BIDIRECTIONAL PLANT CANOPY REFLECTION MODELS DERIVED FROM THE RADIATION TRANSFER EQUATION

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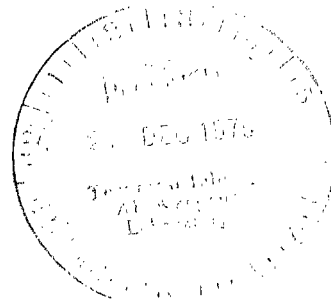| 1. Report No. NASA TN D-8106	2. Government Accession No.	3. Recipient's Catalog No.
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16. Abstract

A collection of bidirectional canopy reflection models is obtained from the solution of the radiation transfer equation for a horizontally homogeneous canopy. A phase function is derived for a collection of bidirectionally reflecting and transmitting planar elements characterized geometrically by slope and azimuth density functions. The assumption is that the canopy can be adequately approximated by such a collection of elements. Two approaches to solving the radiation transfer equation for the canopy are presented. One approach factors the radiation transfer equation into a solvable set of three first-order linear differential equations by assuming that the radiation field within the canopy can be initially approximated by three components: uniformly diffuse downwelling, uniformly diffuse upwelling, and attenuated specular. The solution to these equations, which can be iterated to any degree of accuracy, is then used to obtain overall canopy reflection from the formal solution to the radiation transfer equation. A programable solution to canopy overall bidirectional reflection is given for this approach. The special example of Lambertian leaves with constant leaf bidirectional reflection and scattering functions is considered, and a programable solution for this example is given. The other approach to solving the radiation transfer equation, a generalized Chandrasekhar technique, is presented in the appendix. The technique is exact, but it issues in a set of simultaneous integral equations for which a programable solution may be impractical.

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DERIVED FROM THE RADIATION TRANSFER EQUATION

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SUMMARY

The basic approach to canopy modeling adopted in this paper is to solve the radiation transfer equation under assumptions appropriate for a canopy and thus to predict canopy-reflected radiation under arbitrary conditions of illumination. Consideration of the radiation transfer equation and its formal solution along a path shows that the basic problem of canopy modeling has two parts. The physical part of the problem is to define the phase function by determining the scattering behavior of a volume element that contains a collection of scattering elements. The mathematical part of the problem is to solve the radiation transfer equation for the medium as a whole, not merely along a simple path as is done in the formal solution. Solving the problem for the medium as a whole requires the solution of an infinite set of integrodifferential equations. The phase function is defined by developing a technique to sum the scattering behavior of a collection of planar scattering elements. Coordinate transformations are developed to enable proper treatment of the potential bidirectional scattering behavior of the scattering elements. Canopy geometry is expressed in terms of probability density functions over azimuth and zenith angles.

Two approaches are presented for solving the mathematical problem. One approach is an extension of a Chandrasekhar technique. The approach concludes with eight integrodifferential equations that are reducible to six by transmission reciprocity. Because no programable solution is presented, an alternate approach is emphasized. The basic assumption for this second approach is that the field can be partitioned into a uniformly diffuse upwelling component, a uniformly diffuse downwelling component, and an attenuated specular component. This approach reduces the radiation transfer equation to three easily solvable coupled, first-order linear differential equations of the Shuster or Kubelka-Munk type. The solution to these three equations gives an approximation to the field within the canopy and is used in the formal solution of the radiation transfer equation to obtain canopy bidirectional reflection.

INTRODUCTION

The objective for developing a reflective canopy model is to predict the upwelling monochromatic intensity from a canopy under specified conditions of illumination, preferably in terms of physically measurable characteristics of the canopy.

These characteristics are the geometrical and optical properties of the components that make up a canopy. If these canopy component optical and geometric properties can be related to canopy type, stress, maturity, and yield, the model can be used to relate canopy-reflected radiation to these characteristics. This relationship would imply potential application to the acquisition and analysis of signature statistical properties.

The approach presented in this paper results in three canopy models. The models vary in form and approach depending primarily on the combination of simplifying assumptions made. All models assume horizontally homogeneous canopies. In Model I and Model II, it is assumed that a first-order approximation of the field within the canopy is obtainable from modified Kubelka-Munk equations. In Model I, the acceptance of non-Lambertian leaves is possible if the non-Lambertian character of the leaves is known. Further, an azimuthal anisotropic distribution of leaves will be accepted if this anisotropic distribution is known. In Model II, the further simplifying assumption that the leaves are Lambertian scatterers is made. This assumption results in considerable simplification of the approach used and the calculated expressions obtained. From the further assumption in Model II that the two sides of the leaves are optically indistinguishable, an interesting result is obtained: that all geometric factors affecting scattering of the diffuse field are multiplied by the difference between radiation reflected and radiation transmitted. This result would imply that changes in canopy geometry would have little effect in regions of canopy optical thickness sufficient to ensure that soil background has little effect. For example, wind-induced data scattering in these regions would be minimal.

In Model III, which is the most general model, non-Lambertian leaves and azimuthal anisotropy are accepted and the Kubelka-Munk approximation is not required. Although Model III is exact in principle, difficulties in programing make it, for the most part, a theoretical curiosity. However, the effect of such assumptions as Lambertian leaves, azimuthal isotropy, and some well-defined zenith distribution density functions is not explored in this paper. Perhaps Model III could be made quite workable for some special situations, but this possibility is left for further investigation.

SYMBOLS

A_{ij} Kubelka-Munk equation expansion coefficients

$A_{i\alpha}$ expansion coefficients for expressing E_i in terms of eigenfunction F_α as defined in equation (77)

$a = a_{11} = a_{22}$

a,b,c,d integration frame limits

a_{ij} Kubelka-Munk equation expansion coefficients

B	expression containing canopy geometric characteristics for canopies with Lambertian leaves and optically indistinguishable sides
$b = a_{12} = a_{21}$	
$\mathcal{C}, \mathcal{D}, \mathcal{R}$	transformation matrices
$c = a_{13}$	
$c' = a_{23}$	
$D = \dfrac{d}{dz}$	derivative operator
D'	average value of $\dfrac{1}{\pi} G^2(\theta_n)$
$D(\varphi_n, \nu)$	azimuthal density function
dA	element of horizontal area
da	magnitude of $\overline{da}$
$\overline{da}$	vector surface element
$<da>$	average projected area of a collection of surface elements
da	cross section of cylindrical volume element
$da\hat{n} = \overline{da}$	
$dI(\ell, \hat{k})$	differential change in intensity along ℓ in the direction of $\hat{k}$
ds	differential vector area on the surface of volume element ΔV
dV	volume element
$d\mu'\, d\varphi' = -d\Omega'$	
$d\Omega'$	differential solid angle
$d\Omega_i$	differential solid angle containing incident radiation
$d\Omega_s$	differential solid angle containing scattered radiation

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E_b irradiance of the specular beam measured perpendicular to the beam

E_s horizontal irradiance resulting from reduced specular flux

$E_s(0)$ irradiance on a horizontal plane at the upper surface of a canopy

$E_{\lambda\alpha}(d,z)$ upwelling irradiance at level z in the α layer

$E_{\lambda\alpha}(-d,z)$ downwelling irradiance at level z in the α layer

$E_{\lambda\alpha}(s,z)$ irradiance at level z resulting from the reduced incident specular flux

E_+ upwelling irradiance

E_- downwelling irradiance

$E_{\pm} = \pi L_{\pm}$

F fraction of beam obscured while passing through ΔV

F_α Shuster or Kubelka-Munk equation eigenfunction

$F_{\pm}$ magnitude of incident intensity

$\bar{F}$ variable defined in equation (274)

$\oint (t)$ an arbitrary function defined in equation (174)

$\oint_{\pm}$ variable defined in equations (99) and (100)

$G = \dfrac{\pi}{2}(1 - \cos \theta)$

$g = (a^2 - b^2)^{\frac{1}{2}}$

$\pm h$ integration over upper or lower hemispheres

I intensity where the functional dependence is suppressed

$I(\hat{k})$ upwelling intensity at the upper surface of a canopy

$I(\ell,\hat{k})$ intensity at ℓ along a path direction determined by $\hat{k}$

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$I_d(\tau;\mu,\varphi)$	downwelling diffuse intensity
I_i	incident intensity
$I_i(\hat{k}')$	downwelling intensity at the upper surface of a canopy
$I_i(\tau;\mu,\varphi)$	incident intensity on a layer at optical depth τ
I_s	scattered intensity
$I_s(\hat{k})$	scattered intensity from $\overline{da}$
$I_s(\tau;\mu,\varphi)$	scattered portion of the intensity field at τ
$I_{sa}(\hat{k})$	intensity outgoing from $\Delta\Omega_s$
I_{si}	incident intensity scattered into $\Delta\Omega_s$
$I_t(\ell',\hat{k})$	total intensity field
$I_t(\tau,\hat{k})$	total intensity field at τ
$I_u(\tau;\mu,\varphi)$	upwelling diffuse intensity
$I_{\lambda\alpha}(s,z)$	reduced incident intensity
$I_{\rho L}$	radiance reflected from a lower surface
$I_{\rho U}$	radiance reflected from an upper surface
$I_{\tau L}$	radiance transmitted from a lower surface
$I_{\tau U}$	radiance transmitted from an upper surface
$I_\pm$	intensity directed upward or downward
$J(\ell',\hat{k})$	intensity scattered (per unit length per unit density) into the beam at ℓ in a direction given by $\hat{k}$
$k = a_{33}$	
$\hat{k}$	unit vector; a function of θ,φ

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$\hat{k}'$	unit vector in the direction of incident intensity
$\hat{k}_j$	unit vector with polar angles θ_j and φ_j
$\hat{k}_0(\theta_0,\varphi_0)$	unit vector in the direction of solar or specular intensity
$\hat{k}_{\pm}$	upwelling or downwelling $\hat{k}$
k	beam attenuation function
L	lower leaf side of interest
L'	lower hemisphere
$L_{\lambda\alpha}(d,z)$	uniformly diffuse (Lambertian) upwelling intensity at level z in the α layer
$L_{\lambda\alpha}(-d,z)$	uniformly diffuse (Lambertian) downwelling intensity at level z in the α layer
$L_{\pm}$	uniformly diffuse upwelling and downwelling radiance
ℓ	path
$M = 1,2,3,4$	integration limits as defined in table IV
N	number of elements in ΔV or in dV
n	number density of scattering volume elements
$\hat{n}$	unit vector perpendicular (normal) to an area
n da	differential leaf area index
$\hat{n}$ da	vector area of magnitude da
nJ	intensity scattered into the beam
$nJ(\ell,\hat{k})$	intensity scattered per unit length into the beam at ℓ in a direction given by $\hat{k}$
$\hat{n}\cdot\hat{k}_0 = \cos\gamma_0$	
n ΔV	number of elements in a volume element
$-n\bar{\sigma}I$	intensity removed from the beam by scattering or absorption
$P(\hat{k},\hat{k}')$	phase function relating incident to scattered intensity for a volume element of material

p	power
p_a	power outgoing from $\overline{da}$ into $\Delta\Omega_s$
p_i	total power incident on volume element ΔV
p_s	total power scattered from the volume element
$p_\pm$	upwelling or downwelling power due to downwelling incident intensity
$p'_\pm$	upwelling or downwelling power due to upwelling incident intensity
$\langle p_\pm \rangle$	average power
Q	number determined from boundary conditions in equations (95) to (98)
R	number determined from boundary conditions in equations (95) to (98)
R'	average value of $\frac{1}{\pi}G(\theta_n)$
S	surface of volume element ΔV
$S(\hat{k},\hat{k}')$	bidirectional reflection function for the canopy as a whole
S_L, S_U	lower and upper bidirectional reflection functions for a layer τ' thick
$S(\theta_n,\eta)$	leaf zenith angle density function
T_L, T_U	lower and upper bidirectional transmission functions for a layer τ' thick
$T_\pm$	positive and negative regions of $\oint(t)$
U	upper leaf side of interest
U'	upper hemisphere
x, y, z	coordinate frames related by rotation matrices
z	horizontal canopy level
z_{soil}	distance from the top of the canopy to the soil

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α fraction of intercepted incident power scattered

$\alpha_{\pm} = \pm g$

β fraction of total power scattered into $\Delta\Omega_s$

$\Gamma = \sin\theta \sin\varphi$

γ angle between unit vectors

ΔA_i cross section shared by ΔV and the incident beam

ΔA_s cross section shared by ΔV and the scattered intensity contained in $\Delta\Omega_s$

Δp_i power incident on the volume element from intensity contained in the solid angle $\Delta\Omega_i$

Δp_s total power scattered into the solid angle $\Delta\Omega_i$

ΔV volume element

$\Delta\Omega$ element of solid angle

$\Delta\Omega_i$ element of solid angle containing incident intensity

$\Delta\Omega_s$ element of solid angle containing scattered intensity

$\Delta\Omega_0$ solid angle that is intercepted by the Sun

$\Delta^2 p_i$ amount of Δp_i actually intercepted

$\delta = \varphi_i - \varphi_{n0}$

δ_{ij} Kronecker delta function

$\delta(\mu - \mu_{\pm}), \delta(\varphi - \varphi_{\pm})$ Dirac delta functions

$\hat{\varepsilon}_x, \hat{\varepsilon}_y, \hat{\varepsilon}_z$ unit vectors defined in equation (120)

θ polar zenith angle

θ_n polar zenith angle of the unit normal $\hat{n}$

μ	cosine of the polar zenith angle in a given coordinate frame
ν, η	combined parameters indicating the functional dependence of the density function
$\rho(\hat{k}, \hat{k}')$	bidirectional reflection function
$\rho_{soil}(\hat{k}, \hat{k}')$	soil bidirectional reflection function
$\bar{\sigma}$	average scattering cross section of scattering centers
τ	optical depth
τ'	specific optical depth
$\tau(\hat{k}, \hat{k}')$	bidirectional transmission function
τ_0	upper surface of the canopy
φ	polar azimuth angle
φ_n	polar azimuth angle of the unit normal $\hat{n}$
φ_{n0}	azimuth angle of $\hat{n}$
ψ	integration variable used in equation (183)

Subscripts:

d	downwelling
i	incident direction
i	summing index for ρ, τ
i	summing index for $1, 2, 3$
j	summing index for U, L
j	summing index for $1, 2, 3$
L	lower surface incidence
m	multilayer constituent index
r	radial
s	scattered direction

T	integration interval
t	total
U	upper surface incidence
u	upwelling
α	layer index
λ	wavelength
0	quantities associated with the solar beam
1	class that is dependent on φ_n
2	class that is not dependent on φ_n
$\pm$	upwelling or downwelling

Mathematical notations:

*	leaf coordinate frame
$\{\ \}$	choose the greater of two quantities contained therein
$[\ ]$	choose the lesser of two quantities contained therein

GENERAL APPROACH

All analytic canopy modeling begins implicitly or explicitly with the radiation transfer equation that describes changes in monochromatic intensity or radiance. The radiation transfer equation is written

$$\frac{dI(\ell,\hat{k})}{d\ell} = -n(\ell)\bar{\sigma}(\ell,\hat{k})I(\ell,\hat{k}) + n(\ell)J(\ell,\hat{k}) \tag{1}$$

The equation is a statement of conservation of energy by which radiation transfer through any medium is governed. The change in intensity $I(\ell,\hat{k})$ along a path element $d\ell$ in the direction of the unit vector $\hat{k}$ is the difference between the intensity absorbed and scattered out of the beam and the intensity scattered into the beam. The symbol n is the number density of scattering centers, and $\bar{\sigma}$ is the average scattering cross section of scattering centers. Intensity per unit volume scattered

into the beam at ℓ in a direction given by $\hat{k}$ is represented by the expression $nJ(\ell,\hat{k})$. Integrating factors gives a formal solution

$$I(\ell,\hat{k}) = I\left(\ell_0,\hat{k}\right)e^{-T\left(\ell,\ell_0\right)} + \int_{\ell_0}^{\ell} \frac{J(\ell',\hat{k})e^{-T(\ell,\ell')}}{\overline{\sigma}} \, d\ell' \tag{2}$$

where

$$T\left(\ell,\ell_0\right) \equiv \int_{\ell_0}^{\ell} n\overline{\sigma} \, d\ell \tag{3}$$

and the J term is given by

$$J(\ell',\hat{k}) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} I_t(\ell',\hat{k})P(\hat{k},\hat{k}')d\mu' \, d\varphi' \tag{4}$$

The phase function $P(\hat{k},\hat{k}')$ relates incident to scattered intensity for a volume element of material. The expression $d\mu' \, d\varphi'$ is $-d\Omega'$ where $d\Omega'$ is a differential solid angle, and $\mu' = \cos\theta'$ where θ' is the polar zenith angle and φ' is the polar azimuth angle. Because the expression $I_t(\ell',\hat{k})$ represents the total intensity field at $d\ell$, $I_t(\ell',\hat{k})$ couples the solution along one path with the solution along all other paths; therefore, the formal solution for a single path is not the solution for the medium as a whole, and the solution for the medium as a whole involves an infinite set of coupled integrodifferential equations for which a completely general solution is not known.

The problem of radiation transfer in any medium has two main parts. The physical problem is to find the phase function, which will be expressed in terms of the scattering and absorptive behavior of the individual scattering components or scattering centers in a volume element. The mathematical problem is to solve the system of coupled integrodifferential equations.

Specification of the phase function for a given volume element will be the same regardless of the simplifying assumptions subsequently made to solve the mathematical problem. That is, the phase-function development is the same regardless of whether parallel homogeneous layers are later assumed or whether canopy row effects are considered. The scattering and absorptive behavior of a volume element

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of scattering components is a type of sum of the scattering behavior of each compo-
nent. The problem is to assemble this sum in terms of the scattering and absorption
properties of each component.

The first assumption is that any individual scattering component, such as a
leaf, may be represented by a collection of small planar elements. This assumption
is based on the fact that any surface with any curvature can be represented to any
desired degree of accuracy by a collection of planar elements. These planar ele-
ments will be designated by the vector surface element $\overline{da}$. (A vector surface
element is a vector of magnitude da proportional to the area of $\overline{da}$ in a direction
perpendicular outward to $\overline{da}$.) This equation will be written

$$\overline{da} = da\,\hat{n} \tag{5}$$

where $\hat{n}$ is a unit vector normal from da in the outward direction. Further, the
number volume density of surface elements is designated by n. For a canopy inte-
grated over the height of the canopy, n da will give the leaf area index and thus
can be called the differential leaf area index.

Each element can be characterized by a slope and an azimuth in a z-vertical
coordinate system. Because the slope is the same as the zenith angle of $\overline{da}$, two
angle parameters are associated with $\overline{da}$: the polar zenith angle of the normal θ_n
and the polar azimuth angle of the normal φ_n. The collection of elements in a vol-
ume element will have some distribution of slopes and azimuths that can be described
by density functions. The quantities volume number density, element area, and
slope and azimuth density functions constitute the set of input parameters henceforth
referred to as phase-function geometry. (The product of density and leaf area is
the one-sided leaf area per unit volume.) When combined with the spatial variation
of the phase function over the canopy, these quantities constitute the canopy geomet-
ric inputs. These geometric inputs are a set of morphological descriptors determin-
ing canopy reflectivity; they can be labeled spectral morphological descriptors.

A planar scatterer can be parameterized by bidirectional reflection and trans-
mission functions. For a leaf, this procedure will generally require a bidirectional
reflection function defined on the upper surface, a bidirectional reflection function
defined on the lower surface, and a bidirectional transmission function. These three
bidirectional functions completely specify the scattering and absorptive behavior
of the canopy component and are referred to in this paper as the component optical
properties. Thus, the physical inputs to the phase function, and hence the model,
are these optical and geometric quantities.

The output of the model is the bidirectional scattering behavior of the canopy
as a whole; it is expressed in terms of canopy bidirectional reflection and transmis-
sion functions. These functions can be labeled the canopy optical properties. Once
the canopy optical properties are known, the upwelling intensity is obtainable for

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any specified illumination conditions from a relationship of the form

$$I(\hat{k}) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} I_i(\hat{k}')S(\hat{k},\hat{k}')\mu' \, d\mu' \, d\varphi' \tag{6}$$

The expression $I(\hat{k})$ represents the upwelling intensity at the upper surface of the canopy; $I_i(\hat{k}')$ is the downwelling intensity at the upper surface of the canopy; and $S(\hat{k},\hat{k}')$ is the canopy bidirectional reflection (upwelling, scattering) function for the canopy as a whole. Thus, once $S(\hat{k},\hat{k}')$ is known for a canopy, the downwelling intensity field must be known to obtain the upwelling intensity.

In summary, for operational use of the model, the input or independent variables are the canopy geometry, the component optical properties, and the illumination conditions. The output or dependent variable is the upwelling intensity.

No solution is known for a medium for which the phase function varies arbitrarily as a function of position. Such a variation might be a horizontal variation, such as row effects, or a vertical variation in which density is a function of height. To solve this problem, simplifying assumptions must be made. These assumptions may be made in terms of (1) canopy geometry, (2) component optical properties, or (3) first-order approximations of the field behavior within the canopy. The plant canopy reflection models developed in this paper vary in form and approach depending on the combination of simplifying assumptions taken. The assumption that the canopy can be approximated by homogeneous layers is made in all models. Although the explicit development is for single-layer, single-scattering-component canopies, a method for generalizing to multilayer, multiconstituent canopies is given.

Model I and Model II are closely related. The mathematical operations in both models are made tractable by assuming that the field within the canopy can be initially approximated by a uniformly diffuse downwelling component, a uniformly diffuse upwelling component, and an attenuated incident or specular component. The attenuated specular component is defined as the portion of incident radiation at any level within the canopy that has not intercepted a canopy component. Under this assumption, the radiation transfer equation reduces to easily solvable modified Kubelka-Munk or Shuster equations. Model I is explicitly derived for leaves that need not have constant bidirectional scattering functions; that is, it is not restricted to the special situation of Lambertian leaves. However, Model II is derived solely for uniformly diffuse or Lambertian leaves. Although the development techniques have some variations, Model II is in essence an application of Model I to Lambertian leaves. In Model III, the most general model, the assumption is only that the canopy is representable by horizontally homogeneous layers. This treatment generalizes a method of Chandrasekhar (ref. 1) to generate a set of eight integrodifferential equations containing only the phase functions and the canopy reflection and transmission functions. The eight equations are reduced to six by transmission reciprocity. Model III terminates with these six equations, for which no general solution is known.

All three models are alike in that they begin from the radiation transfer equation. Model I and Model II are different from Model III because a different mathematical approach is used to solve the radiation transfer problem for the canopy as a whole. Identical accuracy should not be expected for all models. Model III should be the most accurate, and Model I should be more accurate than Model II. In particular, Model II becomes inaccurate at wavelengths of less than approximately 0.55 micrometer, where the assumption of constant bidirectional component scattering functions (Lambertian leaves) is not good. This breakdown in the high absorption regions is also noted by Smith (ref. 2) in the model in which he uses this assumption. (Note: In a region of high absorption, scattering tends to become more specular because multiple scattered contributions diminish and leave primarily the more specular Fresnel contributions as reported by Breece and Holmes (ref. 3).)

FORMULATION OF THE PROBLEM

The canopy radiation transfer problem is formulated by a discussion of the phase function and the formal solution to the radiation transfer equation.

The Phase Function

Consider a beam of cross section ΔA within a canopy (or any scattering medium) along a path ℓ. An element of length $d\ell$ defines a cylindrical volume element ΔV. The change in intensity I in $d\ell$ along ℓ gives the equation of transfer with all functional dependence suppressed.

$$\frac{dI}{d\ell} = -n\bar{\sigma}I + nJ \tag{7}$$

The term $-n\bar{\sigma}I$ represents the intensity removed from the beam by scattering or absorption. The term nJ represents the intensity scattered into the beam from radiation incident on the volume element from directions other than along ℓ. This term could also include emission into the beam; however, emission is ignored in this discussion.

The quantity n is the number density of scattering components in ΔV of a given type. Then, the fraction of the beam scattered or absorbed will be given by the fraction of ΔA obscured by the components in ΔV. The number of the components in ΔV is $n \Delta V$, and k is the beam·attenuation function. Thus, the fraction intercepted is

$$k = \frac{n \Delta V \bar{\sigma}}{\Delta A} \tag{8}$$

where $\bar{\sigma}$ is the average cross section of components in the direction ℓ. However, along a cylindrical path

$$\frac{\Delta V}{\Delta A} = d\ell \qquad (9)$$

Thus, for the change in intensity due to intensity removed from the beam

$$\frac{dI}{d\ell} = -n\bar{\sigma}I \qquad (10)$$

A particular form for the average cross section $\bar{\sigma}$ will be developed subsequently. Meanwhile, it is sufficient to note that $\bar{\sigma}$ will be a function of the direction of ℓ, a complication not encountered in the simpler problem of atmospheric radiation transfer.

