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SUMMARY

This report describes the derivation of the equations of motion for a multi-blade rotor. The present analysis advances on current capabilities for calculating rotor responses by introducing a high twist capability and coupled flatwise-edgewise assumed normal modes instead of uncoupled flatwise and edgewise assumed normal modes. The torsion mode is uncoupled as before. Features inherited from previous work include the support system models, consisting of complete helicopters in free flight, or grounded flexible supports, arbitrary rotor-induced inflow, and arbitrary vertical gust model. These representations are described in previous published texts.

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DEFINITIONS OF SYMBOLS

a	acceleration, ft/sec^2. Also matrix (3.17)
a_{ji}	generalized mass in normal mode coordinates, (3.58). Also direction cosine (See 7.28, 7.29)
a_p	acceleration of a point P relative to the origin of the $\bar{X}_5$ axis, Fig. 15
a_o	reverse flow parameter used with steady airfoil data, (9.14), (9.15)
a	speed of sound, ft/sec
a_{11}^*, a_{12}^*, a_{21}^*, a_{22}^*	coefficients in expression for correction to modal stiffness to account for change in pitch (see Sec. 4.2).
$(a_{o5}^{(1)})_i$ $(a_{o5}^{(5)})_i$	i-th components of 3-component acceleration vector evaluated at O_5 and referred to $\bar{X}_1$ and $\bar{X}_5$ axes, respectively, ft/sec^2
a_{mn}^x	general form of radially independent coefficient arising from A_x, A_y, or A_z components of acceleration (See Sec. 7.1c and 7.1d)
A	blade cross-sectional area. Also, bending stiffness matrix (3.10). Also pitch horn attachment point, Fig. 24.
A_x, A_y, A_z	linear acceleration terms in acceleration components, ft/sec^2 (See 7.75 and 7.77).
A_β, A_δ, A_ϵ, A_θ, $A_{\theta e}$, $A_{-\theta s}$, $A_{-\theta'}$, A_{λ_1}, A_{λ_2}, $A_{-\phi s}$, $A_{-\phi'}$, A_ψ, $A_{-\psi s}$, $A_{-\psi'}$	orthogonal transformation matrices defined in (5.34) to (5.47), and (9.46). Subscripts indicate the rotation transformation argument angles.
A_{1s}	cyclic pitch control input, rad. Coefficient of cosine term
$A_{mj}^{(1)}$, $A_{imj}^{(1)}$, $A_{kimj}^{(1)}$, $A_{mj}^{(2)}$, $A_{imj}^{(2)}$, $A_{mj}^{(3)}$, $A_{imj}^{(3)}$	Modal integrals defined in Table 2.

b_{ji} generalized stiffness in normal mode coordinates, (4.9). Also direction cosine (9.89).

$b_{mn}{}^x$ general form of radially independent coefficient arising from B_x, B_y, or B_z components of acceleration (See Sec. 7.1c and 7.1d)

B torsion stiffness matrix, (3.10)

B_x, B_y, B_z angular acceleration coefficients multiplying y_{10} terms in acceleration components, rad/sec^2. See 7.75 and 7.77.

$B_{mj}{}^{(1)}$, $B_{imj}{}^{(1)}$, $B_{1imj}{}^{(1)}$, $B_{mj}{}^{(2)}$, $B_{imj}{}^{(2)}$, $B_{mj}{}^{(3)}$, $B_{imj}{}^{(3)}$

Modal integrals defined in Table 2.

c blade chord, ft. Also, cosine. Also, structural damping coefficient

c_{crit} critical value of structural damping coefficient

$c_{mn}{}^x$ general form of radially independent coefficient arising from C_x, C_y or C_z components of acceleration (See Sec. 7.1c and 7.1d).

c_d section drag coefficient (See 9.11)

c_l section lift coefficient (See 9.10)

c_{jk} coefficient in blade modal damping term, (4.70).

C centrifugal stiffness matrix, (3.10)

C_x, C_y, C_z angular acceleration coefficients multiplying z_{10} terms in acceleration components, rad/sec^2 (See 7.75 and 7.77)

C_{LD} lag damper constant, lb-ft-sec/rad

d section aerodynamic drag, lb/ft, (9.11)

ds fiber length extending between adjacent beam faces after elastic deformation

ds_o fiber length extending between adjacent beam faces before elastic deformation

D	mass matrix, (3.10)
e	blade hinge offset, ft
e_A	distance of local elastic centroid from local elastic axis in the positive y_{10} direction, ft.
E	modulus of elasticity, lb/ft^2
$\bar{E}$	hinge offset vector, (5.20)
EI_z	flatwise blade bending stiffness, $lb\text{-}ft^2$ (6.21)
EI_y	edgewise blade bending stiffness, $lb\text{-}ft^2$ (6.22)
EI_1, EI_2 $\equiv$	EI_y, EI_z
F	three component vector of generalized forces in loading equilibrium equation, (4.1). $F^T = F_1, F_2, F_3$. Also, force acting on a blade section, lb.
F_1, F_2, F_3	components of generalized force in force vector in loading equilibrium equation, defined by (6.63) to (6.65).
g	acceleration of gravity, ft/sec^2
g*	modal damping ratio, $= 2c/c_{crit}$
GJ	blade torsional stiffness, $lb\text{-}ft^2$
$\bar{i}_1$, $\bar{i}_5$	unit vectors in x_1 and x_5 direction
I_B	blade mass moment of inertia, $lb\text{-}ft\text{-}sec^2$
$\bar{j}_1$, $\bar{j}_5$	unit vectors in the y_1 and y_5 directions
k_A	structural radius of gyration measured from the origin of the y_{10}-z_{10} axis (elastic axis) in the positive y_{10} direction, ft, (6.23)
k_{y10}, k_{z10}	mass radii of gyration measured in the y_{10} and z_{10} directions from the origin of the y_{10}-z_{10} axis (elastic axis), ft, (8.13)
k_1, k_2 $\equiv$	k_{y10}, k_{z10}

$\bar{k}_1$, $\bar{k}_5$	unit vectors in the z_1 and z_5 directions
K_D, K_F, K_P	blade leading, flapping and torsion root spring constants, ft-lb/rad
l	section aerodynamic lift, lb/ft, (9.10)
m	blade mass per unit length slug/ft (8.13). Also aerodynamic moment per unit length, lb.
m_d	quasi-steady section damping moment per unit length, lb, (9.13)
m_o	blade reference mass per unit length, slug/ft
M	local Mach number for flow perpendicular to blade section. Also moment acting on a blade section, ft-lb. Also, number of modes.
M_1, M_2	flatwise and edgewise structural reaction moments, ft-lb
p	loading per unit length of blade. lb/ft
P	distance from the origin of the $\bar{X}_5$ axis, ft, Fig. 19. Also denotes point of pushrod attachment (Sec. 12.1). Also intercept of elastic axis in x-z plane, Fig. 11.
q	column vector of modal amplitudes, (4.19). Also moment loading per unit length, lb.
q_i	modal amplitude of i-th mode.
Q	torsion reaction moment, ft-lb. Also position of elastic axis in x-z plane after bending displacement, Fig. 11.
Q_j	generalized force in j-th modal equation
$Q_j^{(1)}$, $Q_j^{(2)}$, $Q_j^{(3)}$	contributions to generalized force Q_j defined by (4.26) to (4.28). Also modal integrals from these contributions (Sec. 8 and Appendix 14.3)
Q_j^A	aerodynamic contribution to Q_j
Q_j^D	inertial contribution to Q_j

Q_j^P point load (lag damper) contribution to Q_j

Q_j^{A1}, Q_j^{A2}, Q_j^{A3}
aerodynamic contributions to generalized forces $Q_j^{(1)}$, $Q_j^{(2)}$, $Q_j^{(3)}$, respectively (See 4.32 to 4.35).

Q_j^{D1}, Q_j^{D2}, Q_j^{D3}
inertial contributions to generalized forces $Q_j^{(1)}$, $Q_j^{(2)}$, $Q_j^{(3)}$, respectively (See 4.32 to 4.35).

Q_j^{P1}, Q_j^{P2}, Q_j^{P3}
point load (lag damper) contributions to generalized forces $Q_j^{(1)}$, $Q_j^{(2)}$, $Q_j^{(3)}$, respectively (See 4.32 to 4.35).

r radial distance in the x_5 direction from hinge, ft (Also denoted by x)

r_A x_5 coordinate of pitch horn attachment, ft. Fig. 24.

r_F hinge offset, ft, Fig. 1

r_p x_5 coordinate of point of attachment of pushrod, ft, Fig. 24

r_T value of r at blade tip, = R-e, ft

r_{ocw} value of r at inboard end of counterweight, ft

R rotor radius = e + r_t or = $r_F + R_1$, ft.

$\bar{R}$ three-component radius vector, (5.20)

R_1 value of x_o at blade tip, = $R - r_F$ ft.

R_1, R_2, R_3...modal integrals, Appendix 14.3

s sine

s_{jk} coefficient of modal acceleration $\ddot{q}_k$ in modal acceleration equation, (4.14)

$s_{jk}^{(1)}$, $s_{jk}^{(2)}$, $s_{jk}^{(3)}$
contributions to s_{jk} from Q_j^{D1}, Q_j^{D2}, and Q_j^{D3}, respectively.

S	square matrix of s_{jk} elements, dimensions of M X M, (4.17)
t	time, sec.
t_j	forcing function on right-side of modal acceleration equation, (4.14).
T	column vector of t_j elements, dimensions M X 1, (4.18).
$\hat{T}$	steady tension acting on a section of blade, lb, (3.5)
u*	longitudinal displacement of the blade elastic center in the x_5 direction, ft, Fig. 8.
U	magnitude of airflow velocity perpendicular to blade section, ft/sec, (9.21). Also when subscripted denotes component of total airflow velocity, $\bar{U}$, ft/sec
U_P	component of $\bar{U}$ parallel to z_{10}', ft/sec, Fig. 18
U_T	component of $\bar{U}$ parallel to y_{10}', ft/sec, Fig. 18
$\bar{U}$	total airflow velocity at a blade section, ft/sec, (9.47) and Fig. 19.
v	edgewise elastic displacement of elastic axis in y direction, Fig. 1, or y_5 direction, Fig. 8, ft. Also, velocity of a point on the blade, ft/sec.
v_i	v-displacement of elastic axis in i-th normal mode.
v_1	sum of blade elastic edgewise and blade rigid body leading displacements of elastic center, ft, (3.6).
v_{1i}	v_1 displacement of elastic axis in i-th normal mode
$(v_{05}^{(1)})_i$	velocity vector evaluated at 0_5 and referred to $\bar{X}_1$ axis, ft/sec
w	flatwise elastic displacement of elastic axis in z direction, Fig. 1, or z_5 direction, Fig. 8, ft.

w_i	w-displacement of elastic axis in i-th normal mode.
w_1	sum of blade elastic flatwise and blade rigid body flapping displacements of elastic center, ft, (3.6).
w_{1i}	w_1 displacement of elastic axis in i-th normal modes
x	coordinate, Figs. 1 and 8. Also vector of displacements θ_e, w_1 and v_1 (See 3.10)
x_i	vector of displacements x in i-th mode
x_o	x-coordinate of a point on the blade in the x-direction of Fig. 1, neglecting longitudinal elastic extension
$\overline{X}_{Io}$	vector of coordinates of hub referred to $\overline{X}_I$ axis
X_1, X_2, X_3	resultant torsion moment per unit length, resultant flatwise loading per unit length, and resultant edgewise loading per unit length, respectively, acting on a blade element.
$\overline{X}$	position vector with components x, y, and z. Also used to designate force, moment, loading and velocity vectors.
y	coordinate, Figs. 1 and 8. Also, vector of root stiffnesses, (3.19)
y_p	y_5 coordinate of point of attachment of pushrod, ft, Fig. (24).
y_{10cg}	y_{10} value of centroid of section mass, ft.
z	coordinate, Figs. 1 and 8.
α	gust inclination, rad, Fig. 20. Also pitch-lag coupling angle, rad. (12.11)
α_r	section angle of attack, rad, Fig. 18
α_1	pitch lag coupling angle, rad.
α_{mn}^x	contribution to a_{mn}^x isolating terms independent of $\ddot{q}_k$.

$\alpha_m^{(1)}, \alpha_{mi}^{(1)}, \alpha_m^{(2)}, \alpha_{mi}^{(2)}, \alpha_m^{(3)}, \alpha_{mi}^{(3)}$

Acceleration coefficients defined in Table 2.

β — blade flapping angle, rad, Fig. 7. Also local pitch angle, rad (in equations for normal modes of vibration), including built-in twist.

β_i — blade flapping angle displacement in i-th mode, rad.

β_D — lead angle, rad, Fig. 1.

β_F — flapping, rad, Fig. 1.

β_P — root end pitch angle, rad, Fig. 1.

β_{mn}^x — contribution to b_{mn}^x isolating terms independent of $\ddot{q}_k$.

$\beta_{km}^{(1)}, \beta_{kmi}^{(1)}, \beta_{kmil}^{(1)}, \beta_{km}^{(2)}, \beta_{kmi}^{(2)}, \beta_{km}^{(3)}, \beta_{kmi}^{(3)}$

Acceleration coefficients defined in Table 2.

γ_{mn}^x — contribution to c_{mn}^x isolating terms independent of $\ddot{q}_k$.

δ — blade lead angle, rad, Fig. 7. Also prefix denoting perturbation variable. Also elastic center displacement due to torsional displacement of a beam element, Fig. 12.

δ_i — lead angle displacement in i-th mode, rad.

δ_{ik} — Kronecker delta.

δ_3 — pitch flap coupling angle, rad, (12.11).

δ_3' — pitch flap coupling angle, rad, (12.11).

δv, δw — elastic center displacements due to torsional displacement of a beam element, (5.73), (5.74).

$(\delta/\delta t)_i$ — differential operator, (7.11).

Δ — vector of elastic center displacements in twisted coordinates, θ_e, Δ_1, Δ_2. Also small length of blade over which the damper moment is applied.

Δ_{eAcw}	distance between chordwise location of counterweight and chordwise location of elastic centroid, e_A, positive in the direction of y_{10}, ft.
Δt	time increment, sec.
Δv, Δw	elastic center displacements accompanying blade torsional displacement, ft, (5.75), (5.76).
Δy^*, Δz^*	rotation arms yielding elastic center displacements due to torsional displacement, (5.67), (5.68), Fig. 11.
Δv_{ik}, Δw_{ik}	contributions to elastic center displacements Δv, and Δw from the i-th flatwise-edgewise and k-th torsion modes, (7.86), (7.88).
$\Delta\theta$	change in pitch angle θ due to flapping and flatwise displacement, rad, (12.2).
Δ_1, Δ_2	elastic center displacements in twisted coordinates, ft, Fig. 3
ϵ	strain. Also denotes a small quantity. Also denotes angle in Fig. 18.
ϵ_0, ϵ_1, ϵ_2, ϵ_3	strain coefficients, (6.10).
ξ	coordinate, Fig. 1. Also see Fig. 5.
η	coordinate, Fig. 1. Also see Fig. 5.
η_c	η value of centroid of mass of blade section.
θ	local blade pitch angle due to control input and built-in twist, rad, $= \theta_c + \theta_t$, Fig. 8.
θ'	angular coordinate, Fig. 5. Also derivative of θ with respect to r.
$\bar{\theta}$	total local blade pitch angle, rad, $= \theta + \theta_e$.
θ_B	local built-in twist angle, rad.
θ_c	sum of built-in twist angle and collective pitch control input, rad, $= \theta_B + \theta_{3/4R}$.
θ_i	blade torsion displacement in i-th normal mode.

θ_e	torsional displacement, rad, Fig. 8.
θ_s	angular coordinate, Fig. 4.
θ_t	time-dependent component of pitch angle due to cyclic control input, pitch-flap, and pitch-lag couplings, (12.1).
θ_{e1}	sum of time-dependent pitch angle θ_t and torsional displacement θ_e, rad ($= \theta_t + \theta_e$).
$\theta_{3/4R}$	collective pitch input angle measured at $r = 3/4R$, rad.
$\theta_B{}^*$	pitch angle used to calculate blade normal modes of vibration, rad.
λ_1, λ_2	elastic angular displacements, rad, Fig. 8
μ_{jk}	correction to modal stiffness accounting for pitch difference, (4.46), (4.66).
$\mu_{jk}{}^*$	contribution to μ_{jk}, (4.67)
$\nu_{mn}{}^x$	contribution to $c_{mn}{}^x$ isolating terms multiplying $\ddot{q}_k$.
ξ	dummy variable. Also coordinate Fig. 1. Also see Fig. 5.
$\bar{\xi}$	position vector with components ξ, η, ζ.
ρ	density, slug/ft^3
ρ_{cw}	density of counterweight, slug/ft.
$\bar{\rho}$	displacement vector O_1O_5, Fig. 15.
$\rho_{mn}{}^x$	contribution to $a_{mn}{}^x$ isolating terms multiplying $\ddot{q}_k$.
$\sigma_{mn}{}^x$	contribution to $b_{mn}{}^x$ isolating terms multiplying $\ddot{q}_k$.
ϕ	blade torsional displacement, rad, Fig. 1. Also inflow angle, rad, (9.20)
ϕ_s	angular displacement, rad, Fig. 4.
ϕ'	angular displacement, rad, Fig. 5.

ψ	blade azimuth angle, rad, Fig. 7.
ψ_R	angle ψ of a reference blade, rad.
$(\psi_R)_o$	angle ψ_R of the reference blade at time $t = t_o$, rad.
ψ_s	angular coordinate, rad, Fig. 4
ψ'	angular coordinate, rad, Fig. 5.
ω	circular frequency of vibration of a normal mode, rad/sec. Also angular velocity, rad/sec.
$\omega_i^{(1)}$	components of angular velocity defined by (7.58).
Ω	rotor rotational speed, rad/sec.

SUBSCRIPTS

c/4, 3c/4	quarter chord, three-quarter chord point.
i, j	dummy subscripts. Denotes direction cosine with i and j ranging from 1 to 3. Also denotes mode number, with i and j ranging from 1 to M.
k, l	dummy subscripts denoting mode number, with k and l ranging from 1 to M.
e	denotes displacement of elastic axis referred to y_6 or z_6 directions, Fig. 8. Also denotes quantity evaluated at hinge.
m	subscript ranging over numbers of elements composing Q_j^{D1*}, Q_j^{D2*}, and Q_j^{D3*}, Table 2. Also first subscript in a_{mn}^x, b_{mn}^x, etc. assuming values 0, 1, or 2 (See Sec. 7.1c and 7.1d).
n	second subscript in a_{mn}^x, b_{mn}^x, etc. assuming values 0 to 6 (See Sec 7.1c and 7.1d).

$x, y, z, x_1, y_1, z_1, x_5, y_5, z_5, x_{10}, y_{10}, z_{10}$
Denotes vector component referred to $x, y, z, x_1, y_1, z_1, x_5, y_5, z_5, x_{10}, y_{10}, z_{10}$ directions

AIR	denotes airflow induced by source independent of rotor motion.
cw	counterweight
H	quantity evaluated at rotor hub.

I, I_1	denotes axis systems.
P_1, P_2	positions of pushrod attachment, Fig. 24.
O_1, O_5	quantities evaluated at points O_1 and O_5 in Fig. 15.
T	denotes quantity evaluated at blade tip.

α, α_R, β, δ, ϵ, θ, θ_e, θ_s, θ', $\bar{\theta}$, λ_1, λ_2, ϕ_s, ϕ', ψ, ψ_s, ψ'

Argument angles in circular functions c and s, or transformation matrices, A, rad.

o	denotes a position before elastic deformation. Also denotes quantity evaluated at hinge.
1-10	denotes axis systems. Also 1 and 2 indicate that the arguments of matrices A, B, C, and D are θ and θ_B*, respectively.

SUPERSCRIPTS

A	aerodynamic term
D	inertial term
P	point load (lag damper) term
T	transpose
X	superscript in a_{mn}^x, b_{mn}^x, etc. denoting x, y, or z (See Sec. 7.1c or 7.1d). Also d/dψ.
D*, D_1*, D_2* D_3*, ( )*	denotes contribution with modal acceleration terms removed.
(0), (1), (2)	denotes zeroeth, first, and second order quantity.
( )	d/dt
( )'	d/dr
($\bar{\ }$)	complex conjugate. Also denotes vector. Also non-dimensional variable.
$\hat{\ }$	equilibrium value. Also running variable (Sec. 5.3).

ABBREVIATIONS

e.a.	elastic axis
e.c.	elastic center

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1. INTRODUCTION

The work described here is a further step in the evolution of methods for calculating the motions and responses of rotor systems. This effort advances beyond the single blade analysis of Ref. (1) and the multi-blade rotor analyses of Ref. (2) and (3), by introducing coupled flatwise-edgewise assumed normal modes and a high twist capability in the multi-blade analysis. Table A compares the features of several rotor response analyses. The present theory is incorporated in Sikorsky computer program Y210 (Ref. 5). The motivation for the introduction of coupled flatwise-edgewise modes is the expectation that a coupled modes analysis will be more accurate than an uncoupled modes analysis when both use the same number of modes, particularly for highly twisted blades. The incorporation of these coupled modes with a high twist capability is expected to provide a superior tool to handle modern high twist rotors and rotors converting to a dual propellor mode of operation.

Apart from these features, the restrictions and scope of the model are similar to those of the previous analyses (Ref. 1 to 3). The present model adheres closely to the assumptions of Ref. (1) regarding approximations to terms involving products of elastic variables and products of small quantities occurring in the equations of motion. These assumptions are listed in the next section and are repeated in the text as required during the derivations. The procedure of coupling the rotor with the supporting system was inherited from Ref. (2). The generalized coordinates and equations of motions for support systems, consisting of a complete helicopter with a rigid fuselage, and a grounded flexible support, are described in Ref. 2 and Ref. 3, respectively. The method of calculating the normal mode shapes and frequencies for coupled flatwise-edgewise-torsion motions of a rotating twisted blade is described in Ref. 4. This is a finite element (transfer-matrix) method based on the Holzer-Myklestad type of treatment. The equations were specialized for the present application to yield coupled flatwise-edgewise modes and an uncoupled torsion mode, as described in Chapter 3. These equations were incorporated in the coupled modes module of program Y210. Also incorporated in program Y210 is a variable inflow calculation module, based on the method of Ref. (5), and the gust representation, described in Ref (2). For completeness we have included in this report the equations introducing the effects of a cylindrical vertical gust superior to that of Ref. (2), but which was not incorporated in the program.

	F-574	Y-200	Y-141 (Y-201)	Y-210
TITLE	Single Blade Aero-elastic Analysis	Multi-Blade Aero-elastic Analysis	Multi-Blade Aero-elastic Analysis with Grounded Flexible Support	Multi-Blade Aero-elastic Analysis with High Twist Capability
Blade Dynamics	Externally calculated Uncoupled Normal Modes 5 flatwise, 3 edgewise 2 torsion	Internally calculated Uncoupled Normal Modes 5 flatwise, 4 edgewise 2 torsion	Same as Y-200	Internally calculated Coupled Normal Modes 7 Modes
Blade Aerodynamics	2D steady-state airfoil data with varying airfoil distribution α, A, B or time delay unsteady aero, no skewed flow	2D steady-state airfoil data, α, A, B unsteady aero, no skewed flow, gust input capability	Same as Y-200	Same as Y-200
Airmass Dynamics	Constant or variable inflow (calculated externally)	Same as F-574	Same as F-574	Constant or variable inflow (calculated internally)
Airframe Dynamics	None	Rigid Body Helicopter or Compound, Free Flight	Same as Y-200 plus option for grounded flexible support (10 modes)	Same as Y-141 (Y-201)
Airframe Aerodynamics	None	Steady-State Aerodynamics	Same as Y-200, no aero on flexible support	Same as Y-141 (Y-201)
Control System Representation	Root spring, no coupling	Same as F-574	Same as F-574	Same as F-574
Rotor Control Input	Collective, 1P and 2P cyclic prescribed hub motion or control inputs	Collective and 1P cyclic	Same as Y-200	Same as Y-200 (Y-201)
Trim Procedure	Trims to specified rotor forces and moments	Trims to specified rotor or aircraft forces and moments	Same as Y-200	Same as Y-200
Computer	UNIVAC 1110	UNIVAC 1110 IBM 360	UNIVAC 1110 IBM 370	IBM 370
Period of Initial Development	1966-1967	1971-1972	1972	1973-1974
Program Running Time (Major Iteration & Transient	[illegible] cpu min. on UNIVAC 1110	12 cpu min. on UNIVAC 1110	12 cpu min. on IBM 370/158	15 cpu min. on IBM 370/158
Rotor Configurations	Articulated, semi-articulated, hingeless	Articulated, semi-articulated, hingeless, teetering	Same as Y-200	Same as Y-200
Primary Use	Blade loads, and deflections, pushrod loads, blade stability	Blade loads and deflections, pushrod loads, rotor hub loads, blade stability, rotor and aircraft gust response	Same as Y-200 plus ground resonance stability	Same as Y-141 plus loads and stability of high twist blades

TABLE A - COMPARISON OF FEATURES OF ROTOR RESPONSE ANALYSES

ORIGINAL PAGE IS OF POOR QUALITY

The present analysis employs Newton's second equation to derive the equations of motion. We were influenced by Ref. (1) in adopting this method, and by a desire to preserve similarities with that derivation, to assist in the development and checking of the equations. The resulting equations are transformed into an approximate uncoupled set of equations using normal mode coordinates by considering the virtual work done by a rotor blade in a virtual displacement of a normal mode. The resulting normal mode differential equations are integrated with respect to time to obtain the response of a rotor blade. Forces and moments induced by the multi-blade rotor are fed to the support and the resulting support motion is fed back to the rotor to obtain the response of the coupled rotor/support system.

This report describes the mathematical model in sufficient detail to enable a reader to understand, check, and modify the theory. The addition of the high twist feature together with hub motion terms caused a large increase in the number of terms and modal integrals in the equations, in comparison with Ref. (1). The proliferation of terms emphasized the need for systematic development of the equations and exposition of the theory, and the text reflects our attempt to satisfy this demand.

