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# A DISCUSSION OF DYNAMIC STABILITY MEASUREMENT TECHNIQUES

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# A DISCUSSION OF DYNAMIC STABILITY MEASUREMENT TECHNIQUES

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## SUMMARY

Techniques for the measurement of the dynamic stability of linear systems are discussed. Particular attention is given to an analysis of the errors in the procedures, and to methods for calculating the system damping from the data. The techniques discussed include: transient decay, moving block analysis, spectral analysis, random decrement signatures, transfer function analysis, and parameter identification methods. The special problems of rotorcraft dynamic stability testing are discussed.

## INTRODUCTION

The flutter testing of an airplane or helicopter requires a method for measuring the dynamic stability of the aeroelastic system. The objective of the test is to establish the stability level throughout the operating range of the aircraft. The capability to accurately determine the stability trends is necessary to safely conduct such a test. Thus a method is required to reliably determine the frequency and damping of all important modes of the system, using measurements of the system response to existing or prescribed inputs. Dynamic stability measurement is a particularly difficult task for rotorcraft, where many degrees of freedom are involved and the process and measurement noise levels can be very high.

A number of techniques have been implemented or suggested for measuring dynamic stability. These techniques may be usefully classified as transient, no-input, input-output, and parameter identification methods. Transient methods use the decaying transient response of the system to determine the dynamic stability. The moving block analysis is a particular procedure for analyzing transient data. The no-input methods use the response due to existing random disturbances of the system (such as aerodynamic turbulence), with averaging of the response to reduce the errors in the estimates. Examples of no-input methods

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are spectral analysis, correlation, and random decrement signatures. The input-output methods use the response due to a measureable external input to determine the system characteristics alone, as well as to reduce the error. Examples of input-output methods are cross-correlation and transfer function analyses. Parameter identification methods use statistical estimation procedures to obtain a best estimate of the system characteristics from the measured response. All of these methods have been used in rotorcraft or airplane flutter testing, with varying degrees of success depending on the aircraft involved and the test environment.

The stability measurement procedure must perform with some level of process and measurement noise present. A primary concern is how such noise reduces the accuracy of the measurements, hence introduces error into the determination of the system damping. This report presents an error analysis of the various stability measurement methods. Some of the results are new (such as the random decrement signature error analysis and the discussion of parameter identification techniques for rotorcraft), while others are available in the standard literature. A unified treatment of the error analysis allows a consistent comparison of the accuracy of the available stability measurement techniques. Thus this report provides a guide for choosing the technique appropriate to a particular experiment.

The basic stability measurement procedure in all cases gives only an estimate of the response, in either the time or the frequency domain. Further processing of the data is required to determine the system parameters from the response, in particular to obtain the damping ratio which gives the quantitative level of stability. Thus for each procedure some alternatives for data processing are discussed. The following stability measurement techniques are studied in detail: transient decay, spectral analysis, random decrement signatures, transfer function analysis, and parameter identification methods. First however, a mathematical definition is given for the aeroelastic system considered.

## LINEAR SYSTEM DYNAMICS

To examine the dynamic stability measurement techniques, a linear system is considered, with excitation by various control inputs and external

disturbances. The response of the system is measured, perhaps with significant measurement noise such as vibration due to rotor imbalance or vibration from the engine. The excitation of the system can be provided by existing unknown disturbances such as aerodynamic turbulence, or by a measureable external input applied specifically to determine the dynamic stability. The system motion is therefore described by linear, time-invariant differential equations, of the form:

$$\dot{x} = Fx + Gu$$

$$y = Hx + v$$

where x is the state vector and y the observation. The vector v is random measurement noise; and u is a random input exciting the system, either an existing disturbance or an external input. The matrices F, G, and H are constant since the system is time-invariant. The solution for the response to excitation by u, with initial conditions at $t_o$, is

$$x(t) = e^{F(t-t_o)} x(t_o) + \int_{t_o}^{t} e^{F(t-\tau)} G u \, d\tau$$

$$y(t) = Hx(t) + v$$

The stability of the system is determined by the eigenvalues of F. The eigenvalues usually occur in complex conjugate pairs, of the form

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

where $\omega_n$ is the natural frequency and $\zeta$ the damping ratio of the mode. The mode is stable if $\zeta > 0$, and is an exponentially decaying oscillation for $0 < \zeta < 1$. It is usually the low damped modes of the system which are of interest, i.e. $0 < \zeta < 0.1$. Note that if $\Lambda$ is the diagonal matrix of the eigenvalues of F, and M the matrix whose columns are the corresponding eigenvectors, then $F = M\Lambda M^{-1}$. For further information on the dynamics of linear systems, the reader is directed to references 1-3.

Regardless of the stability measurement procedure used, a basic requirement is that the system be controllable and observable. This means that the input to the system must sufficiently excite the modes of interest, and the modes must be observable in the response of the variables measured.

## TRANSIENT DECAY

The transient motion of a dynamic system is composed of exponentially decaying oscillations of each mode. A particular measurement is often dominated by one, low-damped mode. From the oscillation period and the decay rate of the trace, the frequency and damping of the mode may be estimated. The typical procedure involves establishing a large amplitude sinusoidal motion by means of an external excitation at the natural frequency of the particular mode of interest. The excitation is stopped, and then the subsequent transient motion is analyzed (often by hand) to determine the damping.

Consider the linear, time-invariant system

$$\dot{x} = Fx + Gu$$

$$y = Hx + v$$

where x is the state and y the measurement; u is process noise such as aerodynamic turbulence, and v is measurement noise. It is assumed that u and v are random disturbances with zero mean. The motion following time $t_o$, where the initial conditions $x(t_o)$ are established by the external excitation, is

$$x(t) = e^{F(t-t_o)}x(t_o) + \int_{t_o}^{t} e^{F(t-\tau)}Gu\,d\tau$$

$$y(t) = Hx(t) + v$$

The first term in x is the transient we wish to observe. The second term is the noise in the trace due to the disturbances occuring after $t_o$. The influence of this noise is the primary concern with this method.

### Statistics

The expected value of the observation is just the transient response:

$$Ey = He^{F(t-t_o)}x(t_o)$$

The external excitation must establish a large, non-zero initial condition for the modes of interest. The variance of the observation is

$$V_y = E(y-Ey)^2 = Ey^2 - (Ey)^2$$
$$= H \int_{t_o}^{t} e^{F(t-\tau)} G Q G^T e^{F^T(t-\tau)} d\tau \, H^T + R$$

where it is assumed that u is white noise with $Eu(\tau_1)u^T(\tau_2) = Q\delta(\tau_1-\tau_2)$, so also $u(t>t_o)$ is uncorrelated with $x(t_o)$. The measurement noise correlation is $Ev(\tau_1)v^T(\tau_2) = R(\tau_1-\tau_2)$. It has been assumed that the initial conditions $x(t_o)$ are deterministic, not random, variables.

