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THE ROLE OF MULTIPLE SCATTERING IN ONE-DIMENSIONAL RADIATIVE TRANSFER

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SUMMARY

The usual methods of solving the radiative transfer equation yield answers which embrace all orders of scattering and thus shed little light on the underlying physical process. The present analysis examines the contributions of the various orders of scattering to the one-dimensional transfer of radiation. In the one-dimensional case an exact analytical solution exists and the problem reduces to that of expanding this exact solution in powers of the albedo for single scattering. Formulas are given which permit the calculation of any order of scattering in an atmosphere of arbitrary optical thickness, particle albedo, and asymmetry parameter. The results should aid in identifying those physical situations where only the lowest orders of scattering play a significant role and where appropriate approximate methods might provide results of acceptable accuracy.

INTRODUCTION

If one seeks to determine the radiation reflected from or transmitted through a plane parallel homogeneous atmosphere (slab) by solving, in the usual way, the radiative transfer equation, the answer one obtains embraces all orders of scattering. Ordinarily, this is all that is needed; however, such an approach provides little understanding of the underlying physical process. The investigation reported herein provides a more detailed insight into the problem to the extent of identifying the contributions of the various orders of scattering to the transfer of radiation through a slab.

Apart from this, the investigation has certain practical connotations. In some of the more complex problems of radiative transfer involving (a) striated atmospheres in which scattering properties vary with depth, (b) scattering particles comparable in size with wavelength, and (c) polarization, methods of successive approximation may have to be utilized. In several of such methods (refs. 1 to 5), the successive iterations correspond to the contributions of successive orders of scattering. In applying these methods, knowledge of the role of multiple scattering would clearly aid in making a preliminary assessment of the rapidity of convergence of the procedure and hence the time and effort that would be needed to achieve a solution. Even in the case of simpler problems, sometimes it suffices to consider only the lowest orders of scattering (primary, secondary, or tertiary). For these situations one can arrive at an answer of acceptable accuracy with a considerable saving of labor. Here also a knowledge of the role of multiple scattering would aid in deciding where the use of such approximate methods is warranted.

In the case of the transfer of radiation through a homogeneous plane parallel atmosphere, even in the simplest of cases involving isotropic scattering, an analytical approach cannot be carried beyond secondary scattering (ref. 6), and in order to determine the contributions of the higher modes of scattering, numerical procedures are required. Such approaches do not lend themselves to a systematic parametric analysis to determine the dependence of the different orders of scattering on slab thickness, albedo for single scattering, and asymmetry factor. In view of this fact, consideration has been limited to the one-dimensional radiative transfer problem. For this case exact solutions exist and the problem reduces to that of expressing the exact solutions as power series in the albedo for single scattering. The formulas so derived permit the determination of the contribution of any order of scattering for any slab thickness, for arbitrary albedo, and for arbitrary asymmetry parameter.

The nature of the radiative transfer problem is such that the results of this one-dimensional analysis might be expected to have more than simply qualitative relevance. In this regard it is to be borne in mind that the two-beam approximation (refs. 7 and 8), which is tantamount to a one-dimensional approach, has proven to be of considerable value in providing results of acceptable accuracy to many three-dimensional problems of radiative transfer.

The one-dimensional transfer problem has special significance in the case of high-energy corpuscular radiation where secondary production is in predominantly near forward angles and transfer calculations are made as if all secondaries proceed in the forward direction (straightahead approximation). (See ref. 9.) The multiple-scattering series performs a special function in this case since it reduces the usual integro-differential transfer equation into a set of coupled differential equations. (See refs. 10 and 11.) Particular advantage of the multiple-scattering series has been noted in application to charged-particle transfer calculations where, because of ionization energy loss, the series has nearly converged after only the double scattering term. (See ref. 11.) This rapid convergence is in spite of the high "albedo" (multiplicities near 3) in these scattering events.

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SYMBOLS

$D(\tau)$	$= F^{\downarrow}(\tau) + F^{\uparrow}(\tau)$	(proportional to photon density)
$D_n(\tau)$	$= \dfrac{F_n^{\downarrow}(\tau) + F_n^{\uparrow}(\tau)}{F^{\downarrow}(\tau) + F^{\uparrow}(\tau)}$	(fraction of photons at depth τ of nth order)
$F(\tau)$	$= F^{\downarrow}(\tau) - F^{\uparrow}(\tau)$	(net downward flux)
$F^{\uparrow}(\tau)$	upward flux of diffuse radiation	
$F^{\downarrow}(\tau)$	downward flux of diffuse radiation	
$F_n^{\uparrow}(0)$	contribution of nth order photons to reflected flux divided by incident flux	
$F_n^{\uparrow}(\tau), F_n^{\downarrow}(\tau)$	contributions of nth order photons to upward and downward flux divided by incident flux, respectively	
$F_n^{\downarrow}(\tau_0)$	contribution of nth order photons to transmitted flux divided by incident flux	
$\tilde{F}_n^{\uparrow}(0)$	contribution of nth order photons to reflected flux divided by total reflected flux	
$\tilde{F}_n^{\downarrow}(\tau_0)$	contribution of nth order photons to transmitted flux divided by total transmitted flux	
F_0	incident flux	
g	asymmetry parameter	
$I(\tau,\mu,\phi)$	specific intensity of diffuse radiation	
$I^{(0)}(\tau,\mu)$	$= \int_0^{2\pi} I(\tau,\mu,\phi)\, d\phi$	
$I_1(\tau)$	specific intensity at depth τ in direction of incident radiation	
$I_2(\tau)$	specific intensity at depth τ in direction opposite to incident radiation	

k	$= \left[\left(1 - \omega_0\right)\left(1 - g\omega_0\right)\right]^{1/2}$
$p(\mu, \phi; \mu', \phi')$	phase function
$p^{(0)}(\mu, \mu')$	$= \frac{1}{2\pi} \int_0^{2\pi} p(\mu, \phi; \mu', \phi')\, d\phi$
r_0	$= \frac{k - 1 + \omega_0}{k + 1 - \omega_0}$
δ	Dirac delta function
θ	angle to upward vertical
μ	$= \cos\theta$
μ_0	cosine of angle of incident beam with downward vertical
τ	normal optical depth
τ_0	normal optical depth of homogeneous plane parallel atmosphere
ϕ	azimuthal angle
ϕ_0	azimuthal angle of incident beam
ω_0	albedo for single scattering (elastic scattering probability)

Primes denote variables of integration.

ANALYSIS

In considering the contribution of various orders of scattering to the radiative transfer through a homogeneous slab, the following recursive approach is invariably adopted. From the intensity distribution of the incident radiation, one calculates the intensity distribution of the first-order scattered photons. With this as an input, one then determines the intensity distribution of second-order scattered photons and so on. When one endeavors to apply this procedure in practice, problems arise. Thus, in the transfer through a plane parallel atmosphere, even in the simple case of isotropic scattering, successive integrals become analytically unmanageable beyond the second order. (See ref. 6.) Even in the one-dimensional problem, although the successive integrations involve only elementary functions, the expressions become excessively complicated after a few terms. In view of this complication, an alternative approach has been adopted in the present paper. For a particular (albeit restricted) class of phase functions, the radiative transfer equations can be solved exactly. The problem then reduces to the mathematical one of developing the exact solutions as power series in the particle albedo. The successive terms of such a series development give, of course, the contributions of successive orders of scattering. The more detailed development of this line of attack is pursued herein.

The following equation for the transfer of diffuse radiation through a homogeneous slab illuminated from above by a parallel beam making an angle $\cos^{-1}\mu_0$ with the downward vertical is derived in reference 12:

$$\mu \frac{dI(\tau,\mu,\phi)}{d\tau} = I(\tau,\mu,\phi) - \frac{1}{4\pi}\int_{-1}^{1} d\mu' \int_0^{2\pi} p(\mu,\phi;\mu',\phi')\, I(\tau,\mu',\phi')\, d\phi' - \frac{F_0}{4}\, p\left(\mu,\phi;-\mu_0,\phi_0\right) e^{-\tau/\mu_0} \tag{1}$$

In this equation $\cos^{-1}\mu$ is measured from the upward vertical and the phase function is normalized in the sense that

$$\int_0^{2\pi}\int_{-1}^{1} p(\mu,\phi;\mu',\phi')\, d\mu\, d\phi = 4\pi\omega_0$$

Integrating equation (1) over ϕ yields

$$\mu \frac{dI^{(0)}(\tau,\mu)}{d\tau} = I^{(0)}(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p^{(0)}(\mu,\mu')\, I^{(0)}(\tau,\mu')\, d\mu' - \frac{\pi F_0}{2}\, p^{(0)}\left(\mu,-\mu_0\right) e^{-\tau/\mu_0} \tag{2}$$

where

$$I^{(0)}(\tau,\mu) = \int_0^{2\pi} I(\tau,\mu,\phi)\, d\phi$$

and

$$p^{(0)}(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\mu,\phi;\mu',\phi')\, d\phi$$

Bear in mind that the phase function depends only on the cosine of the angle of scattering $\phi - \phi'$.

One now further restricts the phase function to the form

$$p^{(0)}(\mu,\mu') = \omega_0\Big[(1 + g)\delta(\mu' - \mu) + (1 - g)\delta(\mu' + \mu)\Big] \tag{3}$$

where ω_0 is the albedo for single scattering.[1]

By substituting this expression for $p^{(0)}(\mu,\mu')$ into equation (2), one obtains

$$\mu \frac{dI^{(0)}(\tau,\mu)}{d\tau} = I^{(0)}(\tau,\mu) - \frac{\omega_0}{2}\Big[(1 + g)\, I^{(0)}(\tau,\mu) + (1 - g)\, I^{(0)}(\tau,-\mu)\Big]$$

$$- \frac{\pi F_0 \omega_0}{2}\Big[(1 + g)\delta\left(-\mu_0 - \mu\right) + (1 - g)\delta\left(-\mu_0 + \mu\right)\Big] e^{-\tau/\mu_0} \tag{4}$$

It is to be noted that the forcing function (last term of eq. (4)) involves Dirac delta functions, and this same characteristic must perforce be imposed on the solutions. Denoting the intensity in the direction of the incident radiation by $I_1(\tau)$ and in the opposite direction by $I_2(\tau)$ yields

$$\left.\begin{aligned} I^{(0)}(\tau,-\mu) &= I_1(\tau)\, \delta\left(\mu - \mu_0\right) \qquad (\mu > 0) \\ I^{(0)}(\tau,\mu) &= I_2(\tau)\, \delta\left(\mu - \mu_0\right) \qquad (\mu > 0) \end{aligned}\right\} \tag{5}$$

By using equations (4) and (5), the equations giving the downward and upward intensities are, respectively:

[1]In the "one-speed approximation" in neutron transport theory, ω_0 would be the average multiplicity.

$$-\mu \frac{dI_1(\tau)}{d\tau} \delta\left(\mu - \mu_0\right) = I_1(\tau)\, \delta\left(\mu - \mu_0\right) - \frac{\omega_0}{2}\left[(1 + g)I_1(\tau)\, \delta\left(\mu - \mu_0\right) + (1 - g)I_2(\tau)\, \delta\left(\mu - \mu_0\right)\right]$$

$$- \frac{\pi F_0 \omega_0}{2}\left[(1 + g)\delta\left(\mu - \mu_0\right) + (1 - g)\delta\left(-\mu_0 - \mu\right)\right] e^{-\tau/\mu_0} \tag{6}$$

$$\mu \frac{dI_2(\tau)}{d\tau} \delta\left(\mu - \mu_0\right) = I_2(\tau)\, \delta\left(\mu - \mu_0\right) - \frac{\omega_0}{2}\left[(1 + g)I_2(\tau)\, \delta\left(\mu - \mu_0\right) + (1 - g)I_1(\tau)\, \delta\left(\mu - \mu_0\right)\right]$$

$$- \frac{\pi F_0 \omega_0}{2}\left[(1 + g)\delta\left(-\mu_0 - \mu\right) + (1 - g)\delta\left(-\mu_0 + \mu\right)\right] e^{-\tau/\mu_0} \tag{7}$$

Integrating over their respective hemispheres yields

$$\mu_0 \frac{dI_1(\tau)}{d\tau} = -I_1(\tau) + \frac{\omega_0}{2}\left[(1 + g)I_1(\tau) + (1 - g)I_2(\tau)\right] + \frac{\pi F_0 \omega_0}{2}(1 + g)e^{-\tau/\mu_0} \tag{8}$$

and

$$\mu_0 \frac{dI_2(\tau)}{d\tau} = I_2(\tau) - \frac{\omega_0}{2}\left[(1 + g)I_2(\tau) + (1 - g)I_1(\tau)\right] - \frac{\pi F_0 \omega_0}{2}(1 - g)e^{-\tau/\mu_0} \tag{9}$$

By introducing downward and upward fluxes as defined by

$$\mu_0 I_1(\tau) = \pi F^{\downarrow}(\tau)$$

and

$$\mu_0 I_2(\tau) = \pi F^{\uparrow}(\tau)$$

into equations (8) and (9), one arrives finally at the pair of first-order linear differential equations

$$\mu_0 \frac{dF^{\downarrow}}{d\tau} = -F^{\downarrow} + \frac{\omega_0}{2}\left[(1 + g)F^{\downarrow} + (1 - g)F^{\uparrow}\right] + \frac{\mu_0 F_0 \omega_0}{2}(1 + g)e^{-\tau/\mu_0} \tag{10}$$

and

$$\mu_0 \frac{dF^{\uparrow}}{d\tau} = F^{\uparrow} - \frac{\omega_0}{2}\left[(1 + g)F^{\uparrow} + (1 - g)F^{\downarrow}\right] - \frac{\mu_0 F_0 \omega_0}{2}(1 - g)e^{-\tau/\mu_0} \tag{11}$$

In addition, the following boundary conditions are to be imposed:

$$F^{\downarrow}(0) = 0 \tag{12}$$

$$F^{\uparrow}\left(\tau_0\right) = 0 \tag{13}$$

Since the postulated phase function as given by equation (3) requires that all photons be scattered in either the forward or the backward direction, not unexpectedly equations (10) to (13) are precisely those governing one-dimensional radiative transfer. Exact solutions for these equations have been derived by Sobolev (ref. 13). For the convenience of the reader, these solutions are rederived in appendix A.

It is of interest to note that the phase function as given by equation (3) with $g \approx 1$ approximates Mie scattering from large smooth spheres and with $g \approx -1$ scattering from opaque particles with rough faces oriented at random (ref. 14).

Throughout the subsequent analysis, μ_0 is set equal to unity (that is, normal incidence). However, by simply replacing τ by τ/μ_0 in any of the results derived below, one obtains answers corresponding to the case in which the slab is obliquely illuminated at an angle $\cos^{-1}\mu_0$ to the downward vertical.

