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Effect of Adhesive Layer Elasticity on the Fracture Mechanics of a Blister Test Specimen

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ABSTRACT

An analytical model of a blister type specimen for evaluating adhesive bond strength is developed. Plate theory with shear deformation is used to model the deformation of the plate, and elastic deformation of the adhesive layer is taken into account. It is shown that the inclusion of the elastic deformation of the adhesive layer can have a significant influence in the energy balance calculations of fracture mechanics.

INTRODUCTION

The increasing use of adhesives in the development of high strength composite materials and in the joining of structural components has brought about the need for standard methods of testing the strength properties of adhesives. One such test proposed for use under a variety of environmental conditions involves a blister specimen subjected to pressure loading [1], [2]. This test employs a circular plate, usually of uniform thickness, bonded through a concentric annular adhesive layer to a flat supporting surface. During testing, the plate is loaded by means of fluid pressure applied to the unbonded concentric center portion of the plate. Figure 1 shows a sketch of a circular plate blister specimen of radius b bonded over the annular area $a \leq x \leq b$ to the rigid supporting surface. After a test to initial debonding or to failure, the specific adhesive fracture energy γ_a of the bond is calculated by means of an energy balance, which relates γ_a to the pressure loading at initiation of debonding and the elastic and geometric properties of the specimen.

The curves given in [2] for determining γ_a are calculated by either plate theory or the finite element method. In these calculations it is assumed that the adhesive layer is so thin that the plate may be considered to be bonded rigidly to the support neglecting the compliance of the adhesive layer. Such an assumption is reasonable when the elastic modulus of the plate has the same order of magnitude as that of the adhesive and the thickness of the adhesive layer is small in comparison with the plate thickness. However, when the circular plate is metal, the adhesive can have a much lower elastic modulus. In such a case the compliance of the adhesive layer may no longer be neglected in the energy balance calculations.

The purpose of this paper is to investigate the effect of the adhesive layer compliance on the fracture mechanics of thin plate blister specimens. Since linear plate theory is employed, the results are restricted to small values of h/a. Also, the maximum plate deflection must be somewhat less than the plate thickness. Because of the similarities of analysis of circular plate bending and cylindrical bending under plane strain conditions, both cases are treated in this paper.

ANALYSIS

The present analysis treats the plate using equations of a plate theory which includes deformations of bending, in-plane extension, and transverse shear. For the one dimensional problems of cylindrical bending (plane strain) or of axisymmetric deformation of a plate the displacement field is described completely by means of the variables u, w, and β. The stresses in the plate are related to the stress resultants and stress couples N_1, N_2, Q, M_1, and M_2 through the usual formulas of plate theory, where the in-plane normal stress varies linearly across the thickness, the transverse shearing stress varies parabolically, and the transverse normal stress is considered to be small. The independent variable x is shown in Figure 1. Plate theory equilibrium equations are

$$dN_1/dx = \tau + c_1(N_2 - N_1)/x \qquad (1)$$

$$dQ/dx = -q - c_1 Q/x \qquad (2)$$

$$dM_1/dx = m + Q + c_1(M_2 - M_1)/x \qquad (3)$$

where c_1 is equal to 0 for the plane strain case and equal to 1 for the axisymmetric case. The distributed moment m is given by

$$m = \bar{h}\tau \qquad (4)$$

where $\bar{h}$ us equal to h/2 if coupling between bending and extension is considered and is 0 if this effect is neglected. The strain displacement equations are

$$\epsilon_2 = c_1 u/x \quad , \quad du/dx = \epsilon_1 \qquad (5)$$

$$dw/dx = \beta - \gamma \tag{6}$$

$$\kappa_2 = c_1\beta/x \quad , \quad d\beta/dx = \kappa_1 \tag{7}$$

These equations are coupled through the stress-strain relations

$$N_1 = K(\varepsilon_1 + \nu\varepsilon_2) \tag{8}$$

$$N_2 = K(\varepsilon_2 + \nu\varepsilon_1) \tag{9}$$

$$\gamma = c_2(Q + m/6)/C \tag{10}$$

$$M_1 = D(\kappa_1 + \nu\kappa_2) \tag{11}$$

$$M_2 = D(\kappa_2 + \nu\kappa_1) \tag{12}$$

where the plate rigidities K, C, and D are given by

$$K = Eh/(1 - \nu^2) \tag{13}$$

$$C = 5Gh/6 = 5(1 - \nu)K/12 \tag{14}$$

$$D = Kh^2/12 \tag{15}$$

and the constant c_2 is to be set equal to 1 if transverse shear deformation is included or set equal to 0 if it is ignored.