The report is organized on the following lines. Chapters 3 and 4 give an overview of the approach. It is necessary to read these chapters to understand the subsequent text. If only general features of the model are of interest, a reading of the text may be confined to these two chapters. We derive in Chapter 3 the differential equations satisfied by the normal modes of vibration and establish the orthogonality properties of the modes. These equations and their orthogonality properties provide the key to the derivation of equations of motion in terms of normal mode coordinates from the differental equations for the multi-blade rotor. Chapter 4 expands the overview, and introduces conventions to describe general terms in the normal mode differential equations, which determine the organization of subsequent chapters, directed to the evaluation of these terms in the general equation.

Chapter 5 introduces the rectangular axis transformations employed for the description of the displacement of a point on a blade. These transformations consisting of linear displacements and rotations, are the generalized coordinates in the equations of motion in physical space (in contrast to modal space, which is reached after the transformation to normal coordinates).

Chapter 6 yields moment and loading equilibrium equations similar to those presented in Ref. (1), to define general expressions for the generalized forces in physical space. These general expressions guide further the organization of the subsequent text, which is dedicated to the systematic evaluation of blade element loads in these expressions.

Chapter 7 begins the evaluation of the terms in the expressions of Chapter 6 for generalized force by obtaining the acceleration of a point on the blade, scheduled for substitution in the inertia contribution to generalized force. In this chapter four different versions of the acceleration are elaborated to ease and systematize the subsequent derivation of inertia loads. Chapter 8 completes the derivation of the working form for the inertia contribution to the generalized force in modal space.

Chapter 9 yields all the working equations needed to calculate the aerodynamic contribution to the generalized force. This chapter includes expressions for the relative flow velocity at the blade for general hub motions and gust-induced velocities.

The text is completed by Chapters 10 to 12 covering the following short subjects. These comprise derivations of

1) lag damper contribution to generalized force;
2) shears and moments affecting the blade and fixed systems, employed to display the response, and required for the analysis of the coupled rotor/support system;
3) expression for contribution to blade pitch induced by cyclic control inputs and pitch-flap and pitch-lag couplings;
4) non-dimensionalization rules employed to reduce the normal mode differential equations to non-dimensional form in the program code.

2. ASSUMPTIONS

The following major assumptions are made to derive the equations of motion.

1. In general, the rotor hub may be in accelerated motion. This state includes uniform motion as a special case.

2. The blade elastic axis is straight before elastic deformation has occurred, and coincides with the feathering axis before elastic deformation.

3. The blade is assumed to have structural and inertial symmetry about a major principal axis located on the aerodynamic chord.

4. Blade flap and lag hinges and feathering bearing are coincident.

5. The blade may have a built-in twist of arbitrary magnitude and distribution along the blade span.

6. The blade angle of impressed collective pitch may have an arbitrary magnitude.

7. Control system flexibility and blade root restraints are represented by root springs. Control circuit mass and damping effects are not modeled.

8. Structural damping ratio is the same in all modes of vibration.

9. Blade element theory describes the aerodynamic forces, and radial flow effects are neglected.

10. In addition to these assumptions, the following assumptions are employed to calculate the normal modes of free vibration of the blade.

   a) There is no mass unbalance
   b) The elastic centroid coincides with the elastic axis
   c) Coriolis terms are neglected
   d) Free vibrations occur about a position of zero steady displacement.

11. The following quantities and their derivatives, where applicable, are assumed to be small in comparison to unity.

a) Flap and lead angles, in radians
b) Ratios of translational elastic displacements to rotor radius, and torsional displacement in radians
c) Ratios to rotor radius of chord, center of gravity, hinge offset, mass and structural radii of gyration, and elastic centroid
d) Ratios of hub linear accelerations to $\Omega^2 R$; hub angular accelerations to Ω^2
e) Ratio of gravity acceleration to $\Omega^2 R$
f) Time-dependent component of pitch angle, in radians induced by cyclic control inputs and pitch-flap and pitch-lag couplings.

12. We list below products of terms neglected as higher order terms in the bending equilibrium and section velocity equations. Small quantities considered in these products comprise translational elastic displacements, torsional displacement, chordwise distances, flap angle, lead angle, time-dependent component of pitch angle, hinge offset, hub accelerations and angular velocities, and gravity acceleration. We neglect in

a) Flatwise and edgewise bending equations

   (1) Second and higher order products of elastic displacements
   (2) Third and higher order products of small quantities

b) Torsional equation

   (1) Third and higher order products of elastic displacements
   (2) Fourth and higher order products of small quantities

c) Section velocity equations

   (1) Second and higher order products of elastic displacements
   (2) Third and higher order products of small quantities.

3. Differential Equations of Motion for System Employed to Calculate Modes of Free Vibration

In this chapter we derive the differential equations of motion satisfied by the system employed to calculate the modes of free vibration, and we establish properties of these equations. These differential equations and their properties play a central part in the derivation of modal differential equations from the multi-blade rotor differential equations. It is recalled that the equations used to calculate the modes are finite element equations involving transfer matrices and are not presented as differential equations in Reference (4). Consequently, it is important to verify the forms of the differential equations corresponding to the coupled modes, and to establish properties subsequently used to derive modal equations from the differential equations for the multi-blade rotor.

Purposes of this chapter are to:

1) verify the forms of the differential equation satisfied by the system yielding the blade modes of vibration,
2) demonstrate briefly the property of orthogonality which is used subsequently to derive single degree of freedom uncoupled modal differential equations from coupled differential equations,
3) set out the procedure for deriving uncoupled modal differential equations from the system of coupled equations, which is the basis for the later derivation of modal equations from the equations of motion for the multi-blade rotor system,
4) indicate the assumptions under which the differential equations for the system employed to calculate the modes are valid, to indicate restrictions like those required to uncouple the torsion motion from the flatwise/edgewise motions, and to show consistency with the assumptions underlying the multi-blade rotor differential equations,
5) compare the modal equations of the present method with the equations employing the uncoupled modes of Reference (1). We show that our modal equations include modes with a predominant rigid body (flap or lag) character, and, hence, that the flap and lag angle equations of Reference (1) need not be separately derived.

Figure 1 illustrates axes systems used for the calculation of the coupled modes. The numbering system is the same as that used in Reference (4) except that primes are added to some subscripted axes to prevent confusion with axis systems subsequently used for the response analysis. The axes apply to a configuration with no steady displacement and coincident hinges.

Consideration of the equilibrium of forces and moments acting on an element of beam yields the following beam element equilibrium equations.

$$\begin{aligned}\delta X_1 = {} & \left[\delta Q + (v' c_\beta + w' s_\beta)\hat{M}_1 + (v' s_\beta - w' c_\beta)\hat{M}_2\right]' \\ & - v'(c_\beta \hat{M}_1 + s_\beta \hat{M}_2)' + w'(-s_\beta \hat{M}_1 + c_\beta \hat{M}_2)' \\ & - v'\hat{q}_y + w'\hat{q}_z + \delta q_x + \hat{p}_y w - \hat{p}_z v \\ = {} & 0\end{aligned} \tag{3.1}$$

$$\begin{aligned}\delta X_2 = {} & (-v'\hat{Q} - \phi s_\beta \hat{M}_1 + c_\beta \delta M_1 + \phi c_\beta \hat{M}_2 + s_\beta \delta M_2)'' \\ & -(w'\hat{T})' + \delta q_y' + (p_x w)' - \delta p_z \\ = {} & 0\end{aligned} \tag{3.2}$$

$$\begin{aligned}\delta X_3 = {} & (w'\hat{Q} - \phi c_\beta \hat{M}_1 - s_\beta \delta M_1 + \phi s_\beta \hat{M}_2 + c_\beta \delta M_2)'' \\ & -(v'\hat{T})' + \delta q_z' + (\hat{p}_x v)' - \delta p_y \\ = {} & 0\end{aligned} \tag{3.3}$$

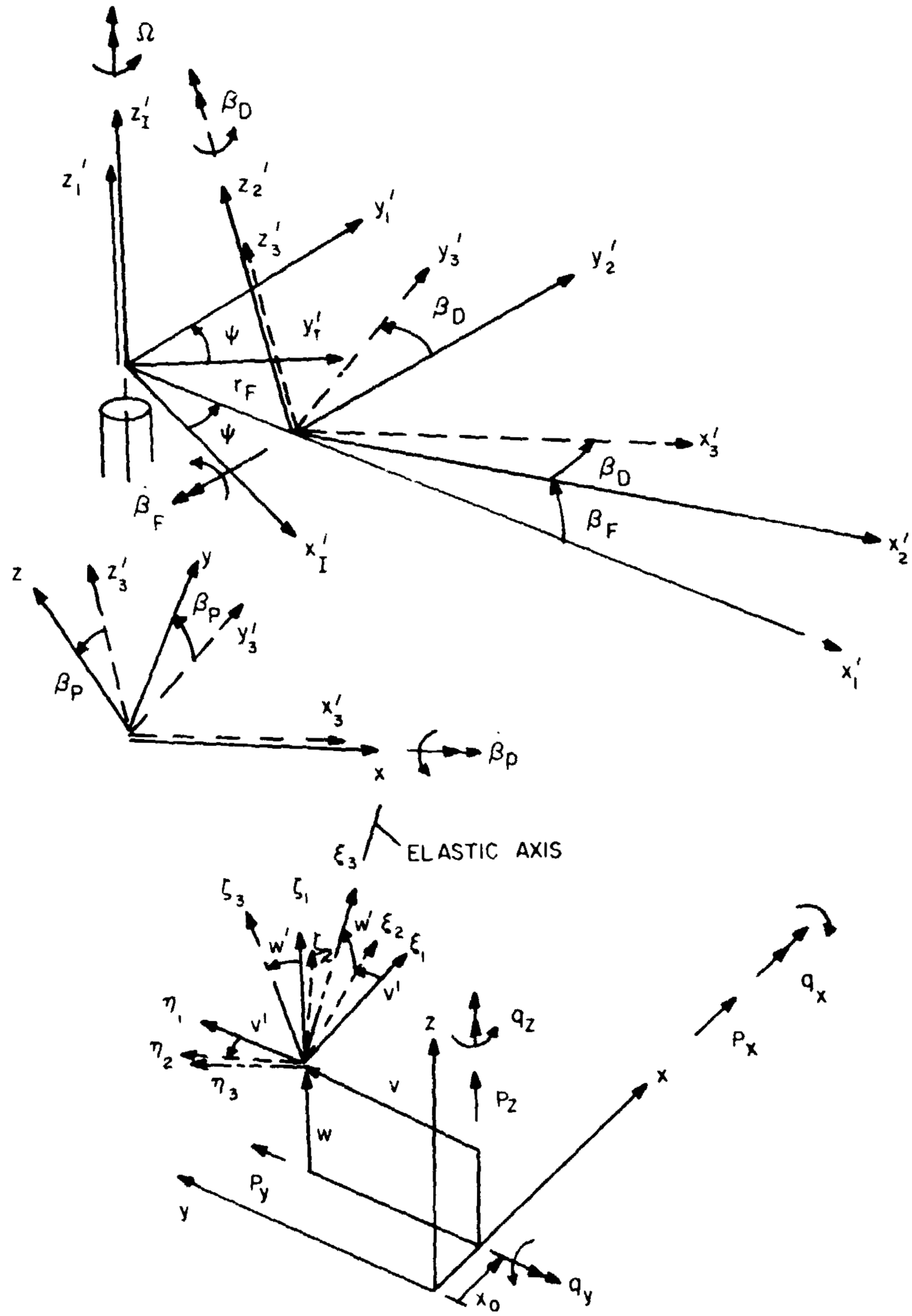
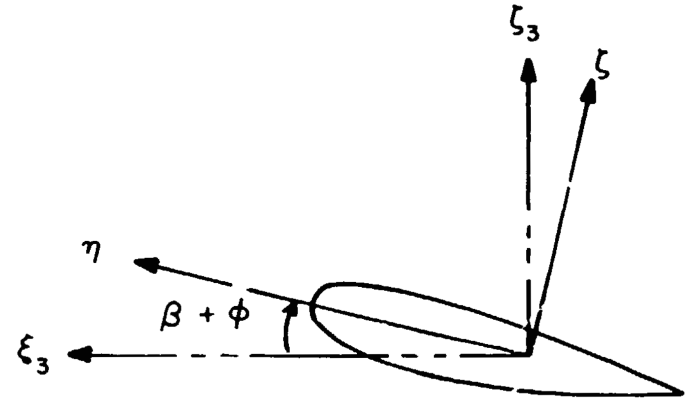


Figure 1. Axes System for Calculation of Coupled Modes.



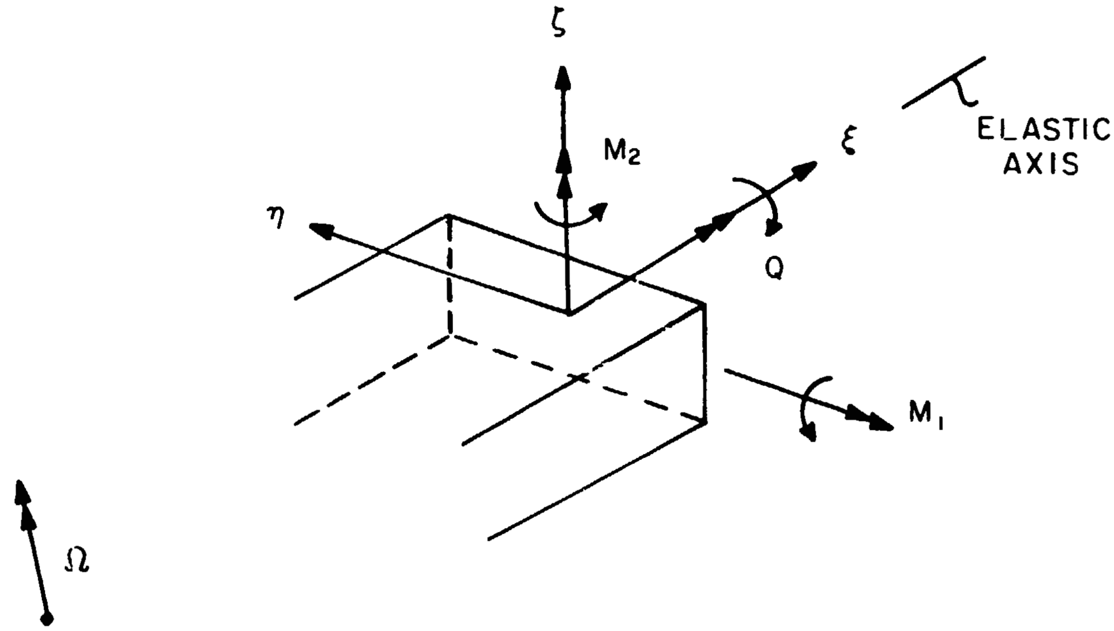


Figure 1. Concluded.

Structural reaction moments are Q, M_1, M_2 illustrated in Figure 1,e. Applied inertia forces per unit beam length are p_x, p_y, p_z. Applied inertia moments per unit beam length are q_x, q_y, q_z. Steady quantities are denoted with a hat, and perturbations are prefixed with δ (δp_x δp_y, etc.), or are untreated (u, v, w). The local pitch angle is β and is not to be confused with the flap angle of the multi-blade rotor response equations. These equations could have been derived also from the beam element equilibrium equations of Houbolt and Brooks, equations (17) and (11), Reference (6), by dividing quantities into steady and pertubation values.

The structural reaction moments may be expressed in terms of section stiffness properties and displacement derivatives by means of equations (15), (16), and (17) of Reference (6).

The inertia loads p_x, p_y, p_z, q_x, q_y, q_z derive from

$$
\begin{aligned}
p_x &= -\int_A a_x \rho \, dA \\
p_y &= -\int_A a_y \rho \, dA \\
p_z &= -\int_A a_z \rho \, dA \\
q_x &= \int_A \left[-(a_y z) + (-a_z y)\right] \rho \, dA \\
q_y &= \int_A -a_x z \rho \, dA \\
q_z &= -\int_A (-a_x y \rho \, dA)
\end{aligned}
\tag{3.4}
$$

In these expressions, y and z are displacements of an elemental mass in a blade section after elastic displacements occur, and a_x, a_y, a_z are acceleration components of the elemental mass. Neglecting blade longitudinal extension u, the accelerations apply to a point distance x_o from the origin of the x-y-z axes.

Figure 2 illustrates inertia force and moment positive conventions, and location of the point where forces, p_x, p_y, p_z and moments, q_x, q_y, q_z are evaluated. This point is on the x-axis and is not at the elastic center of the blade which, in general, is displaced to some other point. The purpose of the illustration is to emphasize that the applied forces and moments of the coupled modes equations are defined differently from the forces and moments of the response analysis, also designated p_x, p_y, p_z, q_x, q_y, q_z, which are evaluated at the elastic center and resolved to the 5 system of the response equations. Also, q_y in the response equations is opposite in sense to the q_y of the coupled modes equation.

Employing the assumptions listed in Chapter 2, we derive below from Reference (4) results for inertia forces and moments. Included in these assumptions are the following assumptions consistent with those of the response equations.

1) Fourth order products of elastic displacements and small quantities, like radii of gyrations, are neglected in the torsion equation

2) Third order products of elastic displacements and small quantities are neglected in the flatwise and edgewise equations.

In addition, the following assumptions are made to uncouple the torsion equation from the flatwise - edgewise equations, and to produce real eigensolutions:

3) $\eta_c = 0$, no mass unbalance

4) $e_A = 0$, elastic centroid coincident with elastic axis

5) Coriolis terms are not included in the inertia forces. This is an assumption of Reference (4).

6) Free vibrations occur about a position of zero steady displacement.

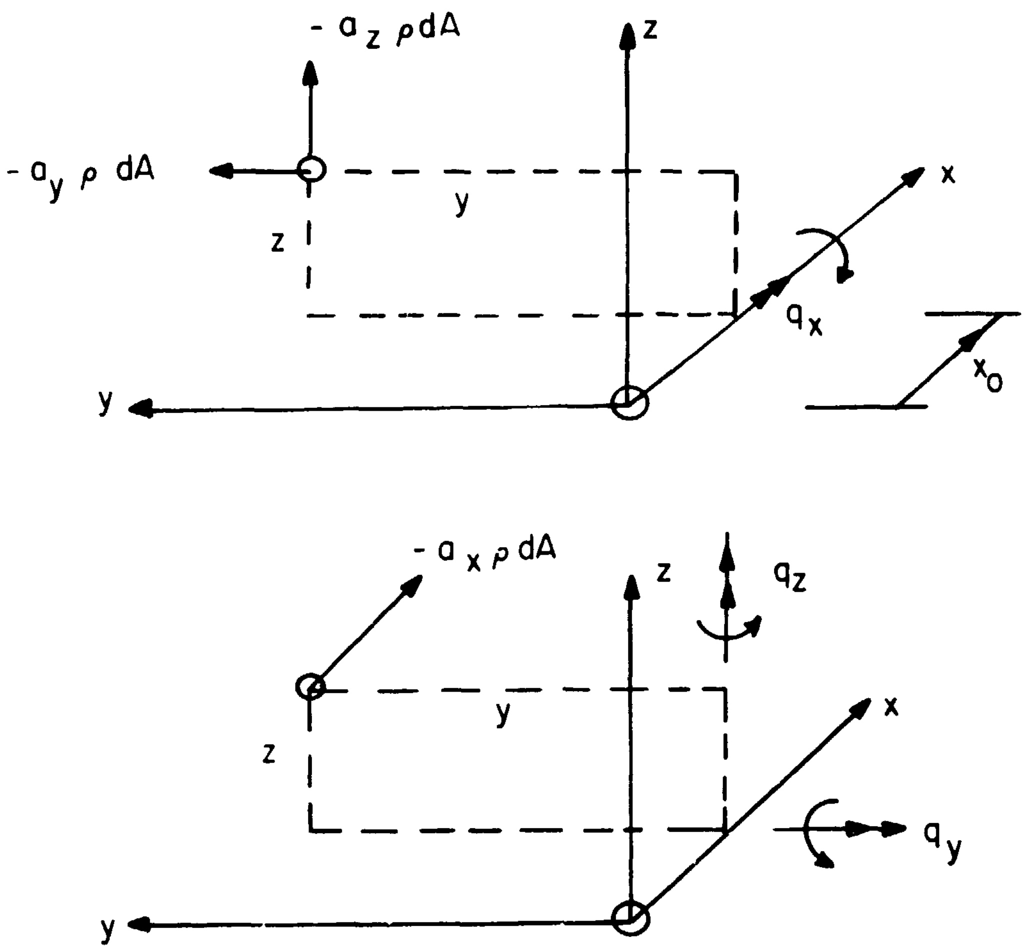


Figure 2. Inertia Loads Conventions for Coupled Modes Analysis.

The following results are obtained:

(3.5)

$$\hat{M}_1 = 0$$
$$\hat{M}_2 = 0$$
$$\hat{Q} = 0$$
$$\hat{T} = \int_r^{R_1} m\Omega^2(\xi + r_F)\,d\xi$$
$$\delta M_1 = EI_1(-v'' s_\beta + w'' c_\beta)$$
$$\delta M_2 = EI_2(v'' c_\beta + w'' s_\beta)$$
$$\delta Q = (GJ + \hat{T} K_A^2)\phi'$$
$$\hat{p}_x = m\Omega^2(x_0 + r_F)$$
$$\hat{p}_y = 0$$
$$\hat{p}_z = 0$$
$$\hat{q}_x = -m\Omega^2(K_2^2 - K_1^2) s_\beta c_\beta$$
$$\hat{q}_y = 0$$
$$\hat{q}_z = 0$$
$$\delta p_x = 0$$
$$\delta p_y = m\Omega^2 v - m\Omega^2 r_F \beta_D - m\ddot{v} - m x_0 \ddot{\beta}_D$$
$$\delta p_z = -m\Omega^2(x_0 + r_F)\beta_F - m\ddot{w} - m x_0 \ddot{\beta}_F$$
$$\delta q_x = -m\Omega^2(K_2^2 - K_1^2)(\phi + \beta_p) - m(K_2^2 + K_1^2)(\ddot{\phi} + \ddot{\beta}_p)$$
$$\delta q_y = -m\Omega^2(x_0 + r_F)w$$
$$\delta q_z = -m\Omega^2(x_0 + r_F)v$$

The coupled modes module can embrace less restrictive models than those satisfying assumptions 1) to 6). To prevent the use of an inconsistent model, inputs like e_A, η_C, e_t, e_y, e_z, EB_1 and EB_2, defined in Reference (4), are overridden by instructions in the program setting these parameters to zero. This has the effect of yielding (3.5) exactly. The differential equations satisfied by the coupled modes are then exactly those derived in this report (see 3.7 to 3.9, below). This observation establishes the consistency of the equations for the natural modes in the program and this report.

Substitution of (3.5) in (3.1) to (3.3) yields the differential equations of motion satisfied by the system employed for the coupled modes analysis, and the substitution

$$
\begin{aligned}
\theta_e &= \phi + \beta_P \\
w_1 &= w + \beta_F x_0 \\
v_1 &= v + \beta_D x_c
\end{aligned}
\tag{3.6}
$$

transforms the equations into

$$
\begin{aligned}
-\delta X_1 = & -\left(\left(GJ + \hat{T}\kappa_A^2\right)\theta_e{}'\right)' \\
& + m\Omega^2(\kappa_2^2 - \kappa_1^2)C_{2\beta}\theta_e + m(\kappa_2^2 + \kappa_1^2)\ddot{\theta}_e \\
= & \ 0
\end{aligned}
\tag{3.7}
$$

$$\delta X_2 = \left\{ (EI_1 c_\beta^2 + EI_2 s_\beta^2) w_1'' + (EI_2 - EI_1) s_\beta c_\beta v_1'' \right\}'' - (w_1' \hat{T})' + m\ddot{w}_1 = 0$$

(3.8)

$$\delta X_3 = \left\{ (EI_2 - EI_1) s_\beta c_\beta w_1'' + (EI_1 s_\beta^2 + EI_2 c_\beta^2) v_1'' \right\}'' - (v_1' \hat{T})' - m\Omega^2 v_1 + m\ddot{v}_1 = 0$$

(3.9)

These equations are expressed in matrix form, to effect the most convenient derivation subsequently of the properties of the system.

(3.10)

$$(Ax'')'' + (Bx')' + Cx + D\ddot{x} = 0$$

$$A = \begin{matrix} 0 & 0 & 0 \\ 0 & EI_1 c_\beta^2 + EI_2 s_\beta^2, & (EI_2 - EI_1) s_\beta c_\beta \\ 0 & (EI_2 - EI_1) s_\beta c_\beta, & EI_1 s_\beta^2 + EI_2 c_\beta^2 \end{matrix}$$

$$B = \begin{matrix} -(GJ + \hat{T} k_A^2) & 0 & 0 \\ 0 & -\hat{T} & 0 \\ 0 & 0 & -\hat{T} \end{matrix}$$

$$C = \begin{matrix} m\Omega^2(k_2^2 - k_1^2)c_{2\beta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m\Omega^2 \end{matrix}$$

$$D = \begin{matrix} m(k_2^2 + k_1^2) & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{matrix}$$

$$x^T = \theta_e', w', v'$$

It is seen from (3.7) to (3.9), or (3.10), and boundary conditions (3.15) and (3.16), Section 3.1, that the torsion motion is confined to (3.7) which does not involve vertical or inplane motions, indicating an uncoupled torsion motion. In general, for non-vanishing pitch, β, vertical and inplane motion terms appear simultaneously in (3.8) and (3.9) indicating that these are coupled. The uncoupled character of the torsion motion and the coupled character of the vertical and inplane motions are consequences of assumptions 1) to 6), listed above. Equation (3.10) does not indicate fully coupled motions and should be viewed as a convenience. Orthogonality results subsequently derived from (3.10), combining torsion and other displacement modes in one expression, are not to be construed as being for fully coupled motions, and apply to uncoupled torsion modes, as justified above. (The orthogonality condition can be broken up into terms only involving torsion modes, and terms only involving vertical and inplane modes, to indicate explicitly the uncoupled character of the torsion mode. This is not done in the large body of text because no advantage derives from doing so for the present exposition. Computational efficiency could be improved, on the other hand, by distinguishing the uncoupled character of the torsion mode, as discussed subsequently.)