In appendix A the following expressions are obtained for the overall intensities of the downward and upward directed beams at a depth τ within a slab of optical thickness τ_0, particle albedo ω_0, and asymmetry factor g:

$$\frac{F^{\downarrow}(\tau)}{F_0} = \frac{e^{k\left(\tau_0-\tau\right)} - r_0^2 e^{-k\left(\tau_0-\tau\right)}}{e^{k\tau_0} - r_0^2 e^{-k\tau_0}} - e^{-\tau} \tag{14}$$

and

$$\frac{F^{\uparrow}(\tau)}{F_0} = r_0 \frac{e^{k\left(\tau_0-\tau\right)} - e^{-k\left(\tau_0-\tau\right)}}{e^{k\tau_0} - r_0^2 e^{-k\tau_0}} \tag{15}$$

where

$$k = \sqrt{\left(1 - \omega_0\right)\left(1 - g\omega_0\right)} \tag{16}$$

and

$$r_0 = \frac{k - 1 + \omega_0}{k + 1 - \omega_0} \tag{17}$$

Contribution of Successive Orders of Scattering to Reflection From a Semi-Infinite Slab

By setting $\tau_0 = \infty$ in equations (14) and (15), the following expressions for the intensities as a function of optical depth are obtained:

$$\frac{F^{\downarrow}(\tau)}{F_0} = e^{-k\tau} - e^{-\tau} \tag{18}$$

and

$$\frac{F^{\uparrow}(\tau)}{F_0} = r_0 e^{-k\tau} \tag{19}$$

The reflected flux, divided by the flux of the incident beam, is thus

$$\frac{F^{\uparrow}(0)}{F_0} = r_0$$

By using the series development of r_0 as given in equation (B20), one obtains

$$\frac{F^{\uparrow}(0)}{F_0} = \sum_{n=1}^{\infty} \left(\frac{1 + g}{4} \omega_0\right)^n \left(\frac{1 + g}{1 - g}\right) \sum_{j=0}^{\frac{n}{2}, \frac{n+1}{2}} \frac{(2n - 2j)!}{j!(n - j)!(n - 2j + 1)!} \left[- \frac{4g}{(1 + g)^2}\right]^j \tag{20}$$

Hence, the contribution of nth order scattering to the reflected beam (normalized by dividing by the incident flux) is given by

$$\omega_0{}^n\left(\frac{1+g}{4}\right)^n\left(\frac{1+g}{1-g}\right)\sum_{j=0}^{\frac{n}{2},\frac{n+1}{2}}\frac{(2n-2j)!}{j!(n-j)!(n-2j+1)!}\left[-\frac{4g}{(1+g)^2}\right]^j \tag{21}$$

By setting $n = 1, 2,$ and 3, one obtains the following contributions of the first three orders of scattering:

First order:

$$\frac{\omega_0}{2^2}(1-g) \tag{22}$$

Second order:

$$\frac{\omega_0{}^2}{2^3}(1+g)(1-g) \tag{23}$$

Third order:

$$\frac{\omega_0{}^3}{2^4}\left[(1+g)^2(1-g)+\frac{1}{4}(1-g)^3\right] \tag{24}$$

These expressions not only define the contribution of successive orders of scattering but, in addition, reveal something of the nature of the scattering processes that are involved. Thus, the presence of the factor $1 - g$ in expression (22) is a mathematical manifestation of the rather obvious fact that the first-order photons in the reflected beam have all undergone a single rearward scattering. In the case of the second-order photons, the presence of the factor $(1 + g)(1 - g)$ in expression (23) indicates that these have all undergone one forward and one rearward scattering (although not necessarily in that order). The third-order photons in the reflected beam are of two kinds corresponding to the two terms of expression (24). The first kind embraces those which have undergone two forward scatterings and one rearward scattering and the second kind, those which have been subjected to three rearward scatterings. Note that in the emerging beam the former preponderates over the latter by a factor of four.

Contribution of Different Orders of Scattering to Photon Population at any Depth Within a Semi-Infinite Slab

The density of photons (D) at a depth τ is proportional to $F^{\downarrow}(\tau) + F^{\uparrow}(\tau)$. Thus, from equations (18) and (19),

$$D = \text{Constant}\left[\left(1 + r_0\right)e^{-k\tau} - e^{-\tau}\right] \tag{25}$$

However, from equations (B14) and (B20)

$$e^{-k\tau} = e^{-\tau}\left(1 + \sum_{n=1}^{\infty} A_n \omega_0^{\,n}\right) \tag{26}$$

and

$$r_0 = \sum_{n=1}^{\infty} B_n \omega_0^{\,n} \tag{27}$$

where

$$A_n = \left[-\frac{(1 + g)}{2}\right]^n \sum_{r=1}^{n} \tau^r \sum_{m=0}^{r} \frac{(-)^m}{m!(r - m)!} \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{m(m - 2)\ .\ .\ .\left[m - 2(n - j - 1)\right]}{j!(n - 2j)!}\left[\frac{2g}{(1 + g)^2}\right]^j \tag{28}$$

$$B_n = \left(\frac{1 + g}{4}\right)^n \left(\frac{1 + g}{1 - g}\right) \sum_{j=0}^{\frac{n}{2},\frac{n+1}{2}} \frac{(2n - 2j)!}{j!(n - j)!(n - 2j + 1)!}\left[-\frac{4g}{(1 + g)^2}\right]^j \tag{29}$$

Substituting equations (26) and (27) into equation (25) yields

$$\begin{aligned} D &= \text{Constant}\left[e^{-\tau}\left(1 + \sum_{n=1}^{\infty} A_n \omega_0^{\,n}\right)\left(1 + \sum_{n=1}^{\infty} B_n \omega_0^{\,n}\right) - e^{-\tau}\right] \\ &= \text{Constant} \sum_{n=1}^{\infty} e^{-\tau}\left(A_n + B_n + \sum_{j=1}^{n-1} A_j B_{n-j}\right)\omega_0^{\,n} \end{aligned}$$

Dividing by equation (25) yields

$$1 = \sum_{n=1}^{\infty} D_n \omega_0{}^n \tag{30}$$

where D_n defines the fraction of photons at depth τ which are of nth order and is given by

$$D_n = \frac{A_n + B_n + \sum_{j=1}^{n-1} A_j B_{n-j}}{\left(1 + r_0\right) e^{(1-k)\tau} - 1} \tag{31}$$

Contribution of Various Orders of Scattering to Reflection From and Transmission Through a Finite Slab

The reflected flux (normalized with respect to the incident flux) is obtained on setting $\tau = 0$ in equation (15)

$$\frac{F^{\uparrow}(0)}{F_0} = r_0 \frac{1 - e^{-2k\tau_0}}{1 - r_0{}^2 e^{-2k\tau_0}} \tag{32}$$

The transmitted flux (normalized as above) is obtained on setting $\tau = \tau_0$ in equation (14)

$$\frac{F^{\downarrow}\left(\tau_0\right)}{F_0} = \frac{\left(1 - r_0{}^2\right) e^{-k\tau_0}}{1 - r_0{}^2 e^{-2k\tau_0}} - e^{-\tau_0} \tag{33}$$

Consider the case $\omega_0 = 1$ (that is, conservative or pure elastic scattering). From equations (16) and (17) $k = 0$ and $r_0 = 1$ and the expressions (32) and (33) for the reflected and transmitted intensities, assume the indeterminate form $0/0$.

In this case let $\omega_0 = 1 - \epsilon$ where ϵ is supposed to be sufficiently small to justify the neglect of all terms but those involving the lowest order in ϵ. Thus,

$$\left.\begin{aligned} k &= \sqrt{1 - g}\,\epsilon^{1/2} \\ r_0 &= 1 - \frac{2}{\sqrt{1 - g}}\epsilon^{1/2} \end{aligned}\right\} \tag{34}$$

The introduction of these expressions into equation (32) yields

$$\frac{F^{\uparrow}(0)}{F_0} = \frac{\left(1 - \frac{2}{\sqrt{1 - g}}\,\epsilon^{1/2}\right)\left(2\sqrt{1 - g}\,\epsilon^{1/2}\,\tau_0\right)}{1 - \left(1 - \frac{4}{\sqrt{1 - g}}\,\epsilon^{1/2}\right)\left(1 - 2\sqrt{1 - g}\,\epsilon^{1/2}\,\tau_0\right)}$$

$$= \frac{2\sqrt{1 - g}\,\epsilon^{1/2}\,\tau_0}{\frac{4}{\sqrt{1 - g}}\,\epsilon^{1/2} + 2\sqrt{1 - g}\,\epsilon^{1/2}\,\tau_0}$$

$$= \frac{(1 - g)\tau_0}{2 + (1 - g)\tau_0} \tag{35}$$

The introduction of equations (34) into equation (33) yields

$$\frac{F^{\downarrow}(\tau_0)}{F_0} = \frac{\frac{4}{\sqrt{1 - g}}\,\epsilon^{1/2}\left(1 - \sqrt{1 - g}\,\epsilon^{1/2}\,\tau_0\right)}{1 - \left(1 - \frac{4}{\sqrt{1 - g}}\,\epsilon^{1/2}\right)\left(1 - 2\sqrt{1 - g}\,\epsilon^{1/2}\,\tau_0\right)} - e^{-\tau_0}$$

$$= \frac{\frac{4}{\sqrt{1 - g}}\,\epsilon^{1/2}}{\frac{4}{\sqrt{1 - g}}\,\epsilon^{1/2} + 2\sqrt{1 - g}\,\epsilon^{1/2}\,\tau_0} - e^{-\tau_0}$$

$$= \frac{2}{2 + (1 - g)\tau_0} - e^{-\tau_0} \tag{36}$$

By the substitution of equations (26) and (27) into equations (32) and (33), one can, by straightforward procedures similar to those used in the preceding subsection, express $\frac{F^{\uparrow}(0)}{F_0}$ and $\frac{F^{\downarrow}(\tau_0)}{F_0}$ as a power series in ω_0. Thus,

$$\frac{F^{\uparrow}(0)}{F_0} = \sum_{n=1}^{\infty} F_n^{\uparrow}(0)\,\omega_0^{\,n} \tag{37}$$

$$\frac{F^{\downarrow}(\tau_0)}{F_0} = \sum_{n=1}^{\infty} F_n^{\downarrow}(\tau_0)\, \omega_0^{\,n} \tag{38}$$

By dividing equation (37) by equation (32) $(\omega_0 \neq 1)$ or by equation (35) $(\omega_0 = 1)$, one obtains

$$1 = \sum_{n=1}^{\infty} \tilde{F}_n^{\uparrow}(0)\, \omega_0^{\,n}$$

where $\tilde{F}_n^{\uparrow}(0) = \dfrac{F_n^{\uparrow}(0)}{F^{\uparrow}(0)}$ and thus defines the fractional contribution of the nth order photons to the reflected beam.

Similarly, by dividing equation (38) by equation (33) (for $\omega_0 \neq 1$) or by equation (36) (for $\omega_0 = 1$), one obtains

$$1 = \sum_{n=1}^{\infty} \tilde{F}_n^{\downarrow}(\tau_0)\, \omega_0^{\,n}$$

where

$$\tilde{F}_n^{\downarrow}(\tau_0) = \frac{F_n^{\downarrow}(\tau_0)}{F^{\downarrow}(\tau_0)}$$

and thus defines the fractional contribution of nth order photons to the transmitted beam.

NUMERICAL RESULTS

The formulas of the preceding section have been used to make calculations bearing on the role played by the various orders of scattering on reflection from semi-infinite slabs and reflection from, and transmission through, slabs of finite thickness. In this section the results of these calculations are presented and discussed. The effects of variation in slab thickness, particle albedo, and asymmetry factor are considered.

It is of interest to note that, in practice, both the albedo for single scattering ω_0 and the asymmetry factor g are subject to wide variation. Thus, the albedo for single scattering can be as high as 0.9999 in clouds. Insofar as the asymmetry parameter is concerned, this value approaches unity for Mie scattering from large smooth spheres and minus unity for opaque particles with rough faces oriented at random.

Overall Reflection From Semi-Infinite Slab

In figure 1 the reflection coefficient is plotted against the asymmetry factor for various values of particle albedo ω_0. As the asymmetry factor increases (the scattering increasing in the forward direction), the impinging photons penetrate deeper and, thus, for particle albedos other than unity, the absorption increases and the reflection decreases. When the scattering is entirely in the forward direction (g = 1), the reflected flux must, of course, fall to zero. At a particle albedo of unity, there is no absorption and, since the steady case is being treated and there is no accumulation of photons within the slab interior, there must be as many photons emerging as are entering. In other words, the reflection coefficient is unity independent of the value assigned to the asymmetry factor. In view of this effect, one would expect the reflection to become insensitive to asymmetry factor at high particle albedos. However, it is interesting to note how large an albedo is needed to produce a reasonable measure of insensitivity to the asymmetry factor. Thus, even with a particle albedo as high as 0.99, the reflected flux at $g = 0.8$ is only about 75 percent of that at $g = -0.8$.

Overall Reflection From and Transmission Through Finite Slabs

For the case of conservative (elastic) scattering $(\omega_0 = 1)$, as many photons leave the slab as enter it, since the time-independent problem is being considered. Thus,

$$F_0 = F_0 e^{-\tau_0} + F^{\uparrow}(0) + F^{\downarrow}(\tau_0)$$

The first term appearing on the right-hand side corresponds to those photons which pass through the slab without undergoing a single scattering. The remaining two terms correspond to those photons which are diffusively reflected from and transmitted through the slab, respectively. In the case of thick slabs, the first term becomes negligibly small and the preceding relation then becomes

$$1 \approx \frac{F^{\uparrow}(0)}{F_0} + \frac{F^{\downarrow}(\tau_0)}{F_0}$$

Figure 2 shows dependence of reflection from and transmission through a slab of optical thickness 0.1 on asymmetry factor for values of particle albedo of 0.2 (0.2) 1.0. Similar plots for slabs of optical thickness 2 and 20 are given in figures 3 and 4, respectively.

In figure 4(a), note how sensitive the reflection coefficient from a thick slab becomes to particle albedo as conservative scattering $(\omega_0 = 1)$ is approached. Thus, for isotropic scattering (g = 0), an increase of particle albedo from 0.9 to 1.0 results in almost a doubling of the intensity of the reflected beam. Transmission through a slab of optical thickness 20 only becomes significant at particle albedos in excess of 0.9. Note how sensitive transmission is to particle albedo at these high values. Thus, again considering the case of isotropic scattering, an increase of particle albedo from 0.99 to 1.0 results in a doubling of the intensity of the transmitted beam. Note also the sensitivity of transmission to asymmetry factor in the neighborhood of g = 1. In the case of conservative scattering, an increase in the asymmetry factor from 0.8 to 1.0 results in a threefold increase in transmission.

Contribution of the First 10 Orders of Scattering to Reflection From a Semi-Infinite Slab

The contribution of the first 10 orders of scattering to reflection from a semi-infinite slab is plotted against particle albedo over the range 0.8 to 1.0 for various values of the asymmetry factor in figure 5. Orders of scattering in excess of 10 only play a significant role at the higher albedos. Thus, at a particle albedo of 0.8, the first 10 orders of scattering contribute more than 90 percent to the reflected beam for all values of the asymmetry factor. However, at an albedo of unity and predominantly forward scattering (g = 0.9), photons which have been scattered more than 10 times contribute 80 percent to the reflected beam.

