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EARTH-MOON SYSTEM: DYNAMICS AND PARAMETER ESTIMATION

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By

W.J. Breedlove, Jr.

Semiannual Progress Report

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
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Under
Grant NSG 1152
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A UNIFIED SPECIAL PERTURBATION MODEL FOR THE MOTION OF THE EARTH-MOON SYSTEM

By

W.J. Breedlove, Jr.[1]

SUMMARY

This report contains a theoretical development of the equations of motion governing the Earth-Moon system. The Earth and Moon are treated as finite rigid bodies and a mutual potential is utilized. The Sun and remaining planets are treated as particles. Relativistic, non-rigid, and dissipative effects are not included.

The translational and rotational motion of the Earth and Moon are derived in a fully coupled set of equations. Euler parameters are used to model the rotational motions.

The mathematical model developed herein is intended for use with data analysis software to estimate physical parameters of the Earth-Moon system using primarily LURE type data.

The Appendix contains two program listings. Program ANEAMØ computes the translational/rotational motion of the Earth and Moon from analytical solutions. Program RIGEM numerically integrates the fully coupled motions as described above.

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INTRODUCTION

The Lunar Laser Ranging Experiment (LURE), (ref. 1) has resulted in the placement of three ranging retroreflectors on the Moon. A series of measurements of the distance of these retroreflectors from several Earth-based observatories began in August 1969. These measurements, at present, allow the determination of the distance to the Moon with an accuracy of ± 8 cm. A resolution of ± 2 to 3 cm is expected within the next few years. Overall, a ± 10-cm accuracy over a 10-year period will soon be available.

The LURE data, in combination with other data types, can be used to determine parameters related to the internal composition of the Earth and Moon (ref. 24). Also, checks of current gravitational theories may be made (ref. 1). For example, data accuracies of ± 3 cm would make feasible the determination of the following parameters (refs. 1 and 2):

A. Geometrical and Orbital Parameters:

station coordinates
retroreflector coordinates
Earth and Moon orbital constants of integration

B. Geophysical Parameters:

station drift
polar wobble
rotation variations
Earth tide
universal time determination
orbital acceleration

C. Selenophysical Parameters:

physical librations
free librations
Moon tide

D. Systematic Error Sources:

fixed bias
zenith-distance bias
arbitrary periodic biases.

The original mathematical model (October 1973) of the LURE team (ref. 1) involved (1) the numerical integration of the Moon and major planets as point masses including relativistic effects, (2) the utilization of an analytical lunar physical libration theory based on Eckhardt's work, plus certain additive and planetary terms from the Improved Lunar Ephemeris and 3rd and 4th order terms in the lunar potential (refs. 19 and 23), and (3) determining the angular position and pole of the Earth from BIH data. Earth tides and dissipative effects were not modeled (ref. 2). Lunar orbital-rotational coupling was not fully modeled (ref. 22). Finally, the BIH data imposes limits on the accuracy achieveable from this model.

The model described above reached its current state by appending additional effects to existing models. For example, Eckhardt's original libration theory did not include the 3rd and 4th order terms in the lunar potential. Also, the additive and planetary terms were appended to this original theory.

This model provided (October 1973) rms residuals in range of ± 3 meters. An improved libration theory would considerably reduce this value. The LURE team suggested that a numerical integration of Euler's equations holds promise for future gains in accuracy for the rotational motion of the Moon (ref. 19). The above residuals imply the existence of unmodeled effects or modeling inaccuracies (ref. 2).

The determination of geometrical and orbital parameters, geophysical parameters, selenophysical parameters, and systematic error sources to an accuracy compatible with the observational accuracy thus awaits the development of a rigorous model of the Earth-Moon system. Previous models have been attempted in a piecewise fashion. Thus, there is a need for a consistent theoretical, mathematical model incorporating Earth and Moon rotational, translational, and deformational motions in a coupled sense. This model should allow for an inhomogeneous Earth and Moon, dissipation effect, general relativity effects, and planetary perturbations. Secondly, there is a need for a "special" numerical model incorporating pertinent effects from the above theoretical model to be used in the parameter estimation process.

The "special" model envisioned at this point (although subsequent investigations may modify it) is of the following form.

1. Treat Sun and planets as perturbing point masses,
2. treat Moon as a tidally deformed body,
3. treat Earth as a tidally deformed body,
4. consider the coupled orbital-rotational motions of the Earth and Moon (ref. 22), and
5. consider the effects of relativity (refs. 25 and 26).

The governing equations of motion for this "special" model are to be numerically integrated (refs. 4 and 9).

The appropriate numerical integration routine to be used in this model should be investigated. One candidate is an extremely accurate Cowell type routine used by Oesterwinter and Cohen in a determination of planetary masses (ref. 7). This routine has been developed over a period of years by Cohen and Hubbard and has been used primarily for solutions to the planetary n-body problem. This scheme is based on the use of a 16th-order set of predictor-corrector formulae for integrating accelerations. Herrick (ref. 27) also espouses the use of numerical integration schemes that integrate accelerations directly.

GENERAL SYMBOLS AND NOMENCLATURE

vector

universal gravitational constant

$(\dot{\ })$, $(\ddot{\ })$	first and second time derivatives
$\vec{r}_i$	radius vector from solar system barycenter to mass center of body i $(i = 1,11)$
$\vec{r}_i'$	radius vector from Sun to mass center of body i $(i = 1,11)$
$\vec{\nabla}_j$	gradient operator with respect to coordinates of mass center of body j $(j = 1,11)$
U	work function
m_j	mass of body j $(j = 1,11)$
β_i'	Euler parameters locating Earth body axes with respect to Earth reference axes $(i = 0,1,2,3)$
ω_i	absolute angular velocity components of Earth resolved along body axes $(i = 1,2,3)$
ω_i'	absolute angular velocity components of Moon resolved along lunar body axes $(i = 1,2,3)$
β_i'''	Euler parameters locating lunar body axes with respect to lunar reference axes $(i = 0,1,2,3)$
r,λ,ϕ	components of spherical polar coordinate system
$\{\ \}$	vector-column
$[\ ]$	matrix
$\{\ \}^T$	transposed vector-row
$\{(\ )^T\}$	translating reference frame
$\lfloor\ \rfloor$	vector-row
$(\bar{\bar{\ }})$	dyadic

PHYSICAL MODEL

For the purposes of this study, the Sun, Mercury, Venus, Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto are modeled as particles. The Earth is modeled as a triaxial rigid body and the Moon as an asymmetric rigid body. The Sun, Moon, and planets interact gravitationally with translational and rotational motions fully coupled. Non-rigid, dissipative, and relativistic effects are not considered here but anticipated for future inclusion in the model.

MATHEMATICAL MODEL

A system of 41 second-order ordinary differential equations has been derived to represent the physical model described in the previous section. These may be summarized as follows:

A. Motion of Sun with respect to center of mass of Solar System

$$\ddot{\vec{r}}_1 = G \sum_{j=2}^{11} m_j \frac{\vec{r}_j}{r_{1j}^{\,3}} \tag{1}$$

B. Motion of Moon and planets with respect to the Sun

$$\ddot{\vec{r}}_i + G(m_1 + m_i) \frac{\vec{r}_i}{r_{i1}^{\,3}} = \sum_{\substack{j=2 \\ j\neq i}}^{11} m_j \vec{\nabla}_j U_{ij} \quad (i = 2,3,\ldots 11) \tag{2}$$

C. Rotational motion of the Earth

$$\begin{aligned} \{\ddot{\beta}'\} &= \frac{1}{2} [\dot{\beta}'] \left(\begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} - \{f(t)\} \right) \\ &\quad + \frac{1}{2} [\beta'] \left(\begin{Bmatrix} 0 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} - \{f(\dot{t})\} \right) \end{aligned} \tag{3}$$

D. Rotational motion of the Moon

$$\{\dot{\beta}'''\} = \frac{1}{2}\,[\dot{\beta}''']\left(\begin{Bmatrix} 0 \\ \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \end{Bmatrix} - [c(\beta''')]_A \begin{Bmatrix} 0 \\ -\dot{\lambda}s\phi \\ \dot{\phi} \\ \dot{\lambda}c\phi \end{Bmatrix}\right)$$

$$+ \frac{1}{2}\,[\beta''']\left(\begin{Bmatrix} 0 \\ \dot{\omega}_1' \\ \dot{\omega}_2' \\ \dot{\omega}_3' \end{Bmatrix} - [c(\beta''')]_A \qquad (4)$$

$$\times \frac{d}{dt}\begin{Bmatrix} 0 \\ -\dot{\lambda}s\phi \\ \dot{\phi} \\ \dot{\lambda}c\phi \end{Bmatrix}\right)$$

The "short-hand" notation $s\phi = \sin(\phi)$ and $c\phi = \cos(\phi)$ has been utilized in the above equation.

A full discussion of the above equations including rationale and derivations appears later in this report.

Equations (1) represent the motion of the Sun with respect to the mass center of the Solar System as forced by the Moon and planets.

Equations (2) represent the motion of the Moon and planets with respect to the Sun. The gradient is to be taken with respect to the coordinates locating each body with respect to the Sun. The force function U_{ij} may be written as

$$U_{ij} = U_{ij}^P + U^I \qquad (5)$$

where U_{ij}^P represents the mutual gravitational interaction between all masses treated as particles and U^I is the mutual potential of the Earth and Moon regarded as finite rigid bodies.

Equations (3) represent the rotational motion of the Earth with respect to a defined "reference" coordinate frame. The Euler parameters $\{\beta'\}^T = \{\beta_0', \beta_1', \beta_2', \beta_3'\}$ represent the angular deviations of the Earth from this "reference" frame. The forcing torques enter through the angular acceleration terms,

$$\begin{Bmatrix} 0 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix}$$

The function $\{f(t)\}$ defines the reference frame. A solution of these equations represents the deviation of the Earth from a uniform rotation about a fixed axis in space, i.e., precession and nutation.

Equations (4) represent the rotational motion of the Moon with respect to a defined "reference" coordinate frame. The Euler parameters $\{\beta'''\}^T = \{\beta_0''', \beta_1''', \beta_2''', \beta_3'''\}$ represent the angular deviations of the earth from this "reference" frame. The forcing torques enter through the angular acceleration terms,

$$\begin{Bmatrix} 0 \\ \dot{\tilde{\omega}}_1 \\ \dot{\tilde{\omega}}_2 \\ \dot{\tilde{\omega}}_3 \end{Bmatrix} .$$

The rotation matrix $[c(\beta''')]_A$ and the terms in $-\dot{\lambda}s\phi$, $\mathring{\phi}$, $\dot{\lambda}c\phi$ and their derivatives account for the motion of the "reference" frame. A solution of these equations represents the optical plus the physical librations.

Rationale for Development of Equations

The various existing analytical theories for the translational and rotational motion of the Earth and Moon are being looked at more and more critically due to ever increasing observational accuracy (ref. 1).

Many previously neglected effects must now be included in the mathematical models used to reduce the observational data. This, of course, leads to a better knowledge of these small effects and their causes. Currently unknown effects may also be discovered as the observational data is analyzed.

The possibility of using LURE data to determine various geophysical and selenophysical parameters was pointed out in the introduction and in reference 2.

Accordingly, this report describes work undertaken to develop a mathematical model of the motion of the Earth-Moon system that has as few restrictions on accuracy as possible. Thus, the coupled rotational-translational motions of both the Earth and Moon are included in this model.

A more immediate and specialized goal, however, is to be able to solve for the coupled rotational-translational motion of the Moon for use in the reduction of LURE data to estimate the low-order gravitational harmonic coefficients of the Moon. Thus, this problem will be emphasized in this report.

Several recent papers (refs. 1, 3, 4, 5) have pointed out the facts that (1) analytical theories for the lunar translational motion and (2) analytical theories for the lunar rotational motion are not accurate enough to be used in the reduction of LURE-type data. Attempts are therefore being made to numerically integrate (1) the equations of motion for the lunar orbit (ref. 3), and (2) the equations governing lunar rotation (refs. 5,6).

The above facts and the success of Cohen and Oesterwinter (ref. 7) in numerically integrating the motion of the solar system have prompted this attempt at a numerical integration of the equations of motion representing the coupled translational-rotational motions of the Earth and Moon.

The formulation of both the translational and rotational equations of motion as a system of second-order differential equations was dictated by the general observation that Class II (second-order) numerical integration methods are more efficient (ref. 8).

Since the force and torque evaluations at each integration step are very costly in computer time, it was decided to utilize Euler parameters rather than Euler angles in the rotational equations. The relation of the rates of change of those parameters to the angular velocity components is algebraic rather than trigonometric in the case of Euler angle rates. Although two additional second-order equations are thereby added to the system, no trigonometric functions need be evaluated at each step. A time saving is thereby accomplished in the integration of the rotational equations. This approach has been common practice in the simulation of aircraft and gyroscopic motions (refs. 9,10). Advantages arise also in problem formulation and parameter estimation when Euler parameters are utilized (ref. 11).

The reference axes used in the rotational equations were chosen so that the large angular rotation rates of the Earth and Moon with respect to inertial space did not have to be integrated. The reference axis for the Earth spins with respect to an inertial system about a fixed axis with a fixed rate equal to the mean sidereal rotation rate of the Earth. The reference axis for lunar rotation is centered at the Moon's mass center and its primary axis points to the Earth's center of mass. The axes of this system are parallel to the unit vectors of a spherical polar coordinate system that locates the Moon with respect to a mean equator and equinox of 1950.0 rectangular system centered at the Earth. This approach is similar in philosophy to the Enke method of celestial mechanics.

Coordinate Systems

The coordinate systems utilized in this study are standard and are summarized in table 1 and illustrated in figures 1 and 2. Transformation between coordinate systems is accomplished using orthogonal rotation matrices in the sense

$$\{x'\} = [R_{xx'}]\ \{x\}$$

where $[R]$ is a 3 x 3 rotation matrix for a rotation of $\{x\}$ into $\{x'\}$.

Table 1. Coordinate systems.

No.	Origin of Frame	Axis Notation ($i = 1,2,3$)	Fundamental Plane	Fundamental Direction	Secondary Direction	Unit Vector Notation ($i = 1,2,3$)	Remarks
1	Barycenter of Solar System	X_i''	Ecliptic of 1950.0	Intersection of ecliptic of 1950.0 and mean equator of Earth of 1950.0	X_3'' points toward North Pole of ecliptic of 1950.0	$\vec{I}_i''$	Primary inertial reference frame
2	Barycenter of Solar System	X_i	Mean equator of Earth of 1950.0	Same as 1	X_3 points toward North Pole of rotation of Earth of 1950.0	$\vec{I}_i$	Secondary inertial reference frame
3	Center of mass of Sun	X_i'	Same as 2	Same as 2	Same as 2	$\vec{I}_i'$	Translating frame with respect to X_i
4	Center of mass of Earth	Y_i	Same as 2	Same as 2	Same as 2	$\vec{J}_i$	Reference frame for Earth rotation. Rotates at uniform rate of $\dot{\alpha}$ with respect to X_i'.

(cont'd.)

Table 1. Coordinate systems (concluded).

No.	Origin of Frame	Axis Notation (i = 1,2,3)	Fundamental Plane	Fundamental Direction	Secondary Direction	Unit Vector Notation (i = 1,2,3)	Remarks
5	Center of mass of Earth	y_i	Plane of equatorial principal axes	Axis of minimum principal moment of inertia	Axis of minimum principal moment of inertia	$\vec{j}_i$	Earth "body fixed" frame
6	Center of mass of Moon	Z_i	Plane formed by radial and longitud-inal unit vectors of spherical polar coor-dinate sys-tem locating the Moon with res-pect to $X_i^{T'}$ cen-tered at the Earth	Axis points from Moon's mass center to Earth's mass center	Axis points opposite to longitud-inal unit vector as described in column 4	$\vec{K}_i$	Reference frame for lunar rotation. An "orbi-tal ref-erence frame".
7	Center of mass of Moon	z_i	Plane of equatorial principal axes	Axis of minimum principal moment of inertia	Axis of minimum principal moment of inertia	$\vec{k}_i$	Moon "body fixed" frame

The superscript T on a set of axes indicates a frame translating with respect to the unsuper-scripted frame.

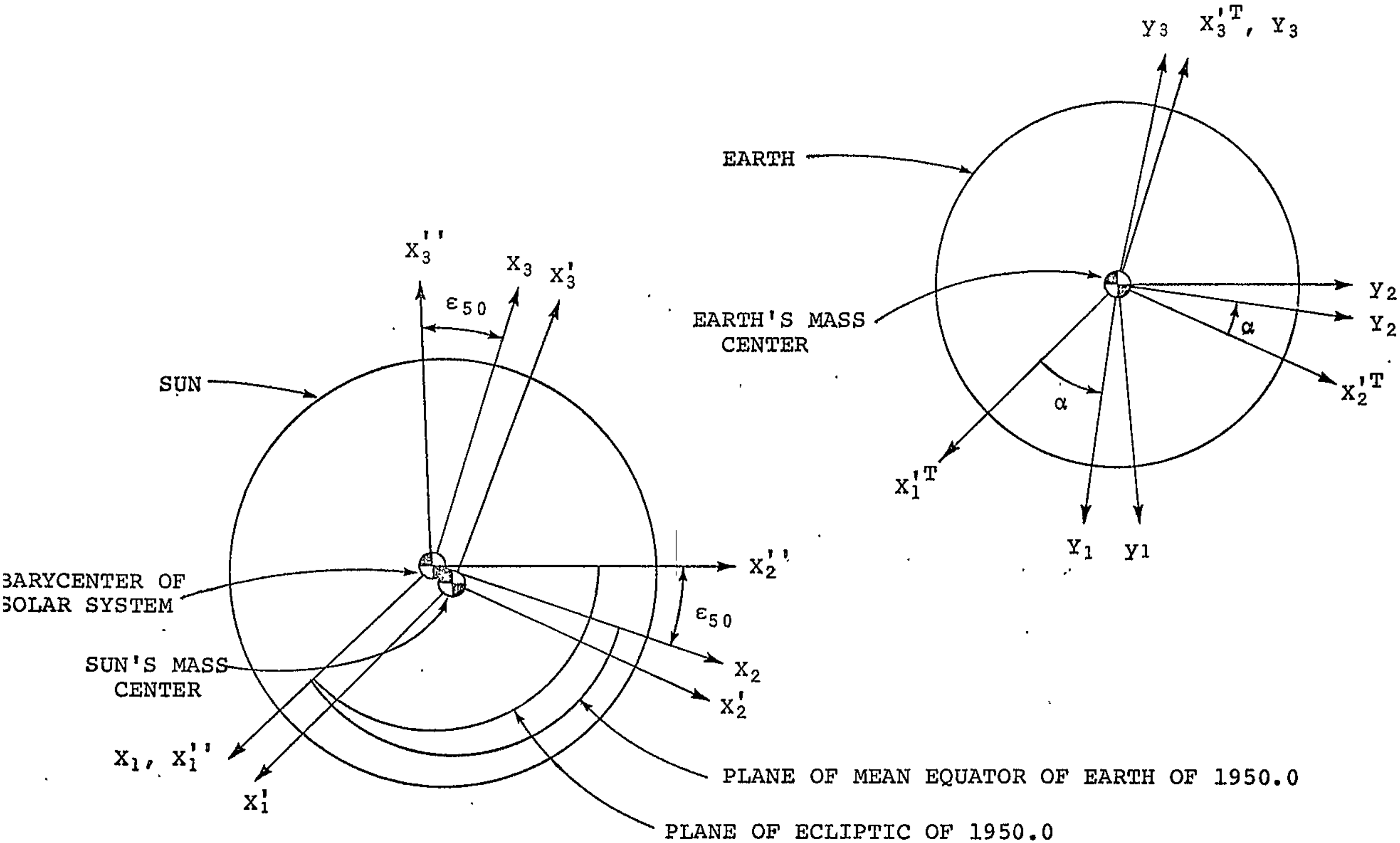

Figure 1. Coordinate reference frames.

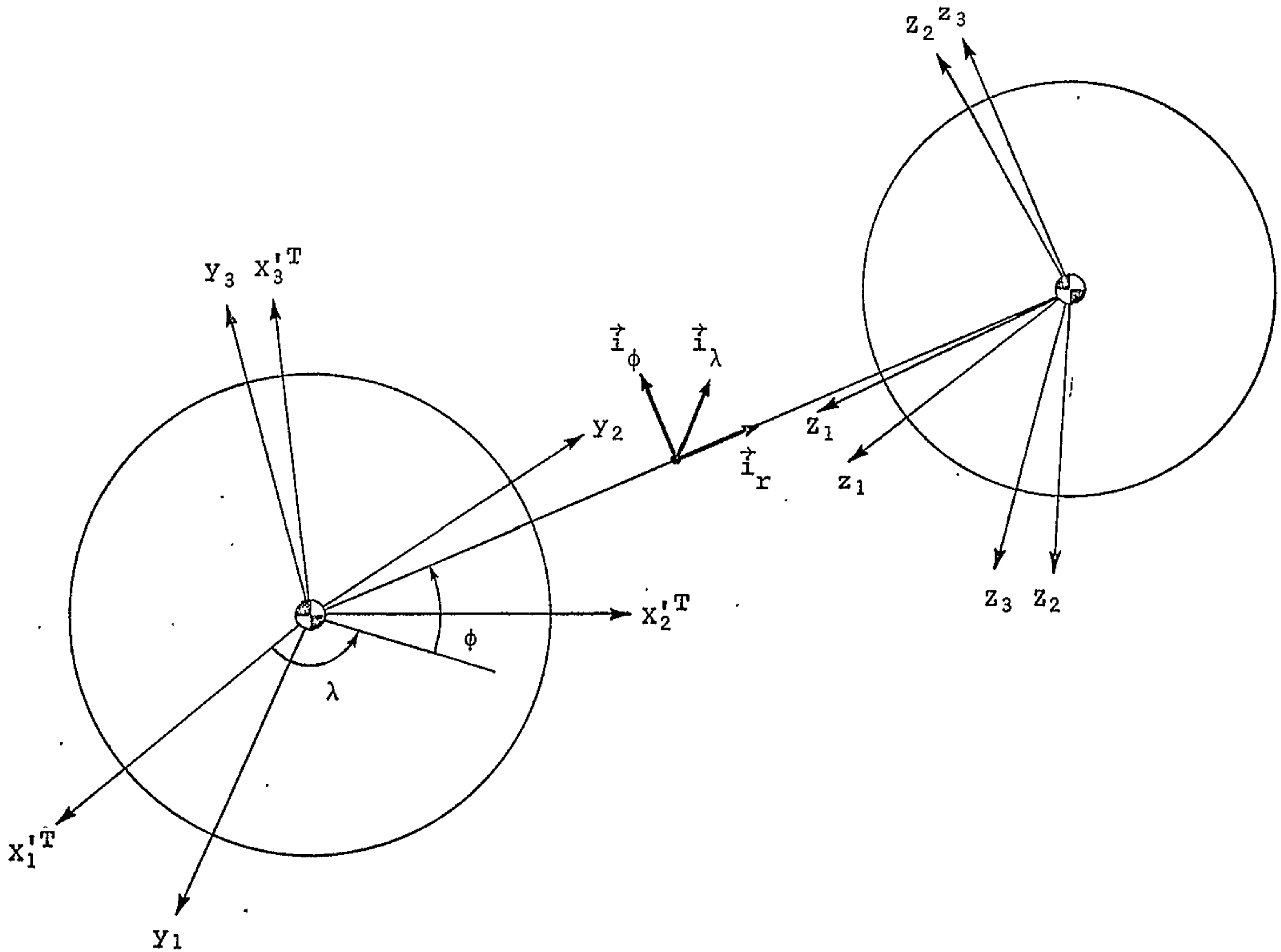

Figure 2. Coordinate reference frames.

Equations of Motion

Translational Equations. Reference 13 provides the equations of motion for the particles and centers of mass with respect to an intertial reference frame, viz.

$$m_i \ddot{\vec{r}}_i = \vec{\nabla}_i \overline{U} \qquad (i = 1,11) \tag{6}$$

where

$$\overline{U} = G \sum_{i>j=1}^{11}\sum \frac{m_i m_j}{r_{ij}} \tag{7}$$

and m_1 is the Sun's mass
m_2 is Mercury's mass
m_3 is Venus' mass
m_4 is Earth's mass
m_5 is the Moon's mass
m_6 is Mars' mass
m_7 is Jupiter's mass
m_8 is Saturn's mass
m_9 is Uranus' mass
m_{10} is Neptune's mass
m_{11} is Pluto's mass.

In equation (6),

$$\vec{r}_i = X_{1i} \vec{I}_1 + X_{2i} \vec{I}_2 + X_{3i} \vec{I}_3 \tag{8}$$

$$\vec{\nabla}_i = \vec{I}_1 \frac{\partial}{\partial X_{1i}} + \vec{I}_2 \frac{\partial}{\partial X_{2i}} + \vec{I}_3 \frac{\partial}{\partial X_{3i}} \tag{9}$$

where X_{1i}, X_{2i}, X_{3i} are the coordinates of mass i.

If the origin of coordinates is now translated to the Sun, the equations of motion for the Moon and planets are

$$\ddot{\vec{r}}_i + G(m_1 + m_i)\frac{\vec{r}_i}{r_{i1}^3} = G \sum_{\substack{j=2 \\ j\neq i}} m_j \left(\frac{\vec{r}_j - \vec{r}_i}{r_{ij}^3} - \frac{\vec{r}_j}{r_{j1}^3} \right) \tag{10}$$

where

$$r_{i1} = \sqrt{(X'_{1i} - X'_{11})^2 + (X'_{2i} - X'_{21})^2 + (X'_{3i} - X'_{31})^2} \ ,$$

$$r_{ij} = \sqrt{(X'_{1j} - X'_{1i})^2 + (X'_{2j} - X'_{2i})^2 + (X'_{3j} - X'_{3i})^2} \ , \text{ and}$$

$$\vec{r}_i = X'_{1i}\,\vec{I}'_1 + X'_{2i}\,\vec{I}'_2 + X'_{3i}\,\vec{I}'_3 \qquad (i = 2,\ldots,11).$$

The terms on the right-hand side of equation (10) arise from the force function U^P_{ij} of equation (5). Reference 12 provides U^P_{ij} as

$$U^P_{ij} = G\left(\frac{1}{r_{ij}} - \frac{\vec{r}_i \cdot \vec{r}_j}{r_{j1}^3}\right) \tag{11}$$

since

$$m_j \vec{\nabla}_i U^P_{ij} = Gm_j\left(\frac{\vec{r}_j - \vec{r}_i}{r_{ij}^3} - \frac{\vec{r}_j}{r_{j1}^3}\right)$$

The equations of motion for the Sun are

$$\ddot{\vec{r}}_1 = G \sum_{j=2}^{11} m_j \frac{\vec{r}_j}{r_{1j}^3} \tag{12}$$

where

$$r_j = X'_{1j} \vec{i}'_1 + X'_{2j} \vec{i}'_2 + X'_{3j} \vec{i}'_3$$

These equations follow directly from equations (6) and (7).

