A TWO PARAMETER SURVEY OF PERIODIC ORBITS IN THE RESTRICTED PROBLEM OF THREE BODIES

(NASA-CR-146066) A TWO PARAMETER SURVEY OF PERIODIC ORBITS IN THE RESTRICTED PROBLEM OF THREE BODIES (Virginia Univ.) 40 p HC $4.00 CSCL 22C

N76-15219

Unclas

G3/13 08682

Peter J. Shelus

Leander McCormick Observatory, University of Virginia,

Charlottesville, Virginia

and

Physics Branch, NASA Manned Spacecraft Center, Houston, Texas

Abstract

Within the context of the restricted problem of three bodies, we wish to show the effects, caused by varying the mass ratio of the primaries and the eccentricity of their orbits, upon periodic orbits of the infinitesimal mass which are numerical continuations of circular orbits in the ordinary problem of two bodies. A recursive power series technique is used to numerically integrate the equations of motion as well as the first variational equations in order to generate a two parameter family of periodic orbits and identify the linear stability characteristics thereof. Seven such families are investigated with equally spaced mass ratios from 0.0 to 1.0 and eccentricities of the orbits of the primaries in a range 0.0 to 0.6. Stable orbits are associated with large distances of the infinitesimal mass from the perturbing primary, nearly circular motion of the primaries and, to a slightly lesser extent, small mass ratios of the primaries. On the other hand, unstable orbits for the infinitesimal mass are associated with small distances from the perturbing primary, highly elliptic orbits of the primaries and large mass ratios.

I. INTRODUCTION

In the restricted problem of three bodies one concerns himself with the motion of a body, having infinitesimal mass, which is subjected to the gravitational influence of two other bodies, called the primaries, having finite mass. The assumption is made that the body of infinitesimal mass exerts no significant gravitational forces upon the two primaries but is only acted upon by them.

The case where the primaries describe circular orbits about their center of mass is usually referred to as the Circular Restricted Problem of Three Bodies and has been studied by many researchers in great detail. Szebehely (1967) provides an excellent text and an extensive reference source for most of the work done through 1966 concerning this problem. Where the primaries describe orbits which are ellipses, the so-called Elliptic Restricted Problem of Three Bodies, research has not nearly been so extensive. See, for example, Szebehely and Giacaglia (1964), Danby (1964a), Bennett (1965), Lanzano (1967), and Broucke (1969). Junqueira (Junqueira and Greene, 1970) has been treating the restricted problem where the motions of the primaries are completely general.

It is the purpose of this paper to present a two-parameter survey of periodic orbits in the restricted problem of three bodies where the primaries are constrained to move on circles or ellipses, and the body of infinitesimal mass has its motion contained within the plane defined by those orbits. In particular, we wish to show the effects caused by varying the parameters m', the mass ratio of the primaries, and e, the eccentricity of the orbit of the primaries, upon periodic orbits of the first kind (Poincare, 1892), i.e., periodic orbits of the body of infinitesimal mass which are numerical continuations of circular orbits in the ordinary problem of two bodies.

Basically, this represents an extension to an earlier paper by Shelus and Kumar (1970), which will hereafter be referred to as Paper A. Since the time of that report, mathematical computations have been improved to eliminate the several inconsistancies which were cited in Paper A. Also, the scope of that previous study has been greatly expanded.

II. EQUATIONS OF MOTION

The equations of motion for the body of infinitesimal mass in the planar elliptic restricted problem of three bodies (Scheibner, 1866; Nechvile, 1926; Szebehely, 1967) are usually presented in the following form:

$$x'' - 2y' = (1 + e \cos v)^{-1} \Omega_x, \qquad \text{(II-1a)}$$

$$y'' + 2x' = (1 + e \cos v)^{-1} \Omega_y, \qquad \text{(II-1b)}$$

where

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{[(x+\mu)^2+y^2]^{1/2}} + \frac{\mu}{[(x-1+\mu)^2+y^2]^{1/2}} . \qquad \text{(II-2)}$$

Dimensionless units are chosen such that the unit of distance is the variable separation between the primaries, the unit of mass is the sum of the masses of the primaries (where μ is taken to be the mass of the less massive of the two primaries, i.e., $\mu \leq \frac{1}{2}$), and the unit of time is picked so that the constant of gravitation will be unity. Primes denote differentiation with respect to the true anomaly, v, of the orbit of the primaries, and e is the eccentricity of that orbit. The subscripts x and y of equations (II-1a,b) represent differentiation of Ω with respect to the indicated variable. Finally, the orthogonal axes (x,y) are chosen so that the primaries are located on the x-axis when $v = 0$.

It should be recognized that the equations (II-1a,b) are defined for a pulsating, nonuniformly rotating coordinate system, where the two primaries occupy fixed positions on the x-axis. The origin of coordinates is at the center of mass of the two primaries and the larger of the two masses is to the left of the origin. That is to say, the coordinates of the mass $1-\mu$ is $(-\mu, 0)$ and those of the mass μ is $(1-\mu, 0)$. Also, when $e = 0$, i.e., when the primaries describe circular orbits, the true anomaly of the orbit of the primaries is identical to the dimensionless time, and the equations cited reduce to those which are commonly used in the planar circular restricted problem of three bodies. Therefore, only one set of equations is needed to describe both the circular and the elliptic restricted problems.

At this time one should also note the fact that the equations of motion for the elliptic restricted problem do not admit the Jacobian integral (Ovenden and Roy, 1960) which has been used so extensively in studies of the circular problem.

III. VARIATIONAL EQUATIONS

In general, the equations of motion (II-1a,b) of the previous section can always be represented by the following system of first order differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n, t) \quad i=1,2,\ldots,n. \qquad \text{(III-1)}$$

The symbol t is used to represent the independent variable instead of v, which was used in Section II. In the elliptic restricted problem, the f_i contain only periodic functions of t.

Let us suppose that the equations (III-1) have periodic solutions:

$$x_i = \phi_i(t) = \phi_i(t+T) \qquad \text{(III-2)}$$

The period T, of course, does not have a uniquely determined value since any integral multiple of T is also a period of the solutions. By allowing

$$x_i = \phi_i(t) + \xi_i \qquad \text{(III-3)}$$

to be a solution differing only slightly from the known periodic solution, we can expand (III-3) in a Taylor's series, and neglecting the squares and higher powers of ξ_i, form the so-called first variational equations of the system (III-1), i.e.,

$$\frac{d\xi_i}{dt} = \frac{\partial f_i}{\partial x_1}\xi_1 + \frac{\partial f_i}{\partial x_2}\xi_2 + \ldots + \frac{\partial f_i}{\partial x_n}\xi_n. \qquad \text{(III-4)}$$

Each of the partial derivatives $\frac{\partial f_i}{\partial x_j}$ is evaluated along the reference solution (III-2).

By the theory of Floquet (Danby, 1964a), the general solution to equations of the form (III-4) is

$$\xi_i = \sum_{j=1}^{n} S_{i,j}(t)\, e^{\alpha_j t} \qquad \text{(III-5)}$$

where the $S_{i,j}$ are periodic functions of t with period T. The α_j's are referred to as characteristic exponents and the $e^{\alpha_j t}$'s are characteristic roots. If all of the characteristic exponents of (III-5) are pure imaginary, the ξ_i can be expressed as the sums and products of purely periodic terms and they will remain finite for all values of t. When there is a real part in any one of the exponents, the ξ_i will become unbounded with t. It should be noted that if all of the characteristic exponents were to have negative real parts, the ξ_i would tend to zero as t became large. However, this cannot happen for Hamiltonian systems since it can be shown (Pollard, 1966) that for those types of systems, if β is a characteristic root, then so must $-\beta$.

Following the treatment by Wintner (1947), the variational equations (III-4) can be expressed in matrix form, i.e.,

$$\dot{\xi}_i = A(t)\, \xi_i, \qquad \text{(III-6)}$$

where A is a matrix periodic in t and the ξ_i and the $\dot{\xi}_i$ are each column vectors. Dots represent differentiation with respect to the independent variable, t. This set of n differential equations will be solved when any set of n linearly independent solutions are known. If each of these n solutions be a column of a matrix X(t), which is called a fundamental matrix or matrizant (Danby, 1964b), it is clear that

$$\dot{X}(t) = A(t)X(t) \qquad \text{(III-7)}$$

Also, any linear combination of the columns of the fundamental matrix is also a solution. That is, Z(t) is also a fundamental matrix if

$$Z(t) = X(t)C \qquad \text{(III-8)}$$

where C is some constant n x n matrix such that det $C \neq 0$.

In particular, let us define one special fundamental matrix, i.e.,

$$\Omega(t_0,t) = X(t)\left[X(t_0)\right]^{-1} \qquad \text{(III-9)}$$

Note that $\Omega(t_0,t_0)$ is the identity matrix I. Setting $X(t_0) = I$, the components of Ω may be found by ordinary numerical integration and

$$X(t) = \Omega(t_0,t)X(t_0). \qquad \text{(III-10)}$$

The unique matrix $\Omega(t_0,t_0+T)$ is called the mondromy matrix of the fundamental matrix X(t) with reference to the given period, T, of A.

The characteristic roots, $e^{\alpha_j T}$, of equation (III-5) are the roots of the characteristic equation

$$|\Omega(t_0,t_0+T) - sI| = 0 \qquad \text{(III-11)}$$

i.e., they are the eigenvalues of the monodromy matrix $\Omega(t_0,t_0+T)$. The characteristic exponents are then

$$\alpha_i = \frac{1}{T}\log s_i = \frac{1}{T}[\log |s_i| + i \arg(s_i+2k\pi)]. \qquad \text{(III-12)}$$

Thus if $|s_i| = 1$, then $\log |s_i| = 0$ and α_i will be pure imaginary. The ξ_i will therefore be bounded if all of the characteristic roots of (III-5) be located on the unit circle in the complex plane.