An expression for the term J will also give an expression for the very important phase function. The expression nJ denotes the radiation scattered into the beam. To derive this expression for nJ, consider an arbitrary volume element ΔV having a cross section ΔA shared by both the beam and the scattering medium. The total power incident on this volume element p_i is found by integrating the defining equation for intensity over S, the surface of ΔV. This integration is written

$$p_i = \int_0^{4\pi} \int_S I_i(\hat{k}') (\hat{k}' \cdot \overline{ds}) d\Omega_i \qquad (11)$$

where $\hat{k}'$ is a unit vector in the direction of incident intensity I_i and where $\overline{ds}$ is a differential vector area on the surface of ΔV. Note that $\hat{k}$ is a function of θ and φ; therefore, a function of $\hat{k}$ and $\hat{k}'$ is also a function of $\theta, \varphi; \theta', \varphi'$. The power incident on ΔV from the intensity contained in $\Delta\Omega_i$, an element of solid angle containing the incident intensity, is designated Δp_i and is written

$$\Delta p_i = I_i \, \Delta\Omega_i \int_S \hat{k}' \cdot \overline{ds} \qquad (12)$$

But

$$\int_S \hat{k}' \cdot \overline{ds} = \Delta A_i \qquad (13)$$

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where ΔA_i is the cross section shared by ΔV and the incident beam. Then

$$\Delta p_i = I_i \, \Delta \Omega_i \, \Delta A_i \qquad (14)$$

Similarly, the total power scattered into the solid angle $\Delta \Omega_s$ is denoted by Δp_s and is written

$$\Delta p_s = I_s \, \Delta \Omega_s \, \Delta A_s \qquad (15)$$

where ΔA_s is the cross section shared by ΔV and the scattered intensity contained in $\Delta \Omega_s$. The amount of Δp_i actually intercepted, $\Delta^2 p_i$, is written

$$\Delta^2 p_i = n \bar{\sigma} \frac{\Delta V}{\Delta A_i} \, \Delta p_i \qquad (16)$$

If the fraction of $\Delta^2 p_i$ scattered is α, then

$$p_s = \alpha \, \Delta^2 p_i \qquad (17)$$

Furthermore, if the fraction of p_s scattered into $\Delta \Omega_s$ is denoted by β, then

$$\frac{\Delta p_s}{\Delta \Omega_s} = \beta p_s \qquad (18)$$

Using equations (14), (16), and (17), the following is obtained.

$$\frac{\Delta p_s}{\Delta \Omega_s} = \alpha \beta n \bar{\sigma} \frac{\Delta V}{\Delta A_i} I_i \, \Delta \Omega_i \, \Delta A_i \qquad (19)$$

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The expression Δp_s is obtained from equation (15). Thus

$$\frac{I_{si}}{\left(\dfrac{\Delta V}{\Delta A_s}\right)} = \alpha \beta n \bar{\sigma} \, \Delta\Omega_i I_i \tag{20}$$

where the subscript i on I_s denotes the scattered intensity resulting from the scattering of I_i contained in $\Delta\Omega_i$. Let $P(\hat{k},\hat{k}')$ be defined by

$$P(\hat{k},\hat{k}') = 4\pi\alpha\beta\bar{\sigma} \tag{21}$$

Then

$$P(\hat{k},\hat{k}') = \frac{4\pi I_{si}(\hat{k})}{n\dfrac{\Delta V}{\Delta A_s} I_i(\hat{k}')\Delta\Omega_i} \tag{22}$$

The phase function $P(\hat{k},\hat{k}')$ relates incoming and outgoing intensity for an arbitrary volume element. This formulation of the phase function is very useful in developing the form of the phase function for the canopy. Note that, in general, the total scattered intensity from a volume element has contributions from incident intensities in all solid angles. Summing equation (20) over all $\Delta\Omega_i$ values yields an integral

$$\frac{I_s(\hat{k})}{\left(\dfrac{\Delta V}{\Delta A_s}\right)} = \frac{n}{4\pi}\int_0^{4\pi} I_i(\hat{k}')P(\hat{k},\hat{k}')d\Omega' \tag{23}$$

where $I_s(\hat{k})$ is the total scattered intensity in the $\hat{k}$ direction as a result of scattering of incident intensity by the volume element ΔV. When the expression for a cylindrical volume element (eq. (9)) is recalled, it is clear from the discussion of equation (1) that equation (23) is also the equation for nJ. Thus, written in full, showing all functional dependence, the expression for J is

$$J(\ell,\hat{k}) = \frac{1}{4\pi}\int_0^{4\pi} I_t(\ell,\hat{k}')P(\hat{k},\hat{k}')d\Omega' \tag{24}$$

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Formal Solution of the Radiation Transfer Equation

The radiation transfer equation is

$$\frac{dI(\ell,\hat{k})}{d\ell} = -n(\ell)\bar{\sigma}(\ell,\hat{k})I(\ell,\hat{k}) + n(\ell)J(\ell,\hat{k}) \tag{1}$$

This equation can be integrated by multiplying each term by $e^{T(\ell,\ell_0)}$ where $T(\ell,\ell_0)$ is defined in equation (3). Note that from a rule of definite integrals

$$T\left(\ell_0,\ell_0\right) = 0 \tag{25}$$

and

$$T\left(\ell,\ell_0\right) = T(\ell,\ell') + T\left(\ell',\ell_0\right) \tag{26}$$

where $\ell_0 < \ell' < \ell$. Then, both sides may be integrated from ℓ to ℓ_0 to obtain equation (2).

$$I(\ell,\hat{k}) = I\left(\ell_0,\hat{k}\right)e^{-T\left(\ell,\ell_0\right)} + \int_{\ell_0}^{\ell} \frac{J(\ell',\hat{k})e^{-T(\ell,\ell')}}{\bar{\sigma}}\, d\ell' \tag{2}$$

with the quantity $J(\ell',\hat{k})$ given by equation (4).

The formal solution gives intensity at ℓ due to scattering along the beam before ℓ_0. If the value of ℓ is at a surface of the scattering medium, the solution gives scattered intensity from the medium. For a canopy, this could be the quantity desired, the upwelling intensity. The problem consists of finding the right-hand terms. The first right-hand term is simply the intensity at some point ℓ_0 on ℓ. This intensity might be determined by boundary conditions and, as such, may not be an insurmountable difficulty. The real problem arises from the second right-hand term and in particular from the integral for J.

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Expressing J requires knowledge of the phase function. As will be shown, to obtain a canopy phase function is difficult but not impossible. However, the J integral also requires knowledge of the value of I at any point along ℓ. If this value were known, there would be no need for the formal solution. Thus, there is an infinite set of coupled integrodifferential equations, a computational problem for which there is no direct solution known to the author.

A canopy phase function that is based on a minimum of assumptions and that is believed to be completely general within these assumptions will be developed in this paper. Two techniques for handling the computational problem will be explored; both techniques begin from the assumption that the scattering medium can be represented by parallel layers. For a canopy, this assumption will mask row effects. One of the computational methods, Model III, a modification of a technique developed by Chandrasekhar (ref. 1) for atmospheres, involves restatement of the problem in terms of other system parameters, namely reflection and transmission functions for the layer as a whole. Then, the radiation transfer equation and its formal solution are used to solve the parameterized problem. This method concludes with a finite set of integral equations. The solution to these equations may provide ultimately the most exact technique for developing a general canopy model. In the second computational method, used for Model I and Model II, the modified Kubelka-Munk equations are used to find a first approximation to the intensity field within the canopy. This technique is simpler and may be found workable from a practical computational point of view. Improvement of the Kubelka-Munk approximation can be obtained by iteration.

BIDIRECTIONAL PHASE FUNCTION FOR A CANOPY

The so-called phase function shown in equation (22) would be better named volume scattering-absorption function; it is basically a ratio of the incident to the scattered intensity for an arbitrary volume element within a medium. The scattering behavior of the volume element is a sum of the scattering behavior of the individual components, or scattering pieces, in the volume element.

Although performing this sum is straightforward in principle, it is fairly complicated in practice. The general summing problem is expressible as the integral sum of sectionally continuous functions in four dimensions. Each dimension is nonzero in half spaces, defined with respect to coordinate frames, each of which is two-dimensionally rotated with respect to the other. The solution of this summation problem, the coordinate transformations, and the integration limits are important at this stage of theory development.

Assumptions and Canopy Visualization

The canopy is viewed as a collection of volume elements, each of which contains an ensemble of canopy components. Each component, such as a leaf, is assumed representable by one or more planar elements. Each planar element, represented by the vector area $\overline{da}$, is characterized by slope, horizontal orientation or azimuth, and optical scattering properties described by upper and lower surface bidirectional reflection functions and a bidirectional transmission function. The slope

is the zenith angle in a z-vertical coordinate frame of the unit normal to $\overline{\overline{da}}$, and the azimuth is the azimuth in this frame of the unit normal to $\overline{\overline{da}}$. The zenith and azimuth distribution for the ensemble of elements in the volume element will be characterized by statistical density functions. These angular density functions, together with the volume number density and the size of area elements, will be called the plant geometry. The bidirectional reflection and transmission functions will be called the component optical properties. The plant geometry and component optical properties are the necessary inputs to the model.

Complete optical properties of canopy components are available for only a very limited number of components. These properties have been measured satisfactorily only by Breece and Holmes (ref. 3) for corn and soybeans. Therefore, in current practical usage, estimates must be made from the numerous directional reflection measurements. This procedure is equivalent to assuming that the components are uniformly diffuse or Lambertian. Analysis of available data suggests that such an approximation is acceptable for wavelengths greater than approximately 0.7 micrometer. However, for wavelengths less than 0.7 micrometer, the model will not be expected to work well without a better estimate of the nonuniform reflection properties of the leaves. This result is borne out by the modeling work done by Smith (ref. 2).

Polarization will be disregarded in this model. However, polarization is an important effect that will likely prove to be a useful diagnostic tool. To account for polarization, the formal theory presented herein can be used with the understanding that the scattering functions are thereby represented in polarization matrices. The aspect of polarization most significant to remote sensing is the existence of a strong front surface quasi-specular reflection for many leaves, notably waxy ones. This reflection or gloss is not greatly affected by the absorptive character of the leaf. For many aspects of spectral signature interpretation, this gloss masks the spectral information and could be called noise. The same effect is seen in an ordinary glossy photograph viewed in glaring light. This glare is often strongly polarized; thus, polarization filtering could be used to remove it. For this reason, polarization research is potentially valuable and could make an important contribution to remote sensing.

Coordinate Frames

The development of the canopy function requires the definition of three coordinate frames and the transformations between them. In the first frame, the observation frame, the z-axis is vertical, the xy-plane is horizontal, and the x-axis is left arbitrary to be chosen in any way that may become convenient later. In the second frame, the upper leaf frame denoted by asterisks, the z*-axis is perpendicular to the planar element; thus, the z*-axis has the same direction as the unit normal $\hat{n}$ to the element $\overline{\overline{da}}$. The x*-axis is chosen along some convenient line of the component, such as the central vein of a leaf. Then, the x*y*-plane is the plane of the element $\overline{\overline{da}}$. The third frame is the lower leaf frame denoted by double asterisks. The upper and lower leaf frames are related by $z^* = -z^{**}$, $y^* = -y^{**}$, and $x^* = x^{**}$. Thus, the double-asterisk frame is related to the asterisk frame only by the direction of the outward-drawn normal.

Each of these frames is conveniently expressed in spherical polar coordinates

using the convention where θ is the zenith angle and φ is the azimuth angle measured counterclockwise from the x-axis when viewed from the upper plus-z hemisphere in the minus-z direction. The assumption is that the planar element is characterized by some slope and azimuth. These are the zenith and azimuth angles of $\hat{n}$, the unit normal to $\overline{da}$, measured outward from the upper surface. Let θ_n be the zenith angle and φ_n be the azimuth angle of $\hat{n}$ in the observation frame.

The transformation equations are obtained by first considering a rotation of φ_n around z by which y is moved to y*, and then taking a rotation of θ_n around y*. The sum of these two rotations will take z into z* and x into x*. The first rotation matrix around z is

$$C = \begin{bmatrix} \cos \varphi_n & \sin \varphi_n & 0 \\ -\sin \varphi_n & \cos \varphi_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

The second rotation matrix around y* is

$$\mathcal{D} = \begin{bmatrix} \cos \theta_n & 0 & -\sin \theta_n \\ 0 & 1 & 0 \\ \sin \theta_n & 0 & \cos \theta_n \end{bmatrix} \tag{28}$$

The product is

$$\mathcal{R} = \mathcal{D}C = \begin{bmatrix} \cos \theta_n \cos \varphi_n & \cos \theta_n \sin \varphi_n & -\sin \theta_n \\ -\sin \varphi_n & \cos \varphi_n & 0 \\ \sin \theta_n \cos \varphi_n & \sin \theta_n \sin \varphi_n & \cos \theta_n \end{bmatrix} \tag{29}$$

Now, consider some radial unit vector in some direction $\hat{k}_r$. Expressed in the observation frame, this term may be written

$$\hat{k}_r = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \qquad (30)$$

The same vector expressed in the asterisk frame is

$$\hat{k}_r^* = \begin{pmatrix} \sin\theta^*\cos\varphi^* \\ \sin\theta^*\sin\varphi^* \\ \cos\theta^* \end{pmatrix} \qquad (31)$$

But

$$\mathcal{R}\hat{k}_r = \hat{k}_r^* \qquad (32)$$

Applying the rotation matrix to $\hat{k}_r$, comparing components with $\hat{k}_r^*$, and solving gives the transformation equations

$$-\varphi^{**} = \varphi^* = \cot^{-1}\left\{ \cos\theta_n \cot\left(\varphi - \varphi_n\right) + \frac{\sin\theta_n \cot\theta}{\sin\left(\varphi - \varphi_n\right)} \right\} \qquad (33)$$

$$\cos\left(\pi - \theta^{**}\right) = \cos\theta^* = \left\{ \cos\theta_n \cos\theta + \sin\theta_n \sin\theta \cos\left(\varphi - \varphi_n\right) \right\} \qquad (34a)$$

$$-\mu^{**} = \mu^* = \cos\theta_n \cos\theta + \sin\theta_n \sin\theta \cos\left(\varphi - \varphi_n\right) \qquad (34b)$$

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Bidirectional Reflection and Transmission Functions

The derivation of a phase function for an ensemble of canopy components consists of two main parts. The first part is to relate incident to scattered intensity for an individual component $\overline{da}$, and the second part is to sum the contributions of all the components in the volume element. The reflection and transmission bidirectional scattering functions are defined by equations of the form

$$I_s(\hat{k}) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \rho(\hat{k},\hat{k}')I_i(\hat{k}')\mu' \, d\mu' \, d\varphi' \tag{35a}$$

where $0 < \mu < 1$, and

$$I_s(\hat{k}) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \tau(\hat{k},\hat{k}')I_i(\hat{k}')\mu' \, d\mu' \, d\varphi' \tag{35b}$$

where $0 > \mu > -1$ and where μ' arises from $\hat{k}' \cdot \hat{n}$ within $\hat{n}$, a unit normal to $\overline{da}$. The expression $I_s(\hat{k})$ is the scattered intensity from $\overline{da}$, and $I_i(\hat{k}')$ is the incident intensity on $\overline{da}$. The quantities $\rho(\hat{k},\hat{k}')$ and $\tau(\hat{k},\hat{k}')$ are the bidirectional *reflection* and *transmission* functions, respectively. An alternate form could also be used.

$$I_s(\hat{k}) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} \rho(\hat{k},\hat{k}')I_i(\hat{k}')d\mu' \, d\varphi' \tag{36a}$$

where $0 < \mu < 1$.

$$I_s(\hat{k}) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} \tau(\hat{k},\hat{k}')I_i(\hat{k}')d\mu' \, d\varphi' \tag{36b}$$

where $-1 < \mu < 0$ and where $\mu = \hat{k} \cdot \hat{n}$. Note that equations (36a) and (36b) are obtained from equations (35a) and (35b) by omitting the μ' within the integral and inserting the $1/4\mu$ before the integral. In both equations, the bidirectional func-

tions so defined are symmetric under interchange of directions. That is

$$\rho(\hat{k},\hat{k}') = \rho(-\hat{k}',-\hat{k}) \qquad (37a)$$

and

$$\tau(\hat{k},\hat{k}') = \tau(-\hat{k}',-\hat{k}) \qquad (37b)$$

Because $\overline{da}$ is a planar element that will generally have different reflective properties depending on the side of incidence, it is convenient to define the total reflection and transmission functions as shown in table I. All the bidirectional functions shown in table I are zero in regions other than those indicated. Note that limits are given in the asterisk or upper leaf frame where the plane of the leaf is the xy-plane. The subscripts U and L designate upper and lower sides of incidence, respectively. Radiation scattered into the hemisphere of incidence and radiation scattered into the hemisphere opposite the hemisphere of incidence are governed by the transmission function $\tau(\hat{k},\hat{k}')$.

In general, the scattered field will be given by

$$I_s(\hat{k}) = I_{\rho U}(\hat{k}^*) + I_{\tau L}(\hat{k}^*) \qquad (38a)$$

where $0 < \mu^* < 1$, and

$$I_s(\hat{k}) = I_{\rho L}(\hat{k}^*) + I_{\tau U}(\hat{k}^*) \qquad (38b)$$

where $-1 < \mu^* < 0$ and where $I_{\rho U}$, $I_{\tau U}$, $I_{\rho L}$, and $I_{\tau L}$ are derived from the equations defining the bidirectional scattering functions (eqs. (35a) and (35b)). The existence of a transmission-function symmetry relationship that makes the distinction between τ_U and τ_L redundant can be shown. However, the upper and lower surface transmission-function symmetry will not be explicitly exhibited in the derivation because omitting it makes checking the derivation less complex. Explicitly,

this derivation is written

$$I_{\rho U}(\hat{k}) = \frac{1}{\pi} \int\limits_{0}^{1} \int\limits_{0}^{2\pi} I_i(\hat{k}') |\hat{k}'\cdot\hat{n}| \rho_U(\hat{k}*,\hat{k}'*) d\mu' \, d\varphi' \tag{39a}$$

$$I_{\tau U}(\hat{k}) = \frac{1}{\pi} \int\limits_{0}^{1} \int\limits_{0}^{2\pi} I_i(\hat{k}') |\hat{k}'\cdot\hat{n}| \tau_U(\hat{k}*,\hat{k}'*) d\mu' \, d\varphi' \tag{39b}$$

$$I_{\rho L}(\hat{k}) = \frac{1}{\pi} \int\limits_{-1}^{0} \int\limits_{0}^{2\pi} I_i(\hat{k}') |\hat{k}'\cdot\hat{n}| \rho_L(\hat{k}*,\hat{k}'*) d\mu' \, d\varphi' \tag{39c}$$

$$I_{\tau L}(\hat{k}) = \frac{1}{\pi} \int\limits_{-1}^{0} \int\limits_{0}^{2\pi} I_i(\hat{k}') |\hat{k}'\cdot\hat{n}| \tau_L(\hat{k}*,\hat{k}'*) d\mu' \, d\varphi' \tag{39d}$$

Note that no terms in these expressions have coordinates with asterisks except the bidirectional scattering functions. The necessity for expressing all scattered intensity in the observation frame (z-vertical) ultimately, even though the bidirectional scattering functions are customarily measured in the leaf frames (with asterisks), is emphasized by this fact. The absolute value on the dot or scalar product in the integral ensures only positive values of reflected intensity. Note that the vector intensity notation indicates the specific μ included in the integral. Remember that intensity may be conveniently expressed as a vector for some purposes; however, it does not always follow the laws of vector addition nor is it a vector field in the same sense that an electrostatic field is a vector field. The intensity field has a magnitude for every direction associated with every point. The electrostatic field has only one magnitude and direction associated with every point. Further, two beams do not add vectorally for purposes of reflection.

Because any incident radiation field on $\overline{da}$ can be viewed as a collection of beams contained within nonoverlapping, small solid angles, the basic problem can be reduced to considering incident radiation in some solid angle $\Delta\Omega_i$. Integration

gives the two pairs

$$I_{\rho U}(\hat{k}) = \frac{1}{\pi} I_i\left(\hat{k}_i\right)\left|\hat{k}_i\cdot\hat{n}\right|\rho_U\left(\hat{k}*,\hat{k}_i^*\right)\Delta\Omega_i \qquad (40a)$$

$$I_{\tau U}(\hat{k}) = \frac{1}{\pi} I_i\left(\hat{k}_i\right)\left|\hat{k}_i\cdot\hat{n}\right|\tau_U\left(\hat{k}*,\hat{k}_i^*\right)\Delta\Omega_i \qquad (40b)$$

if and only if incidence is on the upper surface, and

$$I_{\rho L}(\hat{k}) = \frac{1}{\pi} I_i\left(\hat{k}_i\right)\left|\hat{k}_i\cdot\hat{n}\right|\rho_L\left(\hat{k}*,\hat{k}_i^*\right)\Delta\Omega_i \qquad (40c)$$

$$I_{\tau L}(\hat{k}) = \frac{1}{\pi} I_i\left(\hat{k}_i\right)\left|\hat{k}_i\cdot\hat{n}\right|\tau_L\left(\hat{k}*,\hat{k}_i^*\right)\Delta\Omega_i \qquad (40d)$$

if and only if incidence is on the lower surface, where the nonzero regions are defined in equations (38a) and (38b). The dot product of a unit vector $\hat{k}$ in a direction of interest j and a unit vector $\hat{n}$ normal to $\overline{da}$ expressed in the observation frame without asterisks gives

$$\hat{k}_j\cdot\hat{n} = \cos\gamma_j = \sin\theta_j \sin\theta_n \cos\left(\varphi_j - \varphi_n\right) + \cos\theta_j \cos\theta_n \qquad (41)$$

where γ is the angle between unit vectors. By applying the transformation equations and integrating, the complete scattered intensity field for the individual element $\overline{da}$ is obtained. Note, however, that this field is dependent on the polar angles θ_n and φ_n.

Canopy Geometric Density Functions

The next step is to find the intensity of an ensemble of planar elements, each of which is characterized by a slope and an azimuth. Let N_φ and N_θ describe the number of elements $\overline{da}$ in the volume element ΔV that have a normal between φ_n and $\varphi_n + \Delta\varphi_n$ and θ_n and $\theta_n + \Delta\theta_n$, respectively. Let N equal the total

number of elements in ΔV. Then, density functions may be defined by

$$D\left(\varphi_n, \nu\right) = \lim_{\Delta\varphi_n \to 0} \frac{\Delta N\varphi_n}{N\,\Delta\varphi_n} \tag{42}$$

$$S\left(\theta_n, \eta\right) = \lim_{\Delta\theta_n \to 0} \frac{\Delta N\theta_n}{N\,\Delta\theta_n} \tag{43}$$

where ν and η are combined parameters indicating that $D(\varphi_n, \nu)$ and $S(\theta_n, \eta)$ may depend on such factors as stress, age, wind, and diurnal effects. The density functions $D(\varphi_n, \nu)$ and $S(\theta_n, \eta)$ then compose the fraction of leaves in the intervals φ_n to $\varphi_n + \Delta\varphi_n$ and θ_n to $\theta_n + \Delta\theta_n$ and thus are subject to the normalization condition

$$\int_0^{2\pi} D\left(\varphi_n, \nu\right) d\varphi_n = 1 \tag{44}$$

$$\int_0^{\frac{\pi}{2}} S\left(\theta_n, \eta\right) d\theta_n = 1 \tag{45}$$

The upper limit on $S(\theta_n, \eta)$ implies that the normal to the upper surface of $\overline{da}$ never drops below the horizontal. This simplifying assumption is convenient but not necessary. If this assumption is not true, then values of $\overline{da}$ having a downward-pointing $\hat{n}$ constitute another class of $\overline{da}$ that must be handled similarly to those not having a downward-pointing $\hat{n}$.

