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REQUIREMENTS FOR IMPLEMENTATION OF KÜSSNER AND WAGNER INDICIAL LIFT GROWTH FUNCTIONS INTO THE FLEXSTAB COMPUTER PROGRAM SYSTEM FOR USE IN DYNAMIC LOADS ANALYSES

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16. Abstract

This report describes the objectives of the study, develops elements to evaluate the objectives and summarizes the results. General requirements for dynamic loads analyses are described, indicial lift growth function unsteady subsonic aerodynamic representation reviewed, and the FLEXSTAB CPS evaluated with respect to these general requirements. The effects of residual flexibility techniques on dynamic loads analyses were also evaluated using a simple dynamic model.

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1.0 SUMMARY

This document describes the requirements for implementation of Küssner and Wagner indicial lift growth functions in the NASA-Ames FLEXSTAB Computer Program System (CPS) to represent unsteady aerodynamics for use in dynamic gust loads analyses. This study was performed under NASA contract NAS2-7729, "Development of a FLEXSTAB Computer Program" (task IV, item I).

Use of the Küssner and Wagner unsteady aerodynamic representations is reviewed, the requirements for dynamic loads analyses are outlined, and the applicability to these requirements of the various program segments existing in the NASA-Ames 1.00.xx controls-fixed version of the FLEXSTAB CPS is determined. From this information, the modifications to the existing FLEXSTAB CPS required for creating a dynamic gust loads analysis capability are identified and discussed.

The conclusion is that a large number of modifications and additions to the existing NASA FLEXSTAB CPS would be required before implementation of the Küssner and Wagner indicial lift growth functions into the FLEXSTAB CPS would provide meaningful gust loads analysis capability.

2.0 INTRODUCTION

The NASA has developed the FLEXSTAB CPS (ref. 1) primarily for stability and control analyses of controls fixed elastic flight vehicles using steady state aerodynamics and a low-frequency approximation to unsteady aerodynamics. Other capabilities include the ability to calculate static load distributions (inertia and air loads) and trim control requirements for flexible aircraft.

In order to expand the system to include dynamic loads analyses involving active control systems and to make it attractive for a variety of uses ranging from generalized studies to actual design, it is necessary to have the option of using aerodynamic theories ranging from steady state to unsteady. Reference 2 examined the feasibility of incorporating both the Küssner and Wagner indicial lift growth function unsteady aerodynamic representation and a more exact unsteady aerodynamic theory (Doublet Lattice) into the FLEXSTAB CPS for use in dynamic loads analyses.

This study was limited to examining only the use of Küssner and Wagner indicial lift growth unsteady aerodynamic representations. Specific tasks were to determine:

- The requirements for implementation and use of the Küssner and Wagner indicial lift growth unsteady aerodynamic representations in the NASA-Ames FLEXSTAB 1.00.xx for dynamic loads analyses
- Additional program requirements to complete the preceding task
- Limitations and deficiencies which presently exist in the FLEXSTAB 1.00.xx system for dynamic loads analyses

3.0 GENERAL REQUIREMENTS FOR DYNAMIC LOADS ANALYSES

To evaluate the FLEXSTAB CPS for use in calculating dynamic gust loads, it is first necessary to determine the general requirements of dynamic loads analyses. These requirements were established in reference 2 and are restated here for subsequent use in this study.

A general dynamic aeroelastic loads analysis system includes the following elements:

1. Definition of structural geometry and structural and aerodynamic panels
2. Definition of the structural model
3. Calculation of structural vibration characteristics and determination of the generalized mass and stiffness
4. Definition of the aerodynamic model
5. Definition of the control system model
6. Definition of excitation functions
7. Formulation of the equations of motion and load equations
8. Solution of these equations to determine:
    a. Coordinate and load responses in either the time or frequency domain, and power spectral density (PSD) load parameters
    b. Roots of the characteristic equations

The requirements and levels of sophistication for each of these elements need to be specified in order to establish a base for determining such limitations and deficiencies, presently existing in the FLEXSTAB CPS, as concern dynamic loads analyses capabilities. These general requirements are described in the following paragraphs.

3.1 GEOMETRY AND PANELING

Geometric data describing the aircraft components are required before dynamic aeroelastic loads analyses can be performed. Panel characteristics pertinent to the structural and aerodynamic model representation must be defined; since the aerodynamic and structural panels are seldom identical, separate paneling schemes must be employed for each.

3.2 STRUCTURAL REPRESENTATION

The structure may be represented by beam segments or finite elements for calculation of either flexibility or stiffness matrices. If beam segments are modeled, the mass representation usually consists of lumped masses and inertias which, when summed, equal the total airplane c.g., weight, and rotary inertias. For a finite element structural representation, the mass is usually represented by lumped masses at the finite element nodes corresponding to the weight c.g.'s of small panels, and again total airplane c.g., weight, and rotary inertias are matched. The number of degrees of freedom resulting from these structural and mass representations would be too large for an efficient dynamics solution. Consequently, the mass and stiffness or flexibility matrices are used to formulate an eigenvalue problem whose solution produces structural mode shapes and frequencies which are used to define a smaller number of generalized coordinates than physical coordinates. Thus, the problem can be reduced to a workable size for dynamic loads analyses with little loss of accuracy (if enough modes are used).

3.3 AERODYNAMIC REPRESENTATION

The aerodynamic model is required to calculate the response and excitation air forces that are used to determine generalized air forces for the equations of motion development and to form air load contributions for the load equations. The degree of sophistication required of the aerodynamic representation is dependent upon the type of dynamic aeroelastic loads analysis to be performed. For preliminary design and gust loads, quasi-steady aerodynamics modified with Küssner and Wagner indicial lift growth functions are generally satisfactory because of the large attenuation of the gust forcing function at high frequencies. If mode stabilization, flutter suppression, or stability augmentation systems are considered, then responses at intermediate or higher frequencies are quite important, and a more exact unsteady aerodynamic representation must be used to correctly obtain phase and magnitude responses of the modes.

3.4 CONTROL SYSTEM REPRESENTATION

A control system definition is necessary to assess the effects of active controls responding to arbitrary inputs or feedback signals; e.g., stability augmentation system (SAS) signals. The representation is dependent upon the type of dynamic aeroelastic analysis to be performed. In most cases, the control system can be represented by linear systems amenable to classical control system analysis and synthesis techniques (ref. 3). In some cases (e.g., certain flutter suppression systems), nonlinear analysis techniques should be applied because of electrical component nonlinearities, servo mechanism saturation, or control surface movement limits.

3.5 EXCITATION FUNCTION DEFINITION

Excitation functions may be of several types. They may consist of oscillatory or arbitrary abrupt control surface motion or atmospheric turbulence. For control inputs, the control surface time history or feedback control signal serves to describe the excitation function. Atmospheric turbulence may be described with either a discrete time-dependent or a continuous frequency-dependent model. In the former, a waveform is specified; in the latter, a power spectrum based on a statistical description of random turbulence is defined. Acceptable models of atmospheric turbulence are readily available (refs. 4 and 5).

3.6 EQUATIONS OF MOTION AND LOAD EQUATION FORMULATION

Formulation of matrix coefficients for the equations of motion and load equations requires structural, aerodynamic, and control system data as previously discussed. Provision must be allowed for generation of either constant or frequency-dependent matrices resulting from steady state and unsteady aerodynamics, respectively, and either panel loads or loads at a reference location. There must be a capability to increase matrix size and insert, delete, or change individual matrix elements to incorporate experimental data, additional degrees of freedom, etc. Load equation matrix coefficient generation requires that load stations and an arbitrary load reference axis system be specified if shears, moments, and torsions are to be calculated. In this case, the appropriate panel inertia and aerodynamic forces must be summed to the load station and rotated into the load reference axis system.

3.7 SOLUTION ROUTINES

Solution routines are necessary to solve the equations of motion and load equations in both the time domain (time history solutions) and frequency domain (in order to determine PSD load parameter and to perform time history solutions using the Fourier transform method if unsteady aerodynamics are used). In addition, it is necessary to root the characteristic equation to determine stability characteristics.

The ability to perform dynamic loads analyses is dependent upon satisfying each of these general requirements. The ability to perform satisfactory dynamic loads analyses is determined by the level of technical sophistication of each general requirement of the preceding elements.

4.0 REVIEW OF THE INDICIAL LIFT GROWTH FUNCTION UNSTEADY SUBSONIC AERODYNAMIC REPRESENTATION FOR USE IN DYNAMIC LOADS ANALYSES

The problem of representing unsteady aerodynamic flow assumed importance in the 1930's; subsequently, a number of aerodynamicists began presenting solutions for various formulations of the problem.

Theodorsen first published in the United States a complete solution for the aerodynamic forces on a thin airfoil performing simple harmonic oscillations in a uniform two-dimensional incompressible flow (ref. 6). A solution for the aerodynamic forces on an airfoil subject to a step change in angle of attack was developed by Wagner; the result is reproduced in references 7 and 8. In these two cases, the solution is expressed as a product of the quasi-steady lift and a function known after the investigator's name, which itself may be expressed in terms of Bessel functions. A transform relationship exists between Theodorsen's reduced frequency-dependent function and Wagner's nondimensionalized time-dependent function.