In keeping with the many other studies which deal with the restricted problem of three bodies, we shall use the characteristic roots and the characteristic exponents, as defined above, to classify the periodic orbits which we have computed. Given a periodic orbit in the restricted problem, if all four characteristic roots be located on the unit circle in the complex plane, the orbit will be considered to be of the stable type, i.e., the ξ_i are bounded, in a linear sense. In any other case, the orbit will be considered to be of the unstable type.

Now, returning specifically to the elliptic restricted problem, if we let $\xi_1 = \Delta x$, $\xi_2 = \Delta y$, $\xi_3 = \Delta x'$, and $\xi_4 = \Delta y'$ represent the slight variations from a known periodic solution, we can obtain, as the first variational equations of the system (II-1a,b), the following expressions:

$$\frac{d(\Delta x)}{dt} = \Delta x' \quad \text{(III-13a)}$$

$$\frac{d(\Delta y)}{dt} = \Delta y' \quad \text{(III-13b)}$$

$$\frac{d(\Delta x')}{dt} = 2(\Delta y') + (1 + e \cos v)^{-1} [\ell(\Delta x) + m(\Delta y)] \quad \text{(III-13c)}$$

$$\frac{d(\Delta y')}{dt} = -2(\Delta x') + (1 + e \cos v)^{-1} [m(\Delta x) + n(\Delta y)] \quad \text{(III-13d)}$$

where

$$\ell = 1 - \frac{1-\mu}{[(x+\mu)^2+y^2]^{3/2}} \left[1 - 3\frac{(x+\mu)^2}{(x+\mu)^2+y^2}\right] - \frac{\mu}{[(x+\mu-1)^2+y^2]^{3/2}} \left[1 - 3\frac{(x+\mu-1)^2}{(x+\mu-1)^2+y^2}\right] \quad \text{(III-14a)}$$

$$m = 3\frac{1-\mu}{[(x+\mu)^2+y^2]^{3/2}} \frac{(x+\mu)y}{(x+\mu)^2+y^2} + 3\frac{\mu}{[(x+\mu-1)^2+y^2]^{3/2}} \frac{(x+\mu-1)y}{(x+\mu-1)^2+y^2} \quad \text{(III-14b)}$$

$$n = 1 - \frac{1-\mu}{[(x+\mu)^2+y^2]^{3/2}} \left[1 - 3\frac{y^2}{(x+\mu)^2+y^2}\right] - \frac{\mu}{[(x+\mu-1)^2+y^2]^{3/2}} \left[1 - 3\frac{y^2}{(x+\mu-1)^2+y^2}\right]. \quad \text{(III-14c)}$$

IV. RESULTS

In this study all numerical integrations are performed using a recursive power series technique (Steffensen, 1956; Rabe, 1961; Deprit and Price, 1965; Fehlberg, 1966; Broucke, 1969) as applied to both the equations of motion, (II-1a,b), and the first variational equations, (III-13a,b,c,d). It is usual to assume that the sum of all neglected terms in each series thus computed will be smaller in magnitude than the first term omitted and we have obtained very good results by truncating these series after twenty terms. The last few terms of each series are interrogated to provide a criterion for integration step size. The computations were performed in double precision on a Univac 1108 system at the NASA Manned Spacecraft Center.

The technique which is used to develope the two parameter survey of periodic orbits is a simple and straightforward one and can be outlined in the following manner. As a starting point, one selects a circular two-body orbit for the body of infinitesimal mass about one of the primaries, with a period equal to some integral ratio of the orbital period of the primaries. Essentially, this is a periodic orbit in the circular restricted problem such that $\mu = 0$. The initial conditions of this first orbit are then used as initial conditions for an orbit when μ is now, say, 0.01. This new orbit will, of course, not be periodic under these new conditions; however, it will not be very far from being so.

We use a linear differential correction technique similar to that presented by Deprit and Price (1965) to obtain improved initial conditions. A linearly improved solution can, in general, be specified by the following Taylor's expansion:

$$(x+\Delta x)_t = x_t + \left(\frac{\partial x}{\partial x_0}\right)_t \Delta x_0 + \left(\frac{\partial x}{\partial y_0}\right)_t \Delta y_0 + \left(\frac{\partial x}{\partial \dot{x}_0}\right)_t \Delta \dot{x}_0 + \left(\frac{\partial x}{\partial \dot{y}_0}\right)_t \Delta \dot{y}_0 \quad \text{(IV-1a)}$$

$$(y+\Delta y)_t = y_t + \left(\frac{\partial y}{\partial x_0}\right)_t \Delta x_0 + \left(\frac{\partial y}{\partial y_0}\right)_t \Delta y_0 + \left(\frac{\partial y}{\partial \dot{x}_0}\right)_t \Delta \dot{x}_0 + \left(\frac{\partial y}{\partial \dot{y}_0}\right)_t \Delta \dot{y}_0 \quad \text{(IV-1b)}$$

$$(\dot{x}+\Delta \dot{x})_t = \dot{x}_t + \left(\frac{\partial \dot{x}}{\partial x_0}\right)_t \Delta x_0 + \left(\frac{\partial \dot{x}}{\partial y_0}\right)_t \Delta y_0 + \left(\frac{\partial \dot{x}}{\partial \dot{x}_0}\right)_t \Delta \dot{x}_0 + \left(\frac{\partial \dot{x}}{\partial \dot{y}_0}\right)_t \Delta \dot{y}_0 \quad \text{(IV-1c)}$$

$$(\dot{y}+\Delta \dot{y})_t = \dot{y}_t + \left(\frac{\partial \dot{y}}{\partial x_0}\right)_t \Delta x_0 + \left(\frac{\partial \dot{y}}{\partial y_0}\right)_t \Delta y_0 + \left(\frac{\partial \dot{y}}{\partial \dot{x}_0}\right)_t \Delta \dot{x}_0 + \left(\frac{\partial \dot{y}}{\partial \dot{y}_0}\right)_t \Delta \dot{y}_0 . \quad \text{(IV-1d)}$$

We desire periodicity, i.e., we require that $(x+\Delta x)_T = (x+\Delta x)_0$, $(y+\Delta y)_T = (y+\Delta y)_0$, $(\dot{x}+\Delta\dot{x})_T = (\dot{x}+\Delta\dot{x})_0$, and $(\dot{y}+\Delta\dot{y})_T = (\dot{y}+\Delta\dot{y})_0$, and therefore the following equations can be formed:

$$x_0 - x_T = \left[\left(\frac{\partial x}{\partial x_0}\right)_T - 1\right]\Delta x_0 + \left(\frac{\partial x}{\partial y_0}\right)_T \Delta y_0 + \left(\frac{\partial x}{\partial \dot{x}_0}\right)_T \Delta \dot{x}_0 + \left(\frac{\partial x}{\partial \dot{y}_0}\right)_T \Delta \dot{y}_0 \quad \text{(IV-2a)}$$

$$y_0 - y_T = \left(\frac{\partial y}{\partial x_0}\right)_T \Delta x_0 + \left[\left(\frac{\partial y}{\partial y_0}\right)_T - 1\right]\Delta y_0 + \left(\frac{\partial y}{\partial \dot{x}_0}\right)_T \Delta \dot{x}_0 + \left(\frac{\partial y}{\partial \dot{y}_0}\right)_T \Delta \dot{y}_0 \quad \text{(IV-2b)}$$

$$\dot{x}_0 - \dot{x}_T = \left(\frac{\partial \dot{x}}{\partial x_0}\right)_T \Delta x_0 + \left(\frac{\partial \dot{x}}{\partial y_0}\right)_T \Delta y_0 + \left[\left(\frac{\partial \dot{x}}{\partial \dot{x}_0}\right)_T - 1\right]\Delta \dot{x}_0 + \left(\frac{\partial \dot{x}}{\partial \dot{y}_0}\right)_T \Delta \dot{y}_0 \quad \text{(IV-2c)}$$

$$\dot{y}_0 - \dot{y}_T = \left(\frac{\partial \dot{y}}{\partial x_0}\right)_T \Delta x_0 + \left(\frac{\partial \dot{y}}{\partial y_0}\right)_T \Delta y_0 + \left(\frac{\partial \dot{y}}{\partial \dot{x}_0}\right)_T \Delta \dot{x}_0 + \left[\left(\frac{\partial \dot{y}}{\partial \dot{y}_0}\right)_T - 1\right]\Delta \dot{y}_0 \quad \text{(IV-2d)}$$

where each of the partial derivatives are obtained through the numerical integration of the first variational equations. These equations, (IV-2a,b,c,d), can be solved to provide the corrections which are to be applied to the previous initial conditions and the process can be repeated in an iterative way until the initial conditions of the periodic orbit are reproduced to the desired tolerance. For all orbits which are presented here, initial conditions after one period are reproduced to at least 1 part in the twelfth decimal place. Of course, at the same time, we are able to classify this new periodic orbit as being "stable" or "unstable" using the characteristic exponents as described in the previous section. The entire procedure can then be repeated where μ is now increased to, say, 0.02. In this manner periodic orbits are obtained for increasing values of the mass ratio of the primaries. Remember that all orbits so far described are for the circular restricted problem, i.e., $e = 0$.

At any point in the above procedure, after a periodic orbit has been computed, we can hold the mass ratio constant and, instead, increase the eccentricity of the orbit of the primaries from 0.0 to 0.01, say. The same iteration procedure is then invoked and the result is a periodic orbit in the elliptic restricted problem with $e = 0.01$. The eccentricity of the orbit of the primaries can then be increased to 0.02 and the process again repeated. Thus, starting from a circular orbit in the circular restricted problem with $\mu = 0$, we can move along increasing eccentricity of the orbit of the primaries, or increasing μ, to any desired value in this step-by-step iterative manner.

In progressing along either e or μ, once a number of orbits has been computed, it is possible to extrapolate to better approximations of the initial conditions of the next orbit by making use of the initial conditions of previous orbits. As Broucke (1969) has pointed out, this has allowed for a pronounced reduction in the number of iterations which are required to arrive at the next orbit. A different two parameter "family" of periodic orbits is produced by starting from the very beginning of the procedure with another circular two-body orbit and then proceding in a similar way.