Formal Summing of Planar Ensemble Scattering

Having calculated the scattered radiance for an individual element $\overline{da}$, the problem is to sum the scattered radiance of all $\overline{da}$ in volume ΔV. Equations (38a)

and (38b) can be compactly written

$$I_{sa} = \sum_{ij} I_{ija} \tag{46}$$

where $i = \rho, \tau$ and $j = U, L$. The defining equation for intensity is

$$p = \int_0^1 \int_0^{2\pi} I(\hat{k}) |\hat{k} \cdot \overline{da}| \, d\Omega_s \tag{47}$$

Therefore, the power outgoing from $\overline{da}$ into $\Delta\Omega_s$ is

$$p_a = I_{sa}(\hat{k}) |\hat{k} \cdot \overline{da}| \Delta\Omega_s \tag{48}$$

where $I_{sa}(\hat{k})$ is intensity outgoing from $\overline{da}$ into $\Delta\Omega_s$. The total power into $\Delta\Omega_s$ from all $\overline{da}$ in ΔV is

$$p_s = \sum_{a=1}^{N} p_a \tag{49}$$

This sum over all p_a is accomplished by using the density functions $D(\varphi_n, \nu)$ and $S(\theta_n, \eta)$. The power into $\Delta\Omega_s$ of all $\overline{da}$ in ΔV in intervals $d\theta_n$ and $d\varphi_n$ at θ_n and φ_n then is given by

$$dp_a = n \, \Delta V D\left(\varphi_n, \nu\right) S\left(\theta_n, \eta\right) \left[\overline{da} \cdot I_{sa}\right] \Delta\Omega_s \, d\varphi_n \, d\theta_n \tag{50}$$

Thus

$$p_s = n \, \Delta V \, \Delta\Omega_s \int I_{sa}(\hat{k}) |\hat{k}_s \cdot \overline{da}| D\left(\varphi_n, \nu\right) S\left(\theta_n, \eta\right) d\varphi_n \, d\theta_n \tag{51}$$

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where the integration limits over φ_n and θ_n must be carefully defined. These limits will be considered later.

From equation (15), the power scattered from ΔA into $\Delta \Omega_s$ may be expressed by

$$p_s = I_s\left(\hat{k}_s\right)\Delta \Omega_s \ \Delta A \tag{52}$$

Equating expressions for p_s gives

$$I_s\left(\hat{k}_s\right) = \frac{n}{\Delta A}\frac{\Delta V}{}\int I_{sa}\left(\hat{k}_s\right)|\hat{k}_s \cdot \overline{da}|D\left(\varphi_n,\nu\right)S\left(\theta_n,\eta\right)d\varphi_n \ d\theta_n \tag{53}$$

Using equation (22), which defines the phase function, $P(\hat{k},\hat{k}')$ can be written

$$P\left(\hat{k}_s,\hat{k}_i\right) = \frac{4\pi I_s\left(\hat{k}_s\right)}{\frac{\Delta V}{\Delta A} I_i\left(\hat{k}_i\right)\Delta \Omega_i} \tag{54}$$

Then

$$P\left(\hat{k}_s,\hat{k}_i\right) = \frac{4\pi n \int D\left(\varphi_n,\nu\right)S\left(\theta_n,\eta\right)d\varphi_n \ d\theta_n}{I_i \ \Delta \Omega_i} \tag{55}$$

However, from equations (38a) and (38b)

$$I_{sa} = I_{\rho U} + I_{\tau U} + I_{\rho L} + I_{\tau L} \tag{56}$$

Note that

$$I_{sa}\left(\hat{k}_s\right)|\hat{k}_s \cdot \overline{da}| = I_{sa}\left(\hat{k}_s\right)da \ \cos \ \gamma_a \tag{57}$$

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Then, using equations (39a) to (39d) and ignoring integration limits, $P(\hat{k}_s, \hat{k}_i)$ may be compactly written

$$P\left(\hat{k}_s, \hat{k}_i\right) = 4\pi n \int \cos \gamma_i \, \cos \gamma_s \left(\rho_U + \tau_U + \rho_L + \tau_L\right)$$

$$\times D\left(\varphi_n, \nu\right) S\left(\theta_n, \eta\right) d\varphi_n \, d\theta_n \tag{58}$$

where $\cos \gamma_i$ is a function of $\theta_i, \varphi_i; \theta_n, \varphi_n$, and $\cos \gamma_s$ is a function of $\theta_s, \varphi_s;$ θ_n, φ_n. Transformation to the observation frame makes the bidirectional functions dependent on $\theta_i, \varphi_i; \theta_s, \varphi_s; \theta_n, \varphi_n$. Note also that $P(\hat{k}, \hat{k}')$ is symmetric under interchange of directions.

The phase function has been essentially reduced to an integral, the limits of which are not straightforward or obvious and must be carefully defined. The individual bidirectional reflection scattering functions for an individual element are noted as potentially different depending on the side of $\overline{da}$ on which the radiation is incident. In addition, the transformation of the bidirectional scattering functions from the observation frame with asterisks to the observation frame without asterisks has a complication. These functions are defined only in half spaces separated by the plane of $\overline{da}$. Therefore, the integration limits must be carefully specified to cover only the nonzero regions and thereby to prevent the appearance of contributions from noncontributing regions.

Phase-Function Integration Limits

For a given incident and scattered direction, the problem is to specify the integration limits on equation (58) so that integration includes only domains for which the particular bidirectional function is nonzero in the z-vertical observation frame. The quantity $I_{\rho U}$ will be nonzero for leaves both illuminated and viewed from the upper surface; $I_{\rho L}$ will be nonzero for leaves illuminated and viewed from the lower surface. The quantity $I_{\tau U}$ will be nonzero for leaves illuminated on the upper surface and viewed from the lower; $I_{\tau L}$ will be nonzero for leaves illuminated on the lower surface and viewed from the upper. The nonzero radiance for a given leaf depends on the combination of θ_n and φ_n ranges that characterizes the leaf.

The fundamental calculation used to define these various domains is calculation of the conditions for which a direction of interest is incident on the upper surface or incident on the lower surface. This direction of interest will be either the illumination or the view direction. This calculation may be done by considering the dot or scalar product between the outward normal from the upper surface of $\overline{da}$ and a unit vector in the direction of interest. Let $\hat{n}$ be the unit vector perpendicular to $\overline{da}$ and let $\hat{k}_i$ be a unit vector in the direction of interest. The dot

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product between unit vectors gives the cosine of the angle γ between them. This quantity, which was shown in equation (41) for γ_j, is

$$\cos \gamma_i = \sin \theta_i \sin \theta_n \cos \left(\varphi_i - \varphi_n \right) + \cos \theta_i \cos \theta_n \qquad (59)$$

If $\cos \gamma_i$ is positive, incidence is on the upper surface; if it is negative, incidence is on the lower surface. When $\cos \gamma_i$ is zero, $\hat{n}$ is perpendicular to $\hat{k}_i$ and, thus, the planar element is parallel to the direction of interest. When $\cos \gamma_i = 0$

$$\cos \delta = -\cot \theta_i \cot \theta_n \qquad (60)$$

with

$$\delta = \left(\varphi_i - \varphi_{n0} \right) \qquad (61)$$

where φ_{n0} is the azimuth angle of $\hat{n}$ at which $\cos \gamma_i$ changes signs. If $\left| \cot \theta_i \cot \theta_n \right| > 1$, then δ cannot exist and $\cos \gamma_i$ does not change signs. Because $\tan \theta_n$ is considered always positive, this condition may be written

$$\left| \cot \theta_i \right| > \tan \theta_n \qquad (62)$$

which holds if

$$0 < \theta_n < \frac{\pi}{2} - \theta_i \qquad (63a)$$

or if

$$0 < \theta_n < \theta_i - \frac{\pi}{2} \qquad (63b)$$

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Under the conditions of equation (63a), $\cos \gamma_i$ is positive; of equation (63b), negative. Thus, in the regions defined, $\hat{k}_i$ is incident on the upper and lower surfaces, respectively, independently of φ_n.

Requiring that $\cos \gamma_i$ be positive gives

$$\cos \left(\varphi_i - \varphi_n \right) > \cos \delta_i \qquad (64a)$$

where $0 < \theta_i < \pi/2$ or

$$\cos \left(\varphi_i - \varphi_n \right) > -\cos \delta_i \qquad (64b)$$

where $\pi/2 < \theta_i < \pi$. Under these conditions, $\cos \gamma_i$ is positive if

$$\varphi_i + \delta_i < \varphi_n < \varphi_i + 2\pi - \delta_i \qquad (65a)$$

where $0 < \theta_i < \pi/2$ and

$$\varphi_i + \pi - \delta_i < \varphi_n < \varphi_i + \pi + \delta_i \qquad (65b)$$

where $\pi/2 < \theta_i < \pi$.

Similarly, requiring that $\cos \gamma_i$ be negative gives

$$\cos \left(\varphi_i - \varphi_n \right) < \cos \delta_i \qquad (66a)$$

where $0 < \theta_i < \pi/2$ and

$$\cos \left(\varphi_i - \varphi_n \right) < -\cos \delta_i \qquad (66b)$$

where $\pi/2 < \theta_i < \pi$. Thus

$$\varphi_i + \delta_i < \varphi_n < \varphi_i + 2\pi - \delta_i \qquad (65a)$$

where $0 < \theta_i < \pi/2$ and

$$\varphi_i + \pi - \delta_i < \varphi_n < \varphi_i + \pi + \delta_i \qquad (65b)$$

where $\pi/2 < \theta_i < \pi$. The value of δ_i ranges from $\pi/2$ to π.

Equations (63a), (63b), (65a), and (65b) enable construction of table II, which gives six combined ranges of θ_n and φ_n for incident beams and a similar number for scattered beams. These combined ranges are organized by subsets depending on whether the direction of interest is in the upper or lower hemisphere with respect to the horizontal. The primed classes indicate lower hemisphere directions of interest. The symbols U and L indicate whether the incident or scattered beam lies in the upper or lower hemisphere with respect to the plane of the leaf. The subscripts i and s distinguish incident from scattered beam directions. The subscripts 1 and 2 distinguish ranges that are unrestricted in φ_n from those that are restricted in φ_n. Thus, class U_{s2} would give the range of θ_n and φ_n for which the scattered intensity emanates from the upper surface of the leaf and lies in the upper hemisphere with respect to the horizontal.

For a given incident and scattered direction, a leaf can lie only in one incident class and in one viewing or scattered class. The intersection of the six illumination classes with the six viewing or scattered classes then provides the integration limits over the bidirectional functions for all possible combinations of illumination and viewing. These intersections are labeled as classes 1 to 36 in table III, in which the braces { } mean that the greater of the two quantities contained within should be taken, and the brackets [ ] mean that the lesser of the two quantities contained within should be taken.

Canopy Phase Function

The phase function, including integration limits, can now be written using

equation (58) and table III.

$$P_M\left(\hat{k}_s, \hat{k}_i\right) = 4\pi n \ da \left[\int_a \rho_U + \int_b \tau_U + \int_c \rho_L + \int_d \tau_L\right]$$

$$X \cos \gamma_i \cos \gamma_s \ D\left(\varphi_n, \nu\right) S\left(\theta_n, \eta\right) d\varphi_n \ d\theta_n \qquad (67)$$

where the limits a, b, c, and d for the expression M are given in table IV. The entries refer to the numbers of the intersection regions in table III. The factors $\cos \gamma_i$ and $\cos \gamma_s$ are obtained from equation (41) and may be written explicitly

$$\cos \gamma_i = \sin \theta_i \sin \theta_n \cos \left(\varphi_i - \varphi_n\right) + \cos \theta_i \cos \theta_n \qquad (59)$$

$$\cos \gamma_s = \sin \theta_s \sin \theta_n \cos \left(\varphi_s - \varphi_n\right) + \cos \theta_s \cos \theta_n \qquad (68)$$

Note that i and s refer to incident and scattered directions for a given volume element, not for the canopy as a whole. These equations complete the formal specification of the phase function.

MODEL I - SOLUTION OF THE RADIATION TRANSFER EQUATION USING THE KUBELKA-MUNK APPROXIMATION

One approach to solving the radiation transfer equation begins with a technique for approximating the field within the canopy. When approximated, this field can be used to solve the radiation transfer equation using the full anisotropic canopy phase function. The assumption that the field can be factored to first approximation into an upwelling uniformly diffuse component, a downwelling uniformly diffuse component, and a reduced incident flux (remaining unscattered or absorbed incident flux) is made in this approximation technique. This assumption is equivalent to initially assuming a phase function that scatters uniformly across the upper hemisphere and uniformly across the lower hemisphere. The upwelling, downwelling, and reduced incident fluxes can be unambiguously expressed in terms of irradiance (power per unit area) on a horizontal plane at some level z in some layer i. Thus, $E_{\lambda\alpha}(d,z)$ is read as irradiance at z in layer α resulting from the uniformly diffuse upwelling intensity. The expression $E_{\lambda\alpha}(-d,z)$ is read as irradiance at z

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in layer α resulting from the downwelling uniformly diffuse intensity. The expression $E_{\lambda\alpha}(s,z)$ is read as irradiance at z in layer α resulting from the reduced incident specular flux. Each irradiance depends on wavelength λ.

Using $L_{\lambda\alpha}(d,z)$ (L as in Lambertian) for uniformly diffuse upwelling intensity (radiance), $L_{\lambda\alpha}(-d,z)$ for uniformly diffuse downwelling intensity, and $I_{\lambda\alpha}(s,z)$ for reduced incident intensity, the relationship between intensity and irradiance is given by

$$E_{\lambda\alpha}(d,z) = \pi L_{\lambda\alpha}(d,z) \tag{69}$$

$$E_{\lambda\dot\alpha}(-d,z) = \pi L_{\lambda\alpha}(-d,z) \tag{70}$$

$$E_{\lambda\alpha}(s,z) = \Delta\Omega_s I_{\lambda\alpha}(s,z)\mu_s \tag{71}$$

where $\Delta\Omega_s$ is the solid angle subtended by the source of the specular intensity. No confusion should arise from subsequently suppressing the subscripts λ and α. Note that the specular intensity could also be expressed by Dirac delta functions if desired.

Under the two stream plus specular and semi-isotropic phase-function assumptions, the radiation transfer equation can be transformed into modified Shuster or Kubelka-Munk equations. This transformation gives a set of equations

$$DE_i = \sum_{j=1}^{3} a_{ij}E_j \tag{72}$$

where $i = 1,2,3$, $D = \dfrac{d}{dz}$, a_{ij} represents expansion coefficients, and E_i and E_j refer to $E(d,z)$, $E(-d,z)$, and $E(s,z)$ as appropriate.

$$\sum_{j=1}^{3} \left[a_{ij} - \delta_{ij}D \right] E_j = 0 \tag{73}$$

where δ_{ij} is the Kronecker delta function. This equation has solutions if and only if the determinant

$$\det\left|a_{ij} - D\,\delta_{ij}\right| = 0 \qquad (74)$$

This equation gives the eigenvalue equation

$$\left[D - \lambda_\alpha\right]F_\alpha = 0 \qquad (75)$$

Solving for F_α gives

$$F_\alpha = F_\alpha(0)e^{\lambda_\alpha z} \qquad (76)$$

The symbol F_α represents eigenfunctions of the D operator with eigenvalues λ_α. The E_i may then be expressed as linear combinations of F_α.

$$E_i = \sum_{\alpha=1}^{3} A_{i\alpha}F_\alpha \qquad (77)$$

where expansion coefficients $A_{i\alpha}$ are functions of expansion coefficients a_{ij} and the boundary conditions.

The triplet of equations summed in equation (72) indicates that the rate of change in E_i is linear with that of E_j. The a_{ij} are mathematical expansion coefficients. The physical task is to identify the a_{ij} with physical parameters of the system that are measurable and that might vary with conditions to be detected. For example, if a_{ij} values vary with leaf slope, which may be a function of plant moisture stress, then moisture stress can possibly be detected by looking at the a_{ij}. The physics of this approach is contained almost exclusively in the a_{ij}.

Formal Solution of the Kubelka-Munk Equations

Explicitly, the Kubelka-Munk differential equations as given in equation (72)

can be written

$$\frac{dE_+}{dz} = -a_{11}E_+ + a_{12}E_- + a_{13}E_s \qquad (78)$$

$$\frac{dE_-}{dz} = -a_{21}E_+ + a_{22}E_- - a_{23}E_s \qquad (79)$$

$$\frac{dE_s}{dz} = a_{33}E_s \qquad (80)$$

where $E_1 = E_+$, $E_2 = E_-$, and $E_3 = E_s$. Equation (78) indicates that in the plus-z direction, the change of irradiance E_+ arises from E_+ either scattered or absorbed into E_-, which is a negative change requiring the minus sign, or from E_- and E_s scattered into E_+. Equation (79) is similar except that the change is positive in the minus-z direction; thus, the signs are changed. Examination of equation (80) shows that specular radiation once absorbed or scattered is no longer considered specular; thus, $a_{31} = a_{32} = 0$. Remember that E_s changes along the beam path, which is not necessarily along z. This characteristic will be noted in the explicit calculation of a_{33}.

Solutions of the secular equation give the operator equations

$$D - a_{33} = 0 \qquad (81)$$

$$D = \alpha_+ D = \alpha_- \qquad (82)$$

where

$$\alpha_\pm = \frac{a_{22} - a_{11}}{2} \pm \frac{\sqrt{\left(a_{11} + a_{22}\right)^2 - 4a_{12}a_{21}}}{2} \qquad (83)$$

Using equation (77), the following can be written.

$$E_+ = A_{11}e^{\alpha_+ z} + A_{12}e^{\alpha_- z} + A_{13}e^{a_{33}z} \tag{84}$$

$$E_- = A_{21}e^{\alpha_+ z} + A_{22}e^{\alpha_- z} + A_{23}e^{a_{33}z} \tag{85}$$

$$E_s = A_{33}e^{a_{33}z} \tag{86}$$

Substituting these expressions into equations (78) and (79) and equating coefficients of like exponentials gives

$$\left(\alpha_+ + a_{11}\right)A_{11} - a_{12}A_{21} = 0 \tag{87}$$

$$\left(\alpha_- + a_{11}\right)A_{12} - a_{12}A_{22} = 0 \tag{88}$$

$$\left(a_{33} + a_{11}\right)A_{13} - a_{12}A_{23} - a_{13}A_{33} = 0 \tag{89}$$

$$\left(\alpha_+ - a_{22}\right)A_{21} + a_{21}A_{11} = 0 \tag{90}$$

$$\left(a_{22} - \alpha_-\right)A_{22} - a_{21}A_{12} = 0 \tag{91}$$

$$\left(a_{33} - a_{22}\right)A_{23} + a_{21}A_{13} + a_{23}A_{33} = 0 \tag{92}$$

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Solving equations (89) and (92) gives

$$A_{23} = -\frac{a_{21}a_{13} + a_{23}\left(a_{33} + a_{11}\right)}{a_{11}a_{21} + \left(a_{33} + a_{11}\right)\left(a_{33} + a_{22}\right)} A_{33} \tag{93}$$

$$A_{13} = -\frac{a_{12}a_{23} - a_{13}\left(a_{33} - a_{22}\right)}{a_{12}a_{21} + \left(a_{33} - a_{22}\right)\left(a_{33} + a_{11}\right)} A_{33} \tag{94}$$

Solving the remaining four equations gives

$$A_{11} = \left(1 - \beta_+\right)Q \tag{95}$$

$$A_{12} = \left(1 + \beta_-\right)R \tag{96}$$

$$A_{21} = \left(1 + \beta_+\right)Q \tag{97}$$

$$A_{22} = \left(1 - \beta_-\right)R \tag{98}$$

where

$$\beta_+ = \frac{\alpha_+ + a_{11} - a_{12}}{\alpha_+ + a_{11} + a_{12}} \tag{99}$$

$$\beta_- = \frac{\alpha_- - a_{22} + a_{21}}{\alpha_- - a_{22} - a_{21}} \tag{100}$$

The quantities Q and R are determined from the boundary conditions. That radiation is continuous across layer boundaries is assumed. If the only external radiance source is without a diffuse component, then

$$E_-(0) = 0 \tag{101}$$

Further, the upwelling radiation field at the soil is given by

$$E_+\left(z_{soil}\right) = \frac{E_-\left(z_{soil}\right)}{\pi^2}\int_0^1\int_0^1\int_0^{2\pi}\int_0^{2\pi} \rho_{soil}(\hat{k},\hat{k}')\mu\mu'\ d\mu'\ d\varphi'\ d\mu\ d\varphi$$

$$+ E_s \int_0^1\int_0^{2\pi} \rho\left(\hat{k}_s,\hat{k}'\right)\mu'\ d\mu' \tag{102}$$

The quantity A_{33} from equation (86) is clearly the value of specular radiation at $z = 0$, the top of the canopy. The distance from the top of the canopy to the soil is z_{soil}. The reflectivity of the soil is ρ_{soil}. Thus

$$A_{33} = E_s(0) \tag{103}$$

The expression $a_{ij} = a_{ji}$ will be shown to be equivalent to assuming that the two sides of the scattering components (leaves etc.) are optically indistinguishable and are uniformly diffuse (Lambertian) scatterers. Under this assumption, the following equations are obtained where $\alpha_\pm$ is defined as $\pm g$.

$$g = \left(a^2 - b^2\right)^{\frac{1}{2}} \tag{104}$$

where $a = a_{11} = a_{22}$ and $b = a_{12} = a_{21}$.

$$A_{13} = \frac{c'b - c(k - a)}{g^2 - k^2} A_{33} \qquad (105)$$

where $c = a_{13}$, $c' = a_{23}$, and $k = a_{33}$.

$$A_{23} = \frac{cb + c'(k + a)}{g^2 - k^2} A_{33} \qquad (106)$$

$$\delta_{\pm} = \delta = \left(\frac{a - b}{a + b}\right)^{\frac{1}{2}} \qquad (107)$$

$$A_{11} = \left(1 - \delta\right) Q \qquad (108)$$

$$A_{12} = \left(1 + \delta\right) R \qquad (109)$$

$$A_{21} = \left(1 + \delta\right) Q \qquad (110)$$

$$A_{22} = \left(1 - \delta\right) R \qquad (111)$$

The solution to the differential equations from equations (84) to (86) can be written

$$E_{+}(z) = Q(1 - \delta)e^{gz} + R(1 + \delta)e^{-gz} + A_{13}e^{kz} \qquad (112)$$

$$E_{-}(z) = Q(1 + \delta)e^{gz} + R(1 - \delta)e^{-gz} + A_{23}e^{kz} \qquad (113)$$

$$E_s(z) = A_{33}e^{kz} \qquad (114)$$

Generalization to multilayers is accomplished by using the unabbreviated notation for E (eqs. (69) to (71)) and subscripting all expansion coefficients a_{ij} and A_{ij} and their derivative quantities with the layer label α.

Development of Expansion Coefficients a_{11}, a_{12}, a_{21}, and a_{22}

The first step in deriving expansion coefficients a_{11}, a_{12}, a_{21}, and a_{22} is to consider the interaction of a single planar element with a downwelling or up-welling uniformly diffuse field. An edge-on view of such an element is presented in figure 1. The symbols L_+ and L_- designate the upwelling and downwelling uniformly diffuse radiance of the field. These are related to the upwelling and downwelling irradiance evaluated on a horizontal plane by

$$\pi L_\pm = E_\pm \qquad (115)$$

Let $I_{\rho U}(\hat{k})$ be the intensity reflected from the upper surface of da; let $I_{\tau U}(\hat{k})$ be the intensity incident on the upper surface transmitted outward from the lower surface; and let $I_{\rho L}(\hat{k})$ and $I_{\tau L}(\hat{k})$ be the reflected and transmitted intensity, respectively, for intensity incident on the lower surface. Some portion of these reflected and transmitted intensities will be upwelling and some portion downwelling. These intensities are expressed as a function of the direction $\hat{k}$, which may be expressed by the usual θ and φ polar coordinate angles in any chosen coordinate frame.