The gust encounter problem has been approached similarly. Sears published a solution for the aerodynamic forces on a thin airfoil traversed by a sinusoidal gust (ref. 9). A solution for Küssner's problem, which considers an airfoil encountering a sharp-edged gust, is reproduced in references 7 and 8. These solutions involve terms analogous to the Theodorsen and Wagner functions; the Sears and Küssner functions.

The Wagner and Küssner indicial lift growth functions are customarily designated $\phi(\tau)$ and $\psi(\tau)$, respectively. Although these functions have a relatively simple form, they are not expressible in terms of simple, well-known functions. Therefore, approximations written in simple algebraic terms were developed to facilitate use of the indicial functions. The effect of finite span was considered by Jones (ref. 10), and additional approximations of indicial functions were developed for various finite aspect ratios ranging from 3 to infinity. Compressibility effects were considered by Mazelsky and Drischler (refs. 11 through 13), and some approximate indicial function expressions were developed for various subsonic Mach numbers from 0.0 to 0.7.

A general procedure in past dynamic loads analyses for flight conditions in the subsonic regime has been to use two-dimensional incompressible aerodynamic theory modified to approximate finite span effects* (ref. 14); compressibility effects have been represented implicitly through use of compressibility factors (F_c) to modify the lift curve slopes calculated using theoretical expressions for incompressible flow. For representation of unsteady effects, the use of indicial functions has been extended to include arbitrary airfoil motion and gust excitation functions through application of the superposition integral. This approach has been used extensively throughout the aircraft industry to determine dynamic gust design loads for a number of subsonic, relatively high aspect ratio, jet aircraft. Flight test results have been shown to agree reasonably well with theoretical predictions.**

*Richmond, L. D., "A Rational Method of Obtaining Three-Dimensional Unsteady Aerodynamic Derivatives of Intersecting Airfoils in Subsonic Flow," Boeing document D6-7401, 1962.

** Gilley, T. A. and Cast, R. D., "Theoretical and Experimental Frequency Response Functions for the B-52H Airplanes (WFT 1286)," Contract No. AF 34(601)-22257, 1966.

An analogous scheme has been used in this study in establishing the requirements for implementing a dynamic loads capability in the FLEXSTAB CPS. A quasi-steady state aerodynamic representation, formulated using the steady state AIC matrix produced in the FLEXSTAB CPS, is combined with the instantaneous incidence and indicial lift growth functions to develop an approximate unsteady aerodynamic representation.

Unsteady aerodynamic effects are approximated by considering lift growth to occur in accordance with indicial functions of the following forms:

1. Response indicial function approximation

$$1 - a_1 \exp \alpha_1 t - a_2 \exp \alpha_2 t$$

2. Excitation indicial function approximation

$$1 - b_1 \exp \beta_1 t - b_2 \exp \beta_2 t - b_3 \exp \beta_3 t$$

The coefficients a_i, b_i, α_i, and β_i are functions of aspect ratio and subsonic Mach number. For supersonic flow, these coefficients are generally assumed to be zero. The user is free to choose coefficients that tailor the indicial functions for compatibility with individual analysis requirements. In theory, indicial functions can be simply developed from harmonic solutions; in practice, this procedure requires considerable effort.

The equations of motion as formulated in reference 15 and reformulated in reference 2, using the basis of the approximate unsteady aerodynamic representation described previously, are:

(Structural + SAS) + (Response Aerodynamics) = (Gust Excitation Aerodynamics)

$$\left[M1\right]\left\{q\right\} + \left[M2\right]\left\{\dot{q}\right\} + \left[M3\right]\left\{\ddot{q}\right\} + \left[M4\right]\left\{\dot{q}\right\} * \Phi + \left[M5\right]\left\{\ddot{q}\right\} * \Phi = \left\{C3\right\}\dot{\alpha}g * \psi$$

where

M1, M2, M3 are the appropriate structural matrix coefficients.

M4, M5, and C3 are the appropriate aerodynamic matrix coefficients formed using steady state aerodynamics.

q's are generalized coordinates.

α_g is the gust angle (angle formed by the freestream velocity and the resulting freestream velocity with gust disturbances).

Φ is the Wagner indicial lift growth function.

ψ is the Küssner indicial lift growth function.

$*$ indicates convolution.

The load equations follow the same format as the equations of motion.

$$\text{Load} = \left[\overline{M}1\right]\left\{q\right\} + \left[\overline{M}2\right]\left\{\dot{q}\right\} + \left[\overline{M}3\right]\left\{\ddot{q}\right\} + \left[\overline{M}4\right]\left\{\dot{q}\right\} * \Phi + \left[\overline{M}5\right]\left\{\ddot{q}\right\} * \Phi + \left\{\overline{C}3\right\} \dot{\alpha}g * \psi$$

where

$\overline{M}1$, $\overline{M}2$, and $\overline{M}3$ are load matrix coefficients of the generalized coordinate displacement, rate, and acceleration, respectively.

$\overline{M}4$ and $\overline{M}5$ are load matrix coefficients of the generalized coordinate rate and acceleration convoluted with the Wagner function.

$\overline{C}3$ is the load matrix coefficient of the excitation function convoluted with the Küssner function.

The equations of motion, including indicial functions, may be integrated into the form (first-order differential equation) required in the existing FLEXSTAB CPS by transforming the response indicial function into state form (see app. A). Although the matrix size increases significantly, this form allows a general application of the indicial functions. At one extreme, the same form of the indicial function may be applied to all degrees of freedom or, at the opposite limit, each degree of freedom may be modified by a separate approximation of the indicial function as defined in appendix A.

5.0 APPLICABILITY OF FLEXSTAB PROGRAMS FOR DYNAMIC LOADS

The various programs of the NASA FLEXSTAB CPS were evaluated in reference 2 to determine their applicability and adequacy for dynamic loads analyses using aerodynamic representations which range from steady state to frequency-dependent unsteady aerodynamics. The evaluation was based on the general requirements detailed in section 3 together with the level of technical sophistication required for a dynamic analysis. The basis for this study was that evaluation, modified to include only the Küssner and Wagner indicial lift growth unsteady aerodynamic representation. In addition, the evaluation has been expanded in several areas to further clarify the FLEXSTAB limitations.

Figure 1 identifies the FLEXSTAB 1.00.xx programs considered for dynamic loads application, together with the elements listed in section 3.

5.1 GEOMETRY AND PANELING (GD)

The Geometry Definition program (GD) calculates detailed definitions of the aerodynamic geometry and paneling, required in the downstream programs, from the general input geometric data. This aerodynamic paneling scheme is adequate for use in performing dynamic loads analyses.

5.2 STRUCTURAL REPRESENTATION (ISIC, ESIC)

The Internal Structural Influence Coefficient program (ISIC) uses classical beam theory to formulate the structural model of an aircraft. Structural components are represented by beams approximately coincident with the elastic axis. These massless beams are composed of constant stiffness straight line segments joining structural nodes. Mass distribution is represented by discrete lumped masses associated with the structural nodes.

The slender body structural representation is deficient in two respects. The elastic axis is located along the aerodynamic mean centerline (a constant waterline) regardless of the structural geometry, and the lumped masses are considered to lie on this straight line elastic axis. The nature of this representation precludes proper modeling of a body whose structural axis is upswept and of fuselage torsional inertia effects, both of which will affect antisymmetric analyses. Although the structural flexibility matrix retains the slender body torsional degree of freedom, contributions of the body inertia forces to this freedom are not included because of these limitations. Figure 2, comparing the desired and the FLEXSTAB CPS elastic axis-lumped mass representations, illustrates these deficiencies.

The mass representation of lifting surfaces provides for rotary inertia in the torsional degree of freedom, but not in the bending degree of freedom. To define the torsional mass moment of inertia, the user is required to divide each lumped mass into two parts and move the parts in opposite directions perpendicular to the elastic axis, preserving