3.1 Boundary Conditions

Equation (3.5) yields displacement derivatives as functions of structural moments.

$$\phi' = \delta Q / (GJ + \hat{T} K_A^2) \tag{3.11}$$

$$\begin{Bmatrix} w'' \\ v'' \end{Bmatrix} = \frac{1}{EI_1 EI_2} \begin{bmatrix} EI_2 c_\beta & EI_1 s_\beta \\ -EI_2 s_\beta & EI_1 c_\beta \end{bmatrix} \begin{Bmatrix} \delta M_1 \\ \delta M_2 \end{Bmatrix} \tag{3.12}$$

At the hinge, including root springs, we have

$$\delta G = K_P \beta_P \tag{3.13}$$

$$\begin{Bmatrix} \delta M_1 \\ \delta M_2 \end{Bmatrix} = \begin{bmatrix} c_\beta & -s_\beta \\ s_\beta & c_\beta \end{bmatrix} \times \begin{Bmatrix} K_F \beta_F \\ K_D \beta_D \end{Bmatrix} \tag{3.14}$$

Substitution of (3.13) and (3.14) in (3.11) and (3.12), respectively, yields

$$\phi' = K_P \beta_P / (GJ + \hat{T} K_A^2) \tag{3.15}$$

$$\begin{Bmatrix} w'' \\ v'' \end{Bmatrix} = a^{-1} \begin{Bmatrix} K_F \beta_F \\ K_D \beta_D \end{Bmatrix} \tag{3.16}$$

where a is the matrix

$$a = \begin{pmatrix} EI_1 c_\beta^2 + EI_2 s_\beta^2, & (EI_2 - EI_1) s_\beta c_\beta \\ (EI_2 - EI_1) s_\beta c_\beta, & EI_1 s_\beta^2 + EI_2 c_\beta^2 \end{pmatrix} \tag{3.17}$$

For subsequent substitution of the boundary conditions, we note that A of (3.10) also is

$$A = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{a}} \\ 0 & & \end{array}\right] \tag{3.18}$$

Employing (3.16) we obtain

$$(Ax'')_0 = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & & a \end{array}\right] \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & & a^{-1} \end{array}\right] \begin{Bmatrix} 0 \\ K_F \beta_F \\ K_D \beta_D \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ K_F \beta_F \\ K_D \beta_D \end{Bmatrix}$$

This is rewritten as

$$(Ax'')_0 = y \tag{3.19}$$

with

$$y^T = 0,\ K_F \beta_F,\ K_D \beta_D$$

At the tip, $\delta Q = \delta M_1 = \delta M_2 = 0$ and from (3.11) and (3.12), we find $\varphi' = w'' = v'' = 0$. Also, shears, $\delta M_1' = \delta M_2' = 0$ at the tip. Differentiation of (3.12) then yields $w''' = v''' = 0$, at the tip. Using (3.6) we can state the boundary conditions in terms of θ_e, w_1, v_1.

<u>$x_o = 0$ (hinge)</u>

(3.20)

$$\theta_e = \beta_P$$

$$w_1 = v_1 = 0$$

$$\theta_e',\ w_1',\ v_1' = K_P \beta_P / (GJ + \hat{T} k_A^2),\ \beta_F,\ \beta_D$$

$$\begin{Bmatrix} w_1'' \\ v_1'' \end{Bmatrix} = a^{-1} \begin{Bmatrix} K_F \beta_F \\ K_D \beta_D \end{Bmatrix}$$

with a defined by (3.17). From this last equation we obtain

$$(A x'')_0 = y$$

with

$$y^T = 0,\ K_F \beta_F,\ K_D \beta_D$$

and A defined by (3.10).

<u>$x_o = R_1$ (tip)</u>

$$\theta_e' = w_1'' = v_1'' = w_1''' = v_1''' = 0$$

3.2 Orthogonality of Modes

For a system in free vibration at frequency ω, (3.10) becomes

$$(Ax'')'' + (Bx')' + Cx - \omega^2 Dx = 0 \qquad (3.21)$$

To establish orthogonality of modes we evaluate the work done by a mode x_i with frequency ω_1, in a virtual displacement x_j. To evaluate the work quantity, we premultiply (3.21) by x_j^T and integrate from hinge to tip. We obtain

$$\int_0^{R_1} \left[x_j^T (A x_i'')'' + x_j^T (B x_i')' + x_j^T (C x_i - \omega_i^2 x_j^T D x_i \right] dx_0 = 0 \qquad (3.22)$$

Integration by parts twice of the first term yields

$$\int_0^{R_1} x_j^T (A x_i'')'' dx_0 = \left[x_j^T A' x_i'' + x_j^T A x_i''' - x_j^{T'} A x_i'' \right]_0^{R_1} + \int_0^{R_1} x_j^{T''} A x_i'' dx_0 \qquad (3.23)$$

Invoking boundary conditions (3.20) we are left with

$$\int_0^{R_1} x_j^T (A x_i'')'' dx_0 = (x_j^{T'} A x_i'')_0 + \int_0^{R_1} x_j^{T''} A x_i'' dx_0 \qquad (3.24)$$

Again, from (3.20)

$$\begin{aligned} (x_j^{T'} A x_i'')_0 &= (x_j^{T'})_0 y_i \\ &= K_F \beta_{F_i} \beta_{F_j} + K_D \beta_{D_i} \beta_{D_j} \end{aligned} \qquad (3.25)$$

and (3.24) becomes

$$\int_0^{R_1} x_j^T (A x_i'')'' dx_0 = K_F \beta_{F_i} \beta_{F_j} + K_D \beta_{D_i} \beta_{D_j} + \int_0^{R_1} x_j^{T\,''} A x_i'' dx_0 \tag{3.26}$$

Integration by parts of the second term in (3.22) yields

$$\int_0^{R_1} x_j^T (B x_i')' dx_0 = \left[x_j^T B x_i' \right]_0^{R_1} - \int_0^{R_1} x_j^{T\,'} B x_i' dx_0 \tag{3.27}$$

In view of boundary conditions (3.20) and the fact that $\hat{T} = 0$ at $x_0 = R_1$, we find $B = 0$ at $x_0 = R_1$ and (3.27) becomes

$$\int_0^{R_1} x_j^T (B x_i')' dx_0 = -(x_j^T B x_i')_0 - \int_0^{R_1} x_j^{T\,'} B x_i' dx_0 \tag{3.28}$$

The boundary conditions (3.20) yield

$$-(x_j^T B x_i')_0 = \theta_j ((GJ + \hat{T} K_A^2) \theta_i')_0 = K_P \beta_{P_i} \beta_{P_j} \tag{3.29}$$

and (3.28) becomes

$$\int_0^{R_1} x_j^T (B x_i')' dx_0 = K_P \beta_{P_i} \beta_{P_j} - \int_0^{R_1} x_j^{T\,'} B x_i' dx_0 \tag{3.30}$$

Equation (3.22) for the virtual work becomes

$$K_P \beta_{P_i} \beta_{P_j} + K_F \beta_{F_i} \beta_{F_j} + K_D \beta_{D_i} \beta_{D_j} +$$
$$\int_0^{R_1} \left[x_j^{T\,''} A x_i'' - x_j^{T\,'} B x_i' + x_j^T C x_i - \omega_i^2 x_j^T D x_i \right] dx_0 = 0 \tag{3.31}$$

Similarly the work done by the forces of the j-th mode in the i-th virtual displacement is

$$K_P \beta_{P_i} \beta_{P_j} + K_F \beta_{F_i} \beta_{F_j} + K_D \beta_{D_i} \beta_{D_j}$$

$$\int_0^{R_1} \left[x_i^{T''} A x_j'' - x_i^{T'} B x_j' + x_i^T C x_j - \omega_2^2 x_i^T D x_j \right] dx_0 = 0 \tag{3.32}$$

where ω_2 is the frequency of the j-th mode. At this point we note that products like $x_i^{T''} A x_j''$ are scalars. The value of the product is unchanged by taking its transpose. Thus

$$\left(x_i^{T''} A x_j'' \right)^T = x_j^{T''} A^T x_i'' \tag{3.33}$$

Since matrices A, B, C, D are symmetric

$$A^T, B^T, C^T, D^T = A, B, C, D \tag{3.34}$$

and it follows

$$\left(x_i^{T''} A x_j'' \right)^T = x_j^{T''} A x_i'' \tag{3.35}$$

Taking the transpose of (3.32) and using (3.35) and similarly treating other terms in (3.32) and subtracting the transpose of (3.32) from (3.31) we are left with

$$\left(\omega_2^2 - \omega_1^2 \right) \int_0^{R_1} x_j^T D x_i \, dx = 0 \tag{3.36}$$

If eigenvalues and eigenvectors are real and distinct this requires

$$\int_0^{R_1} x_j^T D x_i \, dx_0 = 0 \qquad i \neq j$$
$$\neq 0 \qquad i = j \tag{3.37}$$

This is the statement of the orthogonality condition of the (normal) modes and it applies to the coupled system (torsion is uncoupled) with or without root springs. Expanding (3.37) we find

$$\int_0^{R_1} m\left[\left(k_1^2+k_2^2\right)\theta_i\,\theta_j + w_{1_i} w_{1_j} + v_{1_i} v_{1_j}\right] dx_0 = 0 \tag{3.38}$$

For the rotated system of axes of Figure 3 defined by the transformation

$$x = \begin{matrix}\theta_e \\ w_1 \\ v_1\end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\beta & s_\beta \\ 0 & -s_\beta & c_\beta \end{bmatrix} \times \begin{Bmatrix}\theta_e \\ \Delta_1 \\ \Delta_2\end{Bmatrix} \tag{3.39}$$

$$= A_\beta \Delta \tag{3.40}$$

(3.41)

$$A_\beta = \begin{matrix} 1 & 0 & 0 \\ 0 & c_\beta & s_\beta \\ 0 & -s_\beta & c_\beta \end{matrix}$$

(3.42)

$$\Delta^T = \theta_e, \Delta_1, \Delta_2$$

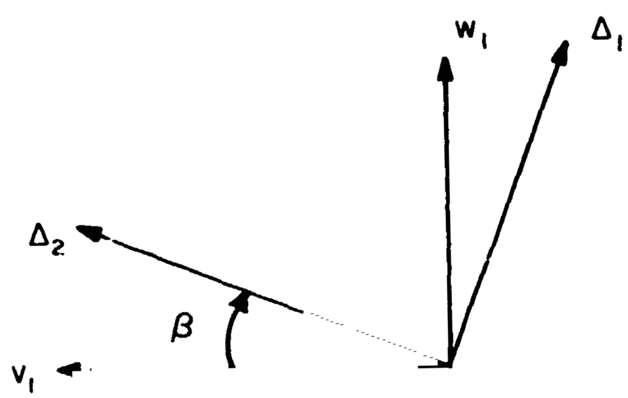


Figure 3. Rotated Axes Defining Elastic Center Displacement in Coupled Modes Analysis.

we find

$$\int_0^{R_1} x_j^T D x_i \, dx_0 = \int_0^{R_1} \Delta^T A_\beta^T D A_\beta \Delta \, dx_0 = \int_0^{R_1} \Delta^T D \Delta \, dx_0 \tag{3.43}$$

Equation (3.37) becomes

$$\int_0^{R_1} x_j^T D x_i \, dx_0 = \int_0^{R_1} \Delta^T D \Delta \, dx_0 =$$

$$\int_0^{R_1} m \left[(\kappa_1^2 + \kappa_2^2) \theta_i \theta_j + \Delta_{1_i} \Delta_{1_j} + \Delta_{2_i} \Delta_{2_j} \right] dx_0 \begin{array}{ll} = 0 & i \neq j \\ \neq 0 & i = j \end{array} \tag{3.44}$$

Equation (3.44) shows that the form of orthogonality condition (3.38) is unchanged under rotation. Equation (3.44) is the form used in the coupled modes module.

3.3 Reality of Eigenvalues and Eigenvectors

The assumption that eigenvectors x and eigenvalues ω^2 are real is proved below. It is recalled that this assumption was used to prove orthogonality of modes. Denote $\lambda = -\omega^2$. Denote complex conjugates by an overbar. Equation (3.21) yields the following forms

$$(A x'')'' + (B x')' + C x + \lambda D x = 0 \tag{3.45}$$

$$(A \bar{x}'')'' + (B \bar{x}')' + C \bar{x} + \bar{\lambda} D \bar{x} = 0 \tag{3.46}$$

Premultiply the first equation by $\bar{x}^T$ and integrate from 0 to R_1. We obtain

$$\int_0^{R_1} \left[\bar{x}^T (A x'')'' + \bar{x}^T (B x')' + \bar{x}^T C x + \lambda \bar{x}^T D x \right] dx_0 = 0 \tag{3.47}$$

Invoking boundary conditions as we did to derive (3.31) and noting that boundary conditions are real, we find

$$K_P \beta_{P_i} \beta_{P_j} + K_F \beta_{F_i} \beta_{F_j} + K_D \beta_{D_i} \beta_{D_j} + \int_0^{R_1} \left[\bar{x}^{T\prime\prime} A x'' - \bar{x}^{T\prime} B x' + \bar{x}^T C x + \lambda \bar{x}^T D x \right] dx_0 = 0 \tag{3.48}$$

Similarly a premultiplication of (3.46) by x^T yields

$$K_P \beta_{P_i} \beta_{P_j} + K_F \beta_{F_i} \beta_{F_j} + K_D \beta_{D_i} \beta_{D_j} + \int_0^{R_1} \left[x^{T\prime\prime} A \bar{x}'' - x^{T\prime} B \bar{x}' + x^T C \bar{x} + \bar{\lambda} x^T D \bar{x} \right] dx_0 = 0 \tag{3.49}$$

In a similar way to the derivation of (3.35) we find

$$\left(x^{T\prime\prime} A \bar{x}'' \right)^T = \bar{x}^{T\prime\prime} A x'' \tag{3.50}$$

Taking the transpose of the terms in (3.49), using (3.50) and similarly treating other terms, and subtracting the transpose of (3.49) from (3.48), we obtain

$$(\lambda - \bar{\lambda}) \int_0^{R_1} \bar{x}^T D x \, dx = 0 \tag{3.51}$$

The integral is not zero in general and consequently

$$\lambda = \bar{\lambda} \tag{3.52}$$

which proves the reality of the eigenvalues. Since matrices A, B, C, D are real in (3.45), the substitution of a real eigenvalue in (3.45) can only yield real eigenvectors. This shows that eigenvectors are real.

3.4 Derivation of the Modal Equation

In this section we illustrate the procedure for deriving the modal equations, which are single degree of freedom uncoupled differential equations, from the set of coupled differential equations (3.10), by invoking orthogonality condition (3.37). The procedure is the basis of the later treatment of the response equations whose purpose is to generate approximately uncoupled modal equations and thereby facilitate and stabilize the integration with time of system response.

Express the displacement in (3.10) as a sum over the number of modes of modal vectors x_i multiplied by modal amplitudes q_i.

$$x = \sum_i x_i q_i = \begin{Bmatrix} \theta_{e1} \\ w_{1} \\ v_{1} \end{Bmatrix} q_1 + \begin{Bmatrix} \theta_{e2} \\ w_{2} \\ v_{2} \end{Bmatrix} q_2 + \cdots \tag{3.53}$$

It is convenient to use the summation convention to contract the algebra.

$$x = x_i q_i \tag{3.54}$$

The appearance of a repeated subscript will always imply summation on that subscript in this report and this convention is extensively employed. We recall that here x_i is a 3 component vector and q_i is a scalar.

Substitute (3.54) in (3.10, premultiply by x_j^T and integrate from 0 to R_1. This yields

$$\int_0^{R_1} \Big[x_j^T (A x_i'')'' q_i + x_j^T (B x_i')' q_i + x_j^T C x_i q_i + x_j^T D x_i \ddot{q}_i \Big] dx_0 = 0$$

(3.55)

Repeat these steps with equation (3.21). This yields

$$\int_0^{R_1} \Big[x_j^T (A x_i'')'' q_i + x_j^T (B x_i')' q_i + x_j^T C x_i q_i - \omega^2 x_j^T D x_i q_i \Big] dx_0 = 0$$

(3.56)

Subtracting (3.56) from (3.55) yields

$$\int_0^{R_1} x_j^T D x_i \, dx_0 (\ddot{q}_i + \omega^2 q_i) = 0 \qquad (3.57)$$

or with

$$a_{ji} = \int_0^{R_1} x_j^T D x_i \, dx_0 \qquad (3.58)$$

we obtain

$$a_{ji} \ddot{q}_i + \omega^2 a_{ji} q_i = 0 \qquad (3.59)$$

Since

$$\begin{aligned} a_{ji} &= 0 \quad j \neq i \\ &\neq 0 \quad j = i \end{aligned} \qquad (3.60)$$

in view of (3.37), equation (3.59) is the desired uncoupled single degree of freedom modal equation.

If equation (3.10) had been

$$(A x'')'' + (B x')' + C x + D\ddot{x} = F \tag{3.61}$$

where F is a 3 component vector representing additional forces, like aerodynamic forces, the modal equation would become

$$a_{ji}\ddot{q}_i + \omega^2 a_{ji} q_i = Q_j \tag{3.62}$$

$$Q_j = \int_0^{R_1} x_j^T F \, dx_0 \tag{3.63}$$

Our approach to the equations to be used to find the response of the rotor is based on the objective of creating a form very similar to (3.62). This is the guiding principle of the subsequent derivation. After modal responses are known, physical displacements, velocities, and accelerations derive from equations similar to (3.54).

3.5 Comparison of Modal Equations

We show here that the set of modal equations of the subject method includes the flap and lead angle equations of Reference (1). Consequently, it is not necessary to derive separate equations expressing the equilibrium of hinge moments (the flap and lead angle equations of Reference (1)), as was done in Reference (1).

Consider a (first) mode which is essentially a rigid body mode. The eigenvector of modal displacements is

$$x_1^T \simeq 0, \beta_{F_1} r, 0 \tag{3.64}$$

$$= 0, r, 0 \tag{3.65}$$

following normalization to $\beta_{F_1} = 1$. The flap angle is

$$\beta_F = \beta_{F_i} q_i \tag{3.66}$$

(repeated suffix indicates summation on i over the modes). Flapping participation is small in the other modes and it follows

$$\beta_F \simeq \beta_{F_1} q_1 = q_1 \tag{3.67}$$

and substitution of (3.65) and (3.67) in (3.62) yields

$$I_B \ddot{\beta}_F + \omega^2 I_B \beta_F = Q_1 \tag{3.68}$$

$$I_B = \int_0^{R_1} m x_0^2 dx_0 \tag{3.69}$$

= blade moment of inertia.

This equation essentially duplicates the flap angle equation, 1-91 of Reference (1). It may be shown similarly that the lead angle equation, 1-96 of Reference (1), is essentially duplicated by a member of (3.62). Thus it is not necessary to derive separate equations expressing the equilibrium of hinge moments, corresponding to the flap and lead angle equations of Reference (1).

4. General Features of Modal Equation for Multi-Blade Rotor System

We discuss here general features of the blade modal equation applicable to the multi-blade rotor system, both to summarize our approach and to organize subsequent chapters, which are directed to the evaluation of terms in this general modal equation.

The differential equation satisfied by the motion of each blade of the multi-blade rotor is assumed to be

$$(A_1 x'')'' + (B_1 x')' + C_1 x + D_1 \ddot{x} = F \tag{4.1}$$

This equation may be verified for the multi-blade rotor by isolation of terms on the left-side from the loading equilibrium equation established in subsequent chapters.

The terms on the left-side are first order structural, inertial, and centrifugal terms that would occur also for a motion of the system in free vibration, considered in Chapter 3, and presented in (3.10). As a result, these terms will exhibit the desired orthogonality properties established for the modes of free vibration, enabling us to form a modal equation like (3.62). The terms on the right side of (4.1) contain all terms except those on the left-side. These terms include aerodynamic and concentrated applied loads, like those due to the lag damper and inertial loads, including Coriolis terms and terms arising from mass unbalance and non-coincidence of elastic centroid and elastic axis.

As a preliminary to the derivation of the equation of motion in modal coordinates, we state that a mode of free vibration satisfies

$$(A_2 x'')'' + (B_2 x')' + C_2 x - \omega^2 D_2 x = 0 \tag{4.2}$$

which is (3.21).

The different subscripts on the matrices in (4.1) and (4.2) distinguish different pitch angles. Subscript 1 signifies that the argument angle in A_1, B_1, C_1 and D_1 (B_1 and D_1 are not functions of pitch angle) is

$$\theta = \theta_c + \theta_t \tag{4.3}$$

with

$$\theta_c = \theta_{\frac{3}{4}R} + \theta_B \tag{4.4}$$

The angle θ_c, is the resultant local pitch, summed from the collective pitch input, $\theta_{3/4R}$, and built-in twist, θ_B. The angle θ_t is a time dependent contribution to pitch angle arising from cyclic control input, and pitch-flap and pitch-lag couplings, such as will be defined in a subsequent chapter.

The subscript 2 in (4.2) signifies that the argument angle in matrices A_2, B_2, C_2, and D_2, is the pitch angle used to find the modes of free vibration, which we shall designate θ_B^*. (The angle β in (3.21)).

The distinction between the pitch distribution used to find the response of the multi-blade rotor and the pitch distribution used to find the modes of free vibration is necessary because these distributions are different, in general. When cyclic inputs and pitch-flap and pitch-lag couplings occur θ cannot be the same as θ_B^* owing to the fact that θ embodies a time-dependent component θ_t while θ_B^* is time-invariant. Even in the case of no cyclic inputs and no pitch couplings, a requirement $\theta_B^* = \theta$ would lead to a recall of the natural frequency calculation module whenever collective angle $\theta_{3/4R}$ changes (such as in a "major iteration"), adversely increasing program execution time. To avoid the calculation of normal modes whenever $\theta_{3/4R}$ is changed, and to allow for time-dependent differences between θ and θ_B^*, the approach will be to distinguish between θ and θ_B^* and properly correct for the effects of these differences. It will be seen that the nature of the correction is an addition to the stiffness (or frequency) of the natural mode, accounting for pitch differences. It is assumed that the normal mode shapes are the same at θ and θ_B^*. This assumption is embodied in (4.5), given immediately below.

Proceeding now to derive the modal equation, we follow the steps of Section (3.4). We express the displacement vector $x^T = \theta_e, w_1, v_1$ as

$$x = x_i q_i \tag{4.5}$$

(see equations 3.53) and (3.54)), premultiply (4.1) and (4.2) by x_j^T, integrate their terms from 0 to r_T and subtract the resulting integrals. We obtain

$$\int_0^{r_T} \Big\{ x_j^T (A_1 - A_2) x_i'' q_i)'' + x_j^T (C_1 - C_2) x_i q_i + x_j^T D_1 x_i (\ddot{q}_i + \omega^2 q_i) \Big\} dr = \int_0^{r_T} x_j^T F dr \tag{4.6}$$

after substituting $B_1 = B_2$, $D_1 = D_2$. Equation (4.6) is re-written

$$a_{ji} \ddot{q}_i + (b_{ji} + \mu_{ji}) q_i = Q_j \tag{4.7}$$

$$a_{ji} = \int_0^{r_T} x_j^T D_1 x_i \, dr \tag{4.8}$$

$$b_{ji} = \omega^2 a_{ji} \tag{4.9}$$

$$\mu_{ji} = \int_0^{r_T} \Big\{ x_j^T ((A_1 - A_2) x_i'')'' + x_j^T (C_1 - C_2) x_i \Big\} dr \tag{4.10}$$

$$Q_j = \int_0^{r_T} x_j^T F \, dr \tag{4.11}$$

Equation (4.7) is the same as (3.62) except for the occurrence of the additional stiffness term μ_{ji}. In view of the orthogonality property of a_{ji}, described in (3.60), we can aver that with μ_{ji} and Q_j excluded, equation (4.7) would be an uncoupled single degree of freedom modal equation. The terms μ_{ji} amd Q_j couple the set of equations (4.7). An important assumption for the success of the time integration of the motion is that these couplings are always sufficiently weak to permit the generalized mass, a_{ji}, and stiffness, b_{ji}, to dominate the motion in modal coordinates, q_i, and hence to preserve essentially the uncoupled character of (4.7).

The interpretation of μ_{ji} is that this is a correction to the stiffness b_{ji} of the coupled modes of vibration, which accounts for differences in structural and centrifugal stiffnesses corresponding to the actual pitch θ and that used to find the modes, θ_B^*. These interpretations are amplified in a subsequent section of this chapter, where μ_{ji} is also reduced to working form.

To use (4.7) in our integration of the motion, we turn it into an equation in q_i, $\dot{q}_i$, and $\ddot{q}_i$ by expressing in Q_j displacements, velocities, and accelerations as modal sums. Typically, flap angle is $\beta = \beta_i q_i$ and vertical elastic displacement is $w = w_i q_i$. Next, (4.7) is re-arranged to place modal acceleration, $\ddot{q}_i$, on one side and modal amplitude, q_i, and velocities, $\dot{q}_i$, on the other side, and this form combined with time-integrations yields the history of motion.

To effect a form of (4.7) with all terms involving $\ddot{q}$ on one side, generalized force Q_j is broken into inertial, aerodynamic, and point loads components. The point loads are loads induced, for example, by the lag damper. With D, A, P denoting inertial, aerodynamic and point load forces this yields

$$Q_j = Q_j^D + Q_j^A + Q_j^P \qquad (4.1?)$$

The only additional source of modal acceleration terms, explicitly occurring in the modal equation, is Q_j^D. For example, terms involving $\ddot{q}$ can come from mass unbalance. The residue of Q_j^D after it is deprived of acceleration terms is written Q_j^{D*}. We then express

$$Q_j^D = Q_j^{D*} - s_{jk}^{(1)} \ddot{q}_k - s_{jk}^{(2)} \ddot{q}_k - s_{jk}^{(3)} \ddot{q}_k \tag{4.13}$$

where $s_{jk}^{(1)}$, $s_{jk}^{(2)}$, $s_{jk}^{(3)}$ are acceleration coefficients deriving from torsion, flatwise, and edgewise equations respectively, explained subsequently in this chapter. Replacing the dummy subscript i by k in (4.7), we re-express (4.7) as

$$s_{jk} \ddot{q}_k = t_j \tag{4.14}$$

$$s_{jk} = a_{jk} + s_{jk}^{(1)} + s_{jk}^{(2)} + s_{jk}^{(3)} \tag{4.15}$$

$$t_j = Q_j^{D*} + Q_j^A + Q_j^P - (b_{jk} + \mu_{jk}) q_k \tag{4.16}$$

Letting M = number of modes, and

$$S = [s_{jk}] \qquad -M \times M \tag{4.17}$$

$$T = \{t_j\} \qquad -M \times 1 \tag{4.18}$$

$$q = \{q_k\} \qquad -M \times 1 \tag{4.19}$$

the matrix equivalent to (4.14) reduces to

$$\ddot{q} = S^{-1} T \qquad M \times 1 \tag{4.20}$$

and this is the desired form for the modal acceleration. In keeping with the expected essential uncoupled character of (4.7), matrix S, although fully populated, will be diagonally dominant.