As will be discussed later, the mutual gravitational potential of the Earth and Moon, treated as finite rigid bodies, may be important to LURE accuracy. Thus, $\overline{U}$ in equations (6) and (7) should be of the form

$$\overline{U} = G \sum_{i>j=1}^{11}\sum^{11} \frac{m_i m_j}{r_{ij}}$$

$$+ \frac{Gm_4 m_5}{r_{45}} \left\{ \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{1}{(r_{45})^n} P_{nm} (s\phi) \right. \tag{13}$$

$$\left. \cdot \left[X_{nm} \cos m\lambda + Y_{nm} \sin m\lambda \right] \right\}.$$

Thus, if the origin is shifted to the sun, the resulting potential may be written as

$$U_{ij} = U_{ij}^{P} + U_{45}^{I}$$

where

$$\left.\begin{aligned} U_{ij}^{P} &= G\left(\frac{1}{r_{ij}} - \frac{\vec{r}_i \cdot \vec{r}_j}{r_{j1}^3}\right) \qquad \begin{matrix} j = 2,\ldots,11 \\ i \neq j \end{matrix} \\ U_{45}^{I} &= \frac{G}{r_{45}} \left\{ \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{1}{n_{45}}\right)^n P_{nm}(s\phi) \right. \\ &\qquad \left. \cdot \left[X_{nm} \cos m\lambda + Y_{nm} \sin m\lambda\right] \right\} \end{aligned}\right\} \tag{14}$$

Rotational Equations for the Earth. The rotational motion of a rigid earth must satisfy Euler's principal axis equations, viz.

$$\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} = \begin{Bmatrix} M_1/A \\ M_2/B \\ M_3/C \end{Bmatrix} - \begin{Bmatrix} k_1\omega_2\omega_3 \\ k_2\omega_1\omega_3 \\ k_3\omega_1\omega_2 \end{Bmatrix} \tag{15}$$

where

$$k_1 = (C - B)/A$$

$$k_2 = (A - C)/B$$

$$k_3 = (B - A)/C \,.$$

In the above, ω_i are inertial angular velocity components in the y_i frames. The moment components M_i likewise are in this frame. A, B, C are the principal moments of inertia of the Earth. In order to orient the Earth with respect to the "reference" axes and the inertial axes the following sets of Euler parameters are introduced:

1) $\{\beta\}$ represents a notation from $\{X_i'\}$ to $\{Y_i\}$,

2) $\{\beta'\}$ represents a notation from $\{Y_i\}$ to $\{y_i\}$, and

3) $\{\beta''\}$ represents a notation from $\{X_i'\}$ to $\{y_i\}$,

where

$$\{\beta\}^T = \lfloor \beta_0, \ \beta_1, \ \beta_2, \ \beta_3 \rfloor \ .$$

Reference 13 provides the following relation between the sets of Euler parameters representing the above successive rotations:

$$\{\beta''\} = [\tilde{\beta}'] \{\beta\} = [\beta] \{\beta'\} \tag{16}$$

where

$$[\tilde{\beta}'] = \begin{bmatrix} \beta_0' & -\beta_1' & -\beta_2' & -\beta_3' \\ \beta_1' & \beta_0' & \beta_3' & -\beta_2' \\ \beta_2' & -\beta_3' & \beta_0' & \beta_1' \\ \beta_3' & \beta_2' & -\beta_1' & \beta_0' \end{bmatrix} \tag{17}$$

and where

$$[\beta] = \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \tag{18}$$

The rotation matrices linking the above coordinate systems are

$$\{Y_i\} = [R_{X'Y}]\ \{X_i'\} = [c(\beta)]\ \{X_i'\} \quad , \tag{19}$$

$$\{y_i\} = [R_{Yy}]\ \{Y_i\} = [c(\beta')]\ \{Y_i\} \ , \text{ and} \tag{20}$$

$$\{y_i\} = [R_{X'y}]\ \{X_i'\} = [c(\beta'')]\ \{X_i'\} \tag{21}$$

The rotation matrices introduced above have the following form where expressed in terms of Euler parameters (ref. 13):

$$[c(\beta)] = \begin{bmatrix} \beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2 & 2(\beta_1\beta_2 + \beta_0\beta_3) & 2(\beta_1\beta_3 - \beta_0\beta_2) \\ 2(\beta_1\beta_2 - \beta_0\beta_3) & \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2 & 2(\beta_2\beta_3 + \beta_0\beta_1) \\ 2(\beta_1\beta_3 + \beta_0\beta_2) & 2(\beta_2\beta_3 - \beta_0\beta_1) & \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2 \end{bmatrix} \tag{22}$$

The matrices $[\beta']$, $[\tilde{\beta}']$, $[c(\beta)]$, etc. are all orthogonal and hence their inverses are their transposes (ref. 13).

The angular velocity of the Earth can now be expressed in terms of the Euler parameters and rates. The inertial angular velocity of the Earth is

$$\vec{\omega} = \omega_1\vec{J}_1 + \omega_2\vec{J}_2 + \omega_3\vec{J}_3 \tag{23}$$

or

$$\vec{\omega} = \vec{\omega}_{Y/X'} + \vec{\omega}_{y/Y} \ . \tag{24}$$

The reference axes, Y_i, rotate at a uniform rate $\dot{\alpha}$ about the $\vec{J}_3$ axis so that

$$\vec{\omega}_{Y/X'} = \dot{\alpha}\ \vec{J}_3 \quad . \tag{25}$$

Reference 13 provides the following relation between the Euler parameters, their rates, and the angular velocity components in the rotating system:

$$\begin{Bmatrix} 0 \\ \omega_{1_{Y/X'}} \\ \omega_{2_{Y/X'}} \\ \omega_{3_{Y/X'}} \end{Bmatrix} = 2[\beta']^{-1} \; \{\dot{\beta}'\} \tag{26}$$

where

$$\{\dot{\beta}'\}^T = \lfloor \dot{\beta}_0 \dot{\beta}_1 \dot{\beta}_2 \dot{\beta}_3 \rfloor$$

To provide a consistent four-parameter representation of the angular velocity of the Earth, the vector $\vec{\omega}_{Y/X'}$ must be projected on the $\{y_i\}$ axes and certain "augmented" matrices must be introduced. Accordingly, the angular velocity of the Earth assumes the form

$$\begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = [c(\beta')]_A \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dot{\alpha} \end{Bmatrix} + 2[\beta']^{-1} \; \{\dot{\beta}'\} \tag{27}$$

where the "augmented" rotation matrix $[c(\beta')]_A$ is of the form

$$[c(\beta')]_A = \left[\begin{array}{c|ccc} 0 & 0 & & 0 \\ \hline 0 & & & \\ 0 & & [c(\beta')] & \\ 0 & & & \end{array}\right] \tag{28}$$

The nature of the "reference" axes, Y_i, provides a very simple form for $[c(\beta')]$, viz.

$$[c(\beta')] = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

where $\alpha = \alpha_o + \dot{\alpha}t$. The combination

$$[c(\beta')]_A \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dot{\alpha} \end{Bmatrix}$$

is now defined as $\{f(t)\}$ and has the form

$$\{f(t)\} = \begin{Bmatrix} 0 \\ 2\dot{\alpha}(\beta_1'\beta_3' - \beta_0'\beta_2') \\ 2\dot{\alpha}(\beta_2'\beta_3' + \beta_0'\beta_1') \\ \dot{\alpha}(\beta_0'^2 - \beta_1'^2 - \beta_2'^2 + \beta_3'^2) \end{Bmatrix} \tag{30}$$

A set of second-order equations can now be formed by differentiating equations (27) and solving for $\{\ddot{\beta}'\}$. These equations assume the form

$$\begin{aligned} \{\ddot{\beta}'\} = {} & \frac{1}{2}[\dot{\beta}']\left(\begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} - \{f(t)\}\right) \\ & + \frac{1}{2}[\beta']\left(\begin{Bmatrix} 0 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} - \{\dot{f}(t)\}\right) \end{aligned} \tag{31}$$

In equation (31),

$$\begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}, \quad \{f(t)\}, \quad \text{and} \quad \begin{Bmatrix} 0 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix}$$

are provided in equations (27), (30), and (15) respectively. The term $\{\dot{f}(t)\}$ is of the form

$$\{\dot{f}(t)\} = \begin{Bmatrix} 0 \\ 2\dot{\alpha}(\beta_1'\dot{\beta}_3' + \dot{\beta}_1'\beta_3' - \beta_0'\dot{\beta}_2' - \dot{\beta}_0'\beta_2') \\ 2\dot{\alpha}(\beta_2'\dot{\beta}_3' + \dot{\beta}_2'\beta_3' + \beta_0'\dot{\beta}_1' + \beta_0'\beta_1') \\ 2\dot{\alpha}(\beta_0'\dot{\beta}_0' - \beta_1'\dot{\beta}_1' - \beta_2'\dot{\beta}_2' + \beta_3'\dot{\beta}_3') \end{Bmatrix} \tag{32}$$

and $[\dot{\beta}']$ is of the form

$$[\dot{\beta}'] = \begin{bmatrix} \dot{\beta}_0' & -\dot{\beta}_1' & -\dot{\beta}_2' & -\dot{\beta}_3' \\ \dot{\beta}_1' & \dot{\beta}_0' & -\dot{\beta}_3' & \dot{\beta}_2' \\ \dot{\beta}_2' & \dot{\beta}_3' & \dot{\beta}_0' & -\dot{\beta}_1' \\ \dot{\beta}_3' & -\dot{\beta}_2' & \dot{\beta}_1' & \dot{\beta}_0' \end{bmatrix} \tag{33}$$

<u>Rotational Equations for the Moon</u>. The derivation of the equations of rotational motion for the Moon proceeds along a path similar to that used for the Earth.

The rotational motion of a rigid Moon must satisfy Euler's equations, viz.

$$\begin{Bmatrix} \dot{\omega}_1' \\ \dot{\omega}_2' \\ \dot{\omega}_3' \end{Bmatrix} = \begin{Bmatrix} M_1'/A' \\ M_2'/B' \\ M_3'/C' \end{Bmatrix} - \begin{Bmatrix} k_1' & \omega_2' & \omega_3' \\ k_2' & \omega_1' & \omega_3' \\ k_3' & \omega_1' & \omega_2' \end{Bmatrix} \tag{34}$$

where all quantities here have analogous definitions to those of equation (15). Primes are used to distinguish variables related to the Moon from those related to the Earth (un-primed).

The inertia rations k_i' in equations (32) have a more familiar notation, viz.

$$\begin{aligned} k_1' &= \alpha \\ k_2' &= -\beta \\ k_3' &= \gamma \quad . \end{aligned} \tag{35}$$

These ratios are related by the constraint

$$\alpha = \frac{\beta - \gamma}{1 - \beta\gamma} \tag{36}$$

Euler parameters are now introduced to orient the Moon with respect to its "reference" axes and the inertial frame. For this purpose, define

$$\{\beta'''\} = \begin{Bmatrix} \beta_0''' \\ \beta_1''' \\ \beta_2''' \\ \beta_3''' \end{Bmatrix}$$

which represents a rotation from $\{Z_i\}$ to $\{z_i\}$. The corresponding rotation matrix is

$$\{z_i\} = [R_{Zz}]\ \{Z_i\} = [c(\beta''')]\ \{Z_i\} \tag{37}$$

The inertial angular velocity of the Moon is

$$\vec{\omega}' = \vec{\omega}_{Z/X'} + \vec{\omega}_{z/Z} \quad . \tag{38}$$

In terms of the Euler parameters $\{\beta'''\}$ and their rates, the components of $\vec{\omega}_{z/Z}$ are

$$\begin{Bmatrix} 0 \\ \omega_{1z/Z} \\ \omega_{2z/Z} \\ \omega_{3z/Z} \end{Bmatrix} = 2[\beta''']^{-1} \ \{\mathring{\beta}'''\} \quad . \tag{39}$$

The angular velocity $\vec{\omega}_{Z/X'}$ of the "reference" frame is defined completely by the translational motion of the Moon with respect to the Earth.

To determine the angular velocity $\vec{\omega}_{Z/X'}$, introduce the spherical polar coordinates r, λ, ϕ as illustrated in figure 2. These are the coordinates of the Moon's center of mass with respect to $\{X_i'\}$.

Now, define the "relative" rectangular coordinates, Δ_i, as

$$\begin{aligned} \Delta_1 &= X'_{15} - X'_{14} = r\cos\phi\cos\lambda \\ \Delta_2 & X'_{25} - X'_{24} = r\cos\phi\sin\lambda \\ \Delta_3 & X'_{35} - X'_{34} = r\sin\phi \quad . \end{aligned} \tag{40}$$

Inverting these expressions provides

$$r = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$$

$$\lambda = \tan^{-1}(\Delta_2/\Delta_1) \quad , \qquad \text{and} \tag{41}$$

$$\phi = \tan^{-1}(\Delta_3/\sqrt{\Delta_1^2 + \Delta_2^2}) \quad .$$

Since the unit vectors $\vec{k}_i$ are related to those of the spherical polar system by

$$\begin{aligned} \vec{k}_1 &= -\vec{\lambda}_r \\ \vec{k}_2 &= -\vec{\lambda}_\lambda \\ \vec{k}_3 &= \vec{\lambda}_\phi \quad , \end{aligned} \tag{42}$$

the inertial angular velocity of the axes $\{Z_i\}$ can be written as

$$\vec{\omega}_{Z/X'} = \dot{\vec{\lambda}} + \dot{\vec{\phi}} = \dot{\lambda}\,\vec{I}_3' - \dot{\phi}\,\vec{\lambda}_\lambda \quad . \tag{43}$$

The above vector may be projected on the $\{Z_i\}$ axes providing

$$\vec{\omega}_{Z/X'} = \dot{\lambda}\left[\cos\phi\,\vec{k}_3 - \sin\phi\,\vec{k}_1\right] + \dot{\phi}\,\vec{k}_2 \quad . \tag{44}$$

The components

$$\begin{aligned} \omega_1' &= -\dot{\lambda}\sin\phi \\ \omega_2' &= \dot{\lambda}\cos\phi \\ \omega_3' &= \dot{\phi} \end{aligned}$$

can be related to the relative position coordinates Δ_i and the relative velocity components $\dot{\Delta}_i$. To do this, differentiate equations (40), solve for $\dot{\Delta}_i$, and put the results in the matrix form

$$\begin{Bmatrix} \dot{\Delta}_1 \\ \dot{\Delta}_2 \\ \dot{\Delta}_3 \end{Bmatrix} = \begin{bmatrix} c\phi c\lambda & -s\lambda & -c\lambda s\phi \\ c\phi s\lambda & c\lambda & -s\lambda s\phi \\ s\phi & 0 & c\phi \end{bmatrix} \begin{Bmatrix} \dot{r} \\ r\dot{\lambda}c\phi \\ r\dot{\phi} \end{Bmatrix} \tag{45}$$

Equation (45) can be inverted to provide

$$\begin{Bmatrix} \dot{r} \\ r\dot{\lambda}c\phi \\ r\dot{\phi} \end{Bmatrix} = \begin{bmatrix} c\phi c\lambda & s\lambda c\phi & c\phi \\ -s\lambda & c\lambda & 0 \\ -c\lambda s\phi & -s\lambda s\phi & c\phi \end{bmatrix} \begin{Bmatrix} \dot{\Delta}_1 \\ \dot{\Delta}_2 \\ \dot{\Delta}_3 \end{Bmatrix} \tag{46}$$

In the above equations, the "short-hand" notation $c\phi \equiv \cos\phi$ and $s\phi \equiv \sin\phi$ has been utilized.

Now, the components of $\vec{\omega}_{\overline{Z/X'}}$ in the $\{z_i\}$ frame are

$$\begin{Bmatrix} \omega_{1Z/X'} \\ \omega_{2Z/X'} \\ \omega_{3Z/X'} \end{Bmatrix} = [c(\beta''')] \begin{Bmatrix} -\dot{\lambda}s\phi \\ \dot{\phi} \\ \dot{\lambda}c\phi \end{Bmatrix} \tag{47}$$

The quantities $-\dot{\lambda}s\phi$, $\dot{\phi}$, $\dot{\lambda}c\phi$ follow from equations (46) and (40) as follows:

$$\dot{\lambda} s\phi = \frac{\Delta_3}{r} \left[\frac{-\dot{\Delta}_1 s\lambda + \dot{\Delta}_2 c\lambda}{\sqrt{\Delta_1^2 + \Delta_2^2}} \right]$$

$$\dot{\phi} = \frac{1}{r} \; [-\dot{\Delta}_1 c\lambda s\phi - \dot{\Delta}_2 s\lambda s\phi + \dot{\Delta}_3 c\phi] \tag{48}$$

$$\dot{\lambda} c\phi = \frac{1}{r} \; [-\dot{\Delta}_1 s\lambda + \dot{\Delta}_2 c\lambda]$$

where

$$\begin{aligned} s\lambda c\phi &= \Delta_2/r \\ s\phi &= \Delta_3/r \\ c\lambda c\phi &= \Delta_1/r \\ c\phi &= \sqrt{\Delta_1^2 + \Delta_2^2}\,/r \\ s\lambda &= \Delta_2/r \;. \end{aligned}$$

The absolute angular velocity of the Moon can now be obtained by adding (39) and (47) to obtain in augmented form,

$$\begin{Bmatrix} 0 \\ \omega_1' \\ \omega_2' \\ \omega_3' \end{Bmatrix} = 2[\beta''']^{1} \{\dot{\beta}'''\} + [c(\ ''')]_A \begin{Bmatrix} 0 \\ -\dot{\lambda} s\phi \\ \phi \\ \dot{\lambda} c\phi \end{Bmatrix} \tag{49}$$

Equation (49) can be solved for $\{\dot{\beta}'''\}$ and differentiated to obtain the second order equation for $\{\beta'''\}$. Thus,

$$\{\ddot{\beta}'''\} = \frac{1}{2}\,[\dot{\beta}''']\left(\begin{Bmatrix} 0 \\ \omega_1' \\ \omega_2' \\ \omega_3' \end{Bmatrix} - [c(\beta''')]_A \begin{Bmatrix} 0 \\ -\dot{\lambda}s\phi \\ \dot{\phi} \\ \dot{\lambda}c\phi \end{Bmatrix}\right)$$

$$+ \frac{1}{2}\,[\beta''']\left(\begin{Bmatrix} 0 \\ \dot{\omega}_1' \\ \dot{\omega}_2' \\ \dot{\omega}_3' \end{Bmatrix} - [c(\beta''')]_A \frac{d}{dt} \begin{Bmatrix} 0 \\ -\dot{\lambda}s\phi \\ \dot{\phi} \\ \dot{\lambda}c\phi \end{Bmatrix}\right. \qquad (50)$$

$$\left. - \frac{d}{dt}\,[c(\beta''')]_A \begin{Bmatrix} 0 \\ -\dot{\lambda}s\phi \\ \dot{\phi} \\ \dot{\lambda}c\phi \end{Bmatrix}\right)$$

In equation (50), $[\beta''']$ is of the form given in equation (18); $[c(\beta''')]_A$ is of the form given in equations (22) and (28); $[\dot{\beta}''']$ is of the form given in equation (33); $\lfloor 0\ \dot{\omega}_1'\ \dot{\omega}_2'\ \dot{\omega}_3' \rfloor$ are given in equations (34); $\lfloor 0\ \omega_1'\ \omega_2'\ \omega_3' \rfloor$ are given in equations (49); and the elements of $[c(\beta''')]_A$ are

$$c_{11_A} = c_{12_A} = c_{13_A} = c_{14_A} = c_{21_A} = c_{31_A} = c_{41_A} = 0$$

$$c_{22_A} = 2(\beta_0\dot{\beta}_0 + \beta_1\dot{\beta}_1 - \beta_2\dot{\beta}_2 - \beta_3\dot{\beta}_3)$$

$$c_{33_A} = 2(\beta_0\dot{\beta}_0 - \beta_1\dot{\beta}_1 + \beta_2\dot{\beta}_2 - \beta_3\dot{\beta}_3) \qquad (51)\ \text{(cont'd.)}$$

$$c_{44_A} = 2(\beta_0\dot{\beta}_0 - \beta_1\dot{\beta}_1 - \beta_2\dot{\beta}_2 + \beta_3\dot{\beta}_3)$$

$$c_{23_A} = 2(\beta_1\dot{\beta}_2 + \beta_1\dot{\beta}_2 + \beta_0\dot{\beta}_3 + \beta_0\dot{\beta}_3)$$

$$c_{24_A} = 2(\beta_1\dot{\beta}_3 + \dot{\beta}_1\beta_3 - \beta_0\dot{\beta}_2 - \dot{\beta}_0\beta_2)$$

$$c_{32_A} = 2(\beta_1\dot{\beta}_2 + \dot{\beta}_1\beta_2 - \beta_0\dot{\beta}_3 - \dot{\beta}_0\beta_3)$$

$$c_{34_A} = 2(\beta_2\dot{\beta}_3 + \dot{\beta}_2\beta_3 + \beta_0\dot{\beta}_1 + \dot{\beta}_0\beta_1)$$

$$c_{42_A} = 2(\beta_1\dot{\beta}_3 + \dot{\beta}_1\beta_3 + \beta_0\dot{\beta}_2 + \dot{\beta}_0\beta_2) \qquad \text{(51) (concl'd.)}$$

$$c_{43_A} = 2(\beta_2\dot{\beta}_3 + \dot{\beta}_2\beta_3 - \beta_0\dot{\beta}_1 - \dot{\beta}_0\beta_1)$$

All β's in the above equations have a triple-prime superscript.

The derivatives of the polar coordinates and their rates in equation (50) are

$$\frac{d}{dt}(\dot{\lambda}c\phi) = \dot{\Delta}_1 \left[-\frac{c\lambda\dot{\lambda}}{r} + \frac{\dot{r}s\lambda}{r^2}\right] + \dot{\Delta}_2 \left[-\frac{s\lambda\dot{\lambda}}{r} - \frac{\dot{r}c\lambda}{r^2}\right] - \ddot{\Delta}_1 \frac{s\lambda}{r} + \ddot{\Delta}_2 \frac{c\lambda}{r} \qquad (52)$$

$$\begin{aligned}\frac{d}{dt}(\dot{\phi}) = {} & \dot{\Delta}_1 \left[\frac{\dot{\lambda}s\lambda s\phi}{r} + \frac{\dot{r}c\lambda s\phi}{r^2} - \frac{\dot{\phi}c\lambda c\phi}{r}\right] \\ & + \dot{\Delta}_2 \left[-\frac{\dot{\lambda}c\lambda s\phi}{r} + \frac{\dot{\phi}s\lambda c\phi}{r} + \frac{\dot{r}s\lambda s\phi}{r^2}\right] \\ & + \dot{\Delta}_3 \left[-\frac{\dot{\phi}s\phi}{r} - \frac{\dot{r}c\phi}{r^2}\right] \\ & - \ddot{\Delta}_1 \frac{c\lambda s\phi}{r} - \ddot{\Delta}_2 \frac{s\lambda s\phi}{r} + \ddot{\Delta}_3 \frac{c\phi}{r}\end{aligned} \qquad (53)$$

$$\frac{d}{dt}(\dot{\lambda}s\phi) = \frac{\dot{\lambda}\dot{\phi}}{c\phi} + \frac{s\phi}{c\phi}\left\{\frac{d}{dt}(\dot{\lambda}c\phi)\right\} \qquad (54)$$

where

$$\dot{\Delta}_i = \dot{X}'_{i5} - \dot{X}'_{i4}$$
$$\ddot{\Delta}_i = \ddot{X}'_{i5} - \ddot{X}'_{i4} \qquad (55)$$

are available from the integration of equations (10).

Forces and Torques

The gravitational forces between a set of particles was given in equations (10). If the Earth and Moon are considered as rigid bodies then mutual gravitational forces and torques arise and must be modeled. Also torques exerted on the Earth and Moon due to a point mass Sun must also be considered. These are developed in this section.

Mutual Gravitational Potential. There are two approaches in the literature for deriving the mutual forces and torques. Approach (A) involves deriving a mutual gravitational potential and then finding the gradient of this potential with respect to the translational and rotational variables to give the forces and torques (refs. 14, 15). Approach (B) involves a direct integration of a differential force and torque over both bodies (ref. 16).

Approach A appears to be more easily developed when higher order gravity harmonics than the second are included for each body. Approach B is more concise than A for the case when only second order terms are retained for either one or the other body. Also, the effect of Earth oblateness on lunar torques can readily be derived using this approach.

For generality and ease of extension to higher orders, Approach A will be followed here. The concise results of Approach B are presented in Appendix A.

For the purposes of this report the Earth and Moon will be modeled as follows:

Earth (ref. 17, $\{y_i\}$)

$c_{20} = -1.082637 \times 10^{-3}$

$c_{21} = s_{21} = 0$

$c_{22} = 1.5362 \times 10^{-5}$

$s_{22} = -8.8149 \times 10^{-7}$

These values provide the following moments of inertia:

$A = .33912\ Ma^2$

$B = .33906\ Ma^2$

$C = .34017\ Ma^2$

if the dynamical flattening $H = (C - A)/c = 3.27293 \times 10^{3}$ is adopted from reference 20.

Moon (refs. 18, 19, $\{z_i\}$)

$c_{20} = -2.0272 \times 10^{-4}$	(ref. 18)
$c_{21} = s_{21} = 0$	(ref. 19)
$c_{22} = 2.221 \times 10^{-5}$	(ref. 18)
$s_{22} = 0.0$	(ref. 19)
$c_{30} = 3.9 \times 10^{-6}$	(ref. 19)
$c_{31} = 28.6 \times 10^{-6}$	(ref. 19)
$c_{32} = 6.0 \times 10^{-6}$	(ref. 19)
$c_{33} = 2.7 \times 10^{-6}$	(ref. 19)
$s_{31} = 8.8 \times 10^{-6}$	(ref. 19)
$s_{32} = 1.8 \times 10^{-6}$	(ref. 19)
$s_{33} = -1.4 \times 10^{-6}$	(ref. 19)
$c_{40} = 23.3 \times 10^{-6}$	(ref. 19)

$c_{41} = 11.1 \times 10^{-6}$ (ref. 19)

$c_{42} = -2.48 \times 10^{-6}$ (ref. 19)

$c_{43} = -0.17 \times 10^{-6}$ (ref. 19)

$c_{44} = -0.25 \times 10^{-6}$ (ref. 19)

$s_{41} = -2.61 \times 10^{-6}$ (ref. 19)

$s_{42} = -3.28 \times 10^{-6}$ (ref. 19)

$s_{43} = -0.45 \times 10^{-6}$ (ref. 19)

$s_{44} = 0.27 \times 10^{-6}$ (ref. 19)

These values provide the following moments of inertia:

$A' = .391753 \; M'a'^2$

$B' = .391842 \; M'a'^2$

$C' = 0.392 \; M'a'^2$ (ref. 18)

if the values of β and γ are taken to be

$\beta = 631.1 \times 10^{-6}$

$\gamma = 226.8 \times 10^{-6}$

as in reference 18.