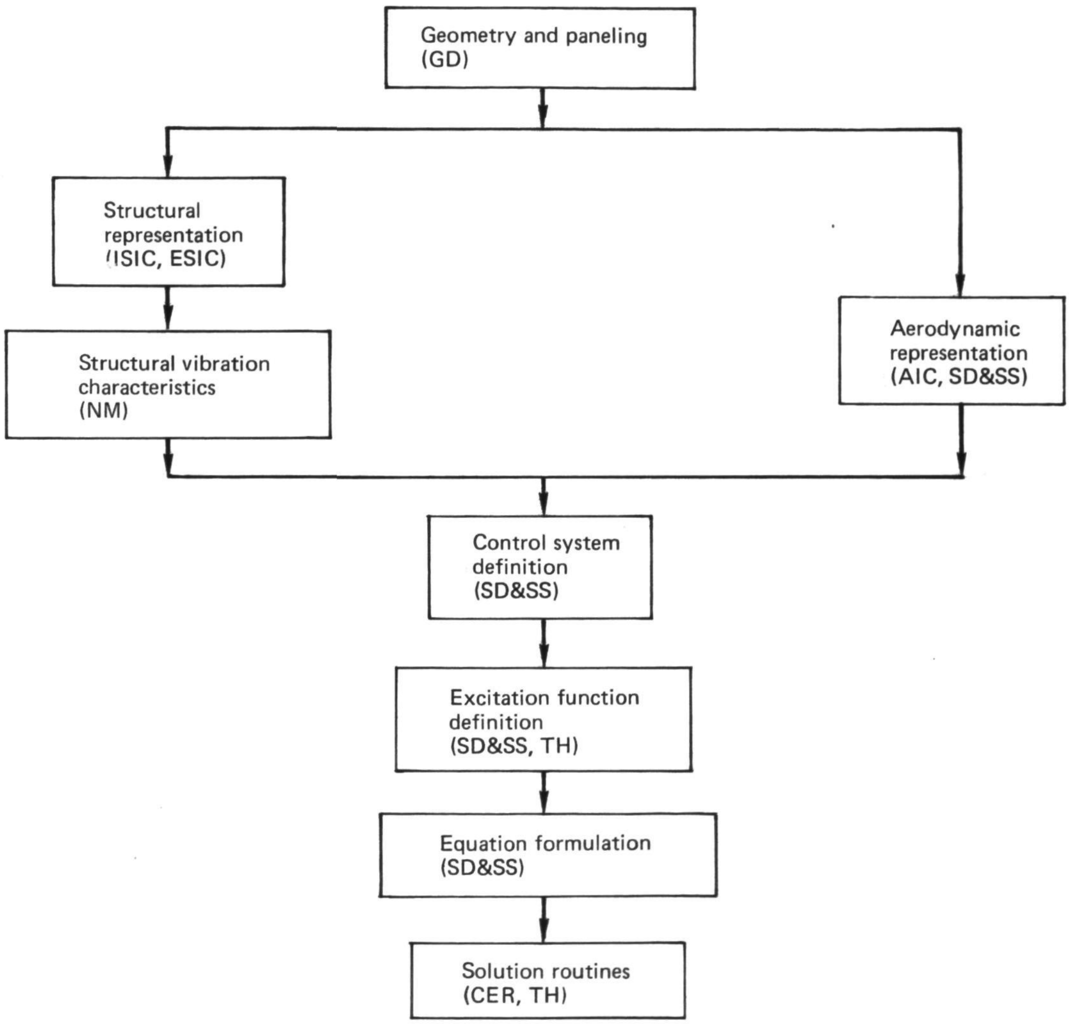
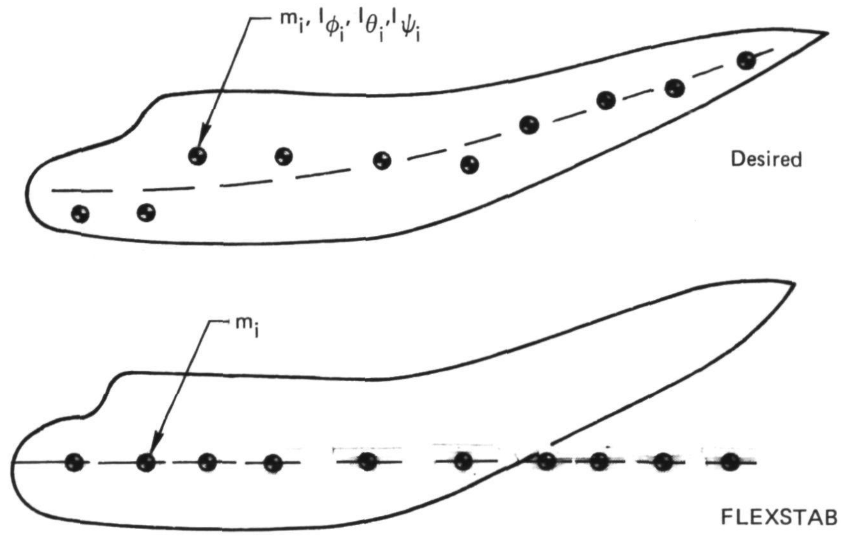


Figure 1.—FLEXSTAB Programs Reviewed for Dynamic Loads Analysis



Where:
m = mass
I_ϕ = pitch moment of inertia about local elastic axis
I_θ = roll moment of inertia about local elastic axis
I_ψ = yaw moment of inertia about local elastic axis
i = ith node

Figure 2.—Slender Body Mass Representation

the c.g. and total mass while modeling the local torsional moment of inertia. This is illustrated in figure 3. However, because there is no allowance in the program to move the masses parallel to the elastic axis, it is impossible to match the bending moments of inertia using this technique.

In general, a rigid airplane analysis requires that the total airplane c.g. location, mass, and mass moments of inertia at the c.g. be adequately defined. For a beam structural model, the use of a lumped mass representation without rotary inertias is, in most cases, satisfactory for rigid airplane analysis. The total rigid airplane analysis requirements can be adequately defined because at the c.g.:

Total mass

$$M = \Sigma m_i$$

Total pitch inertia

$$I_\phi = \Sigma m_i x_i^2 + \Sigma I_{\phi_i} \approx \Sigma m_i x_i^2$$

Total roll inertia

$$I_\theta = \Sigma m_i y_i^2 + \Sigma I_{\theta_i} \approx \Sigma m_i y_i^2$$

Total yaw inertia

$$I_\psi = \Sigma m_i\left(y_i^2 + x_i^2\right) + \Sigma I_{\psi_i} \approx \Sigma m_i\left(y_i^2 + x_i^2\right)$$

C.G. location

$$C.G. = \Sigma m_i x_i / \Sigma m_i$$

Exceptions are:

1. Aircraft in which the x locations of the wing masses coincide very closely to the x location of the total aircraft center of gravity and the body contribution to the pitch inertia is small (examples: flying wing, some current RPV configurations). This will cause an erroneous total $I_{\text{Pitch.}}$

2. Aircraft having small wing masses and large body masses; i.e., no wing fuel together with a large body payload and/or body fuel (example: Century-type fighter aircraft). This will cause an erroneous total $I_{\text{Roll.}}$

A flexible airplane dynamic analysis requires, in addition to the rigid airplane analysis requirements, inclusion of the mass characteristics, mass moments of inertia at the panel c.g., and panel c.g. location of each structural panel used for the dynamic model.

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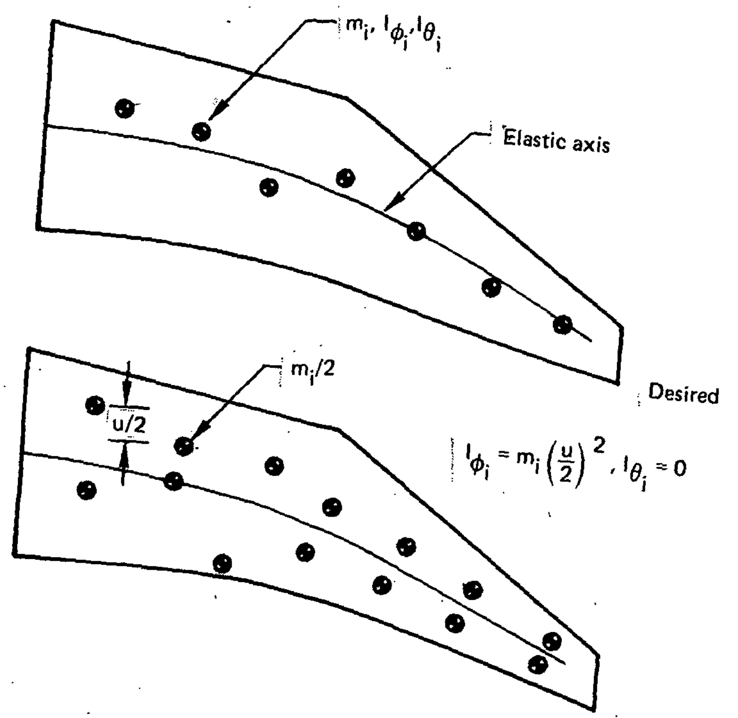


Figure 3.—Thin Body Mass Representation

The calculation of loads using the force summation technique (ref. 7) makes this requirement doubly important; for example, see figure 4 for calculation of wing bending moment just inboard of an outboard store.

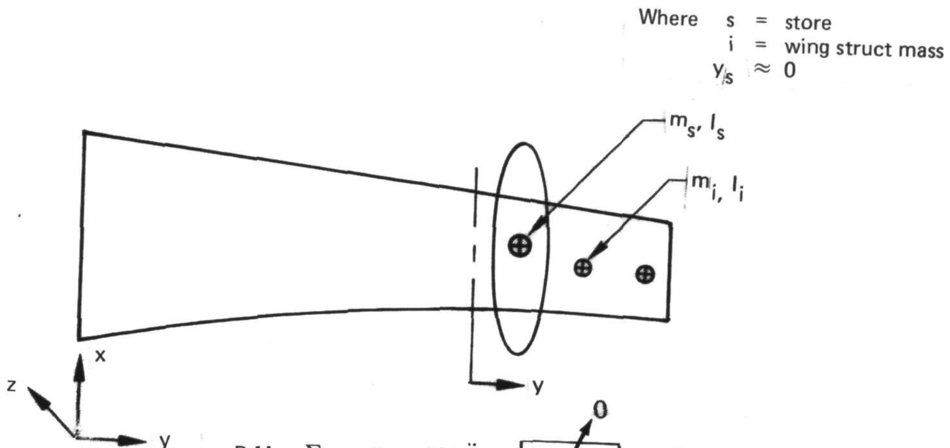
The mass moment of inertia (I_s) of the store is large and the term $I_s \ddot{\theta}_x$ will be of the same order as $\Sigma m_i y_i \ddot{z}_i$ for lightweight structures such as occur at wingtips. Thus, for this example, neglecting the rotary inertia terms $\Sigma I_i \ddot{\theta}_i$ and $I_s \ddot{\theta}_x$ will produce a significant error in the bending moment because of the inertia effects.