This report presents the findings for several of these two parameter families of periodic orbits in the restricted problem of three bodies. It is convenient to divide these families into two groups, i.e., one group of orbits which enclose both of the primaries and a second group of orbits which enclose only one of the primaries (usually the more massive of the two). We shall discuss each of these two groups separately.

a. Orbits encircling both primaries

For the cases where the orbits of the infinitesimal mass encircle both of the primaries, four distinct families have been generated having periods of 4π, 6π, 8π, and 10π, which correspond to periods that are two, three, four, and five times, respectively, the period of the primaries. Therefore, these families have been identified by the ratios 2/1, 3/1, 4/1, and 5/1. Within each family, a particular periodic orbit is identified by e, the eccentricity of the orbit of the primaries, and μ, the mass of m_2, i.e., the primary which is located to the right of the origin. The "stable-unstable" analysis is summarized in Tables I, II, III, and IV wherein the letter S signifies that an orbit is "stable", and U, "unstable". Typical orbits are plotted in Figures 1, 2, 3, and 4; for clarity, these illustrations depict the orbits in a barycentric, inertial coordinate system, not in the rotating-pulsating system which has been used for the numerical integrations.

All of the orbits which were computed for the two families identified by the ratios 2/1 and 3/1 are of the unstable type, in the sense which has been defined in Section III. These two families evolve from circular orbits in the ordinary problem of two bodies which have radii of approximately 1.587... and 2.080..., respectively. To illustrate the nature of these two families of orbits we have deemed it necessary only to compute orbits for mass ratios up to 1/3 (μ <0.25) and eccentricities of the orbits of the primaries in a range 0.0 to 0.1.

The situation is somewhat more interesting when we consider the family of orbits identified by the ratio 4/1. These orbits have evolved from a two-body circular orbit having a radius of approximately 2.519... . They are located farther from the binary system than either of the first two families discussed and orbits have been computed for the full range of mass ratios, i.e., 0.0 to 1.0. For small values of the eccentricity of the orbits of the primaries, the orbits of the infinitesimal mass are linearly stable for all mass ratios. At eccentricity 0.02, those orbits computed for mass ratios of the primaries in excess of about 1/3 ($\mu > 0.25$) prove to be of the unstable type. For an eccentricity of the orbit of the primaries of 0.03, only the orbit for a mass ratio 1/99 ($\mu = 0.01$) is of the stable type. All other orbits which have been computed for this family are linearly unstable.

Finally, we have the family of orbits identified by the ratio 5/1. The orbits of this family have evolved from a circular two-body orbit having a radius of approximately 2.924... . All of these orbits are of the stable type. The full range of mass ratios was considered and the largest value of the eccentricity of the orbit of the primaries was 0.09. It is hypothesized that these orbits will eventually evolve into linearly unstable orbits as the eccentricity is increased further, however, some difficulty has been met in numerically continuing this family of orbits and further work along these lines is required.

b. Orbits encircling one primary

Proceeding now to the cases where the motion of the body of infinitesimal mass encircles only one of the primaries, we are confronted with a slightly different situation. Because it is necessary that the orbital period of the infinitesimal mass be restricted to that which is an integral ratio of the orbital period of the primaries, the body of infinitesimal mass must circulate about one of the primaries (while being perturbed by the other) some integral numbers of times during one orbital period of the primaries. Three such families have been generated having, of course, periods of 2π. We have identified them by the ratios 1/12, 1/6, and 1/3, i.e., the infinitesimal body will complete twelve, six, and three circuits, respectively, about its primary while the primaries complet one circuit.

The "stable-unstable" results for these three families are summarized in Tables V, VI, and VII, where, again, the letter S signifies a "stable" orbit and U an "unstable" one. Typical orbits are plotted in Figures 5, 6, and 7. These illustrations depict the orbits in an inertial system, however, unlike the plots for the previous four families of orbits, the origins here are at the primary rather than at the center of mass.

The orbits of the family identified by the ratio 1/12 evolve from a circular two-body orbit of approximate radius 0.19078..., such that the body of infinitesimal mass encircles its primary exactly twelve times during 2π units of dimensionless time. As is expected, these orbits are stable for the entire range of mass ratios when the eccentricity of the orbit of the primaries is small. Only when this eccentricity reaches 0.49 do some orbits (for large mass ratios of the primaries) become unstable. As the eccentricity continues to increase, orbits become unstable for smaller mass ratios until, at e = 0.53, all mass ratios of the primaries produce unstable orbits for the body of infinitesimal mass.

Continuing to the family of orbits identified by the ratio 1/6, we see that these orbits evolve from a two-body circular orbit of radius 0.30285..., with the infinitesimal mass now encircling its primary exactly six times in 2π units of time. In this case, orbits become unstable at smaller orbital eccentricities (e = 0.17) of the primaries than the previous family discussed, i.e., compare Tables V and VI. Also, note that the dependence of instability upon mass ratio is far more gradual, such that it is only at e = 0.23 that orbits of the infinitesimal mass become unstable for all mass ratios of the primaries.

To conclude this survey we now consider the family of orbits identified by the ratio 1/3. The orbits of this family have evolved from a circular two-body orbit of radius 0.48074... . The body of infinitesimal mass encircles its primary only three times in a period of 2π units of time. All orbits which have been computed prove to be of the unstable type, except for a few cases. One group of orbits, for μ = 0.23 through μ = 0.5, are stable for e = 0.0; these orbits become unstable as e is increased to 0.02 or greater. Another series of orbits, for e = 0.59 to 0.65, are stable for μ = 0.01 only, and all become unstable when μ is increased to 0.02 or more. Some difficulties in obtaining convergence to orbits for large mass ratios and high eccentricities were encountered for this family. These cases are identified in Table VII by the symbol "?".

V. DISCUSSION

The application of the circular restricted problem of three bodies to Solar System dynamics has been most important. Very obvious, of course, has been the study of: 1) the equilibrium points L_4 and L_5 in the Sun-Jupiter-Trojan asteroid configuration; 2) the motion of the Earth-Moon system as perturbed by the Sun; and 3) more recently, the motion of an artificial space vehicle in the Earth-Moon neighborhood. Somewhat less frequently, see for example Huang and Struve (1956) and Kopal (1959), the circular restricted problem has been applied to the dynamics and the evolution of close binary systems.

In the study which has been begun in Paper A and which has been continued here, the philosophy for generating periodic orbits in the restricted problem of three bodies is perhaps somewhat different than that normally assumed in work of this type. We wish to consider this survey of orbits in a discussion of the problem concerning the existence of low mass Earth-like objects in the vicinity of double stars. The components of binary star systems generally describe noncircular orbits and their mass ratios exceed those of Jupiter-Sun ($\sim$0.001) and Moon-Earth ($\sim$0.01). It has been pointed out by Kumar (1967) that it is unlikely that Earth-like objects will survive over long periods of time within such binary star systems; a small Earth-like object may survive in the vicinity of a double star only if it is located close to one or to the other of the two components, or, if it is located far from both stars.

These remarks must also apply to double star systems in which one of the components is a "dark" companion. Table VIII gives a summary of the observed data for these systems. Note the highly elliptic orbits and the large mass ratios of the components. The dark companions in these systems have been identified with stars of very low mass which have become completely degenerate objects, or black dwarfs, without going through normal stellar evolution (Kumar, 1963; 1967). As has been remarked, small Earth-like objects may not survive over long periods of time in systems such as Lalande 21185, unless they are located in very special positions relative to the two primary components.

The results of this survey has at least offered first order support for this hypothesis, to the extent that the dynamics of small masses within binary systems can be satisfactorily modeled by the restricted problem. Seven families have, so far, been obtained. Each has evolved from a circular orbit in the ordinary problem of two bodies, and each orbit encloses one or both primaries. Stable orbits tend to be associated with large distances of the infinitesimal mass from the perturbing primary, nearly circular motion of the primaries and, to a slightly lesser extent, small mass ratios of the primaries. Unstable orbits tend, on the other hand, to be associated with small distances from the perturbing primary, highly elliptic orbits and large mass ratios.

These results, then, are in keeping with the hypothesis (at least in a linear sense) that low mass objects in binary systems will be in "stable" orbits when they are located far from both primary components of the system or when they are located very near to one of the primaries, provided that the eccentricity of the orbit of the primaries is not too large.

ACKNOWLEDGEMENTS

This research has been conducted at the NASA Manned Spacecraft Center under their Resident Academic Program. Financial support is provided by a postdoctoral research associateship from the University of Virginia. My thanks go to Dr. Yoji Kondo and the other members within the Astrophysics Section at NASA/MSC for their hospitality. Also, I wish to acknowlege Dr. Laurence W. Fredrick and Dr. Shiv S. Kumar of the University of Virginia for their continuing interest and support in this project.

REFERENCES

Bennett, A. 1965, Icarus, 4, 177.

Broucke, R. 1969, AIAA Jour., 7, 1003.

Danby, J. M. A. 1964a, Astron. J., 69, 165.

______________ 1964b, AIAA Jour., 2, 16.

Deprit, A. and Price, J. F. 1965, Astron. J., 70, 836.

Fehlberg, E. 1966, NASA TR R-248.

Junqueira, J. L. and Greene, R. W. 1970, Bul. AAS, 2, 247.

Lanzano, P. 1967, Icarus, 6, 114.

Nechvile, V. 1926, Compt. Rend., 182, 310.

Ovenden, M. and Roy, A. E. 1960, Mon. Not. RAS, 123, 1

Poincare, H. 1892, Les Methodes Nouvelles de la Mecanique Celeste, Vol. 1, Gauthier-Villars, Paris.

Pollard, H. 1966, Mathematical Introduction to Celestial Mechanics, Prentice Hall, Englewood, N. J.

Rabe, E. 1961, Astron. J., 66, 500.

Scheibner, W. 1866, Crelle J. Reine Angew. Math., 65, 291.

Shelus, Peter J. and Kumar, S. S. 1970, Astron. J., 75, 315.

Steffenson, J. F. 1956, Mat. Fys. Medd. Dan. Ved. Selsk., 30, No. 18.