Reference 15 provides the general form of the mutual potential between two arbitrarily shaped rigid bodies in the form

$$U^I_{45} = \sum_{\ell=2}^{\infty} U^I_{\ell} = G'r^{-1} \left\{ \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{1}{r^n} P_{nm}(\sin \phi) \cdot [X_{nm} \cos m\lambda + Y_{nm} \sin m\lambda] \right\} , \tag{56}$$

where the term Gr^{-1} has been included in U^p_{ij}, and where $U^I_1 = 0$ due to the choice of coordinate system. Here, M is the mass of the Earth and M' is the mass of the Moon. The X_{nm} and Y_{nm} are functions of a^p, a'^q, c_{pr}, c'_{qs}, s_{pr}, s'_{qs} where a and a' are the mean equatorial radii for masses M and M' and the c's and s's represent the harmonic coefficients for both bodies. Reference 15 provides the lower order values of X_{nm} and Y_{nm} as follows:

$$
\begin{aligned}
X_{2j} &= a^2 c_{2j} + a'^2 c_{2j} \qquad (j = 0,1,2) \\
Y_{2j} &= a\ s_{2j} + a'\ s_{2j} \qquad (j = 1,2) \\
X_{40} &= 6a^2 a'^2 (c_{20} c'_{21} - 2c_{21} c'_{21} - 2s_{21} s'_{21} \\
&\qquad + 2c_{22} c'_{22} + 2s_{22} s'_{22}) \\
X_{41} &= 3a^2 a'^2 (c_{21} c'_{21} + c_{21} c'_{20} - c_{21} c'_{22} \\
&\qquad - c_{22} c'_{21} + s_{21} s'_{22} - s_{22} s'_{21}) \\
Y_{41} &= 3a^2 a'^2 (c_{20} s'_{21} + s_{21} c'_{20} - c_{21} s'_{22} \\
&\qquad - s_{22} c'_{21} + s_{21} c'_{22} + c_{22} s'_{21}) \\
X_{42} &= a^2 a'^2 (c_{20} c'_{22} + c_{22} c'_{20} + c_{21} c'_{21} - s_{21} s'_{21}) \\
Y_{42} &= a^2 a'^2 (c_{20} s'_{22} + s_{22} c'_{20} + c_{21} s'_{21} + s_{21} c'_{21}) \\
X_{43} &= \frac{1}{2} a^2 a'^2 (c_{21} c'_{22} + c_{22} c'_{21} - s_{21} s'_{22} - s_{22} s'_{21}) \\
Y_{43} &= \frac{1}{2} a^2 a'^2 (c_{21} s'_{22} + c_{22} s'_{21} + s_{21} c'_{22} + s_{22} c'_{21}) \\
X_{44} &= \frac{1}{2} a^2 a'^2 (c_{22} c'_{22} - s_{22} s'_{22}) \\
Y_{44} &= \frac{1}{2} a^2 a'^2 (c_{22} s'_{22} + s_{22} c'_{22})
\end{aligned}
\tag{57}
$$

The coordinate systems implicit in equation (56) is illustrated in figure 3.

If $\{y_i\}$ is chosen as the primary reference, then c_{pr}, s_{pr} are the harmonic coefficients of the Earth referenced to $\{y_i\}$ and c'_{qs}, s'_{qs} are those of the Moon referenced to $\{y_i^T\}$. Likewise, r, λ, and ϕ are the spherical polar coordinates of the lunar mass center with respect to $\{y_i\}$. If the $\{z_i\}$ axes are chosen as the primary reference, then the above quantities are referred to $\{z_i\}$ and $\{z_i^T\}$.

The relative orientation of the above axis systems can be expressed as follows:

$$\{y_i\} = \begin{bmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{bmatrix} \{z_i\} = [\ell]\{z_i\} \tag{58}$$

with a more precise definition of the α's, β's, and γ's to be given later.

Force on Moon Due to Earth. In terms of spherical polar coordinates r, λ, ϕ locating the Moon with respect to $\{z_i^T\}$, the vector force may be written as

$$\vec{F}_5^I = m_5 \left[\frac{\partial U_{45}^I}{\partial r} \vec{i}_r + \frac{1}{r \cos \phi} \frac{\partial U_{45}^I}{\partial \lambda} \vec{i}_\lambda + \frac{1}{r} \frac{\partial U_{45}^I}{\partial \phi} \vec{i}_\phi \right]. \tag{59}$$

The general mutual potential may now be specialized to the problem at hand as follows:

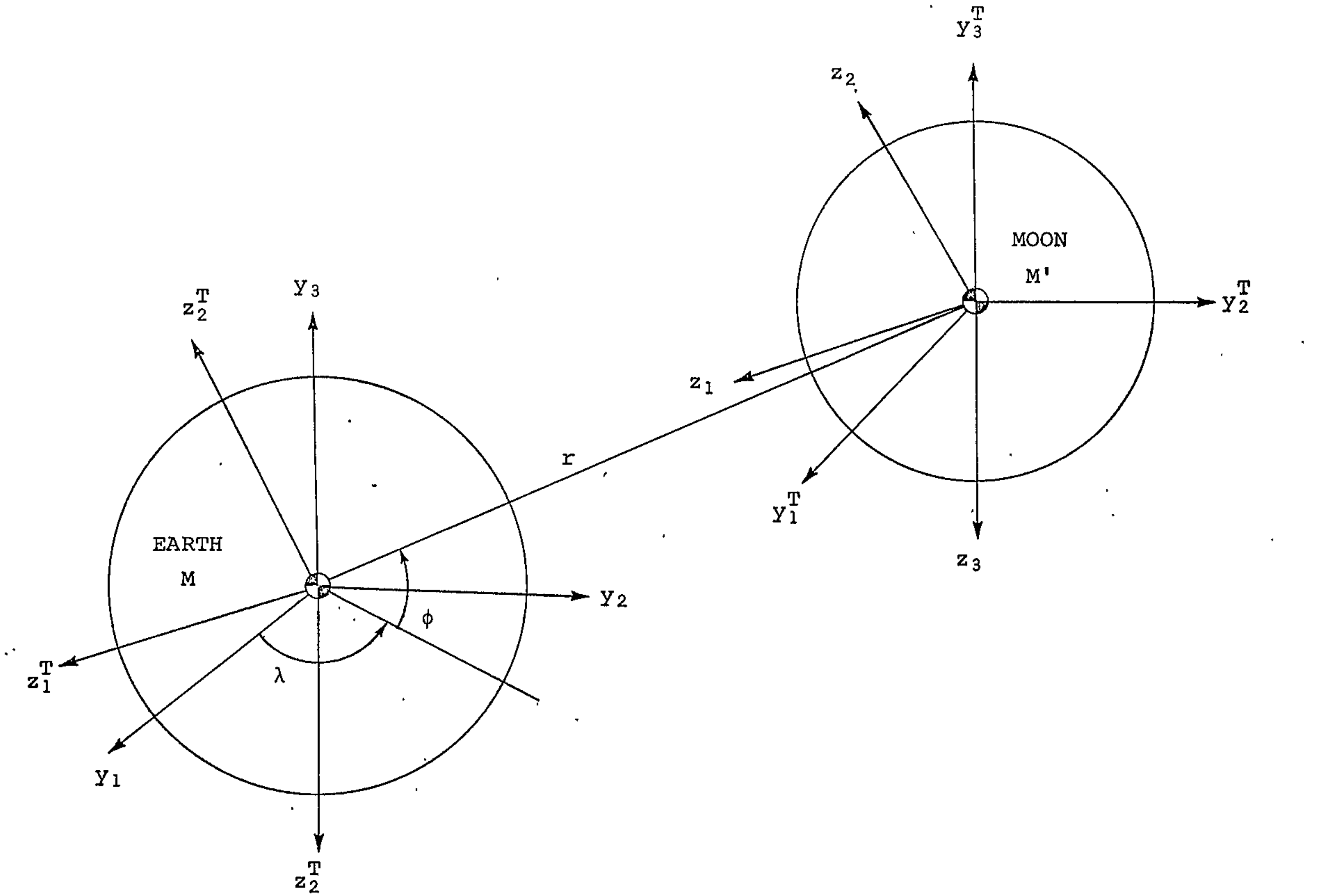

Figure 3. Coordinate systems for mutual gravitational potential.

$$U_2^I = Gr^{-1}\Big[(a/r)^2\Big\{c_{20}P_{20}(s\phi) + P_{21}(c_{21}c\lambda + s_{21}s\lambda) + P_{22}(c_{22}c2\lambda + s_{22}s2\lambda)\Big\} + (a'/r)^2\Big\{c'_{20}P_{20} + P_{22}(c'_{22}c2\lambda + s'_{22}s2\lambda)\Big\}\Big] \tag{60}$$

$$U_3^I = Gr^{-1}\Big[(a'/r)^3\Big\{c'_{30}P_{30} + P_{31}(c'_{31}c\lambda + s'_{31}s\lambda) + P_{32}(c'_{32}c2\lambda + s'_{32}s2\lambda) + P_{33}(c'_{33}c3\lambda + s'_{33}s3\lambda)\Big\}\Big] \tag{61}$$

$$\begin{aligned} U_4^I = Gr^{-1}\Big[&(a'/r)^4\Big\{c'_{40}P_{40} + P_{41}(c'_{41}c\lambda + s'_{41}s\lambda) \\ &+ P_{42}(c'_{42}c2\lambda + s'_{42}s2\lambda) + P_{43}(c'_{43}c3\lambda + s'_{43}s3\lambda) \\ &+ P_{44}(c'_{44}c4\lambda + s'_{44}s4\lambda)\Big\} + (a^2a'^2/r^4)\Big\{6P_{40}(2c_{22}c'_{22} \\ &+ 2s_{22}s'_{22}) + 3P_{41}(c_{21}c'_{20} - c_{21}c'_{22} + s_{21}s'_{22})c\lambda \\ &+ 3P_{41}(s_{21}c'_{20} - c_{21}s'_{22} + s_{21}c'_{22})s\lambda \\ &+ P_{42}(c_{20}c'_{22} + c_{22}c'_{20})c2\lambda + P_{42}(c_{20}s'_{22} + s_{22}c'_{20})s2\lambda \\ &+ (P_{43}/2)(c_{21}c'_{22} - s_{21}s'_{22})c3\lambda + (P_{43}/2(c_{21}s'_{22} \\ &+ s_{21}c'_{22})c3\lambda + (P_{44}/2)(c_{22}c'_{22} - s_{22}s'_{22})c4\lambda \\ &+ (P_{44}/2)(c_{22}s'_{22} + s_{22}c'_{22})s4\lambda\Big\}\Big]. \end{aligned} \tag{62}$$

Now, the partial derivatives of U_{45}^I can be calculated and are

$$\partial U_2^I/\partial r = -3G\,a^2r^{-4}\Big\{c_{20}P_{20} + P_{21}(c_{21}c\lambda + s_{21}s\lambda) + P_{22}(c_{22}c2\lambda + s_{22}s2\lambda)\Big\} - 3G\,a'^2r^{-4}\Big\{c'_{20}P_{20} + P_{22}(c'_{22}c2\lambda + s'_{22}s2\lambda)\Big\} \tag{63}$$

$$\partial U_2^I/\partial\lambda = G\,a^2r^{-3}\Big\{c_{20}P_{20} + P_{21}(-c_{21}s\lambda + s_{21}c\lambda) + 2P_{22}(-c_{22}s\lambda + s_{22}c2\lambda)\Big\} + G\,a'^2r^{-3}\Big\{c'_{20}P_{20} + 2P_{22}(-c'_{22}s2\lambda + s'_{22}c2\lambda)\Big\} \tag{64}$$

$$\partial U_2^I/\partial\phi = G\ a^2 r^{-3}\Big\{c_{20}P_{20_\phi} + P_{21_\phi}(c_{21}c\lambda + s_{21}s\lambda)$$
$$+ P_{22_\phi}(c_{22}c2\lambda + s_{22}s2\lambda)\Big\} + G\ a^2 r^{-3}\Big\{c'_{20}P_{20_\phi}$$
$$+ P_{22_\phi}(c'_{22}c2\lambda + s'_{22}s2\lambda)\Big\} \tag{65}$$

$$\partial U_3^I/\partial r = -4\ G\ a'^3 r^{-5}\Big\{c'_{30}P_{30} + P_{31}(c'_{31}c\lambda + s'_{31}s\lambda)$$
$$+ P_{32}(c'_{32}c2\lambda + s'_{32}s2\lambda) + P_{33}(c'_{33}c3\lambda + s'_{33}s3\lambda)\Big\} \tag{66}$$

$$\partial U_3^I/\partial\lambda = G\ a'^3 r^{-4}\Big\{c'_{30}P_{30} + P_{31}(-c'_{31}s\lambda + c'_{31}c\lambda)$$
$$+ 2P_{32}(-c'_{32}s2\lambda + s'_{32}c2\lambda)$$
$$+ 3P_{33}(-c'_{33}s3\lambda + s'_{33}c3\lambda)\Big\} \tag{67}$$

$$\partial U_3^I/\partial\phi = G\ a'^3 r^{-4}\Big\{c'_{30}P_{30_\phi} + P_{31_\phi}(c'_{31}c\lambda + s'_{31}s\lambda)$$
$$+ P_{32_\phi}(c'_{32}c2\lambda + s'_{32}s2\lambda) + P_{33_\phi}(c'_{33}c3\lambda + s'_{33}s3\lambda)\Big\} \tag{68}$$

$$\partial U_4^I/\partial r = -5G\ a'^4 r^{-6}\Big\{c'_{40}P_{40} + P_{41}(c'_{41}c\lambda + s'_{41}s\lambda)$$
$$+ P_{42}(c'_{42}c2\lambda + s'_{42}\underline{s2}\lambda) + P_{43}(c'_{43}c3\lambda + s'_{43}s3\lambda)$$
$$+ P_{44}(c'_{44}c4\lambda + s'_{44}s4\lambda)\Big\} \quad - 5GMM'a^2a'^2r^{-6}\Big\{6P_{40}\tilde{X}_{40}$$
$$+ 3P_{41}(\tilde{X}_{41}c\lambda + \tilde{Y}_{41}s\lambda) + P_{42}(\tilde{X}_{42}c2\lambda + \tilde{Y}_{42}s2\lambda)$$
$$+ (P_{43}/2)(\tilde{X}_{43}c3\lambda + \tilde{Y}_{43}s3\lambda)$$
$$+ (P_{44}/2)(\tilde{X}_{44}c4\lambda + Y_{44}s4\lambda)\Big\} \tag{69}$$

$$\partial U_4^I/\partial\lambda = G\ a'^4 r^{-5}\Big\{c'_{40}P_{40} + P_{41}(-c'_{41}s\lambda + s'_{41}c\lambda)$$
$$+ 2P_{42}(-c'_{42}s2\lambda + s'_{42}c2\lambda) + 3P_{43}(-c'_{43}s3\lambda + s'_{43}c3\lambda)$$
$$+ 4P_{44}(-c'_{44}s4\lambda + s'_{44}c4\lambda)\Big\} + G\ a^2a'^2r^{-5}\Big\{6P_{40}\tilde{X}_{40}$$
$$+ 3P_{41}(-\tilde{X}_{41}s\lambda + \tilde{Y}_{41}c\lambda) + 2P_{42}(-\tilde{X}_{42}s2\lambda + \tilde{Y}_{42}c2\lambda)$$
$$+ (3P_{43}/2)(-\tilde{X}_{43}s3\lambda + \tilde{Y}_{43}c3\lambda) + 2P_{44}(-\tilde{X}_{44}s4\lambda$$
$$+ \tilde{Y}_{44}c4\lambda)\Big\} \tag{70}$$

$$
\begin{aligned}
\partial U_4^I/\partial\phi = G\, r^{-5} \Big\{ & P_{40_\phi}(a'^4 c'_{40} + 6a^2a'^2\tilde{X}_{40}) \\
& + P_{41_\phi}\Big[(a'^4 c'_{41} + 3a^2a'^2\tilde{X}_{41})c\lambda \\
& + (a'^2 s'_{41} + 3a^2a'^2\tilde{Y}_{41})s\lambda\Big] \\
& + P_{42_\phi}\Big[(a'^4 c'_{42} + a^2a'^2\tilde{X}_{42})c2\lambda \\
& + (a'^4 s'_{42} + a^2a'^2\tilde{Y}_{42})s2\lambda\Big] \\
& + P_{43_\phi}\Big[(a'^4 c'_{43} + \frac{1}{2}\,\tilde{X}_{43})c3\lambda \\
& + (a'^4 s'_{43} + \frac{1}{2}\,\tilde{Y}_{43})s3\lambda\Big] \\
& + P_{44_\phi}\Big[(a'^4 c'_{44} + \frac{1}{2}\,\tilde{X}_{44})c4\lambda \\
& + (a'^4 s'_{44} + \frac{1}{2}\,\tilde{Y}_{44})s4\lambda\Big]\Big\} \;.
\end{aligned}
\tag{71}
$$

In the above equations,

$$\tilde{X}_{40} = 6a^2a'^2[2c_{22}c'_{22} + 2s_{22}s'_{22}]$$

$$\tilde{X}_{41} = 3a^2a'^2[c_{21}c'_{20} - c_{21}c'_{22} + s_{21}s'_{22}]$$

$$\tilde{Y}_{41} = 3a^2a'^2[s_{21}c'_{20} - c_{21}s'_{22} + s_{21}c'_{22}]$$

$$\tilde{X}_{42} = a^2a'^2[c_{20}c'_{22} + c_{22}c'_{20}]$$

$$\tilde{Y}_{42} = a^2a'^2[c_{20}s'_{22} + s_{22}c'_{20}]$$

$$\tilde{X}_{43} = \frac{1}{2}\,a^2a'^2[c_{21}c'_{22} - s_{21}s'_{22}]$$

$$\tilde{Y}_{43} = \frac{1}{2}\,a^2a'^2[c_{21}s'_{22} + s_{21}c'_{22}]$$

$$\tilde{X}_{44} = \frac{1}{2}\,a^2a'^2[c_{22}c'_{22} - s_{22}s'_{22}]$$

$$\tilde{Y}_{44} = \frac{1}{2}\,a^2a'^2[c_{22}s'_{22} + s_{22}c'_{22}]$$

and

$$
\begin{aligned}
P_{20_\phi} &= 3s\phi c\phi \\
P_{21_\phi} &= 3c(2\phi) \\
P_{22_\phi} &= -6s\phi c\phi \\
P_{30_\phi} &= \frac{3}{2}\, c\phi(5s^2\phi - 1) \\
P_{31_\phi} &= \frac{3}{2}\, s\phi + \frac{15}{2}\, s\phi(2c^2\phi - s^2\phi) \\
P_{32_\phi} &= 15c\phi(c^2\phi - 2s^2\phi) \\
P_{33_\phi} &= -45c^2\phi s\phi \\
P_{40_\phi} &= \frac{1}{8}\,(140s^3\phi - 60s\phi)c\phi \\
P_{41_\phi} &= \frac{5}{2}\left[(c^2\phi - s^2\phi)(7s^2\phi - 3) + 14s^2\phi c^2\phi\right] \\
P_{42_\phi} &= \frac{15}{2}\left[14s\phi c^3\phi - 2s\phi c\phi(7s^2\phi - 1)\right] \\
P_{43_\phi} &= 105[c^4\phi - 3s^2\phi c^2\phi] \\
P_{44_\phi} &= -420c^3\phi s\phi
\end{aligned}
\tag{73}
$$

Since the harmonic coefficients c_{ij} and s_{ij} are referred to a coordinate system $\{z_i^T\}$ that rotates with respect to the earth, they are functions of time. These functions may be evaluated by noting the definitions of the c_{ij} and s_{ij} (ref. 21) in terms of the inertia integrals, viz.

$$c_{20} = \frac{1}{a^2M}\left[\frac{I_{11} + I_{22}}{2} - I_{33}\right]$$

$$c_{21} = \frac{1}{a^2M}\ I_{13}$$

$$s_{21} = \frac{1}{a^2M}\ I_{32} \tag{74}$$

$$c_{22} = \frac{1}{4a^2M}\left[I_{22} - I_{11}\right]$$

$$s_{22} = \frac{1}{2a^2M}\ I_{12}$$

and by noting the transformation laws of the inertia matrix, viz.

$$\{z_i\} = [\ell]^T \{y_i\}$$

$$\begin{bmatrix} I_{z_1z_1} & I_{z_1z_2} & I_{z_1z_3} \\ I_{z_2z_1} & I_{z_2z_2} & I_{z_2z_3} \\ I_{z_3z_1} & I_{z_3z_2} & I_{z_3z_3} \end{bmatrix} = [\ell]^T \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} [\ell] \ . \tag{76}$$

The above equations provide the desired functions as follows:

$$a^2Mc_{20} = \frac{1}{2}\left[A + B + C - 3(\alpha''^2A + \beta''^2B + \gamma''^2C)\right]$$

$$a^2Mc_{21} = \alpha\alpha''A + \beta\beta''B + \gamma\gamma''C$$

$$a^2Ms_{21} = \alpha'\alpha''A + \beta'\beta''B + \gamma'\gamma''C \tag{77}$$

$$4a^2Mc_{22} = A(\alpha'^2 - \alpha^2) + B(\beta'^2 - \beta^2) + C(\gamma'^2 - \gamma^2)$$

$$2a^2Ms_{22} = \alpha\alpha'A + \beta\beta'B + \gamma\gamma'C$$

Finally the components of the force on the Moon due to the Earth must be found along the $\{X_i'\}$ axes. Accordingly,

$$F_{5X_i'} = m_5 \left[\frac{\partial U_{45}^I}{\partial r} (\vec{i}_r \cdot \vec{I}_i') + \frac{1}{rc\phi} \frac{\partial U_{45}^I}{\partial \lambda} (\vec{i}_\lambda \cdot I_i') \right.$$

$$\left. + \frac{1}{r} \frac{\partial U_{45}^I}{\partial \phi} (\vec{i}_\phi \cdot I_i') \right] . \tag{78}$$

Torque on Moon Due to Earth. This torque can be derived by first expressing the mutual gravitational potential U_{45}^I in terms of the direction cosines relating the rotational orientation of the moon to the earth; and then by calculating the moment components as follows (ref. 22):

$$\frac{1}{MM'} M_{z1} = \alpha'' U_{\alpha'}^I - \alpha' U_{\alpha''}^I + \beta'' U_{\beta'}^I - \beta' U_{\beta''}^I$$

$$+ \gamma'' U_{\gamma'}^I - \gamma' U_{\gamma''}^I$$

$$\frac{1}{MM'} M_{z2} = \alpha U_{\alpha''}^I - \alpha'' U_{\alpha}^I + \beta U_{\beta''}^I - \beta'' U_{\beta}^I$$

$$+ \gamma U_{\gamma''}^I - \gamma'' U_{\gamma}^I \tag{79}$$

$$\frac{1}{MM'} M_{z3} = \alpha' U_{\alpha}^I - \alpha U_{\alpha'}^I + \beta' U_{\beta}^I - \beta U_{\beta'}^I + \gamma' U_{\gamma}^I - \gamma U_{\gamma'}^I$$

A derivation of the above relations is presented in Appendix C.

In order to derive the torques, the term U_2^I and the second order coupling terms in U_4^I [see eqs. (81) and (82)] will be treated together. Finally the term U_3^I and the remaining terms in $U_4^{\oplus ☾}$ will be treated [see eqs. (86) and (87)].

The reference axes for the second order and coupling terms are $\{y_i\}$. Thus the c'_{ij} and s'_{ij} are functions of the orientation angles. These functions are

$$c'_{20} = \frac{1}{a'^2M'}\left[\frac{I'_{y1y1} + I'_{y2y2}}{2} - I_{y_3y_3}\right]$$

$$= \frac{1}{a'^2M'}\left[\frac{A + B + C}{2} - \frac{3}{2}\ (A'\gamma^2 + B'\gamma'^2 + C'\gamma''^2)\right]$$

$$c'_{21} = \frac{1}{a'^2M'}\, I'_{y1y3}$$

$$= \frac{1}{a'^2M'}\ [\alpha\gamma A' + \alpha'\gamma' B' + \alpha''\gamma'' C']$$

$$s'_{21} = \frac{1}{a'^2M'}\, I'_{y3y2} \tag{80}$$

$$= \frac{1}{a'^2M'}\ [\gamma\beta A' + \gamma'\beta' B' + \gamma''\beta'' C']$$

$$c'_{22} = \frac{1}{4a'^2M'}\Big[A'(\beta^2 - \alpha^2) + B'(\beta'^2 - \alpha'^2)$$

$$+ C'(\beta''^2 - \alpha''^2)\Big]$$

$$s'_{22} = \frac{1}{4a'^2M'}\ [\alpha\beta A' + \alpha'\beta' B' + \alpha''\beta'' C'] \quad .$$

The potential $U_2^I + U^I_{4,\text{coupling}}$ thus assumes the form:

$$U_2^I = G\,r^{-1}\left[(a/r)^2\ \{c_{20}P_{20} + c_{22}P_{22}c2\lambda\} + (a'/r)^2\ \{c'_{20}P_{20} + P_{21}(c'_{21}c\lambda + s'_{21}s\lambda) + P_{22}(c'_{22}c2\lambda + s'_{22}s2\lambda)\}\right] \tag{81}$$

$$\begin{aligned}U^I_{4,\text{coupling}} = G\,a^2a'^2r^{-5}\Big[&P_{40}\ \{6c_{20}c'_{21} + 12c_{22}c'_{22}\}\\ &+ P_{41}\Big(\{-3c_{22}c'_{21}\}c\lambda + \{3c_{20}s'_{21} + 3c_{22}s'_{21}\}\ s\lambda\Big)\\ &+ P_{42}\Big(\{c_{20}c'_{22} + c_{22}c'_{20}\}c2\lambda + \{c_{20}s'_{22}\}\ s2\lambda\Big)\\ &+ P_{43}\Big(\{\tfrac{1}{2}\ c_{22}c'_{21}\}c3\lambda + \{\tfrac{1}{2}\ c_{22}s'_{21}\}\ s3\lambda\Big)\\ &+ P_{44}\Big(\{\tfrac{1}{2}\ c_{22}c'_{22}\}c4\lambda + \{\tfrac{1}{2}\ c_{22}s'_{22}\}\ s4\lambda\Big)\Big]\end{aligned} \tag{82}$$

Now, using equations (79) to (81) the torque components due to the terms in U_2^I are:

$$\begin{aligned}M_{z_1} = GMr^{-3}\ (B' - C')\ \Big[&-3P_{20}\gamma'\gamma'' + P_{21}c\lambda\ (\alpha'\gamma'' + \alpha''\gamma')\\ &+ P_{21}s\lambda(\beta'\gamma'' + \gamma'\beta'') + \frac{P_{22}c2\lambda}{2}\ (\beta'\beta'' - \alpha'\alpha'')\\ &+ \frac{P_{22}s2\lambda}{2}\ (\alpha'\beta'' + \beta'\alpha'')\Big]\end{aligned}$$

(83)
(cont'd.)

$$\begin{aligned}M_{z_2} = GMr^{-3}\ \Big[&(C' - A')\quad 3P_{20}\gamma\gamma'' + P_{21}c\lambda(\alpha\gamma'' + \gamma\alpha'')\\ &+ P_{21}s\lambda(\beta\gamma'' + \gamma\beta'') + \frac{P_{22}c2\lambda}{2}\ (\beta\beta'' - \alpha\alpha'')\\ &+ \frac{P_{22}s2\lambda}{2}\ (\alpha\beta'' + \beta\alpha'')\Big]\end{aligned}$$

$$M_{z3} = GMr^{-3}\ (B' - A')\Big[3P_{20}\gamma\gamma + P_{21}c\lambda(\gamma\alpha' + \alpha\gamma')$$

$$- P_{21}s\lambda(\gamma\beta' + \gamma'\beta) + \frac{P_{22}c2\lambda}{2}(\alpha\alpha' - \beta\beta') \tag{83 (concl'd.)}$$

$$- \frac{P_{22}s2\lambda}{2}(\beta\alpha' + \alpha\beta')\Big]$$

The above components are for an arbitrary orientation of $\{y_i\}$ with respect to $\{z_i\}$. In the derivations presented in the literature, the Earth is treated as a particle so that the relative orientation of $\{y_i\}$ and $\{z_i\}$ is immaterial. To recover those results a special orientation of the $\{y_i\}$ may be taken. If y_i is taken to be pointing at the Moon then $\ell = c\phi c\lambda = \frac{y_1}{r} = 1$, $m = c\phi s\lambda = \frac{y_2}{r} = 0$, and $n = s\phi = \frac{y_3}{r} = 0$. Also, $\ell' = \alpha$, $m' = \alpha'$, and $n' = \alpha''$, where ℓ', m', n' are the direction cosines of the Earth with respect to $\{z_i\}$. This reduces equation (83) to a more recognizable form, viz.

$$M_{z1} = 3GMr^{-3}\ (C' - B')m'n'$$

$$M_{z2} = 3GMr^{-3}\ (A' - C')\ell'n' \tag{84}$$

$$M_{z3} = 3GM\ (B' - A')\ell'm' \ .$$

Actually, equations (83) and (84) are identical as algebraic manipulation will show.