Considering a single degree of freedom system, so the integrations are easily performed, the expected value and variance of the observation are:

$$Ey = hx_o e^{\lambda(t-t_o)}$$
$$V_y = h^2 g^2 q \int_{t_o}^{t} e^{2\lambda(t-\tau)} d\tau + r$$
$$= \frac{h^2 g^2 q}{2\lambda} \left(1 - e^{2\lambda(t-t_o)}\right) + r$$

So the normalized variance is

$$\epsilon^2 = \frac{V_y}{(Ey)^2} = \frac{1}{x_o^2}\left[\left(\frac{g^2 q}{-2\lambda} + \frac{r}{h^2}\right) e^{-2\lambda(t-t_o)} - \frac{g^2 q}{-2\lambda}\right]$$

Note that the steady state variance of the state x with only the excitation u is $V_x = g^2 q/(-2\lambda)$, so

$$\epsilon^2 = \frac{\left(V_x + \frac{r}{h^2}\right) e^{-2\lambda(t-t_o)} - V_x}{x_o^2}$$

At $t = t_o$ the only contribution is from the measurement noise, $\epsilon^2 = r/h^2 x_o^2$. As t increases, the noise due to the disturbance u becomes more important. The transient is decaying, while the motion due to u quickly builds up to the level $V_x$. Thus the ratio of the noise to the transient grows exponentially:

$$\epsilon^2 = \frac{V_x}{x_o^2}\left(e^{-2\lambda(t-t_o)} - 1\right)$$

To determine the damping ratio it is necessary to observe the transient while it decays to a fraction f of the initial value (f = .3 to .5 typically). The maximum error which must be dealt with is thus

$$\epsilon_{max}^2 = \frac{V_x}{x_o^2} (f^{-2} - 1)$$

So $\epsilon_{max}$ is of the order of 2 to 3 times the ratio of the rms response due to u and the initial response amplitude. This result holds for the case of more than one degree of freedom also. Note that the steady state variance Y of the response of the system to the disturbance u alone is given by

$$-(FX + XF^T) = GQG^T$$

from which it follows that $V_y \sim X$.

To reduce the error in this procedure, one can only increase the amplitude of the initial conditions $x(t_o)$, or reduce the noise sources u and v. The measurement noise v is frequently small, or at least under some control by the experimenter (e.g. the rotor 1/rev vibration can be reduced by improving the track). The process noise can also sometimes be controlled, e.g. by conducting a flight test under calm atmospheric conditions only. In wind tunnel experiments however, often there is little which can be done to reduce the turbulence significantly. Increasing the initial excitation may not be practical either; the largest value possible is often used to start with. In summary then, the noise parameter $V_x/x_o^2$ often can not be controlled by the experimenter. This parameter is typically large (say $.5^2$), so that the error in the transient trace is a major problem.

Data Processing

The frequency and damping are often determined by hand, from an oscillograph trace of the response. It is assumed that the motion has the form of a decaying oscillation of a single mode:

$$x = x_o e^{-\zeta\omega_n t} \cos(\omega_n\sqrt{1-\zeta^2}\,t + \phi_o)$$

Then the frequency is given by the period of the oscillation, $\omega = 1/T$ (Hz). To determine the damping ratio, consider two peak amplitude measurements $x_1$

and $x_2$, which are n oscillations apart. Then

$$x_2 = x_1 \exp\left( - \frac{\zeta}{\sqrt{1-\zeta^2}} n 2\pi \right)$$

and

$$\zeta = \frac{\frac{\ln x_1/x_2}{2\pi n}}{\sqrt{1 + \left( \frac{\ln x_1/x_2}{2\pi n} \right)^2}} \cong \frac{\ln x_1/x_2}{2\pi n}$$

For example, $\zeta$ equals .11 divided by the number of cycles to one-half amplitude.

In general, the envelope of the oscillation $x = x_o \exp(-\zeta \omega_n t)$ gives $\ln x = \ln x_o - \zeta \omega_n t$, or

$$\zeta = \frac{-\frac{\partial \ln x / \partial \tau}{2\pi \omega}}{\sqrt{1 + \left( \frac{-\partial \ln x / \partial \tau}{2\pi \omega} \right)^2}} \cong -\frac{\partial \ln x / \partial \tau}{2\pi \omega}$$

where $\omega$ is the oscillation frequency in Hz. The damping ratio then may be calculated from the slope of the log-decrement; the previous method is just a special case, where the slope is calculated from the two points $x_1$ and $x_2$. One data processing procedure is to plot ln x vs. t (removing the mean value first if necessary). The slope of the envelope of this plot gives the damping ratio $\zeta$. In general, determining the envelope of the transient is a difficult task to mechanize however.

A different approach is to find a least-squared-error fit to the data for the curve

$$x = x_o e^{-\sigma t} \cos(\omega t + \phi)$$

Identifying the parameters $\sigma$ and $\omega$ is a nonlinear problem however, so an iterative solution is required. There have been some successful applications of this approach. However, convergence problems are very likely with a high noise level.

Moving Block Analysis

The moving block analysis is a particular data reduction technique for transient motion. The Fourier transform of a block of data from $\tau$ to $\tau+T$ gives a function $\bar{y}(\tau)$. The magnitude of the sprectrum line at the natural frequency $\omega_n$ plotted vs. $\tau$ gives then the damping ratio.

The moving block analysis is defined by the linear operator L:

$$L = \frac{1}{2\pi}\int_{\tau}^{\tau+T} (\cdots)\, e^{-i\Omega(t-\tau)}\,\Delta t$$

$$= \frac{1}{2\pi}\int_{0}^{T} (\cdots)\Big|_{t=t+\tau}\, e^{-i\Omega t}\,\Delta t$$

So $\bar{y}(\tau) = Ly(t)$. The expected value of $\bar{y}$ is

$$E\bar{y} = LEy = LH\, e^{F(t-t_0)}\, x_0$$

$$= HM\, L e^{\lambda(t-t_0)}\, q_0$$

$$= HM\left[\frac{1}{2\pi}\int_0^T e^{\lambda(t+\tau-t_0)-i\Omega\tau}\,\Delta t\right] q_0$$

$$= H e^{F(\tau-t_0)} M\left[\frac{e^{(\lambda-i\Omega)T}-1}{2\pi(\lambda-i\Omega)}\right] q_0$$

The dependence on $\tau$ is $e^{F(\tau-\tau_0)}$, so $\bar{y}$ is still a measure of the transient motion. The last factor in $E\bar{y}$ is independent of $\tau$, and contains the only dependence on the spectrum frequency $\Omega$. This factor is a maximum when the frequency $\Omega$ is equal to the natural frequency of $\lambda$. When $\Omega$ is near the natural frequency $\omega_n$ of a mode, the quantity

$$\frac{e^{(\lambda-i\Omega)T}-1}{2\pi(\lambda-i\Omega)} \cong \frac{1-e^{-\zeta\omega_n T}}{2\pi\zeta\omega_n}$$

is large; away from the resonance it is small. Thus the largest line of the spectrum $\bar{y}$ will be at the natural frequency of the mode dominating the response y. Considering only the line at $\Omega = \omega_n$ therefore, and neglecting all modes but the dominant one (with eigenvector $\vec{u}$), we have:

$$E\bar{y}\big|_{\Omega=\omega_n} \cong e^{\lambda(\tau-t_0)} \frac{1-e^{-\zeta\omega_n T}}{2\pi\zeta\omega_n} (H_n^{\vec{}} q_0)$$

The magnitude of this line is:

$$\left| E\bar{y}\big|_{\Omega=\omega_n} \right| \cong \text{constant} * e^{-\zeta\omega_n(\tau-t_0)}$$

an exponentially decaying function of $\tau$ . The moving block analysis thus results in the envelope of the decaying oscillation directly; this is the principal advantage of the technique.