Modal velocities and amplitudes derive respectively from

$$\dot{q}_i(t+\Delta t) = \dot{q}_i(t) + \ddot{q}_i(t)\Delta t \tag{4.21}$$

$$q_i(t+\Delta t) = q_i(t) + \dot{q}_i(t)\Delta t + \ddot{q}_i(t)\Delta t^2 \tag{4.22}$$

User experience with rotor response equations employing uncoupled modes suggests that (4.22) is superior to a Taylor series, and this is the justification for its use here.

Equation (4.20) is the target of the subsequent text which seeks to derive S and T in (4.20), and which is organized in an orderly manner to do this.

4.1 General Expressions for Generalized Forces

To aid the rational organization of the work required to form the elements of the modal acceleration equation, (4.20), it is of value to break the generalized force into terms of distinct type, namely inertial, aerodynamic, and concentrated load. This section lists the conventions used subsequently to define the components of generalized force.

The generalized force in the modal space, Q_j, is

$$Q_j = \int_0^{r_T} x_j^T F \, dr \tag{4.23}$$

$$= \int_0^{r_T} \left[\theta_j F_1 + w_{1j} F_2 + v_{1j} F_3 \right] dr \tag{4.24}$$

$$\equiv Q_j^{(1)} + Q_j^{(2)} + Q_j^{(3)} \tag{4.25}$$

$$Q_j^{(1)} = \int_0^{r_T} \theta_j F_1 \, dr \tag{4.26}$$

$$Q_j^{(2)} = \int_0^{r_T} w_{1j} F_2 \, dr \tag{4.27}$$

$$Q_j^{(3)} = \int_0^{r_T} v_{1j} F_3 \, dr \tag{4.28}$$

Element $Q_j^{(1)}$ is the work done by torsion moments (per unit length of blade) moving through the torsional component of virtual displacement, θ_j, of the j-th mode. Similarly $Q_j^{(2)}$ and $Q_j^{(3)}$ are the works done by vertical and inplane forces per unit length moving through vertical and inplane displacements, w_{1j}, and v_{1j}, of the j-th mode. Element $Q_j^{(1)}$ may be thought of as a generalized torsion moment applied to the entire blade, and $Q_j^{(2)}$ and $Q_j^{(3)}$ are generalized hinge moments. It is recalled that the forces F_1, F_2, F_3 embody inertial forces in addition to aerodynamic and concentrated loads forces, and these forces are not to be construed as only of external origin.

Decomposition of forces into inertial, aerodynamic and concentrated load components is effected by the following definitions.

$$F_1 = F_1^D + F_1^A + F_1^P \tag{4.29}$$

$$F_2 = F_2^D + F_2^A + F_2^P \tag{4.30}$$

$$F_3 = F_3^D + F_3^A + F_3^D \tag{4.31}$$

Superscripts D, A, and P designate terms of inertial, aerodynamic, and concentrated load origin, respectively. The generalized force components may then be written

$$Q_j^{(1)} = Q_j^{D_1} + Q_j^{A_1} + Q_j^{P_1} \tag{4.32}$$

$$Q_j^{(2)} = Q_j^{D_2} + Q_j^{A_2} + Q_j^{P_2} \tag{4.33}$$

$$Q_j^{(3)} = Q_j^{D_3} + Q_j^{A_3} + Q_j^{P_3} \tag{4.34}$$

where

(4.35)

$$Q_j^{D_1} = \int_0^{r_T} \theta_j F_1^D dr$$

$$Q_j^{D_2} = \int_0^{r_T} w_{ij} F_2^D dr$$

$$Q_j^{D_3} = \int_0^{r_T} v_{ij} F_3^D dr$$

$$Q_j^{A_1} = \int_0^{r_T} \theta_j F_1^A dr$$

$$Q_j^{A_2} = \int_0^{r_T} w_{ij} F_2^A dr$$

$$Q_j^{A_3} = \int_0^{r_T} v_{ij} F_3^A dr$$

$$Q_j^{P_1} = \int_0^{r_T} \theta_j F_1^P dr$$

$$Q_j^{P_2} = \int_0^{r_T} w_{ij} F_2^P dr$$

$$Q_j^{P_3} = \int_0^{r_T} v_{ij} F_3^P dr$$

The interpretation of a typical term is that Q_j^{D2} is the contribution to the generalized force by the vertical inertial loads, F_2^D, moving through the vertical displacement component, w_{1j}, of the j-th mode. The term Q_j^{A2} is a similar contribution deriving from aerodynamic forces.

In view of (4.12), (4.25) and (4.32) to (4.34) we may express inertial, aerodynamic, and point loads contributions to generalized force Q_j as sums of contributions from torsion, vertical, and inplane forces.

$$Q_j^D = Q_j^{D_1} + Q_j^{D_2} + Q_j^{D_3} \tag{4.36}$$

$$Q_j^A = Q_j^{A_1} + Q_j^{A_2} + Q_j^{A_3} \tag{4.37}$$

$$Q_j^P = Q_j^{P_1} + Q_j^{P_2} + Q_j^{P_3} \tag{4.38}$$

The residues of Q_j^{D1}, Q_j^{D2}, and Q_j^{D3} after they are deprived of modal accelerations are written Q_j^{D1*}, Q_j^{D2*}, and Q_j^{D3*}, and the inertial generalized forces may be written

$$Q_j^{D_1} = Q_j^{D_1 *} - S_{jk}^{(1)} \ddot{q}_k \tag{4.39}$$

$$Q_j^{D_2} = Q_j^{D_2 *} - S_{jk}^{(2)} \ddot{q}_k \tag{4.40}$$

$$Q_j^{D_3} = Q_j^{D_3 *} - S_{jk}^{(3)} \ddot{q}_k \tag{4.41}$$

It is seen from these definitions that $s_{jk}^{(1)}$, $s_{jk}^{(2)}$, and $s_{jk}^{(3)}$ are acceleration coefficients whose sources are torsion, vertical, and inplane forces, F_1^D, F_2^D, F_3^D, respectively. The repeated suffix k indicates summation on the number of modes. Defining

$$Q_j^{D*} = Q_j^{D_1 *} + Q_j^{D_2 *} + Q_j^{D_3 *} \tag{4.42}$$

and substituting (4.39) to (4.41) in (4.36), we obtain

$$Q_j^D = Q_j^{D*} - s_{jk}^{(1)} \ddot{q}_k - s_{jk}^{(2)} \ddot{q}_k - s_{jk}^{(3)} \ddot{q}_k \tag{4.43}$$

which is (4.13).

Because the forces F_1, F_2, and F_3 contain a large number of terms, which will appear in subsequent development, reduction of execution time becomes a worthwhile goal. Such a reduction can be achieved by distinguishing the uncoupled character of the torsion mode, in the formation of generalized force Q_j. Recognizing that

$$Q_j = Q_j^{(1)} \tag{4.44}$$

for a torsion mode, and

$$Q_j = Q_j^{(2)} + Q_j^{(3)} \tag{4.45}$$

for a coupled flatwise-edgewise mode, we indicate a means for effecting a reduction of computational effort needed to form Q_j.

4.2 Correction to Modal Frequency to Account for Pitch Differences

In Section 4.0 we showed that a consistent derivation of the modal equation, employing coupled assumed modes calculated for a pitch distribution $\theta_{\S}$, gave rise to a correction μ_{jk} to the modal stiffness b_{jk} to account for the change in pitch to θ during the motion. This correction may be viewed also as a modal frequency correction. This section describes the derivation of the working form of μ_{jk}.

The expression for μ_{jk}, equation (4.10), is the starting point for the derivation of the working form.

$$\mu_{jk} = \int_0^{r_T} \left\{ x_j^T \left[(A_1 - A_2) x_k'' \right]'' + x_j^T (C_1 - C_2) x_k \right\} dr \tag{4.46}$$

Using identities $EI_y = EI_1$, $EI_z = EI_2$, $k_{z10} = k_2$, $k_{y10} = k_1$, we obtain from (3.10)

(4.47)

$$A_1 = \begin{matrix} 0 & 0 & 0 \\ 0 & EI_y C_\theta^2 + EI_z S_\theta^2, & (EI_y - EI_z) S_\theta C_\theta \\ 0 & (EI_y - EI_z) S_\theta C_\theta, & EI_y S_\theta^2 + EI_z C_\theta^2 \end{matrix}$$

(4.48)

$$A_2 = \begin{matrix} 0 & 0 & 0 \\ 0 & EI_y C_{\theta_B^*}^2 + EI_z S_{\theta_B^*}^2, & (EI_y - EI_z) S_{\theta_B^*} C_{\theta_B^*} \\ 0 & (EI_y - EI_z) S_{\theta_B^*} C_{\theta_B^*}, & EI_y S_{\theta_B^*}^2 + EI_z C_{\theta_B^*}^2 \end{matrix}$$

(4.49)

$$C_1 = \begin{matrix} m\Omega^2({}_{z_{10}}^2 - K_{y_{10}}^2) C_{2\theta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m\Omega^2 \end{matrix}$$

(4.50)

$$C_2 = \begin{matrix} m\Omega^2(K_{z_{10}}^2 - K_{y_{10}}^2) C_{2\theta_B^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m\Omega^2 \end{matrix}$$

By means of (3.24) we can show

$$\int_0^{r_T} x_j^T\left[(A_1-A_2)x_K''\right]'' dr = \left[x_j^{T'}(A_1-A_2)x_K''\right]_0 + \int_0^{r_T} x_j^{T''}(A_1-A_2)x_K'' dr \qquad (4.51)$$

From (3.19)

$$(x_K'')_0 = (A_2^{-1})_0 y \qquad (4.52)$$

$$y^T = 0,\ K_F\beta_K,\ K_D\delta_K \qquad (4.53)$$

where β and δ signify flap and lag angles (replacing the notation β_F and β_D of Chapter 3). Equation (4.51) becomes

$$\int_0^{r_T} x_j^T\left[(A_1-A_2)x_K''\right]'' dr = \left[x_j^{T'}(A_1A_2^{-1}-I)y\right]_0 + \int_0^{r_T} x_j^{T''}(A_1-A_2)x_K'' dr \qquad (4.54)$$

With

$$(x_j^T)_0' = (\theta_j)_0',\ \beta_j,\ \delta_j \qquad (4.55)$$

equation (4.54) reduces to

$$\int_0^{r_T} x_J{}^T \left[(A_1 - A_2) x_K'' \right]'' dr =$$

$$K_F \beta_K \beta_j a_{11}^* + K_D \delta_K \beta_J a_{12}^* + k_F \beta_K \delta_j a_{21}^* + K_D \delta_K \delta_j a_{22}^* + \int_0^{r_T} x_J{}^{T''} (A_1 - A_2) x_K'' dr$$

(4.56)

with

(4.57)

$$a_{11}^* = \left\{ \frac{1}{EI_z EI_y} \left[(EI_y c_\theta^2 + EI_z s_\theta^2)(EI_y s^2_{\theta_B *} + EI_z c^2_{\theta_B *}) - (EI_z - EI_y)^2 s_\theta c_\theta c_{\theta_B *} s_{\theta_B *} \right] \right\}^{-1}_{r=0}$$

$$a_{12}^* = \left\{ \frac{1}{EI_z EI_y} \left[(EI_z - EI_y) s_\theta c_\theta (EI_y s^2_{\theta_B *} + EI_z c^2_{\theta_B *}) - (EI_y s_\theta^2 + EI_z c_\theta^2)(EI_z - EI_y) s_{\theta_B *} c_{\theta_B *} \right] \right\}_{r=0}$$

$$a_{21}^* = \left\{ \frac{1}{EI_z EI_y} \left[-(EI_y c_\theta^2 + EI_z s_\theta^2)(EI_z - EI_y) s_{\theta_B *} c_{\theta_B *} + (EI_z - EI_y) s_\theta c_\theta (EI_y c^2_{\theta_B *} + EI_z s^2_{\theta_B *}) \right] \right\}_{r=0}$$

$$a_{22}^* = \left\{ \frac{1}{EI_z EI_y} \left[-(EI_z - EI_y)^2 s_\theta c_\theta s_{\theta_B *} c_{\theta_B *} + (EI_y s_\theta^2 + EI_z c_\theta^2)(EI_y c^2_{\theta_B *} + EI_z s^2_{\theta_B *}) \right] \right\}^{-1}_{r=0}$$

The integral in (4.56) is

$$\int_0^{r_T} x_j^{T''}(A_1 - A_2)\, x_k''\, dr =$$

$$\int_0^{r_T} \Big\{ w_{1_j}''\left[EI_y(c_\theta^2 - c_{\theta_\beta *}^2) + EI_z(s_\theta^2 - s_{\theta_\beta *}^2)\right] w_{1_k}''$$
$$+ w_{1_j}''(EI_z\ EI_y)(s_\theta c_\theta - s_{\theta_\beta} * c_{\theta_\beta} *)\, v_{1_k}''$$
$$+ v_{1_j}''(EI_z - EI_y)(s_\theta c_\theta - s_{\theta_\beta} * c_{\theta_\beta} *)\, w_{1_k}''$$
$$+ v_{1_j}''\left[EI_y(s_\theta^2 - s_{\theta_\beta} *) + EI_z(c_\theta^2 - c_{\theta_\beta}^2 *)\right] v_{1_k}'' \Big\}\, dr$$

(4.58)

The second term in (4.46) is

$$\int_0^{r_T} x_j^T(C_1 - C_2)\, x_k\, dr =$$

$$\int_0^{r_T} m\Omega^2(K_{z_{10}}^2 - K_{y_{10}}^2)(c_{2\theta} - c_{2\theta_\beta} *)\, \theta_j\, \theta_k\, dr$$

(4.59)

Substitution of (4.56) to (4.59) in (4.46) yields the expression for μ_{jk}.

From a computational viewpoint, (4.58) and (4.59) are not efficient because the presence of a time-dependent part θ_t in θ (= $\theta_c + \theta_t$) would require these blade integrations to be performed at each increment of time of the motion. To permit employment of a more efficient calculation of μ_{jk}, integrals are developed below, independent of time, and which need to be evaluated only once before the motion is calculated.

Expressing

$$\theta = \theta_c + \theta_t \tag{4.60}$$

we assume θ_t to be small, specifically of the order of the small quantities, like c.g. offset, and section radii of gyration, and we neglect high order terms contributed by $\mu_{jk} q_k$ to the modal equation of motion. To exploit the different orders of approximations assumed for the torsion loading equilibrium equation in comparison with the flatwise-edgewise loading equilibrium equations, we recognize that the uncoupled character of the torsion mode transforms these approximations into equivalent approximations in the modal space. In the torsion modal equation

$$a_{jk} \ddot{q}_k + (b_{jk} + \mu_{jk}) q_k = Q_j^{(1)} \tag{4.61}$$

$$a_{jk} = \int_0^{r_T} m \left(K_{y_{10}}^2 + K_{z_{10}}^2 \right) \theta_j \theta_k \, dr \tag{4.62}$$

$$b_{jk} = \omega^2 a_{jk}$$

$$\mu_{jk} = \int_0^{r_T} m \Omega^2 \left(K_{z_{10}}^2 - K_{y_{10}}^2 \right) \left(c_{2\theta} \cdot c_{2\theta_\beta *} \right) dr \tag{4.63}$$

(4.64)

$$Q_j^{(1)} = \int_0^{r_T} \theta_j F_1 \, dr$$

we neglect third order products involving elastic variables, and fourth order products of small quantities in $\mu_{jk}q_k$ following the expansion $\theta = \theta_c + \theta_t$. The flatwise-edgewise modal equation

$$a_{jk}\ddot{q}_k + (b_{jk} + \mu_{jk}) q_k = Q_j^{(2)} + Q_j^{(3)} \tag{4.65}$$

is approximated to exclude second order products of elastic variables, and third order products of small quantities, where it is understood that a_{jk}, b_{jk}, and μ_{jk} apply for a zero torsion component θ_j in the flatwise-edgewise mode, j.

The result of these approximations is

$$\begin{aligned} \mu_{jk} &= K_F \beta_k \beta_j a_{11}^* + K_D \delta_k \beta_j a_{12}^* \\ &\quad + K_F \beta_k \delta_j a_{21}^* + K_D \delta_k \delta_j a_{22}^* \\ &\quad + \mu_{jk}^* \end{aligned} \tag{4.66}$$

$$\mu_{JK}^* = (\mu_{jk}^*)_0 + \theta_t (\mu_{jk}^*)_1 \tag{4.67}$$

$$\begin{aligned}(\mu_{jk}^*)_0 = \int_0^{r_T} \Big\{ & m\Omega^2 (k^2_{z_{1.0}} - k^2_{y_{1.0}})(c_{2\theta_B}*)\,\theta_j\,\theta_k \\ & + w_{1_j}'' \left[EI_y (c^2_{\theta_c} - c^2_{\theta_B}*) + EI_z (s^2_{\theta_c} - s^2_{\theta_B}*) \right] w_{1K}'' \\ & + w_{1_j}'' (EI_z - EI_y)(s_{\theta_c} c_{\theta_c} - s_{\theta_B}* c_{\theta_B}*)\, v_{1_K}'' \\ & + v_{1_j}'' (EI_z - EI_y)(s_{\theta_c} c_{\theta_c} - s_{\theta_B}* c_{\theta_B}*)\, w_{1_K}'' \\ & + v_{1_j}'' \left[EI_y (s^2_{\theta_c} - s^2_{\theta_B}*) + EI_z (c^2_{\theta_c} - c^2_{\theta_B}*) \right] v_{1_K}'' \Big\} dr \end{aligned}$$

$$\begin{aligned}(\mu_{jk}^*)_1 = \int_c^{r_T} (EI_z - EI_y) \Big[& w_{1_j}'' s_{2\theta_c} w_{1_K}'' + w_{1_j}'' c_{2\theta_c} v_{1_K}'' \\ & + v_{1_j}'' c_{2\theta_c} w_{1_K}'' - v_{1_j}'' s_{2\theta_c} v_{1_K}'' \Big] dr \end{aligned} \tag{4.68}$$

$$a_{11}^* = (a_{11}^*)_0 + \theta_t (a_{11}^*)_1$$

$$a_{12}^* = (a_{12}^*)_0 + \theta_t (a_{12}^*)_1$$

$$a_{21}^* = (a_{21}^*)_0 + \theta_t (a_{21}^*)_1$$

$$a_{22}^* = (a_{22}^*)_0 + \theta_t (a_{22}^*)_1$$

Coefficients $(a_{11}^*)_0$ to $(a_{22}^*)_0$ are equal to (4.57) with θ_c substituted for θ, and $(a_{11}^*)_1$ to $(a_{22}^*)_1$ are equal to $(\partial a_{11}^*/\partial \theta)_{\theta=\theta_c}$, etc. Appendix 14.1 contains the explicit expressions for the latter coefficients.

Recognizing that (Ax")" are the internal resisting loads due to vertical and inplane bending, the first term in (4.46) contributing to μ_{jk} is seen to be a correction to the modal stiffness, b_{jk} (b_{ji} in (4.7)), for differences in bending stiffnesses at the actual pitch, θ, and the pitch, θ_B^*, used to find the normal modes. Equations (4.56) and (4.58) show explicitly the effects of pitch differences on the bending stiffness correction, μ_{jk}. Equation (4.59) shows that the second term in (4.46) similarly corrects for the difference between centrifugal stiffenings in pitch.

4.3 Addition of Structural Damping to the Modal Equation

To represent the effects of structural damping in any mode, modal equation (4.7) is replaced by the equation

$$a_{jk}\ddot{q}_k + 2c\, c_{jk}\dot{q}_k + (b_{jk} + \mu_{jk})q_k = Q_j \tag{4.69}$$

c = structural damping coefficient

$$c_{jk} = \int_0^{r_T} m\left\{(K_{y_{10}}^2 + K_{z_{10}}^2)\phi_j\phi_k + w_j w_k + v_j v_k\right\} dr \tag{4.70}$$

We show below that this form has the desired properties of structural damping. First it is emphasized that ϕ, w, v are elastic displacements and, consequently are negligible in essential rigid body motions, either pitching, flapping, or leading, with the result that c is negligible in essential rigid body modes of blade motion and the structural damping correctly disappears for such motions. In essential elastic modes with weak rigid body participation

$$a_{jk} = \int_0^{r_T} m \left\{ (k^2_{z_{10}} + k^2_{y_{10}}) \theta_{,j} \theta_{,k} + w_{,j} w_{,k} + v_{,j} v_{,k} \right\} dr$$

$$\simeq \int_0^{r_T} m \left\{ (k^2_{z_{10}} + k^2_{y_{10}}) \phi_j \phi_k + w_j w_k + v_j v_k \right\} dr$$

$$= c_{jk}$$

(4.71)

taking into account the expressions for θ_e, w_1 and v_1 in (3.6). In the free vibration of such an essential elastic mode, (4.69) reduces to the single degree of freedom equation

$$a_{jk} \ddot{q}_k + 2 c a_{jk} \dot{q}_k + \omega^2 a_{jk} q_k = 0 \quad (4.72)$$

in view of the properties of a_{jk}, (3.60), and the damping term is the conventional viscous damping equivalent to structural damping, thereby justifying the term $2cc_{jk}\dot{q}_k$ in (4.69).

A more tractable form for the damping term is one employing the critical damping ratio g^* (twice the value of c/c_{CRIT}),

$$g^* = 2c/c_{CRIT} \quad (4.73)$$

$$c_{CRIT} = \omega \quad (4.74)$$

$$g^* = \frac{2c}{\omega} \quad (4.75)$$

and (4.69) becomes

$$a_{jk} \ddot{q}_k + g^* \omega c_{jk} \dot{q}_k + (b_{jk} + \mu_{jk}) q_k = Q_j \quad (4.76)$$

No provision is made in the program for input of different modal damping ratios, and only one value of g* may be loaded to represent all modes.

With structural damping included element t_j in (4.16) becomes

$$t_j = Q_j^{O*} + Q_j^{A} + Q_j^{P} - (b_{jk} + \mu_{jk}) q_k - g^* \omega c_{jk} \dot{q}_k \tag{4.77}$$

and this is the current form in the program.

4.4 Organization of Subsequent Chapters

The subsequent text is directed to the finding of the S and T matrices in (4.20), and this section describes the organization of this work.

We have already found contributions a_{jk}, b_{jk} and μ_{jk} to the elements s_{jk} and t_j of the S and T matrices. (s_{jk} is defined in (4.15) and t_j is defined in (4.77)). The remaining effort aims at forming Q_j which will complete the s_{jk} and t_j coefficients.

The first step taken is the selection of the generalized coordinates appearing in the equation of motion in physical space, (4.1). These generalized coordinates are the translations and rotations of rectangular axes defining the displacement of a point on the blade. A chapter is allocated to the definitions of rectangular axes.

This is followed by a chapter yielding general expressions for the generalized forces in physical space, F_1, F_2, and F_3 required for the formation of the generalized force in modal space, Q_j ((4.23) or (4.24)). These general expressions for F_1, F_2, and F_3 involve blade element loads, which subsequent chapters are dedicated to finding. Applications of rectangular axis transformations defined in Chapter (5) come into play here to calculate bending strains and bending reaction loads entering moment and loading equilibrium equations. This ground is covered both to establish F_1, F_2 and F_3 and to emphasize that the scope of the analysis embraces large pitch angle and twist rate terms without restrictions.

Distinct chapters are allocated to the formation of inertial, Q_j^D, aerodynamic, Q_j^A, and point loads generalized forces, Q_j^P.

To derive Q_j^D, rectangular axis transformations are used again successively to derive blade accelerations and these yield inertial loads for substitution in the F_1^D, F_2^D and F_3^D expressions.

We then derive aerodynamic forces and relative flow velocities on which these forces depend to form Q_j^A, and this is followed by formation of the point loads generalized force Q_j^P.

5. Rectangular Axes Defining Blade Displacement

We define here the rectangular axes employed for the description of the displacement of a point on the rotor blade, including elastic deformation. Clear definitions of these axes are important because the axis transformations, comprising linear displacements and rotations, are the generalized coordinates in the equations of motion in physical space.

We restrict the text here to the description of each individual axis which we define by means of a transformation matrix giving the position and orientation of the axis in terms of a preceding axis. Successive multiplications of transformation matrices to obtain formulas for displacement, acceleration, aerodynamic velocities, and force components occur in the text of chapters following this one.

A complete description is given here of all coordinates needed in the blade equations of motion. We do not describe the support system generalized coordinates used for the rotor with grounded support, or for the rotor coupled to a rigid body in free flight which are described adequately in References (3) and (2), respectively. It is well to understand that these coordinates together with the blade generalized coordinates are the complete set of coordinates for the coupled problem, and that the present chapter is insufficient as a description of the complete set of generalized coordinates.

Our blade coordinates resemble closely those used in the analysis of rotor blade response employing uncoupled assumed modes (Reference (1)). Similarities and differences are pointed out in the text.

As a preliminary, we note that a compact presentation of definitions of rectangular axes is achieved with matrix notation. A column vector of coordinates is expressed as a capital letter with an overbar. For example, $\overline{X}^T = x, y, z$. Transformation matrices relating one set of coordinates to another are denoted with capital A and are subscripted to show the rotation angle. A typical rotation transformation matrix is

$$A_\beta = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix} \tag{5.1}$$

The symbol $A_{-\beta}$ indicates that the rotation angle argument is $-\beta$. The relation $A_\beta^{-1} = A_{-\beta}$ is extensively used in the text of subsequent chapters, and the subscript notation is valuable both for specificness and for easily determining the inverses required in the derivation. An example of a transformation defining the relation between $\overline{X}_4$ and $\overline{X}_5$ rectangular axes is

$$\overline{X}_4 = A_\beta \overline{X}_5 \tag{5.2}$$

which is equivalent to

$$\begin{aligned} x_4 &= c_\beta x_5 - s_\beta z_5 \\ y_4 &= y_5 \\ z_4 &= s_\beta x_5 + c_\beta z_5 \end{aligned} \tag{5.3}$$

Axis Definitions

The ultimate reference is the stationary inertial rectangular coordinate axis, fixed to the ground, whose vector of coordinates is $\overline{X}_I$, Figure 4. Two different shaft oriented axes are used to treat respectively the rotor with grounded support, and rotor coupled to a rigid body in free flight.

