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A STRICTLY MARKOVIAN EXPANSION
FOR PLASMA TURBULENCE THEORY

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Abstract

The collision operator that appears in the equation of motion for
a particle distribution function that has been averaged over an ensemble
of random Hamiltonians is non-Markovian. It is non-Markovian in that it
involves a propagated integral over the past history of the ensemble averaged
distribution function. All formal expansions of this non-linear collision
operator to date preserve this non-Markovian character term by term yielding
an integro-differential equation that must be converted to a diffusion
equation by an additional approximation. In this note we derive an expansion
of the collision operator that is strictly Markovian to any finite order
and yields a diffusion equation as the lowest nontrivial order. The validity
of this expansion is seen to be the same as that of the standard quasi-
linear expansion.

A Strictly Markovian Expansion for Plasma Turbulence Theory

It has been shown by Misguich and Balescu[1] that the equations of motion for a turbulent Vlasov plasma may be put in a form that is apparently Markovian. Their result was expressed within the framework of renormalized quasi-linear theory[2,3,4] and expressed in terms of highly non-linear integral equations for the appropriate propagators.

It is the purpose of this note to show that the result of Misguish and Balescu[1] may be cast in a form that is readily expandable in powers of the fluctuating operator $L_1(t)$ and the unperturbed or free propagator U_o and that this expansion is strictly Markovian to _any_ _finite_ order. The lowest non-trivial order $(O(L_1^2))$ in this expansion is just the usual quasi-linear equation _including_ the so called adiabatic approximation. An examination of the next order correction terms $(O(L_1^4))$ indicates that in addition to the usual mode-coupling terms generated by the quasi-linear series one obtains an additional term but one that is of the same order of magnitude as the usual terms.

If $f(t) = f(\underset{\sim}{x}, \underset{\sim}{v}, t)$ represents the ensemble averaged distribution function then assuming that the fluctuations in the distribution function vanish at t_o, $f(t)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} + L_0\right) f(t) = A \int_{t_0}^{t} L_1(t) U_A(t,t_1) L_1(t_1) f(t_1) dt_1 \tag{1}$$

where A and U_A are operators introduced by Weinstock[5], A is the averaging operator that ensemble averages everything to its right, and $U_A(t,t_1)$ is the propagator that satisfies the operator equations

$$\left(\frac{\partial}{\partial t} + L_0 + (1-A)L_1(t)\right)U_A(t,t_1) = 0$$

$$U_A(t_1,t_1) = 1. \tag{2}$$

L_0 is the unperturbed, ensemble averaged Liouiville operator considered time independent and $L_0(t)$ is the time dependent, fluctuating Liouiville operator whose statistics are assumed stationary and homogeneous.

If we employ the unperturbed propagator U_0, defined by the equation

$$\left(\frac{\partial}{\partial t} + L_0\right) U_0(t,t_1) = 0 \tag{3}$$

and the boundary condition $U_0(t_1,t_1) = 1$ we may write (2) as
an intergral equation

$$U_A(t,t_0) = U_0(t,t_0) + \int_{t_0}^{t} U_0(t,t_1)(A-1)L_1(t_1)U_A(t_1,t_0)dt_1 \tag{4}$$

Solving by iteration on U_0 one arrives at the formal solution of (4)

$$U_A(t,t_0) = T_L \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \int_{t_0}^{t} (A-1)U_0(t,t_1)L_1(t_1)U_0(t_1,t)dt_1 \right\}^n$$

$$\times U_0(t,t_0). \tag{5}$$

where T_L, the time ordering operator[6,7,8], orders the operator $L_1(t_1)$ from right to left in increasing order of the variable t_1.

We now introduce the tilde notation for quantities propagated by U_0
from the current time t to an intermediate time t_1,τ, etc. For single time operators

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$$\overset{\curvearrowright}{L}_1(t_1) \equiv U_0(t,t_1)L_1(t_1)U_0(t_1,t) \quad ,$$

for   double time propagators

$$\overset{\curvearrowright}{U}(t_1,t_2) \equiv U_0(t,t_1)U(t_1,t_2)U_0(t_2,t)$$

and for distribution functions

$$\overset{\curvearrowright}{f}(t_1) \equiv U_0(t,t_1)f(t_1)$$

Employing this notation we may write for U_A

$$U_A(t_1,t_2) = T_L \exp\left\{\int_{t_2}^{t_1}(A-1)U_0(t_1,t_3)L_1(t_3)U_0(t_3,t_1)dt_3\right\}$$
$$\times\ U_0(t_1,t_2) \tag{6}$$

and

$$\overset{\curvearrowright}{U}_A(t_1,t_2) \equiv U_0(t,t_1)U_A(t_1,t_2)U_0(t_2,t)$$
$$= T_L \exp\left\{(A-1)\int_{t_1}^{t_2}dt_3 \overset{\curvearrowright}{L}_1(t_3)\right\} \tag{7}$$

where we have employed the general operator relation

$$p \exp(q) = \exp(pqp^{-1})\ p \text{ and the fact that } U_0(t_1,t_2) = U_0^{-1}(t_2,t_1).$$