The thin adhesive layer has an elastic modulus smaller than that of the plate; therefore, it is reasonable to ignore the in-plane stress components in the adhesive layer when computing equilibrium of the plate and adhesive layer together. Hence, the adhesive layer is treated as a distributed spring or elastic foundation which transmits normal and shear stresses between the plate and the rigid support. Accordingly, the surface loads on the bonded portion of the plate are related to the plate displacements by means of the expressions

$$q = \sigma = k_N w \tag{16}$$

for the normal component and

$$\tau = k_s(u + \bar{h}\beta) \tag{17}$$

for the shear components. The corresponding surface loads on the unbonded portion of the plate are

$$q = -p \tag{18}$$

and

$$\tau = 0 \tag{19}$$

respectively.

The elastic foundation rigidity constants are

$$k_N = B_o/h_o \tag{20}$$

for the normal stress and

$$k_s = G_o/h_o \tag{21}$$

for the shear stress. The elastic constant B_o is equal to E_o if the normal stress is assumed to be uniaxial, is equal to $E_o/(1 - \nu_o^2)$ if one in-plane stress and one in-plane strain vanish, and is equal to $(1 - \nu_o)E_o/(1 + \nu_o)(1 - 2\nu_o)$ if both in-plane strains are assumed to be zero.

If coupling of bending and extension is neglected ($\bar{h} = 0$) and transverse shear deformation is ignored ($c_2 = 0$), the governing equations of the plane strain problem reduce to

$$w'''' = p/D \tag{22}$$

for the unbonded portion of the plate and to

$$w'''' = -w/\ell^4 \tag{23}$$

in the bonded portion. Here, primes denote differentiation with respect to x. The length ℓ, which characterizes the stiffness ratio of the plate to the elastic foundation, is given by

$$\ell = (D/k_N)^{\frac{1}{4}} \tag{24}$$

The general solutions of (22) and (25) are

$$w = A_1 + A_2x$$

$$+ A_3x^2 + A_4x^3 + px^4/24D \tag{25}$$

and

$$w = A_5e^{\alpha x} \cos\alpha x + A_6e^{\alpha x} \sin\alpha x$$

$$+ A_7e^{-\alpha x} \cos\alpha x + A_8e^{-\alpha x} \sin\alpha x \tag{26}$$

respectively, where the quantities A_i are arbitrary constants and

$$\alpha = \sqrt{2}/\ell \ .$$

The constants A_i are found from boundary conditions

$$\beta = 0 \ , \ Q = 0 \ , \ \text{at } x = 0$$

$$M_1 = 0 \ , \ Q = 0 \ , \ \text{at } x = b$$

and matching conditions of w, β, M_1, and Q at $x = a$. If $(b - a)/\ell \gtrsim 3$, as would be the case for many practical blister test specimens, the constants A_5 and A_6 become negligible and the boundary conditions at $x = b$ have negligible influence on the solution.

Once the solution for $w(x)$ has been found, one can evaluate the maximum normal stress in the adhesive layer using

$$\sigma_{max} = k_N w(a) \tag{27}$$

Following through the calculations one obtains the expression

$$\sigma_{max}/p = (a^2/3\ell^2)\omega_1 \tag{28}$$

where

$$\omega_1 = (1 + 3\sqrt{2}\,\ell/a + 3\ell^2/a^2)/(1 + \sqrt{2}\ell/a)$$

For the axisymmetric problem the system of equations reduces to

$$x^{-1}\{x[x^{-1}(xw')']'\}' = p/D \tag{29}$$

for $0 \leq x < a$ and to

$$x^{-1}\{x[x^{-1}(xw')']'\}' = -w/\ell^4 \tag{30}$$

for $a < x \leq b$ when $\bar{h} = 0$ and $c_1 = 0$.

The general solutions of (29) and (30) are [3]

$$w = B_1 x^2 \log x + B_2 \log x$$
$$+ B_3 x^2 + B_4 + px^4/64D \tag{31}$$

and

$$w = B_5 \, \mathrm{ber}(x/\ell) + B_6 \, \mathrm{bei}(x/\ell)$$

$$+ B_7 \, \mathrm{ker}(x/\ell) + B_8 \, \mathrm{kei}(x/\ell) \qquad (32)$$

respectively, where ber, bei, ker, and kei are zero order Kelvin functions.

The boundary conditions that β and Q vanish at $x = 0$ require that $B_1 = 0$ and $B_2 = 0$. The other constants are found from the boundary conditions

$$M_1 = 0 \quad , \quad Q = 0 \quad \text{at } x = b$$

and matching conditions of w, β, M_1, and Q at $x = a$. If $(b - a)/\ell \gtrsim 3$, the constants B_5 and B_6 may be set equal to zero, and the plate may be treated as infinite; hence, the length b will not appear in the solution. Also, if $a/\ell >> 1$, some simplification is achieved by using one term asymptotic formulas for the Kelvin functions. It is expected that these conditions will be satisfied for many practical blister test specimens.