The coupling terms in U_4^I are handled similarly. Thus,

$$
\begin{aligned}
M_{z_1} = {} & GMr^{-5}a^2(B' - C')\Big[6P_{40}\left\{c_{20}(\alpha''\gamma' + \gamma''\alpha')\right. \\
& \left. + c_{22}(\beta'\beta'' - \alpha'\alpha'')\right\} + 3P_{41}\left\{-c_{22}(\alpha''\gamma' + \gamma''\alpha')c\lambda\right. \\
& \left. + s\lambda(\gamma'\beta'' + \gamma''\beta')(c_{20} + c_{22})\right\} \\
& + \frac{P_{42}}{2}c_{20}\left\{c2\lambda(\alpha'\alpha'' - \beta'\beta'') + s2\lambda(\alpha''\beta' + \alpha'\beta'')\right\} \\
& - 3P_{42}c_{22}c2\lambda\gamma'\gamma'' + \frac{P_{43}}{2}c_{22}\left\{c3\lambda(\alpha''\gamma' + \alpha'\gamma'')\right. \\
& \left. + s3\lambda(\beta''\gamma' + \beta'\gamma'')\right\} - \frac{P_{44}}{2}c_{22}\left\{c4\lambda(\beta'\beta'' - \alpha'\alpha'')\right. \\
& \left. + s4\lambda(\alpha''\beta' + \beta''\alpha')\right\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
M_{z_2} = {} & GMr^{-5}a^2\ (C' - A')\Big[6P_{40}\left\{c_{20}(\gamma''\alpha + \gamma\alpha'')\right. \\
& \left. + c_{22}(\beta\beta'' - \alpha\alpha'')\right\} + 3P_{41}\left\{-c_{22}(\alpha\gamma'' + \gamma\alpha'')c\lambda\right. \\
& \left. + s\lambda(c_{20} + c_{22})(\beta\gamma'' + \gamma\beta'')\right\} + \frac{P_{42}}{2}c_{20}\left\{(\beta\beta''\right. \\
& \left. - \alpha\alpha'')c2\lambda + (\alpha\beta'' + \beta\alpha'')s2\lambda\right\} - 3P_{42}c_{22}c2\lambda\gamma\gamma'' \qquad (85) \\
& + \frac{P_{43}}{2}c_{22}\left\{(\alpha\gamma'' + \gamma\alpha'')c3\lambda + (\beta\gamma'' + \gamma\beta'')s3\lambda\right\} \\
& + \frac{P_{44}}{4}c_{22}\left\{(\beta\beta'' - \alpha\alpha'')c4\lambda + (\alpha\beta'' + \beta\alpha'')s4\lambda\right\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
M_{z_3} = {} & GMr^{-5}a^2\ (B' - A')\Big[6P_{40}\left\{-c_{20}(\alpha'\gamma + \gamma'\alpha)\right. \\
& \left. + c_{22}(\alpha\alpha' - \beta\beta')\right\} + 3P_{41}\left\{+ c_{22}c\lambda(\alpha'\gamma + \gamma'\alpha)\right. \\
& \left. - (c_{20} + c_{22})s\lambda(\beta'\gamma + \gamma'\beta)\right\} + \frac{P_{42}}{2}c_{20}\left\{c2\lambda(\alpha\alpha'\right. \\
& \left. - \beta\beta') - s2\lambda(\alpha\beta' + \beta\alpha')\right\} - 3P_{42}c_{22}c2\lambda\gamma\gamma' \\
& + \frac{P_{43}}{2}c_{22}\left\{-(\alpha'\gamma + \gamma'\alpha)c3\lambda - (\beta'\gamma + \gamma'\beta)s3\lambda\right\} \\
& + \frac{P_{44}}{4}c_{22}\left\{+(\alpha\alpha' - \beta\beta')c4\lambda - s4\lambda(\alpha'\beta + \alpha\beta')\right\}\Big]
\end{aligned}
$$

Finally, the torque components due to the U_3^I potential terms and the U_4^I potential terms not already treated, viz.

$$U_3^I = \frac{G}{r} \left[\left(\frac{a'}{r}\right)^3 \left\{ P_{30}(\phi)c'_{30} + P_{31}(\phi)(c'_{31} \cos \lambda + s'_{31} \sin \lambda) \right.\right.$$

$$+ P_{32}(\phi)(c'_{32} \cos 2\lambda + s'_{32} \sin 2\lambda) \tag{86}$$

$$\left.\left. + P_{33}(\phi)(c'_{33} \cos 3\lambda + s'_{33} \sin 3\lambda) \right\} \right]$$

$$U_4^I = \frac{G}{r} \left[\left(\frac{a'}{r}\right)^4 \left\{ P_{40}c'_{40} + P_{41}(\phi)(c'_{41} \cos \lambda + s'_{41} \sin \lambda) \right.\right.$$

$$+ P_{42}(\phi)(c'_{42} \cos 2\lambda + s'_{42} \sin 2\lambda)$$

$$+ P_{43}(\phi)(c'_{43} \cos 3\lambda + s'_{43} \sin 3\lambda) \tag{87}$$

$$\left.\left. + P_{44}(\phi)(c'_{44} \cos 4\lambda + s'_{44} \sin 4\lambda) \right\} \right]$$

will be derived.

The general approach for calculating torques used earlier will now be modified. The Earth may be considered to be a particle in calculating the torques on the Moon arising from third and fourth degree terms in the lunar potential. Reference 23 provides a simple approach based on this fact that will be followed here.

Consider that the reference axes in equations (86) and (87) are the $\{z_i\}$ axes. Thus the c'_{ij}'s and s'_{ij}'s are constant and r, ϕ, λ are the spherical polar coordinates of the center of mass of the Earth with respect to the axes $\{z_i\}$.

For a point mass Earth, the torque on the finite Moon due to the Earth is equal and opposite to the torque on the Earth due to the Moon. Thus, reference 23 finds the torques to be

$$M_{z_1} = \frac{GM}{A'} \left[z_3 \frac{\partial(U_3^I + U_4^I)}{\partial z_2} - z_2 \frac{\partial(U_3^I + U_4^I)}{\partial z_3} \right].$$

$$M_{z_2} = \frac{GM}{B'} \left[z_1 \frac{\partial(U_3^I + U_4^I)}{\partial z_3} - z_3 \frac{\partial(U_3^I + U_4^I)}{\partial z_1} \right] \tag{88}$$

$$M_{z_3} = \frac{GM}{C'} \left[z_2 \frac{\partial(U_3^I + U_4^I)}{\partial z_1} - z_1 \frac{\partial(U^I + U^I)}{\partial z_2} \right]$$

where

$$\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} = r \begin{Bmatrix} \cos\phi \cos\lambda \\ \cos\phi \cos\lambda \\ \sin\phi \end{Bmatrix} \equiv \begin{Bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{Bmatrix} \tag{89}$$

Reference 23 then provides

$$\begin{aligned} M_{z_1} = \frac{3}{2} \frac{GMM'a'^3r^{-4}}{A'} \Big[& \ell_2(1 - 5\ell_3^2)c_{30} - 10\ell_1\ell_2\ell_3c_{31} \\ & - 10\ell_2(1 + \ell_3^2 - 2\ell_2^2)c_{32} - 60\ell_1\ell_2\ell_3c_{33} \\ & - \ell_3(1 - 5\ell_3^2 + 10\ell_2^2)s_{31} + 20\ell_1(\ell_3^2 - \ell_2^2)s_{32} \\ & + 30\ell_3(\ell_1^2 - \ell_2^2)s_{33} \Big] \end{aligned} \tag{90}$$

(90)
(cont'd.)

$$
\begin{aligned}
M_{z_2} = \frac{3}{2}\,\frac{GMM'a'^3r^{-4}}{B'}\,\Big[& -\ell_1(1 - 5\ell_3^2)c_{30} \\
& + \ell_3(1 - 5\ell_3^2 + 10\ell_1^2)c_{31} \\
& - 10\ell_1(1 + \ell_3^2 - 2\ell_1^2)c_{32} \\
& - 30\ell_3(\ell_1^2 - \ell_2^2)c_{33} + 10\ell_1\ell_2\ell_3 s_{31} \\
& + 20\ell_2(\ell_1^2 - \ell_3^2)s_{32} - 60\ell_1\ell_2\ell_3 s_{33}\Big] \\
& + \frac{3GMM'a'^4r^{-5}}{B'}\left[-\frac{5}{2}\,\ell_1^2 c_{41} + 3s\ell_1^4 c_{43}\right]
\end{aligned}
\tag{90 (concl'd.)}
$$

$$
\begin{aligned}
M_{z_3} = \frac{3}{2}\,\frac{GMM'a'^3r^{-4}}{C'}\,\Big[& -\ell_2(1 - 5\ell_3^2)c_{31} + 40\ell_1\ell_2\ell_3 c_{32} \\
& + 30\ell_2(3\ell_1^2 - \ell_2^2)c_{33} + \ell_1(1 - 5\ell_3^2)s_{31} \\
& - 20\ell_3(\ell_1^2 - \ell_2^2)s_{32} - 30\ell_1(\ell_1^2 - 3\ell_2^2)s_{33}\Big] \\
& + \frac{3GMM'a'^4r^{-5}}{C'}\left[5\ell_1^2 s_{42} - 140\ell_1^4 s_{44}\right]
\end{aligned}
$$

In summary, the force components in the inertial frame $\{X_i'\}$ may be found from equations (78) and the preceding definitions found in equations (59) to (78). The torque components on the Moon due to the Earth resolved along the $\{z_i\}$ axes are the sum of the torques in equations (84), (85), and (90).

Torque on Moon Due to Sun. Since the Sun is treated as a particle, the torque exerted on the Moon is of the same form as equations (84), viz.,

$$M_{z_1} = 3GM_1 r_{14}^{-3} (C' - B')\, m'_{\odot}\, n'_{\odot}$$

$$M_{z_2} = 3GM_1 r_{14}^{-3} (A' - C')\, \ell'_{\odot}\, n'_{\odot} \qquad (91)$$

$$M_{z_3} = 3GM_1 r_{14}^{-3} (B' - A')\, \ell'_{\odot}\, m'_{\odot}$$

where $\ell'_{\odot}$, $m'_{\odot}$, $n'_{\odot}$, are the direction cosines of the Sun with respect to $\{z_i\}$.

Force on Earth Due to Moon. This force is derived in a manner analogous to the force on the Moon due to the Earth presented earlier. The reference axes are chosen to be the $\{y_i^T\}$ set so that the c_{ij} and s_{ij} are constants and the c'_{ij} and s'_{ij} vary. An assumption made in the following is that only second degree terms in the lunar potential are important to the motion of the Earth.

The mutual potential then assumes the form

$$\begin{aligned} U^{I}_{45} = Gr^{-1} \Big[&(a/r')^2 \left\{ c_{20}P_{20}(s\phi) + P_{22}(s\phi)(c_{22}c2\lambda \right. \\ &\left. + s_{22}s2\lambda) \right\} + (a'/r)^2 \left\{ c'_{20}P_{20}(s\phi) + P_{21}(s\phi)(c'_{21}c\lambda \right. \\ &\left. + s'_{21}s\lambda) + P_{22}(s\phi)(c'_{22}c2\lambda + s'_{22}s2\lambda) \right\} \Big] \end{aligned} \qquad (92)$$

where the c'_{ij} and s'_{ij} are functions of the mutual orientation angles as follows:

$$c'_{20} = \frac{1}{a'^2M'} \left[\frac{A + B + C}{2} - \frac{3}{2} (\gamma^2 A + \gamma'^2 B + \gamma''^2 C) \right]$$

$$c'_{21} = \frac{1}{a'^2M'} \left[\alpha\gamma A + \alpha'\gamma' B + \alpha''\gamma'' C \right] \qquad (93)\ \text{(cont'd.)}$$

$$s'_{21} = \frac{1}{a'^2M'} \left[\gamma\beta A + \gamma'\beta' B + \gamma''\beta'' C \right]$$

$$c'_{22} = \frac{1}{4a'^2M'} \left[(\beta^2 - \alpha^2)A + (\beta'^2 - \alpha'^2)B + (\beta''^2 - \alpha''^2)C\right]$$

$$s'_{22} = \frac{1}{2a'^2M'} \left[\alpha\beta A + \alpha'\beta' B + \alpha''\beta'' C\right]$$

The force acting on the Earth projected on the inertial axes $\{X'_i\}$ is

$$F_{4X'_i} = m_4 \left[\frac{\partial U^I_{45}}{\partial r} (\vec{i}_r \cdot \vec{I}'_i) + \frac{1}{rc\phi}\frac{\partial U^I_{45}}{\partial \lambda} (\vec{i}_\lambda \cdot \vec{I}'_i) + \frac{1}{r}\frac{\partial U^I_{45}}{\partial \phi} (\vec{i}_\phi \cdot I'_i)\right] \tag{94}$$

where r, ϕ, λ and the associated unit vectors are the spherical polar coordinates of the Earth with respect to $\{y_i{}^T\}$.

The appropriate partial derivatives are

$$\begin{aligned}\frac{\partial U^I_{45}}{\partial r} = {} & -3Ga^2r^{-4} \Big\{c_{20}P_{20}(s\phi) + P_{22}(s\phi)(c_{22}c2\lambda \\ & + s_{22}s2\lambda)\Big\} -3Ga'^2r^{-4} \Big\{c'_{20}P_{20}(s\phi) \\ & + P_{21}(s\phi)(c'_{21}c\lambda + s'_{21}s\lambda) \\ & + P_{22}(s\phi)(c'_{22}c2\lambda + s'_{22}s2\lambda)\Big\}\end{aligned} \tag{95}$$

$$\begin{aligned}\frac{\partial U^I_{45}}{\partial \lambda} = {} & Ga^2r^{-3} \Big\{c_{20}P_{20} + 2P_{22}(-c_{22}s2\lambda \\ & + s_{22}c2\lambda\Big\} + Ga'^3r^{-3} \Big\{c'_{20}P_{20} \\ & + P_{21}(-c'_{21}s\lambda + s'_{21}c\lambda) \\ & + 2P_{22}(-c'_{22}s2\lambda + s'_{22}c2\lambda)\Big\}\end{aligned} \tag{96}$$

$$\frac{\partial U_{45}^{I}}{\partial \phi} = Ga^2r^{-3} \left\{ c_{20}P_{20_\phi} + P_{22_\phi}(c_{22}c2\lambda + s_{22}s2\lambda) \right\}$$

$$+ Ga'^2r^{-3} \left\{ c'_{20}P_{20_\phi} + P_{21_\phi}(c_{21}c2\lambda + s_{21}s2\lambda) \right. \tag{97}$$

$$\left. + P_{22_\phi}(c'_{22}c2\lambda + s'_{22}s2\lambda) \right\}$$

<u>Torque on Earth Due to Moon and Sun</u>. Again considering only second degree terms this torque is

$$M_{y1} = 3GM_1r_{15}^{-3} (C - B)m_{\odot}n_{\odot}$$

$$+ 3GM_3r_{14}^{-3} (C - B)m_{☾} n_{☾}$$

$$M_{y2} = 3GM_1r_{15}^{-3} (A - C)\ell_{\odot} n_{\odot}$$

$$+ 3GM_4r_{14}^{-3} (A - C)\ell_{☾} n_{☾} \tag{98}$$

$$M_{y3} = 3GM_1r_{15}^{-3} (B - A)\ell_{\odot} m_{\odot}$$

$$+ 3GM_4r_{14}^{-3} (B - A)\ell_{☾} m_{☾}$$

CONCLUDING REMARKS

This report presents a unified development of a physical model and a mathematical model of the Earth-Moon system. The Earth and Moon are considered to be rigid bodies. The equations of motion are formulated in a completely coupled fashion and the mutual potential of the Earth-Moon pair is incorporated in the development.

This model is intended as a basis for a more inclusive theoretical model including relativistic, non-rigid, and dissipative phenomena.

The models are being coded for use in data reduction packages to estimate physical parameters of the Earth-Moon system. The listing for two programs that have been developed to date are provided in Appendix C.

Program ANEAMØ evaluates (1) a truncated form of Brown's lunar theory, (2) Eckhardt's theory for lunar physical librations, and (3) Newcomb's expressions for the rotational motion of the Earth. Program RIGEM numerically integrates the rotational motion of the Earth and Moon and the translational motion of all the planets. More information on these may be found in the listings.

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APPENDIX A

FORCES AND TORQUES BY VECTOR-DYADIC METHOD

Reference 16 provides a derivation of the appropriate equations. These are summarized below as applied to the forces and torques on the Moon due to the Earth.

A. Force on Triaxial Moon Due to Spherical Earth

$$\vec{F} = - \frac{GMM'\vec{i}_r}{r^2} + \frac{3}{2}\frac{GM\Theta}{r^4}$$

$$+ \frac{3GM\vec{i}_r}{r^4} \cdot (\bar{\bar{E}}\Theta - \bar{\bar{I}}) - \frac{15}{2}\frac{GM\vec{i}_r}{r^6}\vec{r} \cdot (\bar{\bar{E}}\Theta - \bar{\bar{I}}) \circ \vec{i}_r \tag{A1}$$

B. Force on Spherical Moon Due to Oblate Earth

$$\vec{F} = - GM'M \left\{ \frac{\vec{i}_r}{r^2} + \frac{Ja'^2}{r^4} \left[\vec{i}_r - 5(\vec{j}_3 \cdot \vec{i}_r)^2 \vec{j}_3 \right.\right.$$

$$\left.\left. + 2(\vec{j}_3 \cdot \vec{i}_r) \vec{j}_3\right]\right\} \tag{a2}$$

C. Torque on Moon Due to Spherical Earth

$$\vec{T} = \frac{-3GM}{r^3} \vec{i}_r \circ \bar{\bar{I}} \times \vec{i}_r \tag{A3}$$

D. Torque on Moon Due to Oblate Earth

$$\vec{T} = \frac{-3GM}{r^3} \vec{i}_r \cdot \bar{\bar{I}} \times \vec{i}_r$$

$$- \frac{5GMJa'^2}{r^5} \left[\left\{ 1 - 7(\vec{j}_3 \cdot \vec{i}_r)^2 \right\} \vec{i}_r \cdot \bar{\bar{I}} \times \vec{i}_r \right.$$

$$+ 2\vec{j}_3 \cdot \vec{i}_r \quad (\vec{j}_3 \cdot \bar{\bar{I}} \times \vec{i}_r + \vec{i}_r \cdot \bar{\bar{I}} \times \vec{j}_3) \tag{A4}$$

$$\left. - \frac{2}{5} \vec{j}_3 \cdot \bar{\bar{I}} \times \vec{j}_3 \right] \quad .$$

In the above equations,

$$\Theta = (A + B + C)/2,$$

$$\bar{\bar{I}} = A\vec{j}_1\vec{j}_1 + B\vec{j}_2\vec{j}_2 + C\vec{j}_3\vec{j}_3$$

$$\bar{\bar{E}} = \vec{j}_1\vec{j}_1 + \vec{j}_2\vec{j}_2 + \vec{j}_3\vec{j}_3$$

APPENDIX B

DERIVATION OF TORQUES FROM THE MUTUAL POTENTIAL

Consider the $\{y_i\}$ frame as the reference frame. The potential at any point of the moon (y_1, y_2, y_3) is given by $\phi\ (y_1, y_2, y_3)$ in its most general form. The total mutual potential may be written as

$$U^I = \int_{M'} \phi\ (y_1, y_2, y_3)\ dM' \quad . \tag{B1}$$

Note that the subscript $_{45}$ on U^I has been omitted here. Now, the force on a particle of mass dM' at point (y_1, y_2, y_3) resolved along the $\{y_i\}$ axes is

$$\vec{f} = f_{y_1}\ \vec{j}_1 + f_{y_2}\ \vec{j}_2 + f_{y_3}\ \vec{j}_3 \tag{B2}$$

where

$$f_{y_i} = \frac{\partial \phi}{\partial y_i} \quad .$$

The components of this force along the axes $\{z_i\}$ are

$$\{f_{z_i}\} = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{bmatrix} \quad \{f_{y_i}\} \tag{B3}$$

The differential torques about the $\{z_i\}$ axes produced by these forces are

$$\begin{aligned} m_{z_1} &= z_2 f_{z_3} - z_3 f_{z_2} \\ m_{z_2} &= z_3 f_{z_1} - z_1 f_{z_3} \\ m_{z_3} &= z_1 f_{z_2} - z_2 f_{z_1} \end{aligned} \tag{B4}$$

These may be written as

$$m_{z_1} = \alpha''\phi_{Y_1}z_2 - \alpha'\phi_{Y_1}z_3$$

$$+ \beta''\phi_{Y_2}z_2 - \beta'\phi_{Y_2}z_3$$

$$+ \gamma''\phi_{Y_3}z_2 - \gamma'\phi_{Y_3}z_3$$

$$m_{z_2} = \alpha\phi_{Y_1}z_3 - \alpha''\phi_{Y_1}z_1$$

$$+ \beta\phi_{Y_2}z_3 - \beta''\phi_{Y_2}z_1 \tag{B5}$$

$$+ \gamma\phi_{Y_3}z_3 - \gamma''\beta_{Y_3}z_1$$

$$m_{z_3} = \alpha'\phi_{Y_1}z_1 - \alpha\phi_{Y_1}z_2$$

$$+ \beta'\phi_{Y_2}z_1 - \beta\phi_{Y_2}z_2$$

$$+ \gamma'\phi_{Y_3}z_1 - \gamma\phi_{Y_3}z_2 \quad .$$

Now, if the differential torques are integrated over the body using

$$M_{z_1} = \int_{M'} m_{z_1}\, dM' \tag{B6}$$

then

$$M_{z_1} = \alpha''\int \phi_{y_1} \frac{\partial y_1}{\partial \alpha'} dM' - \alpha'\int \phi_{y_1} \frac{\partial y_1}{\partial \alpha''} dM'$$

$$+ \beta''\int \phi_{y_2} \frac{\partial y_2}{\partial \beta'} dM' - \beta'\int \phi_{y_2} \frac{\partial y_2}{\partial \beta''} dM' \qquad \text{(B7)}$$

$$+ \gamma''\int \phi_{y_3} \frac{\partial y_3}{\partial \gamma'} dM' - \gamma'\int \phi_{y_3} \frac{\partial y_3}{\partial \gamma''} dM'$$

etc.

Finally,

$$M_{z_1} = \alpha'' U^I_{\alpha'} - \alpha' U^I_{\alpha''}$$

$$+ \beta'' U^I_{\beta'} - \beta' U^I_{\beta''}$$

$$+ \gamma'' U^I_{\gamma'} - \gamma' U^I_{\gamma''} \quad ,$$

(B8)
(cont'd.)

$$M_{z_2} = \alpha U^I_{\alpha''} - \alpha'' U^I_{\alpha}$$

$$+ \beta U^I_{\beta''} - \beta'' U^I_{\beta}$$

$$+ \gamma U^I_{\gamma''} - \gamma'' U^I_{\gamma} \quad ,$$

and

$$M_{z_3} = \alpha' \, U^I_\alpha - \alpha \, U^I_{\alpha'}$$

$$+ \beta' \, U^I_\beta - \beta \, U^I_{\beta'}, \qquad \text{(B8)}$$

(concl'd.)

$$+ \gamma' \, U^I_\gamma - \gamma \, U^I_{\gamma'}$$

APPENDIX C

PROGRAM LISTINGS

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```
                 PROGRAM  RIGEM (INPUT,OUTPUT,TAPE2=INPUT,TAPE3=OUTPUT)
           C
           C                  PROGRAM RIGEM
           C             THIS PROGRAM INTEGRATES THE TRANSLATIONAL MOTICN OF THE
           C     SUN,PLANETS,AND MOON. IT ALSO INTEGRATES THE FULLY COUPLED ROTATIONAL
           C     MOTION OF THE EARTH AND MOON USING SUBROUTINE RA19S
           C
           C
           C     VARIABLES AND PARAMETERS
           C
           C     X VECTER OF COORDINATES
           C     V VECTER OF VELOCITIES
           C     X(1-33) TRANSLATIONAL
           C     X(34-37) EARTH ROTATIONAL
           C     X(38-41) LUNAR ROTATIONAL
           C
           C                     MULTI-CASE OPTION
           C                        ICODE=0  LAST CASE
           C                        ICODE=1  RETURN FOR NEW CASE
           C                NOTE- CURRENTLY CODED TO READ NEW BETA TR. PRIME
           C                        RATES ONLY
           C
           C
000003           DIMENSION X(41),V(41),NORO(2),OMPL(1),D(4,4),F(41)
000003           DIMENSION T(3,3),E(3,3),C(3,3),P(3,3),R(3,3),PP(3,3),DEL(3)
                1,DAL(3),DELV(3),DALV(3),PD(3,3),ED(3,3)
000003           DIMENSION OMM(4),OMDM(4),RT(4)
000003           INTEGER OMPL
000003           COMMON/PARAM/NORO,AO,AD,VJDEP,OMPL,TINC,TMAX
000003           CCMMCN/XOUT/OMM,OMDM,RT
000003           IICODE=0
000004           PI=3.14159265358979
000006           DTR=PI/180.
000010           RTD=180./PI
           C
           C     GET INITIAL CONDITIONS
           C
000011        77 CALL PROB(X,V,TI,TF,NV,NCLASS,NSS,NI,NOR,LL,IICODE)
           C
           C     INTEGRATION
           C
000024         4 CALL RA19S(X,V,TI,TF,XL,LL,NV,NI,NF,NS,NCLASS,NOR,NSS)
```

ORIGINAL PAGE IS OF POOR QUALITY

```
              C
              C                                         OUTPUT SECTION
              C
      000041                TWR=VJDEP+TF
      000043                WRITE(3,100)
      000047                WRITE (3,115) TWR
      000055                WRITE(3,101)VJDEP,NOR,LL,NF,TF
      000073                WRITE(3,105)
      000077                WRITE(3,106)
      000103                II=1
      000104                DO 1 I=1,31,3
      000106                WRITE(3,102)II
      000113                WRITE(3,103)X(I),X(I+1),X(I+2)
      000125                V(I)=V(I)*100.
      000130                V(I+1)=V(I+1)*100.
      000131                V(I+2)=V(I+2)*100.
      000132                WRITE(3,104)V(I),V(I+1),V(I+2)
      000144                II=II+1
      000146              1 CONTINUE
      000150                DO 75 I=1,33
      000151             75 V(I)=V(I)/100.
              C                           EARTH ORIENTATION
      000155                IF (NORO(1).EQ.0) GO TO 76
      000156                WRITE(3,107)
      000161                WRITE (3,115) TWR
      000167                WRITE(3,108)
      000173                WRITE(3,111) (X(I+33),I=1,4)
      000205                WRITE(3,111) (V(I+33),I=1,4)
              C
              C                          EULER PARAMETER TESTS FOR EARTH
              C
      000217                WRITE (3,113)
      000223                TEST1=X(34)**2+X(35)**2+X(36)**2+X(37)**2
      000231                TEST2=X(34)*V(34)+X(35)*V(35)+X(36)*V(36)+X(37)*V(37)
      000240                WRITE (3,112) TEST1,TEST2
              C                           MOON ORIENTATION
      000247             76 WRITE (3,109)
      000253                WRITE (3,115) TWR
      000261                WRITE(3,110)
```

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OF POOR QUALITY