In summary, the moving block analysis involves applying the Fourier transform to the block of data from $\tau$ to $\tau + T$. Usually the FFT algorithm is used, with a sampled time series of data. The magnitude of the line at the natural frequency $\omega_n$ is obtained; if the data is not too noisy, this line will have the largest magnitude in the entire spectrum, and the frequency of the mode may be determined by searching for this peak rather than being input. A plot of $|\bar{y}|$ vs. $\tau$ gives then the envelope of the decaying transient; the slope of $\ln|\bar{y}|$ vs. $\tau$ gives the damping ratio $\zeta$ as discussed above. Since the procedure gives the envelope directly, it is simple to mechanize the calculation of the damping ratio, such as by finding the least-squared-error fit of a straight line to $\ln|\bar{y}|$ vs. $\tau$. This is a significant advantage compared to the processing required to find the damping ratio from the transient motion y.

In choosing the block size T, it can not be too small or the operator reduces to simply $L = 1/2\pi$ ; and if $\zeta\omega_n T$ is small, then the $\Omega = \omega_n$ line of the spectrum does not have the largest magnitude. So there must be a significant decay of the oscillation in the time T. A block length of 3 or 4 periods is probably satisfactory. To increase the frequency resolution, the FFT should not be applied to the block of length T, but to one several times as long, filled with zeros after the data.

Now consider the error involved in the moving block analysis. The linear operator L can have no fundamental influence on the error. The normalized variance of $\bar{y}(\tau)$ must have basically the same value as that of y(t), i.e. error level determined by the parameter $V_x/x_o^2$. Working in the frequency domain helps, since only the noise around the frequency $\omega_n$ is important. This may be expected to reduce the effect of the measurement noise significantly. The operator has little effect on the process noise however. The response of the system to u will be mainly a superposition of oscillations at the natural frequency $\omega_n$, which then is transmitted to $\bar{y}$ along with the transient oscillation.

The variance of $\bar{y}$ is

$$V_{\bar{y}} = H E\left[ L\int_{t_o}^{t} e^{F(t-\sigma_1)} G u \, d\sigma_1 \; L\int_{t_o}^{t} u^T G^T e^{F^T(t-\sigma_2)} d\sigma_2 \right] H^T$$

The evaluation of $V_{\bar{y}}$ is complicated by the operator L. Considering only the dominant mode in the response, the expected value and variance are approximately

$$E\bar{y} \cong H e^{F(\tau-t_o)} \left[ \frac{e^{(\lambda-i\Omega)T}-1}{2\pi(\lambda-i\Omega)} \right] x_o$$

$$V_{\bar{y}} \cong H e^{F(\tau-t_o)} \left[ V_x \left( \frac{e^{(\lambda-i\Omega)T}-1}{2\pi(\lambda-i\Omega)} \right)^2 \right] e^{F^T(\tau-t_o)} H^T$$

The normalized error is

$$\epsilon^2 = \frac{V_{\bar{y}}}{(E\bar{y})^2} \cong \frac{V_x}{x_o^2}$$

which is indeed essentially the same result as for the normalized error of the original transient trace. The minor improvement in the error statistics due to the moving block operator is not the reason for its use. The principal advantage of the procedure is that it extracts the frequency and exponential envelope from the decaying oscillation, which greatly facilitates the mechanization of the data processing task.

## SPECTRAL ANALYSIS

To improve the accuracy of the measurement of the system response, the data must be averaged in some fashion. Spectral analysis is a fundamental technique for measuring the random response of a system. The excitation of the system is provided by existing, random disturbances, such as aerodynamic turbulence. Only the response y of the system is measured. Let $S_y^{(k)}$ be the autospectrum of a particular measured time series of data. Then an estimator of the true spectrum of the response is given by the ensemble average over K calculations of the spectrum:

$$\hat{S}_y(\omega) = \frac{1}{K} \sum_{k=1}^{K} S_y^{(k)}(\omega)$$

The spectrum $\hat{S}_y$ may be obtained by either digital or analog processing.

In general, the process noise exciting the system may consist of multiple inputs (e.g. several gust components), so the true output spectrum is related to the input spectra by the expression

$$S_y = \sum_i |H_i|^2 S_i$$

where $H_i$ is the transfer function of the response y to the i-th input, and $S_i$ is the (unknown) input autospectrum. For the linear system under consideration, $H_i$ is a rational function, and the denominator $D(\omega)$ is the same for all inputs (it is the characteristic equation). Thus

$$S_y = \sum_i \left|\frac{N_i}{D}\right|^2 S_i = \frac{\sum_i |N_i|^2 S_i}{|D|^2} = \frac{N_2(\omega)}{|D(\omega)|^2}$$

We are primarily interested in the poles of the characteristic equation, which give the damping of the system. The output autospectrum $S_y$ is a direct measure of D. There will be a resonant peak of $S_y$ near the natural frequency of a low-damped mode, as long as there is no corresponding zero of $N_2$ near the pole. Such a numerator zero may arise if the mode

is not observable in y, or if the input spectrum is varying greatly in the vicinity of the resonance. Therefore the mode must contribute significantly to the measured response y, as usual; and it is necessary that the input spectrum be reasonably flat near the resonance. The latter restriction is not too severe since the resonances of low-damped modes are very sharp.

Spectral analysis and correlation analysis are entirely equivalent, the former in the frequency domain and the latter in the time domain. Thus the correlation techniques will not be discussed here; their statistics are similar to those of spectral analysis. With a digital processor and hardware FFT units, the spectral analysis is more convenient; it is also better suited for data reduction, such as finding the damping ratio.

Statistics

An analysis of the statistics of the estimator $\hat{S}_v$ of the autospectrum is available in the literature (see references 4 and 5). The expected value and variance of $\hat{S}$ are

$$E\,\hat{S}(\omega) \cong S(\omega) + \frac{\Delta\omega^2}{24} S''(\omega)$$

$$V\,\hat{S}(\omega) \cong \frac{S^2(\omega)}{K}$$

where $\Delta\omega$ is the frequency resolution in the spectrum, and K is the number of averages. These results are essentially independent of the statistical properties of the input and output, hence are applicable to the response to general random disturbances. The normalized bias error is

$$\epsilon_b = \frac{\Delta\omega^2 S''}{24 S}$$

The most critical case is at a resonant peak, where the greatest accuracy is required, and S'' is largest. Assuming that in the vicinity of the peak the spectrum may be approximated by that of a single degree of freedom

system, one obtains $S/S'' = \frac{1}{2}\zeta^2\omega_n^2$. So for a given bias error, the following frequency resolution is required:

$$\Delta\omega = \sqrt{12\epsilon_b}\ \zeta\omega_n$$

Then $\Delta\omega = \frac{1}{2}\zeta\omega_n$ gives $\epsilon_b = .02$, which is essentially negligible bias error (twice that resolution is probably satisfactory). Note that the half-power bandwidth of the peak is $\Delta\omega_2 = 2\zeta\omega_n$, so this resolution corresponds to covering the bandwidth of the peak with 5 spectral lines.