Calculations of the maximum normal stress in the adhesive layer show that its value is given by

$$\sigma_{max}/p = (a^2/8\ell^2)\omega_2 \qquad (33)$$

where

$$\omega_2 = [f_3 - 4(\ell/a)f_1 + 8(\ell/a)^2 f_2]/[f_2 - 2(\ell/a)f_4]$$

$$f_1 = \mathrm{ker}^2(a/\ell) + \mathrm{kei}^2(a/\ell)$$

$$f_2 = \mathrm{ker}(a/\ell)\mathrm{kei}'(a/\ell) - \mathrm{kei}(a/\ell)\mathrm{ker}'(a/\ell)$$

$$f_3 = \mathrm{ker}(a/\ell)\mathrm{ker}'(a/\ell) + \mathrm{kei}(a/\ell)\mathrm{kei}'(a/\ell)$$

$$f_4 = \mathrm{ker}'^2(a/\ell) + \mathrm{kei}'^2(a/\ell)$$

$$(b - a)/\ell \geq 3$$

For cases where coupling of extension and bending effects is included and transverse shear deformation is considered, numerical solutions to the governing system of differential equations and boundary conditions may be obtained by means of any one of the numerical integration techniques which have been applied successfully to the analysis of shells of revolution. Results reported later in this paper were obtained by using a multi-segment numerical integration technique [4] to calculate the plate displacements, stress results, stress couples, and elastic foundation reactions at discrete x values.

CALCULATION OF $\partial U/\partial A$

Bennett, Devries, and Williams [1] have proposed the adhesive fracture criteria

$$\partial U/\partial A = \gamma_a \tag{34}$$

for a blister test specimen under pressure loading. Here U is the total strain energy in the elastic system and A is the debonded area, which is equal to the length a for a plane specimen of unit width and equal to πa^2 for the circular specimen. The parameter γ_a is the specific adhesive fracture energy, representing the energy required to debond a unit area.

For a linearly elastic system the strain energy U may be computed from values of the pressure loading and the displacements of the pressurized portion of the plate according to

$$2U = \int_0^a pw \, dx \tag{35}$$

for the plane problem and

$$2U = 2\pi \int_0^a pw \, x \, dx \tag{36}$$

for the axisymmetric problem. The displacement w must be found by solving a boundary value problem. In [1], classical plate theory for thin specimens and a finite element method for thicker specimens are used to calculate $\partial U/\partial A$ for a plate bonded rigidly to a rigid support; i.e. the compliance of the adhesive layer is neglected. Using deflection expressions of classical plate theory the values of $\partial U/\partial A$ turn out to be

$$\partial U/\partial A = p^2a^4/18D \tag{37}$$

for the plane specimen and [1]

$$\partial U/\partial A = p^2a^4/128D \tag{38}$$

for the circular plate. In order to apply the fracture criterion (34) to specimens with compliant adhesive it is necessary to calculate $\partial U/\partial A$ for the plate on elastic foundation model.

Although the evaluation of integrals (35) and (36) for U and the differentiation with respect to A could in principle be carried out numerically for the plate on an elastic adhesive layer, a much simpler method for computing $\partial U/\partial A$ for this case does exist. Its derivation is given in the Appendix.

Closed form expressions for $\partial U/\partial A$ may be obtained from (28) and (33) together with (A-8) of the Appendix ($\tau = 0$). The results are

$$\partial U/\partial A = (\omega_1 + 3\ell^2/a^2)^2 p^2 a^4/18D \tag{39}$$

for the plane strain case and

$$\partial U/\partial A = (\omega_2 + 8\ell^2/a^2)^2 p^2 a^4/128D \tag{40}$$

for the axisymmetric case. Here ω_1 and ω_2 are functions of ℓ/a defined previously in (28) and (33).

RESULTS AND DISCUSSION

Results of calculations of σ_{max}/p using (28) and (33) are given in Figure 2. The effect of the adhesive layer compliance on the calculated value of σ_{max}/p is indicated by the dependence on the dimensionless parameter ℓ/a. For small values of ℓ/a the values of σ_{max}/p reduce to

$$\sigma_{max}/p \approx (a/\ell)^2/n \tag{41}$$

where n = 8 for the axisymmetric case and n = 3 for the plane strain case. In terms of given dimensions and elastic constants of the plate and adhesive layer the ratio ℓ/a is

$$\ell/a = [E/(1 - \nu^2)B_0]^{\frac{1}{4}}[h^3 h_0/12a^4]^{\frac{1}{4}} \tag{42}$$