```
000265          WRITE(3,111)(X(I+37),I=1,4)
000277          WRITE(3,111)(V(I+37),I=1,4)
000311          WRITE (3,116)
        C
        C          ROUTINE TO CALCULATE EARTHS SELENOGRAPHIC COORDINATES
        C
000315          D(2,2)=X(38)**2+X(39)**2-X(40)**2-X(41)**2
000323          D(3,3)=X(38)**2-X(39)**2+X(40)**2-X(41)**2
000331          D(4,4)=X(38)**2-X(39)**2-X(40)**2+X(41)**2
000336          D(2,3)=2.  *(X(39)*X(40)+X(38)*X(41))
000342          D(2,4)=2.  *(X(39)*X(41)-X(38)*X(40))
000346          D(3,2)=2.  *(X(39)*X(40)-X(38)*X(41))
000352          D(3,4)=2.  *(X(40)*X(41)+X(38)*X(39))
000356          D(4,2)=2.  *(X(39)*X(41)+X(38)*X(40))
000362          D(4,3)=2.  *(X(40)*X(41)-X(38)*X(39))
000366          DAL(1)=X(13)-X(10) $ DAL(2)=X(14)-X(11) $ DAL(3)=X(15)-X(12)
000374          RRR =SQRT(DAL(1)*DAL(1)+DAL(2)*DAL(2)+DAL(3)*DAL(3))
000402          CC=DAL(1)/RRR $ CS=DAL(2)/RRR $ SPH=DAL(3)/RRR
000406          CPH=SQRT(DAL(1)**2+DAL(2)**2)/RRR
000414          CL=CC/CPH $ SL=CS/CPH
000417          CALL FORCE (X,V,TF,F)
000422          SS=SQRT(D(2,2)**2+D(3,2)**2)
000430          SLONG=ATAN2(D(3,2),D(2,2))
000433          SLAT=ATAN2(D(4,2),SS)
000436          SLONG=SLONG*RTD $ SLAT=SLAT*RTD
000440          WRITE (3,117) SLONG,SLAT
        C
        C       TEMPORARY OUTPUT
        C
000450          WRITE (3,9000) OMM(1),OMM(2),OMM(3),OMM(4)
000464     9000 FORMAT (5X,*CHECK*,4E20.12)
000464          WRITE (3,9000) OMDM(1),OMDM(2),OMDM(3),OMDM(4)
000500          WRITE (3,9000) RT(1),RT(2),RT(3),RT(4)
        C
        C          ROUTINE TO CALCULATE PHYSICAL LIBRATIONS
        C
000514          T(1,1)=-CC   $  T(1,2)=-CS     $  T(1,3)=-SPH
000521          T(2,1)=SL    $  T(2,2)=-CL     $  T(2,3)=0.
000525          T(3,1)=-CL*SPH $ T(3,2)=-SL*SPH  $ T(3,3)=CPH
000531          TEP=(VJDEP+TF-2415020.0)/36525.     $ TEP2=TEP*TEP $ TEP3=TEP2* TEP
000537          EPS=(23.452294  -.0130125  *TEP-.00000164  *TEP2+.000000503  *TEP3
               1)*DTR
000546          E(1,1)=1.  $ E(1,2)=0.  $ E(1,3)=0.
```

```
000551         E(2,1)=0.  $ E(2,2)= COS(EPS)       $ E(2,3)=- SIN(EPS)
000556         E(3,1)=0.  $ E(3,2)= SIN(EPS)       $ E(3,3)= COS(EPS)
000564         C(1,1)=D(2,2) $ C(1,2)=D(2,3)  $ C(1,3)=D(2,4)
000570         C(2,1)=D(3,2) $ C(2,2)=D(3,3)  $ C(2,3)=D(3,4)
000575         C(3,1)=D(4,2) $ C(3,2)=D(4,3)  $ C(3,3)=D(4,4)
000601         TT=VJDEP+TF-2433282.5
000604         XKAP=.063107   *TT*DTR/3600.
000610         OMEG=.063107   *TT*DTR/3600.
000612         XNU=.0548757   *TT*DTR/3600.
000614         P(1,1)=- SIN(XKAP)* SIN(OMEG)+ COS(XKAP)* COS(OMEG)* CCS(XNU)
000633         P(1,2)=- COS(XKAP)* SIN(OMEG)-SIN( XKAP)* COS(OMEG)* COS(XNU)
000652         P(1,3)=- COS(OMEG)* SIN(XNU)
000660         P(2,1)= SIN(XKAP)* COS(OMEG)+ COS(XKAP)* SIN(OMEG)* CCS(XNU)
000677         P(2,2)= COS(XKAP)* COS(OMEG)- SIN(XKAP)* SIN(OMEG)* COS(XNU)
000715         P(2,3)=- SIN(OMEG)* SIN(XNU)
000722         P(3,1)= COS(XKAP)* SIN(XNU)
000730         P(3,2)=- SIN(XKAP)* SIN(XNU)
000735         P(3,3)= COS(XNU)
000740         DO 50 K=1,3
000741         DO 51 J=1,3
000742         R(K,J)=0.
000745         DO 52 I=1,3
000746         DO 53 L=1,3
000747      53 R(K,J)=R(K,J)+E(I,K)*P(I,L)*T(J,L)
000764      52 CONTINUE
000766      51 CONTINUE
000770      50 CONTINUE
        C
        C         MATRIX PP BECOMES THE PRODUCT E(TR)*P     *T(TR)*C(TR) TR=TRANSPOSE
        C
000772         DO 70 I=1,3
000774         DO 71 J=1,3
000775         PP(I,J)=0.
001000         DO 72 L=1,3
001001      72 PP(I,J)=PP(I,J)+R(I,L)*C(J,L)
001014      71 CONTINUE
001016      70 CONTINUE
001020         APHI=ATAN2(-PP(3,1),-PP(3,2))
001026         ST= SQRT(PP(3,1)**2+PP(3,2)**2)
001033         ATH= ATAN2(ST,PP(3,3))
001036         APSI= ATAN2(-PP(1,3),PP(2,3))
001042         APHI=APHI*RTD  $  ATH=ATH*RTD  $ APSI=APSI*RTD
001045         APHI=AMOD(APHI,360.)
```

```
001050          IF (APHI.LT.0.) APHI=APHI+360.
001053          APSI=AMOD(APSI,360.)
001056          IF (APSI.LT.0.) APSI=APSI+360.
001060          WRITE (3,120) APHI,ATH,APSI
001072          AOMEG=259.183275-0.0529539222*(36525.*TEP)+0.0001557*(3.6525
               1 **2)*TEP2+0.00000005*(3.6525**3)*TEP3
001106          AMOON=270.434358+13.1763965268*(36525.*TEP)-0.000085*(3.6525
               1 **2)*TEP2+.000000039*(3.6525**3)*TEP3
001122          AI=5549.3/3600.
001124          AOMEG=AMOD(AOMEG,360.)
001127          IF (AOMEG.LT.0.) AOMEG=AOMEG+360.
001131          AMOON=AMOD(AMOON,360.)
001134          IF (AMOON.LT.0.   ) AMOON=AMOON+360.
001137          WRITE (3,121) AMOON,AI,AOMEG
001151          RHO=ATH-AI
001153          SIG=APSI-AOMEG
001155          TAU=APHI-180.-AMOON+APSI
001161          IF (ABS(TAU).GT.10.) TAU=TAU-360.
001166          WRITE (3,118)
001172          WRITE (3,119) RHO,SIG,TAU
        C
        C                  EULER PARAMETER TESTA FOR MOON
        C
001204          TEST3=X(38)**2+X(39)**2+X(40)**2+X(41)**2
001212          TEST4=X(38)*V(38)+X(39)*V(39)+X(40)*V(40)+X(41)*V(41)
001221          CON1=-(V(38)*V(38)+V(39)*V(39)+V(40)*V(40)+V(41)*V(41))
001226          CON2=X(38)*F(38)+X(39)*F(39)+X(40)*F(40)+X(41)*F(41)
001236          DCON=CON1-CON2
001240          WRITE (3,113)
001243          WRITE (3,112) TEST3,TEST4,DCON
001255          IF (TF.GE.TMAX)   GO TO 2
001260          TI=TF
001261          TF=TF+TINC
001263          GO TO 4
001263      100 FORMAT(*1*,40X,*TRANSLATIONAL MOTION*,//)
001263      101 FORMAT(* *,*EPOCH=*,E19.12,5X,*ORDER=*,I2,2X,*LL=*,I2,2X,*NF=*
               A,3X
               1,I5,*DAYS PAST EPOCH=*,E15.7,//)
001263      102 FORMAT(* *,*PLANET*,I4)
001263      103 FORMAT(* *,3(E19.12,2X))
001263      104 FORMAT(* *,3(E19.12,2X)//)
001263      105 FORMAT (9X,*X(AU)*,14X,*Y(AU)*,14X,*Z(AU)*,/)
001263      106 FORMAT (9X,*XD(AU/100D)*,9X,*YD(AU/100D)*,9X,*ZD(AU/100D)*,/)
```

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```
001263     107 FORMAT(*1*,40X,*EARTH ROTATIONAL MOTICN*,//)
001263     108 FORMAT(20X,*EULER PARAMETERS AND RATES,BETA PRIME*,/)
001263     109 FORMAT(*1*,40X,*MOON ROTATIONAL MOTION*,//)
001263     110 FORMAT(20X,*EULER PARAMETERS AND RATES,BETA TR.PRIME*,/)
001263     111 FORMAT (9X,4E19.12)
001263     112 FORMAT (10X,3(E19.12,3X),//)
001263     113 FORMAT (/,20X,*EULER PARAMETER TESTS*)
001263     115 FORMAT (* *,40X,*JULIAN DATE=*,E19.12,//)
001263     116 FORMAT (//,20X,*EARTHS SELENOGRAPHIC COORDINATES*,/)
001263     117 FORMAT (* *,9X,*LONG.=*,F19.12 ,5X,*LAT.=*,E19.12,//)
001263     118 FORMAT(35X,*LUNAR PHYSICAL LIBRATIONS*,/)
001263     119 FORMAT (2X,*RHO=*,E19.12,5X,*SIG=*,E19.12,5X,*TAU=*,E19.12,//)
001263     120 FORMAT (35X,*EULER ANGLES*,//,5X,*PHI=*,E20.12,2X,*THETA=*,E20.12
              1 ,2X,*PSI=*,E20.12,//)
001263     121 FORMAT (35X,*FUNDAMENTAL ANGLES*,//,5X,*MOON=*,E20.12,2X,*I=*
              1 ,E20.12,2X,*OMEGA*,E20.12,//)
001263     122 FORMAT (I5)
001263       2 READ (2,122) ICODE
001271         IF (ICODE.EQ.1) GO TO 77
001273         STOP
001275         END
```

```
                    SUBROUTINE RA19S(X,V,TI,TF,XL,LL,NV,NI,NF,NS,NCLASS,NOR,NSS)
                 C. PROGRAM BY E. EVERHART, PHYSICS AND ASTRONOMY DEPT. UNIVERSITY OF DENVER.
                 C  DENVER, COLORAD 80210.  PHONE (303)-753-2238  OR 753-2362
                 C  INTEGRATOR FOR ORDERS 7, 11, 15, 19.  SINGLE PRECISION VERSION
                 C  THIS IS A VERSION OF INTEGRATOR RADAU
                 C  NV IS THE NUMBER OF DEPENDENT VARIABLES
                 C  NCLASS IS 1 FOR 1ST-ORDER DIFF EQ, AND 2 FOR 2ND ORDER DIFF EQ.
                 C  IF FIRST DERIVATIVES ARE NOT PRESENT (CLASS IIS), THEN USE NCLASS=-2.
                 C  X(NV) IS THE INITIAL POSITION VECTOR.  IT RETURNS AS THE FINAL VALUE
                 C  V(NV) IS THE INITIAL VELOCITY VECTOR.  IT RETURNS AS THE FINAL VALUE
                 C  IN THE CASE NCLASS IS UNITY, THEN V IS SIMPLY ZEROED
                 C  TI IS INITIAL TIME, TF IS FINAL TIME, NF IS NUMBER OF FUNCTION EVALUATIONS
                 C  NS IS THE NUMBER OF SEQUENCES.
                 C  PROGRAM SET UP FOR A MAXIMUM OF 18 SIMULTANEOUS EQUATIONS.
                 C  LL CONTROLS SEQUENCE SIZE. THUS SS=10.**(-LL) IS DESIRED SIZE OF A TERM.
                 C  AS IN  CONTROL SYSTEM I.
                 C  IF LL.LT.0, THEN XL IS THE SPECIFIED CONSTANT SEQUENCE SIZE
                 C  WILL INTEGRATE IN A NEGATIVE DIRECTION IF TF.LT.TI
  000020            DIMENSION X(1),V(1),F1(41),FJ(41),C(36),D(36),R(36)
                   1 ,XI(36),Y(41),Z(41),B(9,41),BE(9,41),H(10),W(9),U(9)
                   2 ,BT(9,41),HH(24)
  000020            DIMENSION MC(8),NW(10),NXI(36)
  000020            LOGICAL J2,NPQ,NSF,NPER,NCL,NES
  000020            DATA NW/0,0,1,3,6,10,15,21,28,36/
  000020            DATA MC/1,9,16,22,27,31,34,36/
  000020            DATA ZERO,ONE/0.,1./
  000020            DATA NXI/2,3,4,5,6,7,8,9,3,6,10,15,21,28,36,4,10,20,35,56,84,5,15,
                   X 35,70,126,6,21,56,126,7,28,84,8,36,9/
  000020            DATA  HH/.212340538239152,  .590533135559265,  .911412040487296,     1
                   X.098535085798826426,  .304535726646363905,  .562025189752613855,   2
                   X.801986582126391827,  .960190142948531257,  .056262560536922146,   3
                   X.180240691736892364,  .352624717113169637,  .547153626330555383,   4
                   X.734210177215410531,  .885320946839095768,  .977520613561287501,   5
                   X.036257812883209460,  .118078978789998700,  .237176984814960385,   6
                   X.381882765304705975,  .538029598918989065,  .690332420072362182,   7
                   X.823883343837004718,  .925612610290803955,  .985587590351123451/   8
  000020            KD=(NOR-3)/2
  000022            KD2=KD/2
  000024            KE=KD+1
  000026            KF=KD+2
  000030            PW=ONE/FLOAT(KD+3)
  000033            NPER=.FALSE.
```

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```
000034          NSF=.FALSE.
000035          NCL=NCLASS.EQ.1
000040          NPQ=NCLASS.LT.2
000043          SR=1.5
000045          IF(NV.EQ.1) SR=1.2
000050          NES=LL.LT.0
000053          TDIF=TF-TI
000055          DIR=TDIF/ABS(TDIF)
000057          IF(NES) XL= ABS(XL)*DIR
000062          NCLASS=IABS(NCLASS)
000064          LA=KD2*KD2-1
000067          DO 14 N=2,KF
000071          LA=LA+1
000073          H(N)=HH(LA)
000075          W(N-1)=ONE/FLOAT(N+N**2*(NCLASS-1))
000103     14   U(N-1)=N+1
000107          DO 22 K=1,NV
000111          IF(NCL) V(K)=ZERO
000115          DO 22 L=1,KE
000117          BT(L,K)=ZERO
000122     22   B(L,K)=ZERO
000131          W1=ONE/FLOAT(NCLASS)
000134          DO 939 J=1,KD
000135          M=MC(J)
000137          JD=J+1
000141          DO 939 L=JD,KE
000142          XI(M)=FLOAT(NXI(M))*W(J)/W(L)
000152    939   M=M+1
000161          C(1)=-H(2)*W(1)
000163          D(1)=H(2)/W(2)
000165          R(1)=ONE/(H(3)-H(2))
000170          LA=1
000171          LC=1
000172          DO 73 K=3,KE
000173          LB=LA
000175          LA=LC+1
000176          LC=NW(K+1)
000200          JD=LC-LA
000201          C(LA)=-H(K)*C(LB)
000205          C(LC)=(C(LA-1)/W(JD)-H(K))*W(JD+1)
000216          D(LA)=H(2)*D(LB)*W(K-1)/W(K)
000224          D(LC)=(D(LA-1)*W(K-1)+H(K))/W(K)
000234          R(LA)=ONE/(H(K+1)-H(2))
```

```
000240          R(LC)=ONE/(H(K+1)-H(K))
000246          IF(K.EQ.3) GO TO 73
000250          DO 72 L=4,K
000251          LD=LA+L-3
000254          LE=LB+L-4
000256          JDM=LD-LA
000260          C(LD)=W(JDM+1)*C(LE)/W(JDM)-H(K)*C(LE+1)
000270          D(LD)=(D(LE)+H(L-1)*D(LE+1))*W(K-1)/W(K)
000301       72 R(LC)=ONE/(H(K+1)-H(L-1))
000311       73 CONTINUE
000314          SS=10.**(-LL)
000320          NL=NI+30
          C  SET IN A REASONABLE ESTIMATE TO T BASED ON EXPERIENCE.  (SAME SIGN AS TF-TI)
000323          IF(.NOT.NES) TP=((FLOAT(NOR)/11.)*0.5**(0.4*FLOAT(LL)))*DIR
000337          IF(NES) TP=XL
000342          IF(TP/TDIF.GT.0.5  ) TP=0.5*TDIF
000347          NF=0
000350          NCOUNT=0
000351     4000 NS=0
000352          TM=TI
000353          SM=1.E4
000355          CALL FORCE(X,V,TM,F1)
000357          NF=NF+1
          C  LOOP 58 FINDS THE BETA-VALUES FROM THE CORRECTED B-VALUES, USING D-C EFF
000361      722 DO 58 K=1,NV
000366          BE(KE,K)=B(KE,K)/W(KE)
000374          DO 58 J=1,KD
000375          JD=J+1
000377          BE(J,K)=B(J,K)/W(J)
000404          DO 58 L=JD,KE
000406          N=NW(L)+J
000411       58 BE(J,K)=BE(J,K)+D(N)*B(L,K)
000431          T=TP
000432          TVAL=ABS(T)
000434          T2=T**NCLASS
          C  LOOP 175 IS THE ITERATION LOOP WITH NL=NI PASSES AFTER THE FIRST SEQ ENCE
000440          DO 175 M=1,NL
000442          J2=.TRUE.
000443          DO 174 J= 2,KF
000445          JD=J-1
000447          LA=NW(JD)
000451          JDM=J-2
000453          S=H(J)
```

```
000454          Q=S**(NCLASS-1)
000462          IF(NPQ) GO TO 5100
000464          DO 1300 K=1,NV
000465          RES=B(KE,K)
000470          TEMP=RES*U(KE)
000473          DO 7340 L=1,KD
000474          JR=KE-L
000476          RES=B(JR,K)+S*RES
000503    7340  TEMP=B(JR,K)*U(JR)+S*TEMP
000512          Y(K)=X(K)+Q*(T*V(K)+T2*S*(F1(K)*W1+S*RES))
000531    1300  Z(K)=V(K)+S*T*(F1(K)+S*TEMP)
000543          GO TO 5200
000543    5100  DO 1400 K=1,NV
000545          RES=B(KE,K)
000550          DO 2340 L=1,KD
000552          JR=KE-L
000554    2340  RES=B(JR,K)+S*RES
000564    1400  Y(K)=X(K)+Q*(T*V(K)+T2*S*(F1(K)*W1+S*RES))
000604    5200  CONTINUE
000604          CALL FORCE(Y,Z,TM+S*T,FJ)
000612          NF=NF+1
000614          IF(J2) GO TO 702
000621          DO 471 K=1,NV
000623          TEMP=BE(JD ,K)
000626          RES=(FJ(K)-F1(K))/S
000632          N=LA
000633          DO 134 L=1,JDM
000635          N=N+1
000637     134  RES=(RES-BE(L,K))*R(N)
000647          BE(JD,K)=RES
000652          TEMP=RES-TEMP
000654          B(JD,K)=B(JD,K)+TEMP*W(JD)
000660          N=LA
000661          DO 471 L=1,JDM
000663          N=N+1
000665     471  B(L,K)=B(L,K)+C(N)*TEMP
000700          GO TO 174
000700     702  J2=.FALSE.
000701          DO 271 K=1,NV
000703          TEMP=BE(1,K)
000706          RES=(FJ(K)-F1(K))/S
000711          BE(1,K)=RES
000714     271  B(1,K)=B(1,K)+(RES-TEMP)*W(1)
```

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000724     174  CONTINUE
000727          IF(M.LT.NI) GO TO 175
000731          HSUM=0.
000732          VAL=TVAL**(-KE)
000737          DO 635 K=1,NV
000741     635  HSUM=HSUM+B(KE,K)**2
000751          HSUM=VAL*SQRT(HSUM)
000754          IF(  NSF  ) GO TO 175
000761          IF(ABS((HSUM-SM)/HSUM).LT.0.01) GO TO 176
000766          SM=HSUM
000766     175  CONTINUE
              C  THIS NEXT PART FINDS THE PROPER STARTING VALUE FOR T
000771     176  IF(  NSF  ) GO TO 180
000773          IF(.NOT.NES) TP=(SS/HSUM)**PW*DIR
001002          IF(NES) TP=XL
001005          IF(NES) GO TO 170
001006          IF(TP/T.GT.ONE) GO TO 170
001013       8  FORMAT(2XI2,2E18.10)
001013          TP=0.8  *TP
001014          NCOUNT=NCOUNT+1
001016          IF(NCOUNT.GT.10) RETURN
001021          PRINT 8,KD,T,TP
001033          GO TO 4000
001037     170  PRINT 8,KD,T,TP
001051          NSF=.TRUE.
              C  FIND POSITION (AND VELOCITY FOR CLASS II AND IIS) AT THE END OF THE SEQUENCE.
001052     180  DO 35 K=1,NV
001057          RES=B(KE,K)
001062          DO 34 L=1,KD
001064      34  RES=RES+B(L,K)
001072          X(K)=X(K)+V(K)*T+T2*(F1(K)*W1+RES)
001105          IF(  NCL  ) GO TO 35
001107          RES=B(KE,K)*U(KE)
001113          DO 33 L=1,KD
001115      33  RES=RES+B(L,K)*U(L)
001126          V(K)=V(K)+T*(F1(K) +RES)
001133      35  CONTINUE
001136          TM=TM+T
001140          NS=NS+1
001141      74  IF(  NPER  ) RETURN
              C  CONTROL ON SIZE OF NEXT SEQUENCE AND RETURN WHEN TF IS REACHED
001144          CALL FORCE(X,V,TM,F1)
001146          NF=NF+1
```

```
001150          IF(NES) GO TO 341
001155          TP=((SS/HSUM)**PW)*DIR
001163          IF(TP/T.GT.SR) TP=SR*T
001167     341  IF(NES) TP=XL
001172          IF(DIR*(TM+TP).LT.DIR*TF-1.E-10) GO TO 77
001200          TP=TF-TM
001202          NPER=.TRUE.
              C PREDICT B-VALUES FOR NEXT SEQUENCE,
001203      77  Q =TP/T
001205          DO 39 K=1,NV
001207          RES=ONE
001211          DO 39 J=1,KE
001212          IF(NS.GT.1) BT(J,K)=B(J,K)-BT(J,K)
001222          IF(J.EQ.KE) GO TO 740
001224          M=MC(J)
001226          JD=J+1
001230          DO 40 L=JD,KE
001231          B(J,K)=B(J,K)+XI(M)*B(L,K)
001241      40  M=M+1
001245     740  RES=RES*Q
001247          TEMP=RES*B(J,K)
001253          B(J,K)=TEMP+BT(J,K)
001260      39  BT(J,K)=TEMP
001266          NL=NI
001267          GO TO 722
001270          END
```

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```
                 SUBROUTINE PROB   (X,V,TI,TF,NV,NCLASS,NSS,NI,NOR,LL,IICODE)
            C
            C    THIS SUBROUTINE PROVIDES CONSTANTS AND INITIAL
            C    CONDITIONS FOR RIGEM
            C    NORO(1)=0 NEGLECT EARTH ROTATION
            C           =1 COMPUTE EARTH ROTATION
            C    NORO(2)=0 NEGLECT MOON ROTATION
            C           =1 COMPUTE MOON ROTATION
            C
            C    OMPL(1)=1     OMITS  MERCURY,SATURN,URANOUS,NEPTUNE,PLUTO
            C    OMPL(1)=0     OMITS NO PLANETS
            C
000016           DIMENSION XMASS(11),X(1),V(1),XMA(11),NORO(2),OMPL(1)
000016           DIMENSION XS(41),VS(41)
000016           INTEGER OMPL
000016           COMMON/XMAS/XMASS
000016           COMMON/INERT/BEM,GAM,ALPM,BEE,GAE,ALPE,EI1,EI2,EI3,AMI1,AMI2,AMI3
000016           COMMON/PARAM/NORO,AO,AD,VJDEP,OMPL,TINC,TMAX
000016           DATA XMA /1.,5983000.,408522.,332945.56192544,27068807.1301,
                1309870O.,1047.3908,3499.2,22930.,19260.,1812000./
000016           IF (IICODE.NE.0) GO TO 12
000017           READ (2,120) NORO(1),NORO(2),OMPL(1)
000031           READ (2,121) VJDEP,TINC,TMAX
000043           READ (2,123) NI,NOR,LL
000055           PI=3.14159265358979
000056           DTR=PI/180. $ RTD=180./PI
000061           BEM=.00063 $ GAM=.0002 $ ALPM=.00043
000066           ALPE=.00322
000067           BEE=.00327
000071           GAE=.000054
000072           AO=100.075542*DTR
000074           AD=360.985647348*DTR
000076           EI1=5. $  EI2=5. $ EI3=5.
000101           AMI1=5. $ AMI2=5. $AMI3=5.
            C     THIS LOOP PROVIDES PLANETARY I.C. PER TABLE 10 ,P.274,CEL.MECH.
            C             JOURNAL V5,NO.3
000104           XK2=(.01720209895)**2
000106           DO 1 I=1,11
000113         1 XMASS(I)=XK2/XMA  (I)
            C    THIS LOOP PROVIDES PLANETARY I.C. PER TABLE 10,P.274
000117           KK=1
000120           WRITE(3,102)
```

