integrand involves quantities of second or higher degree in $|\underline{r}|$, so that no terms of interest for the purposes at hand are lost through this replacement; and, once the replacement has been made, the portion of the integral displayed explicitly in Eq. (17) can be evaluated readily. Thus, one obtains

$$\underline{F}_{(17,18)} = -\frac{G\bar{m}m}{R^2}\left\langle \underline{a}_1 + \frac{1}{\bar{m}R^2}\left\{\frac{3}{2}\,[\mathrm{tr}(\underline{\bar{I}}) - 5\underline{a}_1\cdot\underline{\bar{I}}\cdot\underline{a}_1]\underline{a}_1 + 3\underline{\bar{I}}\cdot\underline{a}_1 + \ldots\right\}\right.$$

$$\left. + \frac{1}{mR^2}\left\{\frac{3}{2}\,[\mathrm{tr}(\underline{I}) - 5\underline{a}_1\cdot\underline{I}\cdot\underline{a}_1]\underline{a}_1 + 3\underline{I}\cdot\underline{a}_1 + \ldots\right\}\right\rangle \tag{19}$$

where the three dots now represent terms of third or higher degree either in $|\underline{r}|/R$ or in $|\underline{\bar{r}}|/R$ or terms involving the product $(|\underline{r}|/R)^i\,(|\underline{\bar{r}}|/R)^j$, with neither i nor j equal to unity, because $\underline{r}$ and $\underline{\bar{r}}$ are drawn from the mass centers of B and $\bar{B}$. Eq. (1) now follows directly from Eq. (19) together with the definitions in Eqs. (2) and (3).

The relationship between Eqs.(2), (6), and (12) is completely analogous to that between Eqs. (2.3.3), (2.3.10), and (2.3.11); similarly, for Eqs. (3), (7), and (13). Hence, to establish the validity of Eqs. (6), (7), and (13), one can proceed exactly as in the corresponding derivations in Sec. 2.3.

Example: An approximate expression is required for the force exerted by a homogeneous, oblate spheroid $\bar{B}$ on a homogeneous rectangular parallepiped B having the dimensions shown in Fig. 2.4.2. The approximation of $\underline{F}$ denoted by $\tilde{\underline{F}}$ in Eq. (14) is to be employed, and the three additive terms in this expression are to be compared with each other. For purposes of numerical comparison, the values

$$\bar{\alpha} = 6.38 \times 10^6 \text{ m}$$

$$\bar{\beta} = 6.36 \times 10^6 \text{ m}$$

$$\alpha = 16 \text{ m}$$

$$\beta < \alpha$$

$$R = 6.6\bar{\alpha} = 4.2108 \times 10^7 \text{ m}$$

are to be used. The system then roughly approximates that of the earth ($\bar{B}$) and a large, synchronous-altitude* artificial satellite (B).

Eqs. (12) and (13) provide a convenient point of departure for the required comparisons. The principal moments of inertia appearing in these equations are given by

*A satellite in circular orbit at synchronous altitude orbits the earth every twenty-four hours, so that when in an equatorial orbit the satellite remains above a fixed point on earth.

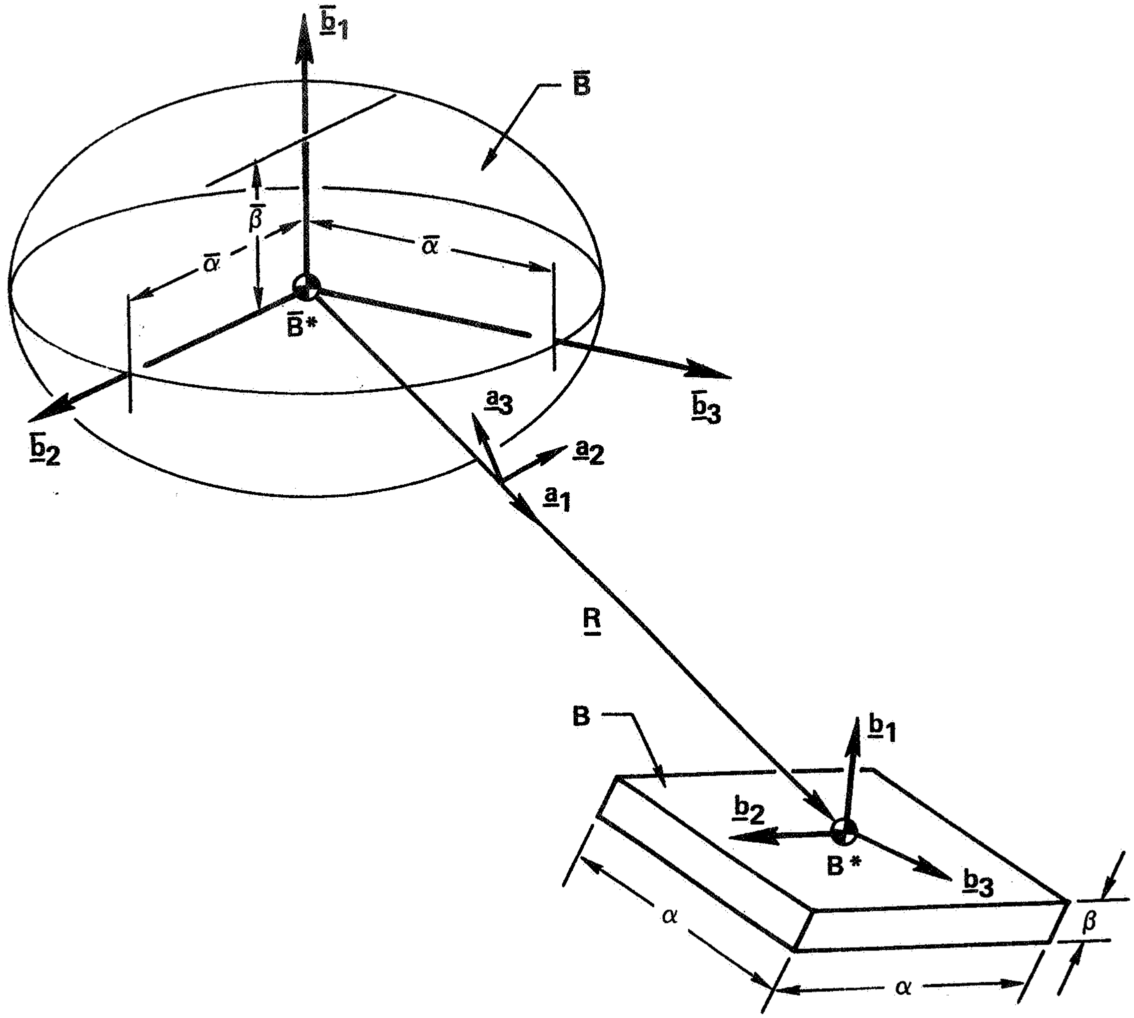

$\bar{\underline{b}}_1$
$\bar{B}$
$\bar{\beta}$
$\bar{\alpha}$
$\bar{\alpha}$
$\bar{B}^*$
$\bar{\underline{b}}_3$
$\bar{\underline{b}}_2$
$\underline{a}_3$
$\underline{a}_2$
$\underline{a}_1$
$\underline{R}$
B
$\underline{b}_1$
$\underline{b}_2$
B *
$\underline{b}_3$
α
β
α

Figure 2.4.2

$$I_1 = \frac{m}{12} (\alpha^2 + \alpha^2) \tag{20}$$

$$I_2 = I_3 = \frac{m}{12} (\alpha^2 + \beta^2) \tag{21}$$

for the rectangular parallelepiped B, and by

$$\bar{I}_1 = \frac{\bar{m}}{5} (\bar{\alpha}^2 + \bar{\alpha}^2) \tag{22}$$

$$\bar{I}_2 = \bar{I}_3 = \frac{\bar{m}}{5} (\bar{\alpha}^2 + \bar{\beta}^2) \tag{23}$$

for the spheroid $\bar{B}$. Later work is facilitated by eliminating β and $\bar{\beta}$ through the introduction of the eccentricity $\bar{\varepsilon}$ of the spheroid $\bar{B}$ with the substitution

$$\bar{\beta}^2 = \bar{\alpha}^2 \ (1-\bar{\varepsilon}^2) \tag{24}$$

and a similar quantity ε for body B with the substitution

$$\beta^2 = \alpha^2 \ (1-\varepsilon^2) \tag{25}$$

The vector $\underline{f}^{(2)}$ then becomes

$$\begin{aligned} \underline{f}^{(2)}_{(12)} = \frac{\alpha^2}{4R^2} \ \{ & [3-3(C_{11}^2 + C_{12}^2 + C_{13}^2) - \frac{\varepsilon^2}{2} (2-3C_{12}^2 - 3C_{13}^2)]\underline{a}_1 \\ & + 2[C_{21}C_{11} + C_{22}C_{12} + C_{23}C_{13} - \frac{\varepsilon^2}{2} (C_{22}C_{12} + C_{23}C_{13})]\underline{a}_2 \\ & + 2[C_{31}C_{11} + C_{32}C_{12} + C_{33}C_{13} - \frac{\varepsilon^2}{2} (C_{32}C_{12} + C_{33}C_{13})]\underline{a}_3 \} \end{aligned} \tag{26}$$

or, after simplification by means of Eq. (1.2.14),

$$\underline{f}^{(2)} \underset{(26)}{=} \frac{\alpha^2\varepsilon^2}{4R^2}\left[\frac{1}{2}(1-3C_{11}^2)\underline{a}_1 + C_{21}C_{11}\underline{a}_2 + C_{31}C_{11}\underline{a}_3\right] \tag{27}$$

By using Eq. (1.2.15), one can now express the magnitude $|\underline{f}^{(2)}|$ of $\underline{f}^{(2)}$ as

$$|\underline{f}^{(2)}| \underset{(27)}{=} \frac{\alpha^2\varepsilon^2}{8R^2}(1 - 2C_{11}^2 + 5C_{11}^4)^{1/2} \tag{28}$$

Similarly,

$$\underline{\bar{f}}^{(2)} = \frac{3\bar{\alpha}^2\bar{\varepsilon}^2}{5R^2}\left[\frac{1}{2}(1 - 3\bar{C}_{11}^2)\underline{a}_1 + \bar{C}_{21}\bar{C}_{11}\underline{a}_2 + \bar{C}_{31}\bar{C}_{11}\underline{a}_3\right] \tag{29}$$

and

$$|\underline{\bar{f}}^{(2)}| = \frac{3\bar{\alpha}^2\bar{\varepsilon}^2}{10R^2}(1 - 2\bar{C}_{11}^2 + 5\bar{C}_{11}^4)^{1/2} \tag{30}$$

Fig. 2.4.3, showing a plot of the function $(1 - 2x^2 + 5x^4)^{1/2}$ versus x in the range $-1 \leq x \leq 1$, can be used to find extremals of $|\underline{f}^{(2)}|$ and $|\underline{\bar{f}}^{(2)}|$ by substituting C_{11} and $\bar{C}_{11}$ for x. By reference to Eqs. (28) and (30) it can thus be concluded that

$$\frac{\alpha^2\varepsilon^2}{4\sqrt{5}R^2} \leq |\underline{f}^{(2)}| \leq \frac{\alpha^2\varepsilon^2}{4R^2} \tag{31}$$

and

$$\frac{3\bar{\alpha}^2\bar{\varepsilon}^2}{5\sqrt{5}R^2} \leq |\underline{\bar{f}}^{(2)}| \leq \frac{3\bar{\alpha}^2\bar{\varepsilon}^2}{5R^2} \tag{32}$$

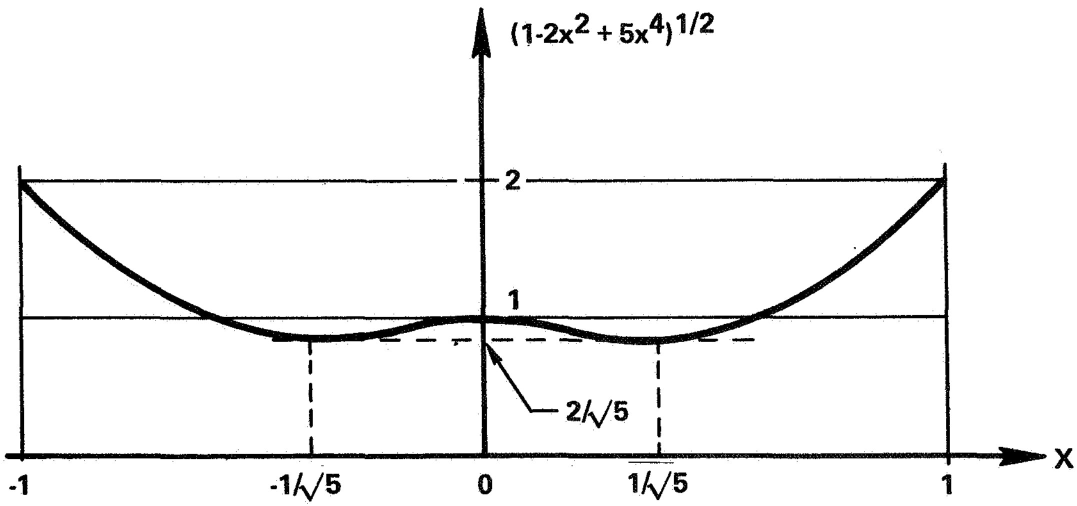


Figure 2.4.3

and substitution of the given numerical values then yields

$$\frac{|\underline{f}^{(2)}|_{max}}{|\underline{\bar{f}}^{(2)}|_{min}} = \frac{5\sqrt{5}}{12}\ \frac{\alpha^2\varepsilon^2}{\bar{\alpha}^2\bar{\varepsilon}^2} \cong \frac{5\sqrt{5}(16)^2\varepsilon^2}{12(6.38)^2\times 10^{12}(.006)} \cong 10^{-9}\varepsilon^2 < 10^{-9} \qquad (33)$$

and

$$\frac{|\underline{\bar{f}}^{(2)}|_{max}}{|\underline{a}_1|} = \frac{3\bar{\alpha}^2\bar{\varepsilon}^2}{5R^2} \cong \frac{3(.006)}{5(6.6)^2} \cong 10^{-4} \qquad (34)$$

Eq. (34) shows that, for a large, synchronous-altitude satellite of the earth, the leading term in the series for $\underline{F}$ in Eq. (1) has a magnitude far exceeding that of $\underline{\bar{f}}^{(2)}$, a vector which reflects the oblateness of the earth through the presence of $\bar{\varepsilon}$ in Eq. (30); and (see Eq. (33)) $|\underline{f}^{(2)}|$ is much smaller than even $|\underline{\bar{f}}^{(2)}|$.

It is noteworthy that the tiny term $\underline{f}^{(2)}$ (see Eq. (27)) depends on C_{11}, C_{21}, and C_{31}, and hence on the orientation of the satellite relative to $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$, whereas $\underline{\bar{f}}^{(2)}$ (see Eq. (29)) involving $\bar{C}_{11}$, $\bar{C}_{21}$, and $\bar{C}_{31}$ depends on the orientation of the satellite's orbital plane relative to the spheroid. For an equatorial orbit, $\underline{\bar{f}}^{(2)}$ reduces to

$$\underline{\bar{f}}^{(2)} \underset{(29)}{=} -\frac{3\bar{\alpha}^2\bar{\varepsilon}^2}{5R^2}\underline{a}_1$$

and the addition of this vector to the term $\underline{a}_1$ in Eq. (14) is of little consequence (see Eq. (34)). By way of contrast, if B^* moves in a non-equatorial orbit, oblateness effects may be significant despite the fact that $\underline{\bar{f}}^{(2)}$ is small in comparison with $\underline{a}_1$, because $\underline{\bar{f}}^{(2)}$ then has components perpendicular to $\underline{a}_1$.

2.5 Centrobaric bodies

As noted in Sec. 2.2, the center of gravity B' of a body B for an attracting particle $\bar{P}$ does not in general coincide with the mass center B^* of B. However, there exist bodies for which the center of gravity and center of mass necessarily coincide. Such bodies are called centrobaric.* Thus a body B of mass m is centrobaric if the force $\underline{F}$ exerted on B by every particle $\bar{P}$ of mass $\bar{m}$ is given by

$$\underline{F} = -\frac{G\bar{m}m}{R^2}\underline{a}_1 \tag{1}$$

where G is the universal gravitational constant, R is the distance between $\bar{P}$ and B^*, and $\underline{a}_1$ is a unit vector directed from $\bar{P}$ toward B^*.

Centrobaric bodies can be found in a variety of interesting shapes and mass distributions (see Prob. 2f); but all such bodies possess the following property: A centrobaric body has the same moment of inertia about every line passing through its mass center. In other words, the inertia ellipsoid of a centrobaric body for the mass center of the body is a sphere. However, not every body possessing this property is centrobaric (see Prob. 2e).

Derivations: The existence of centrobaric bodies is most easily proven by citing a specific case (see the Example, in which a solid sphere is shown to be centrobaric if its mass density at any point is a function only of the distance from the point to the center of the sphere).

To prove that a centrobaric body B has the same moment of inertia about every line passing through its mass center B^*, it is sufficient to show

*In the classical literature, the term center of gravity is sometimes so defined that only a centrobaric body has a center of gravity; then the center of mass coincides with the center of gravity whenever the latter exists.

that the quantity $\underline{a}_1 \cdot \underline{I} \cdot \underline{a}_1$ has a value independent of $\underline{a}_1$, where $\underline{I}$ is the inertia dyadic of B for B^*. Now, taken in conjunction, Eq. (1) and the series expansion for $\underline{F}$ given by Eq. (2.3.1) imply

$$\sum_{i=2}^{\infty} \underline{f}^{(i)} = 0$$

The terms in this summation are independent of each other in the sense that $\underline{f}^{(2)}$ is proportional to R^{-2}, $\underline{f}^{(3)}$ is proportional to R^{-3}, and so forth, and R may be taken to be arbitrarily large (see Eq. (2.3.3). It follows that they vanish separately, that is, that

$$\underline{f}^{(i)} = 0, \quad i = 2, \ldots, \infty$$

Hence

$$\underline{a}_1 \cdot \underline{f}^{(2)} = 0$$

and, using the expression for $\underline{f}^{(2)}$ given in Eq. (2.3.3), one finds that

$$\underline{a}_1 \cdot \underline{I} \cdot \underline{a}_1 = \frac{1}{3} \operatorname{tr}(\underline{I})$$

Since $\operatorname{tr}(\underline{I})$ is an invariant, the moment of inertia $\underline{a}_1 \cdot \underline{I} \cdot \underline{a}_1$ of B about the line that passes through B^* and is parallel to $\underline{a}_1$ has been shown to have the same value for all orientations of $\underline{a}_1$ relative to B.

<u>Example</u>: To show that a solid sphere S of mass m is centrobaric if the mass density ρ at a point P depends only on the distance r between P

and the center S^* of S, the force $\underline{F}$ exerted on S by a particle $\overline{P}$ of mass $\overline{m}$ will be calculated.

The position vector p of P relative to $\overline{P}$, expressed in terms of the spherical polar coordinates r, θ, ψ, the distance R between $\overline{P}$ and S^*, and the unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, shown in Fig. 2.5.1, is given by

$$\underline{p} = (R + r\cos\psi)\underline{a}_1 + r\sin\psi\sin\theta\,\underline{a}_2 + r\sin\psi\cos\theta\,\underline{a}_3 \tag{2}$$

The volume dτ of a differential element of S is

$$d\tau = r^2 \sin\psi\, dr\, d\theta\, d\psi \tag{3}$$

and with $\rho = \rho(r)$ the mass m of S can be expressed as

$$m = \int \rho d\tau = \int_0^a r^2\rho \int_0^\pi \sin\psi \int_0^{2\pi} d\theta\, d\psi\, dr$$

or

$$m = 4\pi \int_0^a r^2 \rho dr \tag{4}$$

where a is the radius of S.

The force $\underline{F}$ exerted by $\overline{P}$ on S is given by

$$\underline{F} \underset{(2.2.2)}{=} - G\overline{m} \int \underline{p}(\underline{p}^2)^{-3/2} \rho d\tau \tag{5}$$

or, in somewhat more explicit terms, by

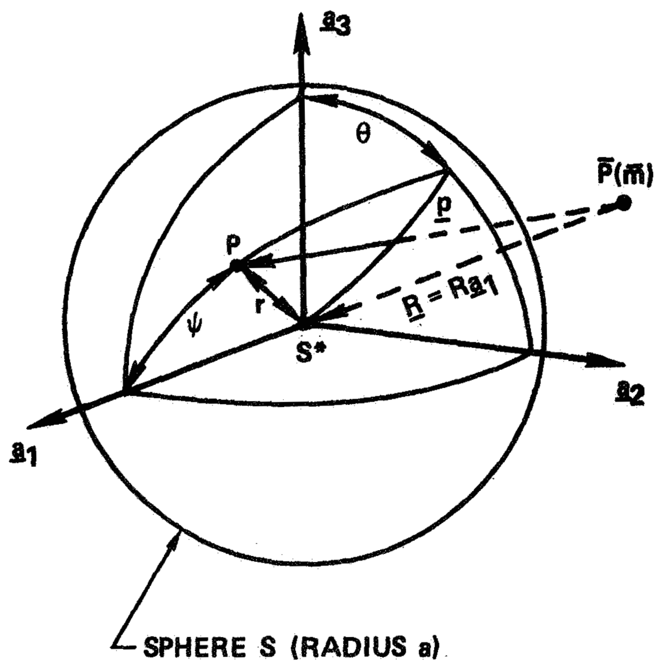

$\underline{a}_3$
θ
$\bar{P}(\bar{m})$
$\underline{\rho}$
P
r
$\underline{R} = R\underline{a}_1$
ψ
S*
$\underline{a}_2$
$\underline{a}_1$
SPHERE S (RADIUS a)

Figure 2.5.1

$$\underline{F} \underset{(2,3,5)}{=} -G\bar{m}\int_0^a r^2\rho\int_0^\pi \sin\psi(R^2+2Rr\cos\psi+r^2)^{-3/2}\int_0^{2\pi}\underline{p}\,d\theta\,d\psi\,dr \tag{6}$$

Using Eq. (2), one obtains for this inner integral in Eq. (6)

$$\int_0^{2\pi}[(R+r\cos\psi)\underline{a}_1 + r\sin\psi\sin\theta\underline{a}_2 + r\sin\psi\cos\theta\underline{a}_3]d\theta - 2\pi(R+r\cos\psi)\underline{a}_1 \tag{7}$$

so that

$$\underline{F} \underset{(6,7)}{=} -2\pi G\bar{m}\int_0^a r^2\rho\int_0^\pi \sin\psi(R^2+2Rr\cos\psi+r^2)^{-3/2}(R+r\cos\psi)d\psi dr\ \underline{a}_1 \tag{8}$$

Integration is now facilitated by the introduction of a new variable v defined such that

$$v^2 \underline{\underline{\Delta}} R^2 + 2Rr\cos\psi + r^2 \tag{9}$$

When this definition and the implied relationships

$$\sin\psi d\psi \underset{(9)}{=} -vdv/(Rr) \tag{10}$$

and

$$R + r\cos\psi \underset{(9)}{=} (v^2 + R^2 - r^2)/(2R) \tag{11}$$

are substituted into Eq. (8), there results

$$\underline{F} = \frac{\pi G\bar{m}}{R^2} \int_0^a r\rho \int_{(R+r)}^{(R-r)} \frac{(v^2+R^2-r^2)}{v^2}\, dv\; dr\; \underline{a}_1$$

$$= -\frac{4\pi G\bar{m}}{R^2} \int_0^a r^2 \rho dr\; \underline{a}_1 \underset{(4)}{=} -\frac{G\bar{m}\; m}{R^2}\, \underline{a}_1 \qquad (12)$$

2.6 Moment exerted on a body by a particle

The system of gravitational forces exerted on a (not necessarily rigid) body B of mass m by a particle $\overline{P}$ of mass $\overline{m}$ produces a moment $\underline{M}$ about the mass center B^* of B, given by

$$\underline{M} = -\underline{R} \times \underline{F} \tag{1}$$

where $\underline{F}$ is the resultant of the system of forces [see Eqs. (2.2.1), (2.2.2), and (2.3.1)] and $\underline{R}$ is the position vector of B^* relative to $\overline{P}$. If the distance R between $\overline{P}$ and B^* substantially exceeds the greatest distance from B^* to any point P of B, this moment can be expressed as

$$\underline{M} = \frac{3G\overline{m}}{R^3} \underline{a}_1 \times \underline{\underline{I}} \cdot \underline{a}_1 + \frac{G\overline{m}m}{R} \sum_{i=3}^{\infty} \underline{m}^{(i)} \tag{2}$$

where $\underline{a}_1$ is a unit vector directed from $\overline{P}$ toward B^*, G is the universal gravitational constant, $\underline{\underline{I}}$ is the inertia dyadic of B for B^*, and the dimensionless vector $\underline{m}^{(i)}$ is a collection of terms of i^{th} degree in $|\underline{r}|/R$, with $\underline{r}$ the position vector of a typical point P of B relative to B^*.

Eq. (2) suggests the approximation

$$\underline{M} \approx \underline{\tilde{M}} \triangleq \frac{3G\overline{m}}{R^3} \underline{a}_1 \times \underline{\underline{I}} \cdot \underline{a}_1 \tag{3}$$

Expressed in terms of scalars I_{ij}, $i,j = 1,2,3$, defined as

$$I_{jk} \triangleq \underline{a}_j \cdot \underline{\underline{I}} \cdot \underline{a}_k \qquad (j,k = 1,2,3) \tag{4}$$

the vector $\underline{\tilde{M}}$ is given by

$$\underline{\tilde{M}} = \frac{3G\bar{m}}{R^3} (I_{21}\underline{a}_3 - I_{31}\underline{a}_2) \tag{5}$$

Alternatively, one can introduce a dextral set of orthogonal unit vectors $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ parallel to the principal axes of inertia of B for B^*, and express $\underline{\tilde{M}}$ in terms of the principal moments of inertia I_1, I_2, I_3 of B for B^* and the direction cosines C_{ij}, $i,j = 1,2,3$ defined respectively as

$$I_j \triangleq \underline{b}_j \cdot \underline{I} \cdot \underline{b}_j \qquad (j = 1,2,3) \tag{6}$$

and

$$C_{ij} \triangleq \underline{a}_i \cdot \underline{b}_j \qquad (i,j = 1,2,3) \tag{7}$$

thus obtaining

$$\underline{\tilde{M}} = \frac{3G\bar{m}}{R^3} [\underline{b}_1(I_3-I_2)C_{12}C_{13} + \underline{b}_2(I_1-I_3)C_{13}C_{11} + \underline{b}_3(I_2-I_1)C_{11}C_{12}] \tag{8}$$

Eq. (5), despite its apparent simplicity, is less useful than Eq. (8), both because the products of inertia I_{21} and I_{31} vary with the orientation of $\underline{a}_1$ relative to B and because the rotational equations of dynamics are generally most easily formulated in terms of the vector basis $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$.

It should be noted that $\underline{\tilde{M}}$, which by Eq. (3) is the first term in Eq. (2), can vanish when $\underline{M}$ does not vanish. Hence, when using $\underline{\tilde{M}}$ as an approximation to $\underline{M}$ one is <u>not necessarily</u> retaining the largest term in the series expansion for $\underline{M}$. Specifically, $\underline{\tilde{M}}$ vanishes whenever $\underline{a}_1$ parallels a principal axis of inertia of B for B^*, but this state of affairs need not produce a zero value for $\underline{M}$ (see Prob. 2g). Moreover, $\underline{\tilde{M}}$ is identically zero for any body with spherical inertia ellipsoid, whereas $\underline{M}$ is identically zero only if that body is also centrobaric (see Sec. 2.5 and Prob. 2e).