The bias error is easily made negligible by a proper choice of the frequency resolution. The remaining error is the normalized variance:

$$\epsilon = \left[\frac{V\hat{S}}{(E\hat{S})^2}\right]^{\frac{1}{2}} = \frac{1}{\sqrt{K}}$$

The error is inversely proportional to the square-root of the number of averages, the standard result for sample means. The estimate of the spectrum may thus be made as accurate as desired, by increasing the number of averages.

An important parameter of the spectral analysis technique is the total record length required:

$$T = \frac{K}{\Delta\omega} = \frac{2}{\zeta\omega_n\epsilon^2}$$

So the total time over which the data is collected must be increased as the damping, frequency, or error is decreased. This is a fundamental result for the amount of data required to define the system from the output due to random disturbances, applicable to all no-input techniques.

Data Processing

In using the output spectrum to determine the system properties, it is generally assumed that the input spectrum is flat in the vicinity of the resonances. Since the low-damped modes are of most interest, the assumption is only required over the narrow frequency range $\Delta\omega/\omega_n = 2\zeta$.

Assuming that the spectrum in the vicinity of a resonance may be approximated by that of a single, second order mode:

$$S_y \cong \frac{\text{constant}}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

it follows that the damping ratio may be determined from the spectrum peak and an integral through the resonant peak:

$$\zeta = \frac{1}{2\pi\omega_n^3} \frac{\int_{-\infty}^{\infty} S\omega^2 d\omega}{S_p} \cong \frac{.34}{\omega_n^3} \frac{\int_{.8\omega_n}^{1.2\omega_n} S\omega^2 d\omega}{S_p}$$

## RANDOM DECREMENT SIGNATURES

The method of random decrement signatures (randomdec) is a procedure developed by Cole (reference 6) for analyzing the response of a linear system to random disturbances. It is designed to give an estimate of the impulse response of the system. The randomdec estimator $\bar{y}$ is defined as the ensemble average of the transient response to existing random disturbances:

$$\bar{y}(t) = \frac{1}{K} \sum_{k=1}^{K} y_k(t)$$

The key to the procedure is to select each ensemble $y_k$ with the same initial conditions by an appropriate triggering method. Then all records have an identical transient due to the initial conditions, while the subsequent noise averages out.

Consider a linear system excited by the unknown random disturbance u; the response measurement is y, and the measurement noise is v. The system is described by the differential equations:

$$\dot{x} = Fx + Gu$$

$$y = Hx + v$$

The solution using the initial conditions at time $t_o$ is

$$x(t) = e^{F(t-t_o)} x(t_o) + \int_{t_o}^{t} e^{F(t-\tau)} G u \, d\tau$$

$$y(t) = Hx(t) + v$$

The observation y(t) is a random process, depending on the random variables u, v, and $x(t_o)$. The expected value of the randomdec signature is

$$E\bar{y} = \frac{1}{K} \Sigma E y_k$$

$$= \frac{1}{K} \Sigma \left[ H \left\{ e^{F(t-t_o)} E x_k(t_o) + \int_{t_o}^{t} e^{F(t-\tau)} G E u_k \, d\tau \right\} + E v_k \right]$$

$$= H e^{F(t-t_o)} x_o$$

where it has been assumed that u and v have zero mean, and that the initial conditions are stationary random variables with expected values $x_o = Ex(t_o)$. Thus y is an unbiased estimator of the system transient response.

In order for a particular mode to be analyzed, it must be observable in y, and excitable by u. The initial conditions $x(t_o)$ are due to the random disturbance u over the time up to $t_o$, so the frequency content of u must be sufficient to excite the modes of interest. This is a fundamental requirement of methods using existing random disturbances of the system, arising here through the initial conditions at time $t_o$. Furthermore, in order for a particular mode to be present in $\bar{y}$, it is necessary that the initial condition for that mode have a non-zero average value over all the ensembles, i.e. $Ex(t_o) \neq 0$. The response x(t) is a random process with mean

$$Ex = F^{-1}G(Eu) = 0$$

So it is not possible to arbitrarily select the ensembles. The result would be $x_o=0$ and hence $E\bar{y} = 0$. It is necessary to trigger on the response in such a manner that $Ex(t_o) \neq 0$ for the modes of interest. Note that triggering on y=0 is not possible either, since it gives zero initial conditions (it would be possible to trigger on y=0, with positive slope only, and hence obtain a non-zero average value of the initial velocity; the variance of the initial velocity would be large however, leading to a large error in $\bar{y}$ as discussed below). The usual procedure is to trigger the start of the ensemble on a fixed, non-zero level of the response y, with either positive or negative slope.

As usual, the ergodic hypothesis is used, and the random decrement signature is obtained by averaging over records which are sequential in time:

$$\bar{y}(t) = \frac{1}{K} \sum_{k=1}^{K} y(t + t_o^{(k)})$$

It is assumed that the records are uncorrelated, which is generally satisfied if there is no overlap (ref. 6). With digital sampling and

processing of the data, the requirement that each record be triggered on the basis of the response means that the records must be non-overlapping. Using records with some overlap would reduce the total sample time some (a factor of 2 is about the maximum possible reduction), but increases the correlation of the records.

Statistics

The variance of the random decrement signature $\bar{y}$ is

$$V_{\bar{y}} = E\bar{y}^2 - (E\bar{y})^2$$

$$= E\left[\frac{1}{K}\sum\left\{H\left(e^{F(t-t_o)}x_k(t_o) + \int_{t_o}^{t} e^{F(t-\tau)}Gu_k\,d\tau\right) + v_k\right\}\right]^2$$

$$- \left(He^{F(t-t_o)}x_o\right)^2$$

The ensembles are uncorrelated, except for the initial conditions. The initial condition $x(t_o)$ is a random variable with mean $Ex(t_o) = x_o$ and variance $V_{x_o} = Ex(t_o)x^T(t_o) - x_o^2$, determined by triggering on the response y in some fashion. Note that the response of the system to the random disturbance u has zero mean and variance $V_x = Ex(t)x^T(t)$, which are not the same as $x_o$ and $V_{x_o}$ due to the action of the procedure for selecting the initial time $t_o$.