In this coordinate-frame-independent notation, the bidirectional reflection and transmission functions may be written $\rho_j(\hat{k},\hat{k}')$ and $\tau_j(\hat{k},\hat{k}')$ with $j = U,L$. The symbols U and L refer to the upper and lower surface of incidence, respectively. Thus, the defining equations for $\rho_j(\hat{k},\hat{k}')$ and $\tau_j(\hat{k},\hat{k}')$ may be written

$$I_{\rho j}(\hat{k}) = \frac{1}{\pi} \int_M I_i(\hat{k}')\rho_j(\hat{k},\hat{k}')\hat{k}'\cdot\hat{n}\, d\Omega' \qquad (116)$$

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and

$$I_{\tau j}(\hat{k}) = \frac{1}{\pi} \int_M I_i(\hat{k}')\tau_j(\hat{k},\hat{k}')\hat{k}'\cdot\hat{n} \, d\Omega' \tag{117}$$

The quantity I_i refers to incident intensity, and integration is over the upper or lower hemisphere as defined by the plane of $\overline{da}$.

Although the subscripts on τ, U and L, will be retained as a simplifying measure, note that reciprocity requires the transmission function obtained for upper surface incidence to be related to the one obtained for lower surface incidence by simple interchange of direction. That is, because

$$\tau(\hat{k},\hat{k}') = \tau(-\hat{k}',-\hat{k}) \tag{37b}$$

and because $\hat{k}$ and $\hat{k}'$ always lie in opposite hemispheres for transmission, then

$$\tau_L(-\hat{k}',-\hat{k}) = \tau_U(-\hat{k}',-\hat{k}) \tag{118}$$

Clearly, the objective of this development is to obtain the total upwelling and the total downwelling power resulting from the scattering of L_+ and L_- by the single planar element da. The coordinate frame for performing the integrations may be chosen in any way that simplifies the calculation. The coordinate frame used in this portion of the model derivation is illustrated in figure 1 and defined in table V. In this frame, the x-axis is down the line of maximum slope of the leaf and the y-axis is perpendicular to da. That is, $\hat{n} = \hat{\varepsilon}_y$. In this frame

$$\hat{n}\cdot\hat{k} = \sin\theta \sin\varphi \tag{119}$$

Customarily, the measured bidirectional reflection and transmission functions are given in the upper leaf frame with $\hat{n} = \hat{\varepsilon}_z^*$ and $\hat{\varepsilon}_x$ down the central line of symmetry. When the central line of symmetry is the line of maximum slope, the following coordinate transformations are obtained; the quantities that have asterisks are the

usual spherical polar angles in the upper leaf frame.

$$\left.\begin{aligned}
\hat{\varepsilon}_y &= \hat{\varepsilon}^*_z \\
\hat{\varepsilon}_x &= \hat{\varepsilon}^*_x \\
\hat{\varepsilon}_z &= \hat{\varepsilon}^*_y
\end{aligned}\right\} \tag{120}$$

$$\left.\begin{aligned}
\sin\theta\,\sin\varphi &= \cos\theta^* \\
\sin\theta\,\cos\varphi &= \sin\theta^*\,\cos\varphi^* \\
\cos\theta &= \sin\theta^*\,\sin\varphi^*
\end{aligned}\right\} \tag{121}$$

$$\left.\begin{aligned}
\theta^* &= \cos^{-1}\left[\sin\theta\,\sin\varphi\right] \\
\varphi^* &= \cot^{-1}\left[\tan\theta\,\cos\varphi\right] \\
\theta &= \cos^{-1}\left[\sin\theta^*\,\sin\varphi^*\right] \\
\varphi &= \cot^{-1}\left[\tan\theta^*\,\cos\varphi^*\right]
\end{aligned}\right\} \tag{122}$$

In the integration frame, space may be partitioned as shown in table V and figure 1. All L_- radiance incident on the upper surface of da in this partitioning is contained in region a, all L_- radiance incident on the lower surface of da is contained in region b, all L_+ radiance incident on the upper surface is contained in region c, and all L_+ radiance incident on the lower surface is contained in region d. Further, all upwelling intensity emerging from the upper surface is contained in region a, all downwelling intensity emerging from the upper surface is contained in region c, all upwelling intensity emerging from the lower surface is contained in region b, and all downwelling intensity emerging from the lower surface is contained in region d. (Emerging intensity is that either reflected from the side or transmitted from the opposite side.) These regions give the integration frame limits whereby the scattered intensity can be written explicitly and, hence, the upwelling and downwelling scattered power can be obtained, as shown in the calculations that follow. The scattered intensities resulting from the downwelling diffuse intensity L_- are given from equations (116) and (117) by using the integration limits a and

b; thus

$$I_{\rho U}(\hat{k}) = \frac{L_-}{\pi} \int_a \rho_U(\hat{k},\hat{k}') |\hat{n} \cdot \hat{k}'| \, d\Omega' \tag{123}$$

$$I_{\tau U}(\hat{k}) = \frac{L_-}{\pi} \int_a \tau_U(\hat{k},\hat{k}') |\hat{n} \cdot \hat{k}'| \, d\Omega' \tag{124}$$

$$I_{\rho L}(\hat{k}) = \frac{L_-}{\pi} \int_b \rho_L(\hat{k},\hat{k}') |\hat{n} \cdot \hat{k}'| \, d\Omega' \tag{125}$$

$$I_{\tau L}(\hat{k}) = \frac{L_-}{\pi} \int_b \tau_L(\hat{k},\hat{k}') |\hat{n} \cdot \hat{k}'| \, d\Omega' \tag{126}$$

The subscripts ρ and τ indicate reflected and transmitted intensity, respectively. Similarly, L and U indicate lower and upper surface incidence, respectively.

Let I_+ denote that portion of $I_{\rho U}$, $I_{\rho L}$, $I_{\tau U}$, and $I_{\tau L}$ directed upward, and let I_- denote that portion directed downward. Then, from the defining equation for intensity, the upward-directed power p_+ is given by

$$p_+ = da \int_0^{2\pi} I_+(\hat{k}) |\hat{n} \cdot \hat{k}| \, d\Omega \tag{127}$$

and the downward-directed power p_- is given by

$$p_- = da \int_0^{2\pi} I_-(\hat{k}) |\hat{n} \cdot \hat{k}| \, d\Omega \tag{128}$$

Thus, p_+ and p_- can be written as

$$p_+\left(\theta_n,\varphi_n\right) = da\int_a\left[I_{\rho U}\left(\hat{k};\theta_n,\varphi_n\right)|\hat{n}\cdot\hat{k}| + I_{\tau L}\left(\hat{k};\theta_n,\varphi_n\right)|\hat{n}\cdot\hat{k}|\right]d\Omega$$

$$+ da\int_b\left[I_{\rho L}(\hat{k})|\hat{n}\cdot\hat{k}| + I_{\tau U}(\hat{k})|\hat{n}\cdot\hat{k}|\right]d\Omega \tag{129}$$

and

$$p_-\left(\theta_n,\varphi_n\right) = da\int_c\left[I_{\rho U}\left(\hat{k};\theta_n,\varphi_n\right)|\hat{n}\cdot\hat{k}| + I_{\tau L}(\hat{k})|\hat{n}\cdot\hat{k}|\right]d\Omega$$

$$+ da\int_d\left[I_{\rho L}\left(\hat{k};\theta_n,\varphi_n\right)|\hat{n}\cdot\hat{k}| + I_{\tau U}(\hat{k})|\hat{n}\cdot\hat{k}|\right]d\Omega \tag{130}$$

The quantities p_+ and p_- are the upwelling and downwelling power from a single scattering component in ΔV. This component is characterized by a slope θ_n. The ensemble of elements in ΔV has some distribution of slopes. Thus, a function $S(\theta_n,\eta)$ may be defined so that $S(\theta_n,\eta)d\theta_n$ is the fraction of components in ΔV having slopes between θ_n and $\theta_n + \theta_n$ $(0 < \theta_n < \pi/2)$. The quantity $S(\theta_n,\eta)$ is a probability density function that describes the distribution of slopes and azimuths. Thus, average power $<p_+>$ and $<p_->$ are defined by

$$<p_+> = \int_0^{\frac{\pi}{2}} p_+\left(\theta_n\right)S\left(\theta_n,\eta\right)d\theta_n \tag{131}$$

$$<p_-> = \int_0^{\frac{\pi}{2}} p_-\left(\theta_n\right)S\left(\theta_n,\eta\right)d\theta_n \tag{132}$$

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where $S(\theta_n, \eta)$ is normalized by

$$1 = \int_0^{\frac{\pi}{2}} S\left(\theta_n, \eta\right) d\theta_n \qquad (133)$$

Note that p_+ and p_- are not irradiances (power per unit area) evaluated on a horizontal plane but power per component element directed upward or downward. To obtain the power per unit area across a horizontal plane, calculation of the number of elements per unit of horizontal area is necessary. A volume element is

$$dV = dz\ dA \qquad (134)$$

where dA is an element of horizontal area. Then, the number of elements in dV is

$$N = n\ dV = n\ dz\ dA \qquad (135)$$

Then, the upwelling power from dV is simply

$$dp_+ = N<p_+> = n\ dz\ dA<p_+> \qquad (136)$$

Similarly, the downwelling power from dV is

$$dp_- = N<p_-> = n\ dz\ dA<p_-> \qquad (137)$$

However, dp_+/dA is simply the irradiance on a horizontal plane resulting from scattering upward by the collection of elements in dV, and dp_-/dA is, similarly, the irradiance on a horizontal plane resulting from scattering downward by the com-

ponents in dV. Thus

$$dE_+ = \frac{dp_+}{dA} = n\langle p_+\rangle dz \qquad (138)$$

$$dE_- = \frac{dp_-}{dA} = n\langle p_-\rangle dz \qquad (139)$$

From equations (138) and (139), the change in irradiance on a horizontal plane resulting from scattering of downwelling L_- by the components in the layer dz can be written

$$\frac{dE_+}{dz} = n\langle p_+\rangle \qquad (140)$$

$$\frac{dE_-}{dz} = n\langle p_-\rangle \qquad (141)$$

Similarly, the scattering of the upwelling uniformly diffuse field can be obtained. Let the primes indicate quantities arising from L_+. Then

$$I'_{\rho U}(\hat{k}) = \frac{L_+}{\pi} \int_c \rho_U(\hat{k},\hat{k}')\hat{n}\cdot\hat{k}' \; d\Omega' \qquad (142)$$

$$I'_{\tau L}(\hat{k}) = \frac{L_+}{\pi} \int_d \tau_L(\hat{k},\hat{k}')\hat{n}\cdot\hat{k}' \; d\Omega' \qquad (143)$$

$$I'_{\rho L}(\hat{k}) = \frac{L_+}{\pi} \int_d \rho_L(\hat{k},\hat{k}')\hat{n}\cdot\hat{k}' \; d\Omega' \qquad (144)$$

$$I'_{\tau U}(\hat{k}) = \frac{L_+}{\pi} \int_c \tau_U(\hat{k},\hat{k}')\hat{n}\cdot\hat{k}' \; d\Omega' \qquad (145)$$

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The limits c and d were given in table V.

Dividing the radiation field into upward and downward power flow gives

$$p'_+ = da \int_a \left[I'_{\rho U}(\hat{k}) |\hat{n} \cdot \hat{k}| + I'_{\tau L}(\hat{k}) |\hat{n} \cdot \hat{k}| \right] d\Omega$$

$$+ da \int_b \left[I'_{\rho L}(\hat{k}) |\hat{n} \cdot \hat{k}| + I_{\tau U}(\hat{k}) |\hat{n} \cdot \hat{k}| \right] d\Omega \qquad (146)$$

$$p'_- = da \int_c \left[I'_{\rho U}(\hat{k}) |\hat{n} \cdot \hat{k}| + I_{\tau L}(\hat{k}) |\hat{n} \cdot \hat{k}| \right] d\Omega$$

$$+ da \int_d \left[I'_{\rho L}(\hat{k}) |\hat{n} \cdot \hat{k}| + I_{\tau U}(\hat{k}) |\hat{n} \cdot \hat{k}| \right] d\Omega \qquad (147)$$

The quantities $<p'_+>$ and $<p'_->$ are then given from the slope density functions by

$$<p'_+> = \int_0^{\frac{\pi}{2}} p'_+\left(\theta_n\right) S\left(\theta_n, \eta\right) d\theta_n \qquad (148)$$

$$<p'_-> = \int_0^{\frac{\pi}{2}} p'_-\left(\theta_n\right) S\left(\theta_n, \eta\right) d\theta_n \qquad (149)$$

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Thus

$$\frac{dE'_+}{dz} = n<p'_+> \tag{150}$$

and

$$\frac{dE'_-}{dz} = n<p'_-> \tag{151}$$

Thus, change in irradiance, both downwelling and upwelling, due to scattered L_+ and L_- has been obtained. The next step in calculating expansion coefficient a_{ij} for $i = 1,2$ and $j = 1,2$ is to calculate the total power incident on $\overline{da}$ from L_+ and L_-. From this quantity, the total power intercepted by both sides of the components in the volume element can be calculated.

The defining equation for intensity or radiance gives the relationship between intensity and power delivered to an area.

$$p_i = \int I_i(\theta,\varphi)\hat{k}\cdot\overline{da}\ d\Omega \tag{152}$$

where $\hat{k}$ is a unit vector in the direction of the intensity. In the coordinate frame shown in table V

$$\hat{n}\cdot\hat{k} = \sin\theta\ \sin\varphi \tag{119}$$

where $\hat{n}$ is a unit vector normal to $\overline{da}$. When power is intercepted from L_- by $\overline{da}$, note that L_- is uniformly diffuse and, thus, constant; however, it is not upwelling. Therefore, it is nonzero only for regions a and b in table V. That intensity contained in region a is incident on the upper side of $\overline{da}$, and that intensity contained in region b is incident on the lower side of $\overline{da}$. Thus, the total power delivered to the upper surface of $\overline{da}$ by L_- is given by

$$p_U = daL_- \int_a \sin^2 \theta \sin \varphi \, d\theta \, d\varphi = \frac{\pi}{2}(1 + \cos \theta) \qquad (153)$$

and the total power delivered by L_- to the lower surface of $\overline{da}$ is given by

$$p_L = daL_- \int_b \sin^2 \theta \sin \varphi \, d\theta \, d\varphi = \frac{\pi}{2}(1 - \cos \theta) \qquad (154)$$

Similarly, using regions c and d, the power delivered to the upper surface by the upwelling L_+ is given by

$$p'_U = daL_+ \int_c \sin^2 \theta \sin \varphi \, d\theta \, d\varphi = \frac{\pi}{2}(1 - \cos \theta) \qquad (155)$$

and the total power delivered to the lower surface of $\overline{da}$ by L_+ is given by

$$p'_L = daL_+ \int_d \sin^2 \theta \sin \varphi \, d\theta \, d\varphi = \frac{\pi}{2}(1 + \cos \theta) \qquad (156)$$

Adding the power intercepted by both sides, the total power from L_- intercepted by $\overline{da}$ is obtained.

$$P_T = p_U + p_L = \pi L_- \, da \qquad (157)$$

The total power from L_+ intercepted by $\overline{da}$ is

$$P'_T = p'_U + p'_L = \pi L_+ \, da \qquad (158)$$

A peculiarity of a uniformly diffuse field is that the total power intercepted by an area is independent of slope or orientation. With equation (158), the derivation has

been completed, and it is possible to write expressions for a_{11}, a_{12}, a_{21}, and a_{22}.

Returning to the Kubelka-Munk equations and their development, note that $a_{11}E_+$ is the portion of E_+ lost over dz because of either absorption or redirection downward of E_+. The total upwelling power intercepted by da is given in equation (158). The amount of power emerging upward from da is given in equation (148). The difference is that power lost by intercepting da. Thus

$$a_{11}E_+ = n\left[L_+\pi \ da \ - <p'_+>\right] \tag{159}$$

Similarly, $a_{22}E_-$ is the change in E_- lost over dz because of either absorption or redirection of E_-. Using a similar argument for the derivation of $a_{11}E_+$ and using equations (132) and (157), $a_{22}E_-$ is given by

$$a_{22}E_- = n\left[L_-\pi \ da \ - <p_->\right] \tag{160}$$

The expression $a_{12}E_-$ is the portion added to E_+ over dz by scattering of E_- into E_+. This quantity is given directly from equation (140) by

$$a_{12}E_- = \frac{dE_+}{dz} = n<p_+> \tag{161}$$

Similarly, $a_{21}E_+$ is the change in E_- over dz resulting from scattering of E_+ into E_-. This quantity is given directly from equation (151) by

$$a_{21}E_+ = n<p'_-> \tag{162}$$

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Note that $E_- = L_-\pi$ and $E_+ = L_+\pi$. The first four a_{ij} can be written

$$a_{11} = n\left[da - \frac{<p'_+>}{L_+\pi}\right] \tag{163}$$

$$a_{12} = \frac{n<p_+>}{L_-\pi} \tag{164}$$

$$a_{21} = \frac{n<p'_->}{L_+\pi} \tag{165}$$

$$a_{22} = n\left[da - \frac{<p_->}{L_-\pi}\right] \tag{166}$$

The preceding a_{ij} are not necessarily equal. Equality of some of the a_{ij} is obtained only under special circumstances to be discussed in a later special-example calculation. The preceding derivation leaves only a_{13}, a_{23}, and a_{33} undeveloped. These coefficients relate changes in the specular or directed component incident on the upper surface of the canopy. Dealing with the specular component is somewhat more complicated because the incident intensity is not uniform over a hemisphere; however, it is simplified somewhat because integrals over beams (differential solid angles) are trivial.

Development of Expansion Coefficient a_{33}

A primary assumption for all models has been that the scattering components (leaves, branches, etc.) are representable by a collection of planar scattering elements $\overline{da}$, each of which has an associated slope θ_n and azimuth φ_n and a unit normal to the surface $\hat{n}$. Then, the slope angle θ_n will be the customary zenith angle θ in a z-axis vertical coordinate frame. Furthermore, density functions may be defined as $D(\varphi_n, \nu)$ (eq. (42)) and $S(\theta_n, \eta)$ (eq. (43)). The expression $D(\varphi_n, \nu)$ represents the fraction of leaves having an azimuth between φ_n and $\varphi_n + d\varphi_n$. The expression $S(\theta_n, \eta)$ represents the fraction of leaves having a zenith angle between θ_n and $\theta_n + d\theta_n$. Using the outward-drawn unit normal, a vector element was defined in equation (5).

Define $I_0(\theta_0, \varphi_0)$ as the specular intensity incident on the upper surface of the canopy; this intensity will be contained in the solid angle $\Delta\Omega_0$ intercepted by the Sun. The expression $\hat{k}_0(\theta_0, \varphi_0)$ is a unit vector in the direction of the solar or specular intensity. Then, the irradiance on a horizontal plane at the upper surface of the canopy would be given by

$$E_s(0) = I_0 \, \Delta\Omega_0 \mu_0 \qquad (167)$$

where $\mu_0 \equiv \cos\theta_0$ and where the irradiance of the specular beam measured perpendicular to the beam is given by

$$E_b = I_0 \, \Delta\Omega_0 \qquad (168)$$

The cross section presented by some $\overline{da}$ to the beam will be given by

$$\overline{da} \cdot \hat{k}_0 = da |\cos\gamma_0| \qquad (169)$$

The expression for $\cos\gamma_0$ is given by considering the dot product of $\hat{k}_0$ and $\hat{n}$ expressed in a z-axis vertical coordinate frame where the x direction (direction of $\varphi = 0$) is left arbitrary. (See eq. (41).) Defining the angle between $\hat{k}_0$ and $\hat{n}$ to be γ_0 gives

$$|\cos\gamma_0| = |\sin\theta_0 \sin\theta_n \cos\left(\varphi_0 - \varphi_n\right) + \cos\theta_0 \cos\theta_n| \qquad (170)$$

The expression for $\cos\gamma_0$ is then a function of θ_n and φ_n explicitly. Using the density functions $D(\varphi_n, \nu)$ and $S(\theta_n, \eta)$, an average cross section may be defined by

$$<da>_{\varphi_n \theta_n} = da \int_0^{2\pi} \int_0^{\frac{\pi}{2}} D\left(\varphi_n, \nu\right) S\left(\theta_n, \eta\right) |\cos\gamma_0| d\varphi_n \, d\theta_n \qquad (171)$$

or

$$<da>_{\varphi_n \theta_n} = da < | \cos \gamma_0 | > \tag{172}$$

where

$$< | \cos \gamma_0 | > = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} D\left(\varphi_n, \nu\right) S\left(\theta_n, \eta\right) | \cos \gamma_0 | d\varphi_n \, d\theta_n \tag{173}$$

where it is assumed that no leaf has a slope $> \pi/2$; that is, elements do not turn over.

Performing the integration over θ_n and φ_n to obtain $< \cos \gamma_0 >$ is completely straightforward if $\cos \gamma_0$ has no negative values. If $\cos \gamma_0$ has a negative value, as would usually be expected, and thus passes through zero, the integral must be broken up into a sum of integrals over regions for which the sign of $\cos \gamma_0$ does not change.

The average value of a function $\oint(t)$ over an interval T is given by

$$<\oint(t)>_T = \frac{1}{T} \int_0^T \oint(t) dt \tag{174}$$

As noted, $< | \oint(t) | > = <\oint(t)>$ if and only if the sign of $\oint(t)$ does not change. If, however, there are regions of the interval T for which $\oint(t)$ possesses a negative sign, the absolute value of the function over the whole interval T is found by taking the negative value of the integral over the regions in which the function is negative and the positive value of the integral over the regions in which the function is positive.

$$< | \oint(t) | >_T = \frac{1}{T} \left[\int_{T_+} \oint(t) dt - \int_{T_-} \oint(t) dt \right] \tag{175}$$

where T_+ denotes the positive regions of $\zeta(t)$ and T_- represents the negative regions of $\zeta(t)$ within the interval T. The boundaries of these regions are marked by the function passing through zero; thus, the boundary is found by solving $\zeta(t) = 0$ to obtain T_0.