Derivations: If B is a continuous distribution of matter, then the resultant $\underline{F}$ of the gravitational forces exerted on B by $\overline{P}$ can be expressed in terms of the force $\underline{dF}$ exerted by $\overline{P}$ on a differential element of B at a generic point P of B as

$$\underline{F} = \int \underline{dF} \tag{9}$$

and, if $\underline{r}$ is the position vector of P relative to the mass center B^* of B, then the moment $\underline{M}$ of the system of gravitational forces about B^* is

$$\underline{M} = \int \underline{r} \times \underline{dF} \tag{10}$$

Substituting

$$\underline{r} = -\underline{R} + \underline{p} \tag{11}$$

where $\underline{R}$ is the position vector of B^* relative to $\overline{P}$ and $\underline{p}$ is the position vector of P relative to $\overline{P}$ (see Fig. 2.6.1), yields

$$\underline{M} \underset{(10,11)}{=} \int (-\underline{R} + \underline{p}) \times \underline{dF} \underset{(9)}{=} -\underline{R} \times \underline{F} + \int \underline{p} \times \underline{dF} \tag{12}$$

But $\underline{dF}$ parallels $\underline{p}$, so that $\underline{p} \times \underline{dF} = 0$ and

$$\underline{M} \underset{(12)}{=} -\underline{R} \times \underline{F} \tag{13}$$

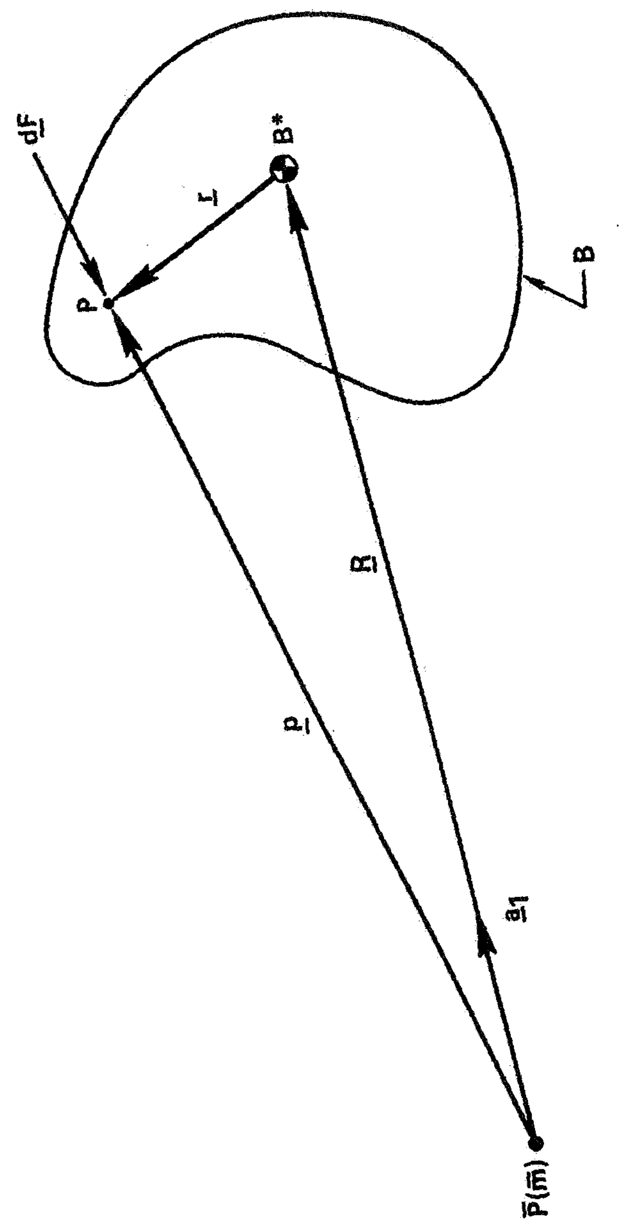


Figure 2.6.1

confirming Eq. (1). Alternatively, this result can also be inferred from Sec. 2.2 and from the fact that two equivalent systems of forces have equal moments about every point. Hence it applies also when B consists of a finite number of particles, $P_1, \ldots, P_N$.

Eq. (2) for $\underline{M}$ is to be proven under the same restrictions on R and on the dimensions of B as apply to the expression for $\underline{F}$ in Eq. (2.3.1). Under these circumstances, substitution from Eqs. (2.3.1) and (2.3.2) into Eq. (13) yields

$$\underline{M} = \frac{G\bar{m}\,m}{R}\,\underline{a}_1 \times \sum_{i=2}^{\infty} \underline{f}^{(i)} = \frac{G\bar{m}'m}{R}\left(\underline{a}_1 \times \underline{f}^{(2)} + \sum_{i=3}^{\infty} \underline{a}_1 \times \underline{f}^{(i)}\right) \tag{14}$$

and use of Eqs. (2.3.3) and (2.3.4) then gives

$$\underline{M} = \frac{3G\bar{m}}{R^3}\,\underline{a}_1 \times \underline{I}\cdot\underline{a}_1 + \frac{G\bar{m}\,m}{R}\sum_{i=3}^{\infty} \underline{a}_1 \times \underline{f}^{(i)} \tag{15}$$

which is equivalent to Eq. (2) if $\underline{m}^{(i)}$ is defined as

$$\underline{m}^{(i)} \triangleq \underline{a}_1 \times \underline{f}^{(i)}\,, \qquad i = 3, \ldots, \infty \tag{16}$$

Verification of the equivalence of Eqs. (5) and (8) with Eq. (3) requires the substitution

$$\underline{a}_1 \underset{(1.2.2)}{=} C_{1i}\underline{b}_i \tag{17}$$

After setting

$$\underline{I} \underset{(4)}{=} I_{ij}\underline{a}_i\underline{a}_j \tag{18}$$

one then arrives at Eq. (5), and using

$$\underset{(6)}{\underline{I}} = I_1\underline{b}_1\underline{b}_1 + I_2\underline{b}_2\underline{b}_2 + I_3\underline{b}_3\underline{b}_3 \tag{19}$$

one obtains Eq. (8).

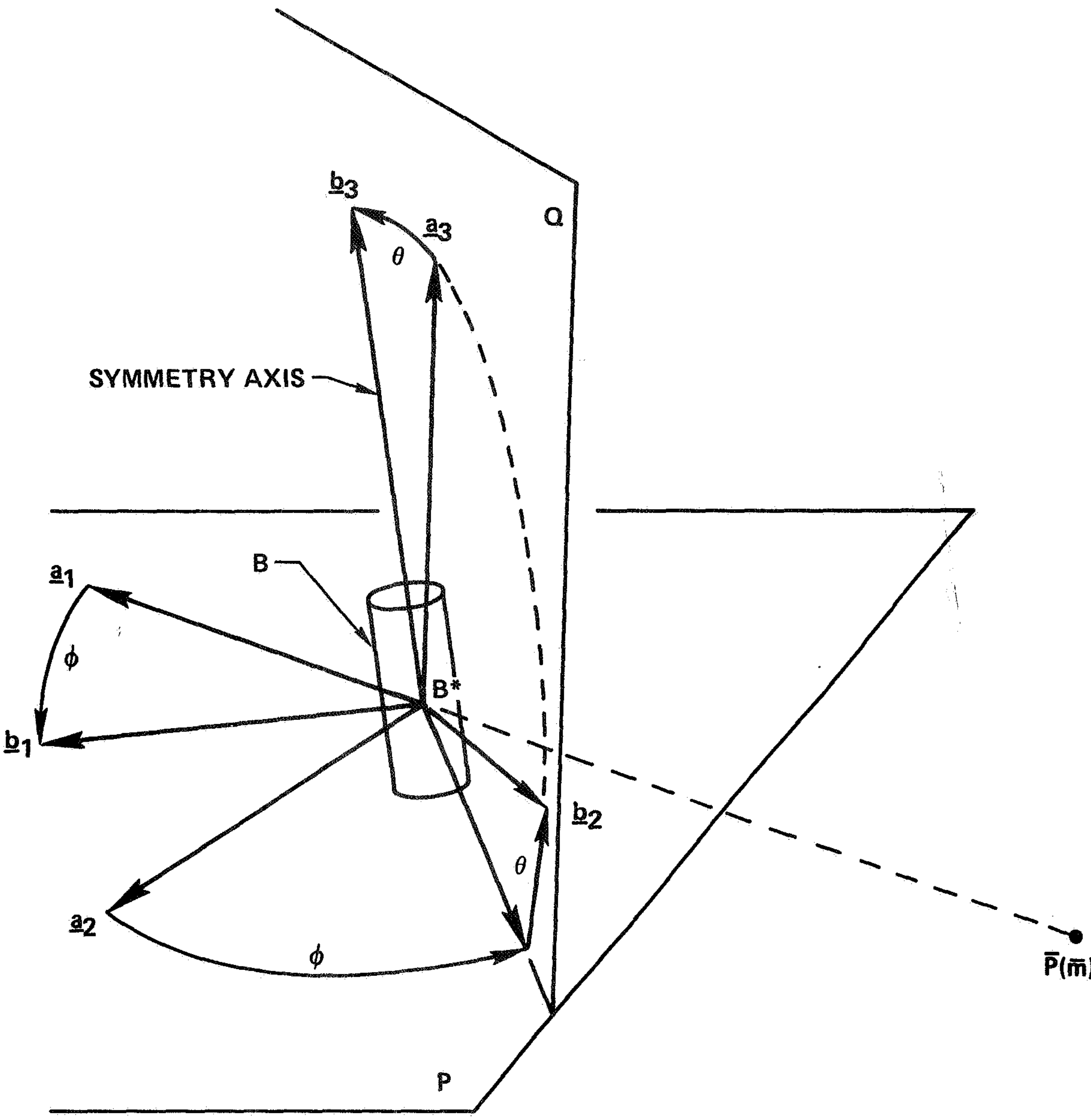
Example: The vector $\underline{\tilde{M}}$ defined in Eq. (3) is to be used to approximate the moment $\underline{M}$ exerted by a particle $\bar{P}$ of mass $\bar{m}$ about the mass center B^* of a homogeneous, right circular cylinder B having moments of inertia J and I, respectively, about the symmetry axis and about any line through B^* and normal to the symmetry axis. The result is to be expressed in terms of two angles, ϕ and θ, used to specify the orientation of the symmetry axis of B relative to a dextral, orthogonal set of unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$. Specifically, as shown in Fig. 2.6.2, θ is the angle between $\underline{a}_3$ and the symmetry axis of B, and ϕ is the angle between $\underline{a}_2$ and the intersection of the plane P passing through B^* and normal to $\underline{a}_3$ with the plane Q determined by the symmetry axis of B and a line passing through B^* and parallel to $\underline{a}_3$.

Of the alternative expressions for $\underline{\tilde{M}}$ given in Eqs. (3), (5) and (8), the last is the most convenient, for suitable unit vectors $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ can be introduced readily, for example as shown in Fig. 2.6.2, where $\underline{b}_1$ is normal to plane Q. Using Eq. (7), one then finds that

$$C_{11} = \cos\phi$$

$$C_{12} = -\cos\theta\sin\phi$$

$$C_{13} = \sin\theta\sin\phi$$



$\underline{b}_3$
$\underline{a}_3$
θ
Q
SYMMETRY AXIS
B
$\underline{a}_1$
ϕ
B*
$\underline{b}_1$
$\underline{b}_2$
θ
$\underline{a}_2$
ϕ
$\bar{P}(\bar{m})$
P

Figure 2.6.2

and, in accordance with Eq. (6),

$$I_1 = I_2 = I , \quad I_3 = J$$

Hence

$$\underset{(8)}{\underline{\tilde{M}}} = \frac{3G\bar{m}}{R^3} (I - J) \sin\theta \sin\phi [\underline{b}_1 \cos\theta \sin\phi + \underline{b}_2 \cos\phi]$$

If $I \neq J$, $\underline{\tilde{M}}$ vanishes only when at least one of the following conditions is fulfilled: $\sin\phi = 0$, $\sin\theta = 0$, or $\cos\theta = \cos\phi = 0$. In all of these cases, the symmetry axis of B is either normal or parallel to the line joining $\bar{P}$ and B^*, and considerations of symmetry indicate that $\underline{M}$ also vanishes under these circumstances. Thus it appears that $\underline{\tilde{M}} = 0$ can imply $\underline{M} = 0$ for a particular body, although this implication is not valid in general.

2.7 Moment exerted on a small body by a small body

When the distance R between the mass centers B^* and $\bar{B}^*$ of two (not necessarily rigid) bodies B and $\bar{B}$ substantially exceeds the greatest distance from the mass center of either body to any point of that body, the system of gravitational forces exerted on B by $\bar{B}$ produces a moment $\underline{M}$ about B^* which can be expressed as

$$\underline{M} = \frac{3G\bar{m}}{R^3} \underline{a}_1 \times \underline{I} \cdot \underline{a}_1 + \frac{Gm\bar{m}}{R} \sum_{i=3}^{\infty} \underline{m}^{(i)} + \frac{Gm\bar{m}}{R} \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \underline{m}^{(ij)} \tag{1}$$

where $\underline{a}_1$ is a unit vector directed from $\bar{B}^*$ toward B^*, $\underline{I}$ is the inertia dyadic of B for B^*, m and $\bar{m}$ are respectively the masses of B and $\bar{B}$, G is the universal gravitational constant, and the dimensionless vector $\underline{m}^{(i)}$ is a collection of terms of i^{th} degree in $|\underline{r}|/R$, while the dimensionless vector $\underline{m}^{(ij)}$ is a collection of terms in the product $(|\underline{r}|/R)^i(|\bar{\underline{r}}|/R)^j$, with $\underline{r}$ and $\bar{\underline{r}}$ respectively the position vectors of generic points of B and $\bar{B}$ relative to B^* and $\bar{B}^*$.

The similarity of Eqs. (1) and (2.6.2) suggests that an approximate relationship similar to Eq. (2.6.3), namely

$$\underline{M} \approx \tilde{\underline{M}} \triangleq \frac{3G\bar{m}}{R^3} \underline{a}_1 \times \underline{I} \cdot \underline{a}_1 \tag{2}$$

may prove useful. The vector $\tilde{\underline{M}}$ thus defined can be expressed as

$$\tilde{\underline{M}} = \frac{3G\bar{m}}{R^3} (I_{21}\underline{a}_3 - I_{31}\underline{a}_2) \tag{3}$$

or as

$$\tilde{\underline{M}} = \frac{3G\bar{m}}{R^3} [\underline{b}_1(I_3-I_2)C_{12}C_{13} + \underline{b}_2(I_1-I_3)C_{13}C_{11} + \underline{b}_3(I_2-I_1)C_{11}C_{12}] \tag{4}$$

where $\underline{a}_i$, $\underline{b}_i$, I_i, I_{ij}, and C_{ij} have the same meaning as in Sec. 2.6, and $\tilde{\underline{M}}$, that is, the first term in Eq. (1), can be dominated by other terms in the series. A significant difference between Eq. (2.6.2) and Eq. (1) is that the former can be replaced with Eq. (2.6.1), whereas the latter cannot be so replaced, even if $\underline{R}$ and $\underline{F}$ are re-defined respectively as the position vector of B^* relative to $\bar{B}^*$ and as the resultant force exerted on B by $\bar{B}$.

<u>Derivations</u>: The validity of Eq. (1) can be established by using Eq. (2.6.2) to represent the moment exerted on B about B^* by a differential element of $\bar{B}$ located at a generic point $\bar{P}$ of $\bar{B}$ (see Fig. 2.7.1) and then integrating over the figure of $\bar{B}$. Specifically, if $\bar{R}$ is the distance between $\bar{P}$ and B^*, $\bar{\underline{a}}_1$ is a unit vector directed from $\bar{P}$ toward B^*, $\bar{\rho}$ is the mass density of $\bar{B}$ at $\bar{P}$, and $d\bar{\tau}$ is the volume of a differential element of $\bar{B}$ at $\bar{P}$, then, from Eq. (2.6.2), with R, $\underline{a}_1$, and $\bar{m}$ replaced by $\bar{R}$, $\bar{\underline{a}}_1$, and $\bar{\rho}d\bar{\tau}$, respectively,

$$\underline{M} = \int\left(\frac{3G}{\bar{R}^3}\,\bar{\underline{a}}_1 \times \underline{I}\cdot\bar{\underline{a}}_1 + \ldots\right)\bar{\rho}d\bar{\tau} \tag{5}$$

where the three dots represent terms of third or higher degree in $|\underline{r}|/R$. Because $\underline{I}$ consists of terms of second degree in $|\underline{r}|/R$, the integration in Eq. (5) can produce only terms of second and higher degree in $|\underline{r}|/R$, and the substitution of $\underline{a}_1$ for $\bar{\underline{a}}_1$ and R for $\bar{R}$ cannot result in the loss of any terms of interest. These substitutions permit the first term in parentheses to be removed from the integrand, leaving as a factor an integral equal to the mass $\bar{m}$ of $\bar{B}$, confirming the first term in Eq. (1); and they yield the

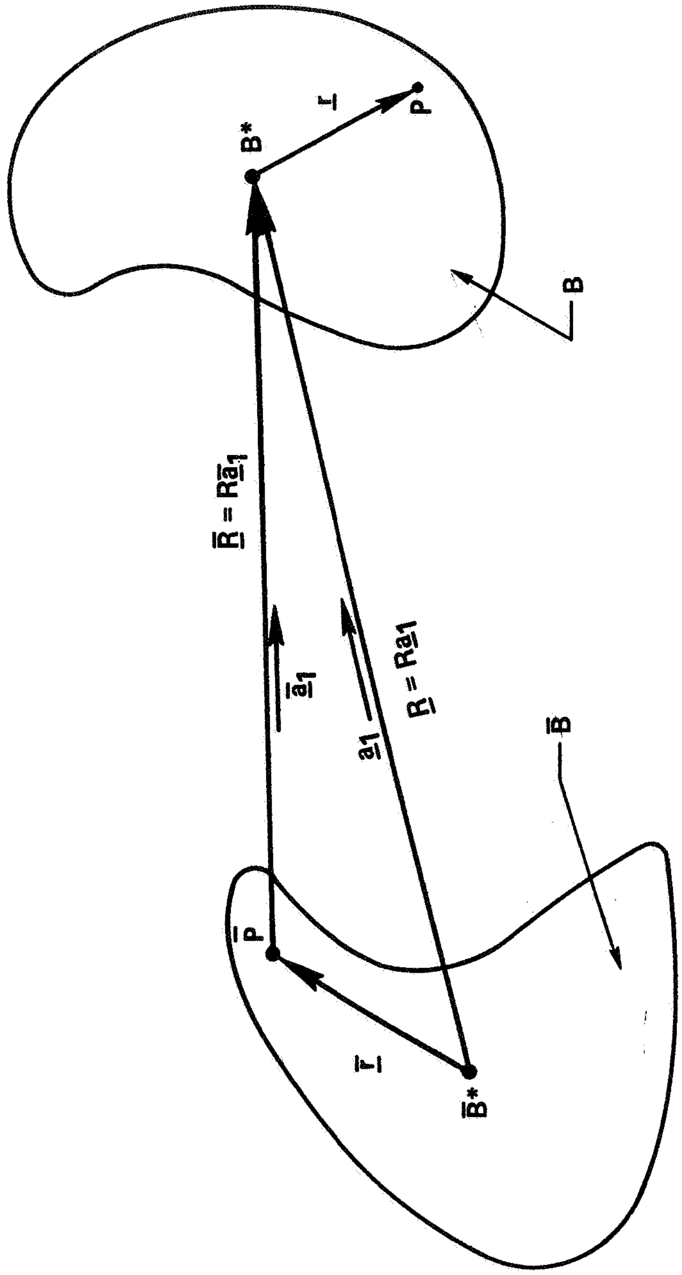


Figure 2.7.1

first series in Eq. (1). The second series in Eq. (1) reflects the deviations of $\underline{a}_1$ from $\overline{\underline{a}}_1$ and R from $\overline{R}$. Every term in this series involves $|\underline{\overline{r}}|/R$, and it does not contain terms linear in $|\underline{\overline{r}}|/R$ because $\underline{\overline{r}}$ is drawn from the mass center of $\overline{B}$.

Eqs. (3) and (4) can be obtained from Eq. (2) by a procedure analogous to that used to derive Eqs. (2.5.5) and (2.5.8) from Eq. (2.5.3).

Example: In the Example of Section 2.4 (see Fig. 2.4.2), an approximate expression was developed for the force exerted by a homogeneous oblate spheroid $\overline{B}$ on a homogeneous rectangular parallelepiped B. Now an approximate expression is to be obtained for $\underline{M}$, the moment exerted by $\overline{B}$ on B about the mass center B^* of B. To this end, $\underline{\tilde{M}}$ as given by Eq. (4) is to be formed, and the magnitude of $\underline{\tilde{M}}$ is to be determined for

$$\overline{m} = 6 \times 10^{24} \text{ kg.}$$

$$m = 160 \text{ kg.}$$

$$\alpha = 16 \text{ m.}$$

$$\beta = 4 \text{ m.}$$

$$R = 4.2108 \times 10^{7} \text{ m.}$$

$$G = 6.6732 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$$

As in the Example of Sec. 2.4, the system roughly approximates the earth $\overline{B}$ and a large, synchronous altitude artificial satellite B.

The symmetry of B permits the substitution of I_2 for I_3 in Eq. (4), furnishing

$$\underline{\tilde{M}} = \frac{3G\overline{m}}{R^3}(I_1 - I_2)C_{11}(C_{13}\underline{b}_2 - C_{12}\underline{b}_3) \tag{6}$$

from which

$$|\tilde{\underline{M}}| = \frac{3G\bar{m}}{R^3}(I_1-I_2)C_{11}(C_{13}^2+C_{12}^2)^{1/2} \underset{(1.2.14)}{=} \frac{3G\bar{m}}{R^3}(I_1-I_2)C_{11}(1-C_{11}^2)^{1/2} \qquad (7)$$

Variations in the magnitude of $\tilde{\underline{M}}$ resulting from changes in the relative orientation of B and $\bar{B}$ thus depend only on changes in the angle between $\underline{a}_1$ and $\underline{b}_1$, and $|\tilde{\underline{M}}|$ attains its maximum value when this angle is equal to $\pi/4$ rad., since the derivative of $|\tilde{\underline{M}}|$ with respect to C_{11} vanishes when C_{11} is $1/\sqrt{2}$. Thus, for any body B with $I_3 = I_2$,

$$|\tilde{\underline{M}}|_{max} = \frac{3G\bar{m}}{2R^3}(I_1 - I_2) \qquad (8)$$

and for the given rectangular parallelepiped, for which

$$I_1 = \frac{m\alpha^2}{6} \qquad (9)$$

and

$$I_2 = I_3 = \frac{m}{12}(\alpha^2 + \beta^2) \qquad (10)$$

$|\tilde{\underline{M}}|_{max}$ becomes

$$|\tilde{\underline{M}}|_{max} = \frac{G\bar{m}m(\alpha^2-\beta^2)}{8R^3} = \frac{\mu m(\alpha^2-\beta^2)}{8R^3} \qquad (11)$$

or, after using the given numerical values,

$$|\tilde{\underline{M}}|_{max} \cong \frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 160 \times (256 - 16)}{8 \times (4.2108)^3 \times 10^{21}} \text{ Nm.}$$

$$\cong 2.6 \times 10^{-5} \text{ Nm.}$$

2.8 Proximate bodies

When dealing with the gravitational interaction of two bodies which are in close proximity, one can at times make effective use of results which, at first glance, may appear to be inapplicable because they were encountered initially in connection with the analysis of the gravitational interaction of widely separated bodies. For instance, consider Eq. (2.7.2), which furnishes an approximation that becomes ever better as the distance R between B^* and $\bar{B}^*$ (see Fig. 2.8.1a) grows in comparison with the largest dimension of either body. Eq. (2.7.2) can be useful also when B and $\bar{B}$ are near each other (see Fig. 2.8.1b), provided $\bar{B}$ does not differ too much from a centrobaric body (see Sec. 2.5) and R is sufficiently large in comparison with the largest dimension of B. For $\bar{B}$ then acts nearly like a particle of mass $\bar{m}$ situated at the mass center $\bar{B}^*$ of $\bar{B}$; the resultant moment about B^* of the forces exerted on B by such a particle is given to a good approximation by Eq. (2.6.3); and this equation is identical with Eq. (2.7.2).

Example: Fig. 2.8.2 shows a particle $\bar{P}$ of mass $\bar{m}$ situated on the axis of revolution of a uniform oblate spheroid B of mass m, at a distance R from the mass center B^* of B. To explore the utility of Eq. (2.3.6) in situations involving proximity of a particle and a body, the ratio $\tilde{F}/F$ is to be plotted versus β for various values of ε, where $\tilde{F}$ is the magnitude of $\underline{\tilde{F}}$ as given in Eq. (2.3.6), F is the magnitude of $\underline{F}$ as given in Eq. (2.2.2), and β and ε are defined as (see Fig. 2.8.2 for a and b)

$$\beta \triangleq b/R \tag{1}$$

and

$$\varepsilon \triangleq [1 - (b/R)^2]^{1/2} \tag{2}$$

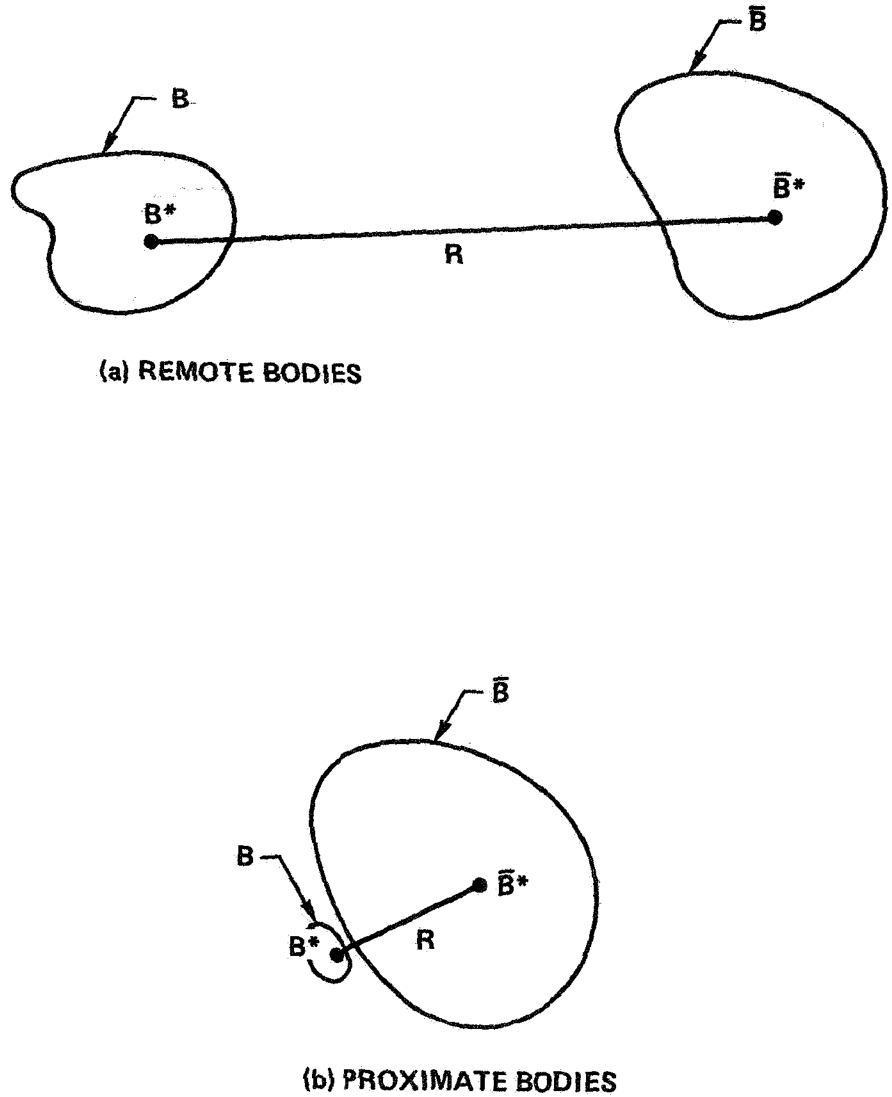

B
B*
$\bar{B}$
$\bar{B}$*
R
(a) REMOTE BODIES
B
B*
$\bar{B}$
$\bar{B}$*
R
(b) PROXIMATE BODIES

Figure 2.8.1

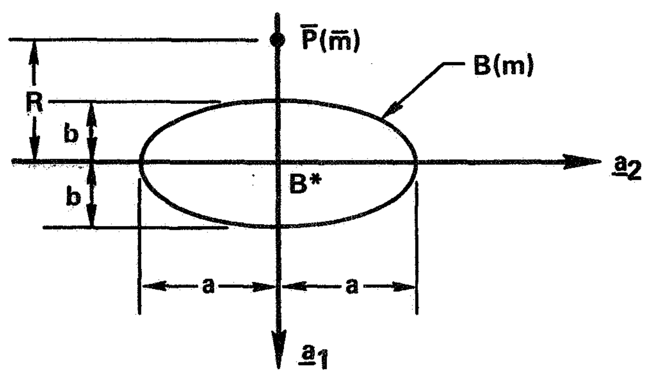


Figure 2.8.2

If $\underline{a}_1$ and $\underline{a}_2$ are unit vectors as shown in Fig. 2.8.2, and $\underline{a}_3 = \underline{a}_1 \times \underline{a}_2$, then the associated moments and products of inertia of B for B^* are given by

$$I_{11} = \frac{2ma^2}{5}, \quad I_{22} = I_{33} = \frac{m}{5}(a^2 + b^2)$$

$$I_{12} = I_{23} = I_{31} = 0$$

Hence

$$\underline{f}^{(2)} \underset{(2.3.10)}{=} -\frac{3}{5R^2}(a^2 - b^2)\,\underline{a}_1$$

$$\underline{\tilde{F}} \underset{(2.3.6)}{=} -\frac{G\bar{m}m}{R^2}\left[1 - \frac{3}{5R^2}(a^2 - b^2)\right]\underline{a}_1$$

$$\underset{(1,2)}{=} -\frac{G\bar{m}m}{R^2}\left(1 - \frac{3}{5}\,\frac{\beta^2\varepsilon^2}{1-\varepsilon^2}\right)\underline{a}_1$$

and

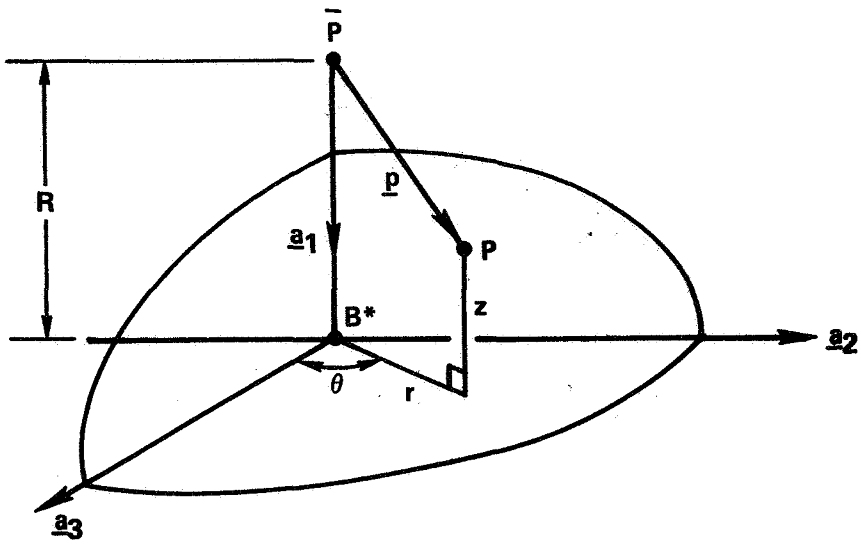
$$\tilde{F} \triangleq |\underline{\tilde{F}}| = \frac{G\bar{m}m}{R^2}\left|1 - \frac{3}{5}\,\frac{\beta^2\varepsilon^2}{1-\varepsilon^2}\right| \qquad (3)$$