Szebehely, V. 1967, Theory of Orbits, Academic Press, New York.

Szebehely, V. and Giacaglia, G. E. O. 1964, Astron. J., 69, 230.

Wintner, A. 1947, The Analytical Foundations of Celestial Mechanics, Princeton Univ. Press, Princeton, N. J.

TABLE I

Linear stability characteristics for orbits of family 2/1

μ \ e	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.1
.01	U	U	U	U	U	U	U	U	U	U	U
.02	U	U	U	U	U	U	U	U	U	U	U
.03	U	U	U	U	U	U	U	U	U	U	U
.04	U	U	U	U	U	U	U	U	U	U	U
.05	U	U	U	U	U	U	U	U	U	U	U
.06	U	U	U	U	U	U	U	U	U	U	U
.07	U	U	U	U	U	U	U	U	U	U	U
.08	U	U	U	U	U	U	U	U	U	U	U
.09	U	U	U	U	U	U	U	U	U	U	U
.1	U	U	U	U	U	U	U	U	U	U	U
.11	U	U	U	U	U	U	U	U	U	U	U
.12	U	U	U	U	U	U	U	U	U	U	U
.13	U	U	U	U	U	U	U	U	U	U	U
.14	U	U	U	U	U	U	U	U	U	U	U
.15	U	U	U	U	U	U	U	U	U	U	U
.16	U	U	U	U	U	U	U	U	U	U	U
.17	U	U	U	U	U	U	U	U	U	U	U
.18	U	U	U	U	U	U	U	U	U	U	U
.19	U	U	U	U	U	U	U	U	U	U	U
.2	U	U	U	U	U	U	U	U	U	U	U
.21	U	U	U	U	U	U	U	U	U	U	U
.22	U	U	U	U	U	U	U	U	U	U	U
.23	U	U	U	U	U	U	U	U	U	U	U
.24	U	U	U	U	U	U	U	U	U	U	U
.25	U	U	U	U	U	U	U	U	U	U	U

TABLE II

Linear stability characteristics for orbits of family 3/1

μ \ e	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.1
.01	U	U	U	U	U	U	U	U	U	U	U
.02	U	U	U	U	U	U	U	U	U	U	U
.03	U	U	U	U	U	U	U	U	U	U	U
.04	U	U	U	U	U	U	U	U	U	U	U
.05	U	U	U	U	U	U	U	U	U	U	U
.06	U	U	U	U	U	U	U	U	U	U	U
.07	U	U	U	U	U	U	U	U	U	U	U
.08	U	U	U	U	U	U	U	U	U	U	U
.09	U	U	U	U	U	U	U	U	U	U	U
.1	U	U	U	U	U	U	U	U	U	U	U
.11	U	U	U	U	U	U	U	U	U	U	U
.12	U	U	U	U	U	U	U	U	U	U	U
.13	U	U	U	U	U	U	U	U	U	U	U
.14	U	U	U	U	U	U	U	U	U	U	U
.15	U	U	U	U	U	U	U	U	U	U	U
.16	U	U	U	U	U	U	U	U	U	U	U
.17	U	U	U	U	U	U	U	U	U	U	U
.18	U	U	U	U	U	U	U	U	U	U	U
.19	U	U	U	U	U	U	U	U	U	U	U
.2	U	U	U	U	U	U	U	U	U	U	U
.21	U	U	U	U	U	U	U	U	U	U	U
.22	U	U	U	U	U	U	U	U	U	U	U
.23	U	U	U	U	U	U	U	U	U	U	U
.24	U	U	U	U	U	U	U	U	U	U	U
.25	U	U	U	U	U	U	U	U	U	U	U

TABLE III

Linear stability characteristics for orbits of family 4/1

u \ e	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.1
.01	S	S	S	S	U	U	U	U	U	U	U
.02	S	S	S	U	U	U	U	U	U	U	U
.03	S	S	S	U	U	U	U	U	U	U	U
.04	S	S	S	U	U	U	U	U	U	U	U
.05	S	S	S	U	U	U	U	U	U	U	U
.06	S	S	S	U	U	U	U	U	U	U	U
.07	S	S	S	U	U	U	U	U	U	U	U
.08	S	S	S	U	U	U	U	U	U	U	U
.09	S	S	S	U	U	U	U	U	U	U	U
.1	S	S	S	U	U	U	U	U	U	U	U
.11	S	S	S	U	U	U	U	U	U	U	U
.12	S	S	S	U	U	U	U	U	U	U	U
.13	S	S	S	U	U	U	U	U	U	U	U
.14	S	S	S	U	U	U	U	U	U	U	U
.15	S	S	S	U	U	U	U	U	U	U	U
.16	S	S	S	U	U	U	U	U	U	U	U
.17	S	S	S	U	U	U	U	U	U	U	U
.18	S	S	S	U	U	U	U	U	U	U	U
.19	S	S	S	U	U	U	U	U	U	U	U
.2	S	S	S	U	U	U	U	U	U	U	U
.21	S	S	S	U	U	U	U	U	U	U	U
.22	S	S	S	U	U	U	U	U	U	U	U
.23	S	S	S	U	U	U	U	U	U	U	U
.24	S	S	S	U	U	U	U	U	U	U	U
.25	S	S	U	U	U	U	U	U	U	U	U
.26	S	S	U	U	U	U	U	U	U	U	U
.27	S	S	U	U	U	U	U	U	U	U	U
.28	S	S	U	U	U	U	U	U	U	U	U
.29	S	S	U	U	U	U	U	U	U	U	U
.3	S	S	U	U	U	U	U	U	U	U	U
.31	S	S	U	U	U	U	U	U	U	U	U
.32	S	S	U	U	U	U	U	U	U	U	U
.33	S	S	U	U	U	U	U	U	U	U	U
.34	S	S	U	U	U	U	U	U	U	U	U
.35	S	S	U	U	U	U	U	U	U	U	U
.36	S	S	U	U	U	U	U	U	U	U	U
.37	S	S	U	U	U	U	U	U	U	U	U
.38	S	S	U	U	U	U	U	U	U	U	U
.39	S	S	U	U	U	U	U	U	U	U	U
.4	S	S	U	U	U	U	U	U	U	U	U
.41	S	S	U	U	U	U	U	U	U	U	U
.42	S	S	U	U	U	U	U	U	U	U	U
.43	S	S	U	U	U	U	U	U	U	U	U
.44	S	S	U	U	U	U	U	U	U	U	U
.45	S	S	U	U	U	U	U	U	U	U	U
.46	S	S	U	U	U	U	U	U	U	U	U
.47	S	S	U	U	U	U	U	U	U	U	U
.48	S	S	U	U	U	U	U	U	U	U	U
.49	S	S	U	U	U	U	U	U	U	U	U
.5	S	S	U	U	U	U	U	U	U	U	U

TABLE IV.

Linear stability characteristics for orbits of family 5/1.

μ \ e	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.1
.01	S	S	S	S	S	S	S	S	S	S	
.02	S	S	S	S	S	S	S	S	S	S	
.03	S	S	S	S	S	S	S	S	S	S	
.04	S	S	S	S	S	S	S	S	S	S	
.05	S	S	S	S	S	S	S	S	S	S	
.06	S	S	S	S	S	S	S	S	S	S	
.07	S	S	S	S	S	S	S	S	S	S	
.08	S	S	S	S	S	S	S	S	S	S	
.09	S	S	S	S	S	S	S	S	S	S	
.1	S	S	S	S	S	S	S	S	S	S	
.11	S	S	S	S	S	S	S	S	S	S	
.12	S	S	S	S	S	S	S	S	S	S	
.13	S	S	S	S	S	S	S	S	S	S	
.14	S	S	S	S	S	S	S	S	S	S	
.15	S	S	S	S	S	S	S	S	S	S	
.16	S	S	S	S	S	S	S	S	S	S	
.17	S	S	S	S	S	S	S	S	S	S	
.18	S	S	S	S	S	S	S	S	S	S	
.19	S	S	S	S	S	S	S	S	S	S	
.2	S	S	S	S	S	S	S	S	S	S	
.21	S	S	S	S	S	S	S	S	S	S	
.22	S	S	S	S	S	S	S	S	S	S	
.23	S	S	S	S	S	S	S	S	S	S	
.24	S	S	S	S	S	S	S	S	S	S	
.25	S	S	S	S	S	S	S	S	S	S	
.26	S	S	S	S	S	S	S	S	S	S	
.27	S	S	S	S	S	S	S	S	S	S	
.28	S	S	S	S	S	S	S	S	S	S	
.29	S	S	S	S	S	S	S	S	S	S	
.3	S	S	S	S	S	S	S	S	S	S	
.31	S	S	S	S	S	S	S	S	S	S	
.32	S	S	S	S	S	S	S	S	S	S	
.33	S	S	S	S	S	S	S	S	S	S	
.34	S	S	S	S	S	S	S	S	S	S	
.35	S	S	S	S	S	S	S	S	S	S	
.36	S	S	S	S	S	S	S	S	S	S	
.37	S	S	S	S	S	S	S	S	S	S	
.38	S	S	S	S	S	S	S	S	S	S	
.39	S	S	S	S	S	S	S	S	S	S	
.4	S	S	S	S	S	S	S	S	S	S	
.41	S	S	S	S	S	S	S	S	S	S	
.42	S	S	S	S	S	S	S	S	S	S	
.43	S	S	S	S	S	S	S	S	S	S	
.44	S	S	S	S	S	S	S	S	S	S	
.45	S	S	S	S	S	S	S	S	S	S	
.46	S	S	S	S	S	S	S	S	S	S	
.47	S	S	S	S	S	S	S	S	S	S	
.48	S	S	S	S	S	S	S	S	S	S	
.49	S	S	S	S	S	S	S	S	S	S	
.5	S	S	S	S	S	S	S	S	S	S	

TABLE V.