Now assume that u and v are uncorrelated with each other. Let $Ev(\tau_1)v^T(\tau_2) = R(\tau_1-\tau_2)$; and assume white noise for u, so $Eu(\tau_1)u^T(\tau_2) = Q\delta(\tau_1-\tau_2)$. Then also $u(t > t_o)$ is uncorrelated with $x(t_o)$. Then the variance of $\bar{y}$ is

$$V_{\bar{y}} = He^{F(t-t_o)}V_{x_o}e^{F^T(t-t_o)}H^T$$

$$+ \frac{1}{K}\left[H\int_{t_o}^{t} e^{F(t-\tau)}GQG^Te^{F^T(t-\tau)}\,d\tau\,H^T + R\right]$$

The term in brackets is the same error as was found for the transient decay trace. It is the response of the system to the random disturbance u subsequent to the time $t_o$ where the initial conditions are determined.

This term is zero at $t = t_o$ (except for the measurement noise contribution), and increases with time until it reaches the steady state level given by $V_x$. The relative size of this term increases exponentially as the transient signal decays. The total normalized error due to this source is approximately

$$\epsilon^2 = \frac{V_{\bar{y}}}{(E\bar{y})^2} \cong \frac{8}{K} \frac{V_x}{x_o^2}$$

So the ensemble averaging reduces this error, inversely proportional to the square-root of the number of averages. This error source can be made as small as required by increasing the number of averages.

The remaining term in $V_{\bar{y}}$ depends upon the variance of the initial conditions, $V_{x_o}$. Both this term and $(E\bar{y})^2$ decrease with time at the same rate, so the normalized error is always about the same value:

$$\epsilon^2 = \frac{V_{\bar{y}}}{(E\bar{y})^2} \cong \frac{V_{x_o}}{x_o^2}$$

Thus the normalized error of y is equal to the normalized error in the initial conditions $x(t_o)$. The averaging process does not reduce this error term, so the accuracy of the signature depends upon keeping the variance of $x(t_o)$ as low as possible.

The variance $V_{x_o}$ depends on the procedure for triggering the start of a sample record. Zero error $V_{x_o} = 0$ requires that the same initial conditions $x_o$ be produced exactly by every triggering event. An important observation concerns the influence of the matrix H in $V_{\bar{y}}$. Since (at $t = t_o$) $V_{\bar{y}} = HV_{x_o}H^T$, it follows that the variance in the initial conditions of the unobserved states do not contribute to $V_{\bar{y}}$. The triggering process, based on the measurement y, can not be expected to produce a specified initial condition for an unobservable state. For such states the initial condition will have a zero mean, and a variance equal to the value in $V_x$. This large variance does not contribute to $V_{\bar{y}}$ however, so it is of no concern in the randomdec procedure. An important example is the velocity of the degrees of freedom of an aeroelastic system. The state variables consist of the displacement and velocity of the degrees of freedom, but generally only the displacements are measured. The velocities are unobservable states

then; the velocity initial conditions will have large variance when the triggering process uses the measured response y, but only the errors in the displacement initial conditions contribute to $V_{\bar{y}}$.

Consider a single degree-of-freedom, second order system, with states $\dot{x}$ and x. The response x is measured, and the sample record is started by triggering on the level of x. Then $V_{\bar{y}}$ is just the variance of the initial condition on x alone. An efficient triggering process should give a small $V_{x_o}$ for the displacement. Therefore this example should have low error in the random decrement signature $\bar{y}$ due to the initial condition errors.

Obviously the triggering procedure is a central element in the calculation of the random decrement signature. A procedure is required to establish the initial conditions with the following properties. The expected value $Ex(t_o) = x_o$ must be large enough for the important modes that they have significant excitation in the measured transient. The variance of the initial conditions $V_{x_o}/x_o^2$ must be small for the observable states; this is essential for low error in the signature. Finally, either $V_x/x_o^2$ must be small, or a large number of averages must be made so that the error due to the random disturbances after $t_o$ is small. Of course $V_x/x_o^2$ can not be very small, since the same random input u produces both $V_x$ and $x(t)$, of which $x(t_o)$ is a member. In summary then, $x_o$ should be large both to reduce the errors and to increase the magnitude of y. If too large a value of $x_o$ is chosen for the triggering process however, the correlation of the records increases. Cole (ref. 6) suggests, on the basis of an analysis of the statistics of the method, that the triggering use a level $y = 1.0$ to $1.6\sigma_y$ (where $\sigma_y$ is the rms value of the response).

Minimizing the error due to the initial conditions variance requires that the triggering process select $t_o$ on the basis of the measurement y such that the response x has the prescribed value. There are two basic reasons why this may be difficult to accomplish, both of which must always be present to some extent: the existence of measurement noise, and systems with more than one mode observable in the measurement. Measurement noise

present in y will give a lower bound on the variance $V_{x_o}$. Unlike the R/K term, this effect of the measurement noise does not average out. Depending on the system, this may be a serious noise source or a negligible one. Note that errors in the triggering process have an effect similar to the measurement noise.

A usual case involves a single measurement of the response of a multi-mode system. If more than one mode is observable in y, then obviously it is not possible to uniquely determine the values of the observable states and thus trigger the record solely based on the single value y(t). In general the initial conditions can not be determined if the dimension of y is less that the dimension of the observable states in $x_o$. In practice, only those states which contribute significantly to y are of concern, and frequently there is only one such state contributing to the measurement. An example of where more than one mode is observable is the measurement of the tip motion of a cantilever beam (e.g. a wing or rotor blade). In such a case the frequency range may be restricted, or only a single mode excited, so that there is no practical difficulty; or in fact it may be necessary to determine the initial conditions of all the modes based on the one measurement. It is possible to obtain an estimate of the state x(t) from limited noisy measurements y, if use is made of the past history of y. This is a classical filtering problem; the solution in this case is the Kalman filter, which depends on the characteristics of the system being investigated. It is possible to identify the system as the measurements proceed, and thus update the filter model. This becomes a rather involved procedure simply to trigger the start of the record.

In summary, it is often possible to trigger on y with low variance of the initial conditions of all observable states, hence with low error in the randomdec signature $\overline{y}$. With a high order system, or with measurement noise the triggering task becomes more difficult, and the error in $\overline{y}$ increases. In the extreme case, the triggering procedure becomes the problem of identifying the state x(t) from noisy measurements of an unknown linear process -- essentially a definition of the entire dynamic stability measurement task.

It should be noted that with multiple states observable in y, all are required to take the prescribed initial conditions at the triggering instant. The time interval between such occurrences will increase as the number of states increases, hence the total testing time (not the sampling time) will increase. Alternatively, for a given testing time the error must be allowed to increase, either by relaxing the trigger criterion and so increasing $V_{x_o}$, or by taking fewer averages.

The above result for $V\bar{y}$ was based on white noise for the random disturbance u. The result will be similar for a general spectrum, $Eu(\tau_1)u^T(\tau_2) = Q(\tau_1 - \tau_2)$, which indeed is more realistic than white noise. The basic influence of averaging in reducing the noise is unchanged, although the number of averages required for a specific value of error will be influenced somewhat by the spectrum of u. If the input spectrum is low at the frequency of the mode to be investigated, it will be necessary to increase the number of averages to compensate for the low signal to noise ratio there. An important consideration for the random decrement procedure is the effect of bandwidth-limited input disturbances on the initial conditions. The random disturbance u is the source of x(t), from which the initial conditions must be chosen. Hence the frequency content of the input has a direct influence on $Ex(t_o)$ and $V_{x_o}$.