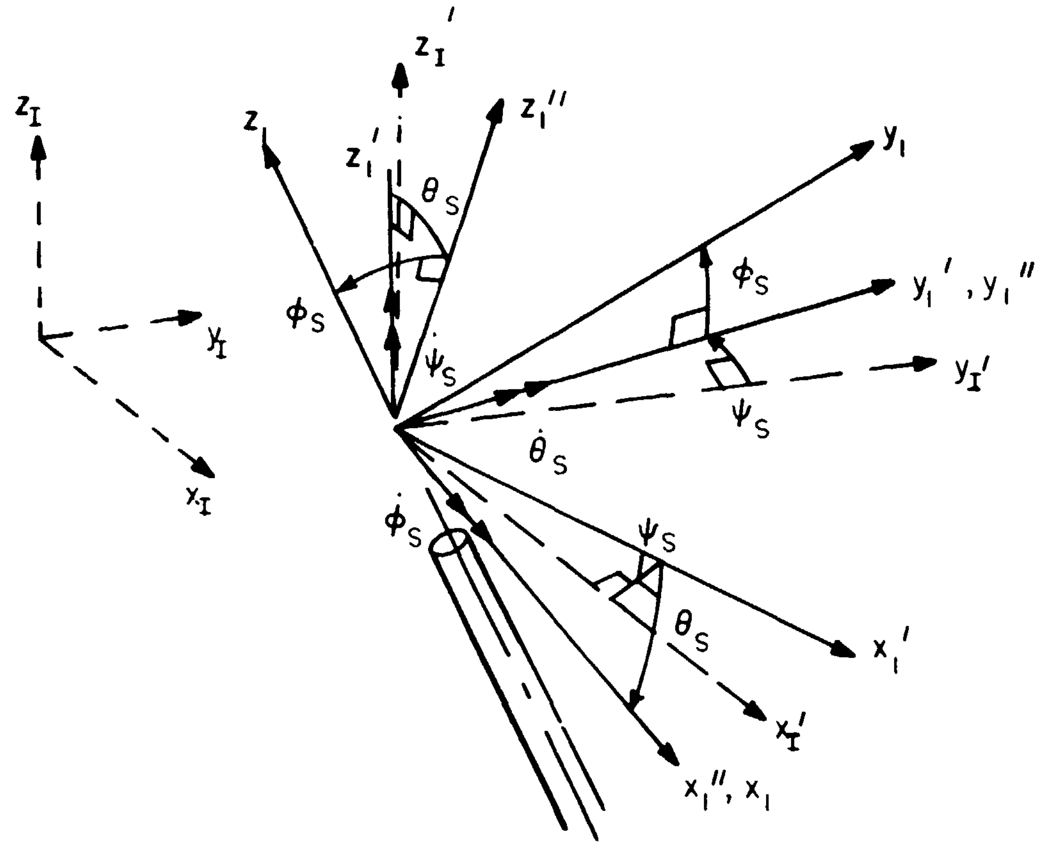


Figure 4. Shaft-Oriented Axes for Analysis of Multi-Blade Rotor With Grounded Support.

1) Grounded Support Shaft Axes, $\bar{X}_1$

Figure 4 illustrates the shaft axes $\bar{X}_1$ used for the analysis of the multi-blade rotor with grounded support. The $\bar{X}_1$ system is reached from one translation and three successive rotations ψ_s, θ_s, ϕ_s, defined by the following transformations

$$\bar{X}_1 = A_{-\phi_s} \bar{X}_1'' \tag{5.4}$$

$$\bar{X}_1'' = A_{-\theta_s} \bar{X}_1' \tag{5.5}$$

$$\bar{X}_1' = A_{-\psi_s} \bar{X}_I' \tag{5.6}$$

$$\bar{X}_I' = \bar{X}_I - \bar{X}_{I_O} \tag{5.7}$$

The $\bar{X}_1$ system is identical to the shaft axes of Reference (1). Because the hub is generally accelerated, $\bar{X}_I'$ is not inertial. Angles θ_s, ϕ_s, ψ_s are the same as those used in Reference (3) for the multi-blade rotor with grounded support. X_{IO} is the location of the hub measured from the X_I origin. Rotation transformations $A_{-\phi_s}$, $A_{-\theta_s}$, $A_{-\psi_s}$ are listed in section 5.1, as are the other rotation transformations given below.

2) Rigid Body in Free Flight Shaft Axes, $\bar{\xi}_1$

Figure 5 illustrates shaft axes $\bar{\xi}_1$ used for the multi-blade rotor coupled to a rigid body in free flight. One translation and three successive rotations yield

$$\bar{\xi}_1 = A_{-\phi'} \bar{\xi}_1'' \tag{5.8}$$

$$\bar{\xi}_1'' = A_{-\theta'} \bar{\xi}_1' \tag{5.9}$$

$$\bar{\xi}' = A_{-\chi'} \bar{\xi}_I' \tag{5.10}$$

$$\bar{\xi}_I' = (A_{\theta'})_{\theta'=\pi} (\bar{X}_I - \bar{X}_{I_O}) \tag{5.11}$$

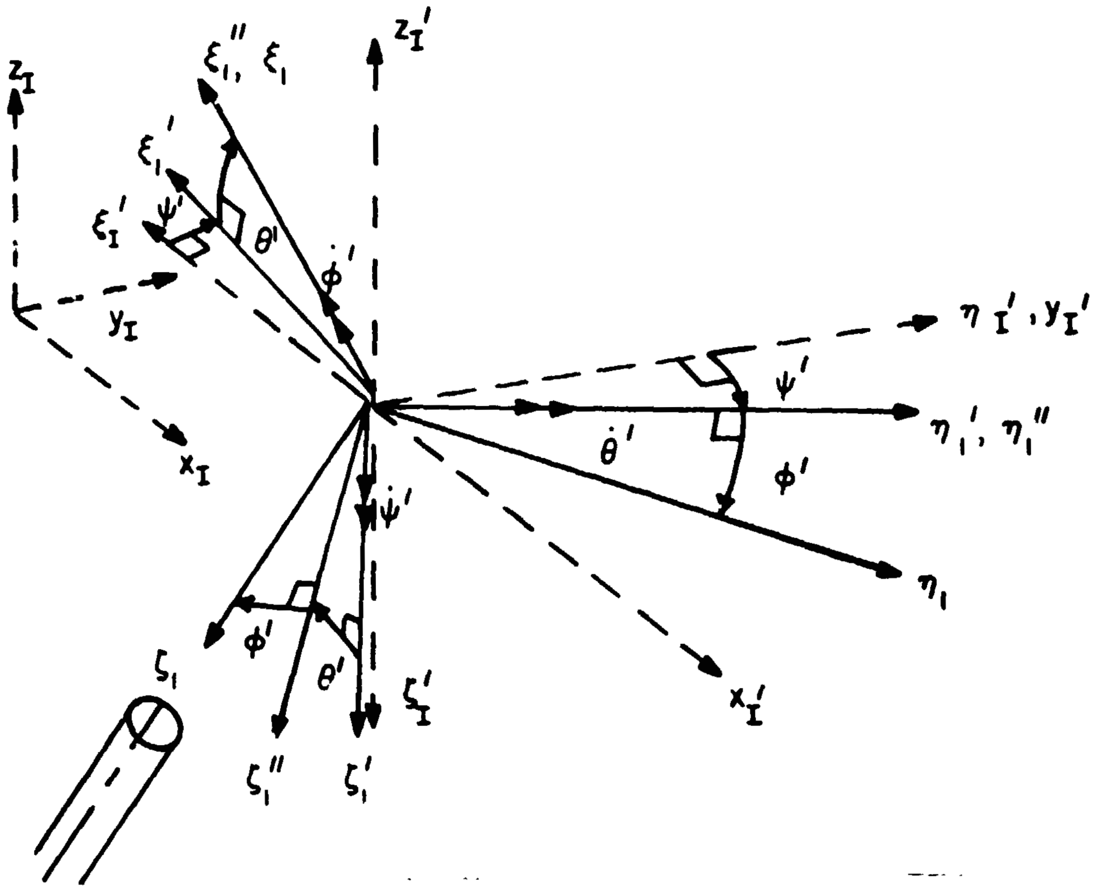


Figure 5. Shaft-Oriented Axes for Analysis of Multi-Blade Rotor Coupled to Rigid Body in Free Flight.

Figure 6 shows the relation between shaft axis $\bar{\xi}_1$ and shaft axis $\bar{X}_1$. The same relation applies between $\bar{X}_I'$ and $\bar{\xi}_I'$. In algebraic terms

$$\bar{X}_1 = (A_{-\theta'})_{\pi} \bar{\xi}_1 \qquad (5.12)$$

$$\bar{X}_I' = (A_{-\theta'})_{\pi} \bar{\xi}_I' \qquad (5.13)$$

Angles θ', ϕ', ψ' defining the altitude of the rigid body are the same as those of Reference (2).

3) <u>Rigid Blade Axes</u>, $\bar{X}_1$ to $\bar{X}_6$

Figure 7 shows these axes. They are described as rigid blade axes because they would yield the displacements of a point on a rigid blade involving azimuth, ψ, hinge offset, e, radial position, r, and rigid blade displacements δ, β, θ, only. These axes are identical to axes with the same names used for the rotor response analysis employing uncoupled assumed modes of Reference (1). They superimpose displacements on those due to shaft displacements and are defined by

$$\bar{X}_1 = A_{\psi} \bar{X}_2 \qquad (5.14)$$

$$\bar{X}_2 = \bar{E} + \bar{X}_3 \qquad (5.15)$$

$$\bar{X}_3 = A_{\delta} \bar{X}_4 \qquad (5.16)$$

$$\bar{X}_4 = A_{\beta} \bar{X}_5 \qquad (5.17)$$

$$\bar{X} = \bar{X}_5 \qquad (5.18)$$

$$= A_{\theta} \bar{X}_6 + \bar{R} \qquad (5.19)$$

$$\bar{E}^T = e, 0, 0 \qquad (5.20)$$

$$\bar{R}^T = r, 0, 0$$

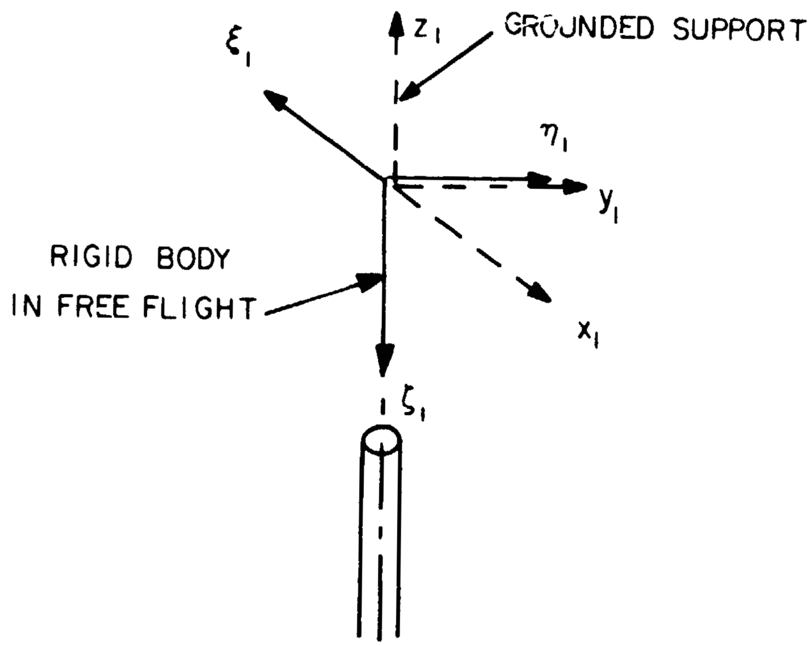


Figure 6. Relationship Between Shaft Axes Used for Grounded Support and Rigid Body in Free Flight.

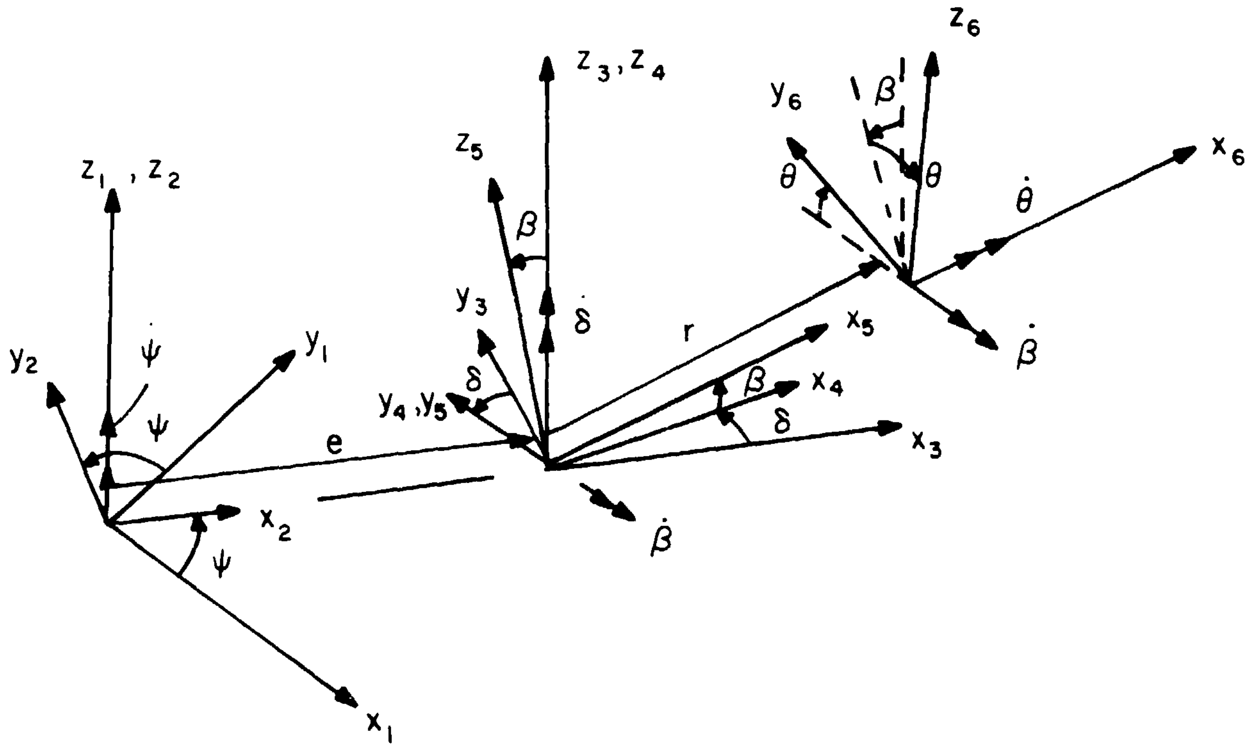


Figure 7. Rigid Blade Displacements for Multi-Blade Rotor Analysis.

4) <u>Blade Elastic Displacements</u>, $\bar{X}_7$ to $\bar{X}_{10}$

Figure 8 shows the axes superposing elastic displacements on all the displacements described above. Elastic displacements are $\overline{W}_e^T = u^*, v_e, w_e$ or alternatives $\overline{W}^T = u^*, v, w$, and elastic twist θ_e. Displacements v_e and w_e, or v and w, cause angular deformations of the blade λ_1 and λ_2 which are expressed in terms of v_e and w_e or v and w in section 5.2. In addition a displacement $\overline{W}^T_9 = 0, v_9, w_9$ accompanies the elastic rotation θ_e of a bent elastic axis, as shown in section 5.3. These $\overline{W}_9$ displacments are shown in the text of subsequent chapters to be equal in importance to terms due to w, v, θ_e alone and are justifiably included.

The first difference between the axes used here and those used for the analysis of Reference (1) employing uncoupled assumed modes is that the new $\bar{X}_7$ axis differs from the uncoupled modes $\bar{X}_{10}$ axis of Reference (1) by a translation $\overline{W}_9$ from the $\bar{X}_9$ axis. The axes are defined by

$$\bar{X}_6 = \bar{X}_7 + \overline{W}_e \tag{5.21}$$

$$\bar{X}_7 = A_{\lambda_2} \bar{X}_8 \tag{5.22}$$

$$\bar{X}_8 = A_{\lambda_1} \bar{X}_9 \tag{5.23}$$

$$\bar{X}_9 = A_{\theta_e} \bar{X}_{10} + \overline{W}_9 \tag{5.24}$$

with

$$\overline{W}_e^T = u^*, v_e, w_e \tag{5.25}$$

$$\overline{W}_9^T = 0, v_9, w_9 \tag{5.26}$$

Coordinates $\overline{W}$ are preferred to $\overline{W}_e$ for the final equations. The latter, it is recalled, were the elastic displacements of Reference (1). Defining

$$\overline{W}^T \quad u^*, v, w \tag{5.27}$$

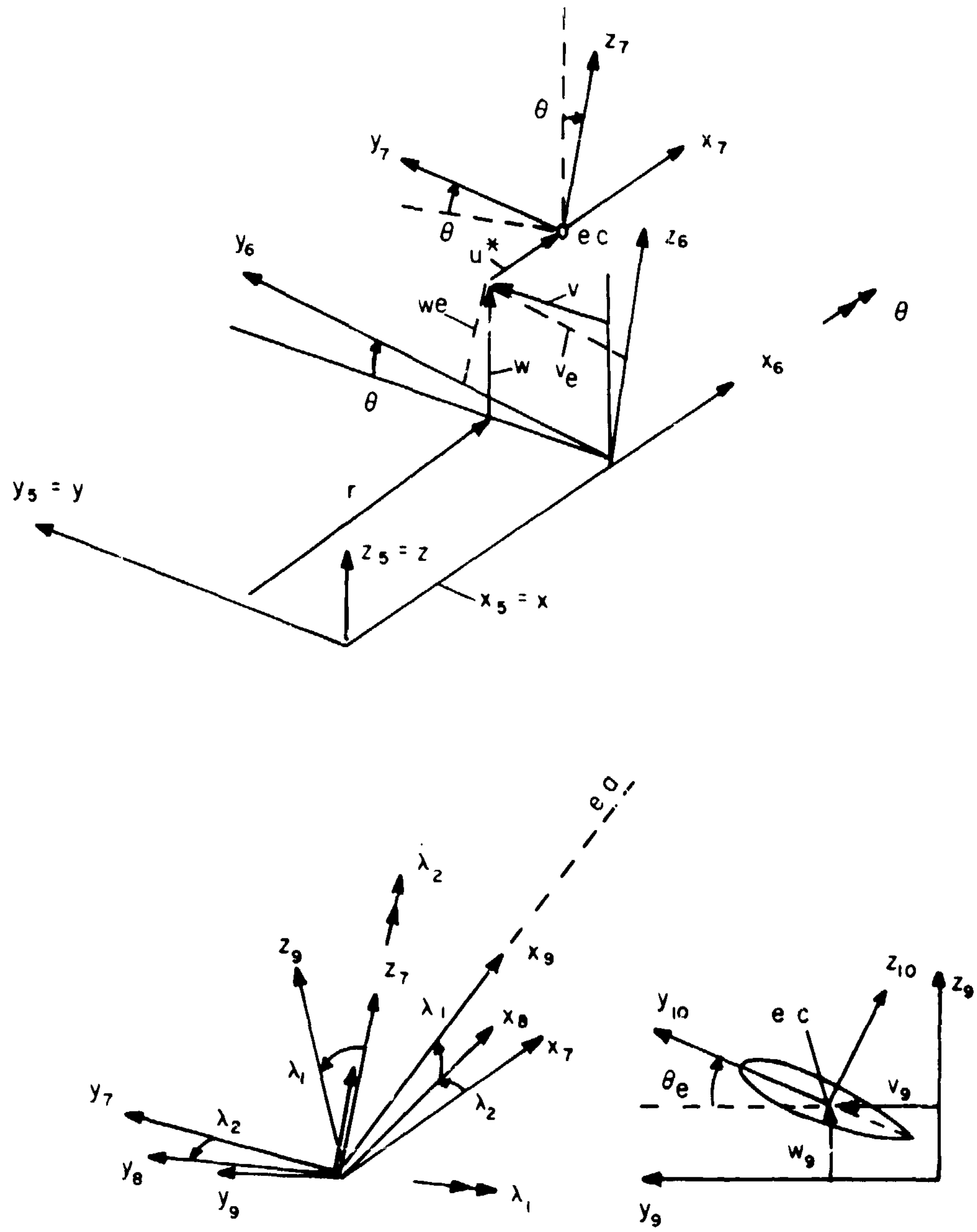


Figure 8. **Blade Elastic Displacements for Multi-Blade Rotor Analysis.**

where v and w are illustrated in Figure 8 and letting

$$\bar{X} = \bar{W} + \bar{R} \tag{5.28}$$

$$\bar{X}_6 = \bar{W}_e \tag{5.29}$$

$$R^T = r, \ 0, \ 0 \tag{5.30}$$

we find from (5.18) that

$$\bar{W} = A_\theta \ \bar{W}_e \tag{5.31}$$

$$\bar{W}_e = A_{-\theta} \ \bar{W} \tag{5.32}$$

and (5.21) may be replaced by

$$\bar{X}_6 = \bar{X}_7 + A_{-\epsilon} \bar{W} \tag{5.33}$$

5.1 List of Rotation Matrices

Listed below are rotation matrices required in equations (5.4) to (5.33),

(5.34)

$$A_{-\psi_s} = \begin{matrix} C\psi_s & S\psi_s & 0 \\ -S\psi_s & C\psi_s & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$A_{-\theta_s} = \begin{matrix} c_{\theta_s} & 0 & -s_{\theta_s} \\ 0 & 1 & 0 \\ s_{\theta_s} & 0 & c_{\theta_s} \end{matrix} \tag{5.35}$$

$$A_{-\phi_s} = \begin{matrix} 1 & 0 & 0 \\ 0 & c\phi_s & s\phi_s \\ 0 & -s\phi_s & c\phi_s \end{matrix} \tag{5.36}$$

$$A_{-\psi'} = \begin{matrix} c\psi' & s\psi' & 0 \\ -s\psi' & c\psi' & 0 \\ 0 & 0 & 1 \end{matrix} \tag{5.37}$$

$$A_{-\theta'} = \begin{matrix} c_{\theta}' & 0 & -s_{\theta}' \\ 0 & 1 & 0 \\ s_{\theta}' & 0 & c_{\theta}' \end{matrix} \tag{5.38}$$

$$A_{-\phi'} = \begin{matrix} 1 & 0 & 0 \\ 0 & c\phi' & s\phi' \\ 0 & -s\phi' & c\phi' \end{matrix} \tag{5.39}$$

$$(A_{-\theta'})_{\pi} = \begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{matrix} \tag{5.40}$$

$$A_\psi = \begin{matrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & \end{matrix} \tag{5.41}$$

$$A_\delta = \begin{matrix} c_\delta & -s_\delta & 0 \\ s_\delta & c_\delta & 0 \\ 0 & 0 & 1 \end{matrix} \tag{5.42}$$

$$A_\beta = \begin{matrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{matrix} \tag{5.43}$$

$$A_\theta = \begin{matrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{matrix} \tag{5.44}$$

$$A_{\lambda_2} = \begin{matrix} 1 & -\lambda_2 & 0 \\ \lambda_2 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \tag{5.45}$$

$$A_{\lambda_1} = \begin{matrix} 1 & 0 & -\lambda_1 \\ 0 & 1 & 0 \\ \lambda_1 & 0 & 1 \end{matrix} \tag{5.46}$$

$$A_{\theta_e} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_e \\ 0 & \theta_e & 1 \end{matrix} \tag{5.47}$$

Matrices $A\lambda_2$, $A\lambda_1$, $A\theta_e$ result from a small angle approximations to angles λ_2, λ_1, θ_e (Figure (8)), such as was made in Reference (1).

5.2 Derivation of Angles λ_1 and λ_2

Figure 9 illustrates angles λ_1 and λ_2. It is seen that

$$\lambda_2 = (dy_7 / dx_7)_{e.a.} \tag{5.48}$$

$$\lambda_1 = (dz_7 / dx_8)_{e\ a} \tag{5.49}$$

From (5.18), (5.33) we find

$$\bar{X} = A_\theta(\bar{X}_7 + A_{-\theta}\bar{W}) + \bar{R} \tag{5.50}$$

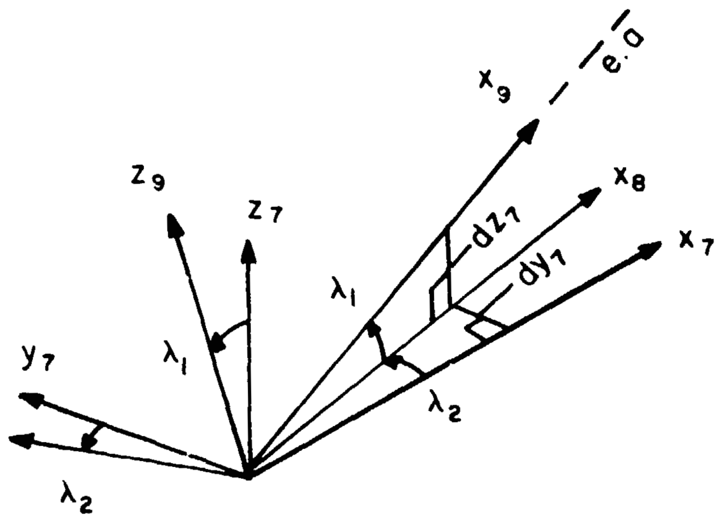
$$= A_\theta \bar{X}_7 + \bar{W} + \bar{R} \tag{5.51}$$

$$\bar{X}_7 = A_\theta^{-1}(\bar{X} - \bar{W} - \bar{R}) \tag{5.52}$$

$$= A_{-\theta}(\bar{X} - \bar{W} - \bar{R}) \tag{5.53}$$

Writing (5.53) out we find

$$x_7 = x - r + u^* \tag{5.54}$$



$$\lambda_2 = (dy_7/dx_7)_{e.a.}$$

$$\lambda_1 = (dz_7/dx_8)_{e.a.}$$

Figure 9. Derivation of Angles λ_1 and λ_2.