For the evaluation of the integral in Eq. (2.2.2), it is convenient to introduce the coordinates r, θ, and z shown in Fig. 2.8.3 and to note that $\underline{p}$, the position vector of P relative to $\bar{P}$, then can be expressed as

$$\underline{p} = (R - z)\underline{a}_1 + r \sin\theta\, \underline{a}_2 + r \cos\theta\, \underline{a}_3$$

while

$$d\tau = r\, d\theta\, dr\, dz$$



$\bar{P}$
R
$\underline{p}$
$\underline{a}_1$
P
z
B*
$\underline{a}_2$
θ
r
$\underline{a}_3$

Figure 2.8.3

As for ρ, the mass density of B, this is given by

$$\rho = \frac{3m}{4\pi ba^2}$$

Hence

$$\underline{F} \underset{(2.2.2)}{=} -\frac{3G\bar{m}m}{4\pi ba^2} \int_{\theta_1}^{\theta_2} \int_{z_1}^{z_2} \int_{r_1}^{r_2} - \frac{(R-z)\underline{a}_1 + r\sin\theta\underline{a}_2 + r\cos\theta\underline{a}_3}{[(R-z)^2+r^2]^{3/2}} \, r \, dr \, dz \, d\theta$$

where

$$\theta_1 = 0, \; \theta_2 = 2\pi; \; z_1 = -b, \; z_2 = b; \; r_1 = 0, \; r_2 = a[1-(z/b)^2]^{1/2}$$

and, after carrying out the indicated integrations and using (1) and (2) to eliminate b/R and a/b, one arrives at

$$F \triangleq |\underline{F}| = \frac{3G\bar{m}m(1-\varepsilon^2)}{R^2\beta^2\varepsilon^2} \left| 1 - \frac{(1-\varepsilon^2)^{1/2}}{2\beta\varepsilon} [H(1) - H(-1)] \right| \tag{4}$$

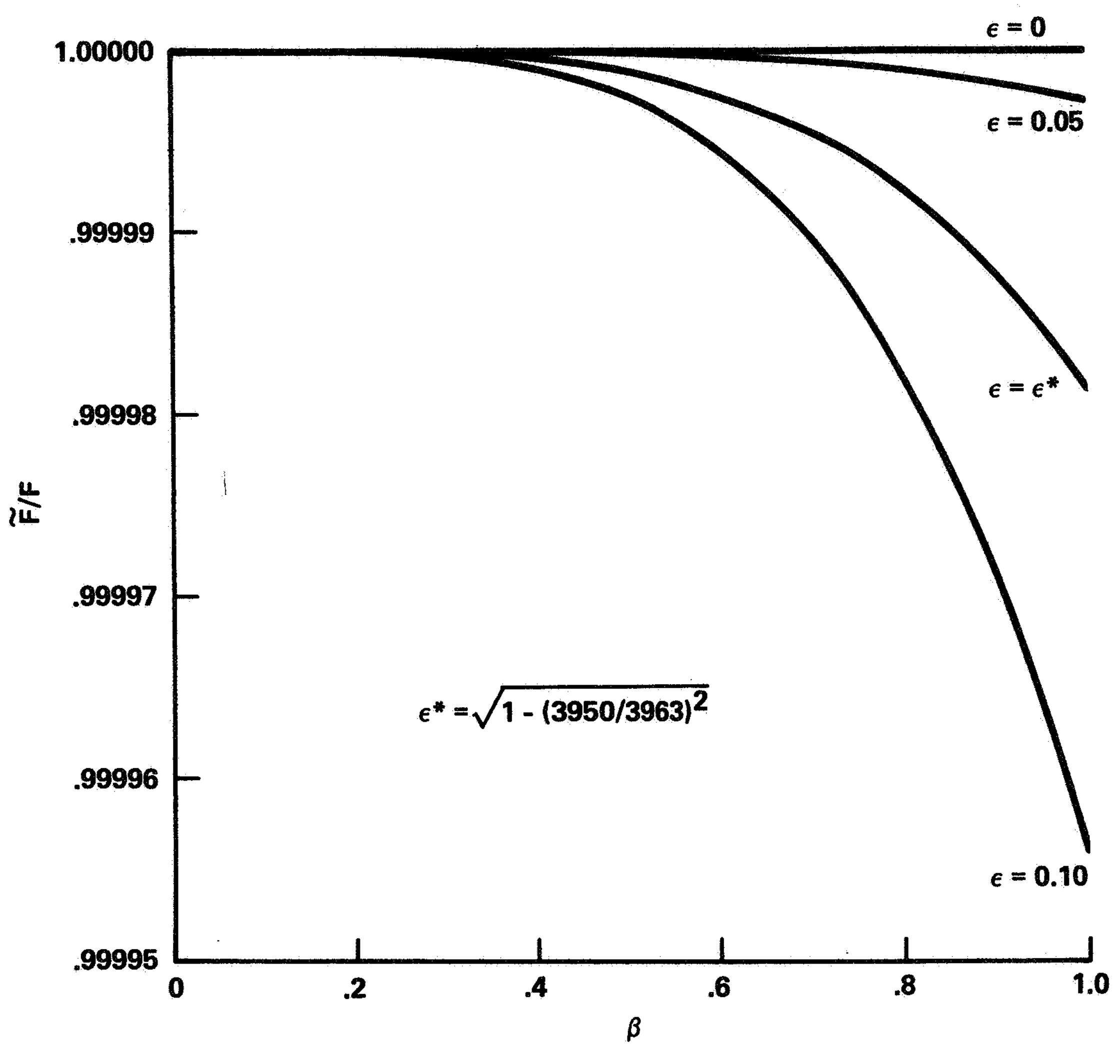
where H is defined as

$$H(\xi) \triangleq \sin^{-1} \frac{1-\varepsilon^2(1-\beta\xi)}{[1-\varepsilon^2(1-\beta^2)]^{1/2}}$$

Consequently,

$$\frac{\tilde{F}}{F} \underset{(3,4)}{=} \frac{\beta^2\varepsilon^2}{3(1-\varepsilon^2)} \frac{\left| 1 - \frac{3}{5}\frac{\beta^2\varepsilon^2}{1-\varepsilon^2} \right|}{\left| 1 - \frac{(1-\varepsilon^2)^{1/2}}{2\beta\varepsilon} [H(1) - H(-1)] \right|} \tag{5}$$

In Fig. 2.8.4, $\tilde{F}/F$ as given by Eq. (5) is plotted versus β for four values of ε. Note that small values of β correspond to placing $\bar{P}$ at a great distance from B^* (see Eq. (1)). Hence it is not surprising that, regardless of the values of ε, $\tilde{F}/F$ approaches unity as β approaches zero. Conversely, values of β near unity represent situations in which the particle comes relatively close to the spheroid; and the error one then makes when using $\tilde{F}$ in place of F can be seen to depend on ε, larger departures of $\tilde{F}/F$ from unity being associated with larger values of ε, that is, with a more pronounced flattening of the ellipsoid (see Eq. (2)). However, it appears that even for $\beta \approx 1.0$, that is, for a particle that is nearly in contact with the spheroid, $\tilde{F}/F$ departs only very slightly from unity so long as $\varepsilon < 0.10$, which is the case, for example, when $\varepsilon = \varepsilon^*$, where ε^* is the eccentricity of a spheroid whose major semi-diameters are equal to the earth's polar and equatorial radii. Consequently, Eq. (2.3.6) may be expected to yield highly accurate results when used in an analysis concerned with a near-earth satellite idealized as a particle.



ϵ = 0
1.00000
ϵ = 0.05
.99999
ϵ = ϵ*
.99998
F̃/F
.99997
ϵ* = √(1 - (3950/3963)²)
.99996
ϵ = 0.10
.99995
0
.2
.4
.6
.8
1.0
β

Figure 2.8.4

2.9 Differentiation with respect to a vector

It is sometimes convenient to employ vector operations on scalar functions in order to express gravitational forces and moments. These operations may be interpreted as ordinary and/or partial differentiations of scalar functions with respect to vector variables. In some cases the vector in question is a position vector, and the ordinary derivative is then a spatial gradient. When the differentiation of a scalar function produces a force, the scalar function is called a force function. In other cases differentiations with respect to other vectors are employed, as in the representation of gravitational moments in terms of derivatives with respect to unit vectors.

To unify the presentation of this subject in Sec. 2.10-2.20, the mathematical tools there employed are first discussed briefly in this Section.

If a scalar quantity F depends on a vector $\underline{v}$, then it is useful to define a vector denoted by $\nabla_{\underline{v}} F$ as follows: Introduce an arbitrary set of mutually perpendicular unit vectors $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$; let $v_i \triangleq \underline{v} \cdot \underline{a}_i$ $(i=1,2,3)$; regard F as a function of v_1, v_2 and v_3; and set

$$\nabla_{\underline{v}} F \triangleq \frac{\partial F}{\partial v_1} \underline{a}_1 + \frac{\partial F}{\partial v_2} \underline{a}_2 + \frac{\partial F}{\partial v_3} \underline{a}_3 \tag{1}$$

The vector $\nabla_{\underline{v}} F$ constructed according to Eq. (1) is invariant with respect to the choice of vector basis $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$. The operation denoted by $\nabla_{\underline{v}}$ may be termed differentiation with respect to vector $\underline{v}$.

Similarly, if a vector quantity $\underline{F}$ depends on a vector $\underline{v}$, and for some arbitrary vector basis $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ one defines $F_i \triangleq \underline{F} \cdot \underline{a}_i$ and $v_i = \underline{v} \cdot \underline{a}_i$ $(i=1,2,3)$, then a scalar and a dyadic may be defined as

$$\underline{\nabla}_{\underline{v}} \cdot \underline{F} \triangleq \frac{\partial F_1}{\partial v_1} + \frac{\partial F_2}{\partial v_2} + \frac{\partial F_3}{\partial v_3} \tag{2}$$

and

$$\underline{\nabla}_{\underline{v}}\, \underline{F} \triangleq \left(\underline{\nabla}_{\underline{v}}\, F_1\right) \underline{a}_1 + \left(\underline{\nabla}_{\underline{v}}\, F_2\right) \underline{a}_2 + \left(\underline{\nabla}_{\underline{v}}\, F_3\right) \underline{a}_3 \tag{3}$$

The quantities defined by Eqs. (2) and (3) do not depend upon the choice of vector basis $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$.

In Eqs. (2) and (3) a vector differentiation concept is introduced without identification of a particular reference frame in which the differentiation occurs; this omission is possible only with the stipulation that the orientations of the unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ do not depend on $\underline{v}$.

Further useful quantities are at times obtained by cascading some of these definitions. For example, when the operation defined in Eq. (2) is performed on the vector defined in Eq. (1), the result is a scalar denoted by $\underline{\nabla}_{\underline{v}}^2\, F$:

$$\underline{\nabla}_{\underline{v}}^2\, F \triangleq \underline{\nabla}_{\underline{v}} \cdot \left(\underline{\nabla}_{\underline{v}}\, F\right) \tag{4}$$

When $\underline{v}$ is the position vector $\underline{p}$ of a generic point relative to some other point, and when the vector $\underline{p}$ can be inferred from the context, the subscript $\underline{v} = \underline{p}$ is often omitted; the quantities $\underline{\nabla}F$, $\underline{\nabla}\cdot\underline{F}$, $\underline{\nabla}\underline{F}$ and $\underline{\nabla}^2 F$ are then called respectively the gradient of F, the divergence of $\underline{F}$, the gradient of $\underline{F}$, and the Laplacian of F.

As a consequence of the definitions given in Eqs. (1)-(4) and of various theorems of the differential calculus of functions of one or more scalar variables, the quantities $\underline{\nabla}_{\underline{v}}\, F$, $\underline{\nabla}_{\underline{v}} \cdot \underline{F}$, and $\underline{\nabla}_{\underline{v}}^2 F$ satisfy many relationships having counterparts in this calculus. For example, if $\underline{v} = \underline{w} + \underline{c}$, where $\underline{c}$ is independent of $\underline{v}$, and if* $F(\underline{v})$ and $G(\underline{w})$ are functions such that $G(\underline{w}) = F(\underline{w}+\underline{c})$, then

$$\underline{\nabla}_{\underline{v}} F = \underline{\nabla}_{\underline{w}} G \tag{5}$$

Or, suppose that $\underline{v}(g_1,\ldots,g_n)$ denotes a vector function of the scalar independent variables $g_1,\ldots,g_n$; suppose, further, that $F(\underline{v})$ and $G(g_1,\ldots,g_n)$ are functions such that $G(g_1,\ldots, g_n) = F[\underline{v}(g_1,\ldots,g_n)]$. Then

$$\frac{\partial G}{\partial g_r} = \underline{\nabla}_{\underline{v}} F \cdot \frac{\partial \underline{v}}{\partial g_r} \qquad (r=1,\ldots,n) \tag{6}$$

Three differentiation formulas involving a unit dyadic $\underline{U}$ and a unit vector $\underline{u}$ having the same direction as a vector $\underline{v}$ will prove useful in the sequel. These are

$$\underline{\nabla}_{\underline{v}}\, \underline{v} = \underline{U} \tag{7}$$

$$\underline{\nabla}_{\underline{v}}\, v = \underline{u} \tag{8}$$

and

$$\underline{\nabla}_{\underline{v}}\, \underline{u} = v^{-1}(\underline{U} - \underline{u}\,\underline{u}) \tag{9}$$

where $v \triangleq \underline{v} \cdot \underline{u}$, so that $\underline{v} = v\,\underline{u}$.

In dealing with gravitational moments, it is at times convenient to invoke the idea of partial differentiation with respect to a vector, in the following sense: suppose that $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, v, $\underline{v}$, $\underline{u}$ and F have the same meanings as heretofore; let $u_i \triangleq \underline{u} \cdot \underline{a}_i$ $(i = 1,2,3)$; and let $\underline{v}(\underline{u},v)$ and $G(\underline{u},v)$ denote respectively a vector function and a scalar function of

*As a notational convenience, a scalar quantity such as F which depends upon the vector $\underline{v}$ is designated $F(\underline{v})$; the functional representation of this quantity (as required by Eqs. (1)-(4)) is written $F(v_1,v_2,v_3)$ or $F(v_1',v_2',v_3')$, where $v_i \triangleq \underline{v} \cdot \underline{a}_i$ and $v_i' \triangleq \underline{v} \cdot \underline{a}_i'$ $(i=1,2,3)$ for arbitrary vector bases $\underline{a}_1,\underline{a}_2,\underline{a}_3$ and $\underline{a}_1',\underline{a}_2',\underline{a}_3'$.

u_1, u_2, u_3, and v, choosing $G(\underline{u}, v)$ such that $G(\underline{u}, v) \triangleq F[\underline{v}(\underline{u}, v)]$. Then, defining $\partial G/\partial \underline{u}$ as

$$\frac{\partial G}{\partial \underline{u}} \triangleq \frac{\partial G}{\partial u_1}\,\underline{a}_1 + \frac{\partial G}{\partial u_2}\,\underline{a}_2 + \frac{\partial G}{\partial u_3}\,\underline{a}_3 \tag{10}$$

one can write

$$\underline{\nabla}_{\underline{v}}\, F = \frac{\partial G}{\partial \underline{u}} \cdot \underline{\nabla}_{\underline{v}}\, \underline{u} + \frac{\partial G}{\partial v}\, \underline{u} \tag{11}$$

or, in view of Eq. (9),

$$\underline{\nabla}_{\underline{v}}\, F = \frac{1}{v}\frac{\partial G}{\partial \underline{u}} + \left(\frac{\partial G}{\partial v} - \frac{1}{v}\frac{\partial G}{\partial \underline{u}} \cdot \underline{u}\right)\underline{u} \tag{12}$$

It follows from this equation and from $\underline{v} = v\underline{u}$ that

$$\underline{v} \times \underline{\nabla}_{\underline{v}}\, F = \underline{u} \times \frac{\partial G}{\partial \underline{u}} \tag{13}$$

Derivations: In characterizing a quantity as a function of a vector, one implies that the quantity is represented in terms of the scalar components of the vector by the same functional relationship, regardless of vector basis. Thus for example if the scalar quantity F is a function of the vector $\underline{v}$, and $v_i \triangleq \underline{v} \cdot \underline{a}_i$ and $v_i' \triangleq \underline{v} \cdot \underline{a}_i'$ $(i=1,2,3)$ for arbitrary orthogonal vector bases $\underline{a}_1, \underline{a}_2, \underline{a}_3$ and $\underline{a}_1', \underline{a}_2', \underline{a}_3'$, then in functional notation

$$F(v_1, v_2, v_3) = F(v_1', v_2', v_3') \tag{14}$$

Using this relationship, one can prove the invariance of $\underline{\nabla}_{\underline{v}}F$, $\underline{\nabla}_{\underline{v}} \cdot \underline{F}$, $\underline{\nabla}_{\underline{v}}\underline{F}$ and $\nabla_{\underline{v}}^2 F$ with respect to the choice of vector basis by establishing the equality of alternative representations of each of these quantities in terms of the arbitrary orthogonal vector bases $\underline{a}_1, \underline{a}_2, \underline{a}_3$and $\underline{a}_1', \underline{a}_2'$, and $\underline{a}_3'$. This objective can be accomplished as follows:

Letting $C_{ij} \triangleq \underline{a}_i \cdot \underline{a}'_j$ (i,j=1,2,3), one can write

$$\underline{a}_i = \underline{a}'_j\, C_{ij} \tag{15}$$

$$F_i = F'_j\, C_{ij} \tag{16}$$

and

$$v'_i = v_j\, C_{ji}$$

from the last of which it follows that

$$\frac{\partial v'_i}{\partial v_j} = C_{ji} \tag{17}$$

Consequently, using familiar differentiation theorems, one can write

$$\frac{\partial F}{\partial v_i} \underset{(14)}{=} \frac{\partial F}{\partial v'_j}\frac{\partial v'_j}{\partial v_i} \underset{(17)}{=} \frac{\partial F}{\partial v'_j}\, C_{ij} \tag{18}$$

and

$$\frac{\partial F}{\partial v_i}\, \underline{a}_i \underset{(18,15)}{=} \frac{\partial F}{\partial v'_j}\, C_{ij}\, \underline{a}'_k\, C_{ik} \tag{19}$$

But

$$C_{ij}\, C_{ik} \underset{(1.2.15)}{=} \delta_{jk} \tag{20}$$

Hence,

$$\frac{\partial F}{\partial v_i}\, \underline{a}_i \underset{(19,20)}{=} \frac{\partial F}{\partial v'_j}\, \underline{a}'_k\, \delta_{jk} = \frac{\partial F}{\partial v'_i}\, \underline{a}'_i$$

which establishes the invariance of $\underline{\nabla}_{\underline{v}}\, F$. For $\underline{\nabla}_{\underline{v}} \cdot \underline{F}$, one has

$$\frac{\partial F_i}{\partial v_i} \underset{(16)}{=} \frac{\partial F'_j}{\partial v_i}\, C_{ij} = \frac{\partial F'_j}{\partial v'_k}\frac{\partial v'_k}{\partial v_i}\, C_{ij}$$

$$\underset{(17)}{=} \frac{\partial F'_j}{\partial v'_k}\, C_{ik}\, C_{ij} \underset{(1.2.15)}{=} \frac{\partial F'_i}{\partial v'_i}$$

and the proofs for $\underline{\nabla}_{\underline{v}}\ \underline{F}$ and $\underline{\nabla}_{\underline{v}} \cdot \underline{\underline{\nabla F}}$ proceed similarly.

When $\underline{F}$ in Eq. (3) is replaced with $\underline{v}$, one has

$$\underline{\nabla}_{\underline{v}}\ \underline{v} = \left(\underline{\nabla}_{\underline{v}}\ v_i\right) \underline{a}_i \tag{21}$$

Now,

$$\underline{\nabla}_{\underline{v}}\ v_i \underset{(1)}{=} \frac{\partial v_i}{\partial v_j}\ \underline{a}_j = \delta_{ij}\ \underline{a}_j = \underline{a}_i \tag{22}$$

Hence

$$\underline{\nabla}_{\underline{v}}\ \underline{v} = \underline{a}_i\underline{a}_i = \underline{\underline{U}} \tag{23}$$

in agreement with Eq. (7).

If $\underline{v} = v\ \underline{u}$, so that $\underline{u}$ has the same direction as $\underline{v}$, then

$$v = (\underline{v} \cdot \underline{v})^{1/2} \tag{24}$$

and

$$\underline{\nabla}_{\underline{v}}\ v \underset{(24)}{=} \underline{\nabla}_{\underline{v}}(\underline{v}\cdot\underline{v})^{\frac{1}{2}} = \tfrac{1}{2}(\underline{v}\cdot\underline{v})^{-\frac{1}{2}}\left[\left(\underline{\nabla}_{\underline{v}}\ \underline{v}\right)\cdot \underline{v} + \underline{v} \cdot \left(\underline{\nabla}_{\underline{v}}\ \underline{v}\right)\right]$$

$$\underset{(23,24)}{=} \frac{1}{2}\ v^{-1}\ (\underline{v}+\underline{v}) = v^{-1}\underline{v} = \underline{u}$$

which proves the validity of Eq. (8).

As for Eq. (9), note that

$$\underline{\nabla}_{\underline{v}}\ \underline{u} = \underline{\nabla}_{\underline{v}}(v^{-1}\underline{v}) = \left(\underline{\nabla}_{\underline{v}}\ v^{-1}\right)\underline{v} + v^{-1}\ \underline{\nabla}_{\underline{v}}\ \underline{v}$$

$$= \left(-v^{-2}\ \underline{\nabla}_{\underline{v}}\ v\right)\ \underline{v} + v^{-1}\ \underline{\nabla}_{\underline{v}}\ \underline{v}$$

$$\underset{(7,8)}{=} \left(-v^{-2}\ \underline{u}\right) v\underline{u} + v^{-1}\ \underline{\underline{U}}$$

$$= v^{-1}\ (\underline{\underline{U}} - \underline{\underline{uu}})$$

Finally, in connection with Eq. (11), note that both u_i and v can be expressed as functions of v_1, v_2, v_3 (where $v_i \triangleq \underline{v} \cdot \underline{a}_i$):

$$u_i = v_i \left(v_1^2 + v_2^2 + v_3^2 \right)^{-\frac{1}{2}} , \quad v = \left(v_1^2 + v_2^2 + v_3^2 \right)^{\frac{1}{2}}$$

It follows that

$$\frac{\partial F}{\partial v_i} = \frac{\partial G}{\partial u_j} \frac{\partial u_j}{\partial v_i} + \frac{\partial G}{\partial v} \frac{\partial v}{\partial v_i} \tag{25}$$

so that

$$\underline{\nabla}_{\underline{v}} F \underset{(1)}{=} \frac{\partial F}{\partial v_i} \underline{a}_i \underset{(2)}{=} \frac{\partial G}{\partial u_j} \frac{\partial u_j}{\partial v_i} \underline{a}_i + \frac{\partial G}{\partial v} \frac{\partial v}{\partial v_i} \underline{a}_i$$

$$\underset{(1)}{=} \frac{\partial G}{\partial u_j} \underline{\nabla}_{\underline{v}} u_j + \frac{\partial G}{\partial v} \underline{\nabla}_{\underline{v}} v$$

$$\underset{(10,8)}{=} \left(\underline{\nabla}_{\underline{v}} u_j \right) \underline{a}_j \cdot \frac{\partial G}{\partial \underline{u}} + \frac{\partial G}{\partial v} \underline{u}$$

$$\underset{(3)}{=} \left(\underline{\nabla}_{\underline{v}} \underline{u} \right) \cdot \frac{\partial G}{\partial \underline{u}} + \frac{\partial G}{\partial v} \underline{u}$$

and this is equivalent to Eq. (11) since $\underline{\nabla}_{\underline{v}}$ u is a symmetric dyadic, as is apparent from Eq. (9).

<u>Example</u>: In Fig. 2.9.1, r, λ and β are spherical coordinates of a point P, and $\underline{b}_1, \underline{b}_2, \underline{b}_3$ are unit vectors pointing respectively in the directions in which P moves when r, λ, β are made to vary, one at a time. The position vector $\underline{p}$ of P relative to 0 may be expressed as

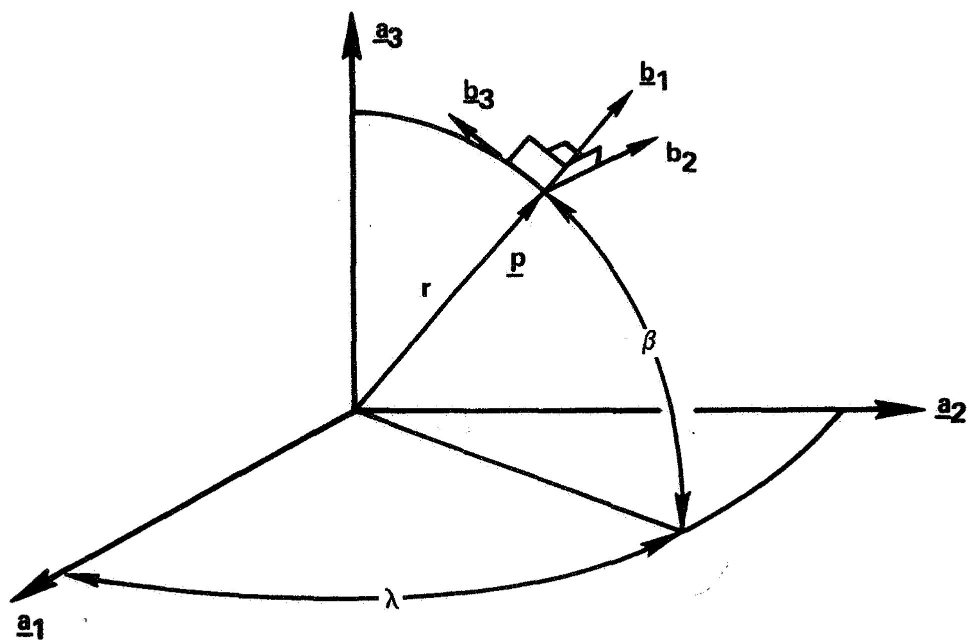
$$\underline{p}\,(r,\lambda,\beta) = r(c\lambda c\beta \underline{a}_1 + s\lambda c\beta \underline{a}_2 + s\beta \underline{a}_3) \tag{26}$$

If $F(\underline{p})$ denotes a function of $\underline{p}$, and if $G(r,\lambda,\beta)$ is defined as

$$G(r,\lambda,\beta) \triangleq F[\underline{p}(r,\lambda,\beta)]$$

then Eq. (6) can be used to express the gradient of F in the following frequently convenient form:

$$\underline{\nabla} F = \frac{\partial G}{\partial r} \underline{b}_1 + \frac{1}{r c\beta} \frac{\partial G}{\partial \lambda} \underline{b}_2 + \frac{1}{r} \frac{\partial G}{\partial \beta} \underline{b}_3 \tag{27}$$



$\underline{a}_3$
$\underline{b}_3$
$\underline{b}_1$
b_2
r
$\underline{p}$
β
$\underline{a}_2$
$\underline{a}_1$
λ

Figure 2.9.1

To this end, one may begin by evaluating the partial derivatives of $\underline{p}$ with respect to r, λ and β:

$$\frac{\partial \underline{p}}{\partial r} \underset{(26)}{=} c\lambda c\beta \underline{a}_1 + s\lambda c\beta \underline{a}_2 + s\beta \underline{a}_3 = \underline{b}_1 \tag{28}$$

$$\frac{\partial \underline{p}}{\partial \lambda} \underset{(26)}{=} r(-s\lambda c\beta \underline{a}_1 + c\lambda c\beta \underline{a}_2) = rc\beta \underline{b}_2 \tag{29}$$

$$\frac{\partial \underline{p}}{\partial \beta} \underset{(26)}{=} r(-c\lambda s\beta \underline{a}_1 - s\lambda s\beta \underline{a}_2 + c\beta \underline{a}_3) = r\underline{b}_3 \tag{30}$$

Next, from Eq. (6), and with $\underline{\nabla}$ written in place of $\underline{\nabla}_{\underline{p}}$,

$$\frac{\partial G}{\partial r} = \underline{\nabla} F \cdot \frac{\partial \underline{p}}{\partial r} \underset{(28)}{=} \underline{\nabla} F \cdot \underline{b}_1 \tag{31}$$

$$\frac{\partial G}{\partial \lambda} = \underline{\nabla} F \cdot \frac{\partial \underline{p}}{\partial \lambda} \underset{(29)}{=} rc\beta\, \underline{\nabla} F \cdot \underline{b}_2 \tag{32}$$

$$\frac{\partial G}{\partial \beta} = \underline{\nabla} F \cdot \frac{\partial \underline{p}}{\partial \beta} \underset{(30)}{=} r\, \underline{\nabla} F \cdot \underline{b}_3 \tag{33}$$

Now, since $\underline{b}_1\underline{b}_1 + \underline{b}_2\underline{b}_2 + \underline{b}_3\underline{b}_3$ is a unit dyadic,

$$\begin{aligned} \underline{\nabla} F &= \underline{\nabla} F \cdot (\underline{b}_1\underline{b}_1 + \underline{b}_2\underline{b}_2 + \underline{b}_3\underline{b}_3) \\ &= \underline{\nabla} F \cdot \underline{b}_1\underline{b}_1 + \underline{\nabla} F \cdot \underline{b}_2\underline{b}_2 + \underline{\nabla} F \cdot \underline{b}_3\underline{b}_3 \end{aligned} \tag{34}$$

Solving Eqs. (31)-(33) respectively for $\underline{\nabla} F \cdot \underline{b}_1$, $\underline{\nabla} F \cdot \underline{b}_2$, and $\underline{\nabla} F \cdot \underline{b}_3$, and substituting into Eq. (34), one arrives at Eq. (27).