We may now write (1) as

$$\left(\frac{\partial}{\partial t} + L_0\right) f(t) = A \int_{t_0}^{t} dt_1 \tilde{L}_1(t) \tilde{U}_A(t,t_1) \tilde{L}_1(t_1) \tilde{f}(t_1) \quad . \tag{8}$$

It should be noted that while $\tilde{U}_A$ is still a propagator its expansion to any finite order via (7) is not, rather it is a completely local operator. All propagation of other quantities is indicated by the tilde notation. Writing $\tilde{U}_A$ as the series indicated by (7) and inserting in (8) produces the standard, non-Markovian quasi-linear series.

To proceed we note that we may rewrite (8) as a function of τ for $t_0 \leq \tau \leq t$ by means of the relation

$$\frac{\partial}{\partial \tau} U_0(t,\tau) = U_0(t,\tau) L_0 = L_0 U_0(t,\tau)$$

as

$$\frac{\partial}{\partial \tau} f(\tau) = A \int_{t_0}^{\tau} dt_1 \tilde{L}_1(\tau) \tilde{U}_A(\tau,t_1) \tilde{L}_1(t_1) \tilde{f}(t_1) \tag{9}$$

where the tilde _still_ implies propagation backward from t not from τ. If $f(t)$ is now considered a boundary value (9) may be integrated immediately to give an integral equation for $\tilde{f}(\tau)$

$$\tilde{f}(\tau) = f(t) + A \int_{t}^{\tau} dt_1 \int_{t_0}^{t_1} dt_2 \tilde{L}_1(\tau) \tilde{U}_A(t_1,t_2) \tilde{L}_1(t_2) \tilde{f}(t_2)$$

$$= f(t) + D(t;\tau) \tilde{f}(\tau) \tag{10}$$

where $D(t;\tau)$ is the integral operator defined in (10).

In (10) t is to be understood as a _fixed_ parameter and $f(t)$ as a _fixed_ boundary value; the integral operator only integrates over the running variable τ.

With this in mind we may now solve (10) by iteration to obtain the formal solution

$$\tilde{f}(\tau) = \sum_{n=0}^{\infty} \{ D(t;\tau) \}^n f(t)$$

$$= \{1 - D(t;\tau)\}^{-1} f(t) \tag{11}$$

Inserting (10) into (8) we now obtain

$$\left(\frac{\partial}{\partial t} + L_0\right) f(t) =$$

$$= \left(A \int_{t_0}^{t} dt_1 \hat{\tilde{L}}_1(t) \hat{U}_A(t,t_1) \tilde{\tilde{L}}_1(t_1) \{1 - D(t;t_1)\}^{-1} \right) f(t) \tag{12}$$

where the operator in curley brackets contains _no explicit propagators_ and is local in time to any finite order. Since D is $O(L_1^2)$ and $\hat{U}_A = 1 + O(L_1^2)$ we have to lowest order

$$\left(\frac{\partial}{\partial t} + L_0\right) f(t) = \left\{ A \int_{t_0}^{t} dt_1 \tilde{\tilde{L}}_1(t) \hat{\tilde{L}}_1(t_1) \right\} f(t) \tag{13}$$

which is just the usual quasi-linear expression _including_ the Markovian or adiabatic approximation.

We see, therefore, that if one considers lowest order quasi-linear theory one may include the Markovian approximation in a rigorous manner, the error involved being formally of a higher order and hence rightly neglected. We shall see in fact that the correctior. obtained by including D to lowest order is of the same order of magnitude as those obtained by the correction to the propagator $\hat{U}_A$ to the same formal order.