Contribution of the First 10 Orders of Scattering to Reflection From and Transmission Through Slabs of Finite Thickness

The contribution of the first 10 orders of scattering to the reflected and transmitted beams is given as a function of slab thickness for various values of particle albedo in figure 6 for the case of predominantly forward scattering (g = 0.8). One notes from figure 6(a) that insofar as the reflected beam is concerned, the modes of scattering in excess of 10 only play a significant role at particle albedos in excess of 0.8 and slab thicknesses in excess of 2. Indeed, for sufficiently thin slabs, only first-order scattering will contribute. For a particle albedo of 0.9 as the slab thickness increases beyond 2, orders of scattering in excess of 10 make an increasing contribution. The contribution tends to a limiting value as a slab thickness of about 14 is approached. At this thickness, the reflected beam is, to all intents and purposes, the same as that for a semi-infinite slab. At a particle albedo of unity, and a slab thickness of 20, photons that have undergone more than 10 scatterings contribute 50 percent to the reflected beam. Further increase of slab thickness will lead to further enhancement of the contribution of these higher modes of scattering. By referring to figure 5, it is to be noted that the limiting contribution in this case is about 66 percent.

By turning to a consideration of transmission (fig. 6(b)), one notes that the orders of scattering in excess of 10 only assume a significant role for slab thicknesses in excess of 2. As the slab thickness increases, there will be, of course, an exponential fall off in the transmitted flux. However, of those photons that do emerge, the higher orders of scattering will make an increasingly important contribution. Thus, the curves, even for small particle albedos, will tend monotonically to zero. This is simply a manifestation of the fact that although for small particle albedos the probability of a photon surviving more than 10 scatterings is low, for sufficiently thick slabs the probability of photons getting through at all with 10 or fewer scatterings is even smaller.

Similar plots are presented for the case of predominantly rearward scattering ($g = -0.8$) in figure 7. Comparison of figures 6(a) and 7(a) and 6(b) and 7(b) indicated, by and large, the more important role of the higher orders of multiple scattering associated with strongly forward scattering. In figure 8 the contributions from orders of scattering in excess of 1 to reflection from and transmission through a slab of optical thickness 0.1 are plotted against particle albedo for the cases of predominantly forward ($g = 0.9$) and rearward ($g = -0.9$) scattering. Even for this thin slab for $g = -0.9$ and a particle albedo in excess of 0.8, orders of scattering of 2 or higher contribute over 40 percent to the transmitted beam. Similar plots are presented in figure 9 for a slab of optical thickness 20. In this case the contributions from orders of scattering in excess of 10 are displayed. It is interesting to note that at a particle albedo of unity and with dominantly rearward scattering, orders of scattering in excess of 10 contribute more than 99.99 percent to the transmitted beam.

Dominance of Various Orders of Scattering With Depth in a Semi-Infinite Slab

In the photon population at any depth, all orders of scattering are presented. As the depth increases, the principal contribution will come from increasingly high orders of scattering. This is exhibited in figure 10 for the case of isotropic scattering in which the domains of dominance of the various orders of scattering are charted within a framework of particle albedo and optical depth. Thus, for particle albedos less than 0.2, first-order scattered photons preponderate down to a depth of 6.6. Between 6.6 and 9.4, second-order scattered photons provide the dominant contribution and so on. As the particle albedo increases, the transition from dominance by one order of scattering to the next as the depth increases becomes increasingly rapid. Thus, at a particle albedo of unity, first-order scattering makes the major contribution down to a depth of 2.7;

however, by the time a depth of 7.5 is reached, the greatest contribution comes from tenth order of scattering.

Dominance of Various Orders of Scattering Within Beams Transmitted Through Finite Slabs

As the slab thickness increases, the principal contribution to the transmitted beam will come from increasingly high orders of scattering. This is illustrated in figure 11 for cases of predominantly forward and rearward scattering, respectively. In each instance the domains of dominance by various orders of scattering are displayed within a framework of particle albedo and slab thickness. In figure 11(b) one notes that although first-order scattering plays a major role for small particle albedos and small slab thicknesses, thereafter dominance is invariably by an even order of scattering. This is a manifestation of the fact that for predominantly rearward scattering, the odd orders of scattering are suppressed in the transmitted beam. Further manifestation of this effect is found in the plots described in the next subsection.

Contribution of the Various Orders of Scattering to Reflection From and Transmission Through Slabs of Finite Thickness

In figure 12, plots are given that show the individual contributions of the first three orders of scattering to reflection from and transmission through a slab of optical thickness 0.1 for albedos of 0.2 and 1.0 with predominantly forward scattering. For this thin slab the reflected and transmitted fluxes consist almost entirely of first-order photons. From figure 13 which corresponds to predominantly rearward scattering, one notes that at high particle albedos second-order photons contribute as much as 30 percent to the transmitted beam.

Figure 14 displays the contributions of the first 10 orders of scattering to the reflection from and transmission through a slab of optical thickness 20 for various particle albedos for the case of predominantly forward scattering ($g = 0.8$). Insofar as reflection is concerned, the contributions of successive orders are monotonically decreasing. However, as the particle albedo increases, the spectrum becomes pronouncedly flatter and subscribes once again to the increasingly important role played by all higher modes of scattering at the higher particle albedos. The curves pertaining to transmission, on the other hand, are peaked. Thus, at a particle albedo of 0.2, third-order scattering contributes most. At a particle albedo of 0.4, it is the seventh order of scattering which preponderates. At particle albedos of 0.6 and higher, the principal contribution comes from orders of scattering in excess of 10.

In figure 15 similar plots are presented for the case of dominantly rearward scattering. In this case it is to be noted that the curves appear to have a jagged or saw-toothed structure. This is a further manifestation of the fact that with predominantly rearward scattering, the even orders of scattering are suppressed in the reflected beam and the odd orders in the transmitted beam. This is strikingly illustrated in figure 16 in which plots are given for a slab of optical thickness 2 and an asymmetry factor of $g = -0.98$.

Orders of Scattering Required for Specified Accuracy

Figure 17 provides plots that define the order of scattering needed to give an accuracy of 1 percent in the calculation of reflected and transmitted fluxes for a slab of optical thickness 0.1. For reflection with predominantly forward scattering and transmission with predominantly rearward scattering, second-order terms must be included at all but the smallest albedos if the desired accuracy is to be achieved.

Similar plots, defining order of scattering needed for 10-percent accuracy, are given in figures 18 and 19 for slabs of optical thickness 2 and 20, respectively, Insofar as reflected flux from slab of optical thickness 2 is concerned, the case of predominantly forward scattering is the severest one since higher orders of scattering are needed to achieve the desired precision. Thus, at a particle albedo as low as 0.2, second order must be included and at high particle albedos, up to fifth order must be considered. Insofar as the transmitted flux is concerned (fig. 18(b)), the roles are reversed and it is the case of predominantly rearward scattering which requires the inclusion of the higher orders of scattering. Very high orders of scattering must be embraced in the case of the thick slab $(\tau_0 = 20)$ if the desired precision is to be achieved. For $g = 0.9$, orders of scattering in excess of 10 are needed for a particle albedo exceeding 0.8 in the computation of reflection and for a particle albedo exceeding 0.33 in the computation of transmission.

CONCLUDING REMARKS

In the present paper expressions for the contributions of various orders of scattering to the overall reflection from and transmission through slabs of arbitrary thickness, particle albedo, and asymmetry parameter have been given for the one-dimensional case and numerical examples have been calculated.

The results presented herein have several applications. In the first place, they will aid in identifying those physical situations – combinations of thickness, albedo, and scattering phase function – in which answers of adequate accuracy can be achieved by considering only first or second orders of scattering. If this is known, solutions of problems in radiative transfer can be obtained with relative ease.

The second area of application relates to those situations where the complexity of the problem necessitates the use of numerical procedures and where the procedure adopted is such that successive iterations define the contributions of the successive orders of scattering. The results contained herein will permit a preliminary assessment to be made of the rapidity of convergence of such series developments and, hence, the effort that will be needed in achieving a solution of the desired precision.

A third area in which methods similar to those discussed herein may find application is to high-energy radiation transfer in which the high-energy secondaries are produced predominantly forward. However, low-energy secondaries are produced and subsequently scatter nearly isotropically. The approximation of low-energy propagation by means of a one-dimensional isotropic scattering law (asymmetry factor zero) would surely produce more realistic results than a strict straightahead approximation (asymmetry factor unity).

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APPENDIX A

EXACT SOLUTION OF RADIATIVE TRANSFER EQUATION IN ONE DIMENSION

It has been shown in the text that by postulating a phase function of the form given by equation (3), that is,

$$p^{(0)}(\mu,\mu') = \omega_0\left[(1 + g)\delta(\mu' - \mu) + (1 - g)\delta(\mu' + \mu)\right] \tag{A1}$$

the equations of radiative transfer in the case of normal illumination $\left(\mu_0 = 1\right)$ are given by the following equations (eqs. (10) to (13)):

$$\frac{dF^{\downarrow}}{d\tau} = -F^{\downarrow} + \frac{\omega_0}{2}\left[(1 + g)F^{\downarrow} + (1 - g)F^{\uparrow}\right] + \frac{F_0\omega_0}{2}(1 + g)e^{-\tau} \tag{A2}$$

$$\frac{dF^{\uparrow}}{d\tau} = F^{\uparrow} - \frac{\omega_0}{2}\left[(1 + g)F^{\uparrow} + (1 - g)F^{\downarrow}\right] - \frac{F_0\omega_0}{2}(1 - g)e^{-\tau} \tag{A3}$$

$$F^{\downarrow}(0) = 0 \tag{A4}$$

$$F^{\uparrow}\left(\tau_0\right) = 0 \tag{A5}$$

The addition and subtraction of equations (A2) and (A3) gives

$$\frac{d}{d\tau}\left(F^{\downarrow} + F^{\uparrow}\right) = -\left(1 - g\omega_0\right)\left(F^{\downarrow} - F^{\uparrow}\right) + F_0 g\omega_0 e^{-\tau} \tag{A6}$$

$$\frac{d}{d\tau}\left(F^{\downarrow} - F^{\uparrow}\right) = -\left(1 - \omega_0\right)\left(F^{\downarrow} + F^{\uparrow}\right) + F_0\omega_0 e^{-\tau} \tag{A7}$$

Denote $F^{\downarrow} + F^{\uparrow}$ (proportional to energy or photon density) by D and $F^{\downarrow} - F^{\uparrow}$ (net downward flux) by F and introduce these quantities into equations (A6) and (A7). The results are

$$\frac{dD}{d\tau} + \left(1 - g\omega_0\right) F = F_0 g\omega_0 e^{-\tau} \tag{A8}$$

$$\left(1 - \omega_0\right) D + \frac{dF}{d\tau} = F_0 \omega_0 e^{-\tau} \tag{A9}$$

These two simultaneous linear equations with constant coefficients can be solved by using standard procedures. A particular solution is, obviously,

$$\left.\begin{aligned} D &= -F_0 e^{-\tau} \\ F &= -F_0 e^{-\tau} \end{aligned}\right\} \tag{A10}$$

By turning to the homogeneous equations, one finds

$$\frac{dD}{d\tau} + \left(1 - g\omega_0\right) F = 0 \tag{A11}$$

$$\left(1 - \omega_0\right) D + \frac{dF}{d\tau} = 0 \tag{A12}$$

Try

$$D = de^{k\tau}$$

$$F = fe^{k\tau}$$

Hence

$$kd + \left(1 - g\omega_0\right) f = 0$$

$$\left(1 - \omega_0\right) d + kf = 0$$

These relations require that

$$\begin{aligned} k^2 &= \left(1 - \omega_0\right)\left(1 - g\omega_0\right) \\ &= 1 - (1 + g)\omega_0 + g\omega_0^{\,2} \end{aligned} \tag{A13}$$

and the general homogeneous solution takes the form

$$\left.\begin{aligned} D &= d_1e^{-k\tau} + d_2e^{k\tau} \\ F &= f_1e^{-k\tau} + f_2e^{k\tau} \end{aligned}\right\} \tag{A14}$$

In order to establish the interrelationships between these four constants, it suffices to substitute them into equation (A12)

$$\left(1 - \omega_0\right)\left(d_1e^{-k\tau} + d_2e^{k\tau}\right) - k\left(f_1e^{-k\tau} - f_2e^{k\tau}\right) = 0$$

Hence

$$f_1 = \frac{1 - \omega_0}{k} d_1$$

and

$$f_2 = -\frac{1 - \omega_0}{k} d_2$$

By substituting these equations into equations (A14) and adding the particular integral, one finds

$$D = F^{\downarrow} + F^{\uparrow} = d_1e^{-k\tau} + d_2e^{k\tau} - F_0e^{-\tau} \tag{A15}$$

$$F = F^{\downarrow} - F^{\uparrow} = \frac{1 - \omega_0}{k}\left(d_1e^{-k\tau} - d_2e^{k\tau}\right) - F_0e^{-\tau} \tag{A16}$$

The addition and subtraction of equations (A15) and (A16) gives

$$F^{\downarrow}(\tau) = d_1\left(\frac{k + 1 - \omega_0}{2k}\right)e^{-k\tau} + d_2\left(\frac{k - 1 + \omega_0}{2k}\right)e^{k\tau} - F_0e^{-\tau} \tag{A17}$$

$$F^{\uparrow}(\tau) = d_1\left(\frac{k - 1 + \omega_0}{2k}\right)e^{-k\tau} + d_2\left(\frac{k + 1 - \omega_0}{2k}\right)e^{k\tau} \tag{A18}$$

The two remaining constants are determined by enforcing the boundary conditions (A4) and (A5),

$$d_1\left(\frac{k + 1 - \omega_0}{2k}\right) + d_2\left(\frac{k - 1 + \omega_0}{2k}\right) - F_0 = 0 \tag{A19}$$

$$d_1\left(\frac{k - 1 + \omega_0}{2k}\right)e^{-k\tau_0} + d_2\left(\frac{k + 1 - \omega_0}{2k}\right)e^{k\tau_0} = 0 \tag{A20}$$

From equation (A20)

$$d_2 = -d_1 r_0 e^{-2k\tau_0} \tag{A21}$$

where

$$r_0 = \frac{k - 1 + \omega_0}{k + 1 - \omega_0} \tag{A22}$$

Substituting equations (A21) and (A22) into equation (A19) yields

$$d_1\left(\frac{k + 1 - \omega_0}{2k}\right) = \frac{F_0}{1 - r_0^2 e^{-2k\tau_0}}$$

Hence,

$$d_2\left(\frac{k + 1 - \omega_0}{2k}\right) = -\frac{r_0 e^{-2k\tau_0}}{1 - r_0^2 e^{-2k\tau_0}} F_0$$

By inserting these constants into equations (A17) and (A18), one finally obtains for the downward and upward fluxes

$$\frac{F^{\downarrow}(\tau)}{F_0} = \frac{e^{k(\tau_0-\tau)} - r_0^2 e^{-k(\tau_0-\tau)}}{e^{k\tau_0} - r_0^2 e^{-k\tau_0}} - e^{-\tau} \tag{A23}$$

and

$$\frac{F^{\uparrow}(\tau)}{F_0} = r_0 \frac{e^{k(\tau_0-\tau)} - e^{-k(\tau_0-\tau)}}{e^{k\tau_0} - r_0^2 e^{-k\tau_0}} \tag{A24}$$

APPENDIX B

EXPANSION OF $e^{-k(\omega_0)\tau}$ AND $r_0(\omega_0)$ AS POWER SERIES IN ω_0

The functions $k(\omega_0)$ and $r_0(\omega_0)$ are defined in equations (A13) and (A22) as follows:

$$k(\omega_0) = \left[1 - (1 + g)\omega_0 + g\omega_0^2\right]^{1/2} \tag{B1}$$

and

$$r_0(\omega_0) = \frac{k - 1 + \omega_0}{k + 1 - \omega_0} \tag{B2}$$

Expansion of $e^{-k(\omega_0)\tau}$ as a Maclaurin Series in ω_0

In the following derivation, use is made of the following expression for the nth differential coefficient of a function of a function.