where the first factor indicates the influence of material properties and the second indicates the influence of geometry. From (41) and (42) one can see that σ_{max}/p becomes singular as $h_0 \to 0$, the order of the singularity being given by of the differential equations were obtained for aluminum plate-epoxy adhesive test specimens. Figure 4 shows typical curves for the stress distribution in the adhesive layer. Results of other cases are summarized in Table 1. In Table 1 the values enclosed in parantheses are based on the closed form solutions. Numerical analysis for cases with no coupling of bending and extension and without plate shear deformation show that the numerical analysis method produces results within one half percent of the closed form solutions. It appears that for specimens in the range included in Table 1 the closed form solutions for the ratio σ_{max}/p are about 10% too high. However, the inclusion of these second order effects appears to make negligible change in the calculated values of $\partial U/\partial A$, even for a ratio h/a as high as 0.25. This does not mean that the results of the closed form solutions are to be considered accurate for thick plates. For thick plates the hypothesis of plane sections, assumed in all the theories employed in this paper, may not be satisfied with sufficient accuracy. In fact, finite element solutions of [1] indicates that considerable deviations from this hypothesis exist for plates with rigid adhesive. Calculations show that the ratio τ_{max}/σ_{max} lies in the range 0.20 - 0.25 for all specimens reported in Table 1.

Consider now how the material properties of the plate and adhesive layer enter into the parameter ℓ/a. From (42) it is seen that the material factor is $[E/(1 - \nu^2)B_0]^{1/4}$, where the elastic constant B_0 for the adhesive layer depends on the Young's modulus, the Poisson's ratio, and an assumption regarding the in-plane stress or strain components within the layer. Of the possible in-plane stress or strain conditions to be assumed, it would seem that the assumption of no in-plane strain would be most appropriate where a low modulus adhesive is used together with a high modulus plate and support. In such a case, the normal strain in the adhesive layer is much larger than the in-plane components. Of course, this assumption is not valid right at the edge of the bond region where one of the stress conditions might be more appropriate; however, at distances on the order of h_0 away from the edge, it is expected that in-plane constraint of the adhesive layer is achieved within reasonable accuracy.

For the case of in-plane constraint of the adhesive, the material factor of the ℓ/a ratio becomes

$$[E/(1 - \nu^2)B_0]^{\frac{1}{4}} = k(E/E_0)^{\frac{1}{4}}$$

where

$$k = [(1 + \nu_0)(1 - 2\nu_0)/(1 - \nu^2)(1 - \nu_0)]^{\frac{1}{4}}$$

The factor k is not very sensitive to Poisson's ratio ν of the plate; however it becomes quite sensitive to Poisson's ratio ν_0 of the adhesive for values of ν_0 close to 0.5 as can be seen in Table 2.

Since the material factor appraoches zero for an incompressible adhesive, ℓ/a also approaches zero. For the case $\ell/a = 0$, the maximum normal stress σ_{max} in the adhesive layer becomes singular with the distributed reaction of the elastic foundation model of the adhesive layer becoming a concentrated line load at the edge. However, since there is no constraint of the shear deformation of the adhesive layer, this layer continues to behave as a distributed shear spring with finite values for τ_{max}.

CONCLUSIONS

A structural mechanics approach has been used to investigate the influence of the elasticity of the adhesive layer on the fracture mechanics parameters of a blister test specimen. It has been demonstrated that in some practical cases the effect of adhesive layer elasticity cannot be neglected. A parameter which serves to indicate the importance of adhesive layer compliance in practical calculations has been successfully identified and is presented in terms of the product of a material factor and a geometry factor.

While the efforts of this paper have concentrated on the blister test specimen under uniform pressure loading, the methods of analysis presented can also be applied in the case of a blister specimen subjected to displacement loading.

Some shortcomings of the present structural mechanics approach are its inability to determine an upper limit of the ratio h/a for which the results are valid and its inability to determine details of the stress distribution at points near the edge of the bond region. For this reason a theory of elasticity study of a blister test specimen with a compliant adhesive layer would be desirable. With these limitations in view, it is hoped that the results presented here will prove useful in the process of standardization of blister specimens for adhesive fracture testing.

ACKNOWLEDGMENT

This problem was suggested to the author by Professor F. Erdogan of Lehigh University. Helpful discussions with Professor Erdogan during the course of the investigation are hereby acknowledged.

