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APPLICATION OF BAYESIAN DECISION THEORY TO AIRBORNE GAMMA SNOW MEASUREMENT

Vernon C. Bissell, *National Weather Service, River Forecast Center, Portland, Oregon 97209**

ABSTRACT

Measured values of several variables are incorporated into the calculation of snow water equivalent as "measured" from an aircraft by snow attenuation of terrestrial gamma radiation. Bayesian decision theory provides a "best" snow water equivalent measurement by taking into account the uncertainties in the individual measurement variables and "filtering" information about the measurement variables through prior notions of what the calculated variable (water equivalent) should be. Generalizations of principles involved in the application are presented in subsequent discussion.

INTRODUCTION

The importance of snow measurements for both water supply forecasting and for flood forecasting is well established. In the case of flood forecasting, the assessment of flood potential based on the water equivalent of snow on the ground provides a lead time for productive action which is not enjoyed during rainfall-caused floods. For example, during the record floods of the spring of 1969 in the upper Mississippi and middle Missouri River drainages, an estimated 137 million dollars in savings was realized through preventive action based on river forecasts. An estimated additional 97 million dollars savings was realized through the proper use of flood control structures based on forecasts [Mondschein, 1971]. Since snow water equivalent is the primary data input to the forecast procedure, the importance of good snow measurements is readily seen.

One very recent snow measurement method, the airborne gamma survey, appears to have great potential in plains areas such as the north-central United States. This method overcomes the difficulties of nonrepresentative sample points and sparseness of samples now encountered in utilizing a ground-based point measurement network. Airborne measurements of gamma radiation coming from the soil in an area provide an excellent indication of the

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amount of snow water shielding the airborne gamma radiation detector from the radiation flux emanating from the ground. Either the flux in the entire energy range ("total count" method), or the flux within a particular energy range ("spectral peak" method) can be used. The use of natural background radiation to measure snow water equivalent was first proposed in the Soviet Union [Kogan *et al.*, 1965]. Dimitriev *et al.* [1972] reported that operational airborne gamma snow surveys are now conducted over several million square kilometers of the U.S.S.R., mainly in regions where the ground hydrometeorological network is extremely sparse. Research in the method has been conducted in the United States since 1969 [Peck *et al.*, 1971; Burson and Fritzsche, 1972; Bissell, 1974; Larsen, 1975].

AIRBORNE GAMMA SNOW MEASUREMENT METHOD

The total gamma radiation near the surface of the earth is dominated by flux of terrestrial origin. As seen in Figure 1, beyond an altitude of about 1500 meters radiation of cosmic origin begins to dominate. The spectral composition of terrestrial gamma radiation flux is shown in Figure 2. Prominent peaks in the spectrum indicate the presence of ^{40}K (1.46 MeV), ^{214}Bi (1.76 MeV, 2.20 MeV), and ^{208}Tl (2.62 MeV) in the lithosphere and, unfortunately, perhaps in the atmosphere as well. Radiation of atmospheric origin is unfortunate because it is a noise source in airborne gamma surveys. The isotope ^{214}Bi is a subsequent decay product of the radioactive noble gas radon which, with a half-life of 3.6 days, can range far into the atmosphere from its origin uranium in the earth's crust. Consequently, much of the ^{214}Bi decay radiation measured during snow surveys may not have been subjected to the hazard of attenuation by snow cover and is therefore noise in the desired radiation signal. The isotope ^{208}Tl is similarly the decay product of a noble radioactive gas (thoron), but is found in the atmosphere in only small quantities due to thoron's much shorter half-life (57 seconds). The primary snow measurement peaks are the ^{40}K spectral peak at 1.46 MeV and the ^{208}Tl spectral peak at 2.62 MeV. The radiation of atmospheric origin is of sufficient magnitude that the usefulness of the "total count" method is considerably reduced. The ^{40}K spectral peak also has an atmospheric contribution to due Compton-scattering of 1.76 and 2.20 MeV photons down to the lower 1.46 MeV energy range, but a correction can be applied to "strip out" this effect by keying on the magnitude of the 1.76 MeV ^{214}Bi peak. Both the ^{40}K and ^{208}Tl peaks have cosmic components which can be "stripped out" by keying on the magnitude of the purely cosmic portion of the spectrum above 3 MeV. Another significant effect which can enter in is the variable attenuation of radiation within the soil itself due to changing soil moisture, and a correction is also applied for this effect based on measured, estimated, or simulated soil moisture.