The calculation of displacements and rotations of points on the lifting surface aerodynamic panels is based on the deformation of the nearest structural node point on the elastic axis. The twist and displacement at this node together with the perpendicular distance from the aerodynamic panel centroid to the elastic axis are used to calculate panel deflections. This approach neglects the increments in bending and torsion deflection due to the elastic axis distance between the intersection of the perpendicular from the panel centroid and the structural node location (Δy of fig. 5).

The External Structural Influence Coefficient program (ESIC) is designed to transform structural matrices calculated using finite element techniques, external to the FLEXSTAB CPS, into structural matrices of the form required by SD&SS. The slender body representation has the same deficiencies as those in ISIC; that is, the structural node points are assumed to lie along the mean aerodynamic centerline. This can result in large localized effects on the structure but the gross effects are small. Consequently, this deficiency has minimal effects on the stability derivative calculations in SD&SS but significantly affects the generation of the generalized inertia and stiffness forces used in the dynamic equations of motion.

The use of cantilever modes is desirable for structural parameter studies and for those design cases where only a few structural components need to be varied (sec. 4 of ref. 7). System modifications required to accommodate cantilever modes would involve major changes in the ISIC and ESIC programs because of the nature of the resulting matrices.

In conclusion, ISIC and, to a lesser extent, ESIC do not meet the structural representation requirements as defined in section 3. The deficiencies that must be corrected before these programs are suitable for dynamic loads analyses are: (1) the addition of mass moment of inertia terms for both slender and thin bodies and (2) for slender bodies, the capability to represent both a curved elastic axis not coincident with the mean aerodynamic axis and masses located off the elastic axis. These requirements would also necessitate the formulation of flexibility matrices which include deflection and slope coefficients due to moments. The capability is also needed either to input cantilever modes into ESIC or to formulate flexibility matrices in ISIC from which cantilever modes can be calculated. Correcting these deficiencies would require a restructuring and recoding effort for both ESIC and ISIC and would, in turn, affect the downstream programs such as NM and SD&SS because of the matrix compositions; i.e., mass matrix is a full matrix.



$$\text{B.M.} = \Sigma\, m_i y_i \ddot{z}_i + \Sigma\, I_i \ddot{\theta}_x + \boxed{\cancelto{0}{m_s y_s z_s}} + I_s \ddot{\theta}_x + \text{moment due to aero forces}$$

Figure 4.—Bending Moment Calculation by Force Summation Technique

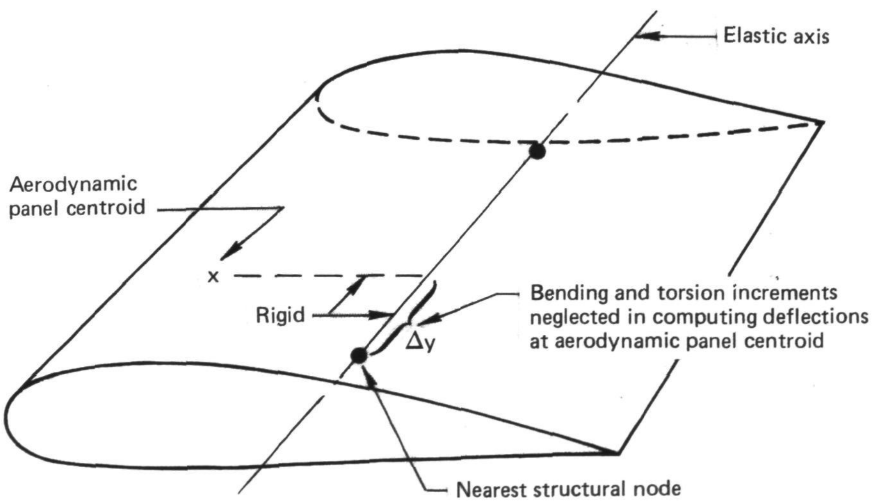


Figure 5.—Thin Body Modal Displacement Reference

5.3 STRUCTURAL MODEL—VIBRATION CHARACTERISTICS (NM)

The normal modes program (NM) requires input from the structural program (ISIC). NM uses this input to formulate and solve an eigenvalue problem to produce structural free-free mode shapes and frequencies that are used to define generalized coordinates and to calculate the generalized inertia and stiffness forces. The structural input data, and eigenvalue problem formulation and solution, are deficient for the purposes of calculating cantilever modes (rigid body modes are not required to calculate cantilever modes) or including moments of inertia in the mass matrix. Consequently, restructuring and recoding of this program are necessary to satisfy the requirements defined in section 3 for a satisfactory dynamic analysis.

5.4 AERODYNAMIC REPRESENTATION (AIC, SD&SS)

The function of the aerodynamic influence coefficient program is to calculate either subsonic or supersonic steady-state aerodynamic influence coefficient (AIC) matrices relating surface pressure to flow incidence, and to produce a low-frequency approximation to an unsteady AIC matrix relating surface pressure to flow incidence time rate of change. The output consisting of elements of the AIC matrices is saved for use in the SD&SS program where the complete AIC matrix is assembled from these elements.

The aerodynamic theory in FLEXSTAB considers linear, potential flow in both subsonic and supersonic regimes. Solutions are written as integral equations involving the strengths of flow singularities distributed over aerodynamic mean surfaces and mean lines. Strengths of the flow singularity distributions are determined using approximations based on those used in the finite element method of Woodward. Thin body and interference body mean surfaces are represented by panels, each with an associated constant strength vortex flow singularity. Slender body mean centerlines are divided into line elements with flow singularities represented by doublets whose strengths vary quadratically. Thickness effects may be represented by sources. The isolated problems are solved and the interference incidence arising from these solutions calculated. This interference incidence is suppressed with a vortex distribution and isolated vortex and source distributions. Finally, the pressure is calculated and the AIC is available from the relation between pressure and surface flow incidence.

The steady state aerodynamic representation may be corrected with experimental data using one of several schemes available in FLEXSTAB. In practice, meaningful corrections of the AIC matrix are difficult to achieve, and the application of corrector schemes has proven generally unsuccessful in the past (ref. 16).

The low-frequency approximation to unsteady aerodynamics used in FLEXSTAB has severe frequency limitations and is generally suitable only for calculation of dynamic stabililty derivatives; FLEXSTAB computes frequency-independent dynamic derivatives corresponding to limit $C_{m_{\dot{\alpha}}}$ as $k \to 0$, etc. The inherent restrictions on frequency and an aspect ratio to Mach number relationship ($k \ll 1$, $k \ll \beta^2/M$, $k|\ln k| \ll 2/AR\sqrt{1 - M^2}$ for subsonic, and $k \ll 1$, $k \ll (M^2 - 1)/M^2$ for supersonic flow) result in an AIC matrix which is not suitable for dynamic loads analyses involving appreciable structural mode response. The envelope of these restrictions for a wing having an aspect ratio of 8.6 (B-52) in subsonic flow is shown in figure 6. Two flight conditions from the B-52 LAMS study are superimposed on this figure to show their relationship to the restrictions.

The use of steady state AIC's and the instantaneous angle of incidence modified with the Wagner indicial lift growth function, and the gust angle modified with the Küssner indicial lift growth function to represent unsteady aerodynamics has been the accepted method used by the aircraft industry for performing dynamic gust loads analyses (refs. 7, 8, 17). The aerodynamic influence coefficient program is suitable for generating steady state AIC's for use in formulating the generalized response and gust aerodynamic forces in a dynamic analysis. The low-frequency aerodynamic approximation is useful for calculating the airplane static and dynamic stability derivatives used in the dynamic equations of motion. It is important that the low-frequency aerodynamic approximation is not used in formulating the generalized aerodynamic forces for the elastic modes (fig. 6), and especially for the gust generalized forces, because the gust spectrum is important at frequencies much above the valid frequency limits of the low-frequency aerodynamic approximation theory.

5.5 CONTROL SYSTEM DEFINITION (SD&SS)

The control system model defined in the SD&SS program can include ailerons, rudders, and elevators. However, if more than one control surface of each type is defined, they must be interdependent; i.e., inboard and outboard ailerons cannot operate independently of each other. Other control surfaces (such as spoilers, tabs, or other active controls such as side force generators) cannot be modeled. Most present aircraft operate with some form of stability augmentation system (SAS); future aircraft will probably have more sophisticated SAS systems functioning via a wide variety of control surfaces. Any program capable of performing dynamic loads analyses must allow for a variety of active control systems.

5.6 EXCITATION FUNCTION DEFINITION (SD&SS, TH)

The excitation function is defined in SD&SS as a time-dependent aerodynamic force at each aerocentroid. These forces are useful in the equations of motion formulation procedure to form generalized excitation forces. Time-dependent shapes of the excitation angle of incidence defined in TH include (1 - cos), step, and square waves. To include unsteady effects, these instantaneous angles of incidence need to be modified with the Küssner indicial lift growth functions.