```
000124         WRITE(3,103)
000130         WRITE(3,104)
000134         WRITE(3,105)
000140         WRITE(3,106)
000144         DO 2 I=1,31,3
000151         READ(2,107)X(I),X(I+1),X(I+2)
000176         XS(I)=X(I) $ XS(I+1)=X(I+1) $ XS(I+2)=X(I+2)
000210         READ(2,107)V(I),V(I+1),V(I+2)
000236         WRITE(3,100) KK,X(I),X(I+1),X(I+2)
000271         WRITE(3,100)  KK,V(I),V(I+1),V(I+2)
000324         V(I)=V(I)/100.
000332         V(I+1)=V(I+1)/100.
000333         V(I+2)=V(I+2)/100.
000335         VS(I)=V(I) $ VS(I+1)=V(I+1) $ VS(I+2)=V(I+2)
000343         KK=KK+1
000345       2 CONTINUE
           C   THIS LOOP PROVIDES INILIAL EULER ANGLES AND RATES FOR EARTH MOON
000347         WRITE(3,108)
000352     108 FORMAT (/,30X,*EARTH INITIAL EULER PARAMETERS AND RATES*,/)
000352         DO 3 L=34,38,4
000357         READ(2,101) X(L),X(L+1),X(L+2),X(L+3)
000412         XS(L)=X(L) $ XS(L+1)=X(L+1) $ XS(L+2)=X(L+2) $ XS(L+3)=X(L+3)
000426         READ(2,101)V(L),V(L+1),V(L+2),V(L+3)
000462         VS(L)=V(L) $ VS(L+1)=V(L+1) $ VS(L+2)=V(L+2) $ VS(L+3)=V(L+3)
000476         WRITE(3,109) X(L),X(L+1),X(L+2),X(L+3)
000532         WRITE(3,109) V(L),V(L+1),V(L+2),V(L+3)
000572         IF(L.EQ.38) GO TO 3
000600         WRITE(3,110)
000603       3 CONTINUE
000611         TI=0.
000611         TF=TINC
000612         NCLASS=+2
000613         NV=41
000614         NSS=0
000615         WRITE (3,125)
000621         WRITE (3,126)
000625         DO 11 I=1,11
000632      11 WRITE (3,127) I,XMASS(I)
000647         WRITE (3,128) ALPE,BEE,GAE
000660         WRITE (3,129) ALPM,BEM,GAM
000672         WRITE (3,130) VJDEP,TINC,TMAX
000704         WRITE (3,131) NORO(1),NORO(2),OMPL(1)
000716         WRITE (3,132) NI,NOR,LL
```

```
                C
                C        MULTI-CASE SEGMENT
                C
 000730                  IF (IICODE.EQ.0) GO TO 14
 000735               12 DO 13 I=1,41
 000737                  X(I)=XS(I)
 000741               13 V(I)=VS(I)
 000745                  READ (2,101) V(38),V(39),V(40),V(41)
 000775                  WRITE (3,133)
 001001                  WRITE (3,110)
 001005                  WRITE (3,109) V(38),V(39),V(40),V(41)
 001041                  TI=0.
 001045                  TF=TINC
 001046              100 FORMAT (3X,I6,3E25.14)
 001046              101 FORMAT (4E20.12)
 001046              102 FORMAT (*1*,40X,*INITIAL CONDITIONS AND PARAMETERS*,//)
 001046              103 FORMAT (30X,*INITIAL POSITION AND VELOCITY OF PLANETS*,/)
 001046              104 FORMAT (30X,*REFERRED TO MEAN EQUATOR AND EQUINOX OF 1950.0*,//)
 001046              105 FORMAT (2X,*PLANETS*,16X,*X*,25X,*Y*,25X,*Z*)
 001046              106 FORMAT (25X,*XD*,25X,*YD*,25X,*ZD*,/)
 001046              107 FORMAT (3E25.14)
 001046              109 FORMAT (20X,4E20.12,/)
 001046              110 FORMAT (30X,*MOON INITIAL EULER PARAMETERS AND RATES*,/)
 001046              120 FORMAT (3I5)
 001046              121 FORMAT (3E20.12)
 001046              123 FORMAT (3I5)
 001046              125 FORMAT (*1*,50X,*PARAMETERS*,///)
 001046              126 FORMAT (40X,*PLANETARY GRAVITY PARAMETERS*,/)
 001046              127 FORMAT (45X,I2,2X,E19.12)
 001046              128 FORMAT (//,40X,*EARTH INERTIA RATIOS*,/,15X,*ALPHA=*,E19.12,3X,
                        1*BETA=*,E19.12,3X,*GAMMA=*,E19.12,/)
 001046              129 FORMAT (40X,*MOON INERTIA RATIOS*,/,15X,*ALPHA=*,F19.12,3X,
                        1*BETA=*,E19.12,3X,*GAMMA=*,E19.12,/)
 001046              130 FORMAT (10X,*EPOCH=*,E19.12,5X,*TINC(DAYS)=*,E19.12,5X,
                        1*TMAX(DAYS)=*,E19.12,//)
 001046              131 FORMAT (35X,*OPTIONS*,/,20X,*NORO(1)=*,I2,2X,*NORO(2)=*,
                        1I2,2X,*OMPL(1)=*,I2,//)
 001046              132 FORMAT (20X,*INTEGRATION PARAMETERS*,/,20X,*NI=*,I3,2X,*NOR=*
                        1,I3,2X,*LL=*,I3,//)
 001046              133 FORMAT (*1*,50X,*NEW CASE*)
 001046               14 IICODE=1
 001050                  RETURN
 001050                  END
```

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```
                SUBROUTINE FORCE(X,V,TM,F)
            C   THIS SUBROUTINE PROVIDES N BODY GRAVITATIONAL
            C              NOTATION
            C      XMASS(1)=SUN            XMASS(6)=MARS
            C      XMASS(2)=MERCURY        XMASS(7)=JUPITER
            C      XMASS(3)=VENUS          XMASS(8)=SATURN
            C      XMASS(4)=EARTH          XMASS(9)=UPANOUS
            C      XMASS(5)=MOON           XMASS(10)=NEDTUUE
            C     XMASS(11)=PLUTO
 000007         DIMENSION X(1 ),V(1),F(1),XMASS(11),Y(66)
 000007         DIMENSION OM(4),B(4,4),BD(4,4),OMD(4),FD(4),CMM(4),BB(4,4),
               1BBD(4,4),DALV(4),D(4,4),      OMDM(4),DD(4,4),FT(4),RD(4),
               1NORO(2),PT(4),DEL(4),DAL(4),C(3,3),OMPL(1)
 000007         INTEGER OMPL
 000007         COMMON/XMAS/XMASS
 000007         COMMON/INERT/BEM,GAM,ALPM,BEE,GAE,ALPE,EI1,EI2,EI3,AMI1,AMI2,AMI3
 000007         COMMON/PARAM/NORO,AO,AD,VJDEP,OMPL,TINC,TMAX
 000007         COMMON/XOUT/OMM,OMDM,RT
 000007         R(XI,XJ,YI,YJ,ZI,ZJ)= SQRT((XJ-XI)**2+(YJ-YI)**2+(ZJ-ZI)**2)
 000035         G=1.
 000037         DO 10 I=1,33
 000040         Y(I)=X(I)
 000042      10 CONTINUE
            C   II IS PLANET COUNTER
 000044         DO 60 I=1,41
 000045      60 F(I)=0.
 000050         IF (OMPL(1).EQ.1) 61,62
 000055      61 NN1=7  $ NN2=19 $ II=3
 000060         GO TO 63
 000061      62 NN1=4  $ NN2=31 $ II=2
 000064      63 DO 2 I=NN1,NN2,3
 000066         RR=R(Y(1),Y(I),Y(2),Y(I+1),Y(3),Y(I+2))
 000101         F(I)=   -G*(XMASS(1)+XMASS(II))*Y(I)/(RR*RR*RR)
 000111         JJ=2
 000112         IF(OMPL(1).EQ.1) JJ=3
 000115         G1=0.
 000116         G2=0.
 000117         DO 3 J=NN1,NN2,3
 000121         IF(II.EQ.JJ)GO TO 33
 000123         G1=G*XMASS(JJ)*(Y(J)-Y(I))/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2),
               NY(J+2))**3+G1
 000146         G2=G*XMASS(JJ)*Y(J)/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),Y(J+2))**3+G2
```

```
000167        33 JJ=JJ+1
000171         3 CONTINUE
000173           F(I)=F(I)+G1-G2
000177           F(I+1)=   -G*(XMASS(1)+XMASS(II))*Y(I+1)/(RR*RR*RR)
000207           JJ=2
000210           IF(CMPL(1).EQ.1) JJ=3
000213           G1=0.
000214           G2=0.
000215           DO 4 J=NN1,NN2,3
000217           IF(II.EQ.JJ)GO TO 44
000221           G1=G1+G*XMASS(JJ)*(Y(J+1)-Y(I+1))/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2)
         B,Y(J+2))**3
000247           G2=G2+G*XMASS(JJ)*Y(J+1)/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),Y(J+2))**3
000271        44 JJ=JJ+1
000273         4 CONTINUE
000275           F(I+1)=F(I+1)+G1-G2
000301           F(I+2)=   -G*(XMASS(1)+XMASS(II))*Y(I+2)/(RR*RR*RR)
000311           JJ=2
000312           IF(CMPL(1).EQ.1) JJ=3
000315           G1=0.
000316           G2=0.
000317           DO 5 J=NN1,NN2,3
000321           IF(II.EQ.JJ)GO TO 55
000323           G1=G1+G*XMASS(JJ)*(Y(J+2)-Y(I+2))/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2)
         B,Y(J+2))**3
000351           G2=G2+G*XMASS(JJ)*Y(J+2)/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),Y(J+2))**3
000373        55 JJ=JJ+1
000375         5 CONTINUE
000377           F(I+2)=F(I+2)+G1-G2
000403         2 II=II+1
      C
      C                   FORCES ON SUN (XMASS(1))
      C
000407           IF (OMPL(1).EQ.1) 64,65
000414        64 NN1=7  $NN2=19  $ JJ=3
000417           GO TO 66
000420        65 NN1=4  $NN2=31  $ JJ=2
000423        66 G1=0.
000424           DC 6 J=NN1,NN2,3
000426           G1=G1+G*XMASS(JJ)*(Y(1)-Y(J))/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),
         BY(J+2))**3
000451         6 JJ=JJ+1
000455           F(1)=-G1
```

```
              C         SUNS ACCELERATIONS SET EQUL TO ZERO FOR TEST
000456                F(1)=0.
000457                IF (OMPL(1).EQ.1) 67,68
000464             67 NN1=7  $NN2=19  $ JJ=3
000467                GO TO 69
000470             68 NN1=4  $NN2=31  $ JJ=2
000473             69 G1=0.
000474                DO 7 J=NN1,NN2,3
000476                G1=G1+G*XMASS(JJ)*(Y(2)-Y(J+1))/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),
                     VY(J+2))**3
000521              7 JJ=JJ+1
000525                F(2)=-G1
000527                F(2)=0.
000530                IF (OMPL(1).EQ.1) 70,71
000534             70 NN1=7  $NN2=19  $JJ=3
000537                GO TO 72
000540             71 NN1=4  $NN2=31  $JJ=2
000543             72 G1=0.
000544                DO 8 J=NN1,NN2,3
000546                G1=G1+G*XMASS(JJ)*(Y(3)-Y(J+2))/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),
                     CY(J+2))**3
000571              8 JJ=JJ+1
000575                F(3)=-G1
000577                F(3)=0.
              C                       ROTATIONAL MOTION OF EARTH
              C
              C                 X(34)=BETA PRIME 0
              C                 X(35)=BETA PRIME 1
              C                 X(36)=BETA PRIME 2
              C                 X(37)=BETA PRIME 3
              C
              C       COMPUTE BETA PRIME AND BETA PRIME DOT MATRICES B ,D(INVERTED)
000600                T=VJDEP-2400000.5
000602                T=T-33282.+TM
000605                PI=3.14159265358979
000606                TWOPI=2.*PI
000610                AL=A0+AD*T
000613                F(37)=0.
000614                OM(4)=0.
000615                DO 109 I=1,3
000616                OM(I)=0.
000617                DEL(I)=0.
000620                F(I+33)=0.
```

```
000622      109 CONTINUE
000623          IF(NORO(1).EQ.0) GO TO 1000
000624          DO 100 I=1,4
000626          BD(I,I)=V(34)
000631      100 B(I,I)=X(34)
000635          B(1,2)=X(35)
000637          B(1,3)=X(36)
000640          B(1,4)=X(37)
000642          B(2,3)=X(37)
000643          B(2,4)=-X(36)
000645          B(3,4)=X(35)
000646          BD(1,2)=V(35)
000650          BD(1,3)=V(36)
000651          BD(1,4)=V(37)
000653          BD(2,3)=V(37)
000654          BD(2,4)=-V(36)
000656          BD(3,4)=V(35)
000657          DO 101 I=2,4
000661          JJ=I-1
000663          DO 102 J=1,JJ
000664          B(I,J)=-B(J,I)
000670          BC(I,J)= -BD(J,I)
000674      102 CONTINUE
000676      101 CONTINUF
          C     CALCULATE O,OMEGA 1,2,3 V$CTOR,OM
000700          DO 103 I=1,4
000701          DO 104 J=1,4
000702      104 OM(I)=2.*B(I,J)*V(J+33)+OM(I)
000713      103 CONTINUE
          C     CALCULATE ANGULAR RATES OF REFERENCE AXIS,CAP Y
000715          FT(1)=0.
000716          FT(2)=X(35)*X(37)-X(34)*X(36)
000723          FT(3)=X(36)*X(37)+X(34)*X(35)
000727          FT(4)=X(34)**2-X(35)**2-X(36)**2+X(37)**2
000735          FT(2)=FT(2)*2.*AD
000740          FT(3)=FT(3)*2.*AD
000741          FT(4)=FT(4)*AD
000742          OM(2)=OM(2)+FT(2)
000744          OM(3)=OM(3)+FT(3)
000746          OM(4)=OM(4)+FT(4)
          C     CALCULATE MOMENTS ACTING ON EARTH PROJECTED ON BODY AXES Y
          C     CALCULATE DIRECTION COSINES OF MOON WRT EARTH CENTERED AXES,
          C     LITTLE Y,DEL
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000750          DAL(1)=X(13)-X(10)
000753          DAL(2)=X(14)-X(11)
000755          DAL(3)=X(15)-X(12)
          C     CALCULATE BETA DOUBLE PRIME
000760          AL= AMOD(AL,TWOPI)
000763          CA= COS(AL/2.)
000767          SA= SIN(AL/2.)
000773          B1=CA*X(34)-SA*X(37)
001001          B2=CA*X(35)-SA*X(36)
001004          B3=SA*X(35)+CA*X(36)
001010          B4=SA*X(34)+CA*X(37)
          C     CALCULATE ELEMENTS OF ROTATION MATRIX,C(BETA DOUBLE PRIME)
001013          C(1,1)=B1*B1+B2*B2-B3*B3-B4*B4
001020          C(1,2)=2.  *(B2*B3+B1*B4)
001024          C(1,3)=2.  *(B2*B4-B1*B3)
001030          C(2,1)=2.  *(B2*B3-B1*B4)
001034          C(2,2)=B1*B1-B2*B2+B3*B3-B4*B4
001040          C(2,3)=2.*(B3*B4+B1*B2)
001044          C(3,1)=2.*(B2*B4+B1*B3)
001050          C(3,2)=2.*(B3*B4-B1*B2)
001054          C(3,3)=B1*B1-B2*B2-B3*B3+B4*B4
001060          DO 105 I=1,3
001062          DO 106 J=1,3
001063      106 DEL(I)=C(I,J)*DAL(J)+DEL(I)
001074      105 CONTINUE
001076          RRR = SQRT(DEL(1)**2+DEL(2)**2+DEL(3)**2)
001104          DEL(1)=DEL(1)/RRR
001105          DEL(2)=DEL(2)/RRR
001106          DEL(3)=DEL(3)/RRR
          C     EM1G,EM2G,EM3G, ARE GRAVITY GRADIENT TERMS
          C     EM1,EM2,EM3 ARE ALL OTHER TORQUES
001107          EM1=0.  $ EM2=0.  $ EM3=0.
001112          EM1G=3.  *DEL(2)*DEL(3)*XMASS(5)*ALPE/(RRR**3)
001120          EM2G=-3.  *DEL(1)*DEL(3)*XMASS(5)*BEE/(RRR**3)
001126          EM3G=3.  *DEL(1)*DEL(2)*XMASS(5)*GAE/(RRR**3)
          C     CALCULATE VALUES OF O,OMEGA1,2,3 DOT VECTOR
001135          OMD(1)=0.
001136          OMD(2)=EM1G+EM1/EI1-ALPE*OM(3)*OM(4)
001143          OMD(3)=EM2G     +EM2/EI2+BEE*OM(2)*OM(4)
001151          OMD(4)=EM3G+EM3/EI3-GAE*OM(2)*OM(3)
001157          FD(1)=0.
001160          FD(2)=X(35)*V(37)+V(35)*X(37)-X(34)*V(36)-V(34)*X(36)
001176          FD(2)=2.  *AD*FD(2)
```

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001201          FD(3)=X(36)*V(37)+V(36)*X(37)+X(34)*V(35)+V(34)*X(35)
001214          FD(3)=2.   *AD*FD(3)
001217          FD(4)=X(34)*V(34)-X(35)*V(35)-X(36)*V(36)+X(37)*V(37)
001232          FD(4)=2.   *AD*FD(4)
001235          DO 107 I=1,4
001236          DO 108 J=1,4
001237          F(I+33)=BD(J,I)*(OM(J)-FT(J))+B(J,I)*(OMD(J)-FD(J))+F(I+33)
001267      108 CONTINUE
001272          F(I+33)=F(I+33)*0.5
001274      107 CONTINUE
001276     1000 CONTINUE
              C              ROTATIONAL MOTION OF MOON
              C
              C         X(38)=BETA TRIPLE PRIME 0
              C         X(39)=BETA TRIPLE PRIME 1
              C         X(40)=BETA TRIPLE PRIME 2
              C         X(41)=BETA TRIPLE PRIME 3

              C     CALCULATE BETA TRIPLE PRIME AND BETA TRIPLE PRIME DOT MATRICES
              C     BB AND BBD (INVERTED)
001276          IF(NORO(2).EQ.0) GO TO 2000
001277          DAL(1)=X(13)-X(10) $ DAL(2)=X(14)-X(11) $ DAL(3)=X(15)-X(12)

              C       NORMALIZATION OF EULER PARAMETERS
001307          XNORM=X(38)*X(38)+X(39)*X(39)+X(40)*X(40)+X(41)*X(41)
001316          XNORM=SQRT(XNORM)
001320          X(38)=X(38) /XNORM $ X(39)=X(39)/XNORM $ X(40)=X(40)/XNORM
001327          X(41)=X(41)/XNORM
001330          DO 200 I=1,4
001331          BBD(I,I)=V(38)
001334      200 BB(I,I)=X(38)
001340          BB(1,2)=X(39)
001342          BB(1,3)=X(40)
001343          BB(1,4)=X(41)
001345          BB(2,3)=X(41)
001346          BB(2,4)=-X(40)
001350          BB(3,4)=X(39)
001351          BBD(1,2)=V(39)
001353          BBD(1,3)=V(40)
001354          BBD(1,4)=V(41)
001356          BBD(2,3)=V(41)
001357          BBD(2,4)=-V(40)
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001361          BBD(3,4)=V(39)
001362          DO 201 I=2,4
001364          JJ=I-1
001366          DO 202 J=1,JJ
001367          BB(I,J)=-BB(J,I)
001373          BBD(I,J)= -BBD(J,I)
001377      202 CONTINUE
001401      201 CONTINUE
          C     CALCULATE O,OMEGA1,2,3 VECTOR FOR MOON
001403          DO 2011 I=1,4
001404     2011 CMM(I)=0.
001407          DO 203 I=1,4
001410          DO 204 J=1,4
001411      204 OMM(I)= 2.  *BB(I,J) *V(J+37) +OMM(I)
001422      203 CONTINUE
001424          RRR=  SQRT(DAL(1)**2 +DAL(2)**2 +DAL(3)**2)
          C
          C     ANGLE PHI DEFINED FROM -PI/2 TO +PI/2
          C
001433          CC=DAL(1)/RRR
001434          CS=DAL(2)/RRR
001436          SPH=DAL(3)/RRR
001437          CPH= SQRT(DAL(1)**2+DAL(2)**2)/RRR
001445          CL=CC/CPH
001446          SL=CS/CPH
001450          DALV(1)=V(13)-V(10)
001455          DALV(2)=V(14)-V(11)
001457          DALV(3)=V(15)-V(12)
          C     R1=0, R2=RDOT, R3=RLAMDOTCPH, R4=RPHIDOT
001462          R1=0.
001463          R2=CC* DALV(1) +CS*DALV(2) +SPH*DALV(3)
001471          R3= -SL*DALV(1) +CL *DALV(2)
001475          R4 = -CL*SPH*DALV(1) -SL*SPH*DALV(2)+CPH* DALV(3)
          C     RT(1)=0,RT(2)=- AMDOTSINPHI,
          C     RT(3)= PHIDOT,RT(4)= AMDOTCOSPHI
001503          RT(1)=R1
001504          RT(2)=-R3*SPH/(RRR*CPH)
001510          RT(3)= R4/RRR
001512          RT(4)=R3/RRR
          C     CALCULATE AUGMENTED ROTATION MATRIX C(BETA TR.PRIME),D
001513          DO 220 I=1,4
001514          DO 205 J=1,4
001515      205 D(I,J)=0.
```

```
001522      220 CONTINUE
001524          D(2,2)=X(38)**2+X(39)**2-X(40)**2-X(41)**2
001532          D(3,3)=X(38)**2-X(39)**2+X(40)**2-X(41)**2
001540          D(4,4)=X(38)**2-X(39)**2-X(40)**2+X(41)**2
001546          D(2,3)=2.  *(X(39)*X(40)+X(38)*X(41))
001553          D(2,4)=2.  *(X(39)*X(41)-X(38)*X(40))
001560          D(3,2)=2.  *(X(39)*X(40)-X(38)*X(41))
001565          D(3,4)=2.  *(X(40)*X(41)+X(38)*X(39))
001572          D(4,2)=2.  *(X(39)*X(41)+X(38)*X(40))
001577          D(4,3)=2.  *(X(40)*X(41)-X(38)*X(39))
001604          DO 206 I=1,4
001605          DO 207 J=1,4
001606      207 OMM(I)=D(I,J)*RT(J)+OMM(I)
001617      206 CONTINUE
              C     CALCULATE MOMENTS ACTING CN MOON
001621          RRR3=RRR**3
001622          RRR2=RRR**2
001623          AMM1=0. $AMM2=0. $AMM3=0.
001626          AMM1G=3.  *D(4,2)*D(3,2)*XMASS(4)*ALPM/RRR3
001633          AMM2G=-3.  *D(4,2)*D(2,2)*XMASS(4)*BEM/RRR3
001640          AMM3G=3.  *D(3,2)*D(2,2)*XMASS(4)*GAM/RRR3
001645          FACT=(13.1763965268*3.14159265358978/180.)**2
001650          FACT=FACT*.9905*(.0025637252**3)/XMASS(4)
001654          AMM1G=AMM1G*FACT $ AMM2G=AMM2G*FACT $ AMM3G=AMM3G*FACT
              C     CALCULATE VALUES OF O,OMEGA1,2,3 DOT VECTOR
001657          OMDM(1)=0.
001660          OMDM(2)=AMM1/AMI1+AMM1G-ALPM*OMM(3)*OMM(4)
001666          OMDM(3)=AMM2/AMI2+AMM2G+BEM*OMM(2)*OMM(4)
001673          OMDM(4)=AMM3/AMI3+AMM3G-GAM*OMM(2)*OMM(3)
              C     CALCULATE VALUES FOR DOT OF RT(I) IE RD(I)
              C     CONVERT RT TO O,-LDOTSINPHI,+PHIDOT,LDOTCOSPHI
              C     CALCULATE TIME DERIVATIVE OF ROTATION MATRIX
001701          RD(1)=0.
001702          RD(2)= DALV(1) *(R2*SL/RRR2-CL*RT(4)/(CPH*RRR))
001712          RD(2)=RD(2) -DALV(2)*(R2*CL/RRR2+SL*RT(4) /(CPH*RRR))
001722          RD(2)=RD(2) -(F(13)- F(10))*SL/RRR
001726          RD(2)= RD(2) +(F(14)- F(11))*CL/RRR
001733          RD(2)= RD(2)*SPH/CPH +RT(3)*RT(4)/( CPH*CPH)
001741          RD(2)=-RD(2)
001742          RD(3)= DALV(1) *(-RT(2)*SL/RRR +R2*CL*SPH/RRR2 -RT(3)*CC/RRR)
001755          RD(3) =DALV(2)*(RT(2)* CL/RRR -RT(3)*CS/RRR +R2*SL*SPH/RRR2) +
               1 RD(3)
001766          RD(3)= DALV(3)*(-RT(3)*SPH/RRR-R2*CPH/RRR2)+RD(3)
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001775          RD(3)= -(F(13)-F(10))* CL*SPH/RRR + RD(3)
002003          RD(3)= -(F(14)-F(11))*SL  *SPH/RRR+RD(3)
002011          RD(3)= (F(15)-F(12)) *CPH/RRR +RD(3)
002016          RD(4)=DALV(1)*(RT(2)* CL/(RRR*SPH)
               1 +R2*SL/RRR2)
002025          RD(4)= DALV(2)* (RT(2)*SL/(RRR*SPH) -R2*CL/RRR2)+RD(4)
002035          RD(4)= -(F(13)-F(10))*SL/RRR+RD(4)
002042          RD(4)=(F(14)- F(11))*CL/RRR +RD(4)
002047          DO 208 I=1,4
002051          DO 209 J=1,4
002052      209 DD(I,J)=0.
002057      208 CONTINUE
002061          DD(2,2)=2.  *(X(38)*V(38)+X(39)*V(39)-X(40)*V(40)-X(41)*V(41))
002076          DD(3,3)=2.  *(X(38)*V(38)-X(39)*V(39)+X(40)*V(40)-X(41)*V(41))
002113          DD(4,4)=2.  *(X(38)*V(38)-X(39)*V(39)-X(40)*V(40)+X(41)*V(41))
002130          DD(2,3)=2.  *(X(39)*V(40)+V(39)*X(40)+X(38)*V(41)+V(38)*X(41))
002145          DD(2,4)=2.  *(X(39)*V(41)+V(39)*X(41)-X(38)*V(40)-V(38)*X(40))
002162          DD(3,2)=2.  *(X(39)*V(40)+V(39)*X(40)-X(38)*V(41)-V(38)*X(41))
002177          DD(3,4)=2.  *(X(40)*V(41)+V(40)*X(41)+X(38)*V(39)+V(38)*X(39))
002214          DD(4,2)=2.  *(X(39)*V(41)+V(39)*X(41)+X(38)*V(40)+V(38)*X(40))
002231          DD(4,3)=2.  *(X(40)*V(41)+V(40)*X(41)-X(38)*V(39)-V(38)*X(39))
              C CALCULATE ACCELERATIONS
002246          DO 2101 I=1,4
002250          F(I+37)=0.
002252     2101 FT(I)=0.
002254          DO 210 I=1,4
002256          DO 211 J=1,4
002257      211 FT(I)=        -D(I,J)*RD(J)-DD(I,J)*RT(J)+FT(I)
002300          FT(I)=FT(I)+OMDM(I)
002303      210 CONTINUE
002305          DO 212 I=1,4
002306          DO 213 J=1,4
002307      213 F(I+37)=F(I+37)+BB(J,I)*FT(J) *0.5
002320      212 CONTINUE
002322          DO 2141 I=1,4
002323     2141 FT(I)=0.
002326          DO 214 I=1,4
002327          DO 215 J=1,4
002330      215 FT(I)=FT(I)+BB(I,J)*V(J+37)
002341      214 CONTINUE
002343          DO 216 I=1,4
002344          DO 217 J=1,4
002345          F(I+37)=F(I+37)+BBD(J,I)*FT(J)
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002353      217 CONTINUE
002355      216 CONTINUE
002357     2000 RETURN
002360          END
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      PROGRAM ANEAMO (INPUT,OUTPUT,PUNCH,TAPE2=INPUT,TAPE3=OUTPUT)
C                            PROGRAM ANEAMO
C     ANALYTIC EARTH AND MOON ROTATION AND TRANSLATION
C
C     THIS PROGRAM CALLS SUBROUTINES LUTH AND LURO PROVIDING RESULTS
C     FROM BROWN'S LUNAR THEORY AND ECKHARDT'S ROTATIONAL THEORY OF THE
C     MOON'S MOTION AT JULIAN DATES VJIN TO VJF IN INCREMENTS OF VJINC
C
C     THIS PROGRAM CALLS SUBROUTINE EARRO WHICH USES STANDARD PRECESSION-
C     -NUTATION FORMULAE AND SIDERAL TIME FORMULAE TO PROVIDE EARTH
C     ORIENTATION
C
C      THIS PROGRAM ALSO PROVIDES I/O FUNCTIONS AND CALLS THE
C       TRANSFORMATION SUBROUTINES ICOND AND AXANG
C                  AXANG- CONVERTS A ROTATION MATRIX INTO AXIS AND
C                         ANGLE OF ROTATION AND THEN INTO EULER
C                         PARAMETERS
C                  ICOND- CONVERTS EULER PARAMETERS BETA DOUBLE PRIME
C                         INTO PARAMETERS BETA PRIME
C
C         ALL ANGLES ARE IN DEGREES, PARALLAX IS IN DEGREES, ALL
C       COORDINATES ARE IN KILOMETERS
C       REFERENCES TO MEAN EQUINOX AND ECLIPTIC OF DATE
C
C              REFERENCES
C        1) IMPROVED LUNAR EPHEMERIS (ILE)
C        2)SAO STANDARD EARTH VOl. 1  L966
C        3) ECKHART A.J.  VOL. 70 NO.7 P.466
C        4) WILLIAMS ET.AL.  LUNAR PHYSICAL LIBRATIONS AND LASER RANGING
C
C          MULTI-CASE OPTION          ICODE=1 READ NEW TITLE AND DATES
C                                           0 STOP
C
C
C           CARD OUTPUT OPTION FOR PARAMETER ESTIMATION
C
C        IPT=0  NO CARD OUTPUT
C        IPT=1  OUTPUT PHYSICAL LIBRATIONS ON CARDS AND OUTPUT INITIAL
C                      VALUES ON PRINTER
C      CARD OUTPUT CONSISTS OF---
C           VJD    RHO    SIGMA    TAU
C      IN FORMAT (1X,4E19.12)
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```
000003        DIMENSION XEC(3),XEQ(3),VV(6),R(3,3),C(3)   ,BETA(4),TT(36,2),
             1 CC(31,2),
             1RR(31,2),NL(29),NLP(29),NF(29),ND(29),RS(3,3),BETDP(4),SBETA(4)
             1 ,S(3,3),P(3,3),XMO(3,3),BETM1(4),BETP1(4),BTPD(4),N(3,3)
             1,LKOK(14),LKOA(14),LKOB(14),PKOK(11),PKOA(11),PKOB(11)
             2,PLARG(23),PLRATE(23)
000003        DIMENSION XIND(21),YDEP(21,4),TDER(1),FDER(1,4),DER1(1,4),
             1DER2(1,4),WK(420)
             2,LATK(26),LATA(26),LATB(26),ADA(12),ADK(12),ADB(12)
000003        REAL LKOK,LKOA,LKOB
000003        COMMON/LUCUN/NL,NLP,NF,ND,PLARG,PLRATE
000003        COMMON/PERT/LKOK,LKOA,LKOB,PKOK,PKOA,PKOB
             1,LATK,LATA,LATB,ADK,ADA,ADB
000003        PI=3.14159265358979
000005        DTR=PI/180.
000007        RTD=180./PI
000010        NSKIP=0
000011        MNPTS=21 $ NDER=21 $ NCVS=4 $ MMAX=1 $ MDER=1 $ IW=-1
        C
        C                     INPUT
        C
        C     INPUT CONSTANTS FOR ECKHARDT'S THEORY
        C
        C      J IS THE BETA-GAMMA INDEX- SEE SUBRIUTINE LURO
        C
000017        J=2
000020        READ (2,119) (TT(I,J),I=1,36)
000033        READ (2,119) (CC(L,J),L=1,31)
000046        READ (2,119) (RR(M,J),M=1,31)
000061        READ (2,119) (PLARG(I),I=1,23)
000073        READ (2,120) (PLRATE(I),I=1,23)
000105        DO 6 LL=1,29
000107      6 READ(2,121) NL(LL),NLP(LL),NF(LL),ND(LL)
        C
        C      INPUT OF SMALL PERIODIC TERMS IN LONGITUDE.LATITUDE AND
        C            PARALLAX FROM THE ILE
        C                 LONGITUDE  LIST (IALPHA)
        C                 PARALAX LIST (IGAMMA)
        C            LATITUDE LIST (IBETA)
        C
        C       INPUT JULIAN DATES,MULTI-CASE CODE AND OUTPUT OPTION
        C
```