The airborne "measured" value of water equivalent is given by the inverse of the relationship expressing the attenuation of soil-originating spectral peak count rates as a function of water equivalent. An exponential form of this relation is given in equations (1) and (2):

$$R_K(W) = \frac{R_o^{(K)}}{1+KS} \exp\left(-\alpha_K W\right) \tag{1}$$

$$R_{Tl}(W) = \frac{R_o^{(Tl)}}{1+KS} \exp\left(-\alpha_{Tl} W\right) \tag{2}$$

where W = effective water equivalent, including air blanket (g/cm^2);
α_K = attenuation coefficient of ^{40}K 1.46 MeV gamma rays in water (cm^2/g);
α_{Tl} = attenuation coefficient of ^{208}Tl 2.62 MeV gamma rays in water (cm^2/g);
S = soil moisture (fraction of dry weight);
K = ratio of gamma attenuation cross section in water to cross section in air, taken to be unity;
$R_o^{(K)}$ = expected value of count rate in ^{40}K decay in soil, with $W=0$ and $S=0$ (counts/second);
$R_o^{(Tl)}$ = expected value of count rate in ^{208}Tl spectral peak due only to ^{208}Tl decay in soil with $W=0$ and $S=0$ (counts/second);

The "stripping" equations (3) and (4) express the "pure" count rates R_K and R_{Tl} as the actual spectral peak count rates obtained minus counts from noise sources:

$$R_K(W) = C_K - B_K + \beta_{BK}\left(C_B - B_B\right) + \beta_{CK}C_C \tag{3}$$

$$R_{Tl}(W) = C_{Tl} - B_{Tl} + \beta_{CT}C_C \tag{4}$$

where C_K = expected count rate in ^{40}K spectral peak window;
B_K = expected value of aircraft background count rate contribution in ^{40}K spectral peak window;
C_B = expected value of count rate in ^{214}Bi spectral peak window;
B_B = expected value of aircraft background count rate contribution in ^{214}Bi (1.76 MeV) spectral peak window;
C_{Tl} = expected count rate in ^{208}Tl spectral peak window;
B_{Tl} = expected value of aircraft background count rate contribution in ^{208}Tl spectral peak window;
C_C = expected value of count rate in cosmic (high energy) spectral window;
$\beta_{BK}, \beta_{CK}, \beta_{CT}$ = "stripping" coefficients, generally negative.

Airborne water equivalent measurements are obtained by estimating C_K, C_B, C_{Tl}, and C_C by the computed count rates N_K/t, N_B/t, N_{Tl}/t, and N_C/t,

where N_K = counts in ^{40}K peak during time t;
N_B = counts in 1.76 MeV ^{214}Bi peak during time t;

N_{Tl} = counts in ^{208}Tl peak during time t;
N_C = counts in cosmic window during time t.

A linear combination of the ^{40}K and ^{208}Tl spectral peak measurements of a 15 cm. water equivalent will have an accuracy of somewhere near one centimeter water equivalent [Bissell, 1974].

Operational use of the gamma snow measurements would require some quality control on the water equivalent calculation. Since automated calculations would be desirable, some mechanism which would automatically impose a constraint of "reasonableness" on the final calculated water equivalent would be in order. This is one area in which Bayesian decision theory seems to fit nicely.

APPLICATION OF BAYESIAN DECISION THEORY

Bayesian decision theory departs from classical statistics in that the distinction between parameters and random variables is not quite so sharp. It is recognized that initial parameter values may not actually be the true values, and some prior, or *a priori*, distribution for parameter values is allowed.

Bayesian decision theory departs from classical estimation theory in that prior knowledge is allowed to play a role in the final decision. The difference between classical estimation theory and Bayesian decision theory is given in Figure 3. It is by introducing the prior distribution of the parameters that the "reasonableness" constraint is imposed. A large variance on this distribution would allow the sample values a greater weight in the final estimate. A very small variance in the *a priori* distribution would cause the estimation function to virtually ignore an outrageous sample.