For the specific instance of obtaining $\langle da \rangle$, the basic problem is averaging $\cos \gamma_0$. Setting $\cos \gamma_0 = 0$ (eq. (170)) gives

$$\sin \theta_n \sin \theta_o \cos(\varphi_n - \varphi_o) + \cos \theta_n \cos \theta_o = 0 \tag{176}$$

or

$$\cos \delta = -\cot \theta_n \cot \theta_o \tag{177}$$

where δ is defined as the value of $\varphi_n - \varphi_o$ for which $\cos \gamma_0 = 0$. Physically, this value corresponds to the value of δ for which the leaf is parallel to the solar beam. Because θ_n and θ_o are $0 \le \theta_n \le \pi/2$ and $0 \le \theta_o \le \pi/2$, respectively, both $\cot \theta_n$ and $\cot \theta_o$ are positive. Thus, because the cosine is never greater than 1, the condition for δ to exist can be expressed as

$$\cot \theta_n \cot \theta_o \le 1 \tag{178}$$

from which the following may be written

$$\frac{\pi}{2} \le \theta_n + \theta_o \le \pi \tag{179}$$

as can be seen from the following trigonometric manipulations.

$$\cot \theta_n \cot \theta_o = \frac{\cos(\theta_n - \theta_o) + \cos(\theta_n + \theta_o)}{\cos(\theta_n - \theta_o) - \cos(\theta_n + \theta_o)} \le 1 \tag{180a}$$

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or

$$\cos\left(\theta_n - \theta_o\right) + \cos\left(\theta_n + \theta_o\right) \leq \cos\left(\theta_n - \theta_o\right) - \cos\left(\theta_n + \theta_o\right) \quad \text{(180b)}$$

or

$$\cos\left(\theta_n + \theta_o\right) \leq 0 \quad \text{(180c)}$$

Thus,

$$\frac{\pi}{2} \leq \theta_n + \theta_o \leq \pi \quad \text{(179)}$$

The average of $\cos \gamma_0$ over φ_n can then be written, using equations (178) and (179), as

$$<|\cos \gamma_0|>_{\varphi_n}{}' = \frac{1}{2\pi} \left\{ \int_0^{\delta+\varphi_o} - \int_{\delta+\varphi_o}^{2\pi-\delta-\varphi_o} + \int_{2\pi-\delta-\varphi_o}^{2\pi} \right\} \cos \gamma_0 \; D\left(\varphi_n\right) d\left(\varphi_n\right) \quad \text{(181)}$$

Then, averaging over θ_n is done by rewriting the existence condition for δ (as defined in eq. (179)) as $\delta \neq 0$ if and only if $\pi/2 - \theta_n \leq \theta_n \leq \pi/2$ and as $\delta = 0$ if and only if $0 \leq \theta_n < \pi/2 - \theta_n$. Thus, two ranges over θ_n are established: that for which δ exists and that for which it does not. Then, the complete expression for the average of $\cos \gamma_0$ over θ_n and φ_n is given by

$$<|\cos \gamma_0|>_{\theta_n \varphi_n} = \frac{1}{2\pi} \left\{ \int_0^{\frac{\pi}{2}-\theta_o} \int_0^{2\pi} + \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}} \left[\int_0^{\delta+\varphi_o} - \int_{\delta+\varphi_o}^{2\pi-\delta-\varphi_o} + \int_{2\pi-\delta-\varphi_o}^{2\pi} \right] \right\}$$

$$\cos \gamma_0 \; S\left(\theta_n\right) D\left(\varphi_n\right) d\theta_n \; d\left(\varphi_n\right) \quad \text{(182)}$$

For the interesting phenomenon of azimuthal symmetry where $D(\varphi_n) = 1$, equation (182) can be simplified by using the general relationship

$$\int_0^{2\pi} \cos\psi \, d\psi = 0 \qquad (183)$$

and the normalization equation for the azimuthal density function (eq. (44)). Performing the integrations gives

$$
<|\cos\gamma_0|>_{\theta_n\varphi_n} = 4\left\{ \frac{\pi}{2}\cos\theta_o \int_0^{\frac{\pi}{2}-\theta_o} S\left(\theta_n\right)\cos\theta_n \, d\theta_n \right.
$$

$$
+ \cos\theta_o \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}} \cos\theta_n \, S\left(\theta_n\right)\left(\delta + \varphi_o - \frac{\pi}{2}\right)d\theta_n
$$

$$
\left. + \sin\theta_o \sin\varphi_o \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}} \sin\theta_n \, S\left(\theta_n\right)\sin\left(\delta + \varphi_o\right)d\theta_n \right\} \qquad (184)
$$

However, equation (177) and the identity $\tan = \sin/\cos$ give

$$\cos\theta_n = -\frac{\sin\theta_n \cos\left(\delta + \varphi_o\right)\sin\theta_o}{\cos\theta} \qquad (185)$$

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Thus, the second and third terms of equation (184) can be combined.

$$
\langle \cos \gamma_0 \rangle_{\theta_n \varphi_n} = 4 \left\{ \frac{\pi}{2} \cos \theta_o \int_0^{\frac{\pi}{2}-\theta_0} S\left(\theta_n\right) \cos \theta_n \, d\theta_n \right.
$$

$$
\left. + \cos \theta_o \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}} \cos \theta_n \, S\left(\theta_n\right) \left[\delta + \varphi_o - \frac{\pi}{2} - \tan\left(\delta + \varphi_o\right)\right] d\theta_n \right\} \tag{186}
$$

The quantity a_{33} can now be developed. The amount of $I_0(\theta_0, \varphi_0)$ intercepted over a path $d\ell$ along $\hat{k}_0(\theta_0, \varphi_0)$ will be proportional to the fraction of the total cross section of the beam obscured by the elements in a volume element along $d\ell$. A volume element dV along $d\ell$ is given by

$$
dV = da \, d\ell \tag{187}
$$

where da is the cross section of the volume element. The number of elements in dV is given by

$$
N = n \, dV \tag{188}
$$

where n is the number density of elements. The total cross section of all elements in dV is then

$$
N\langle da \rangle = n \, dV \langle da \rangle \tag{189}
$$

However, the fraction of the beam obscured is given by

$$
F = \frac{N\langle da \rangle}{da} = n \, d\ell \langle da \rangle \tag{190}
$$

Then

$$
\frac{dE}{d\ell} = \frac{FE_0}{d\ell} = n \langle da \rangle E_0 \tag{191}
$$

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However

$$d\ell = \frac{dz}{\mu_0} \tag{192}$$

Therefore

$$a_{33} = \frac{n<da>}{\mu_0} \tag{193}$$

Now, only the quantities a_{13} and a_{23} must be developed. This development is complicated for several reasons. The specular incident intensity is incident on the top of some leaves and on the bottom of others. Because I_0 depends on φ, azimuthal dependence becomes important. For this reason, all elements, including the bidirectional reflection and transmission functions, must be referenced to a single coordinate frame.

Development of Expansion Coefficients a_{13} and a_{23}

The expansion coefficients a_{13} and a_{23}, which relate the specular intensity to that scattered upward and downward, are obtained by noting $a_{13}E_0$ as the change in upwelling diffuse intensity over dz due to scattering of specular intensity where

$$E_0 = I_0 \, \Delta\Omega_0 \mu_0 \tag{194}$$

Similarly, $a_{23}E_0$ is the change in downwelling diffuse intensity over dz due to scattering of specular intensity.

From the development of the phase function, it was shown that the scattered intensity from a volume element upon which intensity I_0 was incident was given by equation (22).

$$I_s(\hat{k}) = \frac{n}{4\pi} P\left(\hat{k},\hat{k}_0\right)\frac{\Delta V}{\Delta A} I_0 \, \Delta\Omega_0 \tag{195}$$

From equation (15), the power (p_+ for upwelling and p_- for downwelling) scattered from the volume element is given by

$$p_\pm = \Delta A \int_{\pm h} I_s \, d\Omega = \frac{1}{4\pi} \Delta V I_0 \, \Delta\Omega_0 \int_{\pm h} P\left(\hat{k}, \hat{k}_0\right) d\Omega \qquad (196)$$

where $\pm h$ refers to upper or lower hemisphere, respectively. Thus, p_+ is upwelling and p_- is downwelling power. However,

$$\frac{p_\pm}{\Delta V} = \frac{dE_\pm}{dz} \qquad (197)$$

and

$$\frac{dE_\pm}{dz} = \frac{E_0}{4\mu_0\pi} \int_{\pm h} P\left(\hat{k}, \hat{k}_0\right) d\Omega \qquad (198)$$

Therefore, a_{13} and a_{23} are given by

$$a_{13} = \frac{1}{4\pi\mu_0} \int_{+h} P\left(\hat{k}_0, \hat{k}'\right) d\Omega' \qquad (199)$$

$$a_{23} = \frac{1}{4\pi\mu_0} \int_{-h} P\left(\hat{k}_0, \hat{k}'\right) d\Omega' \qquad (200)$$

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Bidirectional Reflection Function for the Canopy as a Whole

The objective of the Kubelka-Munk approach is to obtain an estimate of the radiation field at any level within the canopy. This estimate is then used in the formal solution of the radiation transfer equation to obtain the upwelling field at the upper surface of the canopy. The formal solution to the radiation transfer equation as shown in the appendix equations (269) to (278) may be written

$$\mu \frac{dI(\tau,\hat{k})}{d\tau} = I(\tau,\hat{k}) - F(\tau,\hat{k}) \tag{201}$$

where τ is the optical depth of the canopy, with $F(\tau,\hat{k})$ given by

$$F(\tau,\hat{k}) = \frac{1}{4\pi\overline{\sigma}} \int_{-1}^{1}\int_{0}^{2\pi} P(\hat{k},\hat{k}')I_t(\tau,\hat{k}')d\Omega' \tag{202}$$

The formal solution is given by the method of integrating factors as

$$I\left(\tau,\hat{k}_{\pm}\right) = I\left(\tau_0,\hat{k}_{\pm}\right)e^{-\left|\frac{\tau-\tau_0}{\mu}\right|} + \left|\frac{1}{\mu}\int_{\tau_0}^{\tau} F\left(\tau,\hat{k}_{\pm}\right)e^{-\left|\frac{\tau-\tau'}{\mu}\right|}d\tau'\right| \tag{203}$$

where τ_0 is the upper surface of the canopy and τ' is specific optical depth, with $d\tau = -n <da> dz$ and $\hat{k}_{\pm}$ being the upwelling or the downwelling $\hat{k}$. The function $I_t(\tau,\hat{k}')$ is the total field at τ, which is obtained from the Kubelka-Munk solution. Thus

$$I_t(\tau,\hat{k}) = L_+(\tau) + L_-(\tau) + I_s\left(\tau,\hat{k}_0\right) \tag{204}$$

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However

$$L_+(\tau) = \frac{E_+(\tau)}{\pi} \qquad (205a)$$

$$L_-(\tau) = \frac{E_-(\tau)}{\pi} \qquad (205b)$$

and

$$I_s\left(\tau, \hat{k}_0\right) = I_s\left(0, \hat{k}_0\right) e^{-\left|\frac{\tau-\tau_0}{\mu_0}\right|}$$

$$= \frac{E_s\left(0, \hat{k}_0\right)}{\Delta\Omega_0 \mu_0} e^{-\frac{\left(\tau-\tau_0\right)}{\mu_0}} \qquad (206)$$

Thus, $F(\tau, \hat{k})$ can be explicitly written

$$F\left(\tau, \hat{k}_\pm\right) = \frac{1}{4\pi^2\bar{\sigma}} \left\{ \frac{E_-(\tau)}{\pi} \int_0^1\int_0^{2\pi} P\left(\hat{k}_\pm, \hat{k}'\right) d\mu'\, d\varphi' \right.$$

$$+ \frac{E_+(\tau)}{\pi} \int_{-1}^0\int_0^{2\pi} P\left(\hat{k}_\pm, \hat{k}'\right) d\mu'\, d\varphi'$$

$$\left. + \frac{1}{4\pi} \frac{E_0}{\mu_0} e^{-\left|\frac{\tau-\tau_0}{\mu_0}\right|} P\left(\hat{k}_\pm, \hat{k}_0\right) \right\} \qquad (207)$$

where τ_0 is the upper surface of the canopy.

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From a knowledge of $I(\tau_0, \hat{k})$, it is possible to specify a bidirectional reflection function for the canopy as a whole. The defining equation for the canopy bidirectional reflection function can be written

$$S_U(\tau; \hat{k}, \hat{k}') = 4\pi\mu \; \frac{I(0, \hat{k})}{I_s(\hat{k}_0) \Delta\Omega_0} \tag{208}$$

where $I(0, \hat{k})$ is the upwelling intensity at the top of the canopy, μ is the cosine of the zenith angle of $\hat{k}$, and τ is the optical thickness of the canopy as defined in equation (275) in the appendix. The expression $I_s(\hat{k}_0)$ represents the illumination at the top of the canopy contained in $\Delta\Omega_0$. The canopy bidirectional scattering functions are discussed in greater detail in appendix equation (288) and those that follow.

Once the canopy bidirectional reflection function is known, the upwelling radiation for any condition of illumination may be calculated. Because the canopy bidirectional reflection function is so complicated, a thorough understanding of its characteristics can only be attained by programing the model and doing variational studies. However, valuable insight can be obtained by considering some less complex, special examples.

MODEL II

The purpose of the canopy model is to aid in the interpretation of spectral data. Ideally, the probably unattainable objective for a canopy model would be to provide unique, theoretical signature inverses from which signatures or collections of signatures could be clearly interpreted in terms of type, condition, maturity, yield, and so forth. Realistically, however, the model should improve understanding about how various characteristics affect signatures or how data might best be taken to minimize ambiguities. In short, a realistic goal would be that the model provide understanding and, thus, improve data-taking and analysis techniques.

One problem in using the previously derived model approach is the complexity. It is difficult to gain insight simply from the study of the structure of such complicated systems of equations. Insight can be gained through programing the models and performing parametric variation analysis.

Another practical problem is the paucity of bidirectional scattering data for canopy components. Although some information is available in the NASA Earth Resources Spectral Information System, it is not sufficient. The best available work known to the author was done by Breece and Holmes (ref. 3), who have studied the bidirectional scattering characteristics of green soybean and corn leaves for top incidence.

The obvious approach is to consider some simpler but, hopefully, sufficiently realistic models. At best, the simpler models based on reasonable assumptions (at least for some wavelengths) may produce solid predictions of canopy reflection. At worst, use of such models will simplify the equations so that insights for guiding the parametric equations may be gained by merely studying the structure of the simpler equations.

The most apparent simplifying assumption is that the bidirectional scattering functions are constant; that is, to assume that the components are uniformly diffuse or Lambertian scatterers. Breece and Holmes suggest that this is an excellent assumption for the transmission function at wavelengths from 0.37 to 1 micrometer and almost equally as good for the upper surface reflection function at wavelengths higher than 0.75 micrometer. An array of planar scatterers, each of which is individually Lambertian, does not necessarily give rise to Lambertian scattering for the entire array.

Additional simplifying assumptions for the modified Chandrasekhar approach include assuming the azimuthal density function $D(\varphi_n, \nu)$ to be constant and the slope density function $S(\theta_n, \eta)$ to have some analytic form, such as a Gaussian distribution around some mean slope. A further simplification is obtained by considering only the vertical look direction, which is appropriate to spacecraft-mounted sensor systems.

The first special situation is that of constant leaf bidirectional reflection functions. The reflected and transmitted intensities for upper surface incidence on component da are found from integrating equations (123), (124), (142), and (145); for lower surface incidence, from integrating equations (125), (126), (143), and (144). In the following equations, G is defined as $(\pi/2)(1 - \cos\theta)$ and Γ is defined as $\sin\theta \sin\varphi$.

$$\int_a \Gamma \, d\Omega = \pi - G \tag{209}$$

$$\int_b \Gamma \, d\Omega = G \tag{210}$$

$$\int_c \Gamma \, d\Omega = G \tag{211}$$

$$\int_d \Gamma \, d\Omega = \pi - G \tag{212}$$

where

$$d\Omega = \sin\theta \; d\theta \; d\varphi \tag{213}$$

Thus, for L_-, integrating equations (123) to (126) gives

$$I_{\rho U} = \frac{L_- \rho_U}{\pi} \left[\pi - G\right] \tag{214}$$

$$I_{\tau L} = \frac{L_- \tau_L}{\pi} \left[G\right] \tag{215}$$

$$I_{\rho L} = \frac{L_-}{\pi} \rho_L G \tag{216}$$

$$I_{\tau U} = \frac{L_- \tau_U}{\pi} \left[\pi - G\right] \tag{217}$$

Integrating equations (142) to (145) for L_+ gives

$$I'_{\rho U} = \frac{L_+ \rho_U}{\pi} G \tag{218}$$

$$I'_{\tau L} = \frac{L_+ \tau_L}{\pi} \left(\pi - G\right) \tag{219}$$

$$I'_{\rho L} = \frac{L_+ \rho_L}{\pi} \left(\pi - G\right) \tag{220}$$

$$I'_{\tau U} = \frac{L_+ \tau_U}{\pi} G \tag{221}$$

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These expressions can then be used to integrate equations (129) and (130) to give upwelling power and downwelling power due to downwelling incident intensity.

$$p_+ = \frac{daL_-}{\pi} \left\{ \left[\rho_U (\pi - G) + \tau_L G \right] (\pi - G) + \left[\rho_L G + \tau_U (\pi - G) \right] G \right\} \qquad (222)$$

$$p_- = \frac{daL_-}{\pi} \left\{ \left[\tau_U (\pi - G) + \rho_L G \right] (\pi - G) + \left[\tau_L G + \rho_U (\pi - G) \right] G \right\} \qquad (223)$$

Integrating equations (146) and (147) gives upwelling and downwelling power due to upwelling incident intensity

$$p'_+ = \frac{daL_+}{\pi} \left\{ \left[\tau_L (\pi - G) + \rho_U G \right] (\pi - G) + \left[\tau_U G + \rho_L (\pi - G) \right] G \right\} \qquad (224)$$

$$p'_- = \frac{daL_+}{\pi} \left\{ \left[\rho_L (\pi - G) + \tau_U G \right] (\pi - G) + \left[\rho_U G + \tau_L (\pi - G) \right] G \right\} \qquad (225)$$

Then set

$$\pi D' = \int_0^{\frac{\pi}{2}} G^2 \left(\theta_n \right) S \left(\theta_n, \eta \right) d\theta_n \qquad (226)$$

$$\pi R' = \int_0^{\frac{\pi}{2}} G \left(\theta_n \right) S \left(\theta_n, \eta \right) d\theta_n \qquad (227)$$

Thus, using equations (131), (132), (148), and (149), the average upwelling and downwelling power from scattering of downwelling and upwelling intensity is given by

$$\langle p_+ \rangle = daL_- \left\{ \left[\rho_U + \rho_L - 2\tau \right] \pi D' + 2 \left[\tau - \rho_U \right] \pi R' + \rho_U \pi \right\} \qquad (228)$$

$$\langle p_- \rangle = daL_- \left\{ \left[2\tau - \left(\rho_U + \rho_L \right) \right] \pi D' + \left[\rho_L + \rho_U - 2\tau \right] \pi R' + \tau\pi \right\} \qquad (229)$$

$$\langle p'_+ \rangle = daL_+ \left\{ \left[2\tau - \left(\rho_U + \rho_L \right) \right] \pi D' + \left[\rho_L + \rho_U - 2\tau \right] \pi R' + \tau\pi \right\} \qquad (230)$$

$$\langle p'_- \rangle = daL_+ \left\{ \left[\rho_U + \rho_L - 2\tau \right] \pi D' + \left[2 \left(\tau - \rho_L \right) \right] \pi R' + \rho_L \pi \right\} \qquad (231)$$

Then, using equations (163) to (166), which define the expansion coefficients, a_{11}, a_{12}, a_{21}, and a_{22} may be written

$$a_{11} = n\, da \left\{ 1 - \tau + \left[2\tau - \left(\rho_U + \rho_L \right) \right] D' + \left[\left(\rho_U + \rho_L \right) - 2\tau \right] R' \right\} \qquad (232)$$

$$a_{12} = n\, da \left\{ \left[\left(\rho_U + \rho_L \right) - 2\tau \right] D' + 2 \left[\tau - \rho_U \right] R' + \rho_U \right\} \qquad (233)$$

$$a_{21} = n\, da \left\{ \rho_L + \left[\left(\rho_U + \rho_L \right) - 2\tau \right] D' + 2 \left[\tau - \rho_L \right] R' \right\} \qquad (234)$$

$$a_{22} = n\, da \left\{ 1 - \tau + \left[2\tau - \left(\rho_U + \rho_L \right) \right] D' + \left[\left(\rho_L + \rho_U \right) - 2\tau \right] R' \right\} \qquad (235)$$

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Further, consider the situation in which $\rho_U = \rho_L$; this gives

$$a_{11} = n \ dA\left\{1 - \tau + (\tau - \rho)B\right\} \tag{236}$$

$$a_{12} = n \ dA\left\{\rho + (\rho - \tau)B\right\} \tag{237}$$

$$a_{21} = n \ dA\left\{\rho + (\rho - \tau)B\right\} \tag{238}$$

$$a_{22} = n \ dA\left\{1 - \tau + (\tau - \rho)B\right\} \tag{239}$$

where

$$D' - R' = B = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} G\left(\frac{G}{\pi} - 1\right) S\left(\theta_n, \eta\right) d\theta_n \tag{240}$$

Notice that where $\rho = \tau$, the B expression, which carries all the slope characteristics, is removed from the equation. A preliminary conclusion is that in those spectral regions in which transmission is nearly equal to reflection, the effects of leaf slope are minimized; thus, variations in slope caused by wind should cause decreased statistical variance in $S(\hat{k},\hat{k}')$ and, hence, in signature.

The expansion coefficients a_{13} and a_{23} for constant bidirectional scattering functions can be found in a straightforward manner by noting that the reflection and transmission scattered intensity is proportional to the irradiance. Thus, from the defining equations for the bidirectional scattering functions (eqs. (37b) and (116) to (122) and table V), the intensity due to reflection from the upper surface $I_{\rho U}$, the intensity due to transmission from upper to lower surface $I_{\tau U}$, the intensity due to reflection from the lower surface $I_{\rho L}$, and the intensity due to transmission from

the lower to the upper surface $I_{\tau L}$ can be written

$$I_{\rho U} = \frac{I_0 \, \Delta\Omega_0 \rho_U}{\pi} \left| \hat{k}_0 \cdot \hat{n} \right| \tag{241}$$

$$I_{\tau L} = \frac{I_0 \, \Delta\Omega_0 \tau_L}{\pi} \left| \hat{k}_0 \cdot \hat{n} \right| \tag{242}$$

$$I_{\rho L} = \frac{I_0 \, \Delta\Omega_0 \rho_L}{\pi} \left| \hat{k}_0 \cdot \hat{n} \right| \tag{243}$$

$$I_{\tau U} = \frac{I_0 \, \Delta\Omega_0 \tau_U}{\pi} \left| \hat{k}_0 \cdot \hat{n} \right| \tag{244}$$

where

$$\hat{k}_0 \cdot \hat{n} \equiv \cos \gamma_0 = \sin \theta_0 \sin \theta_n \cos \left(\varphi_0 - \varphi_n \right) + \cos \theta_0 \cos \theta_n \tag{245}$$

If $\cos \gamma_0$ is positive, then incident intensity is on the upper surface of da and $I_{\rho L} = I_{\tau L} = 0$. If $\cos \gamma_0$ is negative, then incidence is on the lower surface of da and $I_{\rho U} = I_{\tau U} = 0$.

In the section entitled "Canopy Geometric Density Functions," it was shown that the conditions for upper and lower incidence can be written as follows. When $\cos \gamma_0 > 0$

$$0 < \theta_n < \frac{\pi}{2} - \varphi_0 \qquad 0 < \varphi_n < 2\pi \tag{246a}$$

or

$$\frac{\pi}{2} - \varphi_0 < \theta_n < \frac{\pi}{2} \qquad \varphi_i - \delta_i < \varphi_n < \varphi_i + \delta_i \tag{246b}$$

When $\cos \gamma_0 < 0$

$$\frac{\pi}{2} - \theta_0 < \theta_n < \frac{\pi}{2} \qquad \varphi_i + \delta_i < \varphi_n < \varphi_0 + 2\pi - \delta_i \tag{247}$$

where, expressed in a z-vertical coordinate system

$$\delta_i = \cos^- \left[\cot \theta_n \cot \theta_0\right] \tag{248}$$

The upwelling power reflected from the upper side of da is given from a considera-
tion of figure 1 by

$$P_{\rho U}^+ = I_{\rho U} \ da \int_a \cos \gamma \ d\Omega \tag{249}$$

where the upwelling power is independent of coordinate representation; therefore,
a convenient representation is chosen: $\hat{n} = \hat{x}$ and $\hat{y}$ is along the slope of da.
Thus,

$$\cos \gamma = \sin \theta \sin \varphi \tag{250}$$

Similarly, the downwelling reflected power from the upper surface is given by

$$P_{\rho U}^- = I_{\rho U} \ da \int_c \cos \gamma \ d\Omega \tag{251}$$

The downwelling transmitted power is given by

$$p_{\tau U}^- = I_{\tau U} \ da \int_d \cos \gamma \ d\Omega \tag{252}$$

and the upwelling transmitted power is given by

$$p_{\tau U}^+ = I_{\tau U} \ da \int_b \cos \gamma \ d\Omega_0 \tag{253}$$

Combining equations (249), (253), (241), and (244) gives the total upwelling power for a scattering element da illuminated on the upper surface.

$$p_U^+\left(\theta_n, \varphi_n\right) = \frac{I_0 \, \Delta\Omega_0}{\pi} \cos \gamma_0 \, da \left\{\rho_U \int_a + \tau_U \int_b\right\} \cos \gamma \, d\Omega \qquad (254)$$

And combining equations (241), (244), (251), and (252) gives the total downwelling power for a scattering element da illuminated on the upper surface.

$$p_U^-\left(\theta_n, \varphi_n\right) = \frac{I_0 \, \Delta\Omega_0}{\pi} \, da \, \cos \gamma_0 \left\{\rho_U \int_c + \tau_U \int_d\right\} \cos \gamma \, d\Omega \qquad (255)$$

Note that these expressions are for a particular da characterized by particular values of θ_n and φ_n.