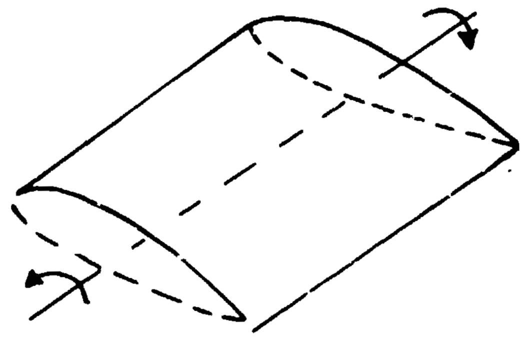


Figure 10. Elementary Twisting Couple Acting on a Blade Element.

$$y_7 = c_\theta (y - v) + S_\theta (z - w) \tag{5.55}$$

$$z_7 = -S_\theta (y - v) + C_\theta (z - w) \tag{5.56}$$

The changes in x_7, y_7 and z_7, for a variation such that y and z lie on the elastic axis, are

$$dx_7 = (1 + u^{*\prime})\, dx \tag{5.57}$$

$$dy_7 = C_\theta\, dv + S_\theta\, dw \tag{5.58}$$

$$dz_7 = -S_\theta\, dv + c_\theta\, dw \tag{5.59}$$

It follows from (5.48) and (5.57) to (5.59) that

$$\lambda_2 = \left(\frac{dy_7}{dx_7}\right)_{e.a} = (c_\theta v' + S_\theta w')(1 - u^{*\prime} + \dots)$$
$$\simeq c_\theta v' + s_\theta w' \tag{5.60}$$

From (5.22)

$$dx_7 \simeq dx_8 \tag{5.61}$$

$$\lambda_1 = \left(\frac{dz_7}{dx_8}\right)_{e.a} \simeq \left(\frac{dz_7}{dx_7}\right)_{e.a} \tag{5.62}$$

$$\simeq -S_\theta v' + C_\theta w' \tag{5.63}$$

In terms of the v_e and w_e coordinates of Reference (1) these results are

$$\lambda_2 = \theta' w_e + v_e' \tag{5.64}$$

$$\lambda_1 = \theta' v_e + w_e' \tag{5.65}$$

which agree with (1.23) and (1.22) of Reference (1)

5.3 Translation $\bar{W}_9$ Accompanying Elastic Rotation θ_e

In this section we derive the translation of a blade section located at $x_5 = x$ accompanying elastic rotations θ_e occurring at stations radially inboard of this section. The translation of the outboard section is represented as a displacement $W_9^{-T} = 0, v_9, w_9$ of the origin of the $\bar{X}_{10}$ axis from the $\bar{X}_9$ axis, (Figure 8).

To calculate v_9 and w_9, imagine the blade to be loaded by a twisting couple acting on a small length of blade, Figure 10. If the inboard section of the blade element is fixed and the outboard section rotates by $d\theta_e$, the entire blade between the tip of the blade and the outboard section of the blade element twists through $d\theta_e$. Rotation occurs about a straight line passing through the local elastic center of the element to which the elementary couple is applied. Superposition of such elementary couples yields arbitrary loadings and corresponding twist distributions needed to calculate elastic center displacements accompanying rotations.

Figure 11 shows the relative positions of a straight line $\hat{x}_9$ directed along the elastic axis at $\hat{x}$, at which the elementary couple is applied, and the elastic center Q at section x. P is the point of intersection of this $\hat{x}_9$ axis with the y - z plane at $x5 = x$. To a high order of approximation the rotation arms about this elastic axis are Δy^* and Δz^*, shown in Figure 11. Neglecting the x- displacement due to rotation, the displacements of the elastic center due to the elementary couples at x are (Figure 12)

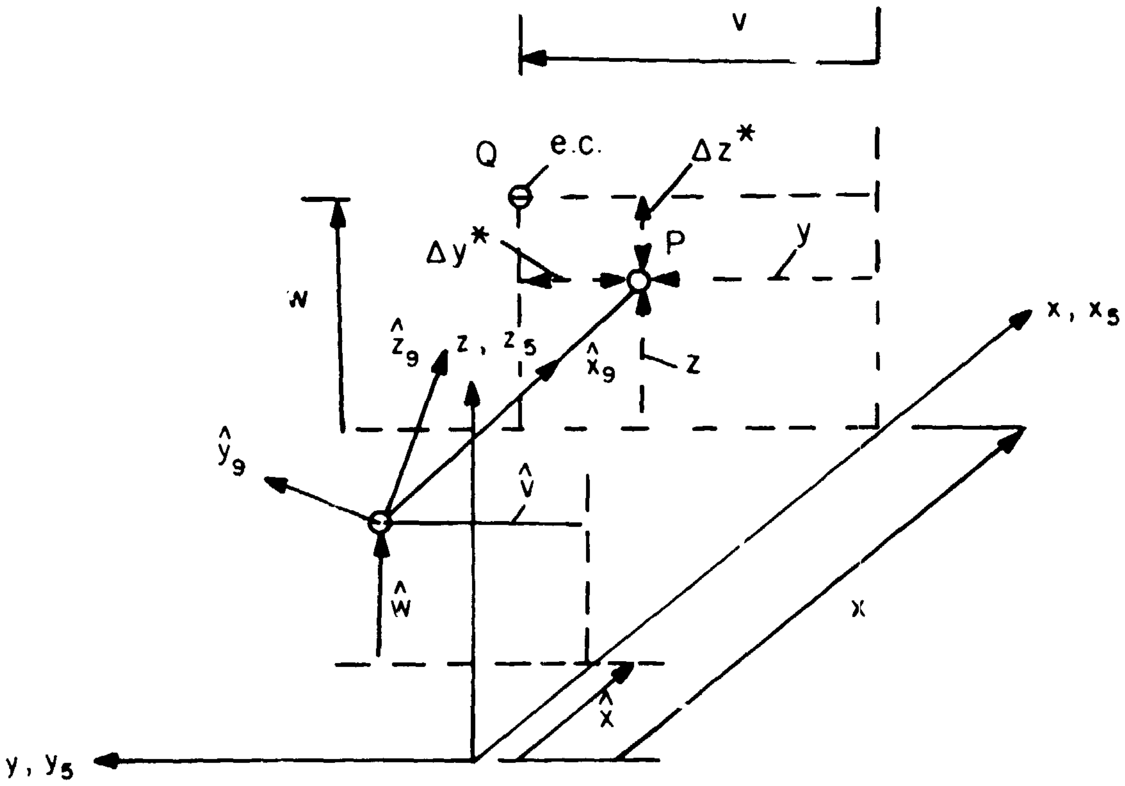


Figure 11. Geometry Used to Calculate Elastic Center Displacement Due to Elastic Rotation.

PROJECTED TO y-z PLANE

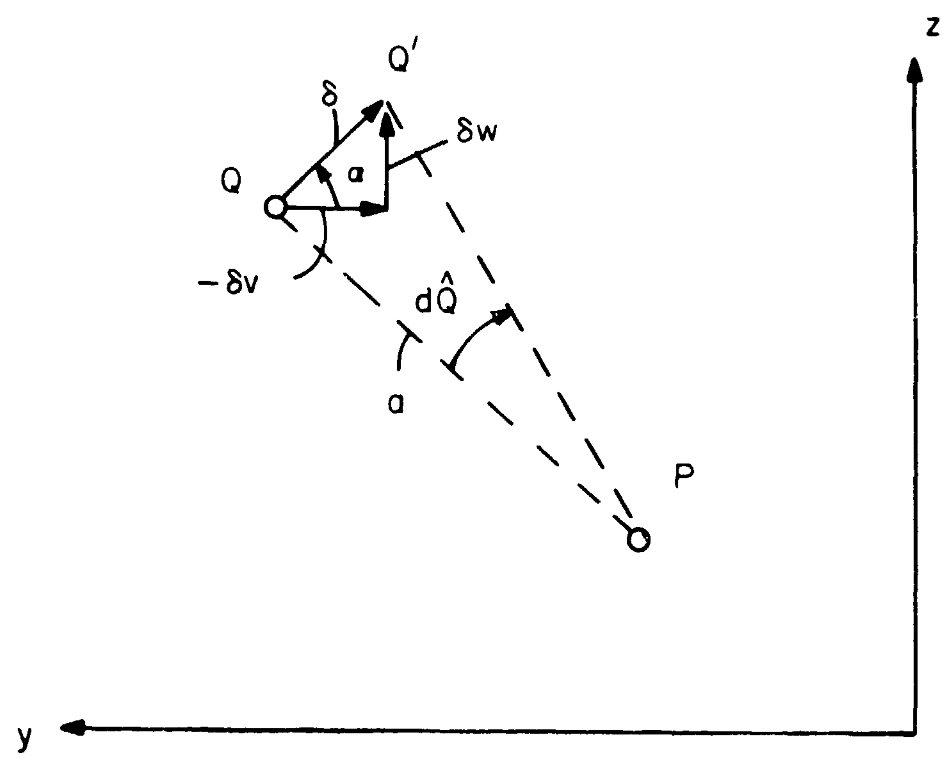


Figure 12. Geometry Used to Calculate Elastic Center Displacement Due to Elastic Rotation - Concluded.

$$\delta = a\, d\hat{\theta}_e \tag{5.66}$$

$$\delta_v = -\delta \sin\alpha = -a\, d\hat{\theta}_e \frac{\Delta z^*}{a} = -d\hat{\theta}_e \Delta z^* \tag{5.67}$$

$$\delta_w = \delta \cos\alpha = a\, d\hat{\theta}_e \frac{\Delta y^*}{a} = d\hat{\theta}_e \Delta y^* \tag{5.68}$$

The coordinates of P are

$$y = \hat{v} + \hat{v}'(x - \hat{x}) \tag{5.69}$$

$$z = \hat{w} + \hat{w}(x - \hat{x}) \tag{5.70}$$

and

$$\Delta y^* = v - y = v - \hat{v} - \hat{v}'(x - \hat{x}) \tag{5.71}$$

$$\Delta z^* = w - z = w - \hat{w} - \hat{w}'(x - \hat{x}) \tag{5.72}$$

Hence (5.67) and (5.68) become

$$\delta_v = -\left[w - \hat{w} - \hat{w}'(x - \hat{x})\right] d\hat{\theta}_e \tag{5.73}$$

$$\delta_w \quad \left[v - \hat{v} - \hat{v}'(x - \hat{x})\right] d\hat{\theta}_e \tag{5.74}$$

Superposing all the elementary couples from 0 to x to generate the actual twist, we find the displacement of the elastic center at x due to blade rotation to be

$$\Delta v = -\int_0^x \left[w - w(r) - w'(r)(x-r) \right] \theta_e'(r)\, dr \tag{5.75}$$

$$\Delta w = \int_0^x \left[v - v(r) - v'(r)(x-r) \right] \theta_e'(r)\, dr \tag{5.76}$$

The corresponding displacements in the y_9 - z_9 plane at x are obtained from a consideration of the transformations relating free vectors in the $\bar{X}_9$ and $\bar{X}$ axis systems. Denoting

$$d\bar{X}^T = 0, \ \Delta v, \ \Delta w \tag{5.77}$$

$$d\bar{X}_q' = u_q, \ v_q, \ w_q \tag{5.78}$$

we find from (5.18), (5.21) to (5.23), and (5.33)

$$d\bar{X} = A_\theta A_{\lambda_2} A_{\lambda_1} d\bar{X}_q \tag{5.79}$$

or using the reversal law for reciprocated products

$$d\bar{X}_q = A_{-\lambda_1} A_{-\lambda_2} A_{-\theta}\, d\bar{X} \tag{5.80}$$

$$A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} = \begin{matrix} 1 & -v' & w' \\ -\lambda_2 & c_\theta & s_\theta \\ -\lambda_1 & -s_\theta & c_\theta \end{matrix} \tag{5.81}$$

from which it follows

$$u_9 = -v'\Delta v + w'\Delta w \simeq 0 \tag{5.82}$$

$$v_9 = S_\theta \Delta w + C_\theta \Delta v \tag{5.83}$$

$$w_9 = C_\theta \Delta w - S_\theta \Delta v \tag{5.84}$$

These are the displacements of the elastic center at x due to elastic rotation of the blade. Hence, the translation $\overline{W}_9$ of the origin of the $\overline{X}_{10}$ axis from the origin of the $\overline{X}_9$ axis is

$$\overline{W}_9^T = 0, v_9, w_9 \tag{5.85}$$

with (5.83) and (5.84) yielding v_9 and w_9.

6. Moment Equilibrium Equations and Expressions for Generalized Forces F_1, F_2, F_3

To form the generalized forces in physical space F_1, F_2, and F_3 required for insertion in the expression for the generalized force in the modal coordinates, Q_j, we first derive internal resisting moments and equilibrate these to the external moments to form moment equilibrium equations. Comparison of derivatives of these moment equilibrium equations with the general equation, (4.1), yields F_1, F_2, F_3.

To calculate the internal resisting moments we first calculate the longitudinal strain of a fiber of beam and then sum the moment effects of the strain in the section. The next section provides a very brief description of the derivation of strain, and this is followed by a section yielding the internal moments. Expressions for F_1, F_2, and F_3 are then calculated and this is followed by a section on expressions for moments and their derivatives occurring either in the F expressions or required for display of the system response.

6.1 Strain Expressions

To calculate the strain component perpendicular to a normal section of the beam, we consider two adjacent sections, separated from each other by a distance $d_{xo} = dr$ in the unstrained state, and calculate the extension of a fiber extending between the faces and passing through points at y_{10} and z_{10} in the faces. The usual assumption that plane sections of the beam remain plane after elastic deformation is contained implicitly in the geometry of deformation assumed, and described in the previous chapter. The expression for strain is

$$\epsilon = \frac{ds - ds_0}{ds_0} \tag{6.1}$$

$$= \frac{\frac{ds}{dx_0} - \frac{ds_0}{dx_0}}{\frac{ds_0}{dx_0}} \tag{6.2}$$

where ds = fiber length after deformation and ds_o = fiber length before deformation. If we express the strain in terms of components of position in the $\bar{X}_5$ (that is, $\bar{X}$) system, then we must evaluate

$$\left(\frac{ds_o}{dx_o}\right)^2 = \left(\frac{dx_o}{dx_o}\right)^2 + \left(\frac{dy_o}{dx_o}\right)^2 + \left(\frac{dz_o}{dx_o}\right)^2 \tag{6.3}$$

$$\left(\frac{ds}{dx_o}\right)^2 = \left(\frac{dx}{dx_o}\right)^2 + \left(\frac{dy}{dx_o}\right)^2 + \left(\frac{dz}{dx_o}\right)^2 \tag{6.4}$$

Defining

$$\frac{d}{dx_o} \equiv \frac{d}{dr} \equiv (\ )'$$

(6.1) becomes

$$\epsilon = \left[(x')^2 + (y')^2 + (z')^2\right]^{\frac{1}{2}} \left[1 + (y_o')^2 + (z_o')^2\right]^{-\frac{1}{2}} - 1 \tag{6.5}$$

Repeated application of the transformations of the previous chapter yields

$$\bar{X} = A_\theta A_{\lambda_2} A_{\lambda_1} A_{\theta_e} \bar{X}_{10} + A_\theta A_{\lambda_2} A_{\lambda_1} \bar{W}_9 + A_\theta \bar{W}_e + \bar{R} \tag{6.6}$$

This expresses the $\bar{X}$ displacements in terms of elastic displacements $\bar{W}_e$, $\bar{W}_9$, and elastic angles λ_1 and λ_2 (related to $\bar{W}_e$ by (5.64) and (5.65)), and section coordinates $\bar{R}$ and $\bar{X}_{10}$.

By assuming all elastic displacements to be absent in (6.6) we obtain the coordinates of the point before elastic deformation.

$$x_0 = r \tag{6.7}$$
$$y_0 = C_\theta y_{10} - S_\theta z_{10}$$
$$z_0 = S_\theta y_{10} + C_\theta z_{10}$$

After elastic deformation, we find

$$x = x_0 + u^* - \lambda_2 y_{10} - \lambda_1 z_{10} \tag{6.8}$$
$$y = y_0 - \theta_e S_\theta y_{10} - \theta_e C_\theta z_{10} + C_\theta v_e - S_\theta w_e$$
$$z = z_0 + \theta_e C_\theta y_{10} - \theta_e S_\theta z_{10} + S_\theta v_e + C_\theta w_e$$

To derive (6.8), 2nd and higher order products of elastic variables were neglected. Carrying out the differentiations of (6.5), we obtain

$$\begin{aligned}\epsilon &= u^{*\prime} + \left(-\lambda_2' + w_e'\theta' + v_e(\theta')^2\right) y_{10} \\ &\quad + \left(-\lambda_1' - v_e'\theta' + w_e(\theta')^2\right) z_{10} \\ &\quad + \theta'\theta_e'\left(z_{10}^2 + y_{10}^2\right)\end{aligned} \tag{6.9}$$

Substitution of the expressions for λ_1 and λ_2 ((5.64) and (5.65)) in (6.9) yields

$$\epsilon = \epsilon_0 + \epsilon_1 y_{10} + \epsilon_2 z_{10} + \epsilon_3 \left(z_{10}^2 + y_{10}^2\right) \tag{6.10}$$

$$\epsilon_0 = u^{*\prime}$$
$$\epsilon_1 = -\left(v_e'' - w_e\theta'' - 2w_e'\theta' - v_e(\theta')^2\right)$$
$$\epsilon_2 = -\left(w_e'' + v_e\theta'' + 2v_e'\theta' - w_e(\theta')^2\right)$$
$$\epsilon_3 = \theta'\theta_e'$$

C 12

6.2 Internal Resisting Forces and Moments

In this section, we calculate the loads of reaction induced by elastic deformation, employing strain expression (6.10). Figure 13 illustrates the positive conventions of these loads. The positive direction of M_{y10} is along y_{10} and is not opposite to y_{10} as was the positive moment employed for the determination of the normal modes.

Referred to the x, y, z axes, the strained fiber has direction cosines x'/s', y'/s', and z'/s'. The corresponding tensile reaction on the face of the beam has components (x'/s') E dA, (y'/s') E dA, and (z'/s') E dA along the x, y, z axes. The corresponding elementary reactions referred to the X_{10} system are

$$\begin{Bmatrix} dF_{x_{10}} \\ dF_{y_{10}} \\ dF_{z_{10}} \end{Bmatrix} = A_{-\theta_e} A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} \begin{Bmatrix} \left(\frac{x'}{s'}\right) \epsilon E dA \\ \left(\frac{y'}{s'}\right) \epsilon E dA \\ \left(\frac{z'}{s'}\right) \epsilon E dA \end{Bmatrix} \tag{6.11}$$

$$A_{-\theta_e} A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} = \begin{bmatrix} 1, & v', & w' \\ -\lambda_2 - \lambda_1 \theta_e, & c_{\bar\theta}, & s_{\bar\theta} \\ -\lambda_1 + \lambda_2 \theta_e, & -s_{\bar\theta}, & c_{\bar\theta} \end{bmatrix} \tag{6.12}$$

with $\bar\theta$ being the total angle, defined in (6.34). From (6.11) we obtain

$$dF_{x_{10}} = \epsilon E dA \tag{6.13}$$

$$dF_{y_{10}} = \epsilon E dA(-\lambda_2 + c_\theta y' + s_\theta z')$$

$$dF_{z_{10}} = \epsilon E dA(-\lambda_1 - s_\theta y' + c_\theta z') \tag{6.14}$$

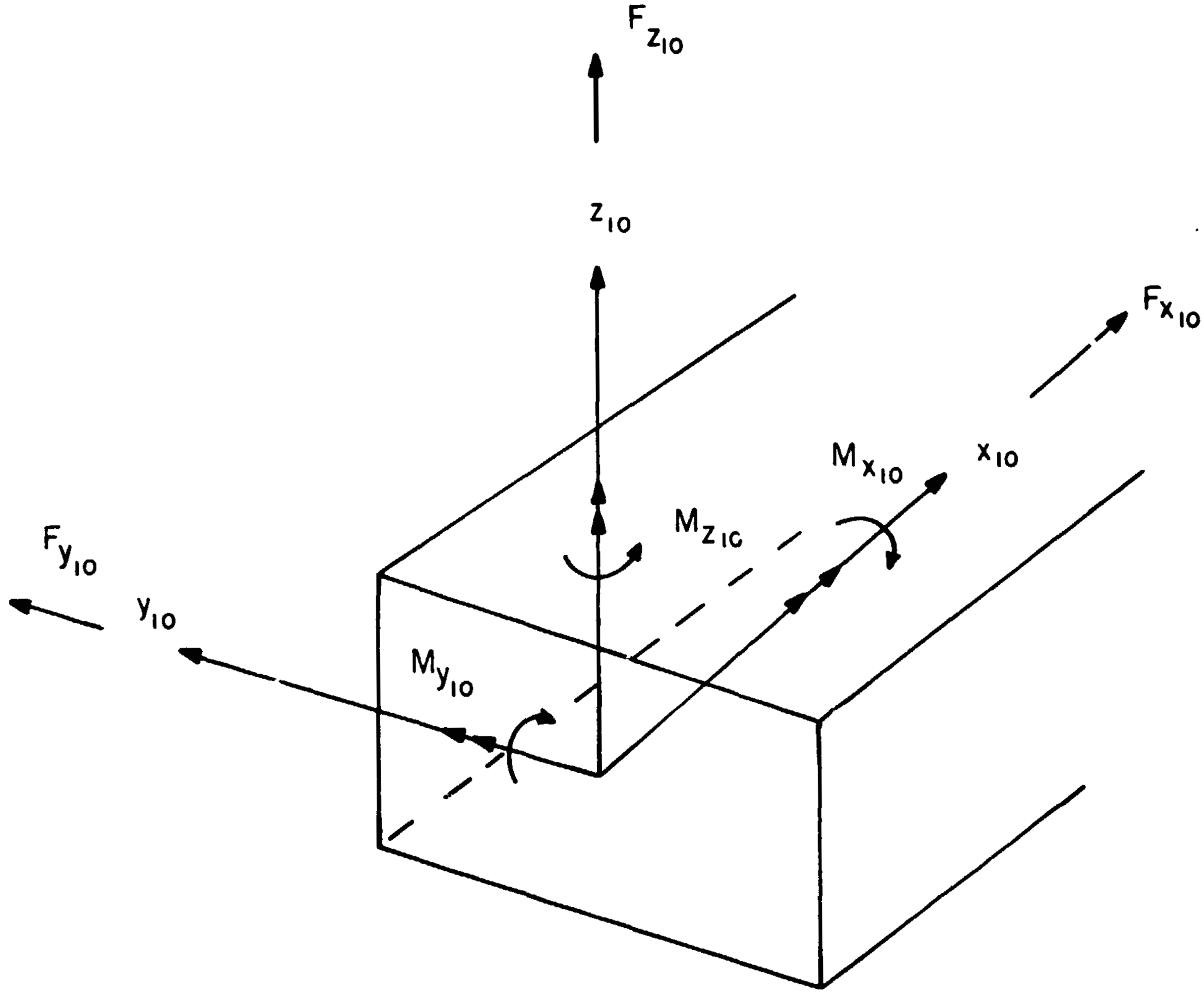


Figure 13. Positive Conventions for Internal Resisting Loads - Multi-Blade Rotor System.

In these expressions second order products of small quantities involving ϵ, y', z', λ_1, λ_2 and θ_e are neglected to obtain (6.13) and third order products are neglected to obtain (6.14).

The section moments are

$$Mx_{10} = GJ\frac{d\theta_e}{dx_{10}} + \int_A -z_{10}\, dFy_{10} + \int_A y_{10}\, dFz_{10} \tag{6.15}$$

$$My_{10} = \int_A z_{10}\, dFx_{10} \tag{6.16}$$

$$Mz_{10} = \int_A -y_{10}\, dFx_{10}$$

Moment Mx_{10} is evaluated at the elastic center with the result that no contributions to moment come from shearing stresses, because the elastic center is defined to be a position where such torque is zero. It is not necessary to approximate dFx_{10}, (6.13), to a higher order than indicated because My_{10} and Mz_{10} neglect second order products of elastic variables, and third order products of small quantities. The approximations to dFy_{10} and dFz_{10} are high enough to permit the consistent determination of Mx_{10} to an order which neglects third order products of elastic variables, and fourth order small quantities in the torsion moment equilibrium equation.

Substitution of ϵ, (6.10), in (6.13) to (6.16) yields

$$Mx_{10} = GJ\theta_e' + (\theta' + \theta_e')K_A^2 Fx_{10} \tag{6.17}$$

$$M_{y_{10}} = \epsilon_2 E I_y$$
$$= -\left(w_e'' + \underline{v_e \theta''} + 2 v_e' \theta' - \underline{w_e (\theta')^2}\right) E I_y \tag{6.18}$$

$$M_{z_{10}} = -e_A F_{x_{10}} - \epsilon_1 E I_z$$
$$= -e_A F_{x_{10}} + \left(v_e'' - \underline{w_e \theta''} - 2 w_e' \theta' - \underline{v_e (\theta')^2}\right) E I_z \tag{6.19}$$

with

$$F_{x_{10}} = \int_A \epsilon E \, dA \tag{6.20}$$

$$E I_y = \int_A z_{10}^2 E \, dA \tag{6.21}$$

$$E I_z = \int_A (y_{10} - e_A)^2 E \, dA \tag{6.22}$$

$$e_A = \frac{1}{EA} \int_A y_{10} E \, dA \tag{6.23}$$

$$K_A^2 = \frac{1}{EA} \int_A y_{10}^2 E \, dA \tag{6.24}$$

The above results were obtained for sections with structural symmetry about the y_{10} axis.

The underlined terms are additional to these present in equations (1-51) to (1-53) of Reference (1), and represent the added effects of large twist rate and non-linear twist, without restrictions.