Linear stability characteristics for orbits of family 1/12.

e	.00	.10	.20	.30	.40	.45	.49	.50	.51	.52	.53	.60
.01	S	S	S	S	S	S	S	S	S	S	U	U
.02	S	S	S	S	S	S	S	S	S	S	U	U
.03	S	S	S	S	S	S	S	S	S	S		U
.04	S	S	S	S	S	S	S	S	S	S		U
.05	S	S	S	S	S	S	S	S	S	S		U
.06	S	S	S	S	S	S	S	S	S	S		U
.07	S	S	S	S	S	S	S	S	S	S		U
.08	S	S	S	S	S	S	S	S	S	S		U
.09	S	S	S	S	S	S	S	S	S	S		U
.1	S	S	S	S	S	S	S	S	S	S		U
.11	S	S	S	S	S	S	S	S	S	S		U
.12	S	S	S	S	S	S	S	S	S	S		U
.13	S	S	S	S	S	S	S	S	S	S		U
.14	S	S	S	S	S	S	S	S	S	S		U
.15	S	S	S	S	S	S	S	S	S	S		U
.16	S	S	S	S	S	S	S	S	S	S		U
.17	S	S	S	S	S	S	S	S	S	S		U
.18	S	S	S	S		S	S	S	S	S		U
.19	S	S	S	S		S	S	S	S	U		U
.20	S	S	S	S		S	S	S	S	U		
.21	S	S	S	S		S	S	S	S	U		
.22	S	S	S	S		S	S	S	S			
.23	S	S	S	S		S	S	S	S			
.24	S	S	S	S		S	S	S	S			
.25	S	S	S	S		S	S	S	S			
.26	S	S	S	S		S	S	S	S			
.27	S	S	S	S		S	S	S	S			
.28	S	S	S	S		S	S	S	S			
.29	S	S	S	S		S	S	S	S			
.30	S	S	S	S		S	S	S	S			
.31	S	S	S	S		S	S	S	S			
.32	S	S	S	S		S	S	S	S			
.33	S	S	S	S		S	S	S	S			
.34	S	S	S	S		S	S	S	S			
.35	S	S	S	S		S	S	S	U			
.36	S	S	S	S		S	S	S	U			
.37	S	S	S	S		S	S	S	U			
.38	S	S	S	S		S	S	S	U			
.39	S	S	S	S		S	S	S				
.4	S	S	S	S		S	S	S				
.41	S	S	S	S		S	S	S				
.42	S	S	S	S		S	S	S				
.43	S	S	S	S		S	S	S				
.44	S	S	S	S		S	S	S				
.45	S	S	S	S		S	S	U				
.46	S	S	S	S		S	S	U				
.47	S	S	S	S		S	S	U				
.48	S	S	S	S		S	S	U				
.49	S	S	S	S		S	S	U				
.5	S	S	S	S		S	S	U				

TABLE VI.

Linear stability characteristics for orbits of family 1/6.

μ \ c	.00	.10	.16	.17	.18	.19	.20	.21	.22	.23	.24	.25	.26	.27	.30	.40
.01	S	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U
.02	S	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U
.03	S	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U
.04	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.05	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.06	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.07	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.08	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.09	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.1	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.11	S	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U
.12	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U	U
.13	S	S	S	S	S	S	S	S	S	S	S	U	U	U	U	U
.14	S	S	S	S	S	S	S	S	S	S	S	U	U		U	U
.15	S	S	S	S	S	S	S	S	S	S	S	U			U	U
.16	S	S	S	S	S	S	S	S	S	S	S	U			U	U
.17	S	S	S	S	S	S	S	S	S	S	S	U			U	U
.18	S	S	S	S	S	S	S	S	S	S	U	U			U	U
.19	S	S	S	S	S	S	S	S	S	S	U	U			U	U
.2	S	S	S	S	S	S	S	S	S	S	U				U	U
.21	S	S	S	S	S	S	S	S	S	S	U				U	U
.22	S	S	S	S	S	S	S	S	S	S	U				U	U
.23	S	S	S	S	S	S	S	S	S	U	U				U	U
.24	S	S	S	S	S	S	S	S	S	U	U				U	U
.25	S	S	S	S	S	S	S	S	S	U	U				U	U
.26	S	S	S	S	S	S	S	S	S	U	U				U	U
.27	S	S	S	S	S	S	S	S	U	U	U				U	U
.28	S	S	S	S	S	S	S	S	U	U					U	U
.29	S	S	S	S	S	S	S	S	U	U					U	U
.3	S	S	S	S	S	S	S	S							U	U
.31	S	S	S	S	S	S	S	U							U	U
.32	S	S	S	S	S	S	S	U							U	U
.33	S	S	S	S	S	S	S	U							U	U
.34	S	S	S	S	S	S	S	U							U	U
.35	S	S	S	S	S	S	U								U	U
.36	S	S	S	S	S	S	U								U	U
.37	S	S	S	S	S	S	U								U	U
.38	S	S	S	S	S	S	U								U	U
.39	S	S	S	S	S	S	U								U	U
.4	S	S	S	S	S	U	U								U	U
.41	S	S	S	S	S	U	U								U	U
.42	S	S	S	S	S	U	U								U	U
.43	S	S	S	S	S	U	U								U	U
.44	S	S	S	S	U		U								U	U
.45	S	S	S	S	U		U								U	U
.46	S	S	S	S	U		U								U	U
.47	S	S	S	S			U								U	U
.48	S	S	S	U			U								U	U
.49	S	S	S	U			U								U	U
.5	S	S	S	U			U								U	U

TABLE VII

Linear stability characteristics for orbits of family 1/3

μ \ e	.0	.01	.02	.1	.2	.3	.4	.5	.6
.01	U	U	U	U	U	U	U	U	S
.02	U	U	U	U	U	U	U	U	U
.03	U	U	U	U	U	U	U	U	U
.04	U	U	U	U	U	U	U	U	U
.05	U	U	U	U	U	U	U	U	U
.06	U	U	U	U	U	U	U	U	U
.07	U	U	U	U	U	U	U	U	U
.08	U	U	U	U	U	U	U	U	U
.09	U	U	U	U	U	U	U	U	U
.1	U	U	U	U	U	U	U	U	?
.11	U	U	U	U	U	U	U	U	?
.12	U	U	U	U	U	U	U	U	?
.13	U	U	U	U	U	U	U	U	U
.14	U	U	U	U	U	U	U	U	U
.15	U	U	U	U	U	U	U	U	U
.16	U	U	U	U	U	U	U	U	U
.17	U	U	U	U	U	U	U	U	U
.18	U	U	U	U	U	U	U	U	U
.19	S	U	U	U	U	U	U	U	U
.2	S	U	U	U	U	U	U	U	U
.21	S	U	U	U	U	U	U	U	U
.22	S	U	U	U	U	U	U	U	U
.23	S	U	U	U	U	U	U	U	U
.24	S	U	U	U	U	U	U	U	U
.25	S	U	U	U	U	U	?	U	U
.26	S	U	U	U	U	U	?	U	U
.27	S	U	U	U	U	U	?	U	U
.28	S	U	U	U	U	U	?	U	U
.29	S	U	U	U	U	U	?	U	U
.3	S	U	U	U	U	?	?	U	U
.31	S	U	U	U	U	?	?	U	U
.32	S	U	U	U	U	?	?	U	U
.33	S	U	U	U	U	?	?	U	U
.34	S	U	U	U	U	?	?	U	U
.35	S	U	U	U	U	U	?	U	U
.36	S	U	U	U	U	U	?	U	U
.37	S	U	U	U	U	U	?	U	U
.38	S	U	U	U	U	U	?	U	U
.39	S	U	U	U	U	U	?	U	U
.4	S	U	U	U	U	U	?	U	U
.41	S	U	U	U	U	U	?	U	U
.42	S	U	U	U	U	U	?	U	U
.43	S	U	U	U	U	U	?	?	U
.44	S	U	U	U	U	U	?	?	U
.45	S	U	U	U	U	U	U	?	U
.46	S	U	U	U	U	U	U	?	U
.47	S	U	U	U	U	U	U	?	U
.48	S	U	U	U	U	U	U	?	U
.49	S	U	U	U	U	U	U	?	U
.5	S	U	U	U	U	U	U	?	U

TABLE VIII

Data for systems containing unseen companions

Star	m' *	e **	Reference
BD +20°2465	1/12	0.6	Reuyl (1943)
Lalande 21185	1/35	0.3	Lippincott (1960)
Barnard's star	0.011	0.75	van de Kamp (1969a)
61 Cygni	1/25	0.5	Deutsch (1960)
Ci 2354	0.094	0.9	Lippincott (1967)
BD +6°398	1/13-1/14	0.6	Lippincott (1969)
η Cas	0.044	-	van de Kamp (1969b)
Ci 2347	0.057	0.2	Bieger (1964)

* Mass ratio of secondary to primary.
** Eccentricity of apparent orbit.

FIGURE CAPTIONS

Figure 1. Typical orbits for family 2/1.

Figure 2. Typical orbits for family 3/1.