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NOMENCLATURE

a	inner dimension (see Figure 1)
A	debonded area
b	outer dimension (see Figure 1)
B_o	elastic constant for adhesive layer
c_1, c_2	constants equal to 0 or 1
C	transverse shear stiffness of plate
D	bending stiffness of plate
E	Young's modulus of plate material
E_o	Young's modulus of adhesive layer
G_o	modulus of rigidity of adhesive layer
h	plate thickness
h_o	thickness of adhesive layer
$\bar{h}$	half thickness of plate or zero
k_N	adhesive layer transverse normal stiffness
k_s	adhesive layer shear stiffness
K	extensional stiffness of plate
ℓ	characteristic length
m	moment of surface loading on plate
M_1, M_2	plate bending moment components
N_1, N_2	plate membrane stress resultants
p	applied normal pressure
q	normal surface loading on plate
Q	transverse shear resultant in plate
u	in-plane displacement at middle surface of plate

NOMENCLATURE (continued)

U	total strain energy
w	transverse displacement of plate
x	coordinate (radius in axisymmetric problems)
β	rotation of normal in plate
γ	transverse shear strain in plate
γ_a	specific adhesive fracture energy
$\varepsilon_1, \varepsilon_2$	membrane strain components in plate
κ_1, κ_2	curvature change components in plate
ν	Poisson's ratio of plate material
ν_o	Poisson's ratio of adhesive layer
τ	shearing stress in adhesive layer
σ	normal stress in adhesive layer
ω_1, ω_2	function of ℓ/a

APPENDIX

Consider two configurations of the blister specimen subjected to the same value of applied pressure p. In configuration I with total strain energy U_I the adhesive layer acts as a continuous spring over the coordinate range $a < x < b$. In configuration II with total strain energy U_{II}, the adhesive layer is debonded over the range $a < x < a + \Delta a$ and acts as a continuous spring over the range $a + \Delta a < x < b$. Let ΔA be new area of debonding in passing from configuration I to configuration II. The derivative $\partial U/\partial A$ is then given by

$$\partial U/\partial A = \lim_{\Delta A \to 0} (U_{II} - U_I)/\Delta A \tag{A-1}$$

Let U^o be the elastic strain energy stored in the circular plate plus the strain energy stored in the continuous spring over the range $a + \Delta a < x < b$. The total elastic strain energy is then given by

$$U = U^o + (\sigma^2/2k_N + \tau^2/2k_s)\Delta A \tag{A-2}$$

where the last term represents the elastic strain energy of the portion of the adhesive layer in the range $a < x < a + \Delta a$. Noting that (σ,τ) have known values $(\sigma_{max},\tau_{max})$ in configuration I and values $(-p,0)$ in configuration II one can write

$$\begin{aligned} U_{II} - U_I = {} & U^o_{II} - U^o_I \\ & + (p^2/2k_N - \sigma_{max}{}^2/2k_N - \tau_{max}{}^2/2k_s)\Delta A \end{aligned} \tag{A-3}$$

The strain energy U^o is a function of the loads acting on the structure consisting of the entire circular plate and the portion of the adhesive layer in the range $a + \Delta a < x < b$. Thus,

$$U^o = U^o(p, F_N, F_s)$$

where F_N and F_s are the normal and tangential components, respectively, of the force transmitted to the structure at $x = a + \varepsilon$. These force components are given by

$$F_N = -\sigma \, \Delta A$$

$$F_s = -\tau \, \Delta A$$

since they are carried to the structure through the portion of the adhesive layer which debonds in passing from configuration I to configuration II. Since the pressure p remains constant during debonding, the change in U^o may be expressed as

$$U^o_{II} - U^o_I = (\partial U^o/\partial F_N)\Delta F_N + (\partial U^o/\partial F_s)\Delta F_s \qquad \text{(A-4)}$$

where the partial derivatives may be evaluated in configuration I. The force increment components in passing from configuration I to configuration II are

$$\Delta F_N = (\sigma_{max} \pm p)\Delta A$$
$$\Delta F_s = \tau_{max} \, \Delta A \qquad \text{(A-5)}$$

Now combining (A-1) through (A-5) and taking the limit $\Delta A \to 0$ results in

$$\partial U/\partial A = (\partial U^o/\partial F_N)(\sigma_{max} + p) + (\partial U^o/\partial F_s)\tau_{max}$$

$$+ p^2/2k_N - \sigma_{max}{}^2/2k_N - \tau_{max}{}^2/2k_s \qquad \text{(A-6)}$$

According to Castigliano's second theorem, the partial derivatives $\partial U^o/\partial F_N$ and $\partial U^o/\partial F_s$ are equal to the plate displacement components at the point of application of F_N and F_s. In turn, these displacement components may be expressed in terms of σ_{max} and τ_{max} through the adhesive layer compliance relationships. Thus,

$$\partial U^o/\partial F_N = \sigma_{max}/k_N$$
$$\partial U^o/\partial F_s = \tau_{max}/k_s \qquad \text{(A-7)}$$

Finally, substituting (A-7) into (A-6) produces the required expression for $\partial U/\partial A$.

$$\partial U/\partial A = (\sigma_{max} + p)^2/2k_N + \tau_{max}{}^2/2k_s \qquad \text{(A-8)}$$

It might be noted that the derivation of (A-8) did not make use of plate theory approximations; therefore, it is also valid when the circular plate is modeled as an elastic continuum instead of a structural plate.