To represent a Sikorsky type counterweight we add to the right-side of (6.19) the term

$$\Delta e_{A_{cw}} \int_{r_{o_{cw}}}^{r} \left(p_{x_{10}}\right)_{cw} d\xi \tag{6.25}$$

where $\Delta_{eA_{cw}}$ is the distance between the chordwise position of the centroid of the counterweight and the elastic centroid at $y_{10} = e_A$, and $r_{o_{cw}}$ is the inner radius of the counterweight. The loading $(px_{10})_{cw}$ is the inertial load of the counterweight per unit span.

$$\left(p_{x_{10}}\right)_{cw} = \int_A \left(-a_{x_{10}}\right) \rho_{cw}\, dA \tag{6.26}$$

$$= \frac{m_{cw}}{m} \left(p_{x_{10}}\right)^{D}$$

In (6.26) m_{cw} is the mass of the counterweight, m is the mass per unit span of a blade section, including the counterweignt and px^{D}_{10} is the inertial load per unit span due to mass m.

With the counterweight term the expression for Mz_{10} is

$$M_{z_{10}} = -e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{o_{cw}}}^{r} \frac{m_{cw}}{m} p_{x_{10}}^{D}\, d\xi + \left(v_e'' - w_e \theta'' - 2 w_e' \theta' - v_e (\theta')^2\right) EI_z \tag{6.27}$$

To be able to identify later the generalized force F_2 and F_3 in vector F of (4.1), whose last two rows express vertical and inplane equilibrium of loads, we transform structural reaction moments to the $\bar{X}_5$ system and using (5.32) we replace w_e and v_e by elastic displacements w and v.

By means of (5.32) we can show

$$C_\theta w'' - S_\theta v'' = -(\theta')^2 w_e + v_e \theta'' + 2\theta' v_e' + w_e'' \quad (6.28)$$

$$S_\theta w'' + C_\theta v'' = -(\theta')^2 v_e - w_e \theta'' - 2\theta' w_e' + v_e'' \quad (6.29)$$

Substitution of (6.28) and (6.29) in (6.18) and (6.27) yields

$$M_{y_{10}} = -EI_y C_\theta w'' + EI_z S_\theta v'' \quad (6.30)$$

$$M_{z_{10}} = EI_z S_\theta w'' + EI_z C_\theta v'' - e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{0cw}}^{r} \frac{m_{cw}}{m} - P_{x_{10}}^{D} \, d\xi \quad (6.31)$$

Moments are related by

$$\bar{M}_5 = A_\theta A_{\lambda_2} A_{\lambda_1} A_{\theta_e} \bar{M}_{10} \quad (6.32)$$

with

$$A_\theta A_{\lambda_2} A_{\lambda_1} A_{\theta_e} = \begin{matrix} 1 & , -\lambda_2 - \lambda_1 \theta_e, & \lambda_2 \theta_e - \lambda_1 \\ v' & , C\bar{\theta} & , -S\bar{\theta} \\ w' & , S\bar{\theta} & , C\bar{\theta} \end{matrix} \quad (6.33)$$

and

$$\begin{aligned} \bar{\theta} &= \theta + \theta_e \\ &= \theta_c + \theta_t + \theta_e \end{aligned} \quad (6.34)$$

Expansion of (6.32) yields

$$-M_{y_s} = -v' M_{x_{10}} + C_{\bar{\theta}} (-M_{y_{10}}) + S_{\bar{\theta}} M_{z_{10}} \quad (6.35)$$

$$M_{z_s} = w' M_{x_{10}} - S_{\bar{\theta}} (-M_{y_{10}}) + C_{\bar{\theta}} M_{z_{10}} \quad (6.36)$$

To obtain (6.35) and (6.36) we neglected second order products of elastic variables and third order products of small quantities.

Substitution of (6.17), (6.30) and (6.31) in (6.35) and (6.36) yields the desired vertical and inplane equations for structural reaction moments in the $\bar{X}_5$ system in terms of displacements w and v. Assembling these equations with torsion equation (6.17), we have

$$-M_{x_{10}} = -\left[GJ\theta_e' + (\theta + \theta_e') K_A^2 F_{x_{10}} \right] \quad (6.37)$$

$$-M_{y_s} = (EI_y C_{\bar{\theta}}^2 + EI_z S_{\bar{\theta}}^2) w'' + (EI_z - EI_y) S_{\bar{\theta}} C_{\bar{\theta}} v''$$
$$+ S_{\bar{\theta}} \left(-e_A F_{x_{10}} + \Lambda e_{A_{cw}} \int_{r_{o_{cw}}}^{r} P_{x_{10}}^{D} \frac{m_{cw}}{m} d\xi \right)$$

(6.38)

(6.39)

$$M_{Z_S} = (EI_Z - EI_y) s_\theta c_\theta w'' + (EI_y s_\theta^2 + EI_Z c_\theta^2) v'' + c_{\bar{\theta}} \left(-e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} \rho x_{10}^{D} \frac{m_{cw}}{m} d\xi \right)$$

The corresponding loading equilibrium equations are obtained by differentiating (6.37) to (6.39).

(6.40)

$$-\left[GJ\theta_e' + (\theta' + \theta_e') K_A^2 F_{x_{10}} \right]' + M_{x_{10}}' = 0$$

(6.41)

$$\left[(EI_y c_\theta^2 + EI_Z s_\theta^2) w'' + (EI_Z - EI_y) s_\theta c_\theta V'' + s_{\bar{\theta}} \left(-e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} \rho x_{10}^{D} \frac{m_{cw}}{m} d\xi \right) \right]'' + M_{y_S}'' = 0$$

(6.42)

$$\left[(EI_Z - EI_y) s_\theta c_\theta w'' + (EI_y s_\theta^2 + EI_Z c_\theta^2) v'' + c_{\bar{\theta}} \left(-e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} \rho_{x_{10}}^{D} \frac{m_{cw}}{m} d\xi \right) \right]'' - M_{Z_S}'' = 0$$

6.3 Generalized Forces F_1 F_2 and F_3

Examination of the rows of the general equation (4.1), which are

$$-\left[(GJ+\hat{T}K_A^2)\theta_e'\right]' + m\Omega^2(k_{z_{10}}^2 - k_{y_{10}}^2)\,C_{2\theta}\,\theta_e + m\Omega^2(k_{z_{10}}^2 + k_{y_{10}}^2)\ddot{\theta}_e = F_1 \qquad (6.43)$$

$$\left[(EI_y C_\theta^2 + EI_z S_\theta^2)w_1'' + (EI_z - EI_y)S_\theta C_\theta v_1''\right]'' + (-\hat{T}w_1')' + m\ddot{w}_1 = F_2 \qquad (6.44)$$

$$\left[(EI_y - EI_z)S_\theta C_\theta w'' + (EI_y S_\theta^2 + EI_z C_\theta^2)v''\right]'' + (-\hat{T}v_1')' - m\Omega^2 V_1 + m\ddot{v}_1 = F_3 \qquad (6.45)$$

indicates that subtraction of (6.40) from (6.43), (6.41) from (6.44), and (6.42) from (6.45), will yield F_1, F_2 and F_3. Doing this and employing the properties $w_1'' = w''$ and $v_1'' = v''$, we obtain

$$F_1 = M_{x_{10}}' - \left[(\hat{T} - F_{x_{10}})K_A^2\theta_e'\right]' + m\Omega^2(K_{z_{10}}^2 - K_{y_{10}}^2)\,C_{2\theta}\,\theta_e + m(K_{z_{10}}^2 + K_{y_{10}}^2)\ddot{\theta}_e + (\theta' K_A^2 F_{x_{10}})' \qquad (6.46)$$

$$F_2 = -My_5'' + (-\hat{T} w_1')' + m\ddot{w}_1 - \left[s_{\bar{\theta}} \left(-e_A Fx_{10} + \Delta e_{A_{cw}} \int_{r_{o_{cw}}}^{r} Px_{10}^{D} \frac{m}{m} c_w \, d\xi \right) \right]'' \quad (6.47)$$

$$F_3 = Mz_5'' + (-\hat{T} v_1')' - m\Omega^2 v_1 + m\ddot{v}_1 - \left[c_{\theta} \left(-e_A Fx_{10} + \Delta e_{A_{cw}} \int_{r_{o_{cw}}}^{r} Px_{10}^{D} \frac{m_{cw}}{m} d\xi \right] '' \quad (6.48)$$

We express Mx_{10}', My_5'', Mz_5'' in terms of local torsion (qx_5, qy_5, qz_5) and direct (px_5, py_5, pz_5) loadings per unit span to reduce further these derivatives to useful forms. In the next section, consideration of equilibrium of a beam element is shown to give

(6.49)

$$Mx_5' = -qx_5 - v' Fz_5 + w Fy_5$$

$$My_5' = -qy_5 - w' Fx_5 + Fz_5$$

$$Mz_5' = -qz_5 + v' Fx_5 - Fy_5$$

where

$$F_{x_5} = \int_r^{r_T} p_{x_5}\, d\xi \qquad (6.50)$$

$$F_{y_5} = \int_r^{r_T} p_{y_5}\, d\xi$$

$$F_{z_5} = \int_r^{r_T} p_{z_5}\, d\xi$$

With

$$\overline{M}_{10} = A_{-\theta_e} A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} \overline{M}_5 \qquad (6.51)$$

and (6.12) for $A-\theta_e$ $A-\lambda_1$ $A-\lambda_2$ $A-\theta$, we obtain

$$M_{x_{10}} = M_{x_5} + v' M_{y_5} + w' M_{z_5} \qquad (6.52)$$

Differentiation of (6.52) yields

$$M_{x_{10}}' = M_{x_5}' + v' M_{y_5}' + w' M_{z_5}' + v'' M_{y_5} + w'' M_{z_5} \qquad (6.53)$$

and substitution of (6.49) in (6.53) yields

$$M_{x_{10}}' = -q_{x_5} - v' q_{y_5} - w' q_{z_5} + v'' M_{y_5} + w'' M_{z_5} \qquad (6.54)$$

Substitution of (6.54) in the expression for F_1, (6.46), and use of the expressions for M_{y_5}, (6.38), and M_{z_5}, (6.39) yields

$$
\begin{aligned}
F_1 = {} & q_{x_5} + v' q_{y_5} + w' q_{z_5} \\
& - \left[(\hat{T} - F_{x_{10}}) K_A^2 \theta_e' \right] \\
& + m \Omega^2 (K_{z_{10}}^2 - K_{y_{10}}^2) C_{Z_\theta} \ddot{\theta}_e \\
& + m (K_{z_{10}}^2 + K_{y_{10}}^2) \ddot{\theta}_e \\
& + (\theta' K_A^2 F_{x_{10}})' \\
& + (v'' S_{\bar{\theta}} - w'' C_{\bar{\theta}}) \left(-e_A F_{x_{10}} + \Delta e_{A_{CW}} \int_{r_{0_{CW}}}^{r} P_{x_{10}}^{D} \frac{m_{CW}}{m} d\xi \right) \\
& - (EI_z - EI_y) \left(w'' v'' C_{Z_\theta} + \frac{1}{2} (w''^2 - v''^2) S_{2\theta} \right)
\end{aligned}
\tag{6.55}
$$

Using the results of the next section that

$$ M_{y_5}'' = -q_{y_5}' - (w' F_{x_5})' - w'' F_{x_5} \tag{6.56} $$

$$ M_{z_5}'' = -q_{z_5}' + (v' F_{x_5})' + v'' F_{x_5} \tag{6.57} $$

in the expressions for F_2, (6.47), and F_3, (6.48), we obtain

S 29 REV D

$$F_2 = q_{y_5}' + (w' F_{x_5})' + P_{z_5} + (-\hat{T} w_1')' + m\ddot{w}_1 - \left[s_{\bar{\Theta}} \left(-e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} P_{x_{10}}^{D} \frac{m_{cw}}{m} d\xi \right) \right]'' \tag{6.58}$$

$$F_3 = -q_{z_5}' + (v' F_{x_5})' + P_{y_5} + (-\hat{T} v_1')' - m\Omega^2 v_1 + m\ddot{v}_1 - \left[c_{\bar{\Theta}} \left(-e_A F_{x_{10}} + \Delta e_{cw} \int_{r_{0_{cw}}}^{r} P_{x_{10}}^{D} \frac{m_{cw}}{m} d\xi \right) \right]'' \tag{6.59}$$

Because we desire to express F_1, F_2 and F_3 in terms of $\bar{X}_5$ system loadings, we ultimately must substitute

$$P_{x_{10}} = P_{x_5} + v' P_{y_5} + w' P_{z_5} \tag{6.60}$$

$$F_{x_{10}} = F_{x_5} + v' F_{y_5} + w' F_{z_5} \tag{6.61}$$

in (6.55), (6.58) and (6.59). Making these substitutions as convenience indicates, expanding

$$\bar{\Theta} = \Theta + \Theta_e \tag{6.62}$$

and deleting negligible terms, as appropriate to torsion, flatwise, and edgewise equations, we obtain

$$
\begin{aligned}
F_1 = {} & q_{x_5} + m\Omega^2 (K^2_{z_{10}} - K^2_{y_{10}})\, c_{2\theta}\, \theta_e + m (K^2_{z_{10}} + K^2_{y_{10}})\, \ddot{\theta}_e \\
& + v' q_{y_5} \\
& + w' q_{z_5} \\
& - ((\hat{T} - F_{x_5}) K_A^2 \theta_e')' \\
& + (\theta' K_A^2 F_{x_{10}})' \\
& + c_\theta e_A F_{x_{10}} w'' \\
& - s_\theta e_A F_{x_{10}} v'' \\
& + v'' s_\theta \Delta e_{A_{cw}} \int_{r_{o_{cw}}}^{r} P^D_{x_{10}} \frac{m_{cw}}{m} d\xi \\
& - w'' c_\theta \Delta e_{A_{cw}} \int_{r_{o_{cw}}}^{r} P^D_{x_{10}} \frac{m_{cw}}{m} d\xi \\
& + \theta_e (v'' c_\theta + w'' s_\theta) \Big(-e_A F_{x_5} \\
& \qquad + \Delta e_{A_{cw}} \int_{r_{o_{cw}}}^{r} P^D_{x_5} \frac{m_{cw}}{m} d\xi \Big) \\
& - (EI_z - EI_y)\left(v'' w'' c_{2\theta} + \tfrac{1}{2}(w''^2 - v''^2)\, s_{2\theta}\right)
\end{aligned}
\qquad (6.63)
$$

$$F_2 = q y_5' + (w' F_{x_5})' + P_{z_5} + (-\hat{T} w_1') + m \ddot{w}_1$$
$$- \Big[S_\theta \Big(-e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} P_{x_{10}}^{D} \frac{m_{cw}}{m} d\xi \Big)$$
$$+ \epsilon_e c_\theta \Big(-e_A F_{x_5} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} P_{x_5}^{0} \frac{m_{cw}}{m} d\xi \Big) \Big]''$$

(6.64)

$$F_3 = -q z_5' + (v' F_{x_5})' + P_{y_5} + (-\hat{T} v_1') - m\Omega^2 v_1 + m \ddot{v}_1$$
$$- \Big[c_\theta \Big(-e_A F_{x_{10}} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} P_{x_{10}}^{D} \frac{m_{cw}}{m} d\xi \Big)$$
$$- \epsilon_e S_\theta \Big(-e_A F_{x_5} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} P_{x_{10}}^{0} \frac{m_{cw}}{m} d\xi \Big) \Big]''$$

(6.65)

The aim of the subsequent chapters is the derivation of the elements on the right-sides of (6.63) to (6.65) in terms of the coordinates of the problem, and the substitution of F_1, F_2 and F_3 in the modal force expressions ($Qj^{(1)}$, $Qj^{(2)}$, $Qj^{(3)}$, (4.26) to (4.28)) to calculate Qj.

The last term in the torsion moment F_1, (6.63), is the torque from bending forces (Compare p. 430ff, Reference (7)). We verify that for $\theta'' = 0$ (linear twist) and negligible $(\theta')^2$, the term is

$$(EI_z - EI_y)\Big(v'' w'' C_{2\theta} + \frac{1}{2}(w''^2 - v''^2) S_{2\theta}\Big) \simeq$$
$$(EI_z - EI_y)\big(v_e'' w_e'' - 2\theta' w_e'' w_e' + 2\theta' v_e'' v_e'\big)$$

(6.66)

which checks with the equivalent term in equation (1-70) of Reference (1).

6.4 Beam Element Equilibrium Equations Used in the Multi-Blade Rotor Analysis

In this section we state briefly the results of considering the equilibrium of an element of beam, to justify expressions for moment derivatives used in the last section, and to derive moment expressions used ultimately for display.

Figure 14 shows the forces and moments acting on an element of beam. All forces and moments are assumed to act at the elastic center to be consistent with the derivation of structural reaction moments, and are resolved to the $\bar{X}_5$ system. The results of force and moment equilibrium considerations are

$$F_{x_5}' + P_{x_5} = 0 \qquad (6.67)$$

$$F_{y_5}' + P_{y_5} = 0$$

$$F_{z_5}' + P_{z_5} = 0$$

$$(6.68)$$

$$M_{x_5}' + q_{x_5} + v' F_{z_5} - w' F_{y_5} = 0$$

$$M_{y_5}' + q_{y_5} + w' F_{x_5} - F_{z_5} = 0$$

$$M_{z_5}' + q_{z_5} - v' F_{x_5} + F_{y_5} = 0$$

Differentiation of the last two equations in (6.68) yields

$$M_{y_5}'' = -q_{y_5}' - (w' F_{x_5})' - P_{z_5} \qquad (6.69)$$

$$M_{z_5}'' = -q_{z_5}' + (v' F_{x_5})' + P_{y_5} \qquad (6.70)$$

which are (6.56) and (6.57) used in the last section to obtain F_2 and F_3.

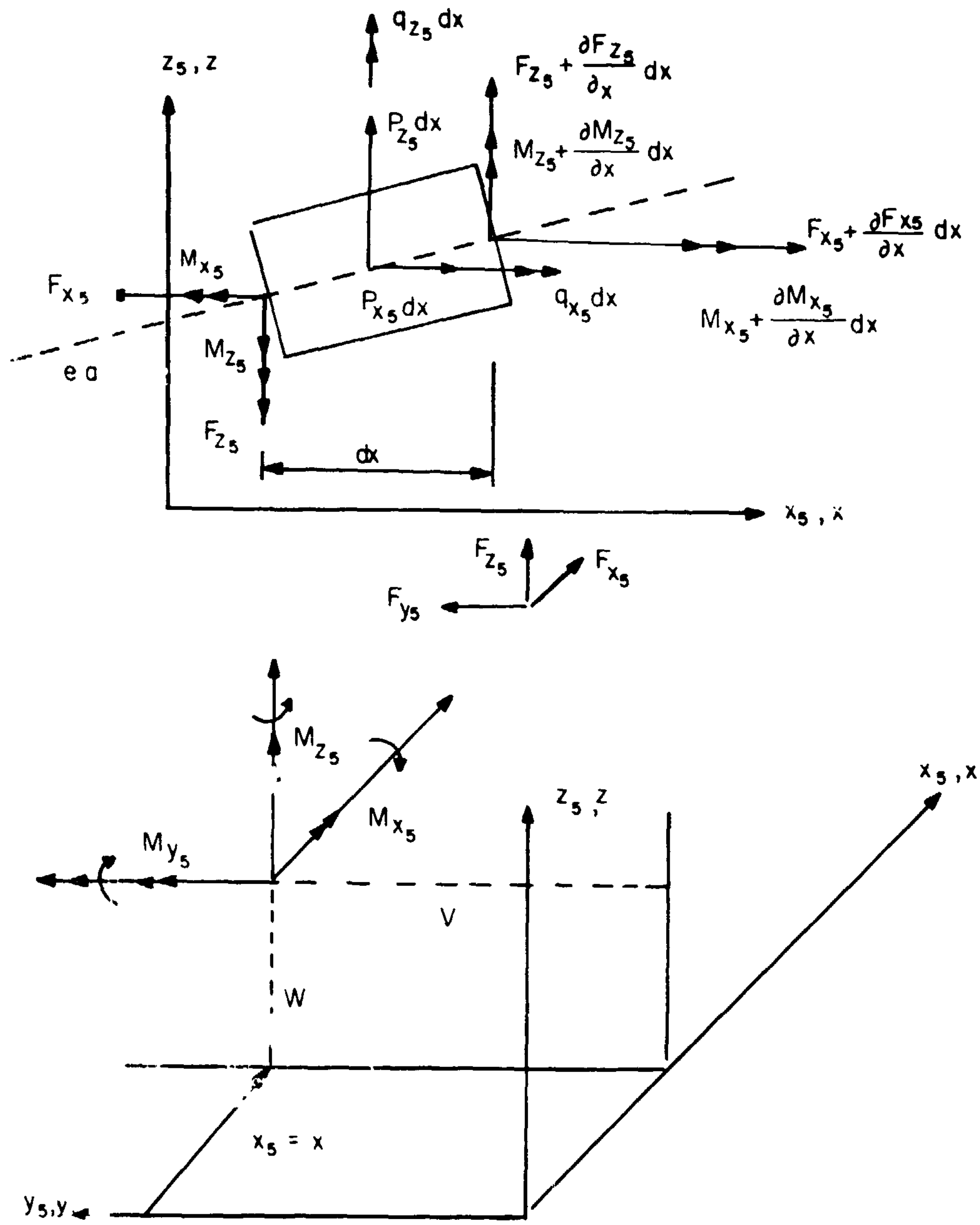


Figure 14. Loads Applied to Beam Element - Multi-Blade Rotor Analysis.

Integration of (6.67) with respect to v and insertion of boundary conditions that tip loads are zero yields

$$F_{x_5} = \int_r^{r_T} p_{x_5} \, d\xi \tag{6.71}$$

$$F_{y_5} = \int_r^{r_T} p_{y_5} \, d\xi$$

$$F_{z_5} = \int_r^{r_T} p_{z_5} \, d\xi$$

which is (6.50), employed in the moment derivative expressions of the last section in the derivation of the F forces.

To obtain moment expressions Mx_5, My_5, and Mz_5, we note the following identities.

$$v'(\xi)\, F_{z_5}(\xi) = \frac{d}{d\xi}\left[(v(\xi) - v(x))\, F_{z_5}(\xi)\right] + \left[v(\xi) - v(x)\right] p_{z_5}(\xi) \tag{6.72}$$

$$-w'(\xi)\, F_{y_5}(\xi) = -\frac{d}{d\xi}\left[(w(\xi) - w(x))\, F_{y_5}(\xi)\right] - (w(\xi) - w(x))\, p_{y_5}(\xi)$$

$$w'(\xi)\, F_{x_5}(\xi) = \frac{d}{d\xi}\left[(w(\xi) - w(x))\, F_{x_5}(\xi)\right] + (w(\xi) - w(x))\, p_{x_5}(\xi)$$

$$-v'(\xi)\, F_{x_5}(\xi) = -\frac{d}{d\xi}\left[(v(\xi) - v(x))\, F_{x_5}(\xi)\right) - (v(\xi) - v(x))\, p_{x_5}(\xi)$$

$$-\int_x^{r_T} F_{z_5}(\xi)\,d\xi = -\int_x^{r_T}(\xi - x)\,P_{z_5}(\xi)\,d\xi + C_1$$

$$\int_x^{r_T} F_{y_5}(\xi)\,d\xi = \int_x^{r_T}(\xi - x)\,P_{y_5}(\xi)\,d\xi + C_2$$

(6.73)

In (6.73) C_1 and C_2 are arbitrary constants. Identities (6.73) may be verified by differentiation.

Integration of (6.68), substitution of (6.72) and (6.73) in the resulting integrals and employment of the condition that tip shears and moments vanish, yields

$$M_{x_5} = \int_x^{r_T} q_{x_5}\,d\xi + \int_x^{r_T}\left[v(\xi) - v(x)\right] P_{z_5}(\xi)\,d\xi - \int_x^{r_T}\left[w(\xi) - w(x)\right] P_{y_5}(\xi)\,d\xi$$

(6.74)

$$M_{y_5} = \int_x^{r_T} q_{y_5}\,d\xi + \int_x^{r_T}\left[w(\xi) - w(x)\right] P_{x_5}(\xi)\,d\xi - \int_x^{r_T}(\xi - x)\,P_{z_5}(\xi)\,d\xi$$

(6.75)

$$M_{z_5} = \int_x^{r_T} q_{z_5}\, d\xi - \int_x^{r_T} \left[v(\xi) - v(x) \right] P_{x_5}(\xi)\, d\xi + \int_x^{r_T} (\xi - x) P_{y_5}(\xi)\, d\xi \tag{6.76}$$

The expressions for local blade oriented moments derive from (6.51) with (6.74) to (6.76) supplying the $\bar{X}_5$ moments.

$$M_{x_{10}} = M_{x_5} + v' M_{y_5} + w' M_{z_5} \tag{6.77}$$

$$M_{y_{10}} = -\lambda_2 M_{x_5} + C_{\bar{\theta}} M_{y_5} + S_{\bar{\theta}} M_{z_5} \tag{6.78}$$

$$M_{z_{10}} = -\lambda_1 M_{x_5} - S_{\bar{\theta}} M_{y_5} + C_{\bar{\theta}} M_{z_5} \tag{6.79}$$

To obtain (6.78) and (6.79) we neglected products of elastic variables, $-\lambda_1 \theta_e M x_5$ and $\lambda_2 \theta_e M x_5$.

Moments (6.77) to (6.79) employing (6.74) to (6.76) are termed the external moments inasmuch as they derive from external sources. Moments (6.17), (6.30) and (6.31) are termed internal moments because they are the equilibrating internal resisting moments.

7. ACCELERATIONS

We derive in this chapter the acceleration of a point on the blade, needed in the expression for the inertia contribution to the generalized force, Q_j^D.

In our approach, the acceleration components are referred to the $\bar{X}_5$ system, and these accelerations are required to involve the linear and angular velocities and accelerations of the hub in the shaft oriented $\bar{X}_1$ system. The parameters of the $\bar{X}_1$ system motion, $\bar{v}_{01}$ and $\bar{\omega}_1$ (Figure 15) and their time derivatives are assumed known. These parameters derive from the solution to the coupled support or coupled rigid body system of equations, obtained according to the following procedure.