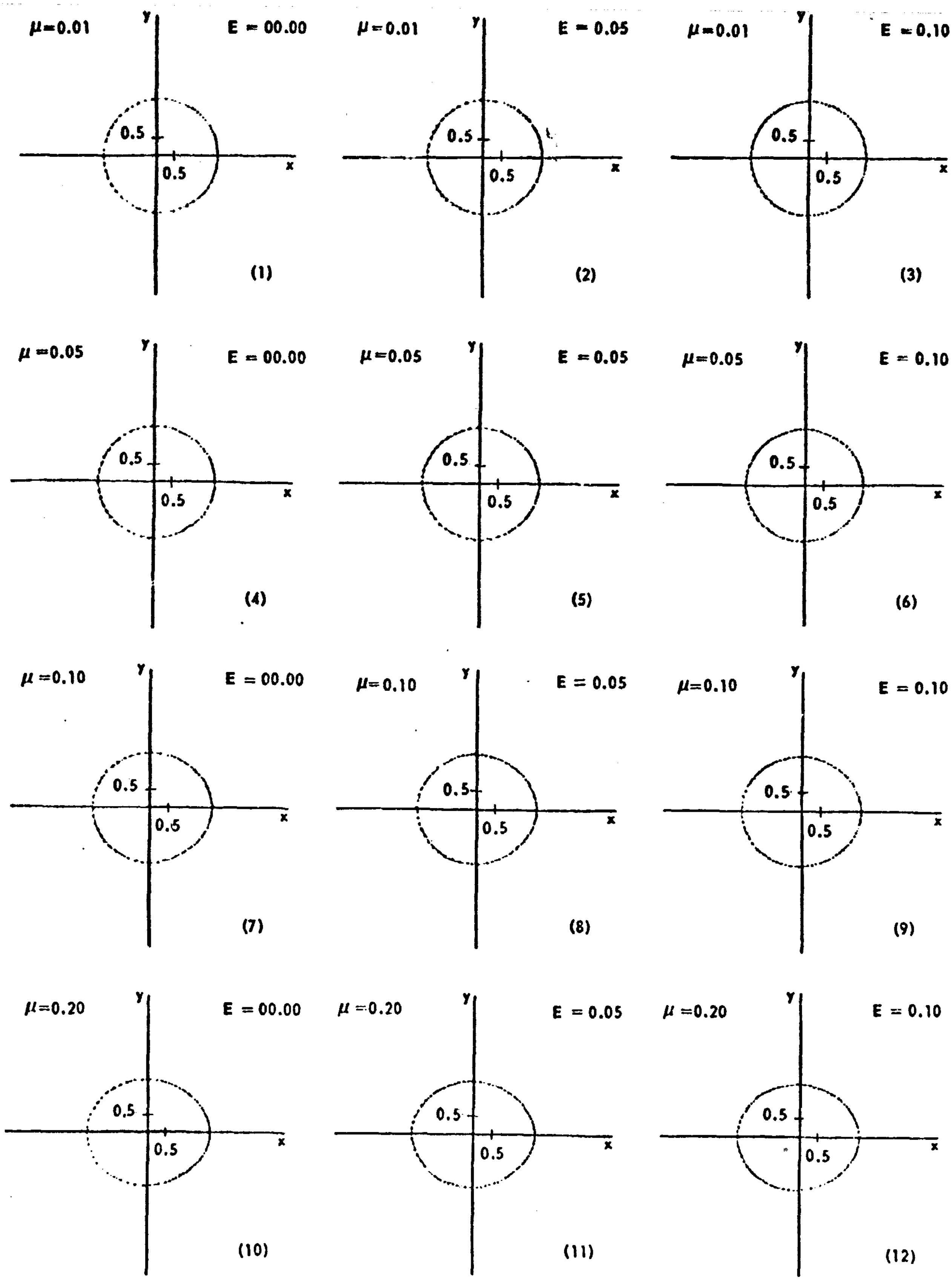
Figure 3. Typical orbits for family 4/1.

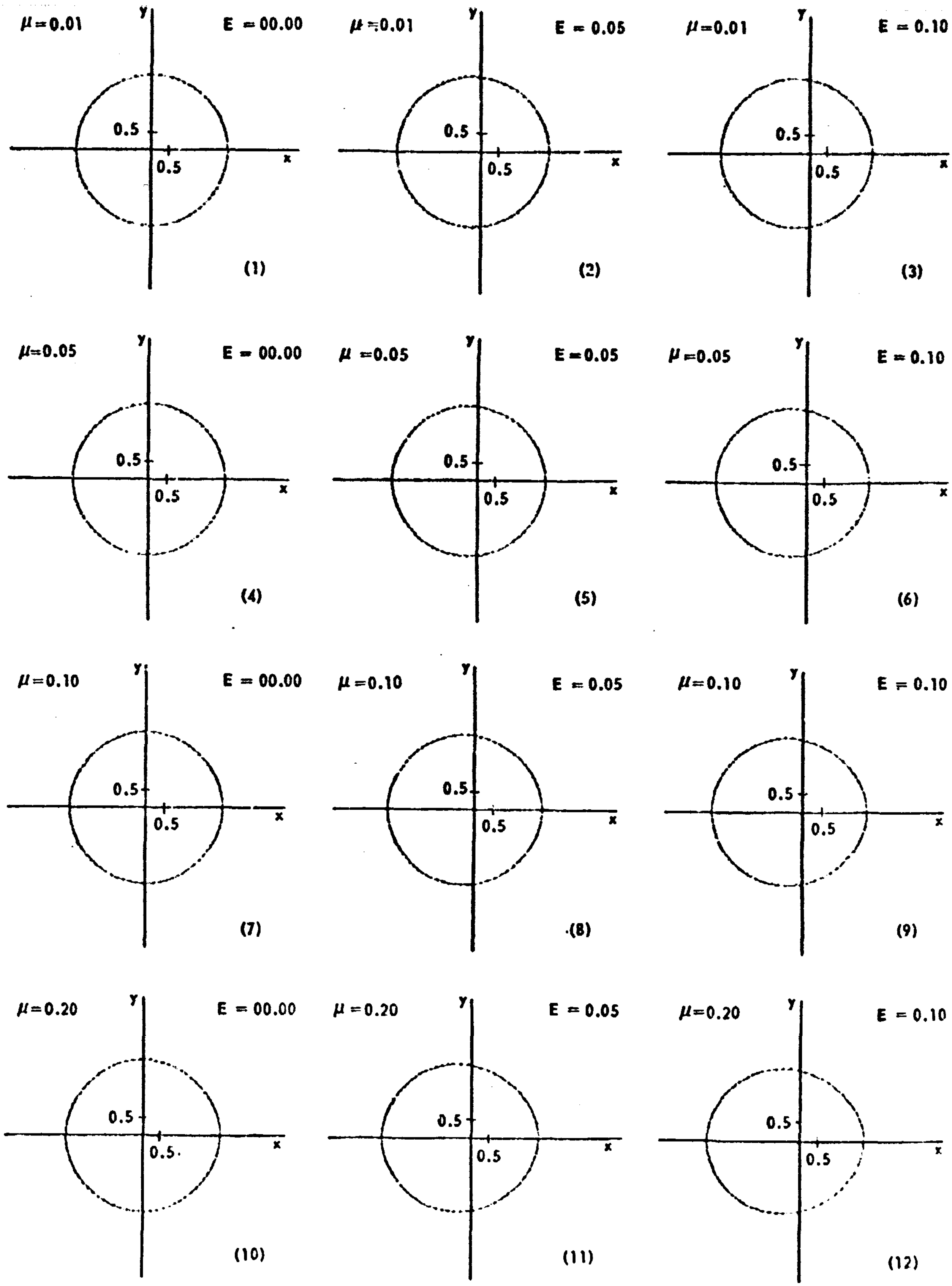
Figure 4. Typical orbits for family 5/1.

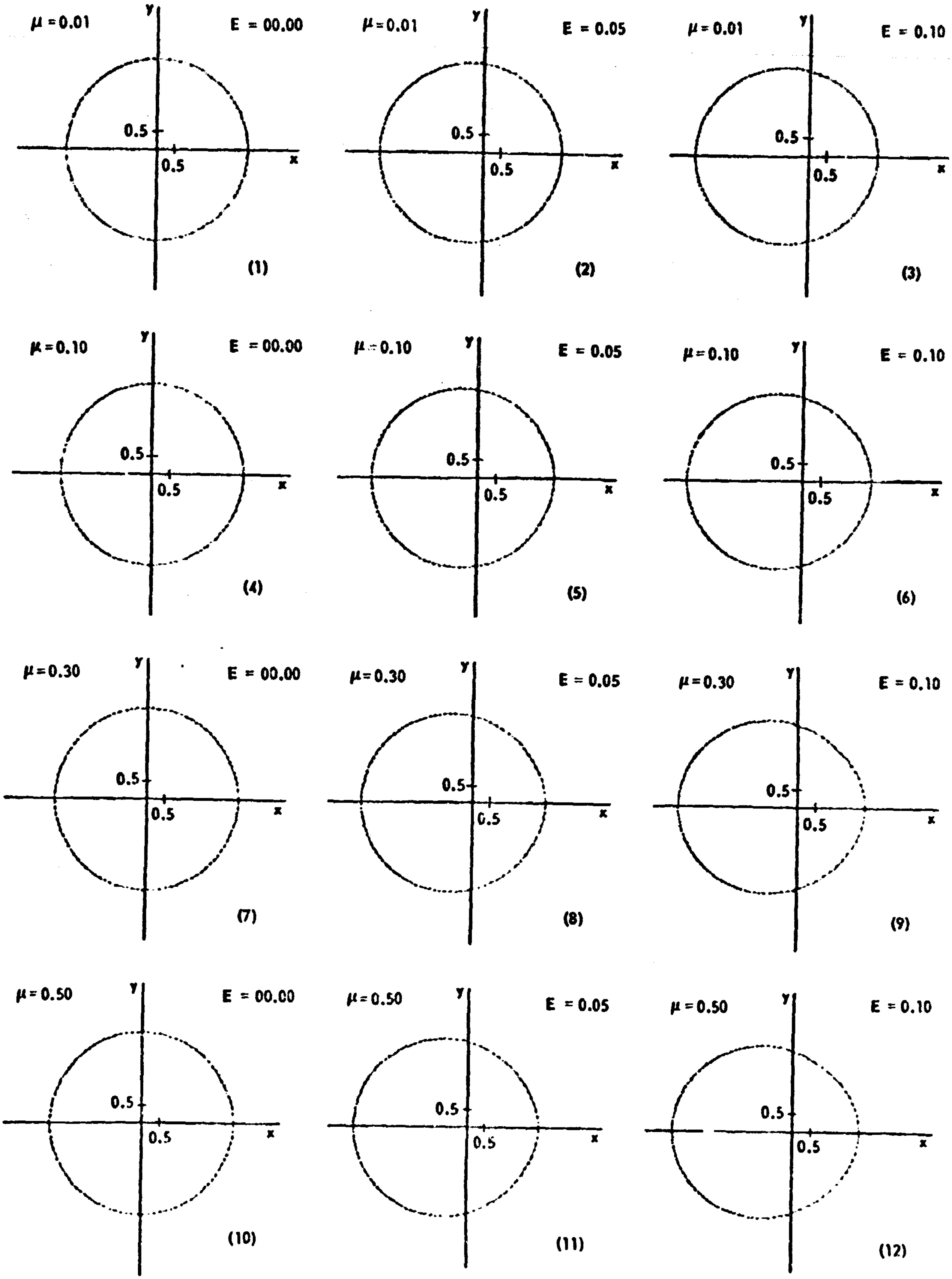
Figure 5. Typical orbits for family 1/12.