The integrals appearing in the preceding expressions have been evaluated in equations (209) to (212) and, thus

$$p_U^+\left(\theta_n, \varphi_n\right) = \frac{I_0 \cdot \Delta\Omega_0 \, \cos \gamma_0 \, da}{\pi} \left[\rho_U \pi + \left(\tau_U - \rho_U\right) G\right] \qquad (256)$$

and

$$p_U^-\left(\theta_n, \varphi_n\right) = \frac{I_0 \, \Delta\Omega_0 \, \cos \gamma_0 \, da}{\pi} \left[\tau_U \pi + \left(\rho_U - \tau_U\right) G\right] \qquad (257)$$

where, by definition, $G = (\pi/2)(1 - \cos \theta)$. When elements are illuminated by I_0 in $\Delta\Omega_0$ incident on the lower side of the element da, they are handled similarly. The reflected intensity is given in equation (243), and the transmitted intensity is given in equation (242). Then, the upwelling reflected power is

$$p_{\rho L}^+ = I_{\rho L} \, da \int_b \hat{k}_r \cdot \hat{n} \, d\Omega \qquad (258)$$

and the downwelling reflected power is

$$p_{\rho L}^{-} = I_{\rho L} \ da \int_{d} \hat{k}_r \cdot \hat{n} \ d\Omega \qquad (259)$$

The upwelling transmitted power is

$$p_{\tau L}^{+} = I_{\tau L} \ da \int_{a} \hat{k}_r \cdot \hat{n} \ d\Omega \qquad (260)$$

and the downwelling transmitted power is

$$p_{\tau L}^{-} = I_{\tau L} \ da \int_{c} \hat{k}_r \cdot \hat{n} \ d\Omega \qquad (261)$$

Using the already evaluated integrals in equations (209) to (212), the total upwelling power for lower incidence from da is

$$p_{L}^{+}\left(\theta_n, \varphi_n\right) = \frac{I_0 \ \Delta\Omega_0 \ \cos \gamma_0}{\pi} \ da \left[\tau_L \pi + \left(\rho_L - \tau_L\right) G\right] \qquad (262)$$

and the total downwelling power for lower incidence is

$$p_{L}^{-}\left(\theta_n, \varphi_n\right) = \frac{I_0 \ \Delta\Omega_0 \ \cos \gamma_0}{\pi} \ da \left[\rho_L \pi + \left(\tau_L - \rho_L\right) G\right] \qquad (263)$$

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The total upward-directed power is obtained by integrating all da over the distributions expressed in the density functions $S(\theta_n,\eta)$ and $D(\varphi_n,\nu)$, where the limits are obtained from the conditions on the sign of $\cos\gamma_0$ given in equations (246) and (247).

The total upward-directed power by all elements in ΔV is given by

$$p^+ = n\,\Delta V\left[\int_0^{\frac{\pi}{2}-\theta_0}\int_0^{2\pi} + \int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}}\int_{\varphi_0-\delta_0}^{\varphi_0+\delta_0}\right]S\left(\theta_n,\eta\right)D\left(\varphi_n,\nu\right)p_U^+\,d\theta_n\,d\varphi_n$$

$$+\,n\,\Delta V\int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}}\int_{\varphi_0+\delta_0}^{\varphi_0+2\pi-\delta_0} p_L^+\left(\theta_n,\varphi_n\right)S\left(\theta_n,\eta\right)D\left(\varphi_n,\nu\right)d\theta_n\,d\varphi_n \qquad (264)$$

and the total downwelling power by all elements in ΔV is given by

$$p^- = n\,\Delta V\left[\int_0^{\frac{\pi}{2}-\theta_0}\int_0^{2\pi} + \int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}}\int_{\varphi_0-\delta_0}^{\varphi_0+\delta_0}\right]p_U^-\left(\theta_n,\varphi_n\right)S\left(\theta_n,\eta\right)D\left(\varphi_n,\nu\right)d\theta_n\,d\varphi_n$$

$$+\,n\,\Delta V\int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}}\int_{\varphi_0+\delta_0}^{\varphi_0+2\pi-\delta_0} p_L^-\left(\theta_n,\varphi_n\right)S\left(\theta_n,\eta\right)D\left(\varphi_n,\nu\right)d\theta_n\,d\varphi_n \qquad (265)$$

As in equations (196) to (200), a_{13} and a_{23} are found from the total upward and downward power by

$$a_{13} = \frac{p^+}{\Delta VE_0} \qquad (266)$$

$$a_{23} = \frac{p^-}{\Delta VE_0} \qquad (267)$$

Because the quantity a_{33} is independent of the bidirectional scattering functions, it is unchanged from the expression given in equation (193). Obtaining the same upper and lower bidirectional reflection functions simply requires removing the subscripts U and L from the previous expressions for expansion coefficients a_{ij}.

Interestingly, in the wavelength regions where $\rho = \tau$, the geometrical factor G disappears. Thus, the effect of leaf slope in those regions may be expected to be diminished inasmuch as leaf slope appears only in a_{33}. Therefore, in these spectral regions, leaf slope variations probably will be dominantly seen in the single scattering component of the specular intensity as expressed in the a_{33} term. In addition, slope will be expected to affect optical depth as defined in equation (275) (appendix), which will then change the effect of lower layers and the soil. However, if the canopy optical depth is a factor of 3 or more (that is, the specular beam has been reduced by a factor of e^{-3}), then slope will have little effect in exposing the soil to either illumination or view. Thus, soil will have little effect on reflected intensity.

MULTILAYER CANOPIES

Heretofore, the canopy model calculations were explicitly written for a single-layer, single-constituent model. For some canopies, these calculations will be sufficient. For others, extension of the model to multilayer, multiconstituent canopies will be necessary. All the fundamental calculations and principles for the multilayer, multiconstituent canopy were developed in the single-layer, single-constituent calculations. An explicit expression of the multilayer, multiconstituent model would require an extremely complicated notation. However, a description of procedures for performing the extension can be easily presented and should be sufficient for any investigator attempting to program the model.

Generalization to multilayers affects the model at essentially two points: the boundary conditions on the Kubelka-Munk equations for a given layer and the calculation (using the formal solution to the radiation transfer equation) to find the upwelling intensity at the surface of the canopy. The boundary conditions for a single-layer canopy are used to solve for the terms Q and R as given in equations (95)

to (98). In the multilayer calculation, each layer is subject to a separate set of Kubelka-Munk equations, constructed for each layer in exactly the same way as for the single-layer canopy. Thus, each layer will have Q and R terms. The set of equations resulting from the equations for Q and R for each layer will be a set of simultaneous algebraic equations that must be solved using the boundary conditions appropriate for the multilayer calculation. The multilayer calculation gives a set of equations in Q_i and R_i where i is the index for the layer. If there are four layers, then i takes the values 1 to 4.

The boundary conditions at the top of the canopy and at the canopy-soil interface are the same for one layer or for many layers. These boundary conditions are given in equations (101) and (102). An additional boundary condition between layers is that the downwelling uniformly diffuse radiation at the bottom of one layer is equal to the downwelling uniformly diffuse radiation at the top of the layer immediately below it. Similarly, the upwelling uniformly diffuse radiation at the top of any one layer is equal to the upwelling uniformly diffuse radiation at the bottom of the layer immediately above it. The attenuated or reduced specular beam intensity at the bottom of one layer is equal to the attenuated specular intensity at the top of the layer immediately below it as well. In short, the total radiation field at any interface between layers has only one value, and that value is shared by both layers.

The other point at which multilayers affect the model calculation is in the use of the formal solution to the radiation transfer equation (eq. (2)) to find the upwelling intensity at the top of the canopy. The problem may be most simply solved in a sequential, layer by layer manner, beginning at the bottom and working upward. First, the radiation transfer equation formal solution is used to find the intensity at the top of the bottom layer by using the Kubelka-Munk-derived fields for the bottom layer in precisely the same way described for the single-layer model. By implication, a phase function for the first layer must be calculated by using precisely the same procedure described for the single-layer model. Once the intensity at the top of the bottom layer is found, this value becomes the same as $I_0(\hat{k})$ for the second layer, and the process is repeated for each successive layer until the value of upwelling intensity at the top of the top layer is found. This value of upwelling intensity in the $\hat{k}$ direction is then used in precisely the same way to determine the total canopy reflection scattering function $S(\hat{k},\hat{k}')$ as was done in the single-layer development.

Generalization to multiconstituent canopies affects the model development in two places. The first place is in the a_{ij}, the expansion coefficients for the Kubelka-Munk equations. The second place is in the development of the phase function. Conceptually, the generalization of the a_{ij} to multiconstituents is simple. Because the a_{ij} are developed from the assumption of single scattering within some volume element, a separate a_{ij} can be developed for each constituent. These a_{ij} are then added. If the constituent index is taken as m, then for an N-constituent layer, the

total a_{ij} for the layer would be given by

$$a_{ij} = \sum_{m=1}^{N} \left(a_{ij}\right)_m \qquad (268)$$

The total a_{ij} would then be used to solve the Kubelka-Munk equations in the same way as for a single-constituent layer. Similarly, the phase function for a multiconstituent layer would be the simple sum of phase functions for each constituent.

Clearly, conceptual generalization of the model to multilayers and multiconstituents is not particularly difficult; however, the notation and computation can become complex. In summary, computation for multilayers and multiconstituents within layers requires the calculation and summing of a set of a_{ij} and the calculation and summing of a phase function for each constituent within each layer. Of course, the physical system could enable some fortuitous simplifications. For example, if the canopy consisted of leaves in which only the volume density varied with height and if reflectivity, transmissivity, and slope and azimuth distributions remained the same, the calculation of the a_{ij} and the phase function for multilayers would not be difficult. Similar simplifications might be effected if only slope or azimuth distributions changed with height or if only reflectivity and transmissivity changed with canopy height.

UNRESOLVED QUESTIONS AND DIRECTIONS FOR FUTURE WORK

Although much has been done, much remains to be done in the fundamental theoretical analysis of canopy reflection. A differential equation approach requires the assumption that a differential volume element can be meaningfully defined. This assumption implies the existence of a volume element that is small compared to the depth of the canopy and that has average values similar to those of adjacent volume elements. The validity of this assumption seems to be related to leaf density in a way that is not quantitatively clear. Thus, one important question is over what dimension can homogeneity be assumed? A related question is what canopy characteristics are necessary for the Kubelka-Munk-type approximation to be valid? These characteristics are probably a function of both density and geometric distribution. Such questions could be usefully investigated in a laboratory using a carefully constructed artificial canopy having translucent plastic arrays.

Another unanswered question concerns the horizontal inhomogeneity or row effects on canopy reflectance. Purely theoretical consideration of this problem is even more difficult than that for the nonrow example considered in this paper. Hopefully, the study of special, less complex situations will enable simplification of the problem to the point at which the horizontally anisotropic situation can be made tractable.

Another important and possibly critical question that is not addressed in this paper or in other current modeling efforts is the theoretical characterization of grain heads such as those found in wheat. The straightforward characterization of a head of wheat as a collection of planar elements may not be possible. This question is important because 50 percent of the top layer volume of mature wheat can be occupied by heads. The solution to this problem may be simpler than it appears; however, the problem remains unsolved.

In the future, a workable canopy model less complex than those presented in this paper probably can be developed. This development should be possible once the dominant effects are clearly established. For example, it may be found that the radiation reflected only once from the leaves is the dominant effect and that the remainder of the upwelling radiation of the canopy can be ignored. Further, because single-surface radiation often is strongly polarized, it is probable that front surface scattering can be isolated and also that polarization may prove to be a powerful remote-sensing diagnostic tool. Even though the models could be generalized to include polarization by writing them in polarization matrix form, this formulation has not been done. Clearly, analysis of the models presented in this paper raises as many questions as it answers.

CONCLUDING REMARKS

An ultimate usefulness of canopy modeling in remote sensing is the identification and quantification of the causes of variation in the statistical properties of canopy signatures. These statistical properties include the mean vector, the covariance matrix, and the correlation matrix or some other measure of separability. Excluding atmospheric effects, the models presented can be used to identify these causes of signature variation. Little can be said quantitatively, however, until models have been programed and experimentally verified, factorial tests have been performed, and limits on applicability have been determined. Nevertheless, some insight can be gained from the structure of the model equations.

The causes of canopy signature variation are identified as variations in component (leaf, stem, tassel, etc.) slope and azimuth distributions as expressed in the slope and azimuth density functions, variations in component density, variations in component bidirectional reflection and transmission, variations in soil reflection properties, and variations in illumination conditions of both solar and sky components. The complexity of the physical situation is evident; thus, the complexity of the canopy model follows. However, in spite of the complexity, analysis of the models shows that the canopy modeling problem is solvable. Because of the complexity of the physical situation, it is probable that a purely experimental program to determine the cause and effect relationships of canopy signatures is impractical and may be impossible. Thus, the use of theoretical models to quantify signature variation cause and effect appears essential in any program. Further, because of the complexity of the physical situation expressed in the models, it is likely that experimental program data and remote-sensing measurements will be taken under circumstances that result in simplifications. One possible simplification is to restrict measurements to that portion of the spectrum in which skylight is minimal and to atmospheric conditions in which cloud contributions are low. Another possible simplification would be to restrict look directions to the vertical, though such a restriction may impose undesirable limits on diagnostic capability.

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A final conclusion from the structure of Model II is that there are probably regions of the spectrum in which variations in slope and azimuth distributions will have minimal effect on signature variation. In these regions, the components are approximately Lambertian and the transmission and reflection of the leaves is approximately equal.

Although this investigation is incomplete, the insights presented will hopefully contribute to raising the quality of future remote-sensing research.

Lyndon B. Johnson Space Center
National Aeronautics and Space Administration
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APPENDIX

MODEL III – THE GENERALIZED CHANDRASEKHAR APPROACH TO SOLVING THE RADIATION TRANSFER EQUATION FOR A PLANAR CANOPY

INTRODUCTION

Canopy modeling must relate incident intensity to upwelling intensity at the upper surface of the canopy. This relationship is obtainable from the solution to the radiation transfer equation for the canopy. Solving the radiation transfer equation has two major aspects. First, a phase function must be developed; second, the total intensity field at every point in the canopy must be found. A technique for solving the radiation transfer equation that is suitable for canopies and that can be approximated by plane-parallel homogeneous layers is discussed in this appendix. The technique, Model III, is a modification of a technique first proposed by Chandrasekhar in his book "Radiative Transfer" (ref. 1). Therefore, the notation used herein will correspond to that used by Chandrasekhar as much as possible and only the details necessary to clarify the modification and the reasons for it will be included. Complete understanding may require familiarity with Chandrasekhar's "Radiative Transfer" (ref. 1), in particular with sections 1 to 9, 13, and 49 to 51.

In the Chandrasekhar technique, the problem is essentially stated in terms of new variables, then the radiation transfer equation is used to solve the reparameterized problem. These new variables are the bidirectional reflection and transmission scattering functions for the canopy layer as a whole.

The primary reason for modifying the Chandrasekhar technique is the dissimilarity between the canopy and the atmospheric phase functions. Unlike atmospheric phase functions, the canopy phase function (1) has no symmetry around the incident direction, (2) has an orientation with respect to the vertical, and (3) does not have mirror symmetry with respect to the horizontal (that is, scattered intensity due to radiation incident from a direction bears no necessary symmetric relationship to radiation scattered from a beam incident from the opposite direction).

RESTATEMENT OF THE RADIATION TRANSFER EQUATION FOR PARALLEL LAYERS

Before restatement of the canopy problem in terms of bidirectional scattering functions, it is useful to reexamine the radiation transfer equation and write it in a form particularly suited for plane-parallel layers. This form has great similarity

to the standard form of the radiation transfer equation for plane-parallel atmospheres. The general radiation transfer equation is

$$\frac{dI(\ell;\mu,\varphi)}{d\ell} + n(\ell)\bar{\sigma}(\ell;\mu,\varphi)I(\ell;\mu,\varphi) = nJ(\ell;\mu,\varphi) \tag{269}$$

Note that the dependence on μ and φ is given explicitly rather than implicitly by using the unit vector $\hat{k}$. The formal solution is obtained by the method of integrating factors and shown to be

$$I(\ell;\mu,\varphi) = I\left(\ell_0;\mu,\varphi\right)e^{-T\left(\ell,\ell_0\right)} + \int_{\ell_0}^{\ell} \frac{J(\ell;\mu,\varphi)}{\bar{\sigma}(\ell;\mu,\varphi)} e^{-T(\ell,\ell')}d\ell' \tag{270}$$

where

$$T\left(\ell,\ell_0\right) = \int_{\ell_0}^{\ell} n(\ell)\bar{\sigma}(\ell;\mu,\varphi)d\ell \tag{271}$$

and $J(\ell;\mu,\varphi)$ is given by

$$J(\ell;\mu,\varphi) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} I_t(\ell;\mu',\varphi')P(\ell;\mu,\varphi;\mu',\varphi')d\Omega' \tag{272}$$

The assumption of homogeneous parallel layers allows change to be expressed in terms of the vertical direction z. Thus,

$$\mu\ d\ell = dz \tag{273}$$

where $-1 < \mu < 1$ and where $\mu \equiv \cos\theta$. The variable F is defined by

$$F = \frac{J}{\bar{\sigma}} \tag{274}$$

and a differential optical depth is defined by

$$d\tau = -n\bar{\sigma}\ dz \tag{275}$$

Then, the radiation transfer equation may be written

$$\frac{\mu\ dI_t(\tau;\mu,\varphi)}{d\tau} = I_t(\tau;\mu,\varphi) - F(\tau;\mu,\varphi) \tag{276}$$

where the subscript t is used to emphasize that the expression applies for the total field. The variable F is then given by

$$F(\tau;\mu,\varphi) = \frac{1}{4\pi} \int_{-1}^{1}\int_{0}^{2\pi} P(\mu,\varphi;\mu',\varphi')I_t(\tau;\mu',\varphi')d\mu'\ d\varphi' \tag{277}$$

This form of the radiation transfer equation is formally solvable using the integrating factor $e^{-(\tau-\tau_0)/\mu}$, which gives the solution

$$I(\tau;\pm\mu,\varphi) = I\left(\tau_0;\pm\mu,\varphi\right)e^{-\left|\frac{\tau-\tau_0}{\mu}\right|} + \left|\frac{1}{\mu}\int_{\tau_0}^{\tau} F(\tau';\pm\mu,\varphi)e^{-\left|\frac{\tau-\tau'}{\mu}\right|}d\tau'\right| \tag{278}$$

where the absolute values ensure that regardless of the direction in which the depth τ is measured, the solution is correct.

Again, the physical meaning of the solution is easily seen. The upwelling intensity at τ, which is $I(\tau;+\mu,\varphi)$, is a result of upwelling intensity at τ_0, which is $I(\tau_0;\mu,\varphi)$, decreased by $e^{-|(\tau-\tau_0)/\mu|}$ plus that scattered into the beam along the path decreased by $e^{-|(\tau-\tau')/\mu|}$, where $\tau > \tau'$ and where $\tau - \tau'$ is the optical distance between the volume element doing the scattering and τ integrated (summed) over all volume elements along the path between τ_0 and τ. Although this solution will be found useful in the modified Kubelka–Munk approach to canopy modeling, it is not in itself a solution for the canopy as a whole.

The S and T Functions for a Planar Medium

Radiation incident on the surface of a layer of some optical thickness will be scattered in part, absorbed in part, and unaffected in part by the layer of material. The absorbed portion is that removed by any process from the wavelength interval of interest and is thus without interest for a reflective model. The scattered portion may be directed back into the hemisphere containing the incident radiation, or it may emerge from the surface opposite that of incidence. These two scattered portions will be designated reflected diffuse radiation and transmitted diffuse radiation, respectively. (Reflected diffuse radiation is equivalent to what Chandrasekhar called scattered diffuse radiation.) In this context, diffuse does not necessarily mean uniformly diffuse, perfectly diffuse, or Lambertian. The radiation that does not encounter scattering centers in the layer will emerge from the other side of the layer unchanged in direction but reduced in magnitude. This portion of radiation is conveniently treated separately and is labeled reduced incident flux.

Let the total scattered or diffuse portion of the field at τ be designated by $I_s(\tau; \mu, \varphi)$. Then

$$I_s(\tau; \mu, \varphi) = I_u(\tau; \mu, \varphi) \tag{279}$$

where $0 < \mu < 1$, and

$$I_s(\tau; \mu, \varphi) = I_d(\tau; \mu, \varphi) \tag{280}$$

where $-1 < \mu < 0$ and where I_u is the upwelling diffuse radiation and I_d is the downwelling diffuse radiation. Let $I_i(\tau; \mu, \varphi)$ represent intensity incident on a layer at τ. By using the identity

$$\int_a^b \oint(x) dx = \int_{-b}^{-a} \oint(-x) dx \tag{281}$$

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the defining equations can be written with all functions specified over the interval $0 < \mu < 1$ to conveniently establish that $+\mu$ in the argument of a function represents radiation in the upward direction and that $-\mu$ represents radiation in the downward direction.

$$I_u(0;\mu,\varphi) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} I_i(0;-\mu',\varphi)S_U(\tau;\mu,\varphi;-\mu',\varphi')d\mu'\ d\varphi' \tag{282}$$

$$I_d(\tau;-\mu,\varphi) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} I_i(0;-\mu',\varphi)T_U(\tau;-\mu,\varphi;-\mu',\varphi')d\mu'\ d\varphi' \tag{283}$$

$$I_u(0;\mu,\varphi) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} I_i(\tau;\mu,\varphi)T_L(\tau;\mu,\varphi;\mu',\varphi')d\mu'\ d\varphi' \tag{284}$$

$$I_d(0;-\mu,\varphi) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} I_i(\tau;\mu,\varphi)S_L(\tau;-\mu,\varphi;\mu',\varphi')d\mu'\ d\varphi' \tag{285}$$

These equations define the relationships of the bidirectional scattering functions S_U, T_U, S_L, and T_L for a layer τ thick. The factor $1/\mu$ is included so that the scattering functions will be symmetric under interchange of directional angles. This operation could also be accomplished by multiplying the integrand by μ'. Then, the equations would be identical to the customary defining equations for the bidirectional scattering (reflection or transmission) used for planar scatterers as found in the work of Love (ref. 4) and Nicodemus (refs. 5 and 6).

At first, the solution to the canopy problem seems to be found in solving for S_U. This premise would be true if the canopy were a single layer; that is, if the canopy could be considered to be infinitely thick or if it were bounded on the lower side by a perfect absorber in the wavelength of interest. If radiation from any source is incident on the lower surface of the layer, then T_L must be known to find the contribution to upwelling intensity at $\tau = 0$. If the upwelling intensity at the lower surface of the layer arises from reflection of the transmitted intensity of the layer, which is likely, then T_U and S_L must be known. Therefore, for multilayers, it is necessary to know the four scattering functions for each layer together with the relationship that describes the attenuation of incident flux to give reduced incident flux. The set of equations that enable solving for these quantities follows.

Boundary Conditions

Because the bidirectional scattering functions for the layer are properties of the layer itself, the functions are independent of the conditions of incident intensity and the bounding conditions outside the layer. Thus, the illumination and boundary conditions will be chosen in the most convenient way.

Incident intensity is taken first as uniform, parallel, downward radiation incident on the upper surface. This incident radiation is represented by Dirac delta functions as

$$I_i(0; -\mu, \varphi) = \pi F_- \delta\left(\mu - \mu_-\right) \delta\left(\varphi - \varphi_-\right) \tag{286}$$

where $F_\pm$ is the magnitude of incident intensity. Incident intensity will then be taken as uniform, parallel radiation incident upward on the lower surface at $\tau = \tau'$. This relationship is also represented by Dirac delta functions as

$$I_i(\tau'; \mu, \varphi) = \pi F_+ \delta\left(\mu - \mu_+\right) \delta\left(\varphi - \varphi_+\right) \tag{287}$$

Using equations (286) and (287) in the defining equations for S_U, T_U, S_L, and T_L gives

$$I(0; \mu, \varphi) = \frac{F_-}{4\mu} S_U\left(\tau'; \mu, \varphi; -\mu_-, \varphi_-\right) \tag{288}$$

$$I(\tau'; -\mu, \varphi) = \frac{F_-}{4\mu} T_U\left(\tau'; -\mu, \varphi; -\mu_-, \varphi_-\right) \tag{289}$$

and

$$I(0; \mu, \varphi) = \frac{F_+}{4\mu} T_L\left(\tau'; \mu, \varphi; \mu'_+, \varphi'_+\right) \tag{290}$$

$$I(\tau'; -\mu, \varphi) = \frac{F_+}{4\mu} S_L\left(\tau'; -\mu, \varphi; \mu_+, \varphi_+\right) \tag{291}$$

It is convenient to assume the layer is bounded by perfect absorbers or infinite space. Then, for either illumination

$$I(0; -\mu, \varphi) = I(\tau'; \mu, \varphi) \equiv 0 \qquad (292)$$

where $0 \leq \mu \leq 1$. That is, there is no scattered, diffuse component incident on the layer either from above or from below. Note the existence of symmetry relationships between upper and lower surface transmission functions T_U and T_L. Thus, it is redundant to distinguish them; however, for simplicity within the derivation, the distinction will be maintained in the notation.