TABLE 1. RESULTS OF NUMERICAL ANALYSIS

(E/E_o = 22.4, ν = 0.333, ν_o = 0.35)

Code	h/a	h_o/a	$\sqrt{2n^2 D(\partial U/\partial A)/a^2 p}$	σ_{max}/p	τ_{max}/p	ℓ/a
Pℓ.0	.125	.010	1.193 (1.206)	71.0 (78.8)	15.6	.071
Pℓ.1	.125	.010	1.222 (1.206)	72.0 (78.8)	16.8	.071
Ax.0	.125	.010	1.248 (1.271)	27.4 (30.4)	5.90	.071
Ax.1	.125	.010	1.290 (1.271)	28.2 (30.4)	6.33	.071
Pℓ.1	.0625	.004	1.081 (1.097)	282. (324.)	76.3	.034
Pℓ.1	.0625	.010	1.126 (1.123)	192. (209.)	40.7	.042
Ax.1	.0625	.004	1.109 (1.123)	108. (124.)	28.7	.034
Ax.1	.0625	.010	1.162 (1.156)	74.0 (80.2)	15.3	.042
Pℓ.1	.25	.010	1.409 (1.368)	28.8 (31.0)	6.80	.119
Ax.1	.25	.010	1.550 (1.482)	11.5 (12.0)	2.55	.119

Code: Pℓ. = Plane Strain Case

Ax. = Axisymmetric Case

0 = Shear Deformation of Plate Ignored

1 = Shear Deformation of Plate Included

n = 3 for plane strain case

n = 8 for axisymmetric case

TABLE 2. VALUES OF k. (ν = 1/3)