In the case at hand, the random variables are the count values received in individual spectral windows. In vector form this is

$$\underset{\sim}{N} = \begin{bmatrix} N_K \\ N_B \\ N_{Tl} \\ N_C \end{bmatrix} \tag{5}$$

The main parameter to be estimated is water equivalent, but other parameters must enter in as well because of their effect on $E\underset{\sim}{N}$. The expected values of the elements of $\underset{\sim}{N}$ are given by

$$\begin{aligned} EN_K &= C_K t \\ &= t\{R_K(W) + B_K - \beta_{BK}(C_B - B_B) - \beta_{CK}C_C\} \\ &= t\{\frac{R_o^{(K)}\exp(-\alpha_K W)}{1+KS} + B_K - \beta_{BK}\frac{(R_o^{(B)}\exp(-\alpha_B W)}{1+KS} + B_B + G) \\ &\quad - \beta_{CK}C_C\} \end{aligned} \tag{6}$$

where G = rate of airborne radon contribution to ^{214}Bi peak count;
$R_o^{(B)}$ = expected value of count rate in ^{214}Bi spectral peak due only to ^{214}Bi decay in soil, with W=0, S=0;
α_B = attenuation coefficient of 1.76 MeV ^{214}Bi gamma rays in water.

$$EN_B = C_B t = t\{\frac{R_o^{(B)} \exp(-\alpha_B W)}{1+KS} + B_B + G\} \tag{7}$$

$$EN_{T1} = C_{T1} t = t\{\frac{R_o^{(T1)} \exp(-\alpha_{T1} W)}{1+KS} + B_{T1} - \beta_{CT} C_C\} \tag{8}$$

$$EN_C = C_C t \tag{9}$$

Looking at equations (6) through (9), the items W (water equivalent), S (soil moisture), G (representing the magnitude of ^{214}Bi concentration in the atmosphere), and C_C (cosmic flux) are the parameters. Thus the parameter vector is given by

$$\underset{\sim}{\theta} = \begin{bmatrix} W \\ S \\ G \\ C_C \end{bmatrix} \tag{10}$$

Now considering $\underset{\sim}{\theta}$ itself as a random variable, the desired result is to develop a posterior distribution for $\underset{\sim}{\theta}$ after the sample results are in [Lindgren, 1968, p. 234]. Let $f(\underset{\sim}{n} \mid \underset{\sim}{\theta})$ be the conditional distribution of $\underset{\sim}{N}$, and $g(\underset{\sim}{\theta})$ be the *a priori* distribution of $\underset{\sim}{\theta}$. The joint distribution of $\underset{\sim}{N}$ and $\underset{\sim}{\theta}$ is

$$f(\underset{\sim}{n},\underset{\sim}{\theta}) = f(\underset{\sim}{n} \mid \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) \tag{11}$$

The marginal (or absolute) distribution of $\underset{\sim}{N}$ is given by

$$h_N(\underset{\sim}{n}) = \int f(\underset{\sim}{n} \mid \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) d\underset{\sim}{\theta} \tag{12}$$

and finally the posterior density of $\underset{\sim}{\theta}$ conditioned on the observation $\underset{\sim}{N}=\underset{\sim}{n}$ is

$$h(\underset{\sim}{\theta} \mid \underset{\sim}{N}) = f(\underset{\sim}{n} \mid \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) / h_N(\underset{\sim}{n}) \tag{13}$$

If the loss function is taken to be $(W-\hat{W})^2$, then the Bayes risk (the posterior expected value of the loss function) is

$$E_h[W-\hat{W}(\underset{\sim}{N})]^2 = \text{Bayes Risk} \tag{14}$$

If we use the posterior expected value of W as the estimate of W,

$$\hat{W} = E_h(W) = \int W h(\underset{\sim}{\theta} \mid \underset{\sim}{N}) d\underset{\sim}{\theta} \tag{15}$$

then this value minimizes the Bayes risk since the Bayes risk written above is a second moment, and the smallest second moment is that taken about the mean of a distribution [Lindgren, 1968, p. 239]. Hence $\hat{W}$ is the Bayes estimate of the water equivalent.