5.7 EQUATIONS OF MOTION AND LOAD EQUATIONS FORMULATION (SD&SS)

The equations of motion are formulated in the SD&SS program using data from the programs defining the geometry, structural, aerodynamic, and control system representations. There is no provision for including the SAS definitions in the equations. Since SD&SS is dependent on data from the previous programs, the degree of

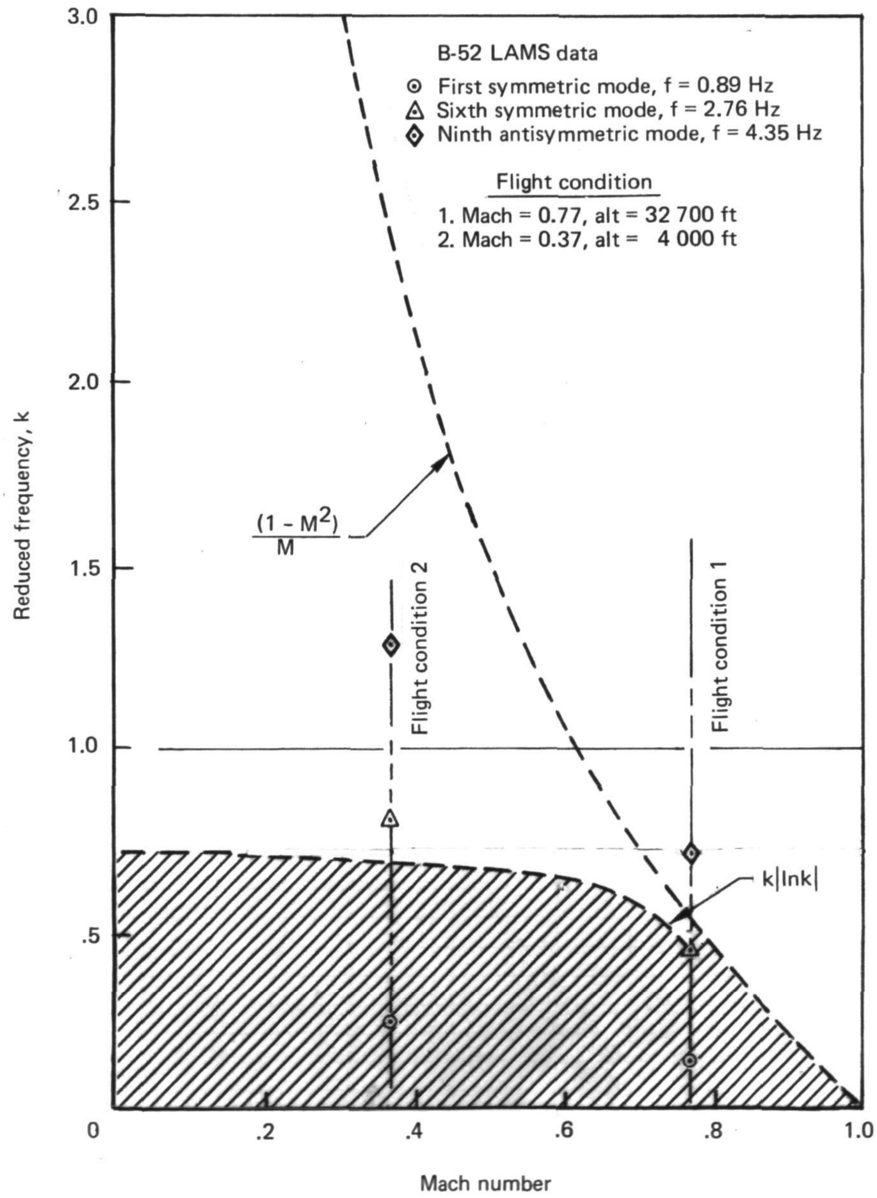


Figure 6.—Reduced Frequency Envelope for the B-52 and for FLEXSTAB Subsonic Aerodynamics

sophistication of the equations of motion is dependent on that of each of the previous programs. Presently, there is no capability to formulate dynamic load equations; without this capability, the FLEXSTAB CPS cannot be used for dynamic loads analyses.

The use of a truncated number of modes plus the residual flexibility of the remaining modes, including the aeroelastic effects as formulated in SD&SS of the FLEXSTAB CPS, is not considered a satisfactory approach for use in dynamic load analysis. It can produce unsatisfactory results and its use is not recommended. The reasons for this recommendation are discussed in more detail in section 6.

5.8 SOLUTION ROUTINES (CER, TH)

The Characteristic Equation Rooting program (CER) is a subprogram of SD&SS and solves for roots of the characteristic equations obtained from the equations of motion. The Time History program (TH) solves the equations of motion in the time domain for several forcing function shapes such as (1 - cos), step, or square waves. No provision is presently available for solution of the equations in the frequency domain.

Since present design gust loads criteria require the determination of loads due to continuous turbulence, a solution routine must be added to solve the equations of motion and load equations using random harmonic analysis techniques. It must be capable of calculating steady state solutions for constant coefficient, linear second-order differential equations. From such steady state solutions, structural dynamic loads due to sinusoidal forcing functions may be determined and statistical characteristics of loads due to continuous turbulence calculated. The equations of motion must be solved for coordinate frequency response functions which are retained for use in calculating load transfer functions (simply load coefficient matrices multiplied by coordinate responses at each frequency). Load power spectra can then be obtained from the product of the gust spectra and the load transfer functions squared. From each load spectrum, root mean square load/root mean square gust velocity ($\overline{A}$) and number of zero crossings (N_0) can be calculated.

6.0 RESIDUAL FLEXIBILITY EFFECTS OF MODES FOR DYNAMIC LOADS ANALYSIS

The formulation and use of "residual flexibility" was first investigated by MacNeal and Schwendler in the early 1960's (ref. 18). Since then, this technique for calculation of stability derivatives and for flutter calculations has been used successfully. However, the use of this technique for flutter, control system analyses, and especially for dynamic loads analysis, can lead to significant errors if care is not exercised.

In this technique, the free vibration modes are partitioned into two sets: the lower frequency explicitly retained modes and the higher frequency residual modes. The basic assumption is that the inertia and damping forces for the higher frequency set of modes are much smaller than the corresponding stiffness forces, and thus can be neglected; i.e., $m\ddot{q} = c\dot{q} << kq$. This can be considered as equivalent to the generalized mass becoming zero for those modes whose residual flexibility effects only are to be considered (ref. 18). Figure 7 shows these effects on the response of a single-degree-of-freedom system and the errors that can occur.

For this example, residual effects are valid when the forcing frequencies are much less than the actual resonance frequency of the mode whose residual flexibility effects only are included. For a multidegree-of-freedom system, where modal coupling occurs, this region is rather ill-defined, but the frequency of interest must be much less than the resonance frequency of the last mode explicitly included. MacNeal and Schwendler recommended that the modes explicitly included should be all those in the frequency range from zero to the maximum frequency of interest plus the next higher mode. However, if the modes are highly coupled or the frequency separation between modes is small, a considerably larger number of modes should be included.

The equations used in dynamic loads analyses must be solved in both the time and frequency domains. If time history solutions are obtained, all the modes explicitly included are excited. The accuracy of the results achieved by including residual effects is dependent upon the amount that each explicitly included mode has been excited and the degree of influence the dynamically deleted modes have on these retained modes. If the frequency of a significantly strong mode of vibration is close to the natural frequencies of the modes whose residual effects only are included, significant errors may be introduced because residual stiffness forces are included without the resulting inertia forces to partially offset them.

To show the effects of residual flexibility and truncated modes approximation on dynamic analyses solved in the frequency domain, a two-degree-of-freedom mass-spring problem without damping was modeled* (fig. 8) and solved for the exact, the approximate residual flexibility, and the approximate truncated mode solutions.

*In conjunction with Dr. Larry Erickson (NASA-Ames)

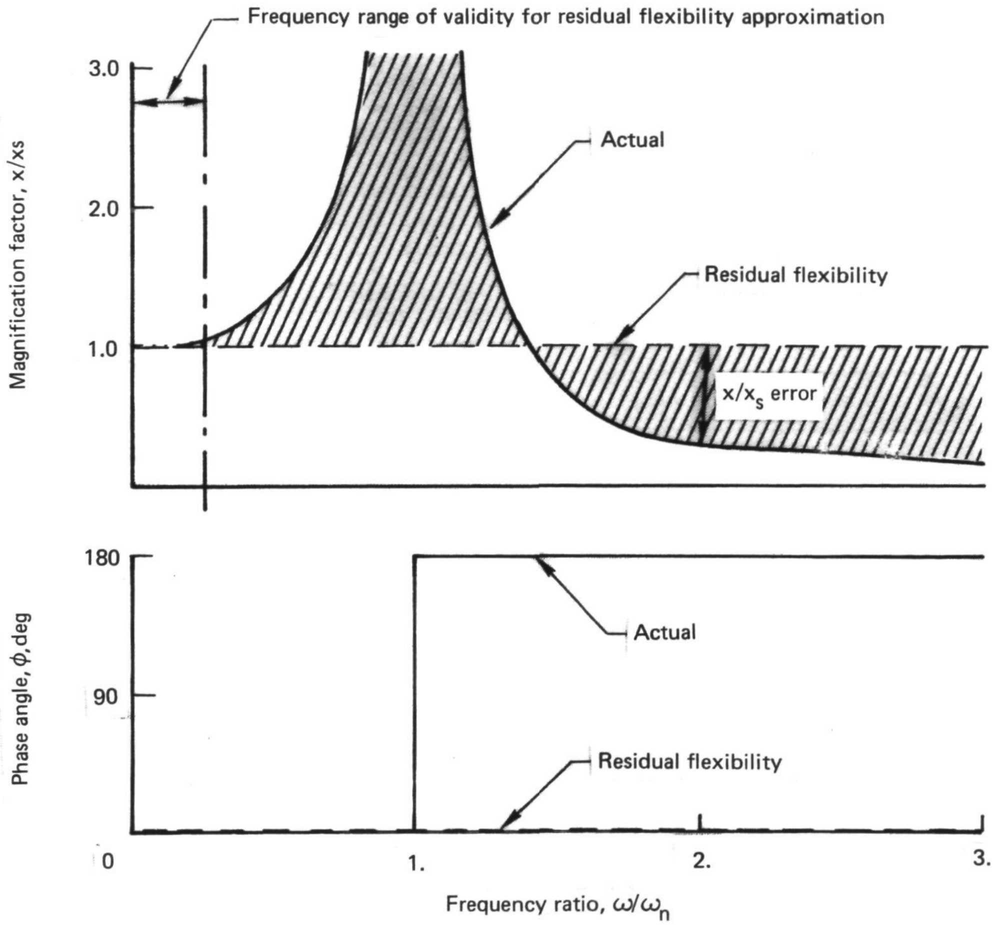


Figure 7.—Dynamic Response for a Single-Degree-of-Freedom System (Undamped)