ν_0	.25	.3	.35	.4	.45
k	.98	.96	.91	.85	.74

ν_0	.48	.49	.499	.4999	.5
k	.60	.51	.29	.16	.00

LIST OF FIGURES

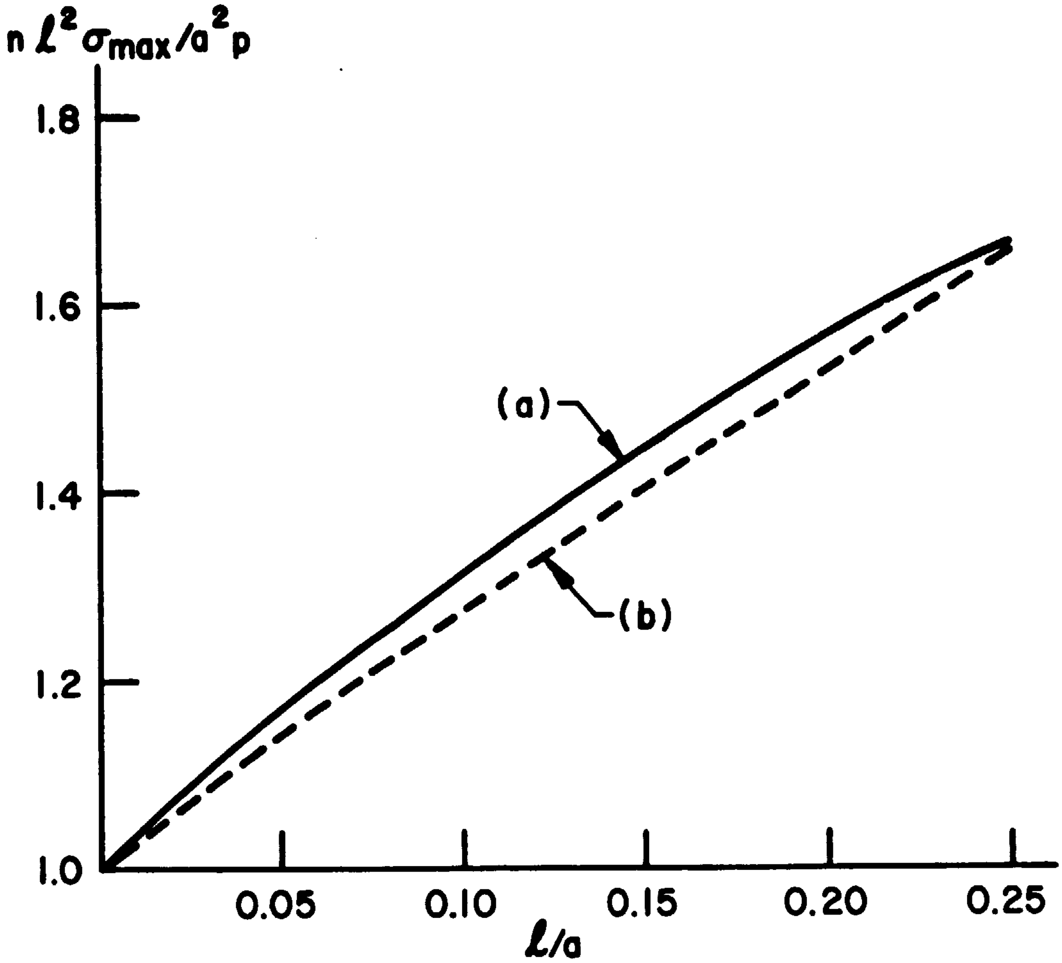

Figure 2. $n\ell^2\sigma_{max}/a^2p$ vs. ℓ/a.
(a) Axisymmetric case (n = 8)
(b) Plane strain case (n = 3)

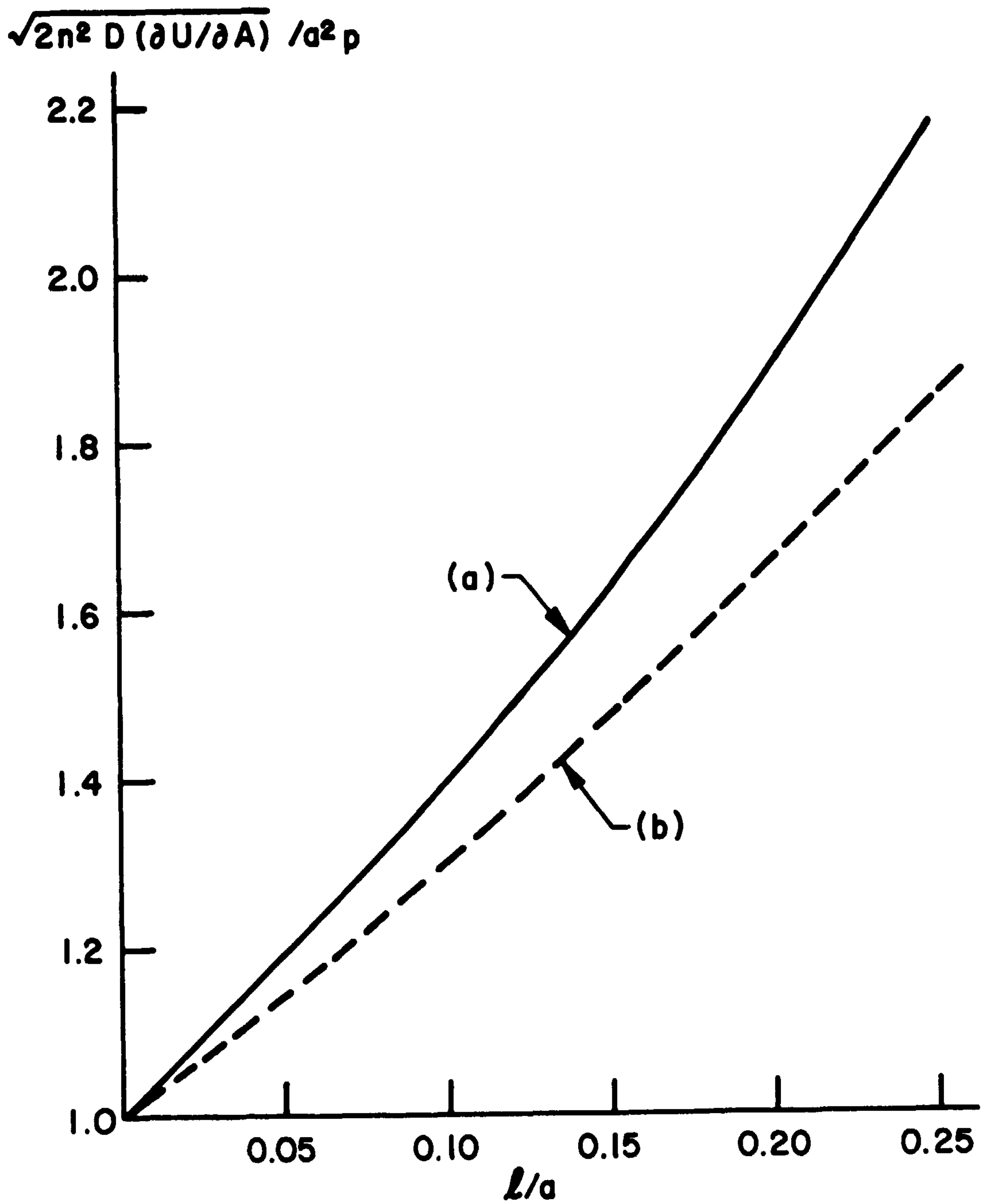

Figure 3. $\sqrt{2n^2 D(\partial U/\partial A)}/a^2 p$ vs. ℓ/a.
(a) Axisymmetric case ($n = 8$)
(b) Plane strain case ($n = 3$)