Finally, consider the sample time required. Cole gives an estimate of T based on the statistics of $\bar{y}$. For the problem of measuring the damping ratio, the result is

$$T = \frac{2}{5\omega_n \epsilon^2}$$

where $\epsilon$ is the normalized error of the damping estimate. This is the same result as for spectral analysis (although the interpretations of $\epsilon$ are somewhat different in the two cases). The amount of data required to extract information about the system with a given accuracy is a fundamental characteristic, independent of the procedure used to extract the information.

As with spectral analysis, for a given T it is necessary to trade off the number of averages and the frequency resolution. The frequency resolution of the randomdec signature is determined by how many periods of the oscillation are measured. If there are n cycles with frequency $\omega_n$ in the signature, then the length of each record is $t = n/\omega_n$ . The number of averages is then

$$K = \frac{T}{t} = \frac{2}{\zeta n \epsilon^2}$$

With $\zeta$ = .01 to .02, $\epsilon$ = .1 to .2, and n = 4, the number of averages required is K = 1000 to 5000. Cole suggests the use of n = 4, and a sample rate of $r = 16\omega_n$ (so the signature consists of 64 samples). Typical applications of the random decrement procedure require K = 4000 to 8000 averages for satisfactory accuracy (references 6 and 7).

Data Processing

The random decrement signature $\bar{y}$ is an estimate of the system transient response. Thus the procedures discussed above for determining the damping ratio from the decaying transient trace are applicable, with improved accuracy due to the averaging. Determining the damping and frequency by means of a least-squared-error fit of a decaying oscillation to the data is more successful than with the unaveraged trace, due to the reduction in the error of the estimate of the transient. Chang (ref. 7) developed such a data processing technique for the randomdec method.

In conclusion, the spectral analysis, correlation, and random decrement techniques are all members of the general class of methods for analyzing the properties of linear dynamic systems utilizing existing (unknown) random disturbances, with averaging to reduce the error. As such, all of these techniques have basically the same statistical characteristics. The choice of method depends on the experiment, the data processing equipment, and as often on the preferences of the experimenter. Correlation is the classical technique. With current digital processors and FFT hardware, it is more useful and convenient to work in the frequency domain with spectral analysis. The random decrement

statistics are somewhat special in their dependence on the variance of the initial conditions, and hence on the triggering process. The sensitivity of $V_{x_o}$ to measurement noise, and the difficulties of the multi-mode case are of greatest concern. The spectrum of the output depends directly on the frequency content of the input, while the randomdec signature depends on the input spectrum indirectly, particularly through $x_o$ and $V_{x_o}$. All procedures are essentially independent of the level of the input, as expected for a linear system. The normalization is implicit in the randomdec triggering, and $S_y$ may be normalized by dividing by $\sigma_y^2 = \int S_y \, d\omega$. All of the procedures require further processing to extract the damping ratio and other parameters. In this regard the spectral analysis is more convenient than correlation or randomdec techniques, which give the response in the time domain.

## TRANSFER FUNCTION ANALYSIS

Stability analysis techniques which rely on existing, unknown random disturbances to excite the system have the basic limitation that the measured response depends on the properties of both the system and the unknown input. By measuring the input as well as the response, it is possible to determine the characteristics of the system alone, as well as reduce the error in the analysis. Usually it is necessary to apply a measurable external input specifically to excite the system. The cross-spectra between the input and output, in addition to the autospectra are found. The ratio gives the transfer function $H(\omega)$ between the input w and the response y, from which the system parameters may be determined.

Consider now a linear system with input and output measurements:

$$\dot{x} = Fx + Gu + Ew$$

$$y = x + v_o$$

$$z = w + v_i$$

The actual system response and input are x and w; y and z are the measured response and input, including random measurement noise. The vector u is the process noise, an unknown random disturbance of the system. The transfer function between x and w is

$$H(\omega) = \frac{x}{w} = (i\omega - F)^{-1}E$$

Let $S_{zy}^{(k)}$ be the cross-spectrum between z and y based on the k-th record of data, and $S_z^{(k)}$ and $S_y^{(k)}$ the corresponding autospecta. Then an estimator of the transfer function $H(\omega)$ is given by the ratio of the averaged cross-spectrum and input autospectrum:

$$\hat{H}(\omega) = \frac{\hat{S}_{zy}}{\hat{S}_z} = \frac{\frac{1}{K}\sum_{k=1}^{K} S_{zy}^{(k)}}{\frac{1}{K}\sum_{k=1}^{K} S_z^{(k)}}$$

Statistics

The analysis of the statistics of the estimator $\hat{H}$ of the system frequency response is available in the literature (see references 4 and 5). For the spectral analysis technique the variance of the spectra estimators $\hat{S}_{zy}$ and $\hat{S}_z$ were discussed. The statistics of their ratio $\hat{H}$ are different however, reflecting the fact that not just the input or output is being measured, but the correlation between the two. Bias errors in $\hat{H}$ are usually much less than the random errors (as long as the frequency resolution is chosen properly), so they are not discussed. The analysis of the variance of $\hat{H}$ gives the following result.

The probability is $\alpha$ that the magnitude of the difference between the estimate $\hat{H}(\omega)$ and the true value $H(\omega)$ is greater than the function $r(\omega)$. Thus with confidence $(1-\alpha)$ H lies within a circle of radius r about $\hat{H}$ on the plane of ImH vs. ReH. So

$$|\hat{H}-H|^2 < r^2$$

where

$$r^2 = \frac{1}{K-1} F_{2,2(K-1);\alpha} \left(1-\hat{\gamma}_{zy}^2\right) \frac{\hat{S}_y}{\hat{S}_z}$$

Here K is the number of averages; $\hat{\gamma}_{zy}$ is the estimate of the coherence function between the input and output

$$\hat{\gamma}_{zy} = \frac{|\hat{S}_{zy}|^2}{\hat{S}_z \hat{S}_y}$$

and F is the value of the F distribution with 2 and 2(K-1) degrees of freedom at the probability level $(1-\alpha)$ (i.e. Prob$(F > F_\alpha) = \alpha$ ). Note that ideally $|H|^2 = S_y/S_z$; and for large F find

$$F_{2,2(K-1);\alpha} \cong -\ln\alpha$$

Hence the normalized error of the estimate of the transfer function is

$$\epsilon^2 = \frac{-\ln\alpha}{K-1} \left(1-\hat{\gamma}_{zy}^2\right)$$

For a confidence level of 95% ( $\alpha = .05$), find $F \cong 3.1$; at the 97.5% and 99% levels, $F \cong 3.8$ and 4.8 respectively. Assuming a Gaussian distribution, the variance (one standard deviation) corresponds to a confidence level of 68%, which gives $F = 1.16$. Hence except for the effect of $\hat{\gamma}_{zy}$, the normalized error is $\epsilon^2 \cong 1/K$, which is the same result as for the variance of the spectrum alone.