With the above background development, now for some simplification. With a little (not unreasonable) bending, the problem can be fit into a multivariate normal framework, within which calculation of conditional means (i.e. the Bayes estimate - equation (5) - when a quadratic loss function - equation (4) is selected) is relatively simple. First it must be recognized that equations (6) through (9) express conditional means of $\underset{\sim}{N}$ given $\underset{\sim}{\theta}$. Assume that $\underset{\sim}{\theta}$ is multivariate normal. Analysis of the individual elements of $\underset{\sim}{\theta}$ indicates that this is a reasonable assumption, with perhaps the normality of G falling the most in question. Also, W is an effective water equivalent which includes both the air blanket and water blanket. Thus even for values of zero snow water equivalent, W is not bumping up against a physical boundary which would disallow negative values.

Let the *a priori* (marginal) distribution of $\underset{\sim}{\Theta}$ be

$$\underset{\sim}{\Theta} \backsim N(\underset{\sim}{\Theta}^*, \underset{\sim}{\Sigma}_{\Theta}) \tag{16}$$

where

$$\underset{\sim}{\Theta}^* = \begin{bmatrix} W^* \\ S^* \\ G^* \\ C_C^* \end{bmatrix}$$

and where $\underset{\sim}{\Sigma}_{\Theta}$ is known beforehand. Now (following Anderson, 1958, p. 28) let's put the random vector $\underset{\sim}{N}$ and the parameter vector $\underset{\sim}{\Theta}$ together into a single augmented vector:

$$\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}^{(1)} \\ \cdots\cdots \\ \underset{\sim}{X}^{(2)} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\Theta} \\ \cdots\cdots \\ \underset{\sim}{N} \end{bmatrix} \tag{17}$$

If $\underset{\sim}{X}$ is multivariate normal, its distribution is designated as

$$f(\underset{\sim}{x}) \backsim N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma}) \tag{18}$$

where

$$\underset{\sim}{\mu} = \begin{bmatrix} \underset{\sim}{\mu}^{(1)} \\ \cdots\cdots \\ \underset{\sim}{\mu}^{(2)} \end{bmatrix}, \qquad \underset{\sim}{\Sigma} = \left[\begin{array}{c:c} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \hdashline \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{array}\right]$$

From (16),

$$\underset{\sim}{\mu}^{(1)} = \underset{\sim}{\Theta}^* \tag{19}$$

and

$$\underset{\sim}{\Sigma}_{11} = \underset{\sim}{\Sigma}_{\Theta} \tag{20}$$

When the distribution (18) is completely specified, calculation of the posterior expected value of $\underset{\sim}{\Theta}$ (our final goal) is a simple matter. First we must determine $\underset{\sim}{\mu}^{(2)}$, $\underset{\sim}{\Sigma}_{12}$ (which is the transpose of $\underset{\sim}{\Sigma}_{21}$) and $\underset{\sim}{\Sigma}_{22}$. These are determined by looking at the conditional mean and covariance of $\underset{\sim}{N}$ based on $\underset{\sim}{\Theta}$. Continuing to follow the notation of Anderson,

$$E(\underset{\sim}{X}^{(2)} \mid \underset{\sim}{x}^{(1)}) = \underset{\sim}{\mu}^{(2)} + \underset{\sim}{\Sigma}_{21}\underset{\sim}{\Sigma}_{11}^{-1}(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)}) = \underset{\sim}{v}(\underset{\sim}{x}^{(1)}) \tag{21}$$

and

$$E\{[\underset{\sim}{X}^{(2)} - \underset{\sim}{v}(\underset{\sim}{x}^{(1)})][\underset{\sim}{X}^{(2)} - \underset{\sim}{v}(\underset{\sim}{x}^{(1)})]^T\}$$

$$= \underset{\sim}{\Sigma}_{22} - \underset{\sim}{\Sigma}_{21}\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{\Sigma}_{12} \overset{\text{def}}{=} \underset{\sim}{\Sigma}_{22.1} \tag{22}$$