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Before examining the first correction terms to the "collision" operator in (12) (they will be of $O(L_1^4)$ since odd order terms are assumed to vanish under the action of the averaging operator A) we shall examine the full Vlasov propagator and the phase space trajectorys generated by them. The Vlasov propagator U satisfies the equation

$$\left(\frac{\partial}{\partial t} + L_0 + L_1(t)\right) U(t,t_0) = 0 \tag{14}$$

with the boundary condition $U(t_0,t_0) = 1$. From the similarity to (2) (only the A operator is missing) it is easy to see that

$$U(t,t_0) = T_L \exp\left\{-\int_{t_0}^{t} dt_1 \tilde{L}_1(t_1)\right\} U_0(t,t_0) \tag{15}$$

and

$$\tilde{U}(t_1,t_2) = T_L \exp\left\{-\int_{t_2}^{t_1} dt_3 \tilde{L}_1(t_3)\right\} . \tag{16}$$

We now introduce the phase functions [5,9],

$$z^\circ(z,t,\tau) \equiv U_0(t,\tau)z$$

$$z^*(z,t,\tau) \equiv U(t,\tau)z$$

where z is a point in n (usually 6) dimensional phase space These functions may be understood as the phase space position at a previous time, τ, of a particle that was at the point z at the current time t and which moved in the intervening time according to the unperturbed or full Liouiville operator respectively.

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The difference between the full Vlasov trajectory ζ^* and the unperturbed trajectory ζ^o is given by

$$\Delta\zeta(\zeta,t,\tau) \equiv \zeta^*(\zeta,t,\tau) - \zeta^o(\zeta,t,\tau)$$

$$= \left(\hat{U}(t,\tau) - 1\right)\zeta^o(\zeta,t,\tau) \quad . \tag{17}$$

If one expands $\hat{U}$ according to (16) it is straightforward to write $\Delta\zeta$ as a series of terms

$$\Delta\zeta = \Delta\zeta^{(1)} + \Delta\zeta^{(2)} + \Delta\zeta^{(3)} + \cdot \cdot \cdot \cdot$$

where

$$\Delta\zeta^{(1)}(t,\tau) = - \int_\tau^t dt_1 \hat{L}_1(t_1)\zeta^o(t_1,\tau)$$

$$\Delta\zeta^{(2)}(t,\tau) = - \int_\tau^t dt_1 \hat{L}_1(t_1)\Delta\zeta^{(1)}(t_1,\tau) \tag{18}$$

etc. where we have omitted the argument ζ. Since to leading order $\Delta\zeta$ is $O(L_1)$:

Since any function of ζ^o or ζ^* satisfies the same equations as ζ^o or ζ^* respectively we have

$$F(\zeta^*(t,\tau)) = \hat{U}(t,\tau)F(\zeta^o(t,\tau)) \quad . \tag{19}$$

But $F(\zeta^*)$ must also be related to $F(\zeta^o)$ by a Taylors series, ie.

$$F(\zeta^*) = F(\zeta^\circ) + \Delta z \frac{\partial}{\partial Z} \cdot F(\zeta^\circ) + \frac{1}{2}(\Delta z)^2 \left(\frac{\partial}{\partial Z} \cdot\right)^2 F(\zeta^\circ) + \cdots$$

so we may relate the expansion of $\hat{\mathcal{U}}$ to a given order to the Taylors series to the same order in $\Delta \zeta$, eg.

$$- \int_\tau^t dt_1 \hat{L}_1(t_1) = \Delta \zeta^{(1)} \frac{\partial}{\partial Z} \cdot \quad ,$$

$$\int_\tau^t dt_1 \hat{L}(t_1) \int_\tau^{t_1} dt_2 \hat{L}(t_2) = \Delta z^{(2)} \frac{\partial}{\partial Z} \cdot + \frac{1}{2}(\Delta z^{(1)})^2 \left(\frac{\partial}{\partial Z} \cdot\right)^2$$

$$\simeq O\left\{\Delta z^2 \left(\frac{\partial}{\partial Z}\right)^2\right\} \tag{20}$$

etc.

With this in mind we now turn to the terms in the collision operator that are $O(L_1^4)$. They are

$$C_4 = \frac{1}{2} A \int_{t_0}^t dt_1 \hat{L}_1(t) T_L (A-1) \int_{t_1}^t dt_2 \hat{L}_1(t_2)(A-1) \int_{t_1}^t dt_3 \hat{L}_1(t_3) \hat{L}_1(t_1)$$

$$+ A \int_{t_0}^t dt_1 \hat{L}_1(t) \hat{L}_1(t_1) A \int_t^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \hat{L}_1(t_2) \hat{L}_1(t_3) \quad , \tag{21}$$

where the first term on the right hand side arises from corrections to the propagator and the second term is due solely to the Markovian formulation.