The coherence function $\hat{\gamma}_{zy}$ is a measure of how well z and y are correlated. With no noise in the system, $\hat{\gamma}_{zy}$ would be exactly equal to 1. So $(1 - \hat{\gamma}_{zy}^2)$ is a measure of the effect of noise on the calculation of the response due to a particular input. For the system described above, assume u, $v_i$, and $v_o$ are uncorrelated random noise; let H be the frequency response of x due to w, and $H_u$ the frequency response of x due to the disturbance u. Then

$$1 - \hat{\gamma}_{zy}^2 = \frac{\hat{S}_z \hat{S}_y - |\hat{S}_{zy}|^2}{\hat{S}_z \hat{S}_y}$$

$$= \frac{S_w (H_u^2 S_u + S_{vo}) + S_{vi} S_y}{(S_w + S_{vi}) S_y}$$

$$\cong \frac{H_u^2 S_u}{S_y} + \frac{S_{vo}}{S_y} + \frac{S_{vi}}{S_z}$$

The first term is the ratio of the system response due to the disturbance u, to the total response; it is always less than 1 since the latter is the sum of the response to u and w. This term may be reduced by increasing the level of the measured input w relative to the unknown input u. The last two terms are the ratio of the measurement noise to signal for the output and input.

It is concluded that for a given number of averages (hence fixed sample time), the error in the transfer function is substantially less than the error in the individual spectra. The error may be directly

reduced by increasing the level of the measured excitation w. The improved error statistics compared to spectral analysis or equivalent techniques is a particularly important factor for full-scale testing, where the frequencies are low and thus the sample times required tend to be long (recall that $T = 2/\zeta\omega_n\epsilon^2$). The other principal advantage of the transfer function analysis is that it produces more information about the system than does the spectral analysis of the response alone. The result is independent of the input spectrum (always assuming that the input sufficiently excites the relevant modes).

Data Processing

Processing of the frequency response data to obtain the damping is similar in some respects to the processing for spectral analysis, although of course the assumption that the input spectrum is flat in the vicinity of the resonance is not required. A typical procedure involves approximating H near the resonance of a low-damped mode by the frequency response of a single, second-order degree of freedom:

$$H = \frac{V_m}{\omega_n^2 - \omega^2 + i\,2\zeta\omega_n\omega}$$

It follows that the damping ratio may be calculated from integrals of the measured transfer function about the resonance:

$$\zeta = \frac{1}{\pi\omega_n}\frac{\left(\int_0^\infty \mathrm{Im}\,H\;\omega\,d\omega\right)^2}{\int_0^\infty |H|^2\omega^2\,d\omega} \cong \frac{.34}{\omega_n}\frac{\left(\int_{.8\omega_n}^{1.2\omega_n} \mathrm{Im}\,H\;\omega\,d\omega\right)^2}{\int_{.8\omega_n}^{1.2\omega_n} |H|^2\omega^2\,d\omega}$$

The greatest effect of other modes on the resonance is usually a phase shift. The assumed single degree-of-freedom response has a phase $\angle H = -90^\circ$ at the natural frequency, but the phase of the measured H can be shifted significantly by nearby modes. Thus the measured frequency response phase should be shifted as follows before applying the above expression to calculate $\zeta$:

$$H_c = H * \left(-i\,\frac{\bar{H}_p}{|H_p|}\right)$$

where $H_p$ is the measured response at the resonant peak.

An alternative approach is to use the methods of Kennedy and Pancu (reference 8). On the ImH vs. ReH plane, the transfer function describes a circle in the vicinity of each resonance. Near the natural frequency, the ideal response of the single degree-of-freedom system is approximately

$$H \cong -\frac{i}{4m\zeta\omega_n^2}\left[1 + e^{-i\frac{2}{\zeta}\left(\frac{\omega}{\omega_n}-1\right)}\right]$$

The radius of this circle thus gives the damping of the mode. The procedure may be implemented as follows. First the data should be shifted to phase $-90^\circ$ at the peak frequency, as indicated above. Then the circle

$$|H|^2 + 2a\,\mathrm{Im}H = b$$

on the H plane is fitted to the measured transfer function, using the least-squared-error method to calculate a and b. Only a few points around the peak should be used, say 3 to 5 on either side; and a weighting function should emphasize the errors for the points nearest the peak. Then the damping coefficient is

$$c_s = 2m\zeta\omega_n = \frac{1}{2a\sqrt{b+a^2}}$$

To find the damping ratio however, it is necessary to further consider the variation of the phase with $\omega$ at the resonance. A least-squared-error fit of the line

$$\angle(iH - a) = -c\left(\frac{\omega}{\omega_n} - 1\right)$$

to the data gives the slope c, and hence the damping ratio $\zeta = 2/c$. A simpler procedure results if the inverse of the transfer function is considered. The ideal, single degree-of-freedom response $H^{-1}$ is exactly a parabola on the complex plane:

$$(\mathrm{Re}\,H^{-1}) + \frac{m}{c_s^2}(\mathrm{Im}\,H^{-1})^2 = m\omega_n^2$$

Thus the curve $(\mathrm{Re}H^{-1}) + a(\mathrm{Im}H^{-1})^2 = b$, may be fitted to the data, using the least-squared-error approach. Then the damping ratio is given by $\zeta = (4ab)^{-\frac{1}{2}}$.

Another approach is to fit the data to a high order rational function, e.g. by least-squared-error methods. The roots of the denominator polynomial then give the stability of all modes of the system observable in the response. The assumed representation of H depends nonlinearly on the parameters however, so an iterative solution is required. Convergence problems are possible with such a procedure, especially if there is significant noise in the measured transfer function.

## PARAMETER IDENTIFICATION

Parameter identification techniques for dynamic stability measurement require further development. It is possible to define the problem for rotorcraft testing however. An algorithm is required to estimate the frequency and damping of all important modes of the system from measurements of the various input and output signals. For rotorcraft there are usually many degrees-of-freedom in the system, and a high noise level in the measurements. It is necessary to identify the structure and statistics of the noise as well as the system for good accuracy. The assumptions about the system must be minimal. It is seldom possible to make assumptions about the model without compromising the effectiveness in identifying unforeseen stability characteristics. In particular the order of the system must be identified, i.e. the number of significant modes involved in the response. The algorithm must give an accurate estimate of the damping of the system, with assured convergence and computational efficiency. It must operate on-line, providing the estimates in a reasonable time after the collection of the data for all input combinations, with the calculations performed by a limited capacity computer.