Under the conditions of illumination specified, the radiation field at any level τ within the layer can be separated into three parts: an upward scattered component $I(\tau; \mu, \varphi)$, a downward scattered component $I(\tau; -\mu, \varphi)$, and an unscattered component composed of the reduced incident flux. The relationship describing the attenuation of F_+ or F_- is obtained from equations (276) and (278) with the omission of the term F.

Integration is immediate to give exponential attenuation by a factor of $e^{-\tau/\mu}$, where τ is the absolute value of the optical thickness of material through which the incident flux passes. Thus, at any level, the reduced incident flux for F_+ will be $\pi F_+ e^{-(\tau'-\tau)/\mu_+}$ and the reduced incident flux for F_- will be $\pi F_- e^{-\tau/\mu_-}$.

Parameterization Principles

The basis for the technique employed to solve for the bidirectional scattering functions was developed by Chandrasekhar in solving for atmospheric transmission. This technique will be modified and expanded for use in canopy modeling. For downward radiation incident on the upper surface F_-, the following four principles can be formulated.

1. For a layer τ' thick, the intensity $I(\tau; \mu, \varphi)$ in the upward direction at any level τ_U results from the reflection of the reduced incident flux $\pi F_- e^{-\tau/\mu_-}$ within the layer, and the reflection of the downward diffuse radiation $I(\tau; -\mu, \varphi)$ (where $0 < \mu < 1$) incident on the surface τ, by the layer of optical thickness $\tau' - \tau$, below τ. The mathematical expression of this principle is

$$I(\tau; \mu, \varphi) = \frac{F_-}{4\mu} e^{-\frac{\tau}{\mu_-}} S_U\left(\tau' - \tau; \mu, \varphi; -\mu_-, \varphi_-\right)$$

$$+ \frac{1}{4\mu\pi} \int_0^1 \int_0^{2\pi} S_U(\tau' - \tau; \mu, \varphi; -\mu', \varphi') I(\tau; -\mu', \varphi') d\mu' \, d\varphi' \qquad (293)$$

2. For a layer τ' thick, the intensity $I(\tau;-\mu,\varphi)$ in the downward direction at any level τ results from the transmission of the incident flux by the layer of optical thickness τ, above the surface at τ, and the reflection at this same surface of the upward diffuse radiation $I(\tau;\mu,\varphi)$ (where $0 < \mu < 1$) incident on τ from below. The mathematical expression of this principle is

$$I(\tau;-\mu,\varphi) = \frac{F_-}{4\mu} T_U\left(\tau;-\mu,\varphi;\mu_-,\varphi_-\right)$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S_L(\tau;-\mu,\varphi;\mu',\varphi')I(\tau;\mu',\varphi')d\mu'\, d\varphi' \qquad (294)$$

3. For a layer τ' thick, the diffuse reflection of the downward incident radiation by the entire layer is equivalent to the reflection by that part of the layer of optical thickness τ above the level τ and the transmission of the diffuse intensity $I(\tau;\mu,\varphi)$ incident on the surface τ from below. The mathematical expression of this principle is

$$\frac{F_-}{4\mu} S_U\left(\tau';\mu,\varphi;-\mu_-,\varphi_-\right) = \frac{F_-}{4\mu} S_U\left(\tau;\mu,\varphi;-\mu_-,\varphi_-\right) + e^{-\frac{\tau}{\mu}} I(\tau;\mu,\varphi)$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T_L(\tau;\mu,\varphi;\mu',\varphi')I(\tau;\mu',\varphi')d\mu'\, d\varphi'$$

$$(295)$$

4. For a layer τ' thick, the diffuse transmission of the incident intensity by the entire layer is equivalent to transmission of the reduced incident flux $\pi F_- e^{-\tau/\mu_-}$ and transmission of the diffuse radiation $I(\tau;-\mu,\varphi)$ (where $0 < \mu < 1$) incident on surface τ by the layer of optical thickness $\tau' - \tau$ below τ. The mathe-

matical expression of this principle is

$$\frac{F_-}{4\mu}\ T_U\left(\tau';-\mu,\varphi;-\mu_-,\varphi_-\right) = \frac{F_-}{4\mu}\ e^{-\frac{\tau}{\mu_-}}\ T_U\left(\tau'-\tau;-\mu,\varphi;-\mu_-,\varphi_-\right)$$

$$+\ e^{-\frac{\tau'-\tau}{\mu}}\ I(\tau;-\mu,\varphi)$$

$$+\ \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T_U(\tau'-\tau;-\mu,\varphi;-\mu',\varphi')I(\tau;-\mu',\varphi')d\mu'\ d\varphi'$$

$$(296)$$

The corollary set of four principles for radiation incident from below F_+ on surface τ' can be formulated as follows.

1. For a layer τ' thick, the intensity $I(\tau;-\mu,\varphi)$ in the downward direction at any level τ within the layer results from the reflection of the reduced incident flux $\pi F_+ e^{-(\tau'-\tau)/\mu_+}$ and the reflection of the upward diffuse intensity $I(\tau;\mu,\varphi)$ (where $0 < \mu < 1$) incident on the surface τ, by the layer of optical thickness τ, above τ. The mathematical expression of this principle is

$$I(\tau;-\mu,\varphi) = \frac{F_+}{4\mu}\ e^{-\frac{\tau'-\tau}{\mu_+}}\ S_L\left(\tau;-\mu,\varphi;\mu_+,\varphi_+\right)$$

$$+\ \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} S_L(\tau;-\mu,\varphi;\mu',\varphi')I(\tau;\mu',\varphi')d\mu'\ d\varphi' \qquad (297)$$

2. For a layer τ' thick, the intensity $I(\tau;\mu,\varphi)$ in the upward direction at any level τ results from the transmission of the incident flux by the layer of optical thickness $\tau' - \tau$ below the surface at the level τ, and the reflection at this same surface of the downward diffuse radiation $I(\tau;-\mu,\varphi)$ (where $0 < \mu \leq 1$) is incident

on τ from above. The mathematical expression of this principle is

$$I(\tau;\mu,\varphi) = \frac{F_+}{4\mu} T_L\left(\tau' - \tau;\mu,\varphi;\mu_+,\varphi_+\right)$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S_U(\tau' - \tau;\mu,\varphi;-\mu',\varphi')I(\tau;-\mu',\varphi')d\mu' \cdot d\varphi' \qquad (298)$$

3. For a layer τ' thick, the diffuse reflection of the upward incident radiation by the entire layer is equivalent to the reflection by that part of the layer of optical thickness $\tau' - \tau$ below the level τ, and the transmission by this layer of the diffuse intensity $I(\tau;-\mu,\varphi)$ (where $0 < \mu \leq 1$) incident on the surface τ from above. The mathematical expression of this principle is

$$\frac{F_+}{4\mu} S_L\left(\tau';-\mu,\varphi;\mu_+,\varphi_+\right) = \frac{F_+}{4\mu} S_L\left(\tau' - \tau;-\mu,\varphi;\mu_+,\varphi_+\right) + e^{-\frac{\tau'-\tau}{\mu}} I(\tau;-\mu,\varphi)$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T_U(\tau' - \tau;-\mu,\varphi;-\mu',\varphi')I(\tau;-\mu',\varphi')d\mu' \, d\varphi'$$

$$(299)$$

4. For a layer τ' thick, the diffuse transmission of the upward incident intensity by the entire layer is equivalent to the transmission of the reduced incident flux $\pi F_+ e^{-(\tau'-\tau)/\mu_+}$ and the transmission of the diffuse radiation $I(\tau;\mu,\varphi)$ (where $0 < \mu \leq 1$) incident on the surface τ by the layer of optical thickness τ above τ.

The mathematical expression of this principle is

$$\frac{F_+}{4\mu} T_L\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right) = \frac{F_+}{4\mu} e^{-\frac{\tau'-\tau}{\mu_+}} T_L\left(\tau;\mu,\varphi;\mu_+,\varphi_+\right) + e^{-\frac{\tau}{\mu}} I(\tau;\mu,\varphi)$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T_L(\tau;\mu,\varphi;\mu',\varphi')I(\tau;\mu',\varphi')d\mu' \, d\varphi'$$

(300)

System of Equations in Functions S and T

The next step is the development of a system of equations for the bidirectional scattering functions. The technique employed was originated by Chandrasekhar and differs from section 51 of his "Radiative Transfer" (ref. 1) only in the addition of subscripts U and L and in the substitution of τ' for τ_1. Equations deriving from principles expressed in equations (293) to (296) will be done simultaneously but separately from those derived from equations (297) to (300). For convenience, set $\mu_+ = -\mu_-$. Then, differentiate equations (293) to (299) with respect to τ, apply the boundary conditions in equation (292), take the limit as $\tau = 0$ for equations (293), (296), (298), and (299), and take the limit as $\tau = \tau'$ for equations (294), (295), (297), and (300). For equations (293) to (296), this procedure gives

$$\left[\frac{dI(\tau;+\mu,\varphi)}{d\tau}\right]_{\tau=0} = \frac{F_-}{4\mu}\left[-\frac{1}{\mu} S_U\left(\tau';\mu,\varphi;-\mu_-,\varphi_-\right) - \frac{\partial S_U\left(\tau';\mu,\varphi;-\mu_-,\varphi_-\right)}{\partial\tau'}\right]$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S_U(\tau';\mu,\varphi;-\mu',\varphi')\left[\frac{dI(\tau;-\mu',\varphi')}{d\tau}\right]_{\tau=0} d\mu' \, d\varphi'$$

(301)

$$\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=\tau'} = \frac{F_-}{4\mu} \frac{\partial T_U\left(\tau';-\mu,\varphi;-\mu_-,\varphi_-\right)}{\partial\tau'}$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S_L(\tau';-\mu,\varphi;\mu',\varphi')\left[\frac{dI(\tau;+\mu',\varphi')}{d\tau}\right]_{\tau=\tau'} d\mu' \, d\varphi'$$

(302)

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$$0 = \frac{F_-}{4\mu} \frac{\partial S_U(\tau';\mu,\varphi;-\mu',\varphi')}{\partial \tau'} + e^{-\frac{\tau'}{\mu}}\left[\frac{dI(\tau';+\mu,\varphi)}{d\tau}\right]_{\tau=\tau'}$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T_L(\tau';\mu,\varphi;\mu',\varphi')\left[\frac{dI(\tau;+\mu',\varphi')}{d\tau}\right]_{\tau=\tau'} d\mu' \, d\varphi' \qquad (303)$$

and

$$0 = \frac{F_-}{4\mu}\left[-\frac{1}{\mu_-}T_U\left(\tau';-\mu,\varphi;-\mu_-,\varphi_-\right) - \frac{\partial T_U\left(\tau';-\mu,\varphi;-\mu_-,\varphi_-\right)}{\partial \tau'}\right]$$

$$+ e^{-\frac{\tau'}{\mu}}\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=0} + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T_U(\tau';-\mu,\varphi;-\mu',\varphi')$$

$$\times \left[\frac{dI(\tau;-\mu',\varphi')}{d\tau}\right]_{\tau=0} d\mu' \, d\varphi' \qquad (304)$$

Similarly, for equations (297) to (300), this procedure gives

$$\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=\tau'} = \frac{F_+}{4\mu}\left[+\frac{1}{\mu_+}S_L\left(\tau';-\mu,\varphi;\mu_+,\varphi_+\right) + \frac{\partial S_L\left(\tau';-\mu,\varphi;\mu_+,\varphi_+\right)}{\partial \tau'}\right]$$

$$+ \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} S_L(\tau';-\mu,\varphi;\mu',\varphi')\left[\frac{dI(\tau;\mu',\varphi')}{d\tau}\right]_{\tau=\tau'} d\mu' \, d\varphi' \tag{305}$$

$$\left[\frac{dI(\tau;+\mu,\varphi)}{d\tau}\right]_{\tau=0} = \frac{F_+}{4\mu}\frac{\partial T\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right)}{\partial \tau'}$$

$$+ \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} S_U(\tau';\mu,\varphi;-\mu',\varphi')\left[\frac{dI(\tau;-\mu',\varphi')}{d\tau}\right]_{\tau=0} d\mu' \, d\varphi' \tag{306}$$

$$0 = -\frac{F_+}{4\mu} \frac{\partial S_L(\tau';-\mu,\varphi;\mu',\varphi')}{\partial\tau'} + e^{-\frac{\tau'}{\mu}}\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=0}$$

$$+ \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T_U(\tau';-\mu,\varphi;-\mu',\varphi')\left[\frac{dI(\tau';-\mu',\varphi')}{d\tau}\right]_{\tau=0} d\mu'\,d\varphi' \quad (307)$$

and

$$0 = \frac{F_+}{4\mu}\left[+\frac{1}{\mu_+}T_L\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right) + \frac{\partial T_L\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right)}{\partial\tau'}\right]$$

$$+ e^{-\frac{\tau'}{\mu}}\left[\frac{dI(\tau;+\mu,\varphi)}{d\tau}\right]_{\tau=\tau'} + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T_L(\tau';\mu,\varphi;\mu',\varphi')$$

$$\times \left[\frac{dI(\tau;+\mu',\varphi')}{d\tau}\right]_{\tau=\tau'} d\mu'\,d\varphi' \quad (308)$$

The derivatives in the preceding equations can be obtained from the previously derived equation of transfer and the expression for F in equations (276) and (277). This procedure gives the set appropriate for equations (293) to (296).

$$\left[\frac{dI(\tau;+\mu,\varphi)}{d\tau}\right]_{\tau=0} = +\frac{1}{\mu}\left[\frac{F_-}{4\mu}S_U\left(\tau';\mu,\varphi;\mu_-,\varphi_-\right) - F_-(0;+\mu,\varphi)\right] \quad (309)$$

$$\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=0} = +\frac{1}{\mu}F_-(0;-\mu,\varphi) \quad (310)$$

$$\left[\frac{dI(\tau;+\mu,\varphi)}{d\tau}\right]_{\tau=\tau'} = -\frac{1}{\mu}F_-(\tau';+\mu,\varphi) \quad (311)$$

$$\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=\tau'} = -\frac{1}{\mu}\left[\frac{F_-}{4\mu}T_U\left(\tau';-\mu,\varphi;\mu_-,\varphi_-\right) - F_-(\tau';-\mu,\varphi)\right] \quad (312)$$

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where F_- is given by

$$F_-(0;\mu,\varphi) = \frac{1}{4} F_- \left\{ P\left(\mu,\varphi;-\mu_-,\varphi_-\right) \right.$$

$$\left. + \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} P(\mu,\varphi;\mu'',\varphi'')S_U\left(\tau';\mu'',\varphi'';\mu_-,\varphi_-\right)\frac{d\mu''}{\mu''} \, d\varphi'' \right\} \quad (313)$$

and

$$F_-(\tau';\mu,\varphi) = \frac{1}{4} F_- \left\{ e^{-\frac{\tau'}{\mu}} P\left(\mu,\varphi;-\mu_-,\varphi_-\right) \right.$$

$$\left. + \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} P(\mu,\varphi;-\mu'',\varphi'')T_U\left(\tau';\mu'',\varphi'';\mu_-,\varphi_-\right)\frac{d\mu''}{\mu''} \, d\varphi'' \right\} \quad (314)$$

The set appropriate for equations (297) to (300) is

$$\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=\tau'} = -\frac{1}{\mu}\left[\frac{F}{4\mu} S_L\left(\tau';-\mu,\varphi;\mu_+,\varphi_+\right) - F_+(\tau';-\mu,\varphi)\right] \quad (315)$$

$$\left[\frac{dI(\tau;-\mu,\varphi)}{d\tau}\right]_{\tau=0} = +\frac{1}{\mu} F_+(0;-\mu,\varphi) \quad (316)$$

$$\left[\frac{dI(\tau;+\mu,\varphi)}{d\tau}\right]_{\tau=\tau'} = -\frac{1}{\mu} F_+(\tau';+\mu,\varphi) \quad (317)$$

$$\left[\frac{dI(\tau;+\mu,\varphi)}{d\tau}\right]_{\tau=0} = \frac{1}{\mu}\left[\frac{F_+}{4\mu} T_L\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right) - F_+(0;\mu,\varphi)\right] \quad (318)$$

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where F_+ is given by

$$F_+(\tau';\mu,\varphi) = \frac{F_+}{4}\left[P\left(\mu,\varphi;\mu_+,\varphi_+\right) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi}S_L\left(\tau';-\mu,\varphi;\mu_+,\varphi_+\right)\right.$$

$$\left. \times P(\mu,\varphi;-\mu',\varphi')\frac{d\mu'}{\mu'}\,d\varphi'\right] \tag{319}$$

$$F_+(0;\mu,\varphi) = \frac{F_+}{4}\left[e^{-\frac{\tau'}{\mu_+}}P\left(\mu,\varphi;\mu_+,\varphi_+\right) + \frac{1}{4\pi}\int_0^1 T_L\left(\tau';\mu',\varphi';\mu_+,\varphi_+\right)P(\mu,\varphi;\mu',\varphi')\frac{d\mu'}{\mu'}\,d\varphi'\right] \tag{320}$$

Combining equations (309) to (314) with equations (301) to (304) and combining equations (315) to (320) with equations (305) to (308) gives the following eight equations in S_U, T_U, S_L, and T_L.

$$\left(\frac{1}{\mu} + \frac{1}{\mu_-}\right)S_U\left(\tau';\mu,\varphi;-\mu_-,\varphi_-\right) + \frac{\partial S_U\left(\tau';\mu,\varphi;-\mu_-,\varphi_-\right)}{\partial\tau'}$$

$$= P\left(\mu,\varphi;-\mu_-,\varphi_-\right)$$

$$+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi}P(\mu,\varphi;\mu'',\varphi'')S_U\left(\tau';\mu'',\varphi'';-\mu_-,\varphi_-\right)\frac{d\mu''}{\mu''}\,d\varphi''$$

$$+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi}S_U(\tau';\mu,\varphi;-\mu',\varphi')P\left(-\mu',\varphi';-\mu_-,\varphi_-\right)\frac{d\mu'}{\mu'}\,d\varphi'$$

$$+ \frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}S_U(\tau';\mu,\varphi;\mu',\varphi')P(-\mu',\varphi';\mu'',\varphi'')$$

$$\times S_U\left(\tau';\mu'',\varphi'';-\mu_-,\varphi_-\right)\frac{d\mu'}{\mu'}\,d\varphi'\,\frac{d\mu''}{\mu''}\,d\varphi'' \tag{321}$$

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$$\frac{\partial S_U\left(\tau';\mu,\varphi;-\mu_-,\varphi_-\right)}{\partial\tau'} = P\left(\mu,\varphi;-\mu_-,\varphi_-\right)\exp\left\{-\tau'\left(\frac{1}{\mu}+\frac{1}{\mu_-}\right)\right\}$$

$$+ \frac{1}{4\pi}e^{-\frac{\tau'}{\mu}}\int_0^1\int_0^{2\pi}P(\mu,\varphi;-\mu'',\varphi'')T_U\left(\tau';-\mu'',\varphi'';-\mu_-,\varphi_-\right)\frac{d\mu''}{\mu''}\ d\varphi''$$

$$+ \frac{1}{4\pi}e^{-\frac{\tau'}{\mu}}\int_0^1\int_0^{2\pi}T_L(\tau';\mu,\varphi;\mu',\varphi')P\left(\mu',\varphi';-\mu_-,\varphi_-\right)\frac{d\mu'}{\mu'}\ d\varphi'$$

$$+ \frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}T_L(\tau';\mu,\varphi;\mu',\varphi')P(\mu',\varphi';-\mu'',\varphi'')$$

$$\times\ T_U\left(\tau';-\mu'',\varphi'';-\mu_-,\varphi_-\right)\frac{d\mu'}{\mu'}\ d\varphi'\ \frac{d\mu''}{\mu''}\ d\varphi'' \qquad (322)$$

$$\frac{1}{\mu}T_U\left(\tau';-\mu,\varphi;-\mu_-,\varphi_-\right) + \frac{\partial T_U\left(\tau';-\mu,\varphi;-\mu_-,\varphi_-\right)}{\partial\tau'}$$

$$= e^{-\frac{\tau'}{\mu}}P\left(-\mu,\varphi;-\mu_-,\varphi_-\right)$$

$$+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi}P(-\mu,\varphi;-\mu'',\varphi'')T_U\left(\tau';-\mu'',\varphi'';-\mu_-,\varphi_-\right)\frac{d\mu''}{\mu''}\ d\varphi''$$

$$+ \frac{e^{-\frac{\tau'}{\mu}}}{4\pi}\int_0^1\int_0^{2\pi}S_L(\tau';-\mu,\varphi;\mu',\varphi')P\left(\mu',\varphi';-\mu_-,\varphi_-\right)\frac{d\mu'}{\mu'}\ d\varphi'$$

$$+ \frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}S_L(\tau';-\mu,\varphi;\mu',\varphi')P(\mu',\varphi';-\mu'',\varphi'')$$

$$\times\ T_U\left(\tau';-\mu'',\varphi'';-\mu_-,\varphi_-\right)\frac{d\mu'}{\mu'}\ d\varphi'\ \frac{d\mu''}{\mu''}\ d\varphi'' \qquad (323)$$

$$\frac{1}{\mu_-} T_U\left(\tau'; -\mu, \varphi; -\mu_-, \varphi_-\right) + \frac{\partial T_U\left(\tau'; -\mu, \varphi; -\mu_-, \varphi_-\right)}{\partial \tau'}$$

$$= e^{-\frac{\tau'}{\mu}} P\left(-\mu, \varphi; -\mu_-, \varphi_-\right)$$

$$+ \frac{e^{-\frac{\tau'}{\mu}}}{4\pi} \int\limits_0^1 \int\limits_0^{2\pi} P(-\mu, \varphi; \mu'', \varphi'') S_U\left(\tau'; \mu'', \varphi''; -\mu_-, \varphi_-\right) \frac{d\mu''}{\mu''} \, d\varphi''$$

$$+ \frac{1}{4\pi} \int\limits_0^1 \int\limits_0^{2\pi} T_U(\tau'; -\mu, \varphi; -\mu', \varphi') P\left(-\mu', \varphi'; -\mu_-, \varphi_-\right) \frac{d\mu'}{\mu'} \, d\varphi'$$

$$+ \frac{1}{16\pi^2} \int\limits_0^1 \int\limits_0^{2\pi} \int\limits_0^1 \int\limits_0^{2\pi} T_U(\tau'; -\mu, \varphi; -\mu', \varphi') P(-\mu', \varphi'; \mu'', \varphi'')$$

$$\times \, S_U\left(\tau'; \mu'', \varphi''; -\mu_-, \varphi_-\right) \frac{d\mu'}{\mu'} \, d\varphi' \, \frac{d\mu''}{\mu''} \, d\varphi'' \qquad (324)$$

$$\left(\frac{1}{\mu} + \frac{1}{\mu_+}\right) S_L\left(\tau; -\mu, \varphi; \mu_+, \varphi_+\right) + \frac{\partial S_L}{\partial \tau'}\left(\tau; -\mu, \varphi; \mu_+, \varphi_+\right)$$

$$= P\left(-\mu, \varphi; \mu_+, \varphi_+\right) + \frac{1}{4\pi} \int\limits_0^1 \int\limits_0^{2\pi} P(-\mu, \varphi; -\mu', \varphi') S_L\left(\tau; -\mu', \varphi'; \mu_+, \varphi_+\right) \frac{d\mu'}{\mu'} \, d\varphi'$$

$$+ \frac{1}{4\pi} \int\limits_0^1 \int\limits_0^{2\pi} S_L(\tau'; -\mu, \varphi; \mu', \varphi') P\left(\mu', \varphi'; \mu_+, \varphi_+\right) \frac{d\mu'}{\mu'} \, d\varphi'$$

$$+ \frac{1}{16\pi^2} \int\limits_0^1 \int\limits_0^{2\pi} S_L(\tau'; -\mu, \varphi; \mu', \varphi') \int\limits_0^1 \int\limits_0^{2\pi} P(\mu', \varphi'; -\mu'', \varphi'')$$

$$\times \, S_L\left(\tau'; -\mu'', \varphi''; \mu_+, \varphi_+\right) \frac{d\mu''}{\mu''} \, d\varphi'' \, \frac{d\mu'}{\mu'} \, d\varphi' \qquad (325)$$