The Laplacian of F may be expressed as

$$\nabla^2 F = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + \sec^2\beta \frac{\partial^2 G}{\partial \lambda^2} + \sec\beta \frac{\partial}{\partial \beta}\left(c\beta \frac{\partial G}{\partial \beta}\right)\right] \tag{35}$$

by using Eq. (4) in conjunction with Eq. (27), as follows:

$$\nabla^2 F \underset{(4)}{=} \nabla \cdot \nabla F \underset{(27)}{=} \left(\frac{\partial}{\partial r} \underline{b}_1 + \frac{1}{rc\beta} \frac{\partial}{\partial \lambda} \underline{b}_2 + \frac{1}{r} \frac{\partial}{\partial \beta} \underline{b}_3 \right) \cdot \underline{\nabla F}$$

$$= \underline{b}_1 \cdot \frac{\partial}{\partial r} (\underline{\nabla F}) + \frac{1}{rc\beta} \underline{b}_2 \cdot \frac{\partial}{\partial \lambda} (\underline{\nabla F}) + \frac{1}{r} \underline{b}_3 \cdot \frac{\partial}{\partial \beta} (\underline{\nabla F}) \tag{36}$$

Evaluating the partial derivative appearing in Eq. (36) but displaying only those terms that will not be eliminated by subsequent dot-multiplications, one has

$$\frac{\partial}{\partial r} (\underline{\nabla F}) \underset{(27)}{=} \frac{\partial^2 G}{\partial r^2} \underline{b}_1 + \ldots$$

$$\frac{\partial}{\partial \lambda} (\underline{\nabla F}) = \frac{\partial G}{\partial r} \frac{\partial \underline{b}_1}{\partial \lambda} + \frac{1}{rc\beta} \frac{\partial^2 G}{\partial \lambda^2} \underline{b}_2 + \frac{1}{r} \frac{\partial G}{\partial \beta} \frac{\partial \underline{b}_3}{\partial \lambda} + \ldots$$

$$= \frac{\partial G}{\partial r} c\beta \underline{b}_2 + \frac{1}{rc\beta} \frac{\partial^2 G}{\partial \lambda^2} \underline{b}_2 - \frac{1}{r} \frac{\partial G}{\partial \beta} s\beta \underline{b}_2 + \ldots$$

$$\frac{\partial}{\partial \beta} (\underline{\nabla F}) = \frac{\partial G}{\partial r} \frac{\partial \underline{b}_1}{\partial \beta} + \frac{1}{r} \frac{\partial^2 G}{\partial \beta^2} \underline{b}_3 + \ldots$$

$$= \frac{\partial G}{\partial r} \underline{b}_3 + \frac{1}{r} \frac{\partial^2 G}{\partial \beta^2} \underline{b}_3 + \ldots$$

and substitution into Eq. (36) then yields

$$\nabla^2 F = \frac{\partial^2 G}{\partial r^2} + \frac{2}{r} \frac{\partial G}{\partial r} + \frac{1}{r^2 c^2 \beta} \frac{\partial^2 G}{\partial \lambda^2} - \frac{s\beta}{r^2 c\beta} \frac{\partial G}{\partial \beta} + \frac{1}{r^2} \frac{\partial^2 G}{\partial \beta^2}$$

which is equivalent to Eq. (35).

2.10 Force function for two particles

The gravitational force $\underline{F}$ exerted on a particle P of mass m by a particle $\bar{P}$ of mass $\bar{m}$ (see Sec. 2.1) can be expressed as

$$\underline{F} = \underline{\nabla}_{\underline{p}} V \triangleq \underline{\nabla} V \tag{1}$$

where $\underline{p}$ is the position vector of P relative to $\bar{P}$ and V is given by

$$V = G\bar{m}mp^{-1} + C \tag{2}$$

with p defined as

$$p \triangleq \left(\underline{p}^2\right)^{\frac{1}{2}} \tag{3}$$

and C an arbitrary constant, while G is the universal gravitational constant.

A scalar function of a vector variable is called a "force function" if the derivative of the function with respect to the variable is equal to a force. Thus V is a force function associated with the gravitational interaction of two particles.

The Laplacian of V (see Sec. 2.9) is zero:

$$\nabla^2 V = 0 \tag{4}$$

Eq. (4) is known as Laplace's equation. This equation arises in many problems of physics, such as electrostatics, magnetostatics, hydrodynamics, and heat flow, as well as in gravitation, and it is one of the classical equations of partial differential equation theory. Any solution of Laplace's equation is called a spherical harmonic, and in the context of gravitational problems a solution is classically called a gravitational potential. For the purpose of characterizing the interaction force $\underline{F}$ between two particles, however, only that special spherical harmonic given by Eq. (2) provides the force function appropriate for substitution into Eq. (1).

<u>Derivations</u>: Differentiation of V with respect to $\underline{p}$ gives

$$\underline{\nabla}V \underset{(2)}{=} G\bar{m}m\underline{\nabla}p^{-1} \underset{(2.9.8)}{=} -G\bar{m}mp^{-2}\,\underline{u} \tag{5}$$

where $\underline{u}$ is the unit vector in the direction of $\underline{p}$, that is,

$$\underline{u} = p^{-1}\underline{p} \tag{6}$$

Consequently,

$$\underline{\nabla}V = -G\bar{m}mp^{-3}\underline{p} \underset{(2.1.1)}{=} \underline{F} \tag{7}$$

The Laplacian of V is

$$\underline{\nabla}^2 V \underset{(2.9.4)}{=} \underline{\nabla} \cdot \underline{\nabla}V \underset{(7)}{=} -G\bar{m}m\underline{\nabla} \cdot \left(p^{-3}\underline{p}\right)$$

$$\underset{\substack{(2.9.8,\\ 2.9.2)}}{=} -G\bar{m}m\left(3p^{-4}\underline{u}\cdot\underline{p} - 3p^{-3}\right) = 0 \tag{8}$$

<u>Example</u>: A particle P of mass m and two particles P_1 and P_2, each of mass $\bar{m}$, are situated as shown in Fig. 2.10.1, where r,λ and β are spherical coordinates of P and $\underline{b}_1,\underline{b}_2,\underline{b}_3$ are unit vectors pointing in the directions in which P moves when r, λ and β are made to increase one at a time. A force function V for the resultant gravitational force $\underline{F}$ acting on P is to be constructed, and this function is to be used to express $\underline{F}$ in terms of components respectively parallel to $\underline{b}_1,\underline{b}_2,\underline{b}_3$.

The forces $\underline{F}_1$ and $\underline{F}_2$ exerted on P respectively by P_1 and P_2 can be expressed as

$$\underline{F}_i = \underline{\nabla}_{\underline{p}_i} V_i \qquad (i=1,2) \tag{9}$$

where $\underline{p}_i$ is the position vector of P relative to P_i and where $V_1(\underline{p}_1)$ and $V_2(\underline{p}_2)$ are given by

$$V_i \underset{(2)}{=} G\bar{m}mp_i^{-1} + C_i \qquad (i=1,2)$$

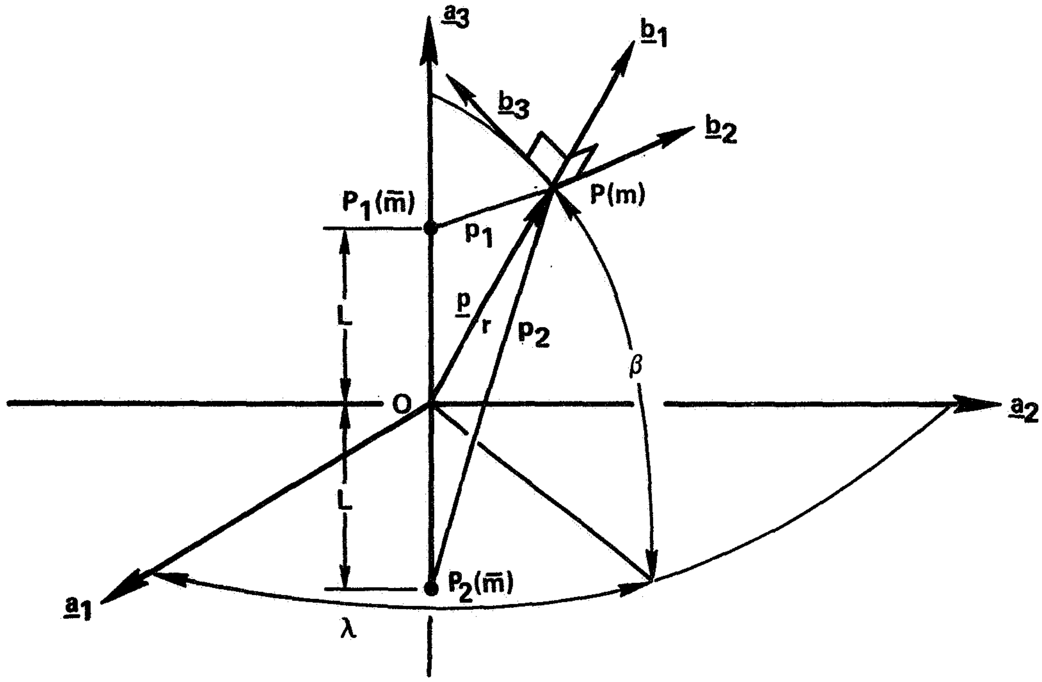

$\underline{a}_3$
$\underline{b}_1$
$\underline{b}_3$
$\underline{b}_2$
$P_1(\bar{m})$
P(m)
p_1
L
$\underline{p}_r$
p_2
β
O
$\underline{a}_2$
L
$\underline{a}_1$
$P_2(\bar{m})$
λ

Figure 2.10.1

with p_i equal to the magnitude of $\underline{p}_i$. Hence

$$\underline{F} = \underline{F}_1 + \underline{F}_2 \underset{(9)}{=} \underline{\nabla}_{\underline{p}_1} V_1 + \underline{\nabla}_{\underline{p}_2} V_2 \tag{10}$$

If $\underline{p}$ is now introduced as the position vector of P relative to O, then $\underline{p}_i = \underline{p} + \underline{c}_i$ (i=1,2) where $\underline{c}_i$ is independent of $\underline{p}_i$; if then $W_i(\underline{p})$ is defined as

$$W_i(\underline{p}) \triangleq V_i(\underline{p} + \underline{c}_i) \qquad (i=1,2) \tag{11}$$

then

$$\underline{\nabla}_{\underline{p}_i} V_i \underset{(2.9.3)}{=} \underline{\nabla}_{\underline{p}} W_i \qquad (i=1,2) \tag{12}$$

Hence

$$\underline{F} \underset{(10,12)}{=} \underline{\nabla}_{\underline{p}} W_1 + \underline{\nabla}_{\underline{p}} W_2 = \nabla_{\underline{p}} (W_1 + W_2) = \underline{\nabla}_{\underline{p}} V \tag{13}$$

where V is defined as

$$V \triangleq W_1 + W_2 \underset{(11)}{=} V_1 + V_2 \underset{(12)}{=} G\bar{m}m\,(p_1^{-1} + p_2^{-1}) + C \tag{14}$$

V is the desired force function.

To express $\underline{F}$ in terms of components parallel to $\underline{b}_1, \underline{b}_2, \underline{b}_3$, note that

$$p_1 = \left(r^2 + L^2 - 2rL\sin\beta\right)^{\frac{1}{2}} \tag{15}$$

and

$$p_2 = \left(r^2 + L^2 + 2rL\sin\beta\right)^{\frac{1}{2}} \tag{16}$$

and define a function W of r, λ, β as

$$W \triangleq G\bar{m}m \left[\left(r^2+L^2-2rL\sin\beta\right)^{-\frac{1}{2}} + \left(r^2+L^2+2rL\sin\beta\right)^{-\frac{1}{2}}\right] + C \tag{17}$$

Then

$$V \underset{(14-17)}{=} W \tag{18}$$

and Eq. (27) of Sec. 29 provides

$$\underline{\nabla}_{\underline{p}}V = \frac{\partial W}{\partial r}\,\underline{b}_1 + \frac{1}{rc\beta}\frac{\partial W}{\partial \lambda}\,\underline{b}_2 + \frac{1}{r}\frac{\partial W}{\partial \beta}\,\underline{b}_3$$

$$\underset{(17)}{=} -G\bar{m}m\Big[\left(r^2+L^2-2rL\sin\beta\right)^{-3/2}\left(r-L\sin\beta\right)$$

$$+\left(r^2+L^2+2rL\sin\beta\right)^{-3/2}\left(r+L\sin\beta\right)\Big]\,\underline{b}_1$$

$$+\,G\bar{m}m\Big[\left(r^2+L^2-2rL\sin\beta\right)^{-3/2}\,L\cos\beta$$

$$-\left(r^2+L^2+2rL\sin\beta\right)^{-3/2}\,L\cos\beta\Big]\,\underline{b}_3 \qquad (19)$$

Substituting from Eq. (19) into Eq. (13) one then arrives at the desired expression for $\underline{F}$.

2.11 Force function for a body and a particle

The resultant gravitational force $\underline{F}$ exerted by a particle $\bar{P}$ of mass $\bar{m}$ on the particles of a (not necessarily rigid) body B (see Sec. 2.2) can be expressed as

$$F = \nabla_{\underline{R}} V \tag{1}$$

where $\underline{R}$ is the position vector of the mass center B^* of B relative to $\bar{P}$ and V is a force function given by

$$V = G\bar{m} \sum_{i=1}^{N} m_i p_i^{-1} + C \tag{2}$$

with

$$p_i \triangleq \left(\underline{p}_i^2\right)^{\frac{1}{2}} \qquad (i=1,\ldots,N) \tag{3}$$

and

$$\underline{p}_i \triangleq \underline{R} + \underline{r}_i \qquad (i=1,\ldots,N) \tag{4}$$

Here B is presumed to consist of particles $P_1,\ldots, P_N$ of masses $m_1,\ldots, m_N$, $\underline{p}_i$ is the position vector of P_i relative to $\bar{P}$, assumed nonzero, $\underline{r}_i$ is the position vector of P_i relative to B^*, C is an arbitrary constant, and G is the universal gravitational constant. If B is a continuous distribution of matter not including the point occupied by $\bar{P}$, then V is given by

$$V = G\bar{m} \int p^{-1} \rho d\tau + C \tag{5}$$

with

$$p \triangleq \left(\underline{p}^2\right)^{\frac{1}{2}} \tag{6}$$

and

$$\underline{p} = \underline{R} + \underline{r}$$

where ρ is the mass density of B at a generic point P of B, $\underline{p}$ is the position vector of P relative to $\bar{P}$, $\underline{r}$ is the position vector of P relative to

B^*, and $d\tau$ is the length, area, or volume of a differential element of the figure (curve, surface, or solid) occupied by B.

The force functions in Eqs. (2) and (5) both satisfy Laplace's equation:

$$\nabla^2 V = 0 \tag{7}$$

<u>Derivations</u>: Differentiation of V with respect to $\underline{R}$ gives

$$\nabla_{\underline{R}} V \underset{(2)}{=} -G\bar{m} \sum_{i=1}^{N} m_i p_i^{-2} \nabla_{\underline{R}}\, p_i \tag{8}$$

Now,

$$\nabla_{\underline{R}} p_i \underset{(3,4)}{=} \nabla_{\underline{R}} \left[(\underline{R}+\underline{r}_i)^2 \right]^{\frac{1}{2}} = \left[(\underline{R}+\underline{r}_i)^2 \right]^{-\frac{1}{2}} (\underline{R}+\underline{r}_i)\ \nabla_{\underline{R}} (\underline{R}+\underline{r}_i)$$

$$\underset{(3,4)}{=} p_i^{-1}\, \underline{p}_i \cdot \nabla_{\underline{R}}\, (\underline{R}+\underline{r}_i) = p_i^{-1} \underline{p}_i \cdot (\underline{U} + 0) = p_i^{-1} \underline{p}_i \tag{9}$$

Hence

$$\nabla_{\underline{R}} V \underset{(8,9)}{=} -G\bar{m} \sum_{i=1}^{N} m_i \underline{p}_i^{-3} \underline{p}_i \underset{(2.2.1)}{=} \underline{F}$$

A parallel derivation shows that Eq. (1) remains valid when Eq. (2) is replaced by Eq. (5).

Eq. (7) follows directly from Eq. (2.10.8), since the order of the operations of evaluating the gradient and integrating over B is interchangeable.

<u>Example</u>: A uniform, thin rod B of length 2L and mass m is subjected to the gravitational attraction of a particle $\bar{P}$ of mass $\bar{m}$. The value of a force function V associated with the resultant gravitational force exerted by $\bar{P}$ on B is to be determined in terms of the cartesian coordinates x,y of $\bar{P}$ for a coordinate system with origin at the mass center B^* of B, as shown in Fig. 2.11.1. The constant C in Eq. (3) is to be chosen such that V approaches zero when y=L and L/x approaches zero.

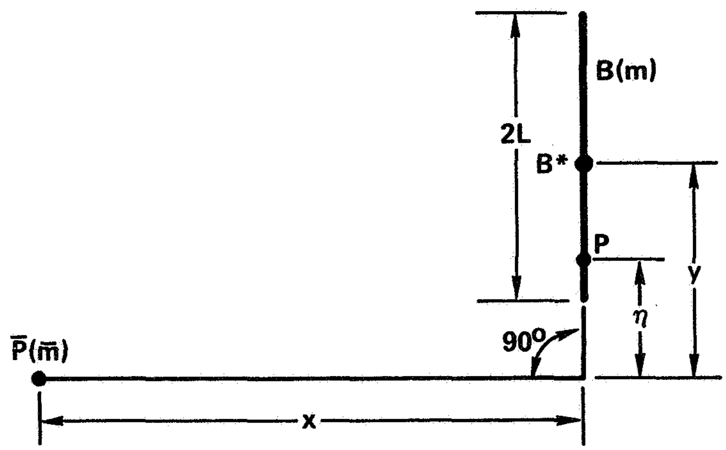


Figure 2.11.1

If the rod is regarded as matter distributed along a straight line segment, then the distance p from $\bar{\bar{P}}$ to a generic point P of the rod can be expressed as

$$p = (x^2 + \eta^2)^{\frac{1}{2}}$$

where η varies from y-L to y+L. The mass density ρ of B at P is equal to m/(2L), and a differential element of B has a length $d\eta$. Hence,

$$V \underset{(3)}{=} \frac{G\bar{m}m}{2L}\int_{y-L}^{y+L}(x^2+\eta^2)^{-\frac{1}{2}}\,d\eta + C$$

$$= \frac{G\bar{m}m}{2L}\ln\left[\frac{y+L+\sqrt{x^2+(y+L)^2}}{y-L+\sqrt{x^2+(y-L)^2}}\right] + C$$

and, when y=L,

$$\lim_{\frac{L}{x}\to 0} V = C$$

which equals zero by hypothesis.

2.12 Force function for a small body and a particle

When a body B is subjected to the gravitational attraction of a particle $\bar{P}$ removed so far from the mass center B^* of B that the largest distance from B^* to any point P of B is substantially smaller than the distance R between $\bar{P}$ and B^*, a useful form of the expressions for the force function V given in Eq. (2.11.2) and (2.11.5) can be found as follows: Replace p_i and p with $\left[(\underline{R} + \underline{r}_i)^2\right]^{\frac{1}{2}}$ and with $\left[(\underline{R} + \underline{r})^2\right]^{\frac{1}{2}}$, respectively, and then expand the integrand in ascending powers of $|\underline{r}_i|/R$ or $|\underline{r}|/R$ to obtain

$$V = \frac{G\bar{m}m}{R}\left(1 + \sum_{i=2}^{\infty} v^{(i)}\right) + C \tag{1}$$

where $v^{(i)}$ is a collection of terms of the i^{th} degree in $|\underline{r}_i|/R$ or $|\underline{r}|/R$, m and $\bar{m}$ are the masses of B and $\bar{P}$, C is an arbitrary constant, and G is the universal gravitational constant. In particular, $v^{(2)}$ is given by

$$v^{(2)} = \frac{1}{2mR^2}\left[\mathrm{tr}(\underline{\underline{I}}) - 3I_{11}\right] \tag{2}$$

where tr $(\underline{\underline{I}})$ is the trace of the inertia dyadic of B for B^*, and I_{11} is the moment of inertia of B about the line connecting $\bar{P}$ and B^*, so that (see Fig. 2.3.1)

$$I_{11} \triangleq \underline{a}_1 \cdot \underline{\underline{I}} \cdot \underline{a}_1 \tag{3}$$

Because I_{11} depends upon the orientation of $\underline{a}_1$ relative to B, Eq. (2) is sometimes less convenient than an alternative form involving central moments and products of inertia of B for an arbitrary vector basis b_1', b_2', b_3', here represented by

$$I_{ij}' \triangleq \underline{b}_i' \cdot \underline{\underline{I}} \cdot \underline{b}_j' \tag{4}$$

The required expression is

$$v^{(2)} = \frac{1}{2mR^2}\left[I'_{11}\left(1-3C'^2_{11}\right) + I'_{22}\left(1-3C'^2_{12}\right) + I'_{33}\left(1-3C'^2_{13}\right)\right.$$

$$\left. -6\left(I'_{12}C'_{11}C'_{12} + I'_{13}C'_{11}C'_{13} + I'_{23}C'_{12}C'_{13}\right)\right] \tag{5}$$

where

$$C'_{ij} \triangleq \underline{a}_1 \cdot \underline{b}'_j$$

When B is a rigid body and b'_1, b'_2, and b'_3 are fixed in B, the scalars I'_{jk} $(j,k=1,2,3)$ in Eq. (5) become constants, whereas I_{11} in Eq. (2) remains a variable.

For the special case in which the body-fixed unit vectors parallel principal axes of B for B^*, past conventions (see Sec. 2.3) suggest the notation

$$v^{(2)} = \frac{1}{2mR^2}\left[I_1\left(1-3C^2_{11}\right) + I_2\left(1-3C^2_{12}\right) + I_3\left(1-3C^2_{13}\right)\right] \tag{6}$$

where I_j and C_{ij} $(j=1,2,3)$ are defined respectively by Eqs. (2.3.8) and (2.3.9).

Eqs. (5) and (6) adopt particularly useful forms when the indicated direction cosines are written in terms of the spherical polar coordinates shown in Fig. 2.12.1. Eqs. (5) and (6) then become respectively,

$$v^{(2)} = \frac{1}{4mR^2}\left[\left(I'_{11} + I'_{22} - 2I'_{33}\right)\left(3\sin^2\beta-1\right)-3\left(I'_{11} - I'_{22}\right)\cos^2\beta\cos 2\lambda\right.$$

$$\left. -6\left(I'_{12}\cos^2\beta\sin 2\lambda + I'_{13}\sin 2\beta\cos\lambda + I'_{23}\sin 2\beta\sin\lambda\right)\right] \tag{7}$$

and

$$v^{(2)} = \frac{1}{4mR^2}\left[\left(I_1 + I_2 - 2I_3\right)\left(3\sin^2\beta -1\right)\right. \tag{8}$$

$$\left. -3\left(I_1-I_2\right)\cos^2\beta\cos 2\lambda\right]$$

It is often convenient to introduce some measure of the dimensions of B into $v^{(2)}$ in order to obtain an expression involving dimensionless

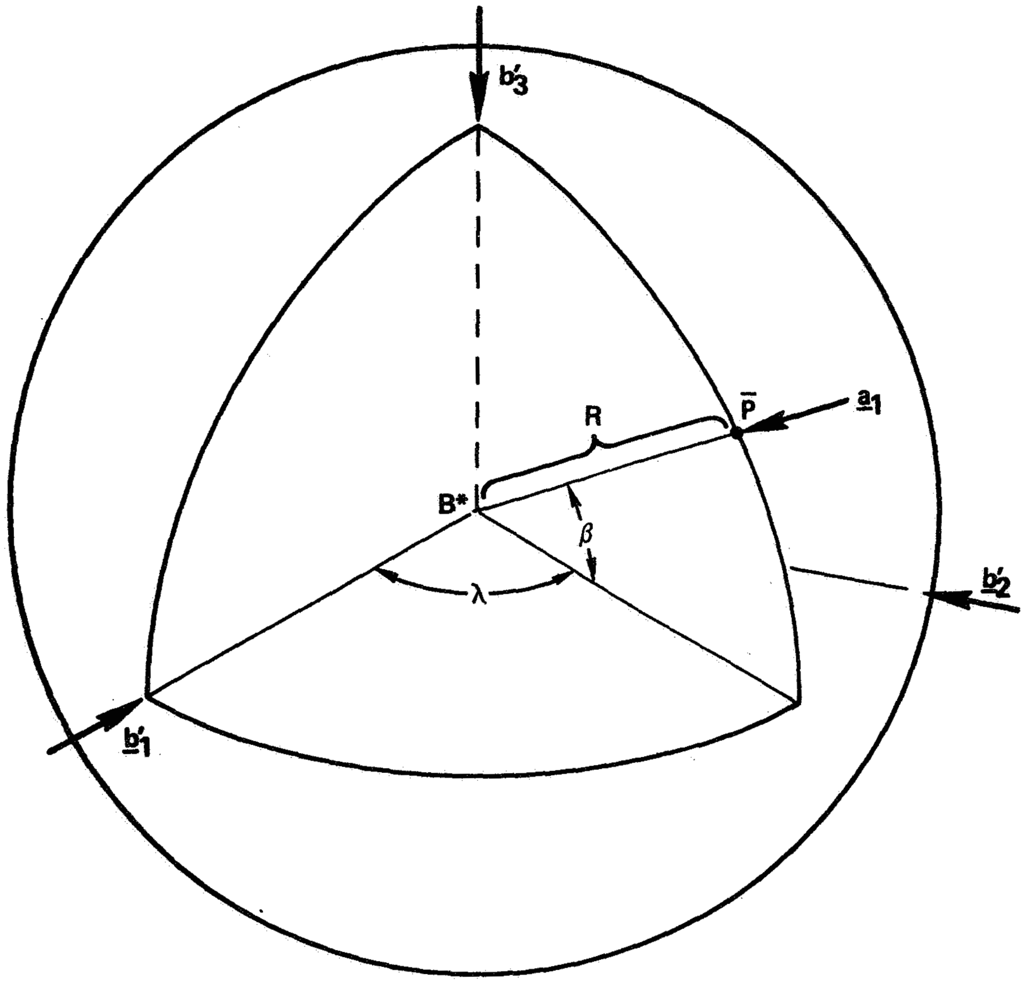

$\underline{b}_3'$
$\bar{P}$
$\underline{a}_1$
R
B*
β
λ
$\underline{b}_2'$
$\underline{b}_1'$

Figure 2.12.1

constants and dimensionless variables. If the symbol R_B is assigned to the selected normalizing dimension, and the symbols

$$J \triangleq \frac{-1}{2mR_B^2} (I_1 + I_2 - 2I_3) \tag{9}$$

and

$$\varepsilon \triangleq -\frac{3}{2mR_B^2} (I_1 - I_2) \tag{10}$$

are introduced, then Eq. (8) becomes

$$v^{(2)} = -\frac{J}{2}\left(\frac{R_B}{R}\right)^2 \left(3\sin^2\beta - 1\right) + \frac{\varepsilon}{2}\left(\frac{R_B}{R}\right)^2 \cos^2\beta\cos 2\lambda \tag{11}$$

If $\tilde{\underline{F}}$ is the approximation to $\underline{F}$ defined in Eq. (2.3.6), then $\tilde{\underline{F}}$ can be expressed as

$$\tilde{\underline{F}} = \underline{\nabla}_{\underline{R}}\tilde{V} \tag{12}$$

where $\tilde{V}$ is given by

$$\tilde{V} = \frac{G\bar{m}m}{R}\left(1 + v^{(2)}\right) + C \tag{13}$$

and $\underline{R}$ is the position vector of B^* relative to $\bar{P}$.

In other words, V is a force function suitable for dealing with the gravitational interaction of a body and a remote particle when one uses $\tilde{\underline{F}}$ to approximate $\underline{F}$.

As previously, the force function V satisfies Laplace's equation; moreover, $\tilde{V}$ also satisfies Laplace's equation.