If $y = f(u)$ and $u = \phi(\omega_0)$, then

$$\frac{1}{n!}\frac{d^n y}{d\omega_0^n} = \sum_{r=1}^{n} \frac{1}{r!}\, {}^nK_r f^{(r)}(u) \tag{B3}$$

where ${}^nK_r$ is the coefficient of h^n in $\left[\phi(\omega_0 + h) - \phi(\omega_0)\right]^r$.

In the present case

$$f(u) = f^{(r)}(u) = e^u \tag{B4}$$

where

$$\left.\begin{aligned} u &= -\left(1 - A\omega_0 - B\omega_0^2\right)^{1/2}\tau \\ A &= 1 + g \\ B &= -g \end{aligned}\right\} \tag{B5}$$

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Determination of ${}^{n}K_r$.- The coefficient ${}^{n}K_r$ appearing in equation (B3) may be determined as follows:

$$\left[\phi\left(\omega_0 + h\right) - \phi\left(\omega_0\right)\right]^r = \tau^r\left\{\left(1 - A\omega_0 - B\omega_0^2\right)^{1/2} - \left[1 - A\left(\omega_0 + h\right) - B\left(\omega_0 + h\right)^2\right]^{1/2}\right\}^r$$

$$= \tau^r\left(1 - A\omega_0 - B\omega_0^2\right)^{r/2}\left[1 - \left(1 - \bar{A}h - \bar{B}h^2\right)^{1/2}\right]^r \qquad \text{(B6)}$$

where

$$\left.\begin{aligned} \bar{A} &= \frac{A + 2\omega_0 B}{1 - A\omega_0 - B\omega_0^2} \\ \bar{B} &= \frac{B}{1 - A\omega_0 - B\omega_0^2} \end{aligned}\right\} \qquad \text{(B7)}$$

The binomial theorem states that

$$(1 + x)^r = 1 + \sum_{p=1}^{\infty} \frac{r(r - 1) \ . \ . \ . \ (r - p + 1)}{p!} x^p \qquad \text{(B8)}$$

and when r is an integer, the series is a terminating one and is given by

$$(1 + x)^r = \sum_{p=0}^{r} \frac{r!}{p!\,(r - p)!} x^p \qquad \text{(B9)}$$

By using equation (B9) in conjunction with equation (B6), one finds

$$\left[\phi\left(\omega_0 + h\right) - \phi\left(\omega_0\right)\right]^r = \tau^r\left(1 - A\omega_0 - B\omega_0^2\right)^{r/2} \sum_{m=0}^{r} \frac{(-)^m\, r!}{m!(r - m)!}\left(1 - \bar{A}h - \bar{B}h^2\right)^{m/2} \qquad \text{(B10)}$$

Applying the binomial theorem for nonintegral exponent (eq. (B8)), one finds

$$\left(1 - \bar{A}h - \bar{B}h^2\right)^{m/2} = 1 + \sum_{p=1}^{\infty} \frac{\left(-\frac{m}{2}\right)\left(1 - \frac{m}{2}\right) \ldots \left(p - 1 - \frac{m}{2}\right)}{p!}\left(\bar{A}h + \bar{B}h^2\right)^p$$

$$= 1 + \sum_{p=1}^{\infty} \frac{\left(-\frac{m}{2}\right)\left(1 - \frac{m}{2}\right) \ldots \left(p - 1 - \frac{m}{2}\right)}{p!}\left(\bar{A}h\right)^p\left(1 + \frac{\bar{B}}{\bar{A}} h\right)^p$$

Making use of equation (B9) results in

$$\left(1 - \bar{A}h - \bar{B}h^2\right)^{m/2} = 1 + \sum_{p=1}^{\infty} \frac{\left(-\frac{m}{2}\right)\left(1 - \frac{m}{2}\right) \ldots \left(p - 1 - \frac{m}{2}\right)}{p!}\left(\bar{A}h\right)^p \sum_{q=0}^{p} \frac{p!}{q!\,(p - q)!}\left(\frac{\bar{B}}{\bar{A}}h\right)^q$$

$$= 1 + \sum_{n=1}^{\infty} h^n \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{\left(-\frac{m}{2}\right)\left(1 - \frac{m}{2}\right) \ldots \left(n - j - 1 - \frac{m}{2}\right)}{j!\,(n - 2j)!} \bar{A}^{n-2j}\bar{B}^j \tag{B11}$$

where the upper bound is $\frac{n - 1}{2}$ or $\frac{n}{2}$ whichever is an integer.

Thus, in the expansion of $\left(1 - \bar{A}h - \bar{B}h^2\right)^{m/2}$, the coefficient of $h^n (n \geqq 1)$ is given by

$$\sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{\left(-\frac{m}{2}\right)\left(1 - \frac{m}{2}\right) \ldots \left(n - j - 1 - \frac{m}{2}\right)}{j!\,(n - 2j)!} \bar{A}^{n-2j}\bar{B}^j \tag{B12}$$

By bearing in mind that ${}^nK_r$ is the coefficient of h^n in $\left[\phi\left(\omega_0 + h\right) - \phi\left(\omega_0\right)\right]^r$, one obtains from equation (B10) and expression (B12)

APPENDIX B

$$^{n}K_r = \tau^r\left(1 - A\omega_0 - B\omega_0^2\right)^{r/2} \sum_{m=0}^{r} \frac{(-)^m r!}{m!\,(r - m)!} \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{\left(-\frac{m}{2}\right)\left(1 - \frac{m}{2}\right)\ldots\left(n - j - 1 - \frac{m}{2}\right)}{j!\,(n - 2j)!} \bar{A}^{n-2j}\bar{B}^{j}$$

$$= \left(-\frac{\bar{A}}{2}\right)^n \tau^r\left(1 - A\omega_0 - B\omega_0^2\right)^{r/2} \sum_{m=0}^{r} \frac{(-)^m r!}{m!\,(r - m)!} \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{m(m - 2)\ldots\left[m - 2(n - j - 1)\right]}{j!\,(n - 2j)!}\left(-\frac{2\bar{B}}{\bar{A}^2}\right)^j \tag{B13}$$

Maclaurin expansion of $e^{-k(\omega_0)\tau}$.- The substitution of equations (B4) and (B13) into equation (B3) yields

$$\frac{1}{n!}\frac{d^ny}{d\omega_0^{\,n}} = e^u\left(-\frac{\bar{A}}{2}\right)^n \sum_{r=1}^{n} \frac{\tau^r\left(1 - A\omega_0 - B\omega_0^2\right)^{r/2}}{r!} \sum_{m=0}^{r} \frac{(-)^m r!}{m!\,(r - m)!} \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{m(m - 2)\ldots\left[m - 2(n - j - 1)\right]}{j!(n - 2j)!}\left(-\frac{2\bar{B}}{\bar{A}^2}\right)^j$$

When $\omega_0 = 0$, then $k = 1$, $u = -\tau$; $\bar{A} = A = 1 + g$ and $\bar{B} = B = -g$

$$\frac{1}{n!}\left(\frac{d^ny}{d\omega_0^{\,n}}\right)_{\omega_0=0} = e^{-\tau}\left[-\frac{(1 + g)}{2}\right]^n \sum_{r=1}^{n} \tau^r \sum_{m=0}^{r} \frac{(-)^m}{m!\,(r - m)!} \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{m(m - 2)\ldots\left[m - 2(n - j - 1)\right]}{j!\,(n - 2j)!}\left[\frac{2g}{(1 + g)^2}\right]^j$$

Thus,

$$e^{-k(\omega_0)\tau} = e^{-\tau}\left(1 + \sum_{n=1}^{\infty} A_n\omega_0^{\,n}\right) \tag{B14}$$

where

$$A_n = \left[-\frac{(1 + g)}{2}\right]^n \sum_{r=1}^{n} \tau^r \sum_{m=0}^{r} \frac{(-)^m}{m!\,(r - m)!} \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{m(m - 2)\ldots\left[m - 2(n - j - 1)\right]}{j!\,(n - 2j)!}\left[\frac{2g}{(1 + g)^2}\right]^j \tag{B15}$$

APPENDIX B

Expansion of $r_0(\omega_0)$ as a Power Series in ω_0

One may rewrite equation (B2) in the form

$$r_0 = \frac{k - (1 - \omega_0)}{k + (1 - \omega_0)}$$

$$= \frac{k^2 + (1 - \omega_0)^2 - 2k(1 - \omega_0)}{k^2 - (1 - \omega_0)^2} \tag{B16}$$

However, from equation (A13)

$$k^2 = (1 - \omega_0)(1 - g\omega_0) \tag{B17}$$

By substituting equation (B17) into equation (B16), one obtains

$$r_0 = \frac{1 - \left(\frac{1 + g}{2}\right)\omega_0 - k}{\left(\frac{1 - g}{2}\right)\omega_0} \tag{B18}$$

Setting $m = 1$ in equation (B11) one obtains

$$k = \left(1 - A\omega_0 - B\omega_0^{\ 2}\right)^{1/2}$$

$$= 1 + \sum_{n=1}^{\infty} \omega_0^{\ n} \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{\left(-\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) \cdots \left(n - j - 1 - \frac{1}{2}\right)}{j!\,(n - 2j)!} A^{n-2j}B^j$$

$$k = 1 - \sum_{n=1}^{\infty} 2\left(\frac{\omega_0}{2}\right)^n \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{(2n - 2j - 2)!}{j!(n - 2j)!(n - j - 1)!}\left(\frac{1 + g}{2}\right)^{n-2j}(-g)^j \tag{B19}$$

The substitution of equation (B19) into equation (B18) yields

$$r_0 = \frac{1 - \left(\frac{1+g}{2}\right)\omega_0 - \left\{1 - \sum_{n=1}^{\infty} 2\left(\frac{\omega_0}{2}\right)^n \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{(2n-2j-2)!}{j!\,(n-2j)!\,(n-j-1)!}\left(\frac{1+g}{2}\right)^{n-2j}(-g)^j\right\}}{\left(\frac{1-g}{2}\right)\omega_0}$$

$$= \frac{\sum_{n=2}^{\infty} 2\left(\frac{\omega_0}{2}\right)^n \sum_{j=0}^{\frac{n-1}{2},\frac{n}{2}} \frac{(2n-2j-2)!}{j!(n-2j)!(n-j-1)!}\left(\frac{1+g}{2}\right)^{n-2j}(-g)^j}{\left(\frac{1-g}{2}\right)\omega_0}$$

The adoption of $s = n - 1$ as the running variable gives

$$r_0 = \sum_{s=1}^{\infty} \left(\frac{1+g}{4}\,\omega_0\right)^s \left(\frac{1+g}{1-g}\right) \sum_{j=0}^{\frac{s}{2},\frac{s+1}{2}} \frac{(2s-2j)!}{j!(s-j)!(s-2j+1)!}\left[-\frac{4g}{(1+g)^2}\right]^j \tag{B20}$$

Case of Isotropic Scattering ($g = 0$)

In this case the only nonvanishing term in the second summation corresponds to $j = 0$ and

$$r_0 = \sum_{s=1}^{\infty} \omega_0{}^s \frac{(2s)!}{2^{2s}\,s!(s+1)!} \tag{B21}$$