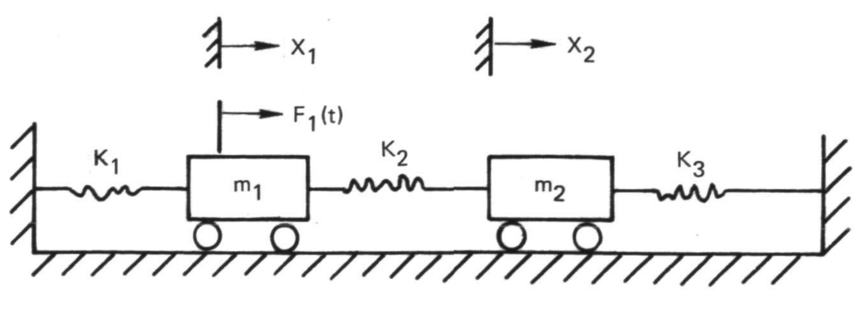


Figure 8.—Two-Degree-of-Freedom Model

Let:

$$K_1 = K_2 = K_3 = K$$

$$m_1 = m_2 = m$$

$$F_1(t) = a_1 \sin \omega t$$

The natural frequencies and mode shapes of the system are:

$$\omega_1{}^2 = K/m$$

$$\omega_2{}^2 = 3K/m = 3\omega_1{}^2$$

$$\phi_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

which corresponds to the generalized coordinates u_j.

The two physical displacements calculated by $X_i = \sum_{j=1}^{2} \phi_{ij} \; u_j$ are:

Exact Solution:

$$\frac{2K}{a_i} X_1 = \left[\frac{1}{1 - (\omega/\omega_1)^2} + \frac{1}{3 - (\omega/\omega_1)^2} \right] \sin \omega t$$

$$\frac{2K}{a_1} X_2 = \left[\frac{1}{1 - (\omega/\omega_1)^2} - \frac{1}{3 - (\omega/\omega_1)^2} \right] \sin \omega t$$

Residual Flexibility Solution: ($\ddot{u}_2 = 0$)

$$\frac{2K}{a_1} X_1 = \left[\frac{1}{1 - (\omega/\omega_1)^2} + \frac{1}{3} \right] \sin \omega t$$

$$\frac{2K}{a_1} X_2 = \left[\frac{1}{1 - (\omega/\omega_1)^2} - \frac{1}{3} \right] \sin \omega t$$

The terms $\pm\frac{1}{3}$ represent the static effects of the u_2 mode.

Modal Truncation: ($u_2 = \ddot{u}_2 = 0$)

$$\frac{2K}{a_1} X_1 = \left[\frac{1}{1-(\omega/\omega_1)^2}\right] \sin \omega t$$

$$\frac{2K}{a_1} X_2 = \left[\frac{1}{1-(\omega/\omega_1)^2}\right] \sin \omega t$$

Although the generalized inertia and damping forces are considered small in the basic assumptions and are neglected for the modes whose residual effects only are included, this does not mean that the accelerations of these modes are zero, but only that they are small. Thus, to obtain accelerations, the displacement solutions can be differentiated twice, yielding the following:

Exact Solution:

$$-\frac{2K}{a_1} \ddot{X}_1 = \left[\frac{\omega^2}{1-(\omega/\omega_1)^2} + \frac{\omega^2}{3-(\omega/\omega_1)^2}\right] \sin \omega t$$

$$-\frac{2K}{a_1} \ddot{X}_2 = \left[\frac{\omega^2}{1-(\omega/\omega_1)^2} - \frac{\omega^2}{3-(\omega/\omega_1)^2}\right] \sin \omega t$$

Residual Flexibility Solution:

$$-\frac{2K}{a_1} \ddot{X}_1 = \left[\frac{\omega^2}{1-(\omega/\omega_1)^2} + \frac{\omega^2}{3}\right] \sin \omega t$$

$$-\frac{2K}{a_1} \ddot{X}_2 = \left[\frac{\omega^2}{1-(\omega/\omega_1)^2} - \frac{\omega^2}{3}\right] \sin \omega t$$

Modal Truncation: ($u_2 = \ddot{u}_2 = 0$)

$$- \frac{2K}{a_1} \ddot{X}_1 = \left[\frac{\omega^2}{1 - (\omega/\omega_1)^2} \right] \sin \omega t$$

$$- \frac{2K}{a_1} \ddot{X}_2 = \left[\frac{\omega^2}{1 - (\omega/\omega_1)^2} \right] \sin \omega t$$

These solutions are shown graphically in figures 9 and 10. When the model includes viscous damping,* the solution becomes considerably more complicated. This problem was formulated and solved on a digital computer, and the results for the displacement and acceleration of mass 1 are shown in figure 11.

The results shown in figures 9 through 11 illustrate the following points.

For static response calculations or at frequencies that may be considered essentially static, the residual flexibility formulation as intended in its original formulation accurately accounts for the residual stiffness effects of the neglected higher order modes in calculating displacements. It can provide considerable improvement over the truncated mode solution.

For frequencies below the first resonance frequency, including the static response, the residual flexibility approximation produces a displacement and acceleration closer to the exact solution than the truncated modes solution. However, when damping is included in the system, the residual flexibility solution begins to depart from the exact solution at a lower frequency than without damping. At frequencies above the second resonance frequency ($\omega_2^2 = 3\omega_1^2$), the displacement of both the exact and truncated modal analysis approaches zero, whereas the residual flexibility approximation approaches a constant. Similarly, the accelerations for the exact and truncated modal analyses approach constants, whereas the acceleration of the residual flexibility approximation diverges as a square of the frequency. However, the residual flexibility solutions are invalid at frequencies which approach or exceed the natural frequencies of the modes not explicitly included.

For random harmonic analysis techniques, the theoretical derivation requires the calculation of solutions for forcing frequencies ranging from zero to infinity; in practice, the upper limit of the frequency range is some finite value. This cutoff frequency is usually established based on the intensity of the forcing function variation with

*Boeing IR&D funds were used to formulate and solve the two-degree-of-freedom model with viscous damping.