Derivations: Replacing p in Eq. (2.11.5) with $\left[(R + r)^2\right]^{\frac{1}{2}}$ provides

$$V = G\bar{m}\int \left(\underline{R}^2 + 2\underline{R}\cdot\underline{r} + \underline{r}^2\right)^{-\frac{1}{2}} \rho\, d\tau + C \tag{14}$$

Introducing $\underline{q}$ as $\underline{q} \triangleq \underline{r}/R$, noting that $\underline{R}/R = \underline{a}_1$, and applying the binomial series, convergent for $|2\underline{R}\cdot\underline{r} + \underline{r}^2| < 1$, find

$$V = \frac{G\bar{m}}{R}\int\left(1+2\underline{a}_1\cdot\underline{q}+\underline{q}^2\right)^{-\frac{1}{2}} \rho d\tau + C$$

$$= \frac{G\bar{m}}{R}\int\left[1-\underline{a}_1\cdot\underline{q}-\frac{1}{2}\underline{q}^2+\frac{3}{2}\left(\underline{a}_1\cdot\underline{q}\right)^2+\ldots\right]\rho\, d\tau + C$$

$$= \frac{G\bar{m}}{R}\left\{\int\rho\, d\tau - \frac{\underline{a}_1}{R}\cdot\int\underline{r}\,\rho\, d\tau - \frac{1}{2R^2}\int\left[\underline{r}^2-3(\underline{a}_1\cdot\underline{r})^2\right]\rho d\tau + \ldots\right\} + C \tag{15}$$

Now,

$$\int\rho\, d\tau = m \tag{16}$$

$$\int\underline{r}\,\rho\, d\tau = 0 \tag{17}$$

and

$$\int\left[\underline{r}^2-3\left(\underline{a}_1\cdot\underline{r}\right)^2\right]\rho\, d\tau = 3\int\left[\underline{r}^2-\left(\underline{a}_1\cdot\underline{r}\right)^2\right]\rho\, d\tau - 2\int\underline{r}^2\rho\, d\tau \tag{18}$$

Furthermore,

$$\int\left[\underline{r}^2-\left(\underline{a}_1\cdot\underline{r}\right)^2\right]\rho\, d\tau = I_{11} \tag{19}$$

and

$$2\int\underline{r}^2\,\rho\, d\tau = tr(\underline{I}) \tag{20}$$

Hence,

$$\int\left[\underline{r}^2-3\left(\underline{a}_1\cdot\underline{r}\right)^2\right]\rho\, d\tau \underset{(18,19,20)}{=} 3I_{11}-tr(\underline{I}) \tag{21}$$

and

$$V \underset{(15,16,17,21)}{=} \frac{G\bar{m}m}{R}\left\{1+\frac{1}{2mR^2}\left[tr(\underline{I})-3I_{11}\right]+\ldots\right\}+C$$

in agreement with Eqs. (1) and (2).

Eq. (5) follows directly from Eq. (2) with the substitution of Eq. (3) for I_{11}, the (summation convention) expansion

$$\underline{I} \underset{(4)}{=} I'_{jk}\underline{b}'_j\underline{b}'_k \tag{22}$$

and the definition of C'_{ij}. With these relationships, Eq. (2) becomes

$$V^{(2)}_{(2,3,4)} = \frac{1}{2mR^2}\left(I'_{ii} - 3\underline{a}_1 \cdot \underline{b}'_i I'_{ij}\underline{b}'_j \cdot \underline{a}_1\right)$$

$$= \frac{1}{2mR^2}\left(I'_{ii} - 3\, C'_{1i}\, I'_{ij}\, C'_{1j}\right)$$

$$= \frac{1}{2mR^2}\Big[I_{11}\left(1-3C'^2_{11}\right) + I'_{22}\left(1-3C'^2_{12}\right) + I'_{33}\left(1-3C'^2_{13}\right)$$

$$-3\left(I'_{12}C'_{11}C'_{12} + I'_{13}C'_{11}C'_{13} + I'_{21}C'_{12}C'_{11} + I'_{23}C'_{12}C'_{13}\right.$$

$$\left.\left. + I'_{31}C'_{13}C'_{11} + I'_{32}C'_{13}C'_{12}\right)\right] \tag{23}$$

With the symmetry relations $I'_{ij} = I'_{ji}$, Eq. (23) confirms Eq. (5). Eq. (6) is merely the special case of Eq. (5) with vanishing products of inertia and new notation, and Eq. (7) and (8) are the special cases of Eqs. (5) and (6) in which the relationship

$$\underline{a}_1 = \cos\beta\cos\lambda\underline{b}'_1 + \cos\beta\sin\lambda\underline{b}'_2 + \sin\beta\underline{b}'_3 \tag{24}$$

from Fig. 2.12.1 has been used to obtain for the required direction cosines the expressions

$$C_{11} = \cos\beta\cos\lambda \quad ; \quad C'_{12} = \cos\beta\sin\lambda \; ; \quad C'_{13} = \sin\beta \tag{25}$$

The combination of Eqs. (5) and (25) produces

$$V^{(2)} = \frac{1}{2mR^2}\Big[I'_{11}\left(1-3\cos^2\beta\cos^2\lambda\right) + I'_{22}\left(1-3\cos^2\beta\sin^2\lambda\right)$$

$$+ I'_{33}\left(1-3\sin^2\beta\right) -6\left(I'_{12}\cos^2\beta\cos\lambda\sin\lambda + I'_{13}\cos\beta\cos\lambda\sin\beta\right.$$

$$\left.\left. + I'_{23}\sin\beta\cos\beta\sin\lambda\right)\right]$$

$$= \frac{1}{2mR^2}\Big\{I'_{11}\left[1-3\cos^2\beta\ (\tfrac{1}{2} + \tfrac{1}{2}\cos2\lambda)\right] + I'_{22}\left[1-3\cos^2\beta\ (\tfrac{1}{2} - \tfrac{1}{2}\cos2\lambda)\right]$$

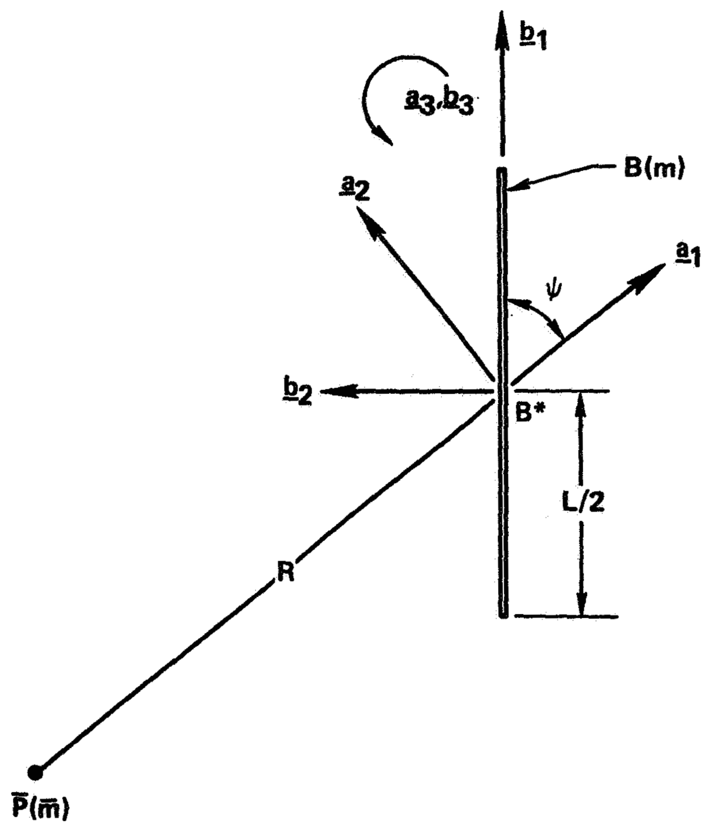


Figure 2.12.2

$$\mathrm{tr}(\underline{I}) = \frac{mL^2}{6}\ , \quad I_{11} = \underline{a}_1 \cdot \underline{I} \cdot \underline{a}_1 = \frac{mL^2}{12}\sin^2\psi$$

after which substitution into Eqs. (2) and (13) gives

$$\tilde{V} = \frac{G\bar{m}m}{R}\left[1 - \frac{L^2}{24R^2}\left(2-3\sin^2\psi\right)\right] + C \tag{31}$$

In order to obtain this result by means of Eq. (11) rather than Eq. (2), one must recognize by comparison of Figs. 2.12.1 and 2.12.2 that in this example $\beta = 0$ and $\lambda = -\psi$. Substituting this result together with the mass properties

$$I_1 = 0 \quad ; \quad I_2 = I_3 = \frac{mL^2}{12} \tag{32}$$

into Eq. (11) provides

$$v^{(2)} \underset{(11)}{=} -\frac{J}{2}\left(\frac{R_B}{R}\right)^2\left(-1\right) + \frac{\varepsilon}{2}\left(\frac{R_B}{R}\right)^2 (1)\cos 2\psi \tag{33}$$

where

$$J \underset{(9)}{=} -\frac{1}{2mR_B^2}\left(0 + \frac{mL^2}{12} - \frac{2mL^2}{12}\right) = \frac{L^2}{24R_B^2} \tag{34}$$

and

$$\varepsilon \underset{(10)}{=} -\frac{3}{2mR_B^2}\left(0 - \frac{mL^2}{12}\right) = \frac{L^2}{8R_B^2} \tag{35}$$

Although it is customary to make some specific choice for R_B (such as $R_B = L/2$), in order to obtain numerical values for J and ε, it is apparent that in combining Eqs. (33 - 35) one can cancel R_B from $v^{(2)}$, so the choice of R_B is irrelevant to $v^{(2)}$. The result then becomes

$$v^{(2)} \underset{(33-35)}{=} \frac{L^2}{R^2}\left[\frac{1}{48} + \frac{1}{16}\cos 2\psi\right]$$

$$= \frac{L^2}{48R^2}\left[1 + 3\left(1-2\sin^2\psi\right)\right] = \frac{L^2}{48}\left(4-6\sin^2\psi\right) \tag{36}$$

Substitution of Eq. (36) into Eq. (13) confirms Eq. (31), as assured by the equivalence of Eqs. (2) and (11). For the orientation shown in Fig. 2.12.2 the calculation from Eq. (2) is more direct, but Eq. (11) offers the advantage of explicit separation of mass properties and general kinematic variables, all in terms of dimensionless quantities.

To verify that Eq. (12) is indeed satisfied, one may proceed as follows:

$$\underline{\nabla}_{\underline{R}}\tilde{V} \underset{(31)}{=} -\frac{G\bar{m}m}{R^2}\,\underline{\nabla}_{\underline{R}}R\left[1+\frac{L^2}{24R^2}\left(2-3\sin^2\psi\right)\right]$$

$$+\frac{G\bar{m}m}{R}\left[-\frac{L^2}{12R^3}\left(2-3\sin^2\psi\right)\underline{\nabla}_{\underline{R}}R-\frac{L^2}{8R^2}\sin 2\psi\,\underline{\nabla}_{\underline{R}}\psi\right] \tag{37}$$

Now,

$$\underline{R} = R\left(\cos\psi\underline{b}_1 - \sin\psi\underline{b}_2\right)$$

Hence

$$\underline{\nabla}_{\underline{R}}\underline{R} = \underline{\nabla}_{\underline{R}}R\left(\cos\psi\underline{b}_1 - \sin\psi\underline{b}_2\right)$$

$$+ R\left(-\sin\psi\,\underline{\nabla}_{\underline{R}}\psi\,b_1 - \cos\psi\,\underline{\nabla}_{\underline{R}}\psi\,\underline{b}_2\right)$$

or, in view of Eqs. (2.10.6) and (2.10.7).

$$\underline{U} = \underline{a}_1\left(\cos\psi\underline{b}_1 - \sin\psi\underline{b}_2\right) - R\,\underline{\nabla}_{\underline{R}}\psi\left(\sin\psi\underline{b}_1 + \cos\psi\underline{b}_2\right)$$

$$= \underline{a}_1\underline{a}_1 - R\,\underline{\nabla}_{\underline{R}}\psi\,\underline{a}_2$$

Dot-multiplication with $\underline{a}_2$ thus gives

$$\underline{a}_2 = -R\,\underline{\nabla}_{\underline{R}}\psi$$

and using $\underline{\nabla}_{\underline{R}}R = a_1$ and $\underline{\nabla}_{\underline{R}}\psi = -\underline{a}_2/R$, one obtains

$$\underline{\nabla}_{\underline{R}}\tilde{V} \underset{(37)}{=} -\frac{G\bar{m}m}{R^2}\left[\underline{a}_1 + \frac{L^2}{8R^2}\left(2-3\sin^2\psi\right)\underline{a}_1 - \frac{L^2}{8R^2}\sin^2\psi\,\underline{a}_2\right] = \underline{\tilde{F}} \tag{38}$$

2.13 Force Function in terms of spherical harmonics

When a body B of mass m is subjected to the gravitational attraction of a particle $\bar{P}$ of mass $\bar{m}$ removed so far from the mass center B^* of B that the largest distance from B^* to any point P of B is substantially smaller than the distance R between $\bar{P}$ and B^*, the force function V given in Eqs. (2.11.2) and (2.11.5) can as an alternative to Eq. (2.12.1) be represented in series form as

$$V = \frac{G\bar{m}m}{R}\left[1 + \sum_{i=2}^{\infty}\left(\frac{R_B}{R}\right)^i C_i\right] + C \tag{1}$$

where G is the universal gravitational constant, C is an arbitrary constant, R_B is an arbitrary distance such that $R_B < R$, and

$$C_i \triangleq \frac{1}{m}\int\left(\frac{r}{R_B}\right)^i P_i\left(\cos\alpha\right)\rho d\tau \tag{2}$$

with r representing the magnitude of the vector from B^* to a generic point P, ρ representing the mass density function of B, and with $P_i(\cos\alpha)$ representing the Legendre polynomial with argument the cosine of the angle α defined by Fig. 2.13.1. In this figure, R,λ,β are the spherical coordinates of $\bar{P}$, r,μ,γ are the sperical coordinates of P, and α is the angle subtended by the two lines joining P and $\bar{P}$ to B^*. The series in Eq. (1) converges for $(r/R)<1$.

By definition, the Legendre polynomials are given by $P_0(y)\triangleq 1$ and

$$P_i(y) \triangleq \frac{1\cdot3\cdot\ \ldots\ (2i-1)}{i!}\left[y^i - \frac{i(i-1)}{(2i-1)\cdot2}\,y^{i-2} + \frac{i(i-1)(i-2)(i-3)}{(2i-1)(2i-3)\cdot2\cdot4}\,y^{i-4} + \ldots\right] \qquad (i=1,\ldots,\infty) \tag{3}$$

with the bracketed series terminating with the y^0 term if i is even and with the y^1 term if i is odd. Thus

$$P_0(y) = 1 \tag{4a}$$

$$P_1(y) = y \tag{4b}$$

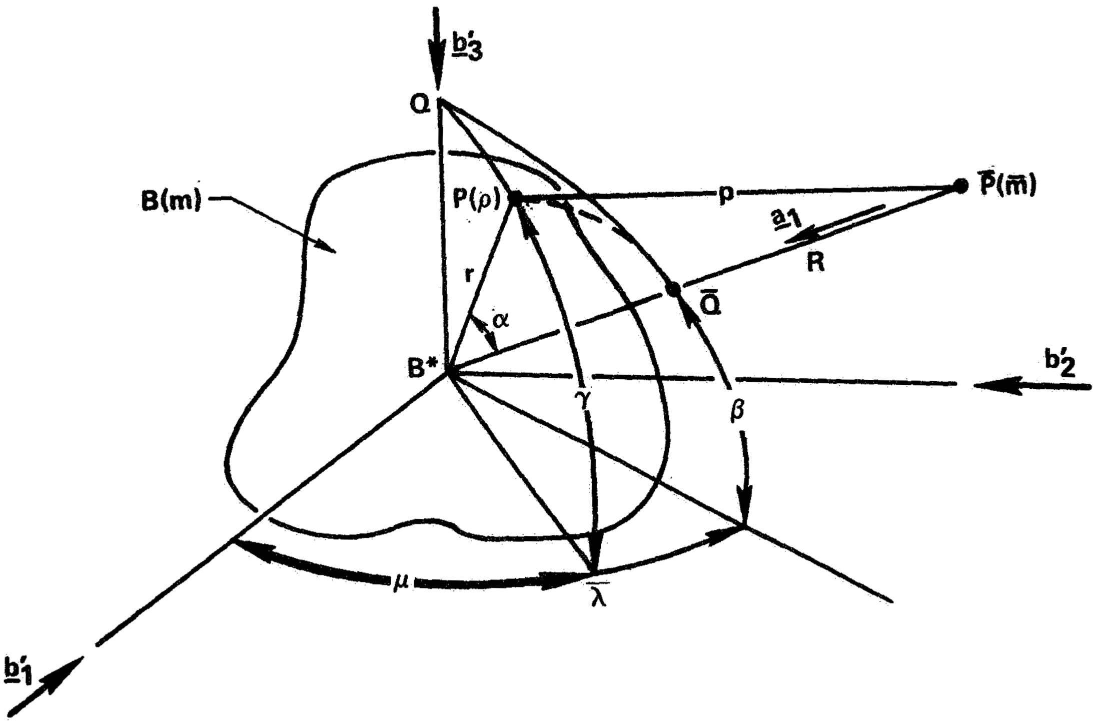


Figure 2.13.1

$$P_2(y) = \tfrac{1}{2}\,(3y^2-1) \tag{4c}$$

$$P_3(y) = \tfrac{1}{2}\,(5y^3-3y) \tag{4d}$$

Eq. (2) for C_i $(i=2,\ldots,\infty)$ can be written in such a way as to separate the properties of B from the location of $\bar{P}$ relative to B by introducing the associated Legendre function of the first kind, $P_i^j(y)$, defined by

$$P_i^j(y) \triangleq \frac{(2i)!\,(1-y^2)^{j/2}}{2^i i!\,(i-j)!}\left[y^{i-j} - \frac{(i-j)(i-j-1)}{(2i-1)\,2}\,y^{i-j-2} + \frac{(i-j)(i-j-1)(i-j-2)(i-j-3)}{(2i-1)(2i-3)\cdot 2\cdot 4}\,y^{i-j-4} + \ldots\right] \quad (i,j=0,1,\ldots,\infty) \tag{5}$$

with the bracketed series terminating with the y^0 term if $i-j$ is even, and with the y^1 term if $i-j$ is odd. For specific indices of particular interest,

$$P_i^0(y) = P_i(y) \tag{6a}$$

and

$$P_1^1(y) = (1-y^2)^{\frac{1}{2}} \;;\; P_2^1(y) = 3y(1-y^2)^{\frac{1}{2}} \;;\; P_2^2(y) = 3(1-y^2) \tag{6b}$$

In terms of associated Legendre functions,

$$C_i = \sum_{j=0}^{i} P_i^j\,(\sin\beta)(C_{ij}\,\cos j\lambda + S_{ij}\sin j\lambda) \tag{7}$$

where

$$C_{i0} \triangleq \frac{1}{m}\int\left(\frac{r}{R_B}\right)^i P_i(\sin\gamma)\;\rho d\tau \qquad (i=2,\ldots,\infty) \tag{8}$$

$$C_{ij} \triangleq \frac{2}{m}\,\frac{(i-j)!}{(i+j)!}\int\left(\frac{r}{R_B}\right)^i P_i^j\;(\sin\gamma)\;\cos j\mu\;\rho d\tau \quad \begin{pmatrix} i=2,\ldots,\infty \\ j=1,\ldots,\infty \end{pmatrix} \tag{9}$$

and

$$S_{ij} \triangleq \frac{2}{m}\,\frac{(i-j)!}{(i+j)!}\int\left(\frac{r}{R_B}\right)^i P_i^j\;(\sin\beta)\;\sin j\mu\;\rho d\tau \begin{pmatrix} i=2,\ldots,\infty \\ j=1,\ldots,\infty \end{pmatrix} \tag{10}$$

so that the constants S_{ij} and C_{ij} (for all indices) establish the properties of B without reference to the location of $\bar{P}$.

When Eqs. (1) and (7) are written in combination as

$$V = \frac{G\bar{m}m}{R}\left[1 + \sum_{i=2}^{\infty}\sum_{j=0}^{i}\left(\frac{R_B}{R}\right)^i P_i^j\,(\sin\beta)(C_{ij}\,\cos j\lambda + S_{ij}\,\sin j\lambda)\right] + C \tag{11}$$

the force function V is said to be in <u>spherical harmonic</u> form. Terms such as P_i^j $(\sin\beta)$ $\cos j\lambda$ and P_i^j $(\sin\beta)$ $\sin j\lambda$ are classically called <u>tesseral</u> harmonics of i^{th} degree and j^{th} order; tesseral harmonics of order zero are called <u>zonal harmonics</u>. A linear combination of tesseral harmonics of i^{th} degree is called a <u>surface spherical harmonic</u> of i^{th} degree, and the product of $R^{-(i+1)}$ with an i^{th} degree surface spherical harmonic is called a <u>solid spherical harmonic</u> of i^{th} degree. Thus Eq. (11) is a representation of V in terms of solid spherical harmonics. The International Astronomical Union has established Eq. (11) as a standard representation of the force function of the earth, choosing the mean equatorial radius of the earth for R_B. This series converges only outside of a sphere containing all of the mass of B, since convergence of the series in Eq. (11) requires $r < R$ for all r; due to the oblateness of the earth, the series does not converge for points in a region of space above the surface of the earth ranging in thickness from zero at the equator to approximately 20 km above the poles.

In comparing the alternative series expansions for V represented by Eqs. (2.12.1), (1), and (11), it is helpful to recognize that

$$C_2 = \frac{1}{2mR_B^2}\left[\text{tr}(\underline{I}) - 3I_{11}\right] \tag{12}$$

where $\underline{I}$ is the inertia dyadic of B for B^*, and I_{11} is the moment of inertia of B about the line joining B^* and $\bar{P}$. The equivalent contribution to Eq. (11) is given by

$$C_2 = \sum_{j=0}^{2} P_2^j\,(\sin\beta)(C_{2j}\cos j\lambda + S_{2j}\sin j\lambda)$$

$$= \tfrac{1}{2}\,(3\sin^2\beta - 1)\,C_{20} + \frac{3}{2}\sin 2\beta\,(C_{21}\cos\lambda + S_{21}\sin\lambda)$$

$$+ 3\cos^2\beta\,(C_{22}\cos 2\lambda + S_{22}\sin 2\lambda) \qquad (13)$$

Physical interpretation can be provided for the constants C_{2j} and S_{2j} for $j = 0,1,2$ by aligning the unit vectors $\underline{b}_1'$, $\underline{b}_2'$, and $\underline{b}_3'$ in Fig. 2.13.1 with principal axes of B for B^*; then

$$C_{20} = \frac{1}{2mR_B^2}\,(I_1 + I_2 - 2I_3) \qquad (14a)$$

$$C_{21} = S_{21} = S_{22} = 0 \qquad (14b)$$

and

$$C_{22} = -\frac{1}{4mR_B^2}\,(I_1 - I_2) \qquad (14c)$$

where I_1, I_2, and I_3 are the principal moments of inertia of B for B^*.

More generally, when $\underline{b}_1'$, $\underline{b}_2'$ and $\underline{b}_3'$ are arbitrary orthogonal unit vectors, then

$$C_{20} = (I_{11}' + I_{22}' - 2I_{33}')\Big/\left(2mR_B^2\right) \qquad (15a)$$

$$C_{21} = -I_{13}'\Big/\left(mR_B^2\right) \quad ; \quad C_{22} = -(I_{11}' - I_{22}')\Big/\left(4mR_B^2\right) \qquad (15b)$$

$$S_{21} = -I_{23}'\Big/\left(mR_B^2\right) \quad ; \quad S_{22} = -I_{12}'\Big/\left(2mR_B^2\right) \qquad (15c)$$

For an axisymmetric rigid body B with its symmetry axis aligned with $\underline{b}_3'$ in Fig. 2.13.1, the constants S_{ij} $(i,j=0,\ldots,\infty)$ and C_{ij} $(i=2,\ldots,\infty;\ j=1,\ldots,\infty)$ in Eq. (11) are zero, and the remaining constants C_{i0} are traditionally relabeled such that

$$C_{i0} \triangleq -J_i \qquad (16)$$

Then V takes the form

$$V = \frac{G\bar{m}m}{R}\left[1 - \sum_{i=2}^{\infty} J_i \left(\frac{R_B}{R}\right)^i P_i(\sin\beta)\right] \tag{17}$$

in which only zonal harmonics appear.

In the notation of Sec. 2.12 (see Eq. 2.12.13), V may be approximated in some cases by

$$\tilde{V} \triangleq \frac{G\bar{m}m}{R}\left[1 + \left(\frac{R_B}{R}\right)^2 C_2\right] + C \tag{18}$$

Laplace's equation is satisfied by both V and $\tilde{V}$, so that

$$\nabla^2 V = 0 \tag{19}$$

and

$$\nabla^2 \tilde{V} = 0 \tag{20}$$

Moreover, Laplace's equation is satisfied by

$$V_i^j \triangleq \left(\frac{1}{R}\right)^{i+1} P_i^j\ (\sin\beta)\left[A_j \cos j\lambda + B_j\ \sin j\lambda\right] \tag{21}$$

for any constants A_j and B_j, so that any truncation of the series expression for V in Eq. (11) consisting of linear combinations of terms given by Eq. (19) is itself an approximation of V which satisfies Laplace's equation.

Derivations: Application of the law of cosines to Fig. 2.13.1 provides

$$p^2 = R^2 - 2rR\cos\alpha + r^2 \tag{22}$$

permitting the representation

$$V \underset{(2.11.5)}{=} G\bar{m} \int p^{-1}\ \rho d\tau + C$$

$$\underset{(22)}{=} \frac{G\bar{m}}{R} \int \left[1 - 2\left(\frac{r}{R}\right) \cos\alpha + \left(\frac{r}{R}\right)^2\right]^{-\frac{1}{2}} \rho d\tau$$

The identity

$$(1-2xy+y^2)^{-\frac{1}{2}} = \sum_{i=0}^{\infty} y^i P_i(x)$$

permits the force function to be expressed in terms of Legendre polynomials (see Eq. (3)) in the form

$$V = \frac{G\bar{m}}{R} \sum_{i=0}^{\infty} \int \left(\frac{r}{R}\right)^i P_i(\cos\alpha)\ \rho d\tau + C \tag{23}$$

whenever*

$$\left|\frac{r}{R}\right| < 1$$

for all r, so that the distance r from B^* to P is substantially less than R for all P in B. The first two terms in Eq. (23) simplify, since

$$P_0\ (\cos\alpha) \underset{(4a)}{=} 1 \tag{24}$$

and

$$P_1(\cos\alpha) = \cos\alpha \tag{25}$$

so that

$$\int \left(\frac{r}{R}\right)^0 P_0(\cos\alpha)\rho d\tau = \int \rho d\tau = m \tag{26}$$

and by virtue of the definition of mass center B^*

$$\int \left(\frac{r}{R}\right)^i P_1(\cos\alpha)\ \rho d\tau = \frac{1}{R} \int r\cos\alpha\rho d\tau = 0 \tag{27}$$

Hence

$$V \underset{(23,26,27)}{=} \frac{G\bar{m}m}{R} \left[1 + \frac{1}{m} \sum_{i=2}^{\infty} \int \left(\frac{r}{R}\right)^i P_i(\cos\alpha)\rho d\tau\right] + C \tag{28}$$

and, by introducing the arbitrary R_B with the identity

*See O.D. Kellogg, Foundations of Potential Theory, Ungar Publishing Co., New York, 1929, p. 128.

$$\left(\frac{r}{R}\right)^i = \left(\frac{R_B}{R}\right)^i \left(\frac{r}{R_B}\right)^i$$

substituted into Eq. (28), one finds Eqs. (1) and (2) confirmed.

Eq. (3) is a definition, which leads directly to Eqs. (4) in special cases. Similarly the definition in Eq. (5) leads directly to Eqs. (6).

Proof of Eq. (7) requires the trigonometric identity

$$\cos\alpha = \sin\beta\sin\gamma + \cos\beta\cos\gamma\cos(\lambda-\mu) \tag{29}$$

which follows from the law of cosines applied to the spherical triangle in Fig. 2.13.1 defined by points B^*, P, Q, and $\bar{Q}$. With Eq. (29) and the lemma*

$$P_i(\cos\alpha) = P_i(\sin\beta)\, P_i(\sin\gamma) + 2\sum_{j=1}^{i} \frac{(i-j)!}{(i+j)!} P_i^j(\sin\beta)\, P_i^j(\sin\gamma)\cos j(\lambda-\mu) \tag{30}$$

Eq. (2) becomes

$$\begin{aligned} C_i \underset{(2,30)}{=} & \frac{P_i(\sin\beta)}{m} \int \left(\frac{r}{R_B}\right)^i P_i(\sin\gamma)\, \rho d\tau \\ & + \frac{2}{m}\sum_{j=1}^{i} \frac{(i-j)!}{(i+j)!} P_i^j(\sin\beta) \left[\cos j\lambda \int \left(\frac{r}{R_B}\right)^i P_i^j(\sin\gamma)\cos j\mu\rho d\tau \right. \\ & \left. + \sin j\lambda \int \left(\frac{r}{R_B}\right)^i P_i^j(\sin\gamma)\sin j\mu\rho d\tau \right] \end{aligned} \tag{31}$$

confirming Eqs. (7) - (10).

Eq. (11) is the direct combination of Eqs. (1) and (7).