Let's see if we can identify some of the terms in (21) and (22) to be something known. First, recognize that equations (6), (7), (8) and (9) express conditional expectations. Assume that our prior knowledge of W and S is good enough that a first order expansion can be made about the prior expected values W^* and S^*. Then the exponential terms in equations (6), (7), and (8) may be expanded. For example,

$$\frac{R_o^{(K)}\exp(-\alpha_K W)}{1+KS} \cong \frac{R_o^{(K)}\exp(-\alpha_K W^*)}{1+KS^*}\left[1 + \alpha_K(W-W^*) - \frac{K(S-S^*)}{1+KS^*}\right] \tag{23}$$

Thus by incorporating the linear expansions, adding and subtracting a few terms in the right places, and rearranging, we get

$$E(\underset{\sim}{N} \mid \underset{\sim}{\Theta})_{4x1} = \underset{\sim}{A}_{4x4}(\underset{\sim}{\Theta}-\underset{\sim}{\Theta}^*)_{4x1} + \underset{\sim}{N}^*_{4x1} \tag{24}$$

where the matrix $\underset{\sim}{A}$ and vector $\underset{\sim}{N}^*$ are constants. Comparing (24) and (21) we get the identification

$$\underset{\sim}{\mu}^{(2)} = \underset{\sim}{N}^* \tag{25}$$

and

$$\underset{\sim}{\Sigma}_{21}\underset{\sim}{\Sigma}_{11}^{-1} = \underset{\sim}{A} \tag{26}$$

Therefore

$$\underset{\sim}{\Sigma}_{21} = \underset{\sim}{A}\underset{\sim}{\Sigma}_{11} = \underset{\sim}{A}\underset{\sim}{\Sigma}_{\Theta} \tag{27}$$

The elements of $\underset{\sim}{\Sigma}_{22.1}$ of equation (22) are determined by physical reasoning. The matrix $\underset{\sim}{\Sigma}_{22.1}$ gives the covariance of the counts vector once $\underset{\sim}{\Theta}$ is specified. But once $\underset{\sim}{\Theta}$ is specified, the variability in the elements N_i is due only to the random nature of nuclear counting. The nature of nuclear counting produces a Poisson random variable which, for sufficiently large expected values, is very well approximated by a normal distribution. This condition is satisfied in gamma snow surveys. Since the Poisson distribution has variance equal to the mean, we take the diagonal elements of $\Sigma_{22.1}$ to be the elements of the conditional mean vector of equation (21). Also, randomness of nuclear counting in one spectral window is independent of nuclear counting in another window during the same time. Thus the off-diagonal elements of $\Sigma_{22.1}$ will be zero, and

$$\Sigma_{22.1} = \begin{bmatrix} v_1(x^{(1)}) & 0 & 0 & 0 \\ 0 & v_2(x^{(1)}) & 0 & 0 \\ 0 & 0 & v_3(x^{(1)}) & 0 \\ 0 & 0 & 0 & v_4(x^{(1)}) \end{bmatrix} \tag{28}$$

Physical reasoning requires that the diagonal elements of $\Sigma_{22.1}$ be positive. If the calculated values are not positive in any particular situation, a different method of computing water equivalent should be used. Finally then Σ_{22} is computed from equation (22) as

$$\begin{aligned} \Sigma_{22} &= \Sigma_{22.1} + \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \\ &= \Sigma_{22.1} + A\Sigma_{11}A^T \end{aligned} \tag{29}$$

It should be noted that Σ_{22} must be a positive definite matrix in order to be a marginal covariance matrix for N. Since $\Sigma_{22.1}$ is positive definite (diagonal, with positive elements), and since Σ_{11} is p.d. (by definition of marginal covariance matrix) it is sufficient to have matrix A with at least as many columns q as rows p, and be of rank p. This means that to compute Σ_{22} in this fashion we must have at least as many parameter values (we have four) as count values (also 4), which condition is satisfied (Anderson, 1958, Appendix A, Theorem 1). The rank of the matrix A would have to be checked in each separate situation.