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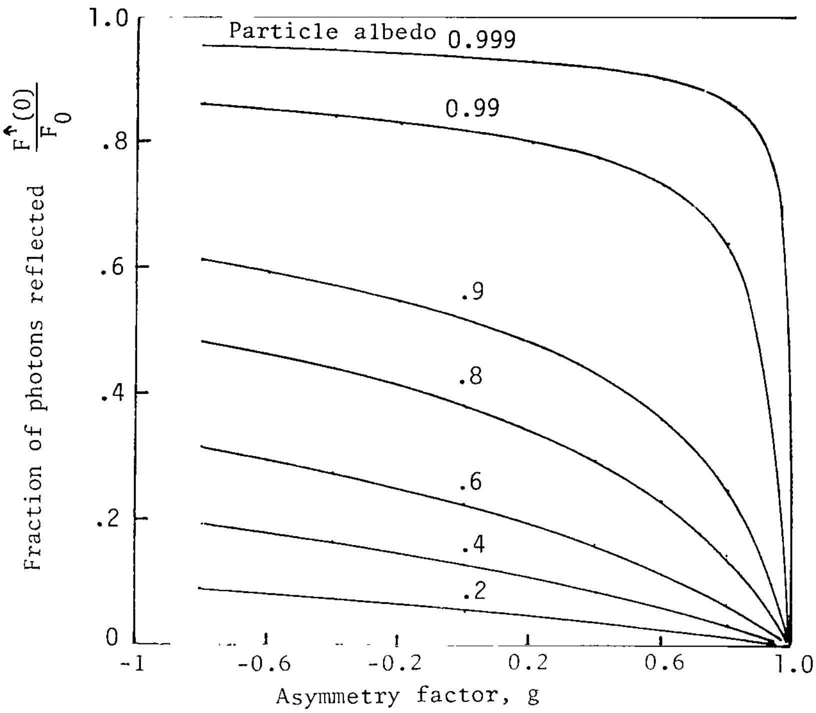
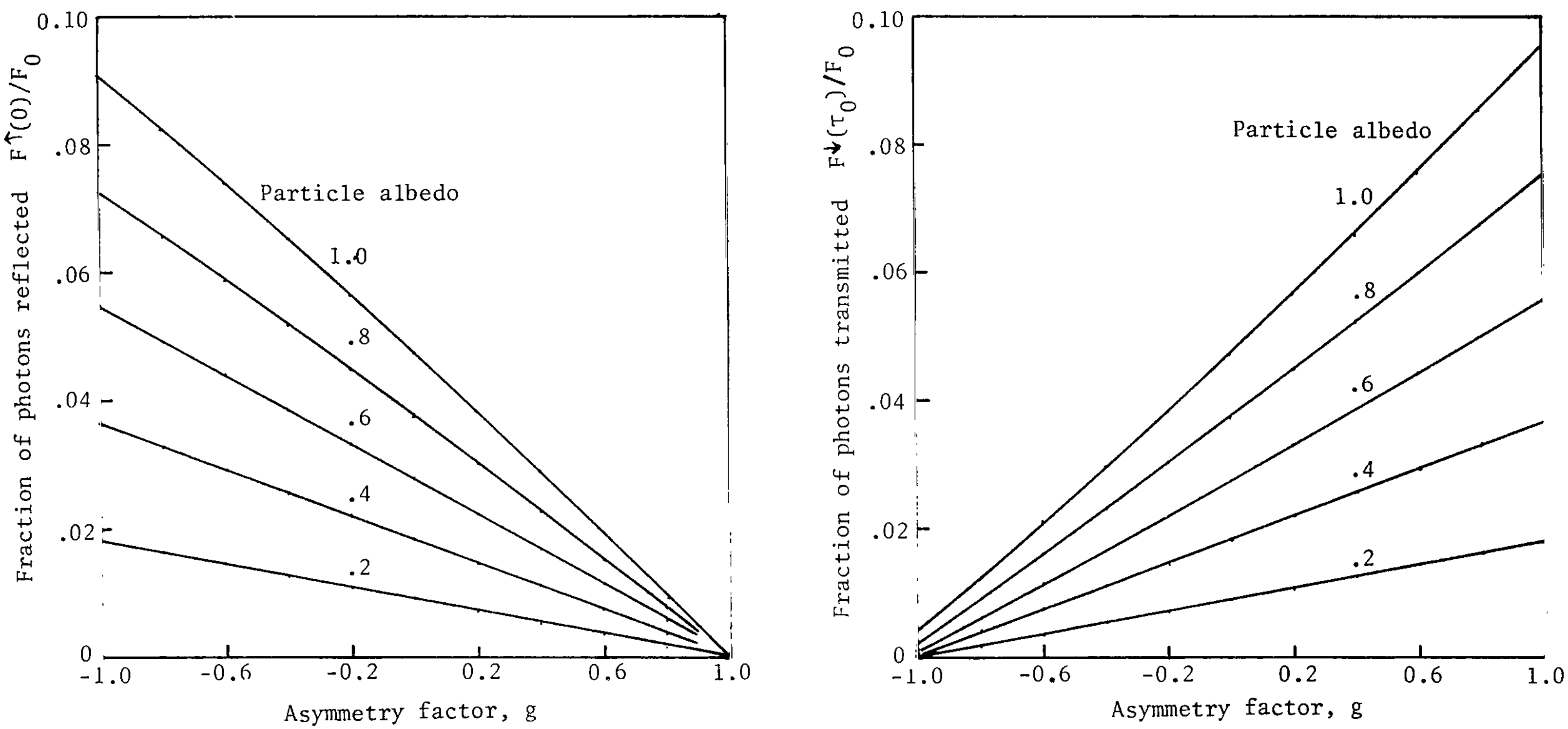
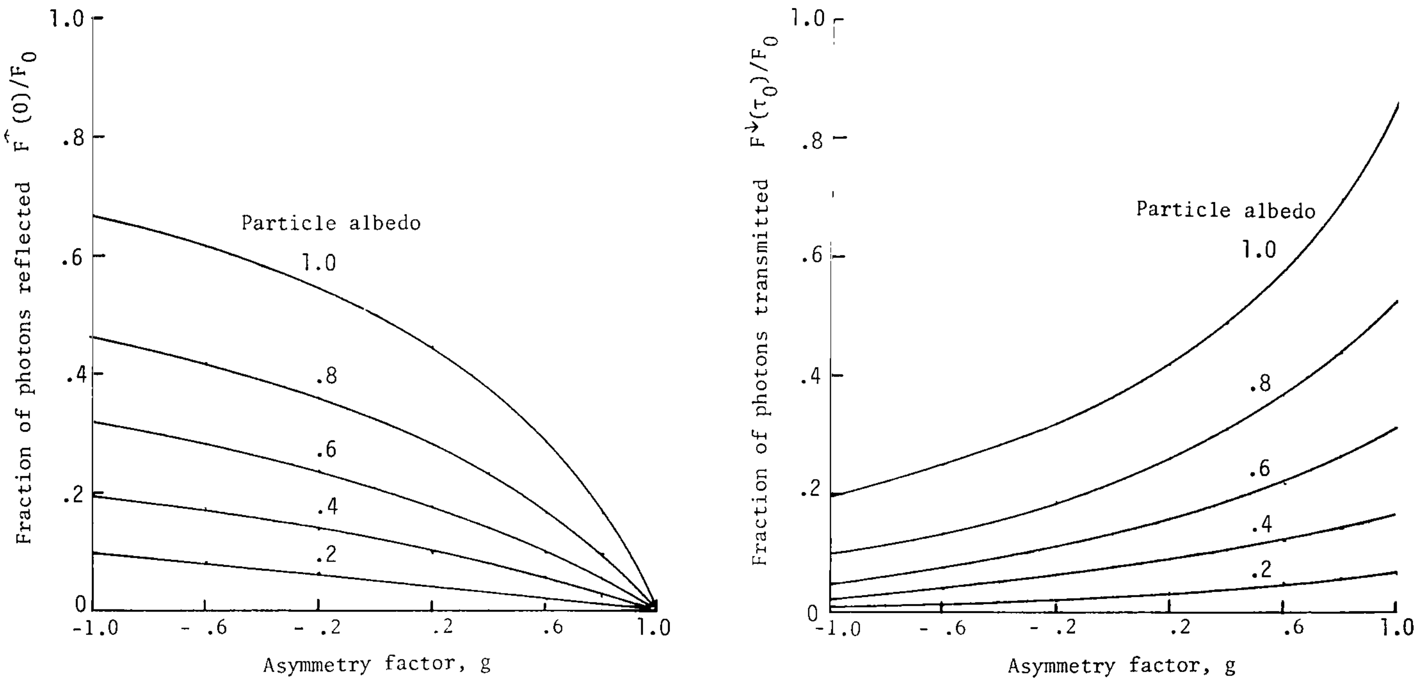


Figure 1.- Reflection coefficient $\left(F^{\uparrow}(0)/F_0\right)$ for semi-infinite slab as a function of asymmetry factor and particle albedo.



(a) Reflection. (b) Transmission.

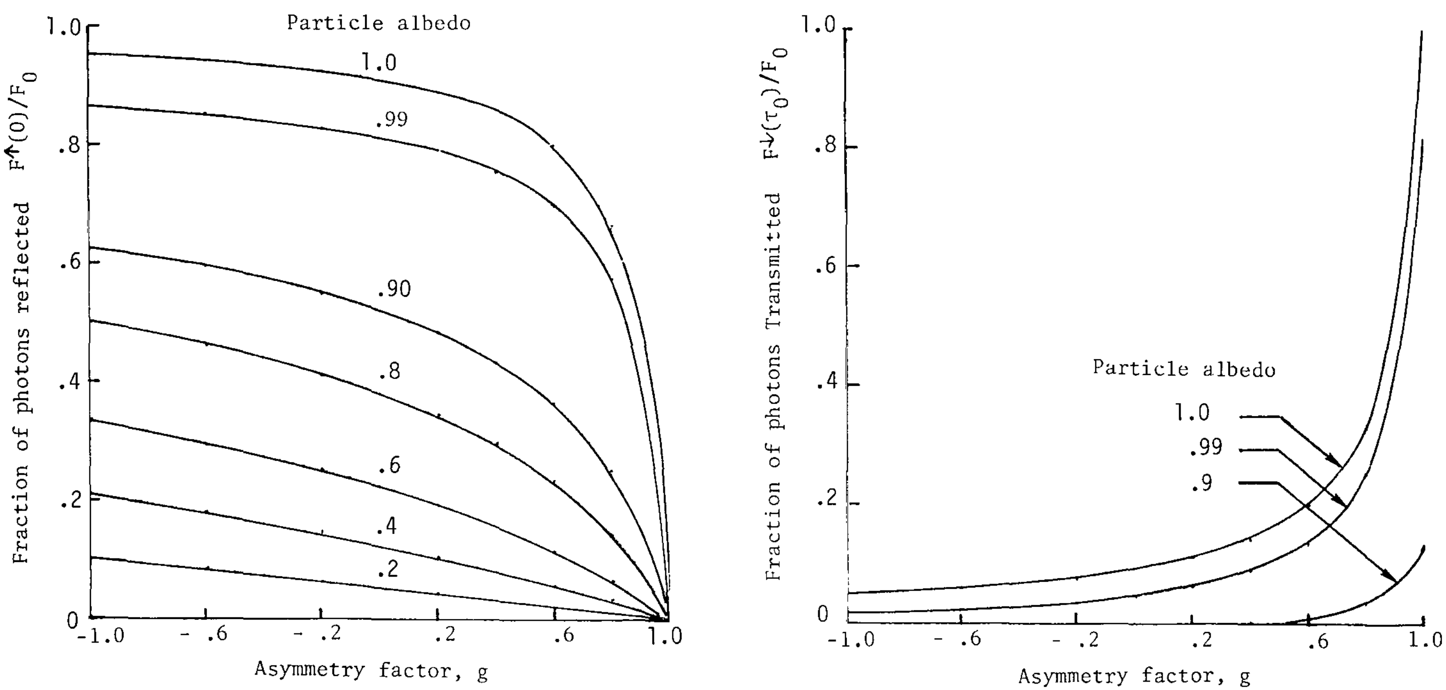
Figure 2.- Reflection and transmission coefficients for a slab of optical thickness 0.1 as a function of asymmetry factor and particle albedo.



(a) Reflection.

(b) Transmission.

Figure 3.- Reflection and transmission coefficients for a slab of optical thickness 2 as a function of asymmetry factor and particle albedo.



(a) Reflection.

(b) Transmission.

Figure 4.- Reflection and transmission coefficients for a slab of optical thickness 20 as a function of asymmetry factor and particle albedo.

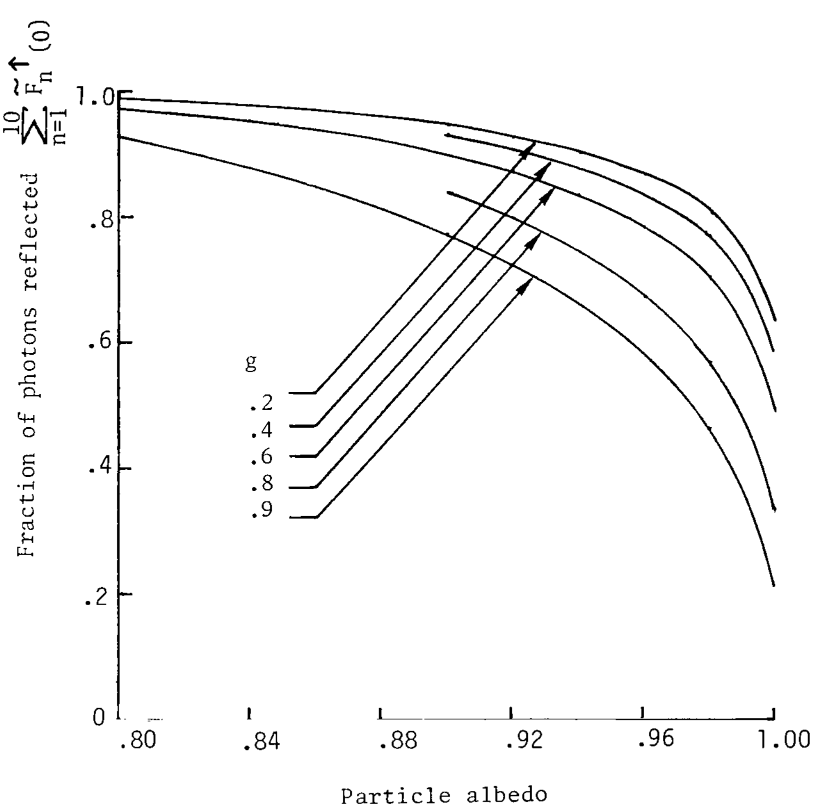
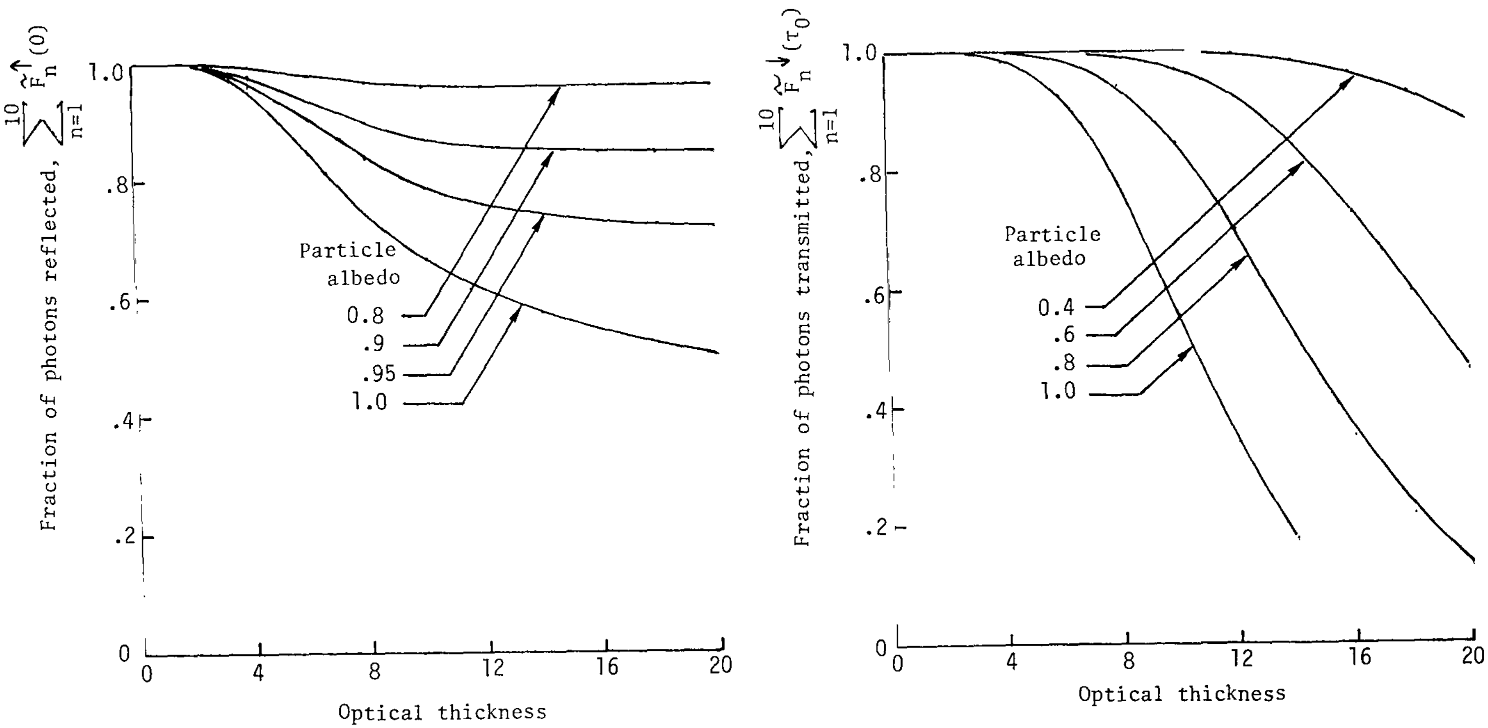


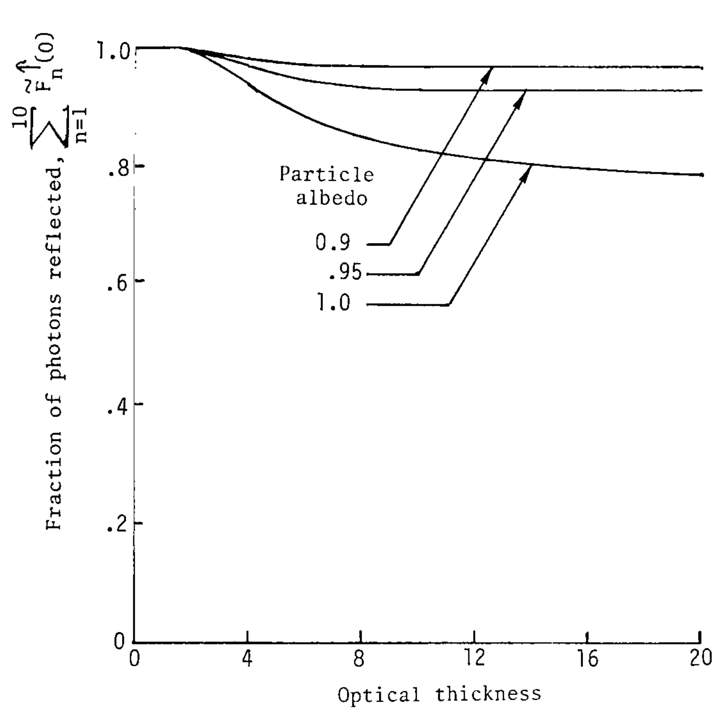
Figure 5.- Fraction of photons reflected from a semi-infinite slab by first 10 orders of scattering.