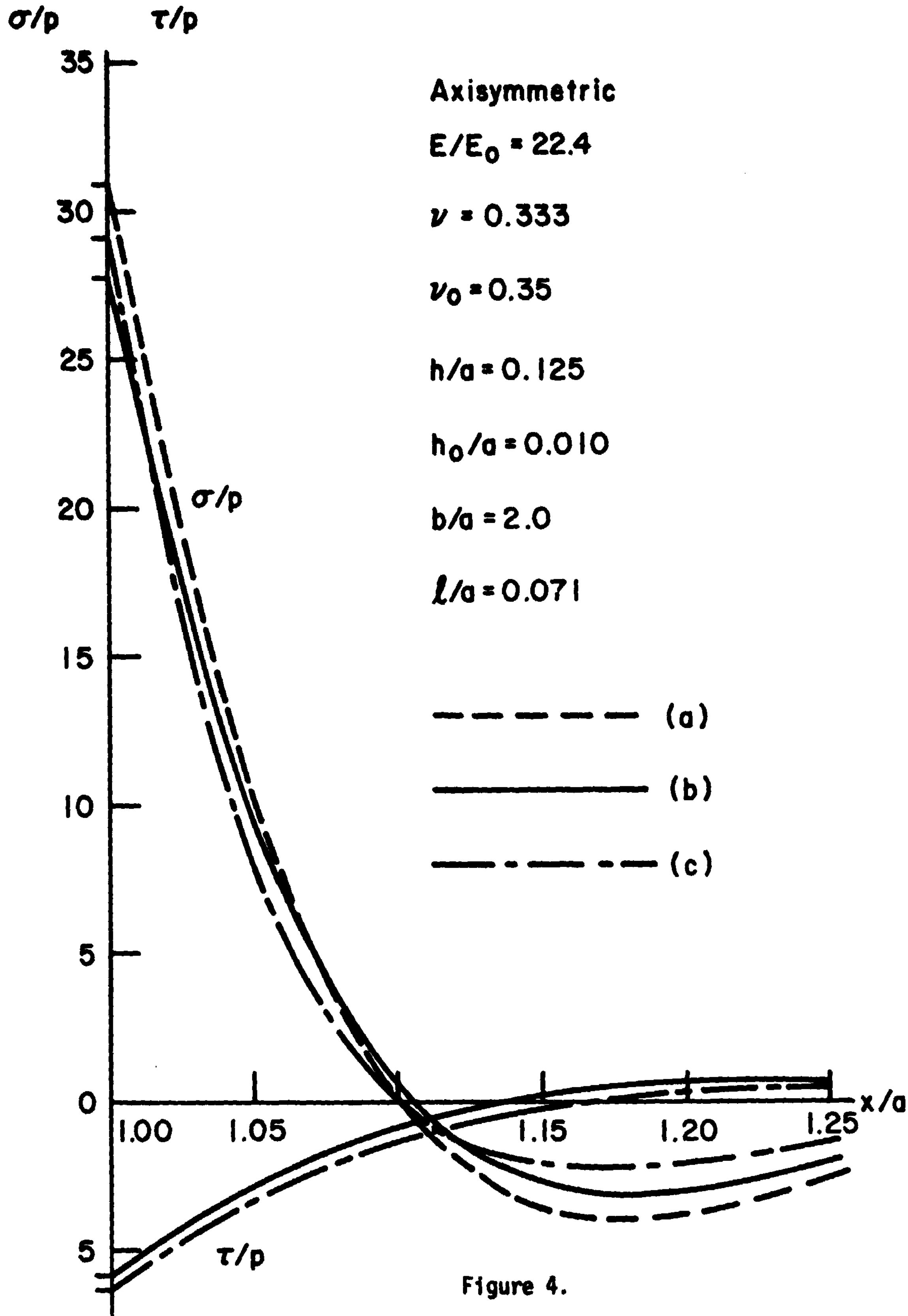

Figure 4.

σ/p and τ/p vs. x/a by numerical integration method for axisymmetric case. Curve (a), Bending only; Curve (b), Bending and Extension only, Transverse Shear Deformation Neglected; Curve (c), Bending, Extension, and Shear.