Figure 6. Typical orbits for family 1/6.

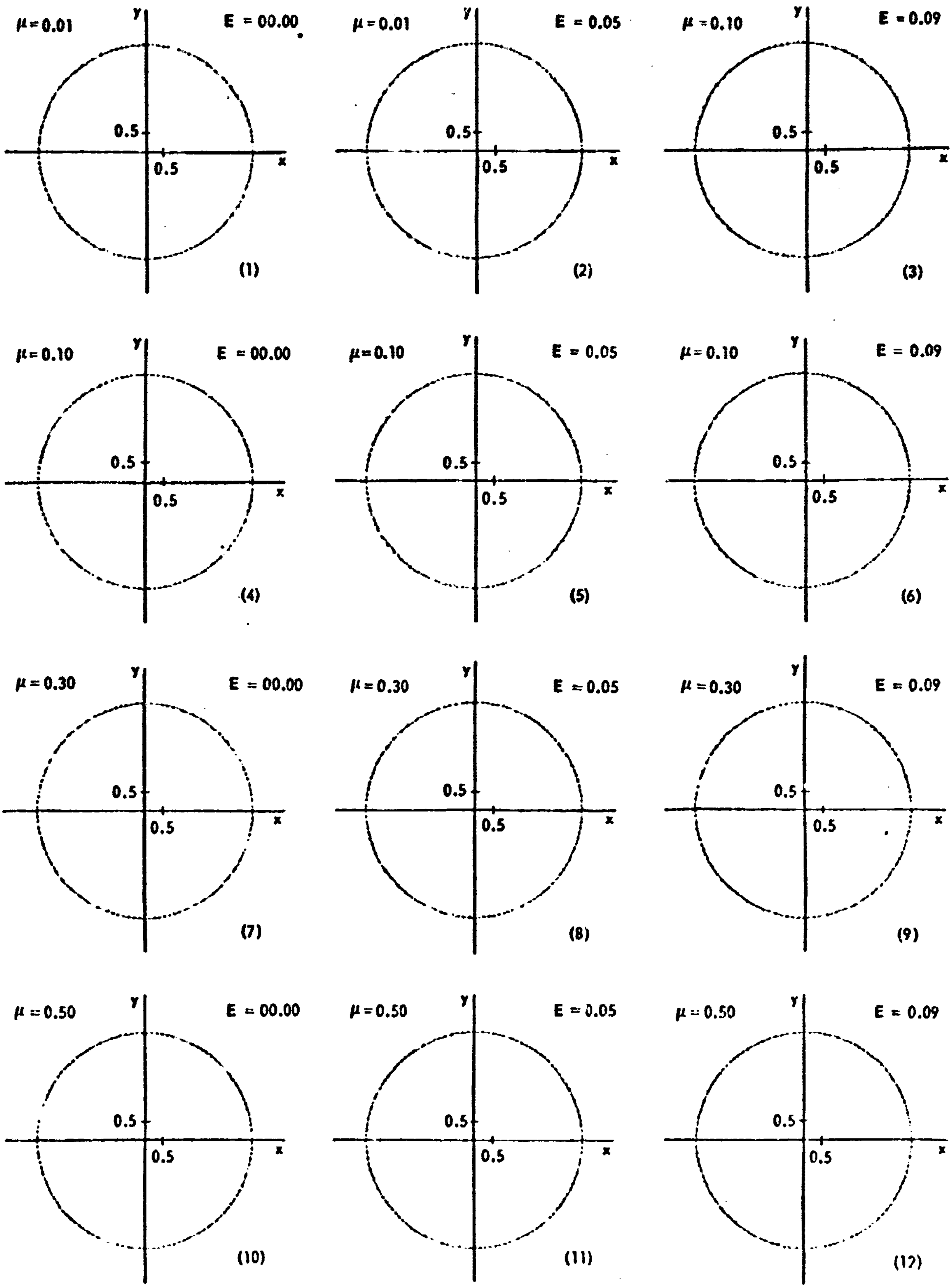
Figure 7. Typical orbits for family 1/3.


μ=0.01
E = 00.00
(1)
μ=0.01
E = 0.05
(2)
μ=0.01
E = 0.10
(3)
μ=0.05
E = 00.00
(4)
μ=0.05
E = 0.05
(5)
μ=0.05
E = 0.10
(6)
μ=0.10
E = 00.00
(7)
μ=0.10
E = 0.05
(8)
μ=0.10
E = 0.10
(9)
μ=0.20
E = 00.00
(10)
μ=0.20
E = 0.05
(11)
μ=0.20
E = 0.10
(12)
x
y
0.5

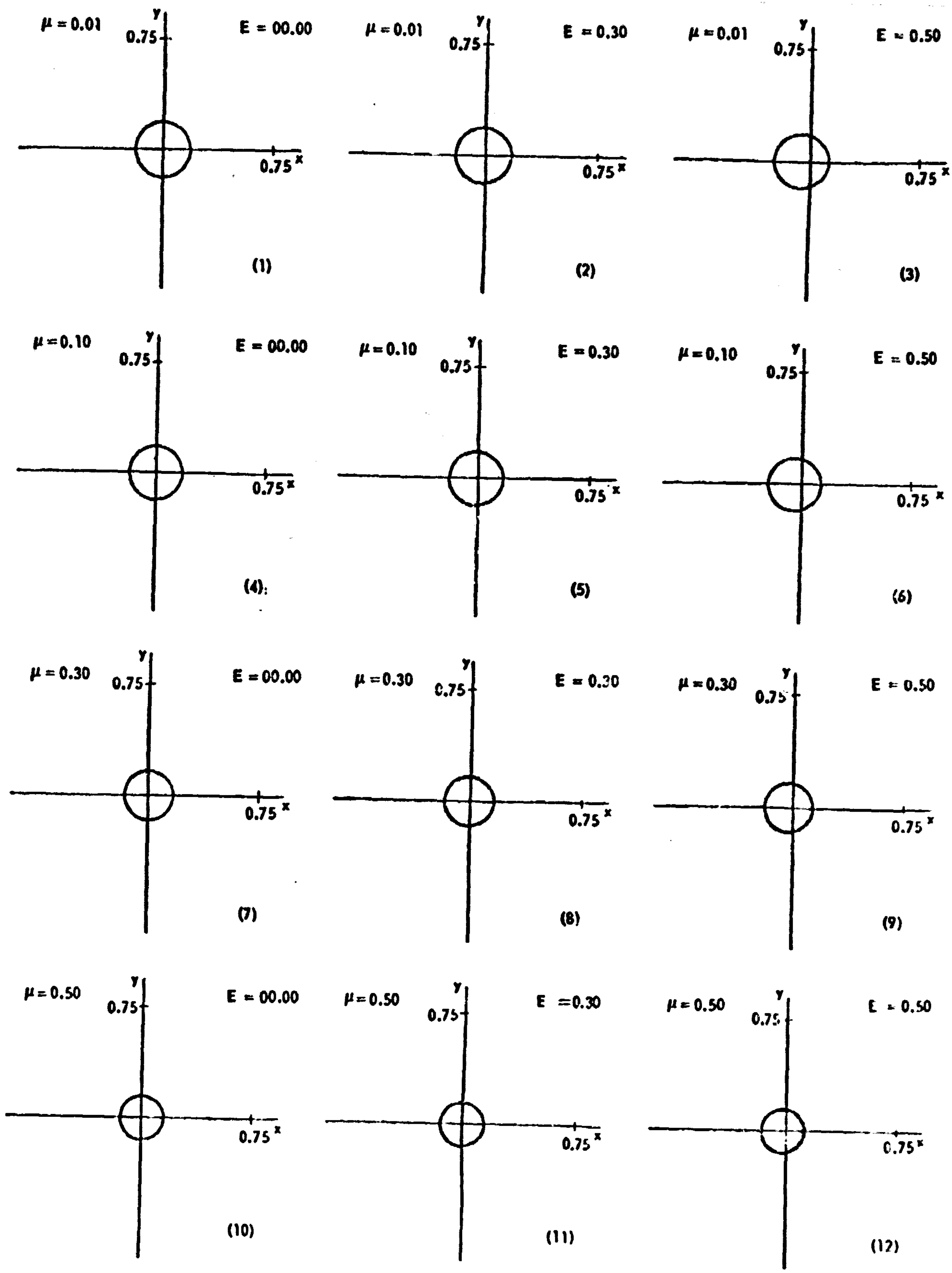

μ=0.01
E = 00.00
(1)
μ=0.01
E = 0.05
(2)
μ=0.01
E = 0.10
(3)
μ=0.05
E = 00.00
(4)
μ=0.05
E = 0.05
(5)
μ=0.05
E = 0.10
(6)
μ=0.10
E = 00.00
(7)
μ=0.10
E = 0.05
(8)
μ=0.10
E = 0.10
(9)
μ=0.20
E = 00.00
(10)
μ=0.20
E = 0.05
(11)
μ=0.20
E = 0.10
(12)
y
x
0.5