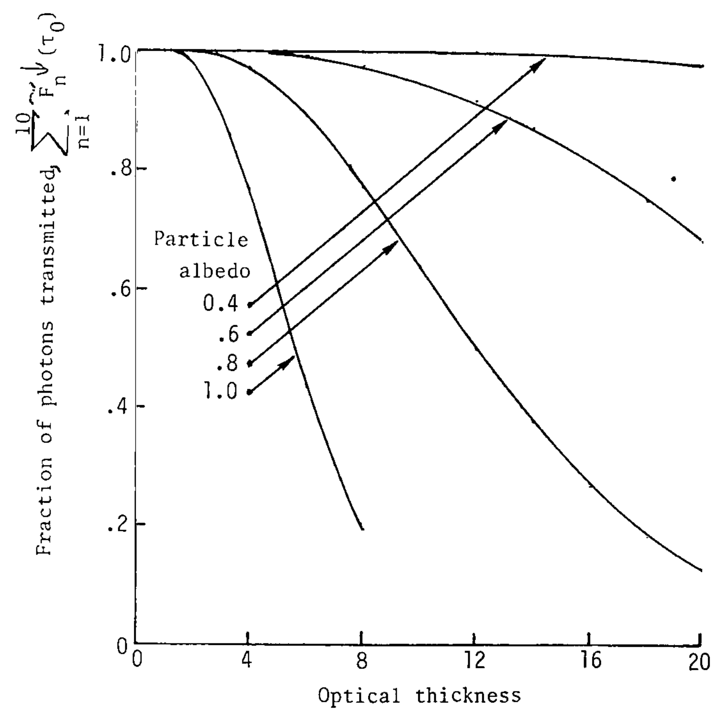
(a) Reflection.

(b) Transmission.

Figure 6.- Contribution of first 10 orders of scattering to reflection from and transmission through finite slabs. (g = 0.8.)

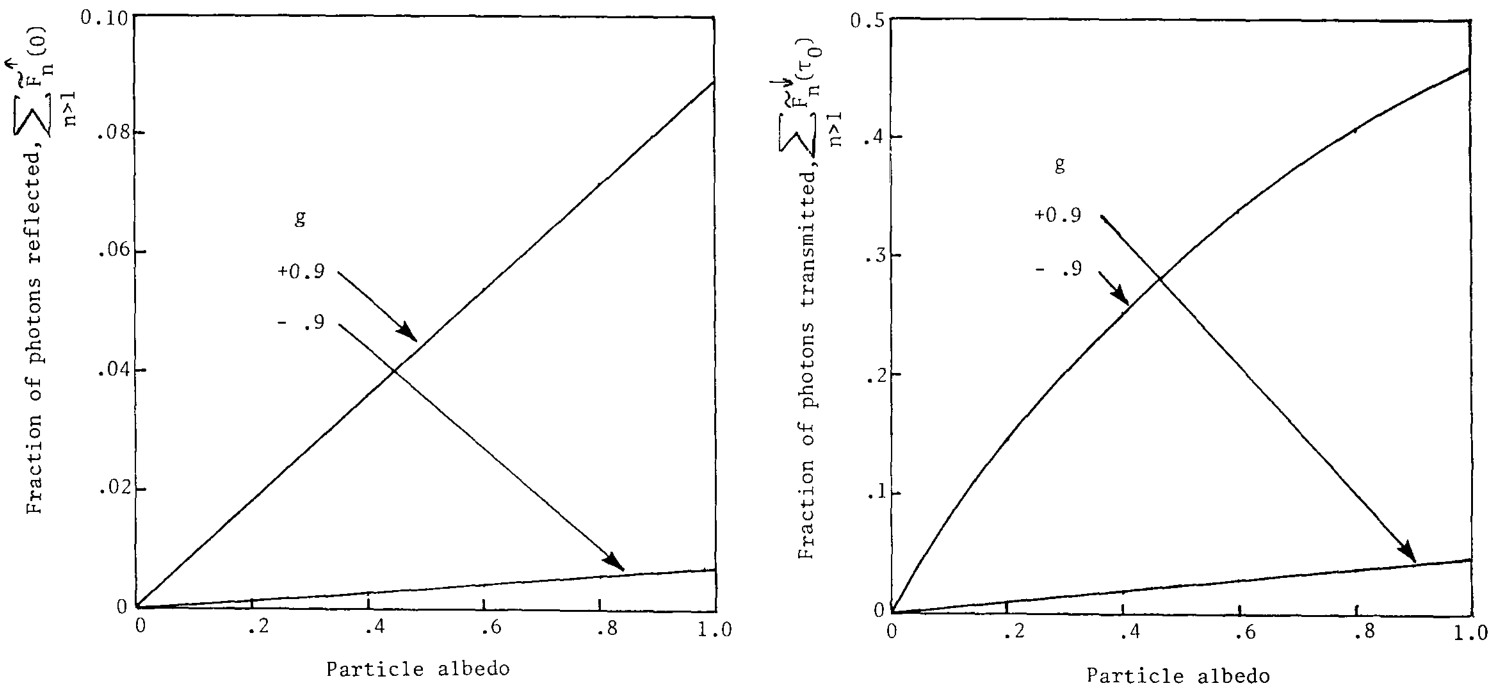


(a) Reflection.



(b) Transmission.

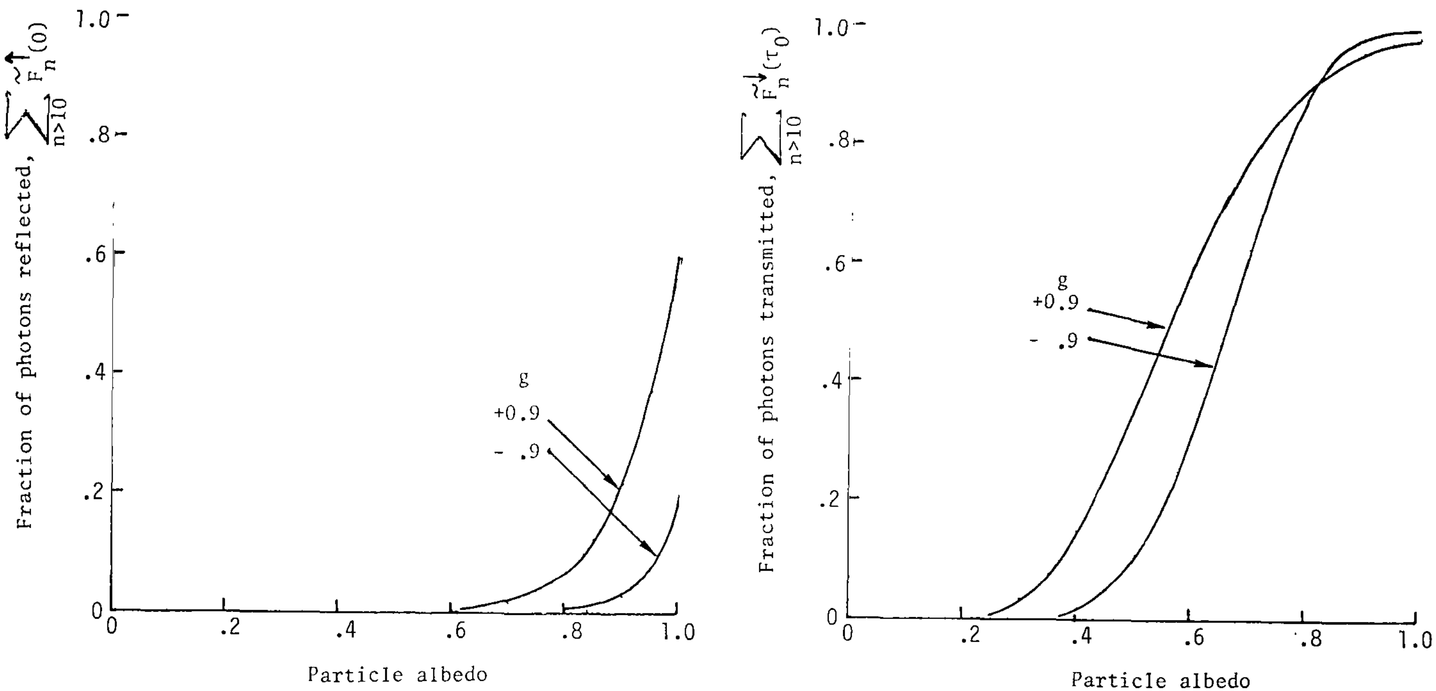
Figure 7.- Contribution of first 10 orders of scattering to reflection from and transmission through finite slabs. (g = -0.8.)



(a) Reflection.

(b) Transmission.

Figure 8.- Contribution of photons that have undergone more than one scattering to reflection and transmission for a slab of optical thickness 0.1.



(a) Reflection.

(b) Transmission.

Figure 9.- Contribution of photons that have undergone more than 10 scatterings to reflection and transmission for a slab of optical thickness 20.

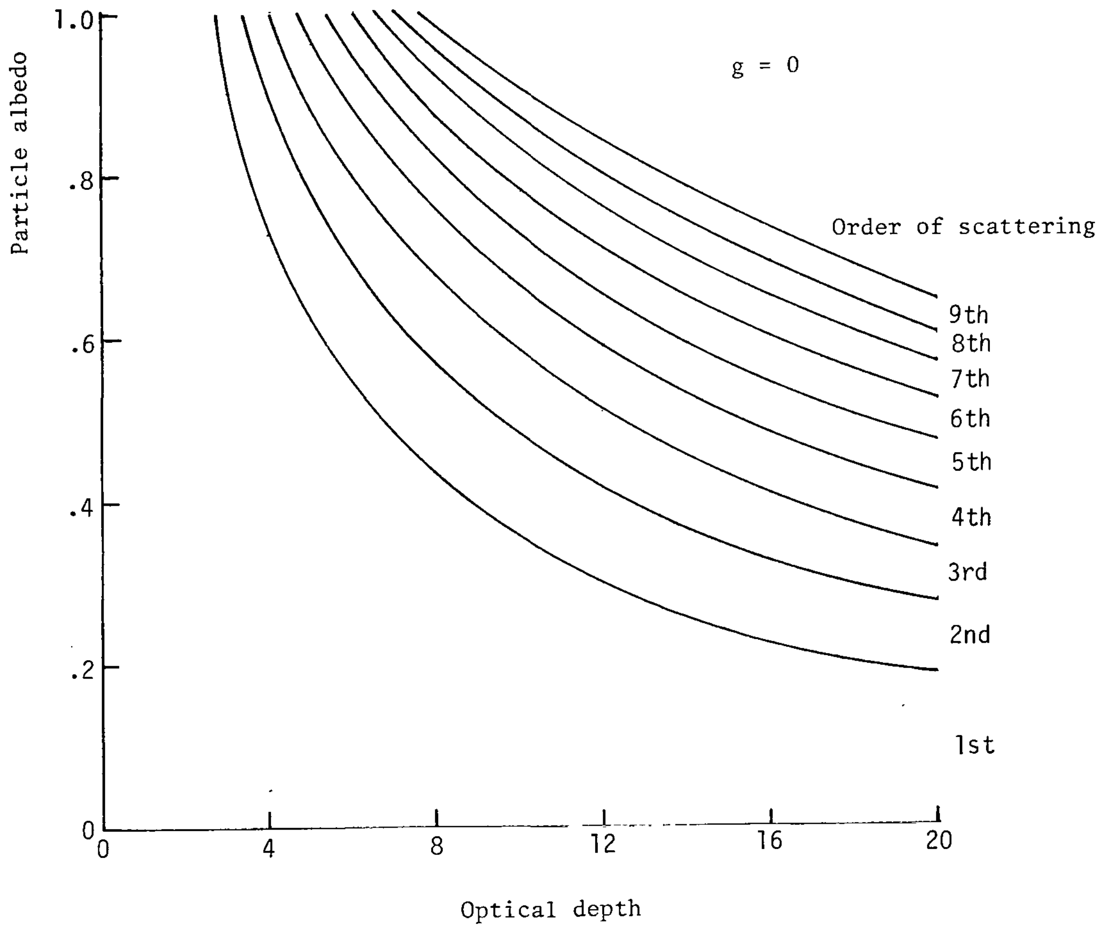
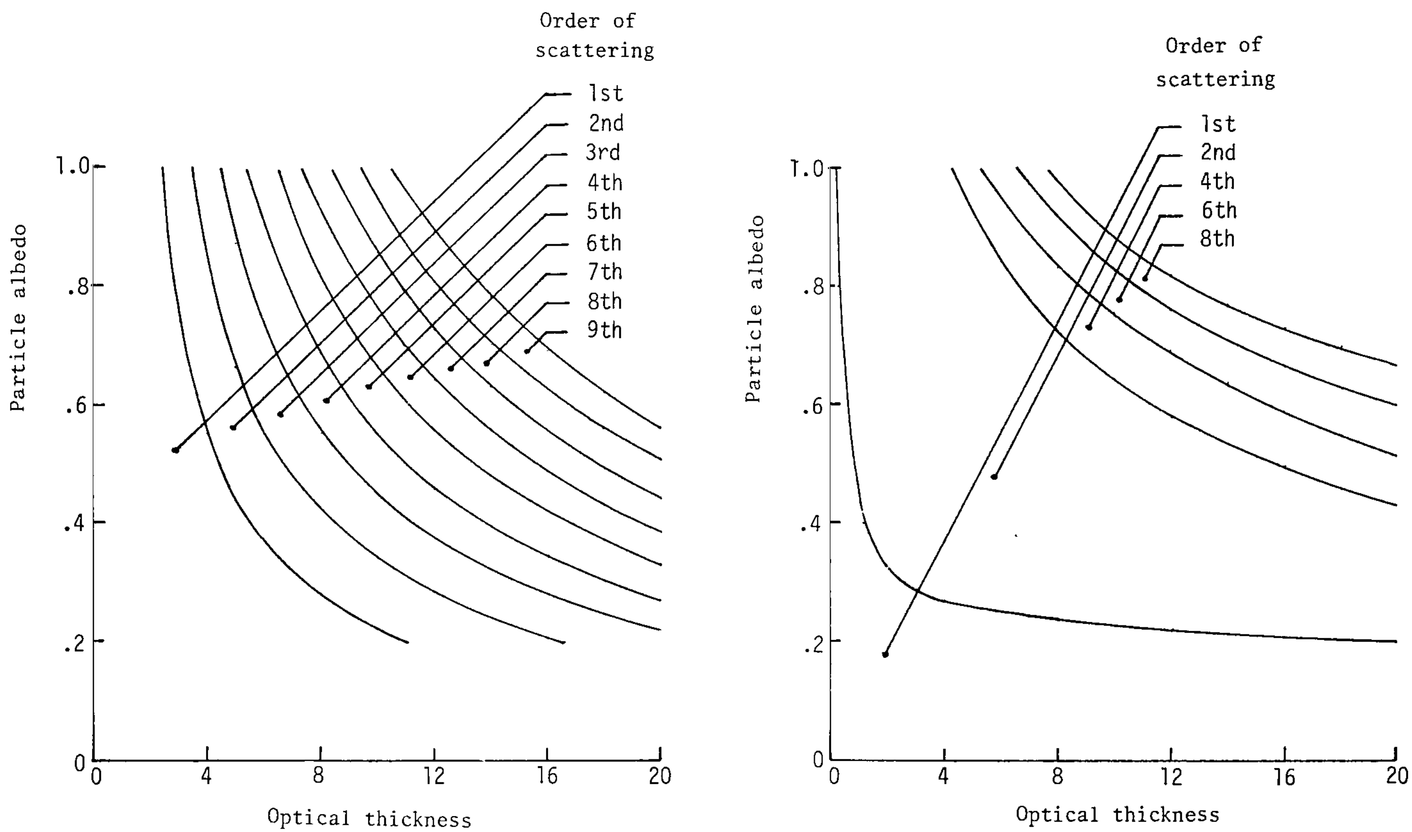