It is assumed that the system is described by linear, time-invariant differential equations, excited by various control inputs and aerodynamic turbulence. The turbulence is defined as the response of a linear process excited by white noise. The system response and input are measured. The measurements can have three kinds of noise: bias error (balance or zero shift), random error (not necessarily white noise), and discrete frequency error (correlated with rotor azimuth, e.g. 1/rev or N/rev vibration). The input signals may often be considered noise-free. Thus the following description of the system may be used. Modal coordinates are used since the system is then defined in minimal form, and directly in terms of the eigenvalues $\lambda$ (the eigenvalues are probably complex however, so actually a real form of the following description is required). The model is:

| | |
|---|---|
| system: | $\dot{q} = \Lambda q + Gu + Ew$ |
| output measurement: | $y = Hq + v_{o_B} + v_{o_R} + v_{o_D}$ |
| input measurement: | $z = w + v_{i_B} + v_{i_R} + v_{i_D}$ |
| turbulence model: | $\dot{u} = Du + v_G$ |

Here y and z are measurements of the response and the input, with noise v (bias, random, and discrete components). The aerodynamic turbulence is u, excited by the white noise $v_G$.

The data consists of the measurements of the response and the input, probably in the form of sampled time series. Sequential measurements are made for the various input combinations (including no input). Then the parameter identification algorithm is applied. It is necessary to identify the system ($\Lambda$, G, E, H, order), the turbulence model (D, order, $v_G$), and the measurement noise model ($v_D$, $v_B$, $v_R$). The eigenvalues $\Lambda$ are of primary concern of course. Clearly the magnitude of the parameter identification task is very great.

There is a serious difficulty with measurements in the rotating frame. With independent blades (i.e. shaft-fixed rotor dynamics) the measurements and the equations defining the system can be in the rotating frame. With coupled motion of the blades and the rotor support, the states and equations must be defined in the nonrotating however, including those for the rotor (otherwise the system is described by periodic coefficient equations, needlessly in this case). For example, the rotor flapping motion should be described by the coning and tip-path-plane tilt motions of the rotor rather than by the flapping of the individual blades. If only measurements in the nonrotating frame are made, there is no difficulty, except that due to limited information. If measurements are available for the motion of all the blades of the rotor, then they can be transformed by analog or digital processing to the nonro ating degrees-of-freedom. But very seldom are rotors instrumented for the stability measurement requirement, so almost always just one blade will be instrumented. It would be very valuable to have an algorithm capable of using the single blade measurements in determining the dynamic stability of the system.

The methods which have been discussed so far (spectral analysis, and transfer function techniques) can not use single blade measurements, except for the case of independent blade dynamics. These techniques require measurements representing the response of a time-invariant system, so they must have measurements from all the blades or they are not able to use the additional information. Parameter identification techniques however have the potential for using the limited information in the single blade measurements. It would not be as accurate as having measurements of the motion of all the blades, but it would be better than ignoring the rotor motion entirely.

Consider the revisions of the measurement representation required to account for having only rotating measurements for a single blade. Assume for now that there is only one measurement y of the blade motion. Let $y_{NR}$ be the corresponding N measurements of the states in the nonrotating frame (N is the number of rotor blades). So $y_{NR}$ is related to the modal coordinates by $y_{NR} = Hq$ as usual. Now y is related to the nonrotating states by a linear (time-varying) transformation $y = h^T y_{NR}$; i.e. the response y is due to the response of all the $y_{NR}$ components. For example, the flap motion of the m-th blade is given by

$$\beta^{(m)}(t) = \beta_0(t) + \sum_{n=1}^{\left[\frac{N-1}{2}\right]} \left(\beta_{nc}(t)\cos n\Psi_m + \beta_{ns}(t)\sin n\Psi_m\right) + \beta_{\frac{N}{2}}(t)(-1)^m$$

where $\Psi_m = \Omega t + 2\pi/N$, and $\beta_{N/2}$ is present only if N is even. Thus h is the 1-by-N vector:

$$h = \begin{bmatrix} 1 \\ \cos n\Psi_m \\ \sin n\Psi_m \\ (-1)^m \end{bmatrix}$$

It follows that y is related to the modal response q by the expression

$$y = h^T Hq + v_o$$

This result is easily extended to the general case. Let the first $n_r$

elements of y be the measurements in the fixed frame, and the last $n_R$ elements of y be the measurements on the rotating blade. Then

$$y = THq + v_o$$

where

$$T = \begin{bmatrix} H & & & 0 \\ & h^T & & \\ & & h^T & \\ 0 & & & \ddots \\ & & & & h^T \end{bmatrix}$$

The matrix h and therefore T are known. The only effect of using the rotating blade measurements is an increase in the dimension of the matrix H which must be identified. Note that H is just the matrix which must be found if all N of the nonrotating rotor states are measured, rather than the rotating degrees-of-freedom of the single blade. The matrix T (i.e. $h^T$) accounts for the reduction in measurements (and information) when only the single blade instrumentation is used.

For rotorcraft dynamic stability measurements, it is also necessary to consider the periodic coefficient case. There are two usual sources of periodic coefficients: the aerodynamics of the rotor in forward flight, i.e. $\mu > 0$ ( $\mu > .5$ in practice); or a two-bladed rotor on an anisotroptic support. In the periodic coefficient case the system is described by the same model as above. The eigenvalues are given by $\lambda$ as usual. Now however the matrices H, G, and E are periodic functions of time (the period is $2\pi/N\Omega$ if in the nonrotating frame, and $2\pi/\Omega$ in the rotating frame, where $\Omega$ is the rotor speed). The parameter identification problem is increased therefore, since these matrices must be found over the entire period now. Undoubtedly the required sample time will be increased. It would probably be desirable to synchronize the sampling of the data with the rotor speed (with the trim rotor speed, since there are cases where the dynamic response involves significant rotor speed perturbations).

## CONCLUDING REMARKS

A number of methods for measuring the dynamic stability of linear systems have been discussed. Particular attention has been given to an analysis of the errors involved. It was found that a significant reduction in the error is achieved when averaging is introduced, and again when measureable external inputs are used. The results are a guide to the design of experiments and the choice of the measurement technique. No universal conclusion about which technique should be used is possible. The choice depends on the experiment, what noise is present, what error is acceptable, and on what data processing equipment is available. For rotorcraft testing the dynamic stability measurement is a critical factor however, so generally the most accurate method possible is desired.

The state-of-the-art of dynamic stability testing in the aircraft industry has been described in a number of surveys (see for example references 9-13). The problem involved is usually the flutter testing of the aircraft wing or tail in flight. Some form of frequency-response analysis procedure is common. There are many variations, but a typical procedure involves swept-sine excitation, digital analysis of the response (analog analysis is also still common), with the damping determined from the circle on the H plane (see the above discussion, and ref. 8).

The techniques discussed may be implemented in many ways. Among the significant options are: the response may be in the time or frequency domain; the input may be swept-sine or random excitation; analog or digital processing may be used; and the data reduction may be accomplished manually, with a special purpose processor, a mini-computer, or with a large capacity computer. The discussions here have dealt with those characteristics which are fundamental to the problem of dynamic stability measurement, and hence are independent of the method of implementing the procedures.

Finally, this report has been limited to an examination of the basic characteristics of the methods for measuring dynamic stability. Further development of techniques for rotorcraft testing requires computer simulations, and additional applications to stability measurements in rotor experiments.

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