Eq. (12) follows from Eq. (2), since

*W.E. Byerly, An Elementary Treatise on Fourier Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, Ginn and Co., Boston, 1893, p. 211.

$$C_2 \underset{(2)}{=} \frac{1}{m}\int\left(\frac{r}{R_B}\right)^2 P_2(\cos\alpha)\rho d\tau$$

$$\underset{(4c)}{=} \frac{1}{2mR_B^2}\int r^2(3\cos^2\alpha-1)\rho d\tau$$

$$= \frac{1}{2mR_B^2}\left[\int 2r^2\rho d\tau - 3\int r^2\sin^2\alpha\rho d\tau\right]$$

$$= \frac{1}{2mR_B^2}\left[\mathrm{tr}(\underline{I}) - 3I_{11}\right]$$

where $\mathrm{tr}(\underline{I})$ is the trace of the inertia dyadic of B for B^* and I_{11} is the moment of inertia of B about the line joining B^* and $\bar{\bar{P}}$.

Eq. (13) is a special case of Eq. (2), with Eq. (4c) and a pair of trigonometric identities substituted.

The physical interpretations in Eqs. (14) and (15) are confirmed by comparing Eq. (13) with Eqs. (2.12.8) and (2.12.7), noting that

$$v^{(i)} \underset{(2.12.1,1)}{=} \left(\frac{R_B}{R}\right)^i C_i \qquad i=2,\ldots\infty \tag{32}$$

and equating coefficients of independent trigonometric functions, such as $(3\sin^2\beta-1)$, $\cos^2\beta\cos 2\lambda$, $\cos^2\beta\sin 2\lambda$, $\sin 2\beta\cos\lambda$, and $\sin 2\beta\sin\lambda$.

Eq. (17) is a special case of Eq. (11) in which the definition in Eq. (16) has been substituted and due to symmetry all dependence on λ has been eliminated.

Eq. (18) follows from the substitution of $v^{(2)}$ from Eq. (32) into Eq. (2.12.13).

Eq. (19) is identical to Eq. (2.11.7), since V as expressed in Sec. 2.13 is equivalent to V as defined in Sec. 2.11.

The statement in Eq. (20) that $\tilde{V}$ also satisfies Laplace's equation was asserted but not proven in Sec. 2.12. Comparison of Eqs. (18) and (13) reveals that

$$\tilde{V} = \frac{G\bar{m}m}{R}\left[1 + \left(\frac{R_B}{R}\right)^2 \sum_{j=0}^{2} P_2^j(\sin\beta)(C_{2j}\cos j\lambda + S_{2j}\sin j\lambda)\right] \tag{33}$$

so that $\tilde{V}$ may be written as a linear combination of terms such as V_i^j defined by Eq. (21). Thus by superposition Eq. (20) is valid if V_i^j satisfies Laplace's equation.

In order to prove that V_i^j is a particular solution of Laplace's equation when written in terms of spherical coordinates, note that $\nabla^2 V = 0$ becomes, from Eq. (2.9.35),

$$\frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \sec^2\beta\frac{\partial^2 V}{\partial\lambda^2} + \sec\beta\frac{\partial}{\partial\beta}\left(\cos\beta\frac{\partial V}{\partial\beta}\right)\right] = 0 \tag{34}$$

To obtain a general solution by separation of variables, adopt the assumed solution

$$V = X(R)Y(\beta)Z(\lambda) \tag{35}$$

and multiply by $R^2\cos\beta/(XYZ)$ to obtain

$$\frac{\cos^2\beta}{X}\frac{d}{dR}\left(R^2\frac{dX}{dR}\right) + \frac{\cos\beta}{Y}\frac{d}{d\beta}\left(\cos\beta\frac{dY}{d\beta}\right) = -\frac{1}{Z}\frac{d^2Z}{d\lambda^2} \tag{36}$$

Since the left side of Eq. (36) does not depend on λ, neither does the right side, so that both sides equal the same constant, here called j^2. Hence

$$\frac{d^2Z}{d\lambda^2} + j^2 Z = 0 \tag{37}$$

and

$$\frac{\cos^2\beta}{X}\frac{d}{dR}\left(R^2\frac{dX}{dR}\right) + \frac{\cos\beta}{Y}\frac{d}{d\beta}\left(\cos\beta\frac{dY}{d\beta}\right) = j^2 \tag{38}$$

Eq. (37) has the solution

$$Z \underset{(37)}{=} A_1\cos j\lambda + B_1\sin j\lambda \tag{39}$$

for arbitrary constants A_1 and B_1.

Eq. (38) can be divided by $\cos^2\beta$ and rewritten as

$$\frac{1}{X}\frac{d}{dR}\left(R^2\frac{dX}{dR}\right) \underset{(38)}{=} \frac{j^2}{\cos^2\beta} - \frac{1}{Y\cos\beta}\frac{1}{d\beta}\left(\cos\beta\frac{dY}{d\beta}\right) \tag{40}$$

The left side of Eq. (40) depends only on R, and the right side only on β; hence both must be a constant, here designated $i(i+1)$. Thus

$$\frac{d}{dR}\left(R^2\frac{dX}{dR}\right) \underset{(40)}{=} i(i+1)X \tag{41}$$

and

$$\frac{1}{\cos\beta}\frac{d}{d\beta}\left(\cos\beta\frac{dY}{d\beta}\right) + \left[i(i+1) - \frac{j^2}{\cos^2\beta}\right]Y = 0 \tag{42}$$

Eq. (41) has the solution

$$X \underset{(41)}{=} A_2R^i + B_2R^{-(i+1)} \tag{43}$$

for arbitrary constants A_2 and B_2, as may be confirmed by substitution.

Eq. (42) adopts a more familiar form when written in terms of

$$y \triangleq \sin\beta \tag{44}$$

so that

$$\frac{d}{d\beta}(\ ) = \frac{d}{dy}(\ )\frac{dy}{d\beta} = \cos\beta\frac{d}{dy}(\ )$$

The resulting equation is

$$\frac{d}{dy}\left[\left(1-y^2\right)\frac{dY}{dy}\right] + \left[i(i+1) - \frac{j^2}{(1-y^2)}\right]Y \underset{(42,44)}{=} 0 \tag{45}$$

or

$$\left(1-y^2\right)\frac{d^2Y}{dy^2} - 2y\frac{dY}{dy} + \left[i(i+1) - \frac{j^2}{(1-y^2)}\right]Y \underset{(45)}{=} 0 \tag{46}$$

Eqs. (42), (45), and (46) are known as <u>associated Legendre equations</u> of degree i and order j. If the symbol $P_i^j(\sin\beta)$ is defined to represent a particular solution to Eq. (42), then Eq. (34) has a particular solution

$$\bar{V}_i^j \underset{(35,39,43)}{=} P_i^j\,(\sin\beta)\left[A_1\cos j\lambda + B_1\sin j\lambda\right]\left[A_2R^i + B_2R^{-(i+1)}\right]$$

which implies a second particular solution

$$V_i^j = \left(\frac{1}{R}\right)^{i+1} P_i^j(\sin\beta)\left[A_j\cos j\lambda + B_j\sin j\lambda\right] \tag{47}$$

thus proving the contention that V_i^j as defined by Eq. (21) satisfies Laplace's equation, subject to the implied supposition that the function $P_i^j(\sin\beta)$ satisfying the associated Legendre equations is the same as the $P_i^j(\sin\beta)$ defined by Eq. (5).

In order to confirm the implied meaning of $P_i^j(\sin\beta)$, it is convenient to rewrite Eq. (46) in terms of W, defined by

$$W \triangleq \left(1-y^2\right)^{-j/2}\ Y \tag{48}$$

so that

$$Y = \left(1-y^2\right)^{j/2} W \tag{49}$$

$$\frac{dY}{dy} = \left(1-y^2\right)^{j/2}\frac{dW}{dy} - jy\left(1-y^2\right)^{j/2-1} \tag{50}$$

and

$$\frac{d^2Y}{dy^2} = \left(1-y^2\right)^{\frac{j}{2}}\frac{d^2W}{dy^2} - 2\,jy\left(1-y^2\right)^{\frac{j}{2}-1}\frac{dW}{dy}$$

$$-\left[1 + \frac{2y^2\left(1-\frac{j}{2}\right)}{1-y^2}\right]j\left(1-y^2\right)^{\frac{j}{2}-1} W \tag{51}$$

Substituting Eqs. (49) - (51) into Eq. (46) and multiplying by $\left(1-y^2\right)^{-\frac{j}{2}}$ yields

$$\left(1-y^2\right)\frac{d^2W}{dy^2} - 2y(j+1)\frac{dW}{dy} + \left[i(i+1) - j^2-j\right] W = 0 \tag{52}$$

Eq. (52) is more meaningful when written in terms of L, as defined by

$$W \triangleq \frac{d^j L}{dy^j} \tag{53}$$

The resulting equation

$$\left(1-y^2\right) \frac{d^{j+2}}{dy^{j+2}} L - 2y(j+1) \frac{d^{j+1}}{dy^{j+1}} L + \left[i(i+1) - j^2-j\right] \frac{d^j L}{dy^j} = 0 \tag{54}$$

can be recognized as the j^{th} y-derivative of Legendre's equation of i^{th} degree,

$$\left(1-y^2\right)\frac{d^2L}{dy^2} - 2y \frac{dL}{dy} + i(i+1) L = 0 \tag{55}$$

(Proof requires only j successive differentiations of Eq. (55) to produce Eq. (54)).

If the symbol $P_i(y)$ designates a particular solution of Eq. (55), then

$$P_i^j(y) \underset{(48,53,55)}{=} \left(1-y^2\right)^{\frac{j}{2}} \frac{d^j}{dy^j} P_i(y) \tag{56}$$

is a particular solution of Eq. (46). In order to confirm Eq. (5), it is now necessary first to establish Eq. (3) for $P_i(y)$. To this end, let

$$\alpha \triangleq \left(y^2-1\right)^i \tag{57}$$

so that

$$\frac{d\alpha}{dy} = 2iy \left(y^2-1\right)^{i-1}$$

and

$$\left(1-y^2\right)\frac{d\alpha}{dy} + 2iy\alpha = 0$$

Differentiate once with respect to y to obtain

$$\left(1-y^2\right) \frac{d^2\alpha}{dy^2} + 2y(i-1) \frac{d\alpha}{dy} + 2i\alpha = 0$$

and differentiate with respect to y i times more to establish

$$\left(1-y^2\right)\frac{d^2}{dy^2}\left(\frac{d^i\alpha}{dy^i}\right) - 2y\frac{d}{dy}\left(\frac{d^i\alpha}{dy^i}\right) + i(i+1)\left(\frac{d^i\alpha}{dy^i}\right) = 0$$

proving that $d^i\alpha/dy^i$ satisfies Legendre's equation. Since $P_i(y)$ has been identified here simply as a particular solution of Legendre's equation, one can define $P_i(y)$ as an arbitrary multiple of $d^i\alpha/dy^i$, and specifically let

$$P_i(y) \underset{(57)}{\triangleq} \frac{1}{2^i i!}\frac{d^i}{dy^i}\left(y^2-1\right)^i \tag{58}$$

This result, known as Rodrigues' formula for the Legendre polynomials, is usually considered as a property of the particular solution $P_i(y)$ satisfying Legendre's equation, but a systematic evaluation of the derivative in Eq. (58) confirms the equivalence of this expression and the series developed in Eq. (3). Substitution of this series into Eq. (56) and repeated differentiation confirms the series expansion for $P_i^j(y)$ developed in Eq. (5).

Example: In conventional practice Eq. (11) is used in application to natural celestial bodies for which mass distribution properties are not known sufficiently well either to calculate C_{ij} and S_{ij} from Eqs. (8) - (10) or to use the series for V presented in Sec. 2.12. The alternative is to deduce values for C_{ij} and S_{ij} (at least for the lower values of i and j) from empirical data on gravitational forces on other bodies.

In the case of the earth, for example, measurements of gravitational forces at the earth's surface and on artifical earth satellites have established the approximate values

$$C_{20} = -1.0827\times10^{-3}$$

$$C_{2j} = S_{2j} = 0 \qquad (j = 1,2)$$

when the normalizing quantity R_B is given the value of the earth's mean equatorial radius, here designated b.

If the earth is assumed to be a homogeneous spheroid, with polar radius c, the empirical data can be used to compute the ratio c/b. Since

$$I_1 = I_2 = m(b^2+c^2)/5$$

and

$$I_3 = 2mb^2/5$$

we have

$$-1.0827\times10^{-3} \underset{(14a)}{=} \frac{1}{2mb^2}\left[\frac{2m}{5}(b^2+c^2) - \frac{4mb^2}{5}\right]$$

$$= \frac{1}{5}\left(\frac{c^2}{b^2} - 1\right).$$

and

$$\frac{c}{b} = (1-5\times1.0827\times10^{-3})^{\frac{1}{2}} = 0.9973$$

2.14 Force function for two small bodies

When the distance R between the mass centers B^* and $\bar{B}^*$ of two (not necessarily rigid) bodies B and $\bar{B}$ exceeds the greatest distance in either body from the mass center to any point of the body, so that the resultant of the system of gravitational forces exerted on B by $\bar{B}$ can be approximated by a force $\tilde{\underline{F}}$ defined as

$$\tilde{\underline{F}} \triangleq -\frac{G m \bar{m}}{R^2}\left(\underline{a}_1 + \underline{f}^{(2)} + \bar{\underline{f}}^{(2)}\right) \tag{1}$$

where G is the universal gravitational constant, m and $\bar{m}$ are the masses of B and $\bar{B}$, $\underline{a}_1$ is a unit vector directed from $\bar{B}^*$ to B^*, and $\underline{f}^{(2)}$ and $\bar{\underline{f}}^{(2)}$ are given by Eqs. (2.4.2) and (2.4.3), then a force function $\tilde{V}$ such that

$$\tilde{\underline{F}} = \nabla_{\underline{R}}\tilde{V} \tag{2}$$

where $\underline{R} = R\underline{a}_1$, is given by

$$\tilde{V} = \frac{G m \bar{m}}{R}\left(1 + v^{(2)} + \bar{v}^{(2)}\right) + c \tag{3}$$

with

$$v^{(2)} = \frac{1}{2mR^2}\left[\mathrm{tr}(\underline{I}) - 3I_{11}\right] \tag{4}$$

and

$$\bar{v}^{(2)} = \frac{1}{2\bar{m}R^2}\left[\mathrm{tr}(\bar{\underline{I}}) - 3\bar{I}_{11}\right] \tag{5}$$

where $\mathrm{tr}(\underline{I})$ and $\mathrm{tr}\ (\bar{\underline{I}})$ are the traces of the inertia dyadics of B for B^* and of $\bar{B}$ for $\bar{B}^*$, while I_{11} and $\bar{I}_{11}$ are the moments of inertia of B and of $\bar{B}$ about the line connecting B^* and $\bar{B}^*$.

Derivation: Differentiating Eq. (3) with respect to $\underline{R}$ and then proceeding as in the derivation of Eq. (2.12.13), one arrives at

$$\nabla_{\underline{R}}\tilde{V} = -\frac{G\bar{m}m}{R^2}\left(\underline{a}_1 + \frac{1}{mR^2}\left\{\frac{3}{2}\left[tr(\underline{I}) - 5\underline{a}_1\cdot\underline{I}\cdot\underline{a}_1\right]\underline{a}_1 + 3\underline{I}\cdot\underline{a}_1\right\}\right.$$

$$\left. + \frac{1}{mR^2}\left\{\frac{3}{2}\left[tr(\underline{I}) - 5\underline{a}_1\cdot\bar{\underline{I}}\cdot\underline{a}_1\right]\underline{a}_1 + 3\bar{\underline{I}}\cdot\underline{a}_1\right\}\right)$$

and use of Eqs. (2.4.2) and (2.4.3) then gives

$$\nabla_{\underline{R}}\tilde{V} = -\frac{G\bar{m}m}{R^2}\left(\underline{a}_1 + \underline{f}^{(2)} + \bar{\underline{f}}^{(2)}\right) = \tilde{\underline{F}}_{(1)}$$

Example: The Example in Section 2.4 deals with an approximation $\tilde{\underline{F}}$ to the gravitational force exerted on a rectangular parallelepiped by an oblate spheroid. To construct a force function $\tilde{V}$ that satisfies Eq. (2), note that the moments of inertia I_{11} and $\bar{I}_{11}$ can be expressed as

$$I_{11} = \frac{m\alpha^2}{12}\left[2 - \varepsilon^2 + (\underline{a}_1\cdot\underline{b}_1)^2\,\varepsilon^2\right]$$

$$\bar{I}_{11} = \frac{2\bar{m}\bar{\alpha}^2}{5}\left[2 - \bar{\varepsilon}^2 + (\underline{a}_1\cdot\bar{\underline{b}}_1)^2\,\bar{\varepsilon}^2\right]$$

while

$$tr\,(\underline{I}) = \frac{m\alpha^2}{6}\,(3 - \varepsilon^2)$$

and

$$tr\,(\bar{\underline{I}}) = \frac{2\bar{m}\bar{\alpha}^2}{5}\,(3 - \bar{\varepsilon}^2)$$

$\tilde{V}$ can then be formed by substituting into Eqs. (3) - (5):

$$\tilde{V} = \frac{G\bar{m}m}{R}\left\{1 + \frac{\alpha^2\varepsilon^2}{24R^2}\left[1 - 3\,(\underline{a}_1\cdot\underline{b}_1)^2\right]\right.$$

$$\left. + \frac{\bar{\alpha}^2\bar{\varepsilon}^2}{5R^2}\left[1 - 3\,(a_1\cdot b_1)^2\right]\right\} + c$$

and it may be verified that differentiation of $\tilde{V}$ with respect to $\underline{R}$ (using Eqs. (2.9.8) and (2.9.9) leads to $\tilde{\underline{F}}$ as given by Eqs. (2.4.14), (2.4.26), and (2.4.29).

2.15 Force functions for a centrobaric body

Since for purposes of dealing with gravitational effects a centrobaric body (see Section 2.5) may be replaced with a particle situated at the mass center of such a body, force functions suitable for dealing with the gravitational interaction of a centrobaric body and a particle, a centrobaric body and any body whatsoever, or a centrobaric body and a remote body can be obtained directly from Eqs. (2.10.2), (2.11.2), (2.11.5),(2.12.1)(2.13.1) or (2.13.11).

Example: The gravitational force $\underline{F}$ exerted by an uniform sphere S of mass M on a dumbell of mass 2m (see Fig. 2.15.1) can be expressed as

$$\underline{F} = \underline{\nabla}_{\underline{R}} V$$

with

$$V \underset{(2.12.2)}{=} GMm \left\{ \left[(\underline{R} + L\underline{n})^2 \right]^{-\frac{1}{2}} + \left[(\underline{R} - L\underline{n})^2 \right]^{-\frac{1}{2}} \right\} + C$$

where $\underline{n}$ is a unit vector directed as shown in Fig. 2.15.1.

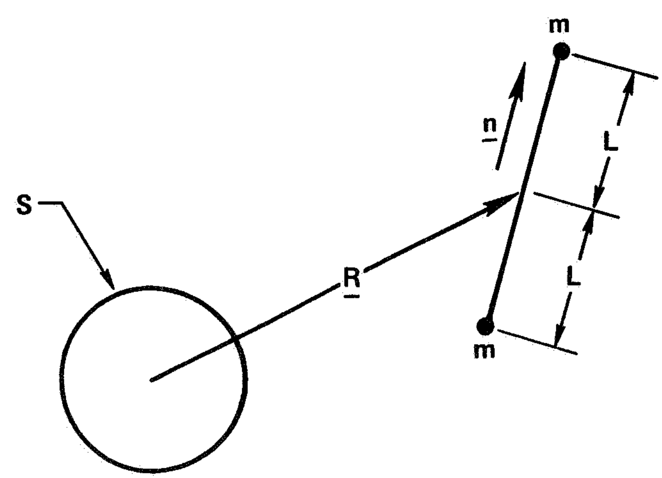


Figure 2.15.1

2.16 Force function for a body and a small body

When the distance between the mass centers B^* and $\bar{B}^*$ of two bodies B and $\bar{B}$ exceeds the greatest distance from B^* to any point of B, and a force function $V(\underline{p})$ for the force exerted by B on a particle of unit mass at a point P situated as shown in Fig. 2.16.1 is available, then there exists a function $V(\underline{R})$, where $\underline{R}$ is the position vector of B^* relative to $\bar{B}^*$, such that the resultant of all gravitational forces exerted by $\bar{B}$ on B can be approximated by a force $\underline{F}$ expressed as

$$\tilde{F} = \underline{\nabla}\tilde{V}(\underline{R}) \tag{1}$$

where the symbol $\underline{\nabla}$ connotes differentiation with respect to $\underline{R}$. The function $\tilde{V}(\underline{R})$ is given by

$$\tilde{V}(\underline{R}) = mV(\underline{R}) - \frac{1}{2}\,\underline{I} : \underline{\nabla}\underline{\nabla}V(\underline{R}) + C \tag{2}$$

where m is the mass of B, $\underline{I}$ is the inertia dyadic of B for B^*, and C is an arbitrary constant.*

Derivations: The gravitational force $d\underline{F}$ exerted by $\bar{B}$ on a differential element of B at P can be expressed as

$$d\underline{F} \underset{(2.11.1)}{=} \underline{\nabla}_{\underline{p}} V \rho d\tau \tag{3}$$

where ρ is the mass density of B at P while $d\tau$ is the volume of the element. Consequently, the resultant $\underline{F}$ of all gravitational forces exerted by $\bar{B}$ on B is given by

$$\underline{F} = \int \underline{\nabla}_{\underline{p}} V \rho d\tau \tag{4}$$

*The double dot product in Eq. (2) is defined such that, for two dyads $\underline{u}_1\underline{u}_2$ and $\underline{v}_1\underline{v}_2$, $(\underline{u}_1\underline{u}_2) : (\underline{v}_1\underline{v}_2) = (\underline{u}_1 \cdot \underline{v}_1)(\underline{u}_2 \cdot \underline{v}_2)$; and it obeys the distributive law when applied to dyadics:

$$\begin{aligned}(\underline{a}_1\underline{a}_2+\underline{b}_1\underline{b}_2+\ldots) : (\underline{A}_1\underline{A}_2+\underline{B}_1\underline{B}_2+\ldots) &= \underline{a}_1\underline{a}_2 : \underline{A}_1\underline{A}_2 + \underline{a}_1\underline{a}_2 : \underline{B}_1\underline{B}_2 + \ldots \\ &+ \underline{b}_1\underline{b}_2 : \underline{A}_1\underline{A}_2 + \underline{b}_1\underline{b}_2 : \underline{B}_1\underline{B}_2 + \ldots \\ &+ \ldots\end{aligned}$$

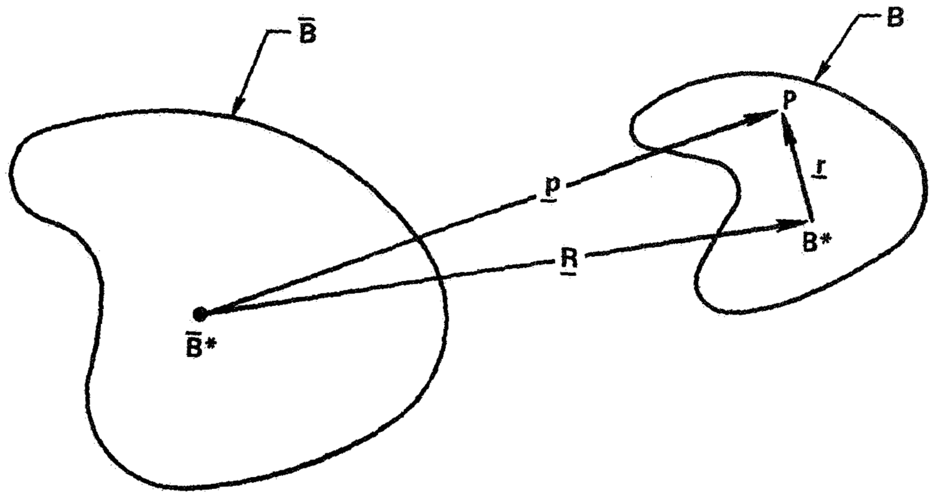


Figure 2.16.1

To obtain the desired approximation $\tilde{\underline{F}}$ to $\underline{F}$, one can expand $V(\underline{p})$ in a Taylor series about $\underline{p} = \underline{R}$, retaining only terms up to and including those of degree three in $|\underline{r}|$, where $\underline{r} \triangleq \underline{p} - \underline{R}$ (see Fig. 2.16.1); differentiate with respect to $\underline{p}$; and then carry out the integration indicated in Eq. (4). These tasks are facilitated by introducing a function $W(\underline{r}) \triangleq V(\underline{r} + \underline{R})$, which permits one to write

$$\underline{\nabla}_p V \underset{(2.9.5)}{=} \underline{\nabla}_r W \tag{5}$$

and

$$\underline{F} \underset{(4.5)}{=} \int \underline{\nabla}_r W \rho d\tau \tag{6}$$

If $\underline{n}_1$, $\underline{n}_2$, $\underline{n}_3$ are any mutually perpendicular unit vectors and $r_i \triangleq \underline{r} \cdot \underline{n}_i$ $(i=1,2,3)$, the Taylor series expansion of W about $\underline{r} = 0$ can be written (using the summation convention)

$$W = W(0) + r_i W_{,i} + \frac{1}{2!} r_i r_j W_{,ij} + \frac{1}{3!} r_i r_j r_k W_{,ijk} + \cdots \tag{7}$$

where

$$W_{,i} \triangleq \left.\frac{\partial W}{\partial r_i}\right|_{\underline{r}=0} (i=1,2,3); \quad W_{,ij} \triangleq \left.\frac{\partial^2 W}{\partial r_i \partial r_j}\right|_{\underline{r}=0} \quad (i,j = 1,2,3) \tag{8}$$

and so forth. Differentiating with respect to r_ℓ, one obtains

$$\begin{aligned}\frac{\partial W}{\partial r_\ell} &\underset{(7,8)}{=} 0 + \delta_{i\ell} W_{,i} + \frac{1}{2!}\left(\delta_{i\ell} r_j + r_i \delta_{j\ell}\right) W_{,ij} \\ &\quad + \frac{1}{3!}\left(\delta_{i\ell} r_j r_k + r_i \delta_{j\ell} r_k + r_i r_j \delta_{k\ell}\right) W_{,ijk} + \cdots \\ &= W_{,\ell} + r_i W_{,i\ell} + \frac{1}{2} r_i r_j W_{,ij\ell} + \cdots\end{aligned}$$

Hence,

$$\underline{\nabla}_r W \underset{(2.9.1)}{=} \left(\underline{\nabla}_r W\right)_{r=0} + \underline{r} \cdot \left(\underline{\nabla}_r \underline{\nabla}_r W\right)_{\underline{r}=0} + \frac{1}{2} \underline{r}\, \underline{r} : \left(\underline{\nabla}_r \underline{\nabla}_r \underline{\nabla}_r W\right)_{r=0} + \cdots$$

Substituting into Eq. (6) only the terms here displayed explicitly, and noting that

$$\int \rho d\tau = m, \quad \int \underline{r}\rho d\tau = 0, \quad \int \underline{r}\underline{r}\rho d\tau = \frac{1}{2}\,\mathrm{tr}\;(\underline{I})\;\underline{U} - \underline{I}$$

where $\underline{U}$ is the unit dyadic, one can now write

$$\underline{\tilde{F}} = m\left(\underline{\nabla}_{\underline{r}}W\right)_{\underline{r}=0} + \frac{1}{2}\left[\frac{1}{2}\,\mathrm{tr}\;(\underline{I})\;\underline{U} - \underline{I}\right] : \left(\underline{\nabla}_{\underline{r}}\underline{\nabla}_{\underline{r}}\underline{\nabla}_{\underline{r}}W\right)_{\underline{r}=0}$$

or, after using Eq. (5),

$$\underline{\tilde{F}} = \left(\underline{\nabla}_{\underline{p}}\left\{mV + \frac{1}{2}\left[\frac{1}{2}\,\mathrm{tr}(\underline{I})\;\underline{U} - \underline{I}\right] : \;\underline{\nabla}_{\underline{p}}\underline{\nabla}_{\underline{p}}V\right\} + C\right)_{\underline{p}=\underline{R}}$$

where C is an arbitrary constant. Now, differentiating $V(\underline{p})$ with respect to $\underline{p}$ and then setting $\underline{p}$ equal to $\underline{R}$ is precisely the same as differentiating $V(\underline{R})$ with respect to $\underline{R}$. Hence, if $\underline{\nabla}$ connotes differentiation with respect to $\underline{R}$, then

$$\underline{\tilde{F}} = \underline{\nabla}\left\{mV(\underline{R}) \;+ \frac{1}{2}\left[\frac{1}{2}\,\mathrm{tr}\;(\underline{I})\underline{U} - \underline{I}\right] : \;\underline{\nabla}\underline{\nabla}V(\underline{R}) + C\right\}$$

Now

$$\underline{U}: \;\underline{\nabla}\underline{\nabla}V(\underline{R}) = \underline{\nabla}\cdot\underline{\nabla}V(\underline{R}) = \underline{\nabla}^2V(R) = 0$$

(2.11.7)

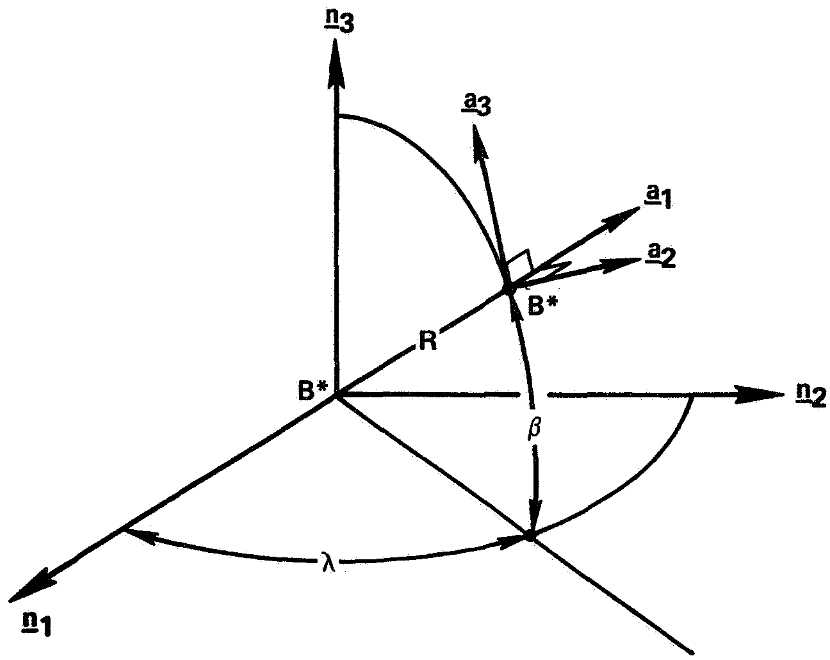
Thus Eq. (1) follows immediately if Eq. (2) is used to form $\tilde{V}(\underline{R})$.