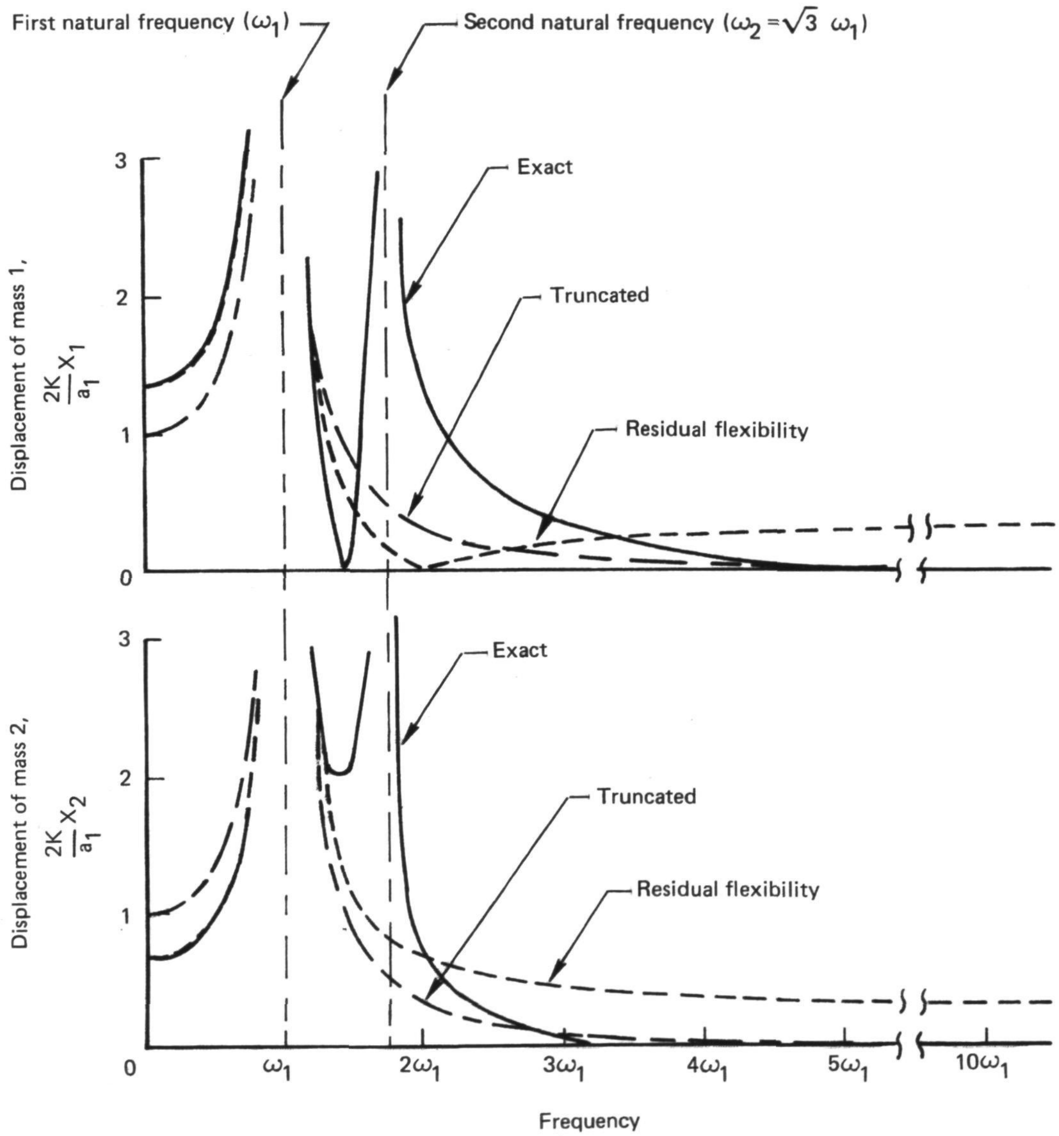


Figure 9.—Displacement Response for a Two-Degree-of-Freedom System

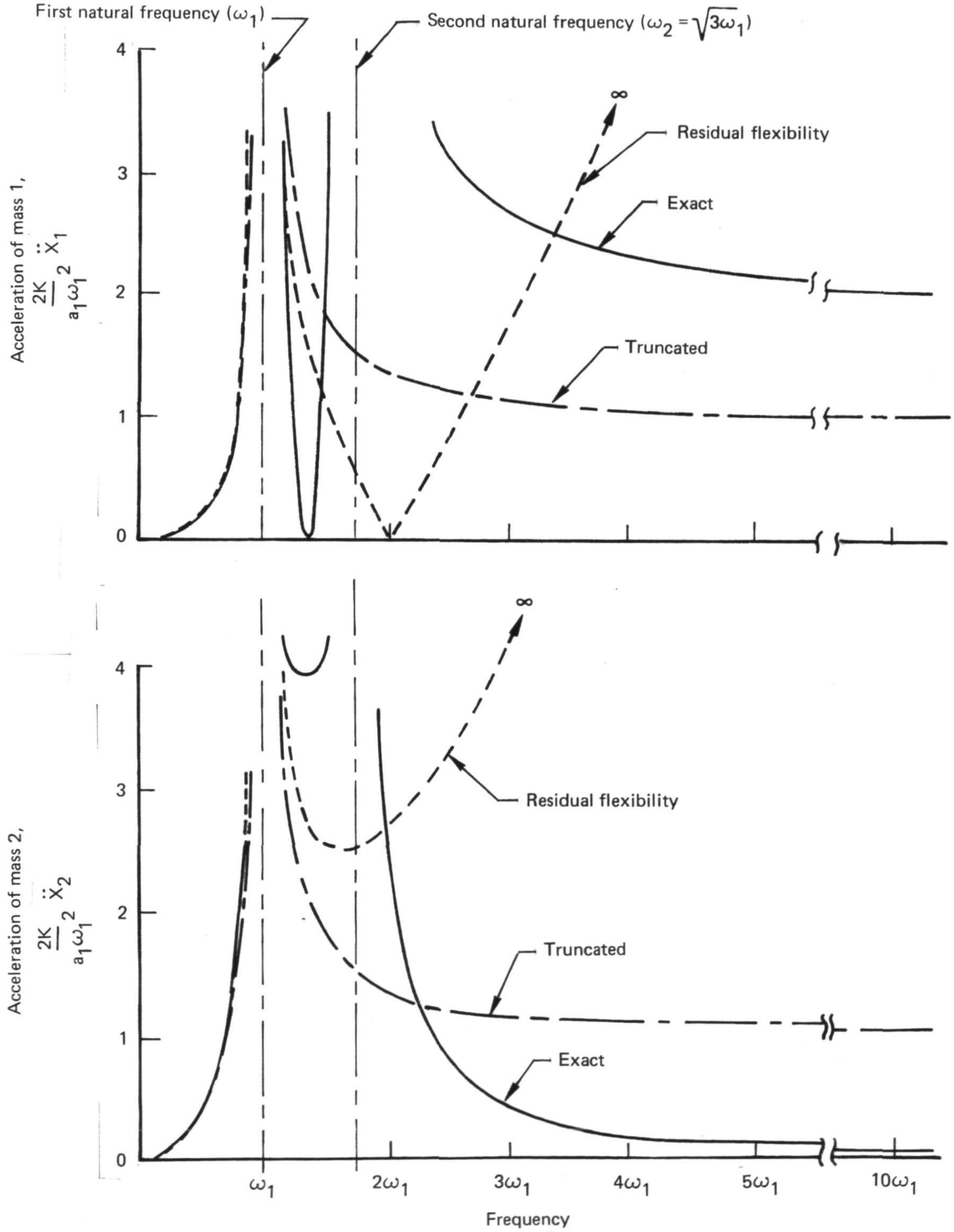


Figure 10.—Acceleration Response for a Two-Degree-of-Freedom System

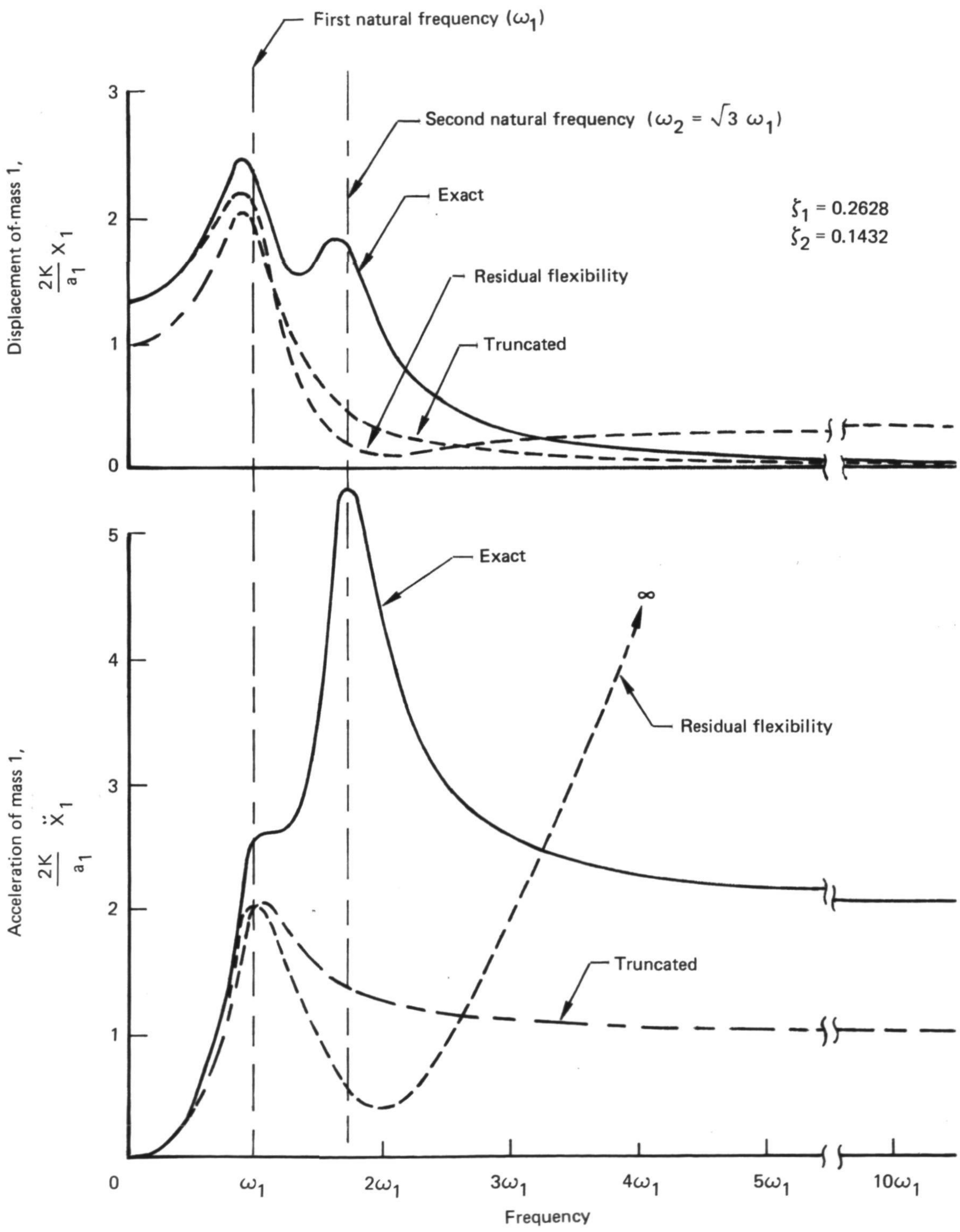


Figure 11.—Response for a Two-Degree-of-Freedom System With Viscous Damping

frequency and also convergence of the PSD parameter (for example, atmospheric turbulence decreases as approximately $(1/\text{freq})^2$). Consequently, modes whose frequencies are slightly higher than this finite cutoff value, as well as those whose frequencies fall in this range, must be included to prevent significant errors from being introduced into the analysis when residual flexibility approximations are included. As can be seen from figures 9, 10, and 11, the error introduced by using residual flexibility approximations is much larger at frequencies above the reasonance frequency than that error if the mode is left out, as in truncated modal analyses. Table 1 summarizes the results of the preceding example as the forcing frequency approaches infinity.