$$\frac{1}{\mu_+} T_L\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right) + \frac{\partial T_L}{\partial \tau'}\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right)$$

$$= e^{-\frac{\tau'}{\mu}} P\left(\mu,\varphi;\mu_+,\varphi_+\right)$$

$$+ \frac{e^{-\frac{\tau'}{\mu}}}{4\pi} \int\limits_0^1\int\limits_0^{2\pi} P(\mu,\varphi;-\mu',\varphi') S_L\left(\tau';-\mu',\varphi';\mu_+,\varphi_+\right)\frac{d\mu'}{\mu'}\ d\varphi'$$

$$+ \frac{1}{4\pi} \int\limits_0^1\int\limits_0^{2\pi} T_L(\tau';\mu,\varphi;\mu',\varphi') P\left(\mu',\varphi';\mu_+,\varphi_+\right)\frac{d\mu'}{\mu'}\ d\varphi'$$

$$+ \frac{1}{(4\pi)^2} \int\limits_0^1\int\limits_0^{2\pi}\int\limits_0^1\int\limits_0^{2\pi} T_L(\tau';\mu,\varphi;\mu',\varphi') P(\mu',\varphi';-\mu'',\varphi'')$$

$$\times\ S_L\left(\tau';-\mu'',\varphi'';\mu_+,\varphi_+\right)\frac{d\mu'}{\mu'}\ d\varphi'\ \frac{d\mu''}{\mu''}\ d\varphi'' \qquad (326)$$

$$\frac{1}{\mu} T_L\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right) + \frac{\partial T_L}{\partial \tau'}\left(\tau';\mu,\varphi;\mu_+,\varphi_+\right)$$

$$= e^{-\frac{\tau'}{\mu_+}} P\left(\mu,\varphi;\mu_+,\varphi_+\right)$$

$$+ \frac{e^{-\frac{\tau'}{\mu_+}}}{4\pi} \int\limits_0^1\int\limits_0^{2\pi} S_U(\tau';\mu,\varphi;-\mu',\varphi') P\left(-\mu',\varphi';\mu_+,\varphi_+\right)\frac{d\mu'}{\mu'}\ d\varphi'$$

$$+ \frac{1}{4\pi} \int\limits_0^1\int\limits_0^{2\pi} P(\mu,\varphi;\mu',\varphi') T_L\left(\tau';\mu',\varphi';\mu_+,\varphi_+\right)\frac{d\mu'}{\mu'}\ d\varphi'$$

$$+ \frac{1}{(4\pi)^2} \int\limits_0^1\int\limits_0^{2\pi}\int\limits_0^1\int\limits_0^{2\pi} S_U(\tau';\mu,\varphi;-\mu',\varphi') P(-\mu',\varphi';\mu'',\varphi'')$$

$$\times\ T_L\left(\tau';\mu'',\varphi'';\mu_+,\varphi_+\right)\frac{d\mu''}{\mu''}\ d\varphi''\ \frac{d\mu'}{\mu'}\ d\varphi' \qquad (327)$$

$$\frac{1}{\mu}\frac{\partial}{\partial\tau'}\,S_L\!\left(\tau';-\mu,\varphi;\mu_+,\varphi_+\right) = e^{-\tau'\left(\frac{1}{\mu}+\frac{1}{\mu_+}\right)}P\!\left(-\mu,\varphi;\mu_+,\varphi_+\right)$$

$$+\,\frac{e^{-\frac{\tau'}{\mu}}}{4\pi}\int_0^1\!\int_0^{2\pi} T_L\!\left(\tau';\mu',\varphi';\mu_+,\varphi_+\right)P(-\mu,\varphi;\mu',\varphi')\frac{d\mu'}{\mu'}\,d\varphi'$$

$$+\,\frac{e^{-\frac{\tau'}{\mu_+}}}{4\pi}\int_0^1\!\int_0^{2\pi} T_U(\tau';-\mu,\varphi;-\mu',\varphi')P\!\left(-\mu',\varphi';\mu_+,\varphi_+\right)\frac{d\mu'}{\mu'}\,d\varphi'$$

$$+\,\frac{1}{(4\pi)^2}\int_0^1\!\int_0^{2\pi}\!\int_0^1\!\int_0^{2\pi} T_U(\tau';-\mu,\varphi;-\mu',\varphi')P(-\mu',\varphi';\mu'',\varphi'')$$

$$\times\; T_L\!\left(\tau';\mu'',\varphi'';\mu_+,\varphi_+\right)\frac{d\mu'}{\mu'}\,d\varphi'\,\frac{d\mu''}{\mu''}\,d\varphi'' \tag{328}$$

The derivatives can be eliminated by simple subtraction, which leaves four integral equations. The solution to these integral equations results in an expression for S_U. Using S_U, upwelling radiation for any specified condition of illumination can then be calculated by means of equation (282).

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TABLE I.- TOTAL REFLECTION AND TRANSMISSION FUNCTIONS

Function	μ	μ'	Schematic
$\rho_U(\hat{k},\hat{k}')*$	$0 < \mu* < 1$	$0 < \mu'* < 1$	
$\tau_U(\hat{k},\hat{k}')*$	$-1 < \mu* < 0$	$0 < \mu'* < 1$	
$\rho_L(\hat{k},\hat{k}')*$	$-1 < \mu* < 0$	$-1 < \mu'* < 0$	
$\tau_L(\hat{k},\hat{k}')*$	$0 < \mu* < 1$	$-1 < \mu'* < 0$	

TABLE II.- SLOPE AND AZIMUTH RANGES FOR PLANAR ELEMENTS
CLASSIFIED BY INCIDENT OR VIEWING SURFACE

Class	Range of θ_i or θ_s	Range of θ_n	Range of φ_n
U_{i1}	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_n < \frac{\pi}{2} - \theta_i$	$0 < \varphi_n < 2\pi$
U_{i2}	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_i < \theta_n < \frac{\pi}{2}$	$\varphi_i - \delta_i < \varphi_n < \varphi_i + \delta_i$
L_i	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_i < \theta_n < \frac{\pi}{2}$	$\varphi_i + \delta_i < \varphi_n < \varphi_i + 2\pi - \delta_i$
U_{s1}	$0 < \theta_s < \frac{\pi}{2}$	$0 < \theta_n < \frac{\pi}{2} - \theta_s$	$0 < \varphi_n < 2\pi$
U_{s2}	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_s < \theta_n < \frac{\pi}{2}$	$\varphi_s - \delta_s < \varphi_n < \varphi_s + \delta_s$
L_s	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_s < \theta_n < \frac{\pi}{2}$	$\varphi_s + \delta_s < \varphi_n < \varphi_s + 2\pi - \delta_s$
U'_i	$\frac{\pi}{2} < \theta_i < \pi$	$\theta_i - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\varphi_i - \pi + \delta_i < \varphi_n < \varphi_i + \pi - \delta_i$
L'_{i1}	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_n < \theta_i - \frac{\pi}{2}$	$0 < \varphi_n < 2\pi$
L'_{i2}	$\frac{\pi}{2} < \theta_i < \pi$	$\theta_i - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\varphi_i + \pi - \delta_i < \varphi_n < \varphi_i + \pi + \delta_i$
U'_s	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_s - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\varphi_i - \pi + \delta_s < \varphi_n < \varphi_s + \pi - \delta_s$
L'_{s2}	$\frac{\pi}{2} < \theta_s < \pi$	$0 < \theta_n < \theta_s - \frac{\pi}{2}$	$0 < \varphi_n < 2\pi$
L'_{s2}	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_s - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\varphi_s + \pi - \delta_s < \varphi_n < \varphi_s + \pi + \delta_s$

TABLE III.– PHASE-FUNCTION INTEGRATION LIMITS

Class	Intersection of classes	Bidirectional function involved	θ_i	θ_s	θ_n	φ_n
1	$U_{i1},\ U_{s1}$	$\rho_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$0 < \theta_n < \frac{\pi}{2} - \begin{Bmatrix}\theta_i\\\theta_s\end{Bmatrix}$	$0 < \varphi_n < 2\pi$
2	$U_{i1},\ U_{s2}$	$\rho_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_s < \theta_n < \frac{\pi}{2} - \theta_i$	$\varphi_s - \delta_s < \varphi_n < \varphi_s + \delta_s$
3	$U_{i2},\ U_{s1}$	$\rho_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_i < \theta_n < \frac{\pi}{2} - \theta_s$	$\varphi_i - \delta_i < \varphi_n < \varphi_i + \delta_i$
4	$U_{i2},\ U_{s2}$	$\rho_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \begin{bmatrix}\theta_s\\\theta_i\end{bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i - \delta_i\\\varphi_s - \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i + \delta_i\\\varphi_s + \delta_s\end{bmatrix}$
5	$U_{i1},\ L_s$	$\tau_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_s < \theta_n < \frac{\pi}{2} - \theta_i$	$\varphi_s + \delta_s < \varphi_n < \varphi_s + 2\pi - \delta_s$
6	$U_{i2},\ L_s$	$\tau_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \begin{bmatrix}\theta_s\\\theta_i\end{bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i - \delta_i\\\varphi_s + \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i + \delta_i\\\varphi_s + 2\pi - \delta_s\end{bmatrix}$
7	$L_i,\ U_{s1}$	$\tau_L(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_i < \theta_n < \frac{\pi}{2} - \theta_s$	$\varphi_i + \delta_i < \varphi_n < \varphi_i + 2\pi - \delta_i$
8	$L_i,\ U_{s2}$	$\tau_L(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \begin{bmatrix}\theta_s\\\theta_i\end{bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i + \delta_i\\\varphi_s - \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i + 2\pi - \delta_i\\\varphi_s + \delta_s\end{bmatrix}$
9	$L_i,\ L_s$	$\rho_L(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \begin{bmatrix}\theta_s\\\theta_i\end{bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i + \delta_i\\\varphi_s + \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i - \delta_i\\\varphi_s - \delta_s\end{bmatrix}$
10	$U_{i1},\ U_s'$	$\rho_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_s - \frac{\pi}{2} < \theta_n < \frac{\pi}{2} - \theta_i$	$\varphi_s - \pi + \delta_s < \varphi_n < \varphi_s + \pi - \delta_s$
11	$U_{i2},\ U_s'$	$\rho_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\begin{Bmatrix}\frac{\pi}{2} - \theta_i\\\theta_s - \frac{\pi}{2}\end{Bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i - \delta_i\\\varphi_s - \pi + \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i + \delta_i\\\varphi_s + \pi - \delta_s\end{bmatrix}$
12	$U_{i1},\ L_{s1}'$	$\tau_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$0 < \theta_n < \begin{bmatrix}\frac{\pi}{2} - \theta_i\\\theta_s - \frac{\pi}{2}\end{bmatrix}$	$0 < \varphi_n < 2\pi$
13	$U_{i1},\ L_{s2}'$	$\tau_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_s - \frac{\pi}{2} < \theta_n < \frac{\pi}{2} - \theta_i$	$\varphi_s + \pi - \delta_s < \varphi_n < \varphi_s + \pi + \delta_s$
14	$U_{i2},\ L_{s1}'$	$\tau_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\frac{\pi}{2} - \theta_i < \theta_n < \theta_s - \frac{\pi}{2}$	$\varphi_i - \delta_i < \varphi_n < \varphi_i + \delta_i$
15	$U_{i2},\ L_{s2}'$	$\tau_U(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\begin{Bmatrix}\frac{\pi}{2} - \theta_i\\\theta_s - \frac{\pi}{2}\end{Bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i - \delta_i\\\varphi_s + \pi - \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i + \delta_i\\\varphi_s + \pi + \delta_s\end{bmatrix}$
16	$L_i,\ U_s'$	$\tau_L(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\begin{Bmatrix}\frac{\pi}{2} - \theta_i\\\theta_s - \frac{\pi}{2}\end{Bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i + \delta_i\\\varphi_s - \pi + \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i + 2\pi - \delta_i\\\varphi_s + \pi - \delta_s\end{bmatrix}$
17	$L_i,\ L_{s1}'$	$\rho_L(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\frac{\pi}{2} - \theta_i < \theta_n < \theta_s - \frac{\pi}{2}$	$\varphi_i + \delta_i < \varphi_n < \varphi_i + 2\pi - \delta_i$
18	$L_i,\ L_{s2}'$	$\rho_L(\hat{k},\hat{k}')$	$0 < \theta_i < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta_s < \pi$	$\begin{Bmatrix}\frac{\pi}{2} - \theta_i\\\theta_s - \frac{\pi}{2}\end{Bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_i + \delta_i\\\varphi_s + \pi - \delta_s\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_i + 2\pi - \delta_i\\\varphi_s + \pi + \delta_s\end{bmatrix}$
19	$L_{i1}',\ U_{s1}$	$\tau_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$0 < \theta_n < \begin{bmatrix}\frac{\pi}{2} - \theta_s\\\theta_i - \frac{\pi}{2}\end{bmatrix}$	$0 < \varphi_n < 2\pi$
20	$L_{i1}',\ U_{s2}$	$\tau_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_s < \theta_n < \theta_i - \frac{\pi}{2}$	$\varphi_s - \delta_s < \varphi_n < \varphi_s + \delta_s$
21	$L_{i2}',\ U_{s1}$	$\tau_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\theta_i - \frac{\pi}{2} < \theta_n < \frac{\pi}{2} - \theta_s$	$\varphi_i + \pi - \delta_i < \varphi_n < \varphi_i + \pi + \delta_i$
22	$L_{i2}',\ U_{s2}$	$\tau_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\begin{Bmatrix}\frac{\pi}{2} - \theta_s\\\theta_i - \frac{\pi}{2}\end{Bmatrix} < \theta_n < \frac{\pi}{2}$	$\begin{Bmatrix}\varphi_s - \delta_s\\\varphi_i + \pi - \delta_i\end{Bmatrix} < \varphi_n < \begin{bmatrix}\varphi_s + \delta_s\\\varphi_i + \pi + \delta_i\end{bmatrix}$
23	$L_{i1}',\ L_s$	$\rho_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\frac{\pi}{2} - \theta_s < \theta_n < \theta_i - \frac{\pi}{2}$	$\varphi_s + \delta_s < \varphi_n < \varphi_s + 2\pi - \delta_s$

TABLE III.- PHASE-FUNCTION INTEGRATION LIMITS - Concluded

Class	Intersection of classes	Bidirectional function involved	θ_i	θ_s	θ_n	φ_n
24	L'_{i2}, L_s	$\rho_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\left\{\begin{matrix}\frac{\pi}{2} - \theta_s \\ \theta_i - \frac{\pi}{2}\end{matrix}\right\} < \theta_n < \frac{\pi}{2}$	$\left\{\begin{matrix}\varphi_s + \delta_s \\ \varphi_i + \pi - \delta_i\end{matrix}\right\} < \varphi_n < \left[\begin{matrix}\varphi_s + 2\pi - \delta_s \\ \varphi_i + \pi + \delta_i\end{matrix}\right]$
25	U'_i, U_{s1}	$\rho_U(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\theta_i - \frac{\pi}{2} < \theta_n < \frac{\pi}{2} - \theta_s$	$\varphi_i - \pi + \delta_i < \varphi_n < \varphi_i + \pi - \delta_i$
26	U'_i, U_{s2}	$\rho_U(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\left\{\begin{matrix}\frac{\pi}{2} - \theta_s \\ \theta_i - \frac{\pi}{2}\end{matrix}\right\} < \theta_n < \frac{\pi}{2}$	$\left\{\begin{matrix}\varphi_s - \delta_s \\ \varphi_i - \pi + \delta_i\end{matrix}\right\} < \varphi_n < \left[\begin{matrix}\varphi_s + \delta_s \\ \varphi_i + \pi - \delta_i\end{matrix}\right]$
27	U'_i, L_s	$\tau_U(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$0 < \theta_s < \frac{\pi}{2}$	$\left\{\begin{matrix}\frac{\pi}{2} - \theta_s \\ \theta_i - \frac{\pi}{2}\end{matrix}\right\} < \theta_n < \frac{\pi}{2}$	$\left\{\begin{matrix}\varphi_s + \delta_s \\ \varphi_i - \pi + \delta_i\end{matrix}\right\} < \varphi_n < \left[\begin{matrix}\varphi_s + 2\pi - \delta_s \\ \varphi_i + \pi - \delta_i\end{matrix}\right]$
28	L'_{i1}, L'_{s1}	$\rho_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$0 < \theta_n < \left[\begin{matrix}\theta_s \\ \theta_i\end{matrix}\right] - \frac{\pi}{2}$	$0 < \varphi_n < 2\pi$
29	L'_{i1}, L'_{s2}	$\rho_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_s - \frac{\pi}{2} < \theta_n < \theta_i - \frac{\pi}{2}$	$\varphi_s + \pi - \delta_s < \varphi_n < \varphi_s + \pi + \delta_s$
30	L'_{i2}, L'_{s1}	$\rho_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_i - \frac{\pi}{2} < \theta_n < \theta_s - \frac{\pi}{2}$	$\varphi_i + \pi - \delta_i < \varphi_n < \varphi_i + \pi + \delta_i$
31	L'_{i2}, L'_{s2}	$\rho_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\left\{\begin{matrix}\theta_i \\ \theta_s\end{matrix}\right\} - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\left\{\begin{matrix}\varphi_i - \delta_i \\ \varphi_s - \delta_s\end{matrix}\right\} + \pi < \varphi_n < \left[\begin{matrix}\varphi_i + \delta_i \\ \varphi_s + \delta_s\end{matrix}\right] + \pi$
32	L'_{i1}, U'_s	$\tau_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_s - \frac{\pi}{2} < \theta_n < \theta_i - \frac{\pi}{2}$	$\varphi_s - \pi + \delta_s < \varphi_n < \varphi_s + \pi - \delta_s$
33	L'_{i2}, U'_s	$\tau_L(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\left\{\begin{matrix}\theta_i \\ \theta_s\end{matrix}\right\} - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\left\{\begin{matrix}\varphi_i + \pi - \delta_i \\ \varphi_s - \pi + \delta_s\end{matrix}\right\} < \varphi_n < \left[\begin{matrix}\varphi_i + \delta_i \\ \varphi_s - \delta_s\end{matrix}\right] + \pi$
34	U'_i, L'_{s1}	$\tau_U(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\theta_i - \frac{\pi}{2} < \theta_n < \theta_s - \frac{\pi}{2}$	$\varphi_i - \pi + \delta_i < \varphi_n < \varphi_i + \pi - \delta_i$
35	U'_i, L'_{s2}	$\tau_U(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\left\{\begin{matrix}\theta_i \\ \theta_s\end{matrix}\right\} - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\left\{\begin{matrix}\varphi_i - \pi + \delta_i \\ \varphi_s + \pi - \delta_s\end{matrix}\right\} < \varphi_n < \left[\begin{matrix}\varphi_i - \delta_i \\ \varphi_s + \delta_s\end{matrix}\right] + \pi$
36	U'_i, U'_s	$\rho_U(\hat{k},\hat{k}')$	$\frac{\pi}{2} < \theta_i < \pi$	$\frac{\pi}{2} < \theta_s < \pi$	$\left\{\begin{matrix}\theta_i \\ \theta_s\end{matrix}\right\} - \frac{\pi}{2} < \theta_n < \frac{\pi}{2}$	$\left\{\begin{matrix}\varphi_i + \delta_i \\ \varphi_s + \delta_s\end{matrix}\right\} - \pi < \varphi_n < \left[\begin{matrix}\varphi_i - \delta_i \\ \varphi_s - \delta_s\end{matrix}\right] + \pi$

TABLE IV.- INTEGRATION LIMITS FOR CANOPY PHASE FUNCTION

μ_i	μ_s	Intersection regions[a] for limit -				M
		a	b	c	d	
$0 < \mu_i < 1$	$0 < \mu_s < 1$	1 to 4	5, 6	9	7, 8	1
$0 < \mu_i < 1$	$-1 < \mu_s < 0$	10, 11	12 to 15	17, 18	16	2
$-1 < \mu_i < 0$	$-1 < \mu_s < 0$	36	34, 35	28 to 31	32, 33	3
$-1 < \mu_i < 0$	$0 < \mu_s < 1$	25, 26	27	23, 24	19 to 22	4

[a]See table III, first column.

TABLE V.- POLAR COORDINATES WITHIN THE
INTEGRATION FRAME

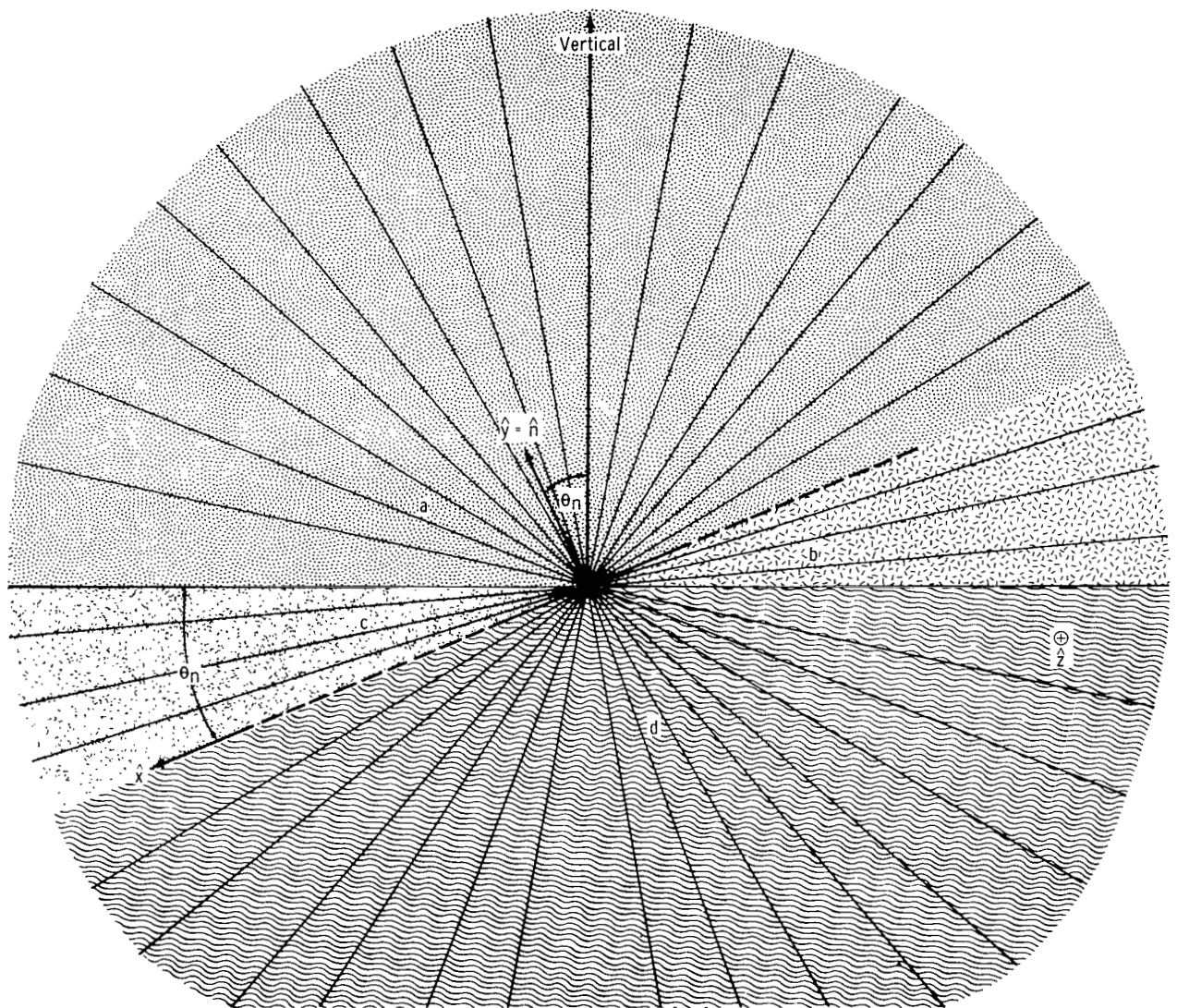| Region	θ	φ
a	$0 < \theta < \pi$	$\pi < \varphi < 2\pi - \theta_n$
b	$0 < \theta < \pi$	$\pi - \theta_n < \varphi < \pi$
c	$0 < \theta < \pi$	$2\pi - \theta_n < \varphi < 2\pi$
d	$0 < \theta < \pi$	$0 < \varphi < \pi - \theta_n$

Figure 1.- Partitioning of space within the integration frame.

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