<u>Example</u>: When $V(\underline{R})$ is available in the form of an explicit function $V^*(R,\lambda,\beta)$ of the spherical coordinates R,λ,β shown in Fig. 2.16.2, where $\underline{n}_1$, $\underline{n}_2,\underline{n}_3$ are any mutually perpendicular unit vectors, one can formulate $\underline{\tilde{F}}$ as follows: Let $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ be mutually perpendicular unit vectors directed as shown in Fig. 2.16.2. Define I_{jk} as

$$I_{jk} \triangleq \underline{a}_j \cdot \underline{I} \cdot \underline{a}_k \quad (j,k = 1,2,3) \tag{9}$$

and form Q_{ii} $(i = 1,2,3)$ and $\tilde{V}^*(R,\lambda,\beta)$ as

$$Q_{11} \triangleq \frac{\partial^2 V^*}{\partial R^2} \tag{10}$$



$\underline{n}_3$
$\underline{a}_3$
$\underline{a}_1$
$\underline{a}_2$
B*
R
B*
$\underline{n}_2$
β
λ
$\underline{n}_1$

Figure 2.16.2

$$Q_{22} \triangleq \frac{1}{R}\frac{\partial V^*}{\partial R} + \frac{\sec^2\beta}{R^2}\frac{\partial^2 V^*}{\partial \lambda^2} - \frac{\tan\beta}{R^2}\frac{\partial V^*}{\partial \beta} \tag{11}$$

$$Q_{33} \triangleq \frac{1}{R}\frac{\partial V^*}{\partial R} + \frac{1}{R^2}\frac{\partial^2 V^*}{\partial \beta^2} \tag{12}$$

and

$$\tilde{V}^*(R,\lambda,\beta) \triangleq mV^* - \frac{1}{2}\left(I_{11}Q_{11} + I_{22}Q_{22} + I_{33}Q_{33}\right) + C \tag{13}$$

Then

$$\underline{\tilde{F}} = \underline{a}_1 \frac{\partial \tilde{V}^*}{\partial R} + \underline{a}_2 \frac{\sec\beta}{R}\frac{\partial \tilde{V}^*}{\partial \lambda} + \underline{a}_3 \frac{1}{R}\frac{\partial \tilde{V}^*}{\partial \beta} \tag{14}$$

For the derivation of Eq. (14), it is helpful to note that, since the indicated partial differentiation is referred to the reference frame in which $\underline{n}_1, \underline{n}_2, \underline{n}_3$ in Fig. 2.16.2 are fixed,

$$\begin{bmatrix} \frac{\partial \underline{a}_1}{\partial R} & \frac{\partial \underline{a}_1}{\partial \lambda} & \frac{\partial \underline{a}_1}{\partial \beta} \\ \frac{\partial \underline{a}_2}{\partial R} & \frac{\partial \underline{a}_2}{\partial \lambda} & \frac{\partial \underline{a}_2}{\partial \beta} \\ \frac{\partial \underline{a}_3}{\partial R} & \frac{\partial \underline{a}_3}{\partial \lambda} & \frac{\partial \underline{a}_3}{\partial \beta} \end{bmatrix} = \begin{bmatrix} 0 & c\beta\underline{a}_2 & \underline{a}_3 \\ 0 & -c\beta\underline{a}_1 + s\beta\underline{a}_3 & 0 \\ 0 & -s\beta\underline{a}_2 & -\underline{a}_1 \end{bmatrix} \tag{15}$$

Also, proceeding as in the Example in Sec. 2.9, one can verify that, if $V(\underline{R}) = V^*(R,\lambda,\beta)$, then

$$\underline{\nabla}V(\underline{R}) = \underline{a}_1 \frac{\partial V^*}{\partial R} + \underline{a}_2 \frac{\sec\beta}{R}\frac{\partial V^*}{\partial \lambda} + \underline{a}_3 \frac{1}{R}\frac{\partial V^*}{\partial \beta} \tag{16}$$

Differentiating once more with respect to $\underline{R}$ and using Eq. (15) one then finds that

$$\underline{\nabla}\underline{\nabla}V(\underline{R}) = Q_{11}\underline{a}_1\underline{a}_1 + Q_{22}\underline{a}_2\underline{a}_2 + Q_{33}\underline{a}_3\underline{a}_3 + \cdots \tag{17}$$

where Q_{11}, Q_{22}, Q_{33} are given by Eqs. (4)-(6) while the dots represent all terms involving the dyads $\underline{a}_i\underline{a}_j$ with $i \neq j$. Similarly, the inertia dyadic $\underline{I}$ can be expressed as

$$\underline{I} \underset{(9)}{} I_{11}\underline{a}_1\underline{a}_1 + I_{22}\underline{a}_2\underline{a}_2 + I_{33}\underline{a}_3\underline{a}_3 + \ldots \tag{18}$$

Hence,

$$\underline{I} : \underline{\nabla\nabla}(\underline{R}) \underset{(17,18)}{=} I_{11}Q_{11} + I_{22}Q_{22} + I_{33}Q_{33}$$

With V^* in place of $V(\underline{R})$, one thus obtains from Eq. (2)

$$\tilde{V}(\underline{R}) = mV^* - \frac{1}{2}\left(I_{11}Q_{11} + I_{22}Q_{22} + I_{33}Q_{33}\right) + C$$

Moreover, defining $\tilde{V}^*(R,\lambda,\beta)$ as the right-hand member of this equations, which yields Eq. (13), one can write [compare with Eq. (16)]

$$\underline{\nabla}\tilde{V}(\underline{R}) = \underline{a}_1 \frac{\partial\tilde{V}^*}{\partial R} + \underline{a}_2 \frac{\sec\beta}{R} \frac{\partial\tilde{V}^*}{\partial\lambda} + \underline{a}_3 \frac{\partial\tilde{V}^*}{\partial\beta}$$

Substitution into Eq. (1) then produces Eq. (14).

2.17 Force function expressions for moment exerted on a body by a particle

Force functions (see Secs. 2.10 - 2.16) can be used also in connection with gravitational moments. For the interaction of a particle and a (not necessarily rigid) body, the moment $\underline{M}$ given by Eq. (2.6.1) can be expressed as

$$\underline{M} = -\underline{R} \times \underline{\nabla}V \tag{1}$$

where $V(\underline{R})$ is given by Eq. (2.11.2) or by Eq. (2.11.5) and $\underline{\nabla}V$ is the derivative of V with respect to $\underline{R}$; or, if W is defined as $W(\underline{a}_1, R) \triangleq V[\underline{R}(\underline{a}_1, R)]$, $\underline{M}$ is given also by

$$\underline{M} = -a_1 \times \frac{\partial W}{\partial \underline{a}_1} \tag{2}$$

Similarly, for $\tilde{\underline{M}}$ as defined in Eq. (2.6.3), one can write

$$\tilde{\underline{M}} = -R \times \underline{\nabla}\tilde{V} \tag{3}$$

where $\tilde{V}$ is given by Eq. (2.12.13); or $\tilde{\underline{M}}$ can be expressed as

$$\tilde{\underline{M}} = -a_1 \times \frac{\partial \tilde{W}}{\partial \underline{a}_1} \tag{4}$$

where $\tilde{W}$ is a function of $\underline{a}_1$ and R defined in terms of the moment of inertia I_{11} of B about the line connecting $\bar{P}$ and B^* as

$$\tilde{W} \triangleq -\frac{3G\bar{m}I_{11}}{2R^3} + C$$

Derivations: Eq. (1) is obtained by substituting from Eq. (2.11.1) into Eq. (2.6.1); and Eq. (2) then follows if one uses Eq. (2.9.12).

Referring to Eqs. (2.3.6) and (2.3.3), one can write

$$-R\,\underline{a}_1 \times \tilde{\underline{F}} = \frac{3G\bar{m}}{R^3}\underline{a}_1 \times \underline{I} \cdot \underline{a}_1 \underset{(2.6.3)}{=} \tilde{\underline{M}}$$

and, replacing $R\,\underline{a}_1$ with R, one then arrives at Eq. (3) after using Eq. (2.12.12) to eliminate $\tilde{\underline{F}}$.

If $\approx{W}$ is defined as

$$\tilde{\tilde{W}}(\underline{a}_1,R) \triangleq \tilde{V}[\underline{R}(\underline{a}_1,R)]$$

then

$$\tilde{\tilde{W}} \underset{(2.12.13)}{=} \frac{G\bar{m}m}{R}\left\{1 + \frac{1}{2mR^2}\left[tr(\underline{I})-3I_{11}\right]\right\} + C$$

and

$$\underline{R} \times \underline{\nabla}\tilde{V} \underset{(2.9.12)}{=} \underline{a}_1 \times \frac{\partial\tilde{\tilde{W}}}{\partial\underline{a}_1}$$

$$= \underline{a}_1 \times \frac{\partial}{\partial\underline{a}_1}\left\langle\frac{G\bar{m}m}{R}\left\{1 + \frac{1}{2mR^2}\left[tr(\underline{I})-3I_{11}\right]\right\} + C\right\rangle$$

or, since tr (I) is independent of $\underline{a}_1$,

$$\underline{R} \times \underline{\nabla}\tilde{V} = \underline{a}_1 \times \frac{\partial}{\partial\underline{a}_1}\left(-\frac{3G\bar{m}I_{11}}{2R^3} + C\right) \underset{(5)}{=} \underline{a}_1 \times \frac{\partial\tilde{W}}{\partial\underline{a}_1}$$

Consequently

$$\tilde{\underline{M}} \underset{(3)}{=} -\underline{a}_1 \times \frac{\partial\tilde{W}}{\partial\underline{a}_1}$$

in agreement with Eq. (4).

Example, in Fig. 2.17.1, $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are dextral sets of orthogonal unit vectors, $\bar{P}$ is a particle of mass $\bar{m}$, and B is a rigid body of mass m. The unit vector $\underline{a}_1$ is chosen such that the position vector $\underline{R}$ of the mass center B^* of B relative to $\bar{P}$ is given by $\underline{R} = R\underline{a}_1$; and $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are fixed in B.

If θ_1, θ_2, θ_3 are body-two orientation angles (see Sec. 1.7) for B in a reference frame in which $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ are fixed, one can express $\underline{R}$ as

$$\underline{R}(R,\theta_1,\theta_2,\theta_3) \underset{(1.7.31)}{=} R(c_2\underline{b}_1 + s_2s_3\underline{b}_2 + s_2c_3\underline{b}_3) \tag{6}$$

Given a force function $V(\underline{R})$ for the gravitational interaction of $\bar{P}$ and B, one can, therefore, define a function $W(R,\theta_1, \theta_2, \theta_3)$ as

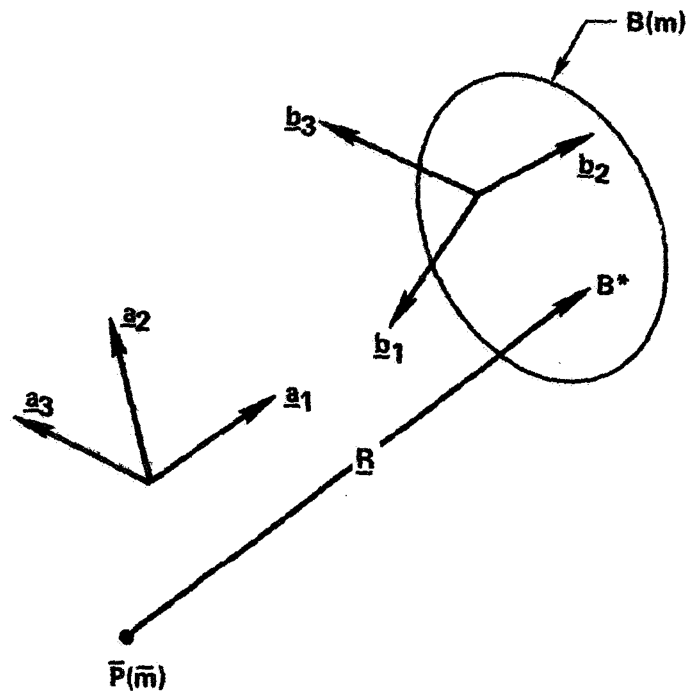


Figure 2.17.1

$$W(R,\theta_1,\theta_2,\theta_3) \triangleq V[\underline{R}(R,\theta_2,\theta_3)] \tag{7}$$

The moment $\underline{M}$ exerted by $\bar{P}$ on B about B^* can then be expressed in terms of partial derivatives of W with respect to θ_2 and θ_3 by proceeding as follows:

Partial differentiation of $\underline{R}$ with respect to R, θ_2, and θ_3 yields

$$\frac{\partial \underline{R}}{\partial R} \underset{(6)}{=} c_2\underline{b}_1 + s_2s_3\underline{b}_2 + s_2c_3\underline{b}_3 \tag{8}$$

$$\frac{\partial \underline{R}}{\partial \theta_2} \underset{(6)}{=} R(-s_2\underline{b}_1 + c_2s_3\underline{b}_2 + c_2c_3\underline{b}_3) \tag{9}$$

$$\frac{\partial \underline{R}}{\partial \theta_3} \underset{(6)}{=} R(s_2c_3\underline{b}_1 - s_2s_3\underline{b}_3) \tag{10}$$

From Eq. (2.9.5)

$$\frac{\partial W}{\partial R} = \underline{\nabla}V \cdot \frac{\partial \underline{R}}{\partial R} \underset{(8)}{=} c_2\, \underline{\nabla}V \cdot \underline{b}_1 + s_2s_3\, \underline{\nabla}V \cdot \underline{b}_2 + s_2c_3\, \underline{\nabla}V \cdot \underline{b}_3 \tag{11}$$

$$\frac{\partial W}{\partial \theta_2} = \underline{\nabla}V \cdot \frac{\partial \underline{R}}{\partial \theta_2} \underset{(9)}{=} R\left(-s_2\, \underline{\nabla}V \cdot \underline{b}_1 + c_2s_3\, \underline{\nabla}V \cdot \underline{b}_2 + c_2c_3\, \underline{\nabla}V \cdot \underline{b}_3\right)$$

$$\frac{\partial W}{\partial \theta_3} = \underline{\nabla}V \cdot \frac{\partial \underline{R}}{\partial \theta_3} \underset{(10)}{=} R\left(s_2c_3\, \underline{\nabla}V \cdot \underline{b}_2 - s_2s_3\, \underline{\nabla}V \cdot \underline{b}_3\right) \tag{13}$$

Eqs. (11)-(13) may be solved for the dot-products of $\underline{\nabla}V$ with $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$; and $\underline{\nabla}V$ can then be expressed as

$$\begin{aligned}
\underline{\nabla}V &= \underline{\nabla}V \cdot \underline{b}_1\underline{b}_1 + \underline{\nabla}V \cdot \underline{b}_2\underline{b}_2 + \underline{\nabla}V \cdot \underline{b}_3\underline{b}_3 \\
&= \left(\frac{\partial W}{\partial R} c_2 - \frac{1}{R}\frac{\partial W}{\partial \theta_2} s_2\right)\underline{b}_1 \\
&\quad + \left(\frac{\partial W}{\partial R} s_2s_3 + \frac{1}{R}\frac{\partial W}{\partial \theta_2} c_2s_3 + \frac{1}{R}\frac{\partial W}{\partial \theta_3}\frac{c_3}{s_2}\right)\underline{b}_2 \\
&\quad + \left(\frac{\partial W}{\partial R} s_2c_3 + \frac{1}{R}\frac{\partial W}{\partial \theta_2} c_2c_3 - \frac{1}{R}\frac{\partial W}{\partial \theta_3}\frac{s_3}{s_2}\right)\underline{b}_3
\end{aligned} \tag{14}$$

Consequently

$$\underset{\sim}{M} \underset{(1)}{=} -\underset{\sim}{R} \times \underset{\sim}{\nabla} V \underset{(6,14)}{=} \frac{\partial W}{\partial \theta_3} \underset{\sim}{b}_1 - \left(\frac{\partial W}{\partial \theta_2} c_3 - \frac{\partial W}{\partial \theta_3} \frac{c_2 s_3}{s_2} \right) \underset{\sim}{b}_2$$

$$- \left(\frac{\partial W}{\partial \theta_3} \frac{c_2 c_3}{s_2} + \frac{\partial W}{\partial \theta_2} s_3 \right) \underset{\sim}{b}_3 \qquad (15)$$

This equation does not involve θ_1 explicitly; but, referred to $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$, rather than to $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$, $\underset{\sim}{M}$ is given by [use (1.17.31)].

$$\underset{\sim}{M} = \left(c_1 \frac{\partial W}{\partial \theta_2} + \frac{s_1}{s_2} \frac{\partial W}{\partial \theta_3} \right) \underset{\sim}{a}_2 + \left(s_1 \frac{\partial W}{\partial \theta_2} - \frac{c_1}{s_2} \frac{\partial W}{\partial \theta_2} \right) \underset{\sim}{a}_3$$

which brings the dependence of $\underset{\sim}{M}$ on θ_1 into evidence.

2.18 Force function expressions for moment exerted on a body by a body

If V is the force function for a body B and a particle $\bar{P}$ of unit mass, then the moment exerted by a body $\bar{B}$ on B about its mass center B^* is given by

$$\underline{M} = -\int_{\bar{B}} \underline{\bar{R}} \times \nabla_{\underline{\bar{R}}} V \, \bar{\rho} d\bar{\tau} \tag{1}$$

where, as in Fig. 2.7.1, $\bar{P}$ is a generic point of $\bar{B}$, $\underline{\bar{R}}$ is the vector from $\bar{P}$ to B^*, $\bar{\rho}$ is the mass density of $\bar{B}$ and $d\bar{\tau}$ is a differential element of volume of $\bar{B}$.

When the distance R between the mass centers B^* and $\bar{B}^*$ of B and $\bar{B}$ is large relative to the largest dimensions of B, then $\underline{M}$ can be approximated by

$$\underline{M} \cong 3G \int_{\bar{B}} \frac{\bar{\underline{a}}_1 \times \underline{I} \cdot \bar{\underline{a}}_1}{\bar{R}^3} \bar{\rho} d\bar{\tau}$$

in which $\bar{R}$ and $\bar{\underline{a}}_1$ are defined by $\underline{\bar{R}} = \bar{R}\, \bar{\underline{a}}_1$, as in Fig. 2.7.1.

When R is large relative to the dimensions of both B and $\bar{B}$, $\underline{M}$ can be approximated by Eqs. (2.17.3) or Eqs. (2.17.4) and (2.17.5) with $\tilde{V}$ from Eqs. (2.12.13), where now R, $\underline{R}$, and $\underline{a}_1$ are to be interpreted as indicated in Fig. 2.7.1, I_{11} is the moment of inertia of B about the line joining $\bar{B}^*$ and B^*, and $\bar{m}$ is the mass of $\bar{B}$.

Derivations:

Eq. (1) follows from the definition of a moment about B^* and the identification of $\nabla_{\underline{\bar{R}}} V \, \bar{\rho} d\bar{\tau}$ as the force exerted on B by the differential element of mass at $\bar{P}$.

As shown in Eqs. (2.12.1) and (2.12.2), the force function for B and an element of unit mass at $\bar{P}$ can be written as

$$V = \frac{Gm}{\bar{R}} \left\{ 1 + \frac{1}{2m\bar{R}^2} \left[\mathrm{tr}(\underline{I}) - 3\bar{\underline{a}}_1 \cdot \underline{I} \cdot \bar{\underline{a}}_1 \right] + \sum_{i=3}^{\infty} v^{(i)} \right\} \tag{3}$$

where $v^{(i)}$ represents terms of i^{th} degree in the ratio $|r|/R$ (see Fig. 2.7.1). Differentiation yields

$$\underset{\substack{(2.10.7)\\(2.10.6)}}{\nabla_{\bar{\underline{R}}} V =} -\frac{Gm}{\bar{R}^2}\,\bar{\underline{a}}_1\left\{1+\frac{1}{2m\bar{R}^2}\left[\mathrm{tr}(\underline{I})-3\bar{\underline{a}}_1\cdot\underline{I}\cdot\bar{\underline{a}}_1\right]+\sum_{i=3}^{\infty}v^{(i)}\right\}$$
$$-\frac{Gm}{\bar{R}}\left\{\frac{\bar{\underline{a}}_1}{m\bar{R}^3}\left[\mathrm{tr}(\underline{I})-3\bar{\underline{a}}_1\cdot\underline{I}\cdot\bar{\underline{a}}_1\right]+\frac{3}{m\bar{R}^3}\left(\underline{U}-\bar{\underline{a}}_1\bar{\underline{a}}_1\right)\cdot I\cdot\bar{\underline{a}}_1\right.$$
$$\left.+\sum_{i=3}^{\infty}\nabla_{\bar{\underline{R}}}v^{(i)}\right\} \quad (4)$$

Substitution of Eq. (4) into Eq. (1) leads to

$$\underline{M} = 3G\int_{\bar{B}}\frac{\bar{\underline{a}}_1\times\underline{I}\cdot\bar{\underline{a}}_1}{\bar{R}^3}\,\bar{\rho}d\bar{\tau} + Gm\int_{\bar{B}}\bar{\underline{a}}_1\times\sum_{i=3}^{\infty}\nabla_{\bar{\underline{R}}}\,v^{(i)}\,\bar{\rho}d\bar{\tau} \quad (5)$$

which justifies the approximation in Eq. (2) when body B is small in dimension relative to R.

Beginning with Eq. (2), one can when $\bar{B}$ is also of small dimension relative to R introduce the approximations

$$\bar{R} = R(1 + \ldots) \quad (6)$$

$$\bar{\underline{a}}_1 = \underline{a}_1 + \ldots \quad (7)$$

where three dots represent terms of first degree or higher in the ratio $|r|/R$ (see Fig. 2.7.1). By substituting Eqs. (6) and (7) into Eq. (2), and using the definition

$$\bar{m} \triangleq \int \bar{\rho}d\bar{\tau} \quad (8)$$

one can confirm the applicability of Eq. (2.7.2), with $\underline{a}_1$ now interpreted as in Fig. 2.7.1; Eq. (2.7.2) has been shown in Section 2.17 to be equivalent to Eqs. (2.17.3) and to the combination of Eqs. (2.17.4) and (2.17.5).

Example

Fig. 2.4.2 in Section 2.4 portrays a homogeneous oblate spheroid $\bar{B}$ and a homogeneous rectangular parallelepiped B, with the mass centers $\bar{B}^*$ and B^* of $\bar{B}$ and B respectively separated by a distance R that substantially exceeds the maximum dimensions of $\bar{B}$ and B.

The moment exerted by $\bar{B}$ on B about B^* is approximated by

$$\tilde{\underline{M}} \underset{(2.17.3)}{=} -\underline{R} \times \underline{\nabla}_{\underline{R}}\tilde{V} \tag{9}$$

where

$$\tilde{V} \underset{(2.12.13)}{=} \frac{G\bar{m}m}{R}\left\{1 + \frac{1}{2mR^2}\left[\mathrm{tr}(\underline{I}) - 3\underline{a}_1 \cdot \underline{I} \cdot \underline{a}_1\right]\right\} \tag{10}$$

As implied by the Example of Section 2.14, Eq. (10) produces

$$\tilde{V} = \frac{G\bar{m}m}{R}\left\{1 + \frac{\alpha^2\varepsilon^2}{24R^2}\left[1 - 3(\underline{a}_1 \cdot \underline{b}_1)^2\right]\right\} + C \tag{11}$$

[This result should be contrasted with V as it actually appears in the Example of Section 2.14; the same distinction exists between the V of Eq. (2.12.13) and the V of Eq. (2.14.3)].

Differentiation of Eq. (11) provides

$$\underline{\nabla}_{\underline{R}}\tilde{V} \underset{\substack{(11, 2.9.8,\\ 2.9.9)}}{=} -\frac{G\bar{m}m}{R^2}\left\{\underline{a}_1 + \frac{\alpha^2\varepsilon^2}{4R^2}\left[\frac{1}{2}\left(1 - 3C_{11}^2\right)\underline{a}_1 + C_{21}C_{11}\underline{a}_2 + C_{31}C_{11}\underline{a}_3\right]\right\} \tag{12}$$

as indicated by the Examples in Sections 2.14 and 2.4.

Substitution and cross multiplication produce

$$\tilde{\underline{M}} \underset{(9,12)}{=} \frac{G\bar{m}m\alpha^2\varepsilon^2}{4R^3} C_{11}\left(C_{21}\underline{a}_3 - C_{31}\underline{a}_2\right) \tag{13}$$

This result should be identical to Eq. (2.7.6) in the Example of Section 2.7. To reconcile these two expressions for $\underline{M}$, one must recognize the identities

$$\left(I_1 - I_2\right) \underset{(2.7.9,\ 2.7.10,\ 2.4.25)}{=} m\alpha^2\varepsilon^2/12 \tag{14}$$

and

$$C_{13}\underline{b}_2 - C_{12}\underline{b}_3 = C_{21}\underline{a}_3 - C_{31}\underline{a}_2 \tag{15}$$

The validity of Eq. (15) follows from the expressions

$$\underline{a}_3 = C_{31}\underline{b}_1 + C_{32}\underline{b}_2 + C_{33}\underline{b}_3 \tag{16}$$

$$\underline{a}_2 = C_{21}\underline{b}_1 + C_{22}\underline{b}_2 + C_{23}\underline{b}_3 \tag{17}$$

and the equality of an element of the direction cosine matrix with its cofactor, both of which are established in Section 1.2.

2.19 Force function expression for the moment exerted on a small body by a body

When the distance between the mass centers B^* and $\bar{B}^*$ of two bodies B and $\bar{B}$ exceeds the greatest distance from B^* to any point of B, and a force function $V(\underline{p})$ for the forces exerted by $\bar{B}$ on a particle of unit mass at a point P situated as shown in Fig. 2.16.1 is available, then the system of gravitational forces exerted by $\bar{B}$ on B produces a moment about B^* that is given approximately by

$$\underline{\tilde{M}} = -\underline{I} \overset{\times}{\cdot} \underline{\nabla}\underline{\nabla} V(\underline{R}) \tag{1}$$

where $\underline{I}$ is the inertia dyadic of B for B^*, $\underline{R}$ is the position vector of B^* relative to $\bar{B}^*$, and $\underline{\nabla}$ connotes differentiation with respect to $\underline{R}$.†

Derivation: The system of gravitational forces exerted by $\bar{B}$ on B produces a moment $\underline{M}$ about B^* such that

$$\underline{M} = \int \underline{r} \times \underline{\nabla}_p V \rho d\tau \tag{2}$$

Expanding $\underline{\nabla}_p V$ in a Taylor series as in Sec. 2.16 following Eq. (2.16.4), one can show that

$$\underline{\nabla}_p V = \underline{\nabla} V(\underline{R}) + \underline{r} \cdot \underline{\nabla}\underline{\nabla} V(\underline{R}) + \ldots$$

so that, retaining only terms displayed explicitly, one arrives at

$$\underline{\tilde{M}} \overset{=}{\underset{(2)}{}} \int \underline{r} \times \underline{\nabla} V(\underline{R})\ \rho d\tau + \int \underline{r} \times \left[\underline{r} \cdot \underline{\nabla}\underline{\nabla} V(\underline{R})\right] \rho d\tau$$

The first integral vanishes because $\underline{r}$ originates at the mass center of B^*; and

$$\underline{r} \times \left[\underline{r} \cdot \underline{\nabla}\underline{\nabla} V(\underline{R})\right] = \underline{r}\,\underline{r} \overset{\times}{\cdot} \underline{\nabla}\underline{\nabla} V(R)$$

† The cross-dot product in Eq. (1) is defined such that, for two dyads $\underline{u}_1\underline{u}_2$ and $\underline{v}_1\underline{v}_2$, $(\underline{u}_1\underline{u}_2) \overset{\times}{\cdot} (\underline{v}_1\underline{v}_2) \triangleq (\underline{u}_1 \times \underline{v}_1)(\underline{u}_2 \cdot \underline{v}_2)$; and it obeys the distributive law when applied to dyadics:

$$(\underline{a}_1\underline{a}_2 + \underline{b}_1\underline{b}_2 + \ldots) \overset{\times}{\cdot} (\underline{A}_1\underline{A}_2 + \underline{B}_1\underline{B}_2 + \ldots) = \underline{a}_1\underline{a}_2 \overset{\times}{\cdot} \underline{A}_1\underline{A}_2 + \underline{a}_1\underline{a}_2 \overset{\times}{\cdot} \underline{B}_1\underline{B}_2 + \ldots + \underline{b}_1\underline{b}_2 \overset{\times}{\cdot} \underline{A}_1\underline{A}_2 + \underline{b}_1\underline{b}_2 \overset{\times}{\cdot} \underline{B}_1\underline{B}_2 + \ldots + \ldots$$

Hence

$$\tilde{\underline{M}} = \left(\int \underline{r}\underline{r}\ \rho d\tau\right) \underset{\bullet}{\times} \underline{\nabla\nabla} V(\underline{R}) = \left[\frac{1}{2}\ \mathrm{tr}(\underline{I})\underline{U} - \underline{I}\right] \underset{\bullet}{\times} \underline{\nabla\nabla} V(\underline{R})$$

But $\underline{\nabla\nabla} V(\underline{R})$ is a symmetric dyadic, and the cross-dot product of $\underline{U}$ and any symmetric dyadic is equal to zero. What remains is Eq. (1).