Now to write the answer. The Bayes estimate of water equivalent under a quadratic loss function is the posterior expected value or, identically, the conditional mean based on the observed sample:

$$\begin{aligned} \hat{\Theta} = \begin{bmatrix} \hat{W} \\ \hat{S} \\ \hat{G} \\ \hat{C}_C \end{bmatrix} &= E(X^{(1)} \mid x^{(2)}) = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)}) \\ &= \Theta^* + \Sigma_\Theta A^T[\Sigma_{22.1} + A\Sigma_\Theta A^T]^{-1}(n - N^*) \end{aligned} \tag{30}$$

which is seen simply to be a regression function on the count values. It is noted that not only is the posterior estimate of water equivalent given, but posterior estimates of soil moisture, atmospheric ^{214}Bi concentrations, and mean cosmic flux are given as well. Furthermore, the posterior covariance is given by a matrix $\Sigma_{11.2}$ computed by exchanging indices in equation (22). Thus the first element in the first row of $\Sigma_{11.2}$ is the posterior variance of the water equivalent measurement $\hat{W}$.

THE KEY TO THE PROBLEM - ENCODING PRIOR KNOWLEDGE

Looking back at the final product (equation (30)) of the previous section, it is seen that two major inputs were necessary: (1) the matrix $\underset{\sim}{A}$ which is a linearized version of the physical laws relating random variables and parameters, and (2) the vector $\underset{\sim}{\Theta}^*$ and matrix Σ_Θ which quantify the description of prior knowledge. The whole key to the Bayesian approach is having enough prior knowledge to make all the gyrations worthwhile. In the present application, the question is "where do we get our prior information?" The present formulation of the National Weather Service's Snow Accumulation and Ablation Model (Anderson, 1973) will provide hydrological simulation of snow cover water equivalent W. Periodic ground measurements by cooperative observers where available assist in keeping the simulation "on track". It is anticipated that an associated soil moisture model will produce estimates of soil moisture S under the snow cover for input to the airborne gamma snow measurements. Development of the covariance matrix Σ_Θ first of all requires special calibrations over a period of time to quantify the performance of the hydrological simulations in updating previous measurements. The airborne ^{214}Bi concentrations certainly follow physical laws, but the inability to model these adequately means that in general some kind of a mean ^{214}Bi level G* will have to be used in conjunction with a large "ignorance factor" (entry on the diagonal of Σ_Θ) for G. The cosmic flux is reasonably well defined as a function of season and barometric pressure.

The most important point to be made in this paper is found in the above paragraph - the usefulness of simulation of physical processes to provide information which may be helpful in interpreting signals received in the remote sensing process. This principle would seem to go beyond the airborne gamma snow survey application and extend to other remote observations of hydrological variables as well. The continuing effort to wring more and more information from remote sensing imagery would seem to require more and more sophistication as more and more automated interpretation processes are applied. A critical consideration is that progress must be made on a coordinated front so that remote sensing products are useful to the simulation practitioner and, similarly, that simulation results are available and useful to the remote sensing interpretation process.

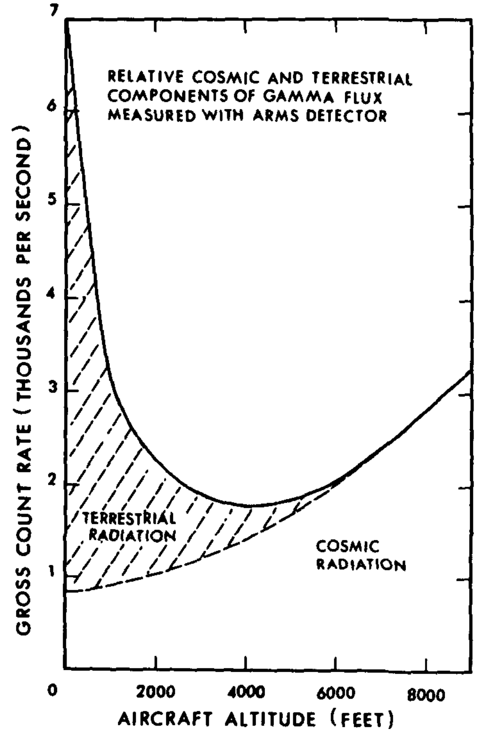

Figure 1. Relative contribution of terrestrial and cosmic gamma flux with altitude.