```
000124          DO 30 I=1,14
000126          READ (2,161)LKOK(I),LKOA(I),LKOB(I)
000137       30 CONTINUE
000141          DO 31 I=1,11
000143          READ (2,161) PKOK(I),PKOA(I),PKOB(I)
000154       31 CONTINUE
000156          DO 32 I=1,26
000160       32 READ (2,166) LATK(I),LATA(I),LATB(I)
        C
        C            INPUT OF ADDITIVE TERMS IN LONGITUDE(I=1,7),NODE(I=8,9)
        C                      AND GAMMA(I=10,12)
        C
000173          DO 33 I=1,12
000175       33 READ (2,167) ADK(I),ADA(I),ADB(I)
000210       60 READ (2,900)
000214          READ (2,100) VJIN,VJINC,VJF,ICODE,IPT
000232          IFLAG=0
000233          VJD=VJIN
        C
        C                      LUNAR THEORY
        C
000235        2 CALL LUTH (VJD,VV,DA,DB,DC,XEC,XEQ,TU)
000245          IF(NSKIP.EQ.1) GO TO 20
000247          IF (IFLAG.EQ.1.AND.IPT.EQ.1) GO TO 20
000255          WRITE (3,901)
000260          WRITE (3,900)
000264          WRITE (3,902)
000270          WRITE(3,101)VJD
000276          WRITE(3,102)VV(1),VV(2),VV(3),VV(4),VV(5),VV(6)
        C       CALCULATE DELAWAY ARGUMENTS
        C       XL=MOON≠S MEAN ANOMALY
        C       XCAPL=SUN≠S MEAN LONGITUDE
        C       XLPR=SUN≠S MEAN ANOMALY
        C       F=MEAN ANOMALY OF MOON+LUNAR ARGUMENT OF PERIGEE
        C
000316       20 XL=VV(1)-VV(3)
000320          XCAPL=VV(1)-VV(5)
000322          XLPR=XCAPL-VV(2)
000324          F=VV(1)-VV(4)
000326          XL=AMOD(XL,360.)
000331          XLPR=AMOD(XLPR,360.)
000334          XCAPL=AMOD(XCAPL,360.)
```

```
000337         F=AMOD(F,360.)
000342         IF(XL.LE.0.) XL=XL+360.
000344         IF(XCAPL.LE.0.)XCAPL=XCAPL+360.
000347         IF(XLPR.LE.0.)XLPR=XLPR+360.
000353         IF(F.LE.0.) F=F+360.
000357         IF(NSKIP.EQ.1) GO TO 21
000361         IF (IFLAG.EQ.1.AND.IPT.EQ.1) GO TO 21
000367         WRITE(3,107)
000372         WRITE(3,103)XL,XCAPL,XLPR,F
000406         WRITE(3,108)
000412         WRITE(3,104)OA,OB,OC
000424         WRITE(3,109)
000430         WRITE(3,105)XEC(1),XEC(2),XEC(3)
000442         WRITE(3,110)
000446         WRITE(3,106)XEQ(1),XEQ(2),XEQ(3)
        C
        C      ECKHARDT'S THEORY FOR LUNAR ROTATION
        C
000460      21 CALL EARRO(VJD,R,THETA,S,N,P)
000464         CALL LURO(XL,XLPR,F,VV,P,J,TAU,CI,RHO,CC,RR,TT,RS,XMO,XEQ,SLONG
              1 ,SLAT,TU)
000506         IF (NSKIP.EQ.1) GO TO 22
000510         IF (IPT-1) 48,49,48
000512      49 PUNCH 50,VJD,RHO,CI,TAU
000526         IF (IFLAG.EQ.0) GO TO 48
000527         GO TO 47
000530      48 IF (NSKIP.EQ.1) GO TO 22
000532         WRITE(3,113)
000536         WRITE (3,101) VJD
000544         WRITE(3,111)
000550         WRITE(3,112)TAU,CI,RHO,J
000564         CALL AXANG(RS,DEL,C,BETA)
000567         WRITE(3,124)
000573         DO 7 I=1,3
000575         L=1
000576         WRITE (3,122) RS(I,L),RS(I,L+1),RS(I,L+2)
000614       7 CONTINUE
000616         WRITE(3,125)
000622         WRITE(3,123)(BETA(I),I=1,4)
000634      22 CALL AXANG (XMO,DEL,C,BETA)
000637         IF(NSKIP.EQ.1) GO TO 23
000641         WRITE(3,160)
000645         WRITE(3,123) (BETA(I),I=1,4)
```

```
000657          DO 10 I=1,4
000661       10 SBETA(I)=BETA(I)
000664          SSLON=SLONG $ SSLAT=SLAT
         C
         C      LOOP TO CALCULATE EULER PARAMETER RATES BETA TRIPLE PRIME BY
         C      NUMERICAL DIFFERENTIATION
         C
000667          NSKIP=1
000670          JKJ=1
000671          IW=-1
000672          DELT=.0005
000674          TDER(1)=VJD $ TSTO=TDER(1)
000676          VJD=VJD-10*DELT
000701          GO TO 2
000702       23 XIND(JKJ)=VJD
000704          DO 24 L=1,4
000706       24 YDEP(JKJ,L)=BETA(L)
000716          JKJ=JKJ+1
000717          VJD=VJD+DELT
000721          IF (JKJ.GT.21) G O TO 25
000724          GO TO 2
000724       25 CALL SPLDER (MNPTS,NDER,NCVS,MMAX,MDER,XIND,YDEP,TDER,FDER,
               1DER1,DER2,IW,WK,IERR)
000742          DO 27 L=1,4
000744       27 BTPD(L)=DER1(1,L)
000752          NSKIP=0
000753          VJD=TSTO
000754          WRITE(3,141)
000760          WRITE(3,142) (BTPD(I),I=1,4)