Example: When $V(\underline{R})$ is available in the form of an explicit function $V^*(R,\lambda,\beta)$ of the spherical coordinates R,λ,β shown in Fig. 2.16.2, one can find $\tilde{\underline{M}}$ as follows:

Differentiate Eq. (2.16.16) to verify that

$$\underline{\nabla\nabla} V(R) = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}\ \underline{a}_i \underline{a}_j$$

where Q_{11}, Q_{22}, and Q_{33} are given in Eqs. (2.16.10 - 2.16.12) and

$$Q_{12} = Q_{21} \triangleq \sec\beta \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial V^*}{\partial \lambda}\right)$$

$$Q_{23} = Q_{32} \triangleq \frac{1}{R^2}\frac{\partial}{\partial \beta}\left(\sec\beta \frac{\partial V^*}{\partial \lambda}\right)$$

$$Q_{31} = Q_{13} \triangleq \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial V^*}{\partial \beta}\right)$$

Next, let

$$I_{jk} \triangleq \underline{a}_j \cdot \underline{I} \cdot \underline{a}_k \qquad (j,k=1,2,3)$$

Then

$$\begin{aligned}\tilde{\underline{M}} \underset{(1)}{=} &\left[I_{12}Q_{31} - Q_{12}I_{31} + (I_{22}-I_{33})Q_{23} - (Q_{22}-Q_{33})I_{23}\right]\underline{a}_1 \\ &+ \left[I_{23}Q_{12} - Q_{23}I_{12} + (I_{33}-I_{11})Q_{31} - (Q_{33}-Q_{11})I_{31}\right]\underline{a}_2 \\ &+ \left[I_{31}Q_{23} - Q_{31}I_{23} + (I_{11}-I_{22})Q_{12} - (Q_{11}-Q_{22})I_{12}\right]\underline{a}_3\end{aligned}$$

2.20 Gravitational potential energy

In the context of Lagrangian or Hamiltonian mechanics, the phrase "potential energy"* refers to a function of the generalized coordinates q_r $(r=1,\ldots,n)$ of a system S (in an inertial reference frame A) such that

$$F_r = -\frac{\partial \Phi}{\partial q_r} \quad (r=1,\ldots,n) \tag{1}$$

where F_r, the generalized force corresponding to q_r, is defined in terms of $\underline{p}_i$, the position vector of a typical particle P_i of S relative to a point fixed in A, and $\underline{F}_i$, the resultant of all contact and body forces acting on P_i, as

$$F_r \triangleq \sum_{i=1}^{N} \frac{\partial \underline{p}_i}{\partial q_r} \cdot \underline{F}_i \qquad (r=1,\ldots,n) \tag{2}$$

Here N is the number of particles and the partial differentiations are performed in reference frame A. In some situations, Φ is simply the negative of one of the force functions discussed in Secs. 2.10 - 2.19. For example, this is the case when S consists of a single particle P of mass m moving under the action of the gravitational force exerted by a particle $\bar{P}$ of mass $\bar{m}$ fixed in A. Φ is then given by

$$\Phi = -(G\bar{m}mp^{-1} + C) \tag{3}$$

where G, p, and C have precisely the same meaning as in Eq. (2.10.2). Eq. (3) also furnishes the potential energy when S consists of two particles moving under the action of their mutual gravitational attraction. In other situations, Φ cannot be found so simply, but some of the force functions of Secs. 2.10-2.14 are, nevertheless, useful, for they provide contributions to Φ. For example, if S consists of a rigid body B of mass m and two particles P_1 and P_2, of masses m_1 and m_2, respectively, and if these three objects move

---

*The symbol Φ is used for potential energy rather than the more familiar V, to avoid confusion with the use of V for the force function in Secs. 2.10-2.19.

under the action of forces of mutual gravitational attraction, then Φ can be expressed as

$$\Phi = -\left(\Phi_0+\Phi_1+\Phi_2\right) \tag{4}$$

with [see Eq. (2.10.2)]

$$\Phi_o = Gm_1m_2p_o^{-1} + C_o \tag{5}$$

and [see Eq. (2.11.5)]

$$\Phi_i = Gm_i\int p_i^{-1}\ \rho d\tau + C_i \qquad (i=1,2) \tag{6}$$

where p_o is the distance between P_1 and P_2, p_i is the distance from P_i to a generic point P of B, ρ is the mass density of B at P, and C_o, C_1 and C_2 are arbitrary constants.

Force functions associated with approximations to gravitational forces and moments can at times be used in the formulation of a potential energy Φ; but it can occur that apparently relevant force functions exist when Φ does not exist. For example, let S consist of a rigid body B of mass m moving under the action of gravitational forces exerted by a particle $\bar{P}$ of mass $\bar{m}$ fixed in A, and suppose that the resultant gravitational force $\underline{F}$ exerted on B by P is approximated with $\tilde{\underline{F}}$ as defined in Eq. (2.3.6), while the moment $\underline{M}$ about the mass center B^* of B of all forces exerted on B by P is approximated with $\tilde{\underline{M}}$ as defined in Eq. (2.6.3). Then the potential energy Φ of S can be expressed as

$$\Phi = -\tilde{V} \tag{7}$$

with $\tilde{V}$ given by Eq. (2.12.13); but if $\underline{F}$ is approximated with $\hat{\underline{F}}$ as defined in Eq. (2.3.5), while $\underline{M}$ is approximated with $\tilde{\underline{M}}$, then (as shown for a specific case in the <u>Example</u> following) there exists no function Φ that satisfies Eq. (1), despite the fact that force functions for $\hat{\underline{F}}$ and $\tilde{\underline{M}}$ do exist in the sense that $\hat{\underline{F}}$ and $\tilde{\underline{M}}$ can be expressed as

$$\hat{\underline{F}} = \underline{\nabla}_{\underline{R}} V_1 \tag{8}$$

and

$$\tilde{\underline{M}} = -\underline{R} \times \underline{\nabla}_{\underline{R}} V_2 \tag{9}$$

where $\underline{R}$ is the position vector of B^* relative to $\bar{P}$ while

$$V_1 = G\bar{m}mR^{-1} \tag{10}$$

and

$$V_2 = G\bar{m}mR^{-1}v^{(2)} \tag{11}$$

with $v^{(2)}$ as given in Eq. (2.12.2). (The non-existence of a potential energy under these circumstances may be an inconvenience, but it does not render the approximations under consideration invalid. They are, in fact, particularly useful, and are, therefore, employed extensively, in the analysis of motions of space vehicles).

<u>Derivations</u>: The resultant of all contact and body forces acting on a single particle P of mass m moving under the action of the gravitational force exerted by a particle $\bar{P}$ of mass $\bar{m}$ is given by

$$\underline{F} \underset{(2.1.1)}{=} -G\bar{m}m\underline{p}(\underline{p}^2)^{-3/2} \tag{12}$$

Consequently

$$F_r \underset{(2)}{=} \frac{\partial \underline{p}}{\partial q_r} \cdot \underline{F} \underset{(12)}{=} -G\bar{m}m \frac{\partial \underline{p}}{\partial q_r} \cdot \underline{p}(\underline{p}^2)^{-3/2}$$

$$= \frac{\partial}{\partial q_r}\left[G\bar{m}m(\underline{p}^2)^{-1/2}\right] = \frac{\partial}{\partial q_r} G\bar{m}mp^{-1}$$

$$\underset{(3)}{=} -\frac{\partial \Phi}{\partial q_r}$$

which shows that Φ as given in Eq. (3) is the potential energy of a system S consisting solely of P. Similarly, if S consists of both P and $\bar{P}$, the resultants $\underline{F}$ and $\bar{\underline{F}}$ of the forces acting on P and on $\bar{P}$ are given by

$$\underline{F} \underset{(2.1.1)}{=} -G\bar{m}m\left(\underline{p}_1-\underline{p}_2\right)\left[\left(\underline{p}_1-\underline{p}_2\right)^2\right]^{-3/2} \tag{13}$$

and by

$$\bar{\underline{F}} \underset{(2.1.2)}{=} -\underline{F} \tag{14}$$

where $\underline{p}_1$ and $\underline{p}_2$ are respectively the position vectors of P and $\bar{P}$ relative to a point fixed in A; and

$$F_r \underset{(2)}{=} \frac{\partial \underline{p}_1}{\partial q_r}\cdot\underline{F} + \frac{\partial \underline{p}_2}{\partial q_r}\cdot\bar{\underline{F}} \underset{(14)}{=} \left(\frac{\partial \underline{p}_1}{\partial q_r} - \frac{\partial \underline{p}_2}{\partial q_r}\right)\cdot\underline{F}$$

$$= \left[\frac{\partial}{\partial q_r}\left(\underline{p}_1-\underline{p}_2\right)\right]\cdot\underline{F} \underset{(13)}{=} -G\bar{m}m\left[\frac{\partial}{\partial q_r}\left(\underline{p}_1-\underline{p}_2\right)\right]\cdot\left(\underline{p}-\bar{\underline{p}}\right)\left[\left(\underline{p}_1-\underline{p}_2\right)^2\right]^{-3/2}$$

or, since $\underline{p}_1-\underline{p}_2 = \underline{p}$,

$$F_r = -G\bar{m}m\,\frac{\partial \underline{p}}{\partial q_r}\cdot\underline{p}\left(\underline{p}^2\right)^{-3/2} = \frac{\partial}{\partial q_r}\left(G\bar{m}mp^{-1}\right) \underset{(3)}{=} -\frac{\partial \Phi}{\partial q_r}$$

so that Φ as given in Eq. (3) is once again the potential energy of S.

When S consists of a rigid body B of mass m and two particles, P_1 and P_2, of masses m_1 and m_2, the resultants $\underline{F}_1$ and $\underline{F}_2$ of all contact and body forces acting on P_1 and P_2 are respectively [see Eqs. (2.1.1), (2.1.2),(2.2.2)]

$$\underline{F}_1 = -Gm_1m_2\,\underline{p}_o\left(\underline{p}_o^2\right)^{-3/2} + Gm_1\int \underline{p}_1\left(\underline{p}_1^2\right)^{-3/2}\rho d\tau \tag{15}$$

and

$$\underline{F}_2 = Gm_1m_2\,\underline{p}_o\left(\underline{p}_o^2\right)^{-3/2} + Gm_2\int \underline{p}_2\left(\underline{p}_2^2\right)^{-3/2}\rho d\tau \tag{16}$$

where $\underline{p}_o$ is the position vector of m_1 relative to m_2, and $\underline{p}_1$ and $\underline{p}_2$ are the position vectors of a generic point P of B relative to P_1 and P_2 as shown in Fig. 2.20.1. The contribution to F_r of forces acting on P_1 and P_2 is given by [see Fig. 2.20.1 for $\underline{R}_1$ and $\underline{R}_2$, and use Eq. (2)].

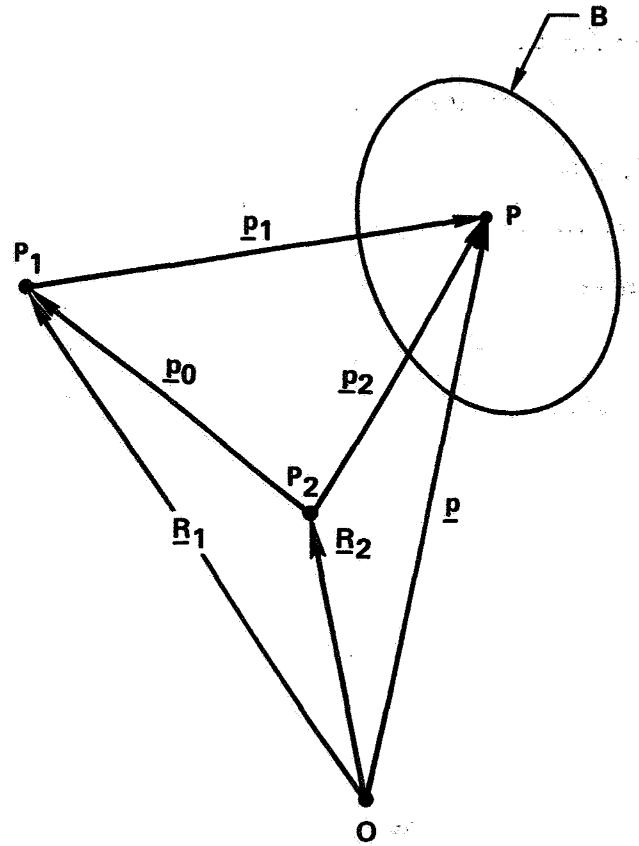

B
P
$\underline{p}_1$
P_1
$\underline{p}_0$
$\underline{p}_2$
P_2
$\underline{R}_1$
$\underline{R}_2$
$\underline{p}$
O

Figure 2.20.1

$$\frac{\partial \underline{R}_1}{\partial q_r} \cdot \underline{F}_1 + \frac{\partial \underline{R}_2}{\partial q_r} \cdot \underline{F}_2 \underset{(15,16)}{=} -Gm_1 m_2 \frac{\partial \underline{p}_o}{\partial q_r} \cdot \underline{p}_o \left(\underline{p}_o^2\right)^{-3/2}$$

$$+ Gm_1 \int \frac{\partial \underline{R}_1}{\partial q_r} \cdot \underline{p}_1 \left(\underline{p}_1^2\right)^{-3/2} \rho d\tau$$

$$+ Gm_2 \int \frac{\partial \underline{R}_2}{\partial q_r} \cdot \underline{p}_2 \left(\underline{p}_2^2\right)^{-3/2} \rho d\tau \qquad (17)$$

As for forces acting on the particles of B, these either are forces exerted by P_1 and P_2, or they are forces exerted by particles of B on each other. Since B is a rigid body, the latter forces make no contribution to F_r (see Kane, Dynamics, Holt, Rinehart and Winston, p. 81), and a typical force of the former kind can be expressed as

$$d\underline{F} \underset{(2.1.1)}{=} -G\left[m_1 \underline{p}_1 \left(\underline{p}_1^2\right)^{-3/2} + m_2 \underline{p}_2 \left(\underline{p}_2^2\right)^{-3/2}\right] \rho d\tau \qquad (18)$$

The contribution of this force to F_r is thus given by [see Fig. 2.20.1 for $\underline{p}$, and use Eq. (2)]

$$\frac{\partial \underline{p}}{\partial q_r} \cdot d\underline{F} \underset{(18)}{=} -G\left[m_1 \frac{\partial \underline{p}}{\partial q_r} \cdot \underline{p}_1 \left(\underline{p}_1^2\right)^{-3/2} + m_2 \frac{\partial \underline{p}}{\partial q_r} \cdot \underline{p}_2 \left(\underline{p}_2^2\right)^{-3/2}\right] \rho d\tau$$

$$= -G\left[m_1 \frac{\partial \underline{R}_1}{\partial q_r} \cdot \underline{p}_1 \left(\underline{p}_1^2\right)^{-3/2} + m_1 \frac{\partial \underline{p}_1}{\partial q_r} \cdot \underline{p}_1 \left(\underline{p}_1^2\right)^{-3/2}\right] \rho d\tau$$

$$-G\left[m_2 \frac{\partial \underline{R}_2}{\partial q_r} \cdot \underline{p}_2 \left(\underline{p}_2^2\right)^{-3/2} + m_2 \frac{\partial \underline{p}_2}{\partial q_r} \cdot \underline{p}_2 \left(\underline{p}_2^2\right)^{-3/2}\right] \rho d\tau \qquad (19)$$

and F_r can now be formulated as

$$F_r \underset{(2)}{=} \frac{\partial \underline{R}_1}{\partial q_r} \cdot \underline{F}_1 + \frac{\partial \underline{R}_2}{\partial q_r} \cdot \underline{F}_2 + \int \frac{\partial \underline{p}}{\partial q_r} \cdot d\underline{F}$$

$$\underset{(17,19)}{=} -Gm_1m_2 \frac{\partial \underline{p}_o}{\partial q_r} \cdot \underline{p}_o\left(\underline{p}_o^2\right)^{-3/2}$$

$$- Gm_1 \int \frac{\partial \underline{p}_1}{\partial q_r} \cdot \underline{p}_1\left(\underline{p}_1^2\right)^{-3/2} \rho d\tau - Gm_2 \int \frac{\partial \underline{p}_2}{\partial q_r} \cdot \underline{p}_2\left(\underline{p}_2^2\right)^{-3/2} \rho d\tau$$

$$= \frac{\partial}{\partial q_r}\left[Gm_1m_2\left(\underline{p}_o^2\right)^{-1/2} + Gm_1 \int \left(\underline{p}_1^2\right)^{-1/2} \rho d\tau + Gm_2 \int \left(\underline{p}_2^2\right)^{-1/2} \rho d\tau\right] \tag{20}$$

or, with

$$p_j \triangleq \left(\underline{p}_j^2\right)^{1/2} \qquad (j = 0,1,2)$$

as

$$F_r \underset{(20)}{=} \frac{\partial}{\partial q_r}\left[Gm_1m_2\, p_o^{-1} + Gm_1 \int p_1^{-1} \rho d\tau + Gm_2 \int p_2^{-1} \rho d\tau\right]$$

$$\underset{(5,6)}{=} \frac{\partial}{\partial q_r}\left(\Phi_o + \Phi_1 + \Phi_2\right) \underset{(4)}{=} - \frac{\partial \Phi}{\partial q_r}$$

Hence, Φ as given by Eq. (4) is the potential energy of S.

To form the generalized force associated with the force system presumed to be acting on a rigid body B in connection with Eq. (17), one can begin by expressing F_r as (see Kane, Dynamics, Holt, Rinehart and Winston, p. 81)

$$F_r = \tilde{\underline{F}} \cdot \frac{\partial}{\partial q_r}\left(R\underline{a}_1\right) + \tilde{\underline{M}} \cdot \frac{\partial \underline{\omega}}{\partial \dot{q}_r} \tag{21}$$

where R and $\underline{a}_1$ are defined as in Sec. 2.13, while $\underline{\omega}$, the angular velocity of B in reference frame A, is regarded as a function of q_r and $\dot{q}_r$ (r=1,...,n).. Using Eqs. (2.3.6) and (2.6.3), one thus obtains

$$F_r \underset{(21)}{=} - \frac{G\bar{m}m}{R^2}\left(\underline{a}_1+\underline{f}^{(2)}\right) \cdot \frac{\partial}{\partial q_r}\left(R\underline{a}_1\right) + \frac{3G\bar{m}}{R^3}\left(\underline{a}_1\times\underline{I}\cdot\underline{a}_1\right) \cdot \frac{\partial \underline{\omega}}{\partial \dot{q}_r} \tag{22}$$

or, equivalently,

$$F_r \underset{(22)}{=} -\frac{G\bar{m}m}{R^2}\left[\underline{a}_1 \cdot \frac{\partial}{\partial q_r}\left(R\underline{a}_1\right) + \underline{f}^{(2)} \cdot \frac{\partial}{\partial q_r}\left(R\underline{a}_1\right)\right]$$

$$+ \frac{3G\bar{m}}{R^3}\left(\frac{\partial \underline{\omega}}{\partial \dot{q}_r} \times \underline{a}_1\right) \cdot \left(\underline{I} \cdot \underline{a}_1\right) \tag{23}$$

Now,

$$\frac{\partial}{\partial q_r}\left(R\underline{a}_1\right) = \frac{\partial R}{\partial q_r}\underline{a}_1 + R\frac{\partial \underline{a}_1}{\partial q_r} \tag{24}$$

and, using a superscript B to denote differentiation in B, one can write

$$\frac{\partial \underline{a}_1}{\partial q_r} = \frac{{}^B\partial \underline{a}_1}{\partial q_r} + \frac{\partial \underline{\omega}}{\partial \dot{q}_r} \times \underline{a}_1 \tag{25}$$

so that

$$\left(\frac{\partial \underline{\omega}}{\partial \dot{q}_r} \times \underline{a}_1\right) \cdot \left(\underline{I} \cdot \underline{a}_1\right) \underset{(25)}{=} \frac{\partial \underline{a}_1}{\partial q_r} \cdot \underline{I} \cdot \underline{a}_1 - \frac{{}^B\partial \underline{a}_1}{\partial q_r} \cdot \underline{I} \cdot \underline{a}_1 \tag{26}$$

or, since $\underline{I}$ is independent of q_r (r=1 ,..., n) in B,

$$\left(\frac{\partial \underline{\omega}}{\partial \dot{q}_r} \times \underline{a}_1\right) \cdot \left(\underline{I} \cdot \underline{a}_1\right) \underset{(26)}{=} \frac{\partial \underline{a}_1}{\partial q_r} \cdot \underline{I} \cdot \underline{a}_1 - \frac{1}{2}\frac{{}^B\partial}{\partial q_r}\left(\underline{a}_1 \cdot \underline{I} \cdot \underline{a}_1\right)$$

$$= \frac{\partial \underline{a}_1}{\partial q_r} \cdot \underline{I} \cdot \underline{a}_1 - \frac{1}{2}\frac{\partial I_{11}}{\partial q_r} \tag{27}$$

where I_{11} is the moment of inertia of B about the line passing through B^* and parallel to $\underline{a}_1$. Substituting from Eqs. (24) and (27) into Eq. (23), one thus arrives at

$$F_r = \frac{-G\bar{m}m}{R^2}\left[\frac{\partial R}{\partial q_r} + R\underline{a}_1 \cdot \frac{\partial \underline{a}_1}{\partial q_r} + \frac{\partial R}{\partial q_r}\underline{a}_1 \cdot \underline{f}^{(2)} + R\frac{\partial \underline{a}_1}{\partial q_r} \cdot \underline{f}^{(2)}\right] \tag{28}$$

$$+ \frac{3G\bar{m}}{R^3}\left(\frac{\partial \underline{a}_1}{\partial q_r} \cdot \underline{I} \cdot \underline{a}_1 - \frac{1}{2}\frac{\partial I_{11}}{\partial q_r}\right) \tag{28}$$

The vector $\partial\underline{a}_1/\partial q_r$ is necessarily perpendicular to $\underline{a}_1$. Consequently the dot-product of this vector with $\underline{a}_1$ vanishes. Expressing $\underline{f}^{(2)}$ as in Eq. (2.3.3), one can thus rewrite Eq. (28) as

$$F_r = -\frac{G\bar{m}m}{R^2}\frac{\partial R}{\partial q_r} - \frac{3G\bar{m}}{2R^3}\left\{\frac{1}{R}\frac{\partial R}{\partial q_r}\left[tr(I)-3I_{11}\right]+\frac{\partial I_{11}}{\partial q_r}\right\}$$

$$\underset{(2.12.2)}{=} \frac{\partial\tilde{V}}{\partial q_r}$$

which shows that Φ as given in Eq. (7) is the potential energy in question.

Example: Fig. 2.20.2 shows a thin rod B of mass m and a particle $\bar{P}$ of mass $\bar{m}$, the same system previously considered in the Examples in Sec. 2.3 and 2.11. If B is constrained to remain in a plane fixed in an inertial reference frame A, and if $\bar{P}$ is fixed in this plane, then the position and orientation of B in A can be described in terms of the generalized coordinates $q_1 = \theta$, $q_2 = R$, and $q_3 = \psi$. Suppose now that $\underline{F}$, the resultant force exerted by $\bar{P}$ on B, is approximated as

$$\underline{F}\approx\underline{\tilde{F}} \underset{(2.3.6)}{=} -\frac{G\bar{m}m}{R^2}\left[\underline{a}_1 + \frac{L}{8R^2}\left(2-3\sin^2\psi\right)\underline{a}_1 - \frac{L^2}{8R^2}\sin 2\psi\,\underline{a}_2\right] \tag{29}$$

while $\underline{M}$, the moment about B^* of the forces exerted on B by $\bar{P}$, is approximated with

$$\underline{M}\approx\underline{\tilde{M}} \underset{(2.6.3)}{=} -\frac{G\bar{m}mL^2}{8R^3}\left(\sin 2\psi\,\underline{a}_3\right) \tag{30}$$

Then the generalized forces corresponding to θ, R, ψ are

$$F_\theta = 0 \tag{31}$$

$$F_R = -\frac{G\bar{m}m}{R^2}\left[1 + \frac{L}{8R^2}\left(2-3\sin^2\psi\right)\right] \tag{32}$$

$$F_\psi = -\frac{G\bar{m}mL^2}{8R^2}\sin 2\psi \tag{33}$$

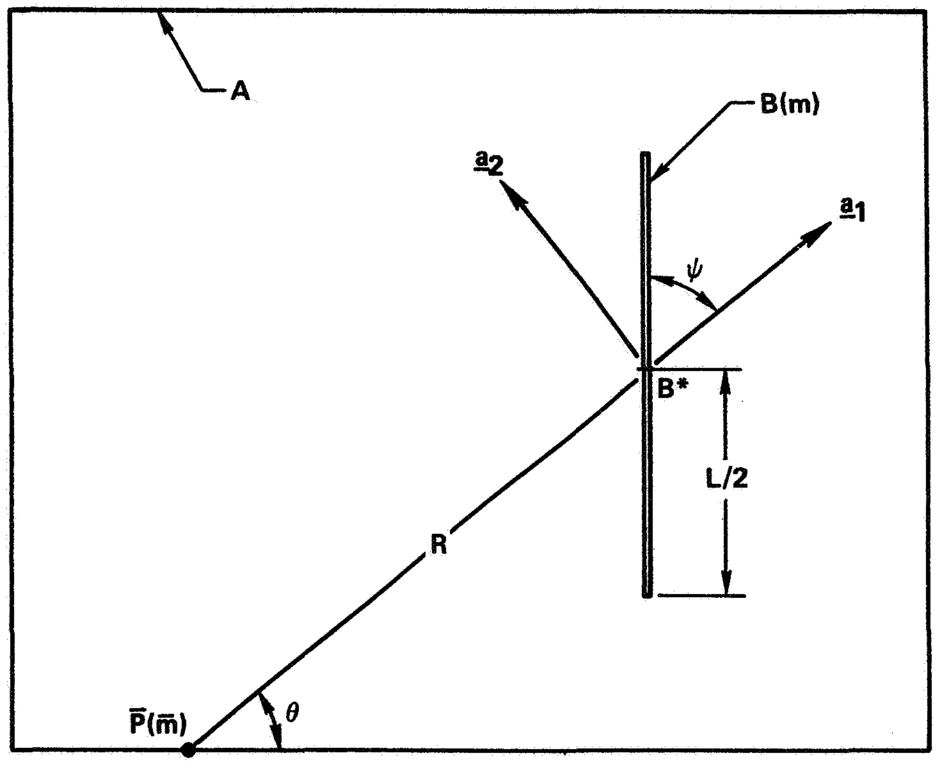


Figure 2.20.2

and, if Φ is taken to be the negative of the function $\tilde{V}$ found in the Example in Sec. 2.11, that is,

$$\Phi = -\frac{G\bar{m}m}{R}\left[1 + \frac{L^2}{24R^2}\left(2-3\sin^2\psi\right)\right] - C$$

Then, as may be verified by carrying out the indicated differentiations,

$$F_\theta = -\frac{\partial\Phi}{\partial\theta},\ F_R = -\frac{\partial\Phi}{\partial R},\ F_\psi = -\frac{\partial\Phi}{\partial\psi} \tag{34}$$

But, if instead of approximating $\underline{F}$ as in Eq. (29) one uses

$$\underline{F} \approx \hat{\underline{F}} = -\frac{G\bar{m}m}{R^2}\underline{a}_1 \tag{35}$$

while continuing to approximate $\underline{M}$ as in Eq. (30), then the generalized forces become

$$F_\theta = F_\psi = -\frac{G\bar{m}mL^2}{8R^3}\sin 2\psi,\ F_R = -\frac{G\bar{m}m}{R^2}$$

and there exists no function Φ of θ, R and ψ such that Eqs. (34) are satisfied simultaneously. Finally, if additional constraints are introduced so as to force B^* to move with a prescribed velocity, which can be accomplished by applying suitable contact forces at B^*, then θ and R cease to be generalized coordinates; F_ψ, the generalized force for what is now a one-degree-of-freedom system, is given by Eq. (33), both when $\underline{F}$ is approximated as in Eq. (29) and when Eq. (35) is used; and a potential energy Φ thus exists in both cases because F_ψ can be expressed as $F_\psi = -\partial\Phi/\partial\psi$ if one lets

$$\Phi = -\frac{G\bar{m}mL^2}{16R^2}\cos 2\psi + C$$

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