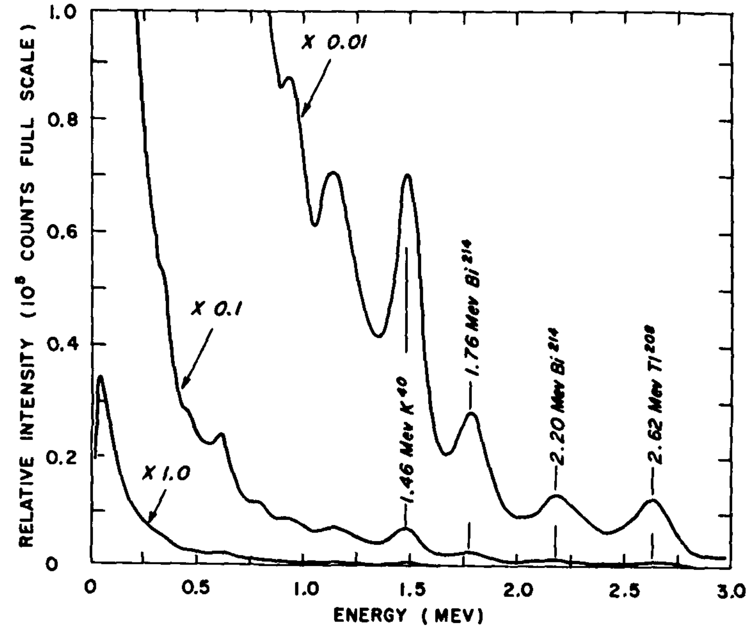

Figure 2. Spectral composition of terrestrial gamma radiation flux.

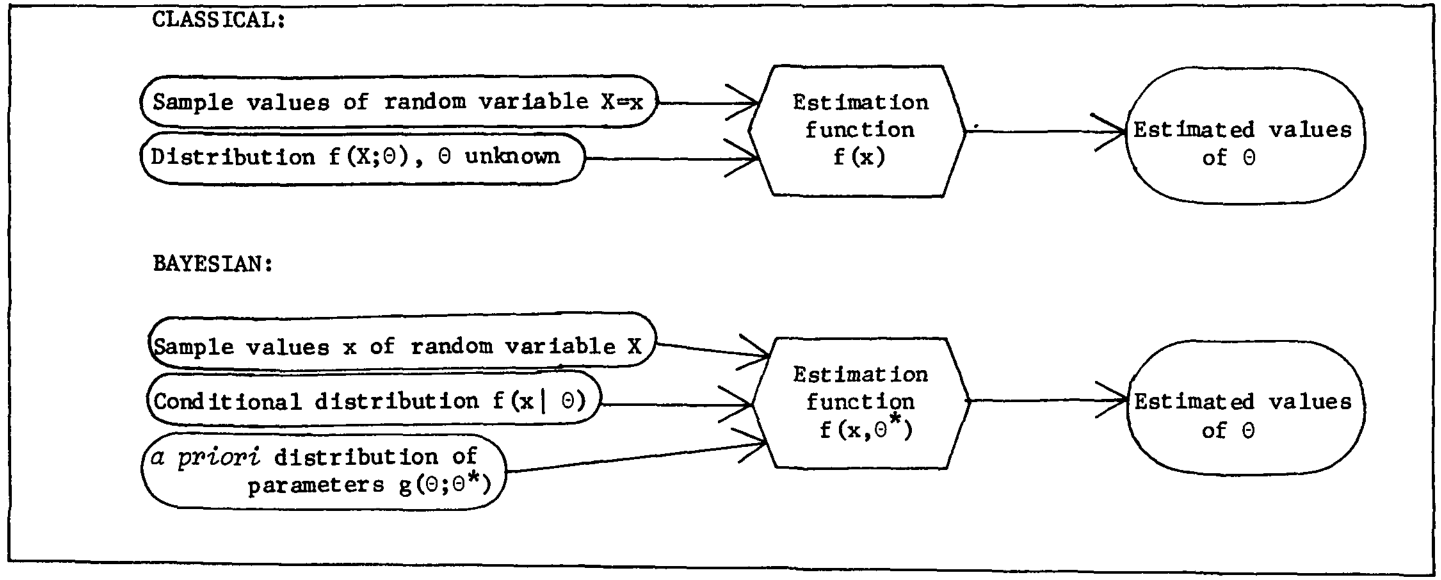

Figure 3. Classical estimation versus Bayesian decision theory to determine value of parameter Θ.

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