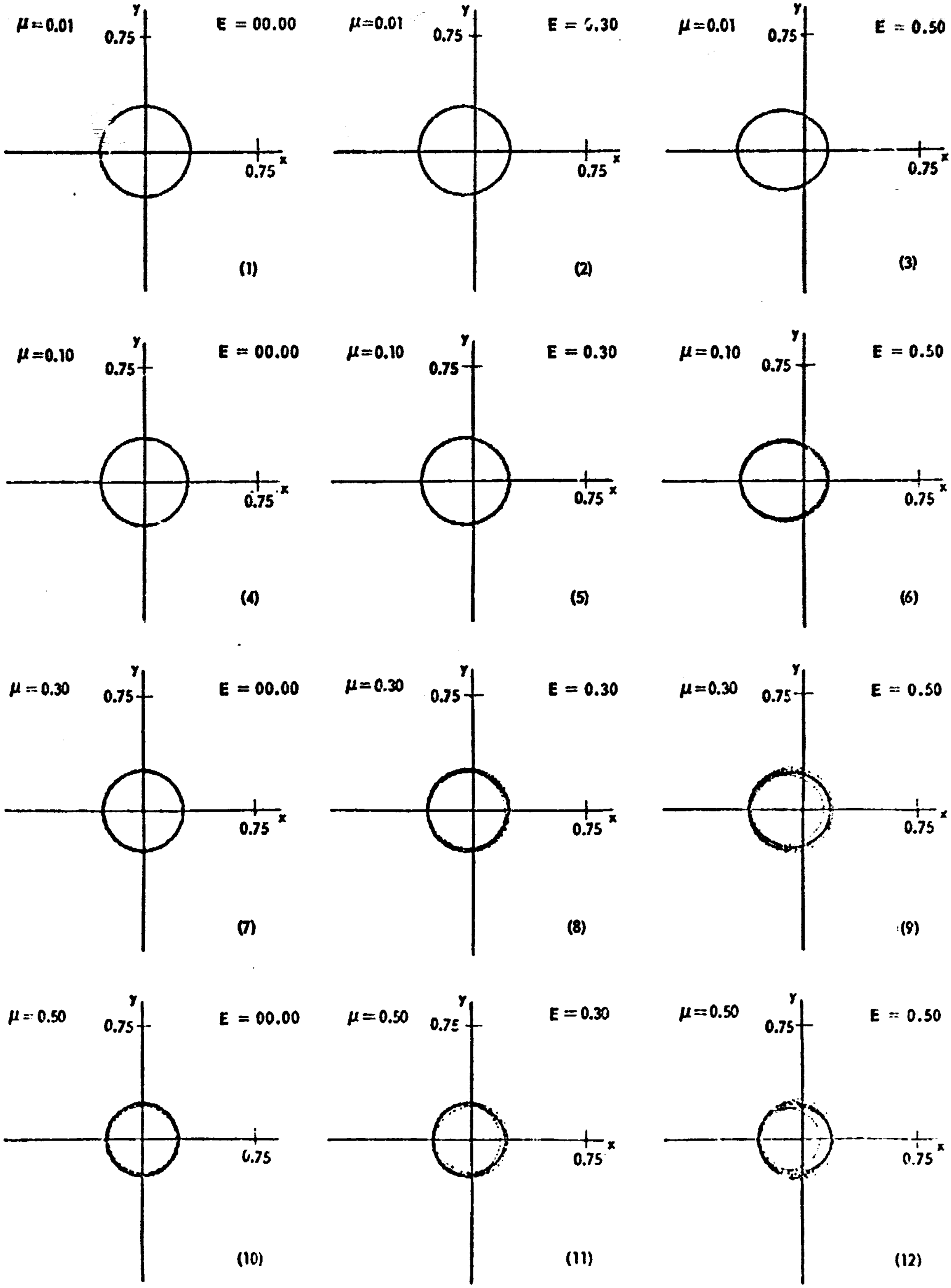


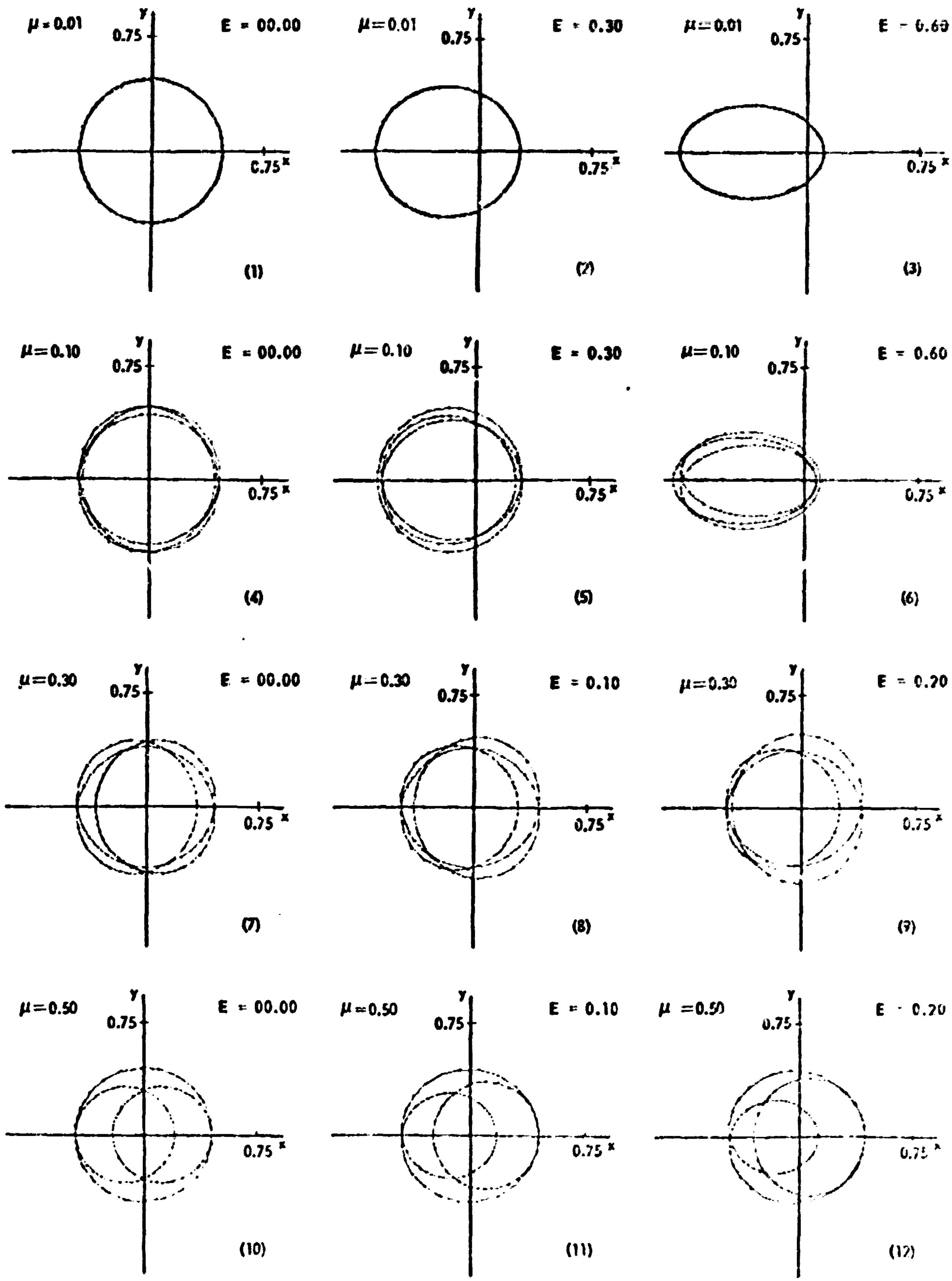
μ = 0.01
E = 00.00
(1)
μ = 0.01
E = 0.05
(2)
μ = 0.01
E = 0.10
(3)
μ = 0.10
E = 00.00
(4)
μ = 0.10
E = 0.05
(5)
μ = 0.10
E = 0.10
(6)
μ = 0.30
E = 00.00
(7)
μ = 0.30
E = 0.05
(8)
μ = 0.30
E = 0.10
(9)
μ = 0.50
E = 00.00
(10)
μ = 0.50
E = 0.05
(11)
μ = 0.50
E = 0.10
(12)
x
y
0.5



μ = 0.01
E = 00.00
(1)
μ = 0.01
E = 0.05
(2)
μ = 0.10
E = 0.09
(3)
μ = 0.10
E = 00.00
(4)
μ = 0.10
E = 0.05
(5)
μ = 0.10
E = 0.09
(6)
μ = 0.30
E = 00.00
(7)
μ = 0.30
E = 0.05
(8)
μ = 0.30
E = 0.09
(9)
μ = 0.50
E = 00.00
(10)
μ = 0.50
E = 0.05
(11)
μ = 0.50
E = 0.09
(12)
x
y
0.5


μ = 0.01
E = 00.00
(1)
μ = 0.01
E = 0.30
(2)
μ = 0.01
E = 0.50
(3)
μ = 0.10
E = 00.00
(4)
μ = 0.10
E = 0.30
(5)
μ = 0.10
E = 0.50
(6)
μ = 0.30
E = 00.00
(7)
μ = 0.30
E = 0.30
(8)
μ = 0.30
E = 0.50
(9)
μ = 0.50
E = 00.00
(10)
μ = 0.50
E = 0.30
(11)
μ = 0.50
E = 0.50
(12)
y
x
0.75


μ=0.01
E = 00.00
0.75
y
x
(1)
μ=0.01
E = 0.30
(2)
μ=0.01
E = 0.50
(3)
μ=0.10
E = 00.00
(4)
μ=0.10
E = 0.30
(5)
μ=0.10
E = 0.50
(6)
μ=0.30
E = 00.00
(7)
μ=0.30
E = 0.30
(8)
μ=0.30
E = 0.50
(9)
μ=0.50
E = 00.00
(10)
μ=0.50
E = 0.30
(11)
μ=0.50
E = 0.50
(12)


μ=0.01
E = 00.00
(1)
μ=0.01
E = 0.30
(2)
μ=0.01
E = 0.60
(3)
μ=0.10
E = 00.00
(4)
μ=0.10
E = 0.30
(5)
μ=0.10
E = 0.60
(6)
μ=0.30
E = 00.00
(7)
μ=0.30
E = 0.10
(8)
μ=0.30
E = 0.20
(9)
μ=0.50
E = 00.00
(10)
μ=0.50
E = 0.10
(11)
μ=0.50
E = 0.20
(12)
y
x
0.75