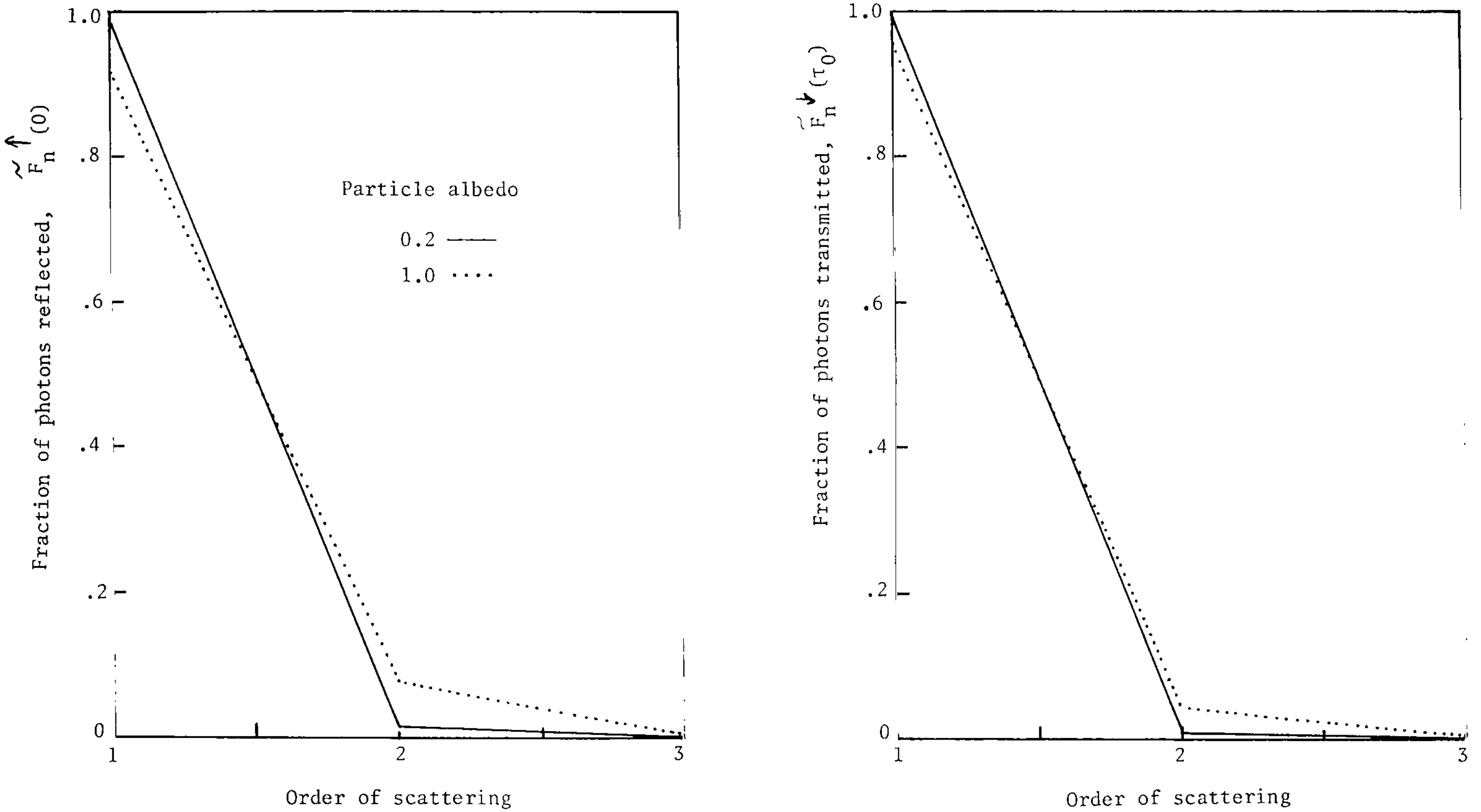
Figure 10.- Term in series expansion contributing most to photon population.



(a) g = 0.8.

(b) g = -0.8.

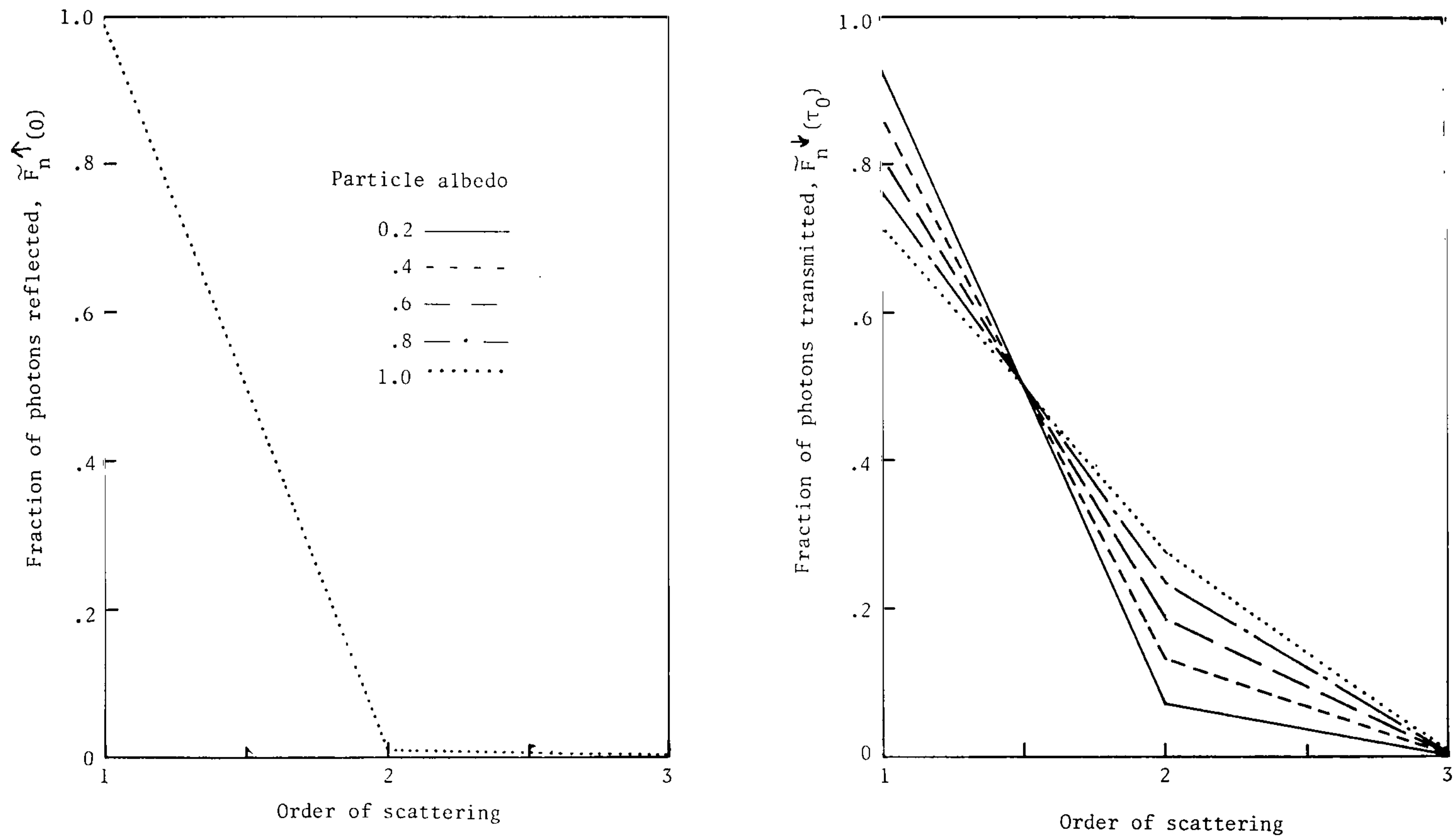
Figure 11.- Term in series expansion contributing most to transmitted beam.



(a) Reflection.

(b) Transmission.

Figure 12.- Contributions of successive orders of scattering to reflection and transmission for a slab of optical thickness 0.1. (g = 0.8.)



(a) Reflection.

(b) Transmission.

Figure 13.- Contributions of successive orders of scattering to reflection and transmission for a slab of optical thickness 0.1. (g = -0.8.)

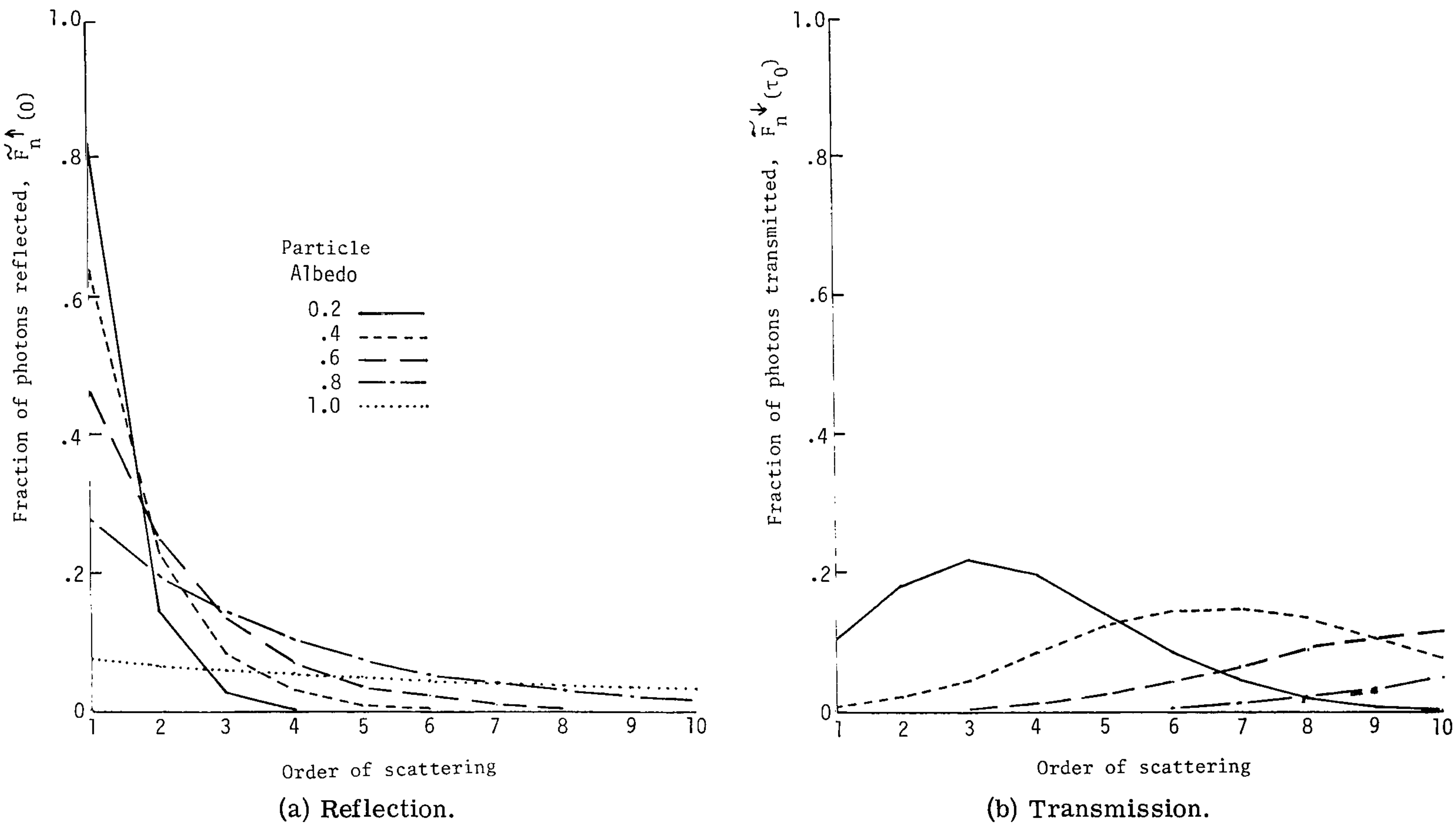
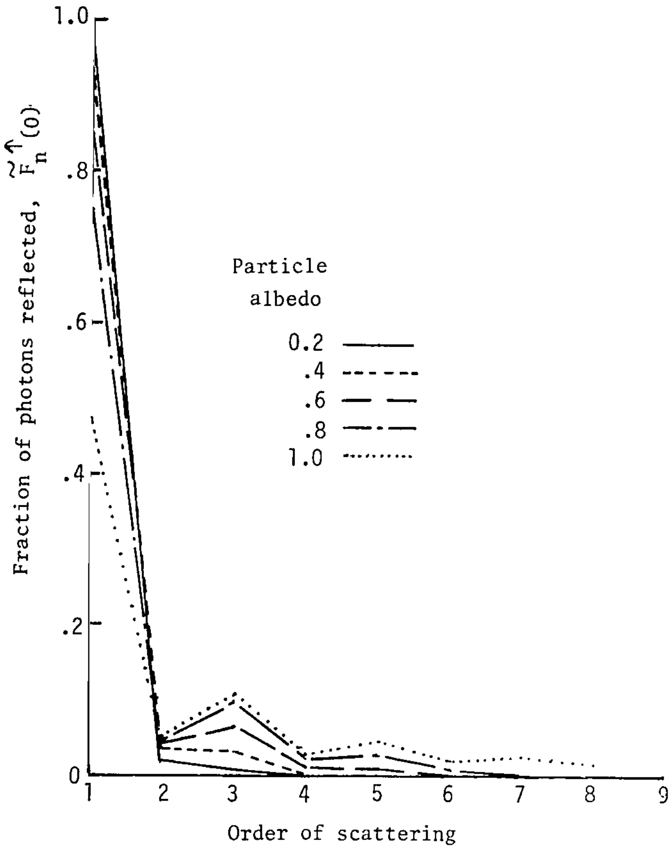
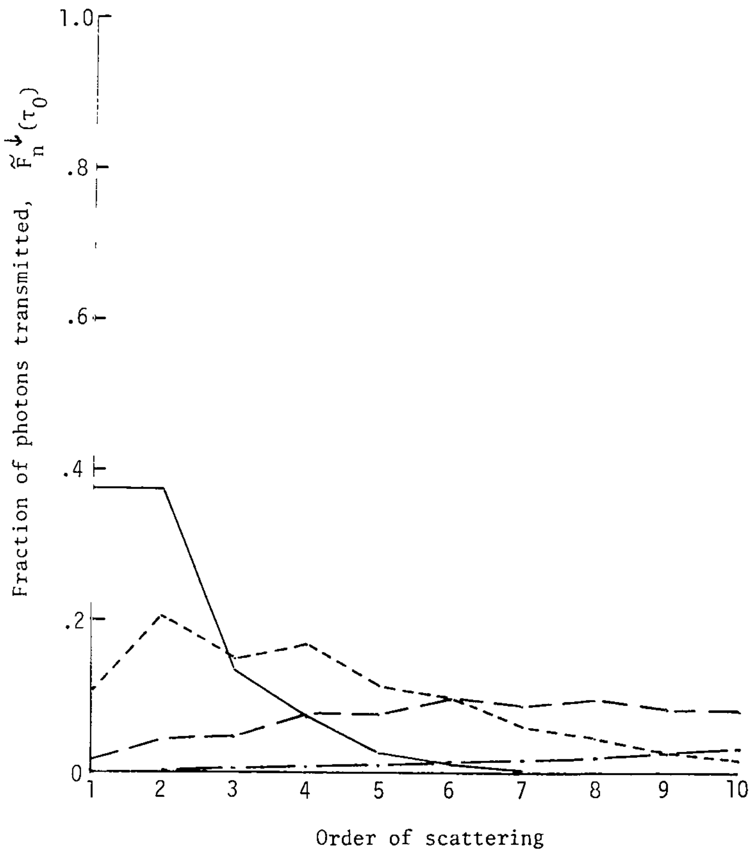


Figure 14.- Contribution of successive orders of scattering to reflection and transmission for a slab of optical thickness 20. (g = 0.8.)