Noting that A operating on an odd number of $\hat{L}_1$'s gives zero, taking account of the effects of T_L explicitly and relabeling some of the integration variables gives

$$C_4 = A\hat{L}_1(t) \int_{t_0}^{t} dt_1 \hat{L}_1(t_1) \int_{t_0}^{t_1} dt_2 \hat{L}_1(t_2) \int_{t_0}^{t_2} dt_3 \hat{L}_1(t_3)$$

$$- A\hat{L}_1(t) \int_{t_0}^{t} dt_1 \hat{L}_1(t_1) A \int_{t_0}^{t_1} dt_2 \hat{L}_1(t_2) \int_{t_0}^{t_2} dt_3 \hat{L}_1(t_3)$$

$$- A\hat{L}_1(t) \int_{t_0}^{t} dt_1 \hat{L}_1(t_1) A \int_{t_1}^{t} dt_2 \hat{L}_1(t_2) \int_{t_0}^{t_2} dt_3 \hat{L}_1(t_3) \tag{22}$$

We wish to compare these terms with

$$C_2 = A\hat{L}_1(t_1) \int_{t_0}^{t} dt_1 \hat{L}_1(t_1) \quad .$$

Since we have seen that

$$- \int_{\tau}^{t} dt_1 \hat{L}(t_1) \; \cdot \; \Delta \zeta^{(1)}(\zeta,t,\tau)\frac{\partial}{\partial z}. \tag{20}$$

we shall estimate the size of such terms by a factor
$\Delta\zeta(T)/z_c$ where $\Delta\zeta(T)$ is the deviation of the Vlasov orbit from the unperturbed
orbit during some effective time T and z_c is a characteristic scale in
phase space. Since the derivative operators operate on everything to their
right we should consider z_c to be characteristic of either the random force
field or the distribution function whichever gives a smaller value.
In the following we shall assume that the characteristic scale of the random
force, the correlation length, is much smaller than the scale of variation
of the distribution function.

In estimating the effective time T over which the orbit deviation
is relevant the averaging operators play a crucial role. We shall here
assume that there exists a characteristic correlation time τ_c for the propagated
operators $\hat{L}_1(t)$ whether due to their being propagated in phase space or

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due to an intrinsic time variation such that if two of these operators are averaged together this average is non-zero only when the two times, eg. t_1 and t_2 are within a τ_c or so of each other. This limits the effective range of many of the integrals in (22).

The first term in (22) is a four point correlation; if we assume that $\tilde{L}_1$ is a Gaussian process this term may be expressed as the sum of the product of all possible (three in this case) pair correlations with unit weight[10]. One of these terms will just be cancelled by the second term in (22). This is an example of the non-appareance of "unconnected diagrams" discussed by Thomson and Benford[11] however, this is only true if $\tilde{L}_1$ is a Gaussian process. If terms like this do not vanish then a pure secularity appears since an examination of the second term in (22) reveals that the integration over t_2 is not limited by any correlation and hence grows with t. Such secularities growing as higher powers of t will appear in all higher orders[12]. If such terms do not appear it is straightforward to verify that all other time integrations are limited to range of a few τ_c by interlocking or nested correlations and we may estimate $T \simeq \tau_c$. With this approximation and noting further that

$$\tilde{L}_1 = \lim_{\tau \to t} \frac{\partial}{\partial \tau}\left(\Delta\zeta(t,\tau)\right) \cdot \frac{\partial}{\partial Z} \simeq \Delta\zeta(t,t-\tau_c)/(\tau_c z_c)$$

we obtain $C_2 \simeq \tau_c^{-1}(\Delta\zeta(\tau_c)/z_c)^2$

and all terms of $C_4 \simeq \tau_c^{-1}(\Delta\zeta(\tau_c)/z_c)^4$

$$\simeq (\Delta\zeta(\tau_c)/z_c)^2 C_2 \text{ where } \Delta\zeta(\tau_c) \equiv \Delta\zeta(t,t-\tau_c).$$

The effective expansion parameter for this theory is just the ratio $\Delta\tilde{Z}(\tau_c)/z_c$. For a more quantitative discussion of this ratio see Jones[13].

We see therefore that the correction terms generated by the Markovian formulation are not only of the same formal order in L_1 as the correction terms to the usual quasi-linear approximation but are of the same order of magnitude. There is therefore no a-priori reason to prefer the integro-differential equation of the usual approach over the far simpler Markovian or adiabatic approximation that is often brought in as a further approximation.

One may notice that the Markovian formulation generates more terms for any order above the lowest and hence might involve greater absolute error to any order than the non-Markovian approach. Such a conclusion is not justified in general, for if one assumes, in addition to being a Gaussian process, that the $\tilde{L}_1$ operators may be freely commuted it is straightforward to demonstrate that the additional term in (22) exactly cancels the other terms to make C_4 exactly zero. This is just a detailed example of the general case shown by Jones and Birmingham[8] namely that Gaussian statistics together with the commutability of the $\tilde{L}_1$ operators at different times leads to exactly soluble models whose solutions are just those given by the quasi-linear approximation with the Markovian or adiabatic approximation.

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