Table 1.—Response of an Undamped Two-Degree-of-Freedom System as the Forcing Function Frequency Approaches Infinity

Solution	$\omega \rightarrow \infty$			
	x_1	x_2	$\ddot{x}_1$	$\ddot{x}_2$
Exact	0	0	F(t)/K	0
Residual flexibility	F(t)/6K	F(t)/6K	∞	∞
Modal truncation	0	0	F(t)/2K	F(t)2K

In general, for dynamic loads analyses, the primary value of residual flexibility effects is in the modeling of all elastic effects in the rigid body response of the aircraft. However, since it is necessary to have modes explicitly included whose frequencies range from zero to a level higher than the maximum frequency of interest which is itself fairly high (order of 15 to 20 Hz), the increased complexity of including residual effects, rather than using truncated modes, is not justified. Because of this increased complexity and uncertainty at what frequencies the solutions become invalid, it is recommended that this technique not be employed for dynamic loads analyses, and that only truncated modes be used with a sufficient number of modes included to encompass the highest frequency of interest in order to have reasonably accurate results. In addition, the modes considered must include at least one bending and one torsion mode for each lifting surface. This is required to obtain the elastic effects of all surfaces, including cases where the structure is dynamically stiff but statically flexible.

7.0 CONCLUSIONS

The following modifications or additions to the FLEXSTAB CPS are necessary to provide the capability to adequately perform dynamic gust loads analyses:

- Modify the slender body elastic axis representation to allow a curved, arbitrarily located elastic axis.
- Modify the slender body mass representation to include mass and rotary inertias off the elastic axis.
- Modify the thin body mass representation to include proper mass distribution and rotary inertias.
- Modify the aerodynamic reference point interpolation routine to include bending and rotation slope changes between structural nodes.
- Modify the flexibility matrix formulation to allow calculation of displacements and rotations due to moments in addition to current displacements and rotations due to forces.
- Include the capability of using cantilever modes in the analysis.
- Modify the program to calculate either cantilever or free-free modes.
- Include the capability to calculate load equations consisting of shears, bending moments, torsion, net panel forces, aerodynamic panel forces, accelerations, velocities, and displacements.
- Add the capability to include the SAS representation in the equations of motion.
- Add a new solution routine to solve the equations of motion and load equations in the frequency domain.
- Add Küssner and Wagner indicial lift growth functions to represent unsteady aerodynamics.
- Add the capability to solve load equations.
- Allow the use of truncated modes without including residual flexibility approximations for the deleted modes when performing dynamic gust loads analyses.

Implementation of these modifications or additions to the FLEXSTAB CPS will make it suitable for dynamic gust loads provided that, if a SAS system is included, it primarily influences only the low-frequency rigid body airplane response. However, the Küssner and Wagner indicial lift growth function representation of unsteady aerodynamics is not satisfactory for analysis of aircraft where mode suppression or flutter suppression with active controls is required. This type of analysis requires accurate phase and magnitude relationships between modes, which requires a more exact unsteady aerodynamic and structural representation as outlined in reference 2. The advantage of the analysis system outlined in reference 2 over this system is that it is satisfactory for dynamic gust loads analyses employing techniques such as the Küssner and Wagner unsteady aerodynamic approach and for both dynamic gust loads and active control analysis using more exact theories. It thus appears that implementation of the system proposed in reference 2 is a much more effective approach than would result from implementation of the required changes determined from this study.

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APPENDIX A

EQUATIONS OF MOTION INCLUDING THE WAGNER INDICIAL LIFT GROWTH FUNCTION IN STATE FORM

When an unsteady aerodynamic representation using indicial lift growth functions is considered, the dynamic equations of motion for an aircraft may be written in the form:

$$[M_1]\{q\} + [M_2]\{\dot{q}\} + [M_3]\{\ddot{q}\} + [M_4]\{\dot{q}\}*\phi + [M_5]\{\ddot{q}\}*\phi = \{C_3\}\dot{\alpha}_g*\psi \qquad \text{(A-1)}$$

where

$[M_1]$ = generalized structural stiffness matrix

$[M_2]$ = generalized damping matrix

$[M_3]$ = generalized inertia matrix

$[M_4]$ = generalized aerodynamic stiffness matrix

$[M_5]$ = generalized aerodynamic damping matrix

$\{q\}$ = generalized coordinates

$\{C_3\}$ = gust forcing matrix

α_g = gust angle

ϕ = Wagner indicial lift growth function

ψ = Küssner indicial lift growth function

$*$ = indicates convolution

$\dot{q}*\phi$ = $q(o)\phi + \int_0^t \dot{q}(\tau)\,\phi\,(t-\tau)d\tau$

Assume initial conditions,

$$q(o) = \dot{q}(o) = \alpha_g(o) = 0$$

Since

$$\mathcal{L}[q(t)]\ \mathcal{L}[F(t)] = \mathcal{L}[\int_0^t q(\tau)\,F(t-\tau)d\tau]$$

Then the Laplace transform of equation (A-1) is:

$$\left[M_1 + sM_2 + s^2M_3 + (M_4 + sM_5)\, s \cdot \bar{\phi}(s)\right]\left\{\bar{q}(s)\right\} = \left\{C_3\right\} \cdot \bar{\alpha}_g(s) \cdot s \cdot \bar{\psi}(s) \qquad \text{(A-2)}$$

where $\bar{\phi}(s) = \mathcal{L}\left[\phi\right]$, etc.

Consider an approximation to the Wagner indicial lift growth function having the form:

$$\phi = a_1 - b_1 e^{-\alpha_1 t} - c_1 e^{-\beta_1 t}$$

then

$$\bar{\phi}(s) = a_1/s - b_1/(s + \alpha_1) - c_1/(s + \beta_1)$$

$$s \cdot \bar{\phi}(s) = a_1 - s \cdot b_1/(s + \alpha_1) - s \cdot c_1/(s + \beta_1)$$

Define vectors $\bar{r}$ to write the Wagner indicial function in state form

$$\begin{aligned} a_1\left\{\bar{r}_1(s)\right\} &= b_1/(s + \alpha_1)\left\{\bar{q}(s)\right\} \\ a_1\left\{\bar{r}_2(s)\right\} &= c_1/(s + \beta_1)\left\{\bar{q}(s)\right\} \end{aligned} \qquad \text{(A-3)}$$

Substitute equation (A-3) into equation (A-2); then equations describing the system are:

$$\begin{aligned} &\left[M_1 + sM_2 + s^2M_3\right]\left\{\bar{q}(s)\right\} + a_1 \cdot \left[M_4 + sM_5\right]\left\{\bar{q}(s) - \bar{r}_1(s) - \bar{r}_2(s)\right\} \\ &\quad = \left\{F_1\right\} \cdot \bar{\alpha}_g(s) \cdot s \cdot \bar{\psi}(s) \end{aligned} \qquad \text{(A-4)}$$

$$\begin{aligned} s \cdot b_1\left\{\bar{q}(s)\right\} &= a_1 \cdot (s + \alpha_1)\left\{\bar{r}_1(s)\right\} \\ s \cdot c_1\left\{\bar{q}(s)\right\} &= a_1 \cdot (s + \beta_1)\left\{\bar{r}_2(s)\right\} \end{aligned} \qquad \text{(A-5)}$$

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where equation (A-4) represents the equations of motion with the Wagner indicial lift growth function in state form, and equation (A-5) relates the indicial function vectors $\bar{r}_1$ (s) to the generalized coordinates $\bar{q}(s)$.

In expanded matrix form, equations (A-4) and (A-5) become

$$s^2 \begin{bmatrix} M_3 & & \\ & & \\ & & \end{bmatrix} \begin{Bmatrix} \bar{q} \\ \bar{r}_1 \\ \bar{r}_2 \end{Bmatrix} + s \begin{bmatrix} M_2 + M_5 & M_5 & M_5 \\ b_1 & a_1 & \\ c_1 & & a_1 \end{bmatrix} \begin{Bmatrix} \bar{q} \\ \bar{r}_1 \\ \bar{r}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} M_1 + M_4 & M_4 & M_4 \\ & a_1\alpha_1 & \\ & & a_1\beta_1 \end{bmatrix} \begin{Bmatrix} \bar{q} \\ \bar{r}_1 \\ \bar{r}_2 \end{Bmatrix} = \begin{Bmatrix} C_3 \\ 0 \\ 0 \end{Bmatrix} \dot{\alpha}_g * \psi$$

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