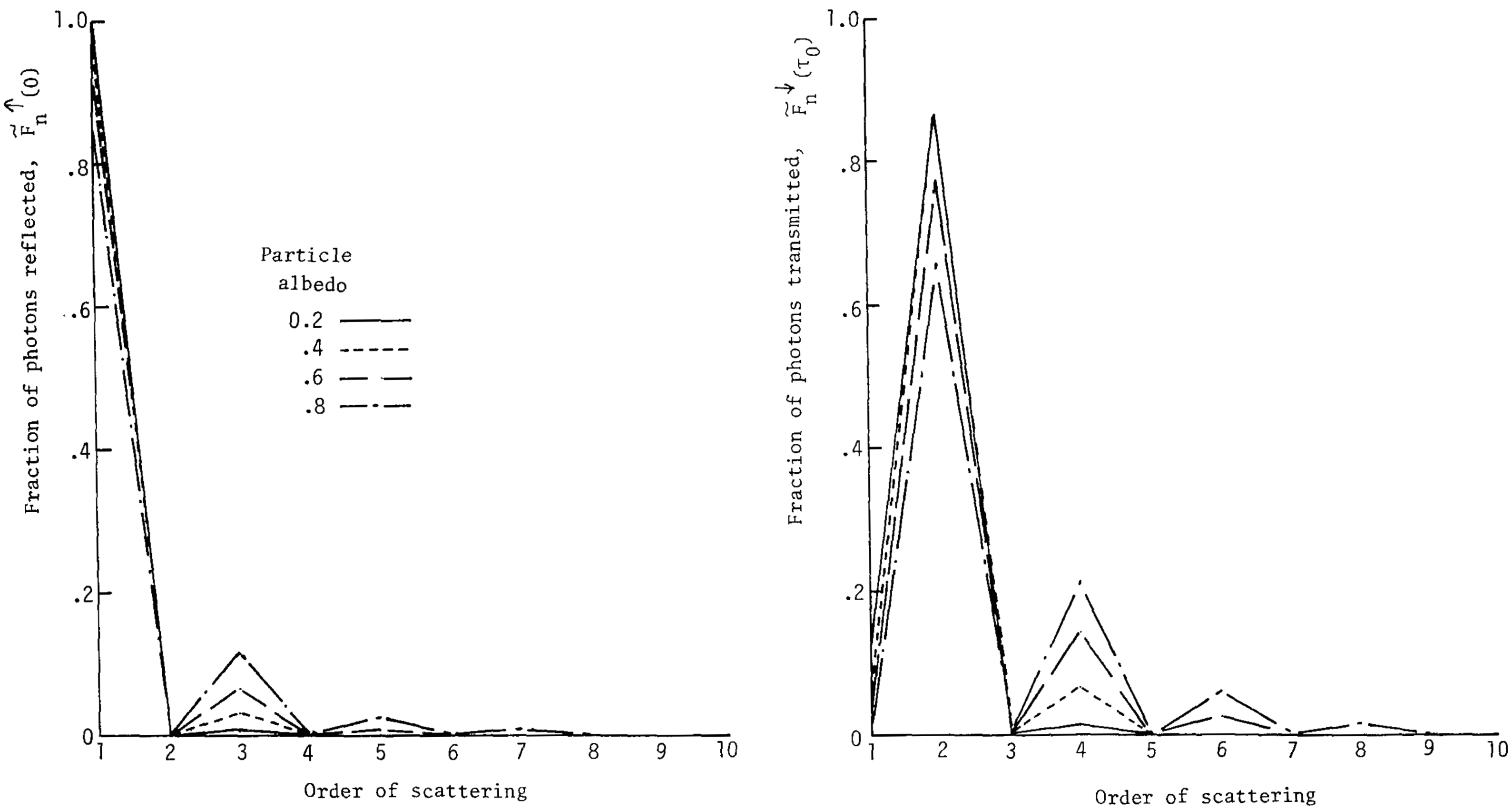


(a) Reflection.



(b) Transmission.

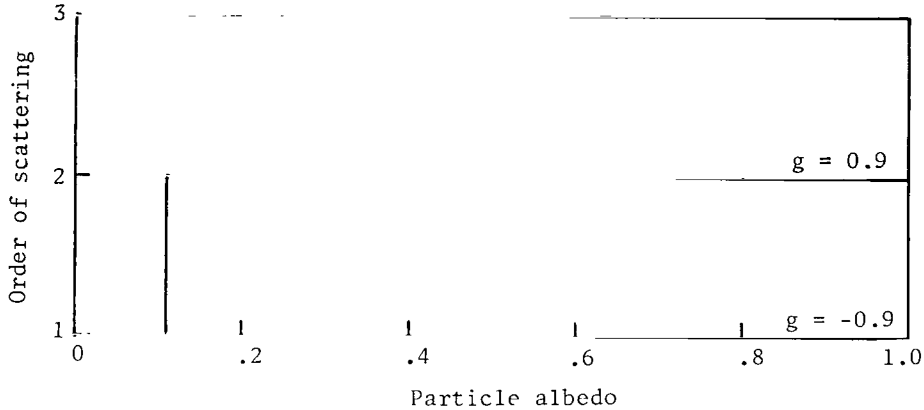
Figure 15.- Contribution of successive orders of scattering to reflection and transmission for a slab of optical thickness 20. (g = -0.8.)



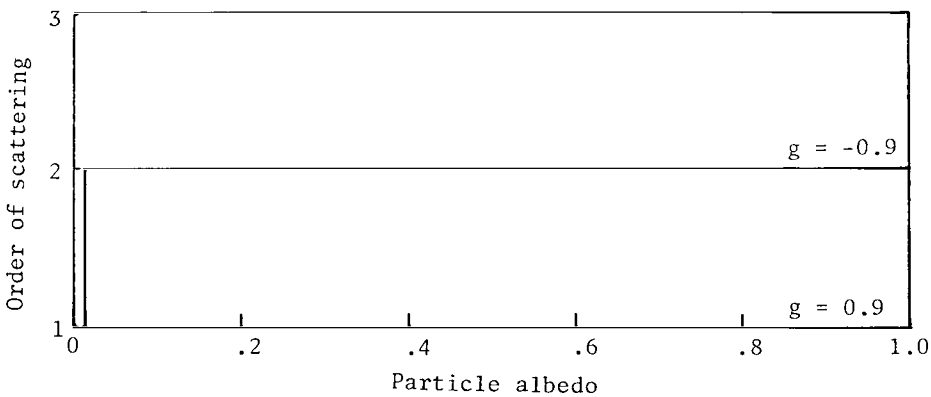
(a) Reflection.

(b) Transmission.

Figure 16.- Contribution of successive orders of scattering to reflection and transmission for a slab of optical thickness 2. (g = -0.98.)

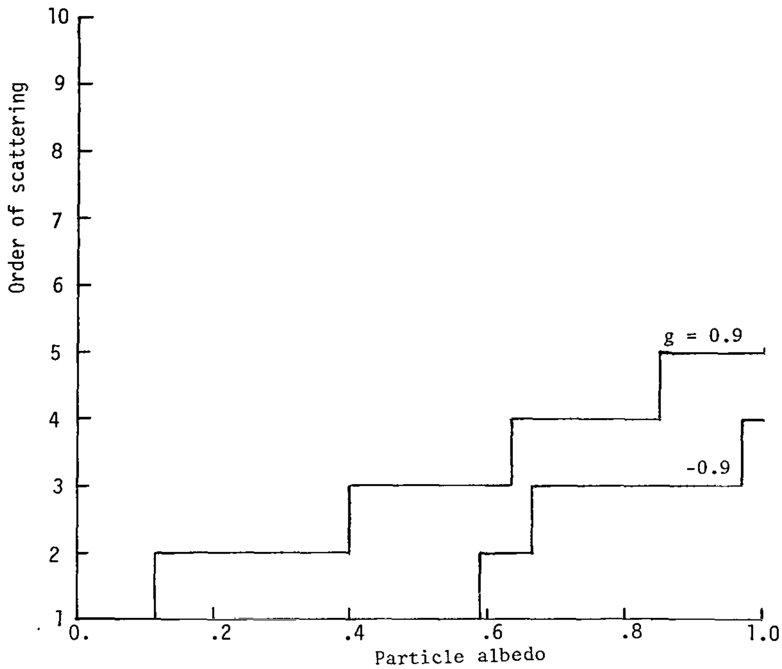


(a) Reflection: optical thickness 0.1.

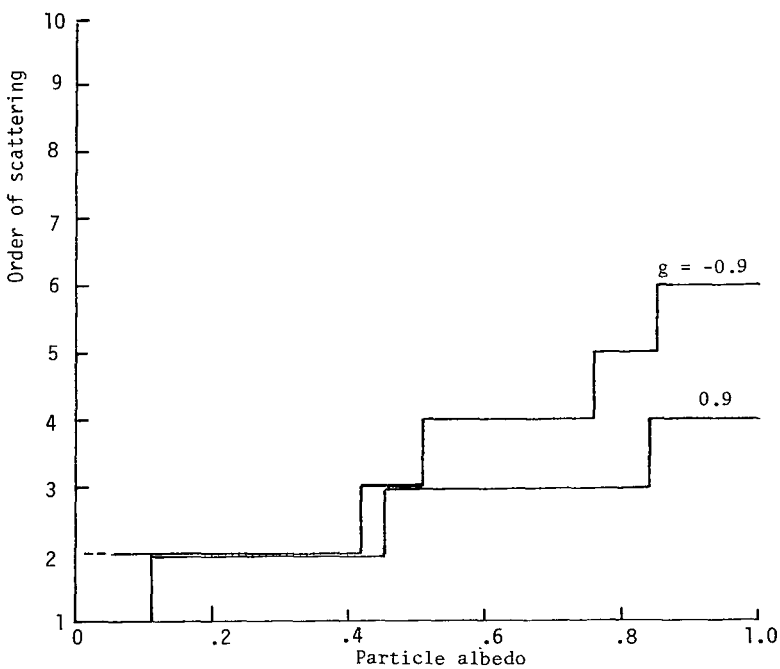


(b) Transmission: optical thickness 0.1.

Figure 17.- Order of scattering needed to give within 1-percent accuracy.

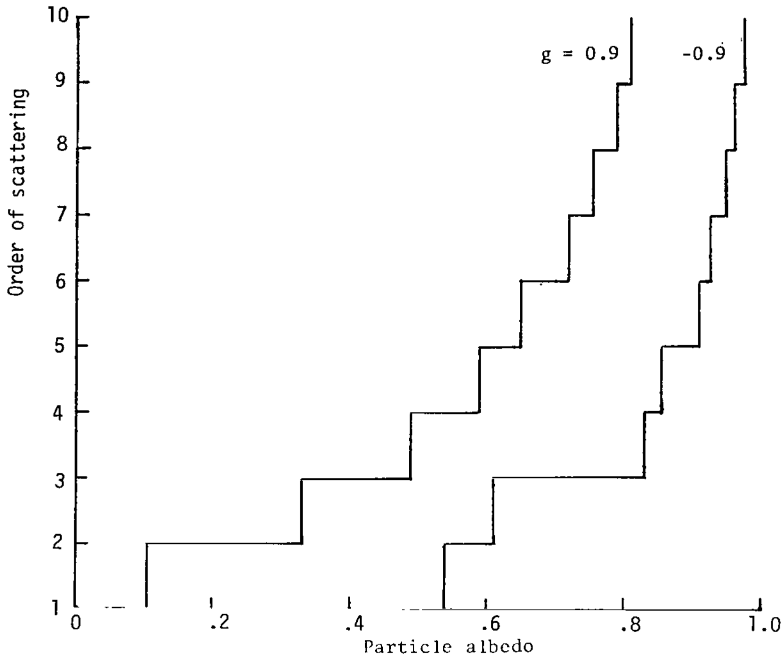


(a) Reflection: optical thickness 2.

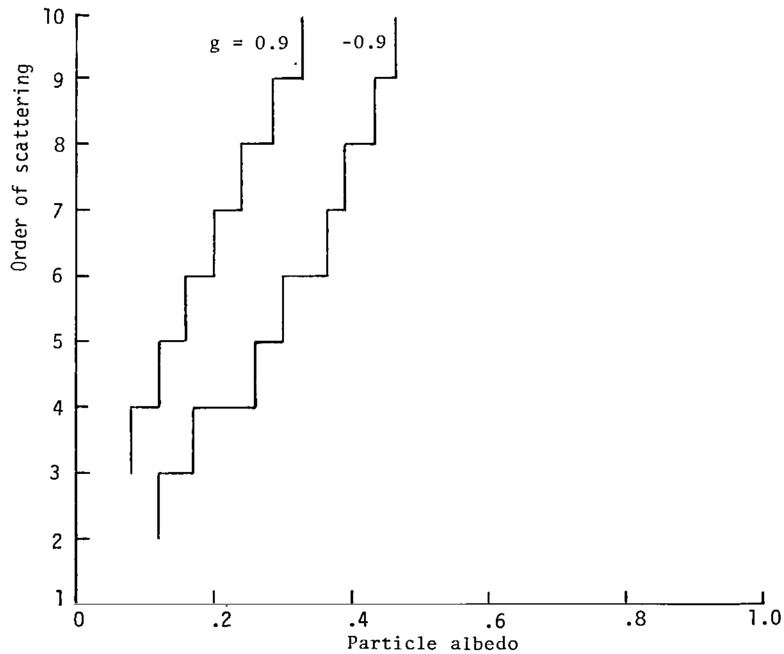


(b) Transmission: optical thickness 2.

Figure 18.- Order of scattering needed to give within 10-percent accuracy.



(a) Reflection: optical thickness 20.



(b) Transmission: optical thickness 20.

Figure 19.- Order of scattering needed to give within 10-percent accuracy.

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