         C     EULER PARAMETER TESTS

000772          TEST1=SBETA(1)**2+SBETA(2)**2+SBETA(3)**2+SBETA(4)**2
001000          TEST2=SBETA(1)*BTPD(1)+SBETA(2)*BTPD(2)+SBETA(3)*BTPD(3)+
               1 SBETA(4)*BTPD(4)
001007          WRITE (3,164)
001012          WRITE (3,165) TEST1,TEST2
001022          WRITE (3,162)
001026          WRITE (3,163) SSLON,SSLAT
         C
         C      PRECESSION-NUTATION CALCULATIONS FOR EARTH ORIENTATION
         C
001036       40 CALL EARRO(VJD,R,THETA,S,N,P)
```

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001042          IF (NSKIP.EQ.1) GO TO 41
001044          WRITE (3,115)
001050          WRITE (3,101) VJD
001056          THETA= THETA*RTD
001060          WRITE(3,140) THETA
001065          THETA=THETA*DTR
001067          WRITE(3,128)
001072          DO 12 L=1,3
001074          K=1
001075          WRITE (3,129) P(L,K),P(L,K+1),P(L,K+2)
001113       12 CONTINUE
001115          WRITE(3,150)
001121          DO 13 L=1,3
001123          K=1
001124          WRITE (3,129) N(L,K),N(L,K+1),N(L,K+2)
001142       13 CONTINUE
001144          WRITE(3,131)
001150          DO 14 L=1,3
001152          K=1
001153          WRITE (3,129) S(L,K),S(L,K+1),S(L,K+2)
001171       14 CONTINUE
001173          WRITE (3,124)
001177          DO 5 L=1,3
001201          K=1
001202          WRITE (3,116) R(L,K),R(L,K+1),R(L,K+2)
001220        5 CONTINUE
001222       41 CALL AXANG(R,DEL,C,BETA)
001225          IF (NSKIP.EQ.1) GO TO 42
001227          WRITE(3,117)
001233          WRITE(3,118) (BETA(I),I=1,4)
001245       42 CALL ICOND(VJD, BETA,      THETA,S,N,P)
001251          DO 11 I=1,4
001253       11 SBETA(I)=BETA(I)
001256          IF (NSKIP.EQ.1) GO TO 43
001260          WRITE(3,130)
001264          WRITE(3,118) (BETA(I),I=1,4)
       C
       C   LOOP TO CALCULATE EULER PARAMETER RATES BETA PRINE DOT BY
       C                       NUMERICAL DIFFERENTIATION
       C
001276          NSKIP=1
001277          JKJ=1
001300          DELT=.00005
```

```
001302          IW=-1
001303          TDER(1)=VJD $ TSTO=TDER(1)
001305          VJD=VJD-10*DELT
001310          GO TO 40
001311       43 XIND(JKJ)=VJD
001313          DO 45 L=1,4
001315       45 YDEP(JKJ,L)=BETA(L)
001325          JKJ=JKJ+1
001326          VJD=VJD+DELT
001330          IF (JKJ.GT.21) GO TO 44
001333          GO TO 40
001333       44 CALL SPLDER(MNPTS,NDER,NCVS,MMAX,MDER,XIND,YDEP,TDER,FDER,
               1 DER1,DER2,IW,WK,IERR)
001351          DO 46 L=1,4
001353       46 BETDP(L)=DER1(1,L)
001361          NSKIP=0
001362          VJD=TSTO
001363          WRITE(3,127)
001367          WRITE(3,126) (BETDP(I),I=1,4)

         C     EULER PARAMETER TESTS

001401          TEST1=SBETA(1)**2+SBETA(2)**2+SBETA(3)**2+SBETA(4)**2
001407          TEST2=SBETA(1)*BETDP(1)+SBETA(2)*BETDP(2)+SBETA(3)*BETDP(3)
               1 +SBETA(4)*BETDP(4)
001416          WRITE (3,164)
001421          WRITE (3,165) TEST1,TEST2
001431       47 IF (VJD.GE.VJF) GO TO 3
001434          VJD=VJD+VJINC
001436          IFLAG=1
001437          GO TO 2
001437        3 IF (ICODE.EQ.1) GO TO 60
001441       50 FORMAT (1X,4E19.12)
001441      100 FORMAT (3E20.10,2I5)
001441      101 FORMAT (* *,44X,*JULIAN DATE= *,E19.12,/  )
001441      102 FORMAT (* *,*MOON= *,F10.5,5X,*GAMMA= *,F10.5,5X,*GAMMA PRIME=*,
               1F10.5,/,* *,*OMEGA= *,F10.5,5X,*D= *,F10.5,5X,*OBLIQUITY= *,F10.5,
               2///)
001441      107 FORMAT (50X,*DELAUNAY ARGUMENTS*,/)
001441      103 FORMAT (* *,*L= *,F10.5,5X,*CAP L= *,F10.5,5X,*L PRIME= *,F10.5,5X
               1,*F= *,F10.5,///)
001441      108 FORMAT (50X,*ECLIPTIC LONG. AND LAT.*,/)
001441      104 FORMAT (* *,*LONGITUDE= *,F10.5,5X,*LATITUDE= *,F10.5,5X,*PARALLAX
```

```
                 1=*,F10.5,///)
 001441       109 FORMAT (50X,*RECTANGULAR ECLIPTIC COORDINATES*,/)
 001441       105 FORMAT (* *,*X= *,F12.4,5X,*Y= *,F12.4,5X,*Z= *,F12.4,///)
 001441       110 FORMAT (50X,*RECTANGULAR EQUATORIAL COORDINATES*,/)
 001441       111 FORMAT (50X,*PHYSICAL LIBRATIONS*,/)
 001441       106 FORMAT (* *,*X= *,F12.4,5X,*Y= *,F12.4,5X,*Z= *,F12.4,///)
 001441       112 FORMAT (* *,*LONG.= *,F12.9,5X,*NODE= *,F12.9,5X,*INCLINATION= *,
                 1F12.9,5X,*BETA GAMMA INDEX= *,I3,///)
 001441       113 FORMAT (*1*,50X,*MOONS ORIENTATION*,///)
 001441       119 FORMAT(7F10.5)
 001441       120 FORMAT (5F15.5)
 001441       121 FORMAT(4I2)
 001441       117 FORMAT (50X,*EULER PARAMETERS-MEQEQ TO BODY*,/)
 001441       124 FORMAT (30X,*DIRECTION COSINES(MEQEQ50 TO BODY)*,/)
 001441       122 FORMAT (20X,3(E19.12,5X))
 001441       125 FORMAT (//,50X,*EULER PARAMETERS-MEQEQ TO BODY*,/)
 001441       123 FORMAT (20X,4(E19.12,5X),/)
 001441       128 FORMAT (50X,*PRECESSION MATRIX*,/)
 001441       115 FORMAT (*1*,50X,*EARTH ORIENTATION*,/)
 001441       116 FORMAT (20X,3(E19.12,5X))
 001441       118 FORMAT (20X,4(E19.12,5X),/  )
 001441       127 FORMAT (//,50X,*EULER PARAMETER RATES-REF TO BODY*,/)
 001441       126 FORMAT (20X,4(E19.12,5X),//)
 001441       129 FORMAT (20X,3(E19.12,5X))
 001441       130 FORMAT (50X,*EULER PARAMETERS-REF. TO BODY*,/)
 001441       131 FORMAT (/,50X,*SPIN MATRIX*,/)
 001441       140 FORMAT (44X,*SIDEREAL TIME=*,E19.12,/)
 001441       141 FORMAT (50X,*EULER PARAMETER RATES REF. TO BODY*,/)
 001441       142 FORMAT (20X,4(E19.12,5X),///)
 001441       150 FORMAT (/,50X,*NUTATION MATRIX*,/)
 001441       160 FORMAT (50X,*EULER PARAMETERS REF. TO BODY*,/)
 001441       161 FORMAT (3E20.12)
 001441       162 FORMAT (50X,*EARTHS SELENOGRAPHIC COORDINATES*,/)
 001441       163 FORMAT (20X,*LONG=*,E19.12,10X,*LAT=*,E19.12,/)
 001441       164 FORMAT (/,50X,*EULER PARAMETER TESTS*,/ )
 001441       165 FORMAT (40X,E22.15,5X,E22.15)
 001441       167 FORMAT (2E10.3,E20.12)
 001441       166 FORMAT (E10.3,E10.3,E20.12)
 001441       900 FORMAT (80H
                 1                                    )
 001441       901 FORMAT (*1*)
 001441       902 FORMAT (* *,50X,*LUNAR ORBIT*,//)
 001441           STOP
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001443 END

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              SUBROUTINE LURO(ARGL,ARGLP,ARGF,VV,P,J,TAU,SIG,RHO,C,R,T,RM,XMO,
             1 XEO,SLONG,SLAT,TU)
        C
        C     THIS SUBROUTINE PROVIDES THE PHYSICAL LIBRATION IN LONGITUDE (TAU),
        C     IN NODE (ISIGMA),AND IN INCLINATION (RHO) FROM ECKHARDT+S LUNAR
        C     LIBRATION TABLES-REF:THE MOON V1 P264.
        C                  ADDITIVE AND PLANETARY TERMS ARE INCLUDED PER REF-
        C       LUNAR PHYSICAL LIBRATIONS AND LASER RANGING, WILLIAMS,ET.AL)
        C
        C               INPUT
        C     T(I,J)=TAU COEFFICIENTS
        C     I=TERM NO.        J=COEFFICIENTS(BETA)
        C     J=1        BETA=0.0006268       GAMMA=0.0002300   (GT 1 ARC-SEC)
        C     J=2        BETA=0.00063         GAMMA=0.0002   +ADDITIVE/PLANETARY
        C                                                     TERMS (GT .3 ARC-SEC)
        C
        C     ARGL=MEAN ANOMALY OF THE MOON=
        C    ARGLP=MEAN ANOMALY OF THE SUN=
        C     ARGF=MOON-OMEGA
        C     ARGD=MEAN ELONGATION OF MOON FROM SUN(D)
        C     C(I,J)=SIGI COEFFICIENTS
        C     R(I,J)=RHO COEFFICIENTS
        C
 000025       DIMENSION T(36,2),C(31,2),R(31,2),NL(29),NLP(29),NF(29),ND(29)
 000025       DIMENSION RM(3,3),EC(3,3),RR(3,3),VV(6),PLARG(23),PLRATE(23)
 000025       DIMENSION XMO(3,3),XEQ(3),XM1(3,3),P(3,3),XEQ5(3)
 000025       COMMON/LUCON/NL,NLP,NF,ND,PLARG,PLRATE
 000025       ARGD=VV(5)
 000026       IF (J.EQ.1)XI=5521.5
 000031       IF (J.EQ.2)XI=5549.3
 000034       PI=3.14159265358979
 000036       TWOPI=2.*PI
 000037       RTD=180./PI
 000041       DTR=PI/180.
 000042       TTT=TU*36525.
 000044       ARGL=ARGL*DTR
 000045       ARGLP=ARGLP*DTR
 000046       ARGF=ARGF*DTR
 000047       ARGD=ARGD*DTR
 000050       TAU=0. $ SIGI=0. $ RHO=0.
 000053       DO 1 I=1,13
 000055       ARG=NL(I)*ARGL+NLP(I)*ARGLP+NF(I)*ARGF+ND(I)*ARGD
```

```
000071        1 TAU=TAU+T(I,J)* SIN(ARG)
000106          IDIV=13
000107          DO 2 I=1,8
000111          ARG=NL(I+IDIV)*ARGL+NLP(I+IDIV)*ARGLP+NF(I+IDIV)*ARGF+ND(I+IDIV)*
               AARGD
000131        2 SIGI=SIGI+C(I,J)* SIN(ARG)
000145          IDIV=IDIV+8
000147          DO 3 I=1,8
000150          ARG=NL(I+IDIV)*ARGL+NLP(I+IDIV)*ARGLP+NF(I+IDIV)*ARGF+ND(I+IDIV)*
               BARGD
000170        3 RHO=RHO+R(I,J)* COS(ARG)
000205          DO 50 I=1,23
000206          AA=PLARG(I)+PLRATE(I)*TTT
000212          AA=AA*TWOPI
000213          AA=AMOD(AA,TWOPI)
000216          IF (AA.LT.0.) AA=AA+TWOPI
000220          TAU=TAU+T(I+13,J)*SIN(AA)
000234          SIGI=SIGI+C(I+8,J)*SIN(AA)
000250       50 RHO=RHO+R(I+8,J)*COS(AA)
000265          TAU=TAU/3600. $ RHO=RHO/3600. $ SIG=(SIGI/XI)*RTD
000273          TH=RHO+XI/3600.
000277          PSI=VV(4)+SIG
000301          PHI=180.+VV(1)-PSI+TAU
000306          PSI =AMOD(PSI,360.) $ PHI=AMOD(PHI,360.) $  TH=AMOD(TH,360.)
000317          IF (PSI.LT.0.) PSI=PSI+360.
000322          IF (PHI.LT.0.) PHI=PHI+360.
000325          IF (TH.LT.0.)  TH=TH+360.
000331          PSI=PSI*DTR $ PHI=PHI*DTR $ TH= TH*DTR
000334          CPH= COS(PHI) $  SPH= SIN(PHI)
000341          CPS= COS(PSI) $  SPS= SIN(PSI)
000345          CTH= COS(TH)  $  STH= SIN( TH)
        C
        C       RM- ECLIPTIC OF DATE TO BODY ROTATION MATRIX
        C
000351          RM(1,1)=CPH*CPS-SPH*CTH*SPS
000356          RM(1,2)=-CPH*SPS-SPH*CTH*CPS $ RM(1,2)=-RM(1,2)
000365          RM(1,3)=-SPH*STH
000367          RM(2,1)=CPS*SPH+CPH*CTH*SPS  $ RM(2,1)=-RM(2,1)
000375          RM(2,2)=-SPS*SPH+CPH*CTH*CPS
000402          RM(2,3)=CPH*STH   $RM(2,3)=-RM(2,3)
000406          RM(3,1)=-SPS*STH
000410          RM (3,2)=CPS*STH
000411          RM(3,3)=CTH
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```
000413          VV(6)=VV(6)*DTR
000421          EC(1,1)=1. $  EC(1,2)=0.   $  EC(1,3)=0.
000424          EC(2,1)=0. $  EC(3,1)=0.   $  EC(2,2)= COS(VV(6))
000435          EC(2,3)= SIN(VV(6)) $ EC(3,2)=- SIN(VV(6))  $ EC(3,3)= COS(VV(6))
000465          DO 12 I=1,3
000467          DO 13 L=1,3
000470       13 RR(I,L)=0.
000475       12 CONTINUE
000477          DO 10 I=1,3
000500          DO 11 L=1,3
000501          DO 15 K=1,3
000502       15 RR(I,L)=RM(I,K)*EC(K,L)+RR(I,L)
000517       11 CONTINUE
000521       10 CONTINUE
           C
           C     MULTIPLY MATRIX RR BY P TO REFER ANGLES TO MEAN EQUATOR AND
           C                    EQUINOX OF 1950.0
           C              RM- MEQE050 TO BODY ROTATION MATRIX
           C
000523          DO 30 I=1,3
000524          DO 31 L=1,3
000525          RM( I,L)=0.
000530          DO 32 K=1,3
000532       32 RM(I,L)=RR(I,K)*P(K,L)+RM(I,L)
000550       31 CONTINUE
000552       30 CONTINUE
000554          DO 33 I=1,3
000555          XEQ5(I)=0.
000556          DO 34 K=1,3
000560       34 XEQ5(I)=P(K,I)*XEQ(K)+XEQ5(I)
000573       33 CONTINUE
000575          DO 35 I=1,3
000576       35 XEQ(I)=XEQ5(I)
           C
           C        CALCULATION OF EULER PARAMETERS FOR ROTATION FROM  UPCASE Z
           C        TO LOWCASE Z FOR USE AS INITIAL CONDITION IN RIGEM
           C
000602          RRR= SQRT(XEQ(1)**2+XEQ(2)**2+XEQ(3)**2)
000612          CC= XEQ(1)/RRR
000614          CS= XEQ(2)/RRR
000615          SPH= XEQ(3)/RRR
000617          CPH= SQRT(XEQ(1)**2+XEQ(2)**2)/RRR
000625          SL= XEQ(2)/(RRR*CPH)
```

```
000630          CL= XEQ(1)/(RRR*CPH)
000631          XM1(1,1)=-CL*CPH $ XM1(1,2)=SL $ XM1(1,3)=-CL*SPH
000636          XM1(2,1)=-CPH*SL $ XM1(2,2)=-CL $ XM1(2,3)=-SL*SPH
000641          XM1(3,1)=-SPH     $ XM1(3,2)=0.  $ XM1(3,3)=CPH
000644          DO 20 I=1,3
000651          DO 21 K=1,3
000652       21 XMO(I,K)=0.
000660       20 CONTINUE
000662          DO 22 I=1,3
000663          DO 23 K=1,3
000664          DO 24 L=1,3
000665       24 XMO(I,K)=XMO(I,K)+RM(I,L)*XM1(L,K)
000704       23 CONTINUE
000706       22 CONTINUE
          C
          C  CALCULATION OF EARTHS SELENOGRAPHIC COORDINATES
          C
000710          SS= SQRT (XMO(1,1)**2+XMO(2,1)**2)
000716          SLONG=ATAN2(XMO(2,1),XMO(1,1))
000724          SLAT=ATAN2 (XMO(3,1),SS)
000732          SLONG=SLONG*RTD  $ SLAT=SLAT*RTD
000735          RETURN
000736          END
```

```
                SUBROUTINE LUTH (VJD,VV,OA,OB,OC,XEC,XEQ,TU)
        C       THIS SUBRUUTINE PROVIDES AN APPROXIMATE VERSION OF BROWN+S LUNAR
        C         THEORY (REF. ILE)
        C                  INPUT
        C       THE CALLING PROGRAM SHOULD PROVIDE THE JULIAN DATE (VJD)
        C               VARIABLES
        C       VV1=MEAN LONGITUDE OF MOON,MEASURED IN ECLIPTIC FROM MEAN EQUINOX
        C       OF DATE TO MEAN ASC. NODE OF LUNAR ORBIT THEN ALONG ORBIT (DEGREES)
        C       (MOON)
        C       VV2=SUN+S MEAN LONGITUDE OF PERIGEE (DEG)   (GAMMA)
        C
        C       VV3=MEAN LONGITUDE OF LUNAR PERIGEE,MEAS. IN ECLIPTIC FROM MGAU EQUINOX
        C       OF DATE TO MEAN ASCENDING NODE OF LUNAR ORBIT THEN ALONG ORBIT
        C        (DEG)    (GAMMA PRIME)
        C       VV4=LONG. OF MEAN ASC. NODE OF LUNAR ORBIT ON ECLIPTIC MEAS. FROM
        C       MEAN EQUINOX OF DATE (DEG)   (OMEGA)
        C       VV5=MEAN ELONG. OF MOON FROM SUN XEC(1,2,3)=ECLIPTIC RECTANGULAR
        C       COORDINATES (DEG)    (D)
        C       OA,OB,OC=MOON+S ECLIPTIC LONGITUDE LATITUDE PARALLAX (DEG.,DEG.,DEG.)
        C
        C       XEQ(1,2,3)=EQUATORIAL RECTANGULAR COORDINATES
        C       V6=OBLIQUITY OF THE ECLIPTIC (DEG)  (OBLIQUITY)
        C
000013          DIMENSION XEC(3),XEQ(3),VV(6),LKOK(14),LKOA(14),LKOB(14),
               1 PKOK(11),PKOA(11),PKOB(11)
               2 ,LATK(26),LATA(26),LATB(26),ADA(12),ADB(12),ADK(12)
000013          REAL LKOK,LKOA,LKOB
000013          COMMON/PERT/LKOK,LKOA,LKOB,PKOK,PKOA,PKOB
               1,LATK,LATA,LATB,ADK,ADA,ADB
000013          DMSTR(D,VM,S)= 3.141592653589793  /180.  *(D+(VM+S/60.  )/60.  )
000027          PI=3.141592653589793
000030          TU=(VJD-2415020.  )/36525.
000033          TU2=TU*TU
000034          TU3=TU2*TU
000035          COR1=1336.  *2.  *PI
000040          COR2=11.  *2.  *PI
000042          COR3=5.  *2.  *PI
000043          COR4=1236.  *2.  *PI
000046        1 V1= DMSTR(270.  ,26.  ,3.69  ) + (COR1 + DMSTR(307.  ,52.  ,59.31
               10))*TU - DMSTR(0.  ,0.  ,4.08  )*TU2 + DMSTR(0.  ,0.  ,.0068  )*
               2TU3
000105          V2= DMSTR(281.  ,13.  ,15.  ) + DMSTR(0.  ,0.  , 6189.03  )*TU
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           1+ DMSTR(0.  ,0.  ,1.63  )* TU2 + DMSTR(0.  ,0.  ,.012  )*TU3
000144      V3= DMSTR(334.  ,19.  ,46.75  ) + (COR2+DMSTR(109.  ,2.  ,2.52  ))
           1*TU- DMSTR(0.  ,0.  ,37.17  )*TU2-DMSTR(0.  ,0.  ,.045  )*TU3
000204      V4= DMSTR(259.  ,10.  ,59.79  ) - (COR3+DMSTR(134.  ,8.  ,31.23  )
           1)*TU + DMSTR(0.  ,0.  ,7.48  )*TU2+ DMSTR(0.  ,0.  ,.008  )*TU3
000244      V5= DMSTR(350.  ,44.  ,15.65  ) + (COR4+ DMSTR(307.  ,6.  ,51.18
           1))*TU - DMSTR(0.  ,0.  ,5.17  )*TU2 + DMSTR(0.  ,0.  ,.0068  )*TU3
000305    3 V6= DMSTR(23.  ,27.  ,8.26  ) - DMSTR(0.  ,0.  ,46.845  )*TU-
           1DMSTR(0.  ,0.  ,.0059  )*TU2 + DMSTR(0.  ,0.  ,.00181  )*TU3
000343      V1=V1*180.  /PI
000346      V2=V2*180.  /PI
000347      V3=V3*180.  /PI
000351      V4=V4*180.  /PI
000353      V5=V5*180.  /PI
000354      V6=V6*180.  /PI
000356      R360=360.
000357      V1=AMOD(V1,360.  )
000362      V2=AMOD(V2,360.  )
000365      V3=AMOD(V3,360.  )
000370      V4=AMOD(V4,360.  )
000373      V5=AMOD(V5,360.  )
000376      V6=AMOD(V6,360.  )
000401      IF(V1.LE.0.)  V1=V1+360.
000404      IF(V2.LE.0.)  V2=V2+360.
000407      IF(V3.LE.0.)  V3=V3+360.
000413      IF(V4.LE.0.)  V4=V4+360.
000417      IF(V5.LE.0.)  V5=V5+360.
000423      VV(1)=V1 $VV(2)=V2 $VV(3)=V3 $ VV(4)=V4 $ VV(5)=V5 $VV(6)=V6
000434      V1=V1*PI/180. $ V2=V2*PI/180. $ V3=V3*PI/180.
000441      V4=V4*PI/180. $ V5=V5*PI/180. $ V6=V6*PI/180.
      C
      C   CALCULATION OF ADDITIVE TERMS
      C
000447      TE1=.53733431  -(10104982.E-12)*TU*36525.
           1 +191.E-16*TU*TU*36525.  **2
000456      TE1=TE1*2.  *PI
000460      ATL=14.27  * SIN(TE1)*PI/(3600.  *180.  )
000467      TE2=.71995354  -(147094228.E-12)*TU*36525.
           1 +43.E-16*TU*TU*36525.  **2
000477      TE2=TE2*2.  *PI
000501      AT0=95.96* SIN(TE2)*PI/(3600.  *180. 0)
000510      TE3=.48398132-(147269147.E-12)*TU*36525.
           1 +43.E-16*TU*TU*36525.  **2
```

```
000520          TE3=TE3*2.  *PI
000522          ATO1=15.58  * SIN(TE3)*PI/(3600.  *180.  )
000531          TE4=.71995354-(147094228.E-12)*TU*36525.
               1+(43.E-16)*TU*TU*36525.**2
000541          TE4=TE4*2.*PI
000543          ATL1=7.261*SIN(TE4)*PI/(3600.*180.)
000552          TE5=.52453688-(147162675.E-12)*TU*36525.
               1 +(43.E-16)*TU*TU*36525.**2
000562          TE5=TE5*2.*PI
000564          ATO2=1.86*SIN(TE5)*PI/(3600.*180)
000573          ATL2=0. $ ATO3=0.  $ DGC=0.
000577          DO 13 I=1,17
000604          ADARG=(ADA(I)+ADB(I)*TU*36525.)*2.*PI
000612       13 ATL2=ATL2+ADK(I)*SIN(ADARG)
000626          DO 14 I=8,9
000627          ADARG=(ADA(I)+ADB(I)*TU*36525.)*2.*PI
000635       14 ATO3=ATO3+SIN(ADARG)*ADK(I)
000651          DO 15 I=10,12
000652          ADARG=(ADA(I)+ADB(I)*TU*36525.)*2.*PI
000660       15 DGC=DGC+ADK(I)*COS(ADARG)
000674          V1=V1+ATL+ATL1+ATL2
000677          V4=V4+ATO+ATO1+ATO2+ATO3
          C
          C       CALCULATION OF PERIODIC TERMS
          C
000704          VA=V1-V3
000706          VB=V1-V5
000710          VC=VB-V2
000712          VD=V1-V4
000714          V5S=  SIN(V5)
000716          V5C=  COS(V5)
000721          V52=2*V5
000723          V52S =  SIN(V52)
000725          V52C=  COS(V52)
000727          V54=4*V5
000732          V54S=  SIN(V54)
000734          V54C=  COS(V54)
000736          VAS=  SIN(VA)
000740          VAC=  COS(VA)
000742          VA2=2*VA
000745          VA2S=  SIN(VA2)
000747          VA2C=  COS(VA2)
000751          VA3=3*VA
```

```
000754      VA3S=  SIN(VA3)
000756      VA3C=  COS(VA3)
000760      VBS=  SIN(VB)
000762      VBC=  COS(VB)
000764      VCS=  SIN(VC)
000766      VCC=  COS(VC)
000770      VDS=  SIN(VD)
000772      VDC=  COS(VD)
000774      VD2=2*VD
000777      VD2S=  SIN(VD2)
001001      VD2C=  COS(VD2)
001003      V111=VA-V52
001005      V11S=  SIN(V111)
001007      V11C=  COS(V111)
001011      V121=VA2-V52
001013      V12S=  SIN(V121)
001015      V131=VA+VC-V52
001020      V13S=  SIN(V131)
001022      V13C=  COS(V131)
001024      V141=VA+V52
001026      V14S=  SIN(V141)
001030      V14C= COS(2*VA+V52)
001036      V14CC= COS(V141)
001040      V151=VC-V52
001042      V15S =  SIN(V151)
001044      V15C=  COS(V151)
001046      V161=VA-VC
001050      V16S=  SIN(V161)
001052      V16C=  COS(V161)
001054      V171=VA+VC
001056      V17S=  SIN(V171)
001060      V17C=  COS(V171)
001062      V181=VD2-V52
001064      V18S=  SIN(V181)
001066      V191=VA+VD2
001070      V19S=  SIN(V191)
001072      V211=VA-VD2
001074      V21C=  COS(V211)
001076      V21S =  SIN(V211)
001100      V311=VA-V54
001102      V31S=  SIN(V311)
001104      V31C=  COS(V311)
001106      V411=VA2-V5*4
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```
001112          V41S=  SIN(V411)
001114          V511=VA-VC-V52
001117          V51S=  SIN(V511)
001121          V611=VC+V52
001123          V61S=  SIN(V611)
001125          V711=VA+VD
001127          V71S=  SIN(V711)
001131          V811=VD-VA
001133          V81S=  SIN(V811)
001135          V911=VD-V52
001137          V91S=  SIN(V911)
001141          V221=VD+V52-VA
001144          V22S=  SIN(V221)
001146          V231=VD+VA-V52
001151          V23S=  SIN(V231)
001153          V241=VD+V52
001155          V24S=  SIN(V241)
001157          V251=VA2+VD
001161          V25S=  SIN(V251)
001163          V261=VD-V52-VA
001166          V26S=  SIN(V261)
001170          V271=VD-VA2
001172          V27S=  SIN(V271)
001174          V281=VC+VD-V52
001177          V28S=  SIN(V281)
001201          VV11=-V161
001203          VV1S=  SIN(VV11)
001205          V291=V611-VA
001207          V29S=  SIN(V291)
001211          V441=V151-VA
001213          V44S=  SIN(V441)
001215          V451=-V211
001217          V45S=  SIN(V451)
001221          V54=V5*4
001224          V54S=  SIN(V54)
001226          A11=V5*2+VA2
001232          A11S=  SIN(A11)
001234          A12=VA-VC+V52
001237          A12S=  SIN(A12)
001241          A13=VA-V5
001243          A13S=  SIN(A13)
001245          A14=VC+V5
001247          A14S=  SIN(A14)
```

```
001251        A15=VA3-V52
001253        A15S=  SIN(A15)
001255        V61C=  COS(V611)
001257        V12C=  COS(V121)
001261        V41C=  COS(V411)
001263        A12C=  COS(A12)
001265        V51C=  COS(V511)
001267        AA1=VA2-VC $ AA2=VA2+VC $ AA3=VA3-V52
001275        AA1C= COS(AA1) $ AA2C= COS(AA2) $ AA3C= COS(AA3)
001303        AA4=VC+V5 $ AA5=VA+V5 $ AA6=VD2-V52
001311        AA4C= COS(AA4) $ AA5C= COS(AA5) $ AA6C= COS(AA6)
001317        OA= 22639.5   *VAS -4586.465   *V11S +2369.912   *V52S +769.016  *
             1 VA2S -668.146   *VCS -411.608   *VD2S -211.656   *V12S -205.962
             2 *V13S +191.953  * V14S -165.145   *V15S +147.687   *V16S -125.
             3  154  *V5S -109.673   *V17S -55.173  *V18S -45.099  *V19S+39.528
             4  *V21S -38.428   *V31S +36.124   *VA3S -30.773   *V41S +28.475   *
             5 V51S -24.420  *V61S
             6+13.902  *V54S  + 14.387   * A11S + 14.577   * A12S + 18.609   *
             7 A13S + 18.023   * A14S - 13.193   * A15S
001411        DO 10 I=1,14
001412        ALKAR=(LKOA(I)+LKOB(I)*TU*36525.   )*2.   *PI
001420     10 OA=OA+LKOK(I)* SIN(ALKAR)
001433        OA=(OA/3600.   )*PI/180.
001436        OA=OA+V1
001440        OC= 3422.70   +186.5398  *VAC +34.3117   *V11C +28.2333  *V52C
             1 +10.1657  *VA2C +3.0861  *V14CC+1.9178  *V15C +1.4437  *V13C
             2 +1.1528  *V16C -.9781   *V5C -.9490  *V17C -.7136  * V21C +.6215
             3     *VA3C +.6008  *V31C
             4 + .2607   * V54C - .3   * V61C - .3997   * VCC + .2833   * V14C
             5 - .3039   * V12C + .3722   * V41C + .2302   * A12C - .2257   *
             1 V51C
001512        OC=OC+.1268  *AA1C-.1038  *AA2C-.1187  *AA3C+.1494  *AA4C
             1 -.1093*AA5C-.1052*AA6C
001527        DO 11 I=1,11
001531        PKARG=(PKOA(I)+PKOB(I)*TU*36525.   )*2.   *PI
001537     11 OC=OC+PKOK(I)* COS(PKARG)
001552        OC=OC*(1.   -4.6747E-5)
001554        OC3=(OC*OC*OC)/(6.   *206265.   **2)
001557        OC=OC+OC3
001560        OBB=-112.79  *V5S+2373.36  *V52S+192.72  *V14S+22609.07  *VAS
             1 -4578.13  *V11S -38.64  *V31S +767.96  *VA2S -152.53  *V12S -34.
             2 07*V41S+50.64  *VA3S -25.10  *V61S -126.98  *VCS -165.06  *V15S
             3 -115.18  *V17S -182.36  *V13S -23.59  *V29S -138.76  *VV1S -31.70
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```
                4   *V44S -52.14  *V18S-85.13  *V45S
001630            DO 12 I=1,22
001631            LARG=(LATA(I)+LATB(I)*TU*36525.)*2.*PI
001641         12 OBB=OBB+LATK(I)*SIN(LARG)
001655            OBB= OBB*PI/(180.  *60.  *60.  )
001660            OBBB= -526.069  *V91S +44.297  *V23S +20.599  *V81S -30.598  *V26S
                1 -24.649  *V27S -22.571  *V28S
001675            DO 16 I=24,26
001676            LARG=(LATA(I)+LATB(I)*TU*36525.)*2.*PI
001706         16 OBBB=OBBB+LATK(I)*SIN(LARG)
001722            S=VD+OBB
001724            SS=SIN(S) $ SS3=SIN(3.*S)  $ SS5=SIN(5.*S)
001736            LARG=(LATA(23)+LATB(23)*TU*36525.)*2.*PI
001746            COK=1.+2.708E-6+139.978*DGC
001752            G1=18519.75*COK
001753            G2=6.241*COK**3
001756            G3=.004*COK**5
001760            SSA=G1*LATK(23)*SIN(LARG)
001765            SSB=G2*SSA/G1
001767            SSC=G3*SSA/G1
001771            SSD=SSA/G1
001772            OB=SSA*SS+SSB*SS3+SSC*SS5+OBBB
002005            OB=OB*PI/(180.  *60.  *60.  )
002010            OC=OC*PI/(180.  *60.  *60.  )
002012          2 HP=6378.16  /OC
002013            OAS=  SIN(OA)
002021            OAC=  COS(OA)
002027            OBS=  SIN(OB)
002035            OBC=  COS(OB)
        C
        C           CALCULATION OF  ECLIPTIC AND MEAN EQUINOS OF DATE COORDINATES
        C
002043            HPX=HP*OBC*OAC
002045            HPY=HP*OBC*OAS
002050            HPZ=HP*OBS
002052            XEC(1)=HPX $ XEC(2)=HPY $ XEC(3)=HPZ
002056            V6S=  SIN(V6)
002060            V6C=  COS(V6)
        C
        C           CALCULATION OF MEAN EQUATOR AND EQUINOX OF DATE COORDINATES
        C
002062            RADGX=HPX
002064            RADGY=HPY*V6C-HPZ*V6S
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002070          RADGZ=( HPY)*V6S+HPZ*V6C
002072          XEQ(1)=RADGX $XEQ(2)=RADGY $ XEQ(3)=RADGZ
002100          OA=OA*180.  /PI
002106          OB=OB*180.  /PI
002107          OC=OC*180.  /PI
002110          OA=AMOD(OA,R360)
002113          OB=AMOD(OB,R360)
002116          OC=AMOD(OC,R360)
002121          RETURN
002122          END
```

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```
                 SUBROUTINE EARRO(VJD,R,THETA,S,N,P)
         C
         C       THIS SUBROUTINE UTILIZES NEWCOMS ANALYTICAL EXPRESSIONS TO CALCULATE
         C       THE PRECESSION MATRIX,P,THE NOTATION MATRIX,N,AND THE SIDERAL
         C       ROTATION MATRIX,S. INPUT IS THE JULIAN DATE,VJD. OUTPUT CONSISTS OF
         C       THE DIRECTION COSINES RELATING THE X AND W AXIS SYSTEMS OF THE
         C       REFERENCE IN THE FORM X=(SNP)W
         C
         C       REF: SAO STANDARD EARTH 1966
         C
 000011          REAL N(3,3)
 000011          DIMENSION S(3,3),        P(3,3),R(3,3)
 000011          PI=3.14159265358979
 000012          TWOPI=2.  *PI
 000014          DTR=PI/180.
 000016          RTD=180.  /PI
         C
         C       CALCULATE MODIFIED JULIAN DATE AND S
         C
 000017          XMJD=VJD-2400000.5
 000021          T=XMJD-33282.0
 000023          T2=T*T
 000024          ARG1=(12.1128  -.052954   *T)*DTR
 000030          ARG2=ARG1*2.
 000032          ARG3=(280.0812  +.985647   *T)*DTR*2.
 000036          ARG4=(64.3824  +13.176398  *T)*DTR*2.
 000043          ARG1= AMOD(ARG1,TWOPI)
 000046          ARG2= AMOD(ARG2,TWOPI)
 000051          ARG3= AMOD(ARG3,TWOPI)
 000054          ARG4= AMOD(ARG4,TWOPI)
 000057          THETA=  (100.075542  +360.985647348  *T+.29E-12*T2-4.392E-3*
                ASIN(ARG1)+.053E-3* SIN( ARG2)-.325E-3* SIN( ARG3)-.05E-3*
                BSIN(  ARG4))*DTR
 000124          THETA=AMOD(THETA,TWOPI)
 000127          CTH= COS(THETA)
 000135          STH= SIN(THETA)
 000143          S(1,1)=CTH $ S(2,2)=CTH $ S(1,2)=STH $ S(2,1)=-STH
 000150          S(1,3)=0.  $ S(3,1)=0.  $ S(3,3)=1.  $ S(2,3)=0. $ S(3,2)=0.
         C
         C       CALCULATE NUTATION MATRIX,N
         C
 000156          R1=ARG1
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```
000157          R2=ARG3
000161          R3=ARG4
000162          DELMU= -76.7E-6* SIN(R1) +.9E-6* SIN(2*R1) -5.7E-6* SIN(R2)-
               1 .9E-6* SIN(R3)
000205          DELNU= -33.3E-6* SIN(R1) +.4E-6* SIN(2*R1) -2.5E-6* SIN(R2) -.4E-6
               1 * SIN(R3)
000227          DELEP= 44.7E-6* COS(R1)- .4E-6* COS(2*R1) +2.7E-6* COS(R2)
               1 +.4E-6* COS(R3)
000251          CNU=COS(DELNU) $ SNU=SIN(DELNU)
000255          CMU= COS(-DELMU) $ SMU= SIN(-DELMU)
000265          CEP= COS(-DELEP) $ SEP= SIN(-DELEP)
000275          N(1,1)=CNU*CMU $ N(1,2)=CNU*SMU $ N(1,3) =-SNU
000306          N(2,1)=CMU*SEP*SNU-SMU*CEP
000311          N(2,2)=SMU*SEP*SNU+CEP*CMU
000314          N(2,3)=SEP*CNU
000316          N(3,1)=CMU*SNU*CEP+SEP*SMU
000322          N(3,2)=SMU*SNU*CEP-SEP*CMU
000325          N(3,3)=CEP*CNU
          C
          C     CALCULATE PRECESSION MATRIX,P
          C
000327          XKAP=0.063107  *T*DTR/3600.
000332          OMEG=0.063107  *T*DTR/3600.
000334          XNU=0.0548757  *T*DTR/3600.
000337          P(1,1)=- SIN(XKAP)* SIN(OMEG)+ COS(XKAP)* COS(OMEG)* COS(XNU)
000400          P(1,2)= - COS(XKAP)* SIN(OMEG)- SIN(XKAP)* COS(OMEG)* COS(XNU)
000443          P(1,3)= - COS(OMEG)* SIN(XNU)
000460          P(2,1)= SIN(XKAP)* COS(OMEG)+ COS(XKAP)* SIN(OMEG)* COS(XNU)
000523          P(2,2)= COS(XKAP)* COS(OMEG)- SIN(XKAP)* SIN(OMEG)* COS(XNU)
000565          P(2,3)= - SIN(OMEG)* SIN(XNU)
000602          P(3,1)= COS(XKAP)* SIN(XNU)
000620          P(3,2)= - SIN(XKAP)* SIN(XNU)
000635          P(3,3)= COS(XNU)
          C
          C     CALCULATE FINAL ROTATION MATRIX,R
          C
000644          DO 1 K=1,3
000646          DO 2 J=1,3
000647          R(K,J)=0.
000652          DO 3 I=1,3
000653          DO4 L=1,3
000654          R(K,J)=R(K,J)+S(K,I)*N(I,L)*P(L,J)
000672        4 CONTINUE
```

```
000675        3 CONTINUE
000677        2 CONTINUE
000701        1 CONTINUE
000703          RETURN
000704          END
```

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```
                SUBROUTINE ICOND(VJD,BETA,      THETA,S,N,P)
          C
          C       THIS SUBROUTINE PROVIDES THE TRANSFORMATION FROM BETA
          C           DOUBLE PRIME EULER PARAMETERS TO BETA PRIME PARAMETERS
          C
          C          BETA   ENTERS AS BETA DOUBLE PRIME
          C          BETA RETURNS AS BETA PRIME
          C
 000011         DIMENSION BETA(4),B(4,4),BT(4,4),BETAS(4),OM(4),BETAD(4),BETDP(4)
               1 ,S(3,3),       P(3,3),OI(3)
 000011         REAL N(3,3)
 000011         XMJD=VJD-2400000.5
 000013         T=XMJD-33282.
 000015         PI=3.14159265358979
 000016         TWOPI=2.  *PI
 000020         DTR=PI/180.
 000022         RTD=180.  /PI
          C
          C     BETA MATRIX,B
          C
 000023         ALPHA=(100.075542  +360.985647348  *T)*DTR
 000027         ALPHA=AMOD(ALPHA,TWOPI)
 000032         ALPHA=ALPHA/2.
 000034         CA2= COS(ALPHA)
 000036         SA2= SIN(ALPHA)
 000040         B(1,1)=CA2 $ B(1,2)=0. $  B(1,3)=0. $  B(1,4)=-SA2
 000044         B(2,1)=0.  $ B(2,2)=CA2 $ B(2,3)=-SA2 $ B(2,4)=0.
 000050         B(3,1)=0.  $ B(3,2)=SA2 $ B(3,3)=CA2  $ B(3,4)=0.
 000054         B(4,1)=SA2 $ B(4,2)=0. $ B(4,3)=0. $ B(4,4)=CA2
          C
          C     TRANSPOSE B TO GET BETA INVERSE,BT
          C
 000060         DO 5 I=1,4
 000065       5 BETAS(I)=BETA(I)
 000071         DO 1 I=1,4
 000072         DO 2 J=1,4
 000073       2 BT(I,J)=B(J,I)
 000104       1 CONTINUE
          C
          C     CALCULATE BETA PRIME
```

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```
000110          BETA(I)=0.
000112          DO 4 J=1,4
000113        4 BETA(I)=BETA(I)+BT(I,J)*BETAS(J)
000125        3 CONTINUE
000127          RETURN
000130          END
```

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```
                    SUBROUTINE AXANG(R,DEL,C,BETA)
              C
              C     THIS SUBROUTINE CALCULATES THE AXIS AND ANGLE OF REVOLUTION FOR
              C     AND ROTATION MATRIX,R. IT ALSO PROVIDES THE EULER PARAMETERS
              C     BETA
              C
              C     REF:KORN AND KORN
              C
 000007             DIMENSION R(3,3),C(3),BETA(4)
 000007             PI=3.14159265358979
 000010             TR=R(1,1)+R(2,2)+R(3,3)
 000013             CDEL=(TR-1.   )/2.
 000016             SDEL= SQRT(1.   -CDEL*CDEL)
 000022             DEL= ATAN2(SDEL,CDEL)
              C
              C     DEL IS SUPLLIED IN RANGE ZERO TO PI
              C
 000031             C(1)=(R(3,2)-R(2,3))/(2.   *SDEL)
 000035             C(2)=(R(1,3)-R(3,1))/(2.   *SDEL)
 000041             C(3)=(R(2,1)-R(1,2))/(2.   *SDEL)
 000045             C(1)=-C(1) $ C(2)=-C(2) $ C(3)=-C(3)
              C
              C    SEQUENCE TO KEEP CALCULATED ROTATION AXIS GENERALLY ALIGNED WITH
              C               BODY ROTATION AXIS
              C
 000051             IF (C(3).GE.0.   ) GO TO 1
 000052             C(1)=-C(1) $ C(2)=-C(2) $ C(3)=-C(3)
 000056             DEL=2.   *PI-DEL
 000060           1 DEL=DEL/2.
              C     CALCULATE EULER PARAMETERS
 000062             BETA(1)= COS(DEL)
 000067             BETA(2)=C(1)* SIN(DEL)
 000075             BETA(3)=C(2)* SIN(DEL)
 000103             BETA(4)=C(3)* SIN(DEL)
 000111             RETURN
 000112             END
```