The response of the support (or coupled rigid body) derives from the input of rotor forces and moments to the support at the hub. Then the support responses, $\bar{v}_{01}$, $\bar{\omega}_1$, $\dot{\bar{v}}_{01}$ and $\dot{\bar{\omega}}_1$ are returned to the rotor blade equations of motion and new rotor forces and moments are calculated. These new forces are fed back again to the support or rigid body, and the process continues with time to obtain the response of the combined system. Figure 16 shows the flow of calculation coupling the rotor and support modules.

In view of this procedure, our plan is to express the blade acceleration in terms of the motion of the hub in the $\bar{X}_1$ system, as well as other motions, and to resolve the acceleration components to the $\bar{X}_5$ system.

Following this, we elaborate four different versions of the acceleration with a mind to easing and systematizing the subsequent derivations of inertia loads and generalized force Q_j^D, and to effect a separation of modal accelerations $\ddot{q}$ from q and $\dot{q}$ in the modal equation.

Including the acceleration of gravity, the acceleration of a particle on the blade is

$$\bar{a} = \frac{d\bar{V}}{dt} + \bar{g} \tag{7.1}$$

where the velocity of the particle may be expressed

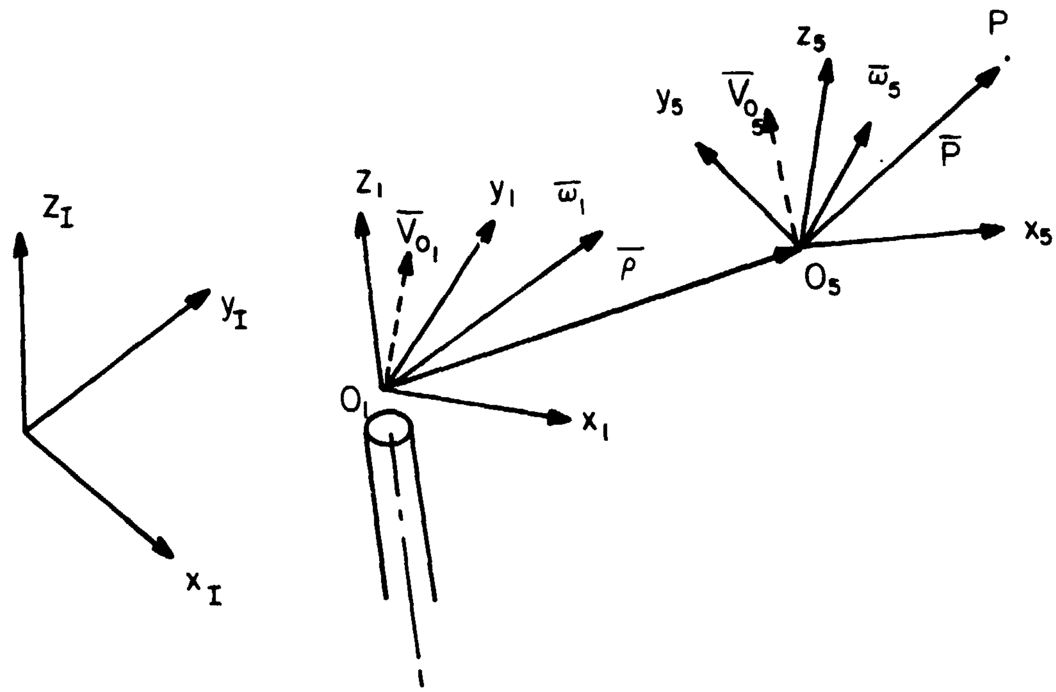


Figure 15. Rectangular Axes Used in the Evaluation of Acceleration.

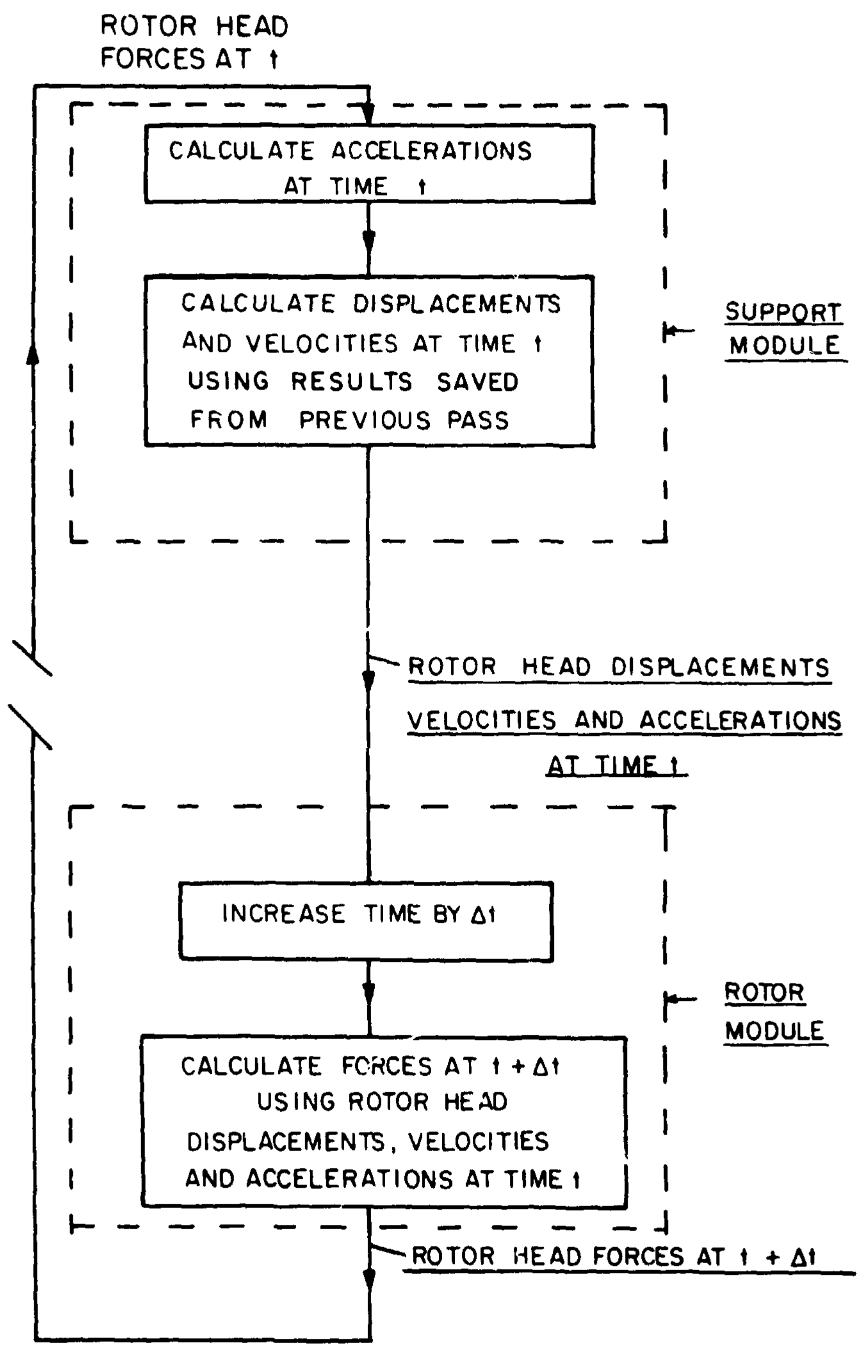


Figure 16. Flow of Calculation Illustrating Rotor/ Support or Rotor/Rigid Body Coupling.

$$V = \overline{V}_{05} + \frac{d\overline{P}}{dt} \tag{7.2}$$

$$\overline{V}_{05} = \overline{V}_{01} + \frac{d\overline{\rho}}{dt} \tag{7.3}$$

In (7.2) and (7.3), $\overline{v}_{01}$ and $\overline{v}_{05}$ are the linear velocities of the origins 0_1 and 0_5 of the $\overline{X}_1$ and $\overline{X}_5$ axes, $\overline{\rho}$ is the displacement vector $\overline{0_1 0_5}$, and $\overline{P}$ is the displacement vector $\overline{0_5 P}$ defining the position of a point on the blade, Figure 15.

Substitute (7.2) and (7.3) in (7.1). We obtain

$$\overline{a} = \overline{a}_{0_5} + \overline{g} + \frac{d^2\overline{P}}{dt^2} \tag{7.4}$$

$$\overline{a}_{0_5} = \frac{d\overline{V}_{05}}{dt} \tag{7.5}$$

$$= \frac{d\overline{V}_{0_1}}{dt} + \frac{d^2\overline{\rho}}{dt^2} \tag{7.6}$$

The grouping in (7.4) eases the evaluation of integrals Q_j^D by distinguishing terms $\overline{a}_{05} + \overline{g}$, independent of blade radius, from the term $d^2\overline{P}/dt^2$, which depends on radius through radial variations of pitch and elastic displacements.

We proceed to evaluate now $\overline{a}_{05}$, $\overline{g}$, and $d^2\overline{P}/dt^2$, which compose the acceleration $\overline{a}$, (7.4).

a) Term $\overline{a}_{05}$

To obtain the acceleration of the origin of the $\overline{X}_5$ system, $\overline{a}_{05}$, we express

$$\bar{v}_{o_5} = \bar{i}_1 (v_{o_5})_{x_1} + \bar{j}_1 (v_{o_5})_{y_1} + \bar{k}_1 (v_{o_5})_{z_1} \tag{7.7}$$

and using (7.5), we find

$$\bar{a}_{o_5} = \frac{d}{dt}\bar{v}_{o_5} = \bar{i}_1 \frac{d}{dt}(v_{o_5})_{x_1} + \bar{j}_1 \frac{d}{dt}(v_{o_5})_{y_1} + \bar{k}_1 \frac{d}{dt}(v_{o_5})_{z_1}$$
$$+ \frac{d\bar{i}_1}{dt}(v_{o_5})_{x_1} + \frac{d\bar{j}_1}{dt}(v_{o_5})_{y_1} + \frac{d\bar{k}_1}{dt}(v_{o_5})_{z_1} \tag{7.8}$$

Employ the properties

$$\frac{d\bar{i}_1}{dt} = \bar{\omega}_1 \times \bar{i}_1 , \quad \frac{d\bar{j}_1}{dt} = \bar{\omega}_1 \times \bar{j}_1 , \quad \frac{d\bar{k}_1}{dt} = \bar{\omega}_1 \times \bar{k}_1 \tag{7.9}$$

where $\bar{\omega}_1 = \bar{i}_1 \omega_{x_1} + \bar{j}_1 \omega_{y_1} + \bar{k}_1 \omega_{z_1}$ (7.10)

Define

$$\left(\frac{\delta}{\delta t}\right)_1 = \bar{i}_1 \frac{d}{dt} + \bar{j}_1 \frac{d}{dt} + \bar{k}_1 \frac{d}{dt} \tag{7.11}$$

We find

$$\bar{a}_{o_5} = \left(\frac{\delta}{\delta t} v_{o_5}\right)_1 + \bar{\omega}_1 \times \bar{v}_{o_5} \tag{7.12}$$

Similarly, from (7.3) we obtain

$$\bar{v}_{o_5} = \bar{v}_{o_1} + \left(\frac{\delta \bar{\rho}}{\delta t}\right)_1 + \bar{\omega}_1 \times_1 \bar{\rho} \tag{7.13}$$

where

$$\bar{v}_{o_1} = \bar{i}_1 v_{o_{x_1}} + \bar{j}_1 v_{o_{y_1}} + \bar{k}_1 v_{o_{z_1}} \tag{7.14}$$

$$\bar{\rho} = \bar{i}_1 \rho_{x_1} + \bar{j}_1 \rho_{y_1} + \bar{k}_1 \rho_{z_1} \tag{7.15}$$

Define

$$\bar{a}_{O_5} = \bar{i}_1 (a_{O_5})_{x_1} + \bar{j}_1 (a_{O_5})_{y_1} + \bar{k}_1 (a_{O_5})_{z_1} \tag{7.16}$$

We obtain from (7.12) and (7.13)

$$(a_{O_5})_{x_1} = (\dot{V}_{O_5})_{x_1} + \omega_{y_1} (V_{O_5})_{z_1} - \omega_{z_1} (V_{O_5})_{y_1} \tag{7.17}$$

$$(a_{O_5})_{y_1} = (\dot{V}_{O_5})_{y_1} - \omega_{x_1} (V_{O_5})_{z_1} + \omega_{z_1} (V_{O_5})_{x_1}$$

$$(a_{O_5})_{z_1} = (\dot{V}_{O_5})_{z_1} + \omega_{x_1} (V_{O_5})_{y_1} - \omega_{y_1} (V_{O_5})_{x_1}$$

$$(V_{O_5})_{x_1} = V_{O_{x_1}} + \dot{\rho}_{x_1} + \omega_{y_1} \rho_{z_1} - \omega_{z_1} \rho_{y_1} \tag{7.18}$$

$$(V_{O_5})_{y_1} = V_{O_{y_1}} + \dot{\rho}_{y_1} - \omega_{x_1} \rho_{z_1} + \omega_{z_1} \rho_{x_1}$$

$$(V_{O_5})_{z_1} = V_{O_{z_1}} + \dot{\rho}_{z_1} + \omega_{x_1} \rho_{y_1} - \omega_{y_1} \rho_{x_1}$$

Substitute (7.18) in (7.17). We obtain

$$\begin{aligned}(a_{O_5})_{x_1} = {} & \dot{V}_{O_{x_1}} + \ddot{\rho}_{x_1} + \dot{\omega}_{y_1} \rho_{z_1} + \omega_{y_1} \dot{\rho}_{z_1} - \dot{\omega}_{z_1} \rho_{y_1} - \omega_{z_1} \dot{\rho}_{y_1} \\ & + \omega_{y_1} (V_{O_{z_1}} + \dot{\rho}_{z_1} + \omega_{x_1} \rho_{y_1} - \omega_{y_1} \rho_{x_1}) \\ & - \omega_{z_1} (V_{O_{y_1}} + \dot{\rho}_{y_1} - \omega_{x_1} \rho_{z_1} + \omega_{z_1} \rho_{x_1})\end{aligned} \tag{7.19}$$

$$\begin{aligned}(a_{O_5})_{y_1} = {} & \dot{V}_{O_{y_1}} + \ddot{\rho}_{y_1} - \dot{\omega}_{x_1} \rho_{z_1} - \omega_{x_1} \dot{\rho}_{z_1} + \dot{\omega}_{z_1} \rho_{x_1} + \omega_{z_1} \dot{\rho}_{x_1} \\ & - \omega_{x_1} (V_{O_{z_1}} + \dot{\rho}_{z_1} + \omega_{x_1} \rho_{y_1} - \omega_{y_1} \rho_{x_1}) \\ & + \omega_{z_1} (V_{O_{x_1}} + \dot{\rho}_{x_1} + \omega_{y_1} \rho_{z_1} - \omega_{z_1} \rho_{y_1})\end{aligned}$$

$$
\begin{aligned}
(a_{05})_{z_1} = {} & \dot v_{0z_1} + \ddot\rho_{z_1} + \dot\omega_{x_1}\rho_{y_1} + \omega_{x_1}\dot\rho_{y_1} - \dot\omega_{y_1}\rho_{x_1} - \omega_{y_1}\dot\rho_{x_1} \\
& + \omega_{x_1}\left(v_{0y_1} + \dot\rho_{y_1} - \omega_{x_1}\rho_{z_1} + \omega_{z_1}\rho_{x_1}\right) \\
& - \omega_{y_1}\left(v_{0x_1} + \dot\rho_{x_1} + \omega_{y_1}\rho_{z_1} - \omega_{z_1}\rho_{y_1}\right)
\end{aligned}
$$

To find $\bar\rho$, we apply successively (5.14) to (5.17) and obtain

$$\bar X_1 = A_\psi\left(\bar E + A_\zeta A_\beta \bar X_5\right) \tag{7.20}$$

Since $\bar\rho = \bar X_1$ for $\bar X_5 = 0$, we get from (7.20)

$$\bar\rho = A_\psi \bar E \tag{7.21}$$

Since $\bar E$ = constant

$$\dot{\bar\rho} = \dot A_\psi \bar E \tag{7.22}$$

$$\ddot{\bar\rho} = \ddot A_\psi \bar E \tag{7.23}$$

We obtain the following results

$$
\begin{aligned}
\rho_{x_1}, \rho_{y_1}, \rho_{z_1} &= \rho c_\psi, \rho s_\psi, 0 \\
\dot\rho_{x_1}, \dot\rho_{y_1}, \dot\rho_{z_1} &= -\rho s_\psi \Omega, \rho c_\psi \Omega, 0 \\
\ddot\rho_{x_1}, \ddot\rho_{y_1}, \ddot\rho_{z_1} &= -\rho c_\psi \Omega^2, -\rho s_\psi \Omega^2, 0
\end{aligned} \tag{7.24}
$$

Equations (7.19) and (7.24) complete the formulas defining $\bar a_{05}$ in the $\bar X_1$ system. To resolve the components of $\bar a_{05}$ to the $\bar X_5$ system, we define

$$\bar a_{05} = \bar i_5 (a_{05})_{x_5} + \bar j_5 (a_{05})_{y_5} + \bar k_5 (a_{05})_{z_5} \tag{7.25}$$

and apply the transformation

$$\begin{pmatrix}(a_{05})_{x_5}\\(a_{05})_{y_5}\\(a_{05})_{z_5}\end{pmatrix} = A_{-\beta}\,A_{-\delta}\,A_{-\psi} \times \begin{pmatrix}(a_{05})_{x_1}\\(a_{05})_{y_1}\\(a_{05})_{z_1}\end{pmatrix} \tag{7.26}$$

Using the repeated suffix convention for summations, we write this

$$\left(a_{05}^{(5)}\right)_i = a_{ij}\left(a_{05}^{(1)}\right)_j \tag{7.27}$$

where the repeated suffix j indicates summation from j = 1 to 3, and where

$$[a_{ij}] = A_{-\beta}\,A_{-\delta}\,A_{-\psi} \tag{7.28}$$

$$= \begin{bmatrix} C_\beta C_\delta C_\psi - C_\beta C_\delta S_\psi, & C_\beta C_\delta S_\psi + C_\beta S_\delta C_\psi, & S_\beta \\ -S_\beta C_\psi - C_\delta S_\psi, & -S_\delta S_\psi + C_\delta C_\psi, & 0 \\ -S_\beta C_\delta C_\psi + S_\beta S_\delta S_\psi, & -S_\beta C_\delta S_\psi - S_\beta S_\delta C_\psi, & C_\beta \end{bmatrix} \tag{7.29}$$

$$\left(a_{05}^{(5)}\right)_i \equiv (a_{05})_{x_5},\ (a_{05})_{y_5},\ (a_{05})_{z_5} \qquad i = 1, 2, 3 \tag{7.30}$$

$$\left(a_{05}^{(1)}\right)_i \equiv (a_{05})_{x_1},\ (a_{05})_{y_1},\ (a_{05})_{z_1} \qquad ''$$

The symbol for the direction cosine, a_{ij} in (7.28) is not to be confused with the generalized mass in normal coordinates, (3.58).

Equation (7.27) defines the components of acceleration of 0_5 in the $\bar{X}_5$ directions, in terms of the motions $\bar{v}_{o1}$ and $\omega 1$ of the $\bar{X}_1$ system, and the generalized coordinates β, δ, and azimuth ψ, and is thus the desired form for $\bar{a}_{o5}$.

b) Term $\bar{g}$

Gravity acceleration components are derived below for each of the two shaft-oriented axes.

Transformations (5.4) to (5.6) yield the relations between free vectors $\bar{X}_1$ and $\bar{X}_I$ in terms of the angles of the grounded support

$$\bar{X}_1 = A_{-\phi_S} \; A_{-\theta_S} \; A_{-\psi_S} \; \bar{X}_I' \qquad (7.31)$$

Identify the free vectors as

$$\bar{X}_1^T = g_{x_1}, \; g_{y_1}, \; g_{z_1} \qquad (7.32)$$
$$\bar{X}_I'^T = 0, \; 0, \; g$$

We obtain

$$\begin{matrix} g_{x_1} \\ g_{y_1} \\ g_{z_1} \end{matrix} = A_{-\phi_S} A_{-\theta_S} A_{-\psi_S} \times \begin{matrix} 0 \\ 0 \\ g \end{matrix} = \begin{matrix} -g\, s\theta_S \\ g\, s\phi_S\, c\theta_S \\ g\, c\phi_S\, c\theta_S \end{matrix} \qquad (7.33)$$

To obtain the $\bar{X}_1$ gravity acceleration components in terms of the angles of the rigid body in free flight, we use (5.12)

$$\bar{X}_1 = (A_{-\theta'})_{\theta' = \pi} \bar{\xi}_1 \qquad (7.34)$$

Replacing $\bar{\xi}_1$ by (5.8) to (5.10) this becomes

$$\bar{X}_1 = (A_{-\theta'})_\pi \; A_{-\phi'} \; A_{-\theta'} \; A_{-\psi'} \; \bar{\xi}_I' \qquad (7.35)$$

and using (5.13) we get

$$\bar{X}_1 = (A_{-\theta'})_\pi \; A_{-\phi'} \; A_{-\theta'} \; A_{-\psi'} \; (A_{\theta'})_\pi \; \bar{X}_I' \qquad (7.36)$$

Use identities (7.32). This yields

$$\begin{Bmatrix} g_{x_1} \\ g_{y_1} \\ g_{z_1} \end{Bmatrix} = (A_{-\theta'}) A_{-\phi'} A_{-\theta'} A_{\psi'} (A_{\theta'})_{\pi} \times \begin{Bmatrix} 0 \\ 0 \\ g \end{Bmatrix} = \begin{Bmatrix} -g s_{\theta'} \\ -g s_{\phi'} c_{\theta'} \\ +g c_{\phi'} c_{\theta'} \end{Bmatrix} \quad (7.37)$$

Identify

$$g_i \equiv g_{x_1}, g_{y_1}, g_{z_1} \qquad i = 1, 2, 3 \quad (7.38)$$

We obtain the $\bar{X}_5$ system components of acceleration from

$$\begin{Bmatrix} g_{x_5} \\ g_{y_5} \\ g_{z_5} \end{Bmatrix} = A_{-\beta} A_{-\zeta} A_{-\psi} \times \begin{Bmatrix} g_{x_1} \\ g_{y_1} \\ g_{z_1} \end{Bmatrix} = \begin{Bmatrix} a_{1j} g_j \\ a_{2j} g_j \\ a_{3j} g_j \end{Bmatrix} \quad (7.39)$$

with a_{ij} defined by (7.29), and the repeated suffix j indicating summation from j = 1 to 3.

Equation (7.39) defines the components of acceleration in the $\bar{X}_5$ system in terms of the $\bar{X}_1$ system gravity acceleration components. The $\bar{X}_1$ system gravity acceleration components derive from (7.33) for the grounded support and (7.37) for the rigid body in free flight.

c) <u>Term $d^2\bar{P}/dt^2$</u>

We derive here the acceleration contribution $d^2\bar{P}/dt^2$ in (7.4), arising from the displacement, $\bar{P}$, of a blade particle from the origin 0_5 of the $\bar{X}_5$ axes.

Defining

$$\bar{P} = \bar{i}_5 P_{x_5} + \bar{j}_5 P_{y_5} + \bar{k}_5 P_{z_5} \quad (7.40)$$

and differentiating this twice, we obtain

$$\frac{d^2\bar{P}}{dt^2} = \bar{i}_5 \frac{d^2 P_{x_5}}{dt^2} + \bar{j}_5 \frac{d^2 P_{y_5}}{dt^2} + \bar{k}_5 \frac{d^2 P_{z_5}}{dt^2}$$
$$+ 2\frac{d\bar{i}_5}{dt}\frac{dP_{x_5}}{dt} + 2\frac{d\bar{j}_5}{dt}\frac{dP_{y_5}}{dt} + 2\frac{d\bar{k}_5}{dt}\frac{dP_{z_5}}{dt}$$
$$+ \left(\frac{d^2\bar{i}_5}{dt^2}\right) P_{x_5} + \left(\frac{d^2\bar{j}_5}{dt^2}\right) P_{y_5} + \left(\frac{d^2\bar{k}_5}{dt^2}\right) P_{z_5} \tag{7.41}$$

To reduce (7.41) to a working form, we evaluate first $d\bar{i}_5/dt$, $d\bar{j}_5/dt$, $d\bar{k}_5/dt$, $d^2\bar{i}_5/dt^2$, $d^2\bar{j}_5/dt^2$, $d^2\bar{k}_5/dt^2$ and this is followed by the expressing of $\bar{P}$ in terms of elastic displacements and pitch angles.

Employ the properties

$$\frac{d\bar{i}_5}{dt} = \bar{\omega}_5 \times \bar{i}_5 , \quad \frac{d\bar{j}_5}{dt} = \bar{\omega}_5 \times \bar{j}_5 , \quad \frac{d\bar{k}_5}{dt} = \bar{\omega}_5 \times \bar{k}_5 \tag{7.42}$$

Define

$$\bar{\omega}_5 = \bar{i}_5 \omega_{x_5} + \bar{j}_5 \omega_{y_5} + \bar{k}_5 \omega_{z_5} \tag{7.43}$$

Use the identities

$$\begin{array}{lll} \bar{i}_5 \times \bar{i}_5 = 0 & \bar{j}_5 \times \bar{i}_5 = -\bar{k}_5 & \bar{k}_5 \times \bar{i}_5 = \bar{j}_5 \\ \bar{i}_5 \times \bar{j}_5 = \bar{k}_5 & \bar{j}_5 \times \bar{j}_5 = 0 & \bar{k}_5 \times \bar{j}_5 = -\bar{i}_5 \\ \bar{i}_5 \times \bar{k}_5 = -\bar{j}_5 & \bar{j}_5 \times \bar{k}_5 = \bar{i}_5 & \bar{k}_5 \times \bar{k}_5 = 0 \end{array} \tag{7.44}$$

We obtain

$$\frac{d\bar{i}_5}{dt} = 0 + \bar{j}_5 \omega_{z_5} - \bar{k}_5 \omega_{y_5} \tag{7.45}$$
$$\frac{d\bar{j}_5}{dt} = -\bar{i}_5 \omega_{z_5} + 0 + \bar{k}_5 \omega_{x_5}$$

$$\frac{d\bar{K}_5}{dt} = \bar{i}_5\, \omega_{y_5} - \bar{j}_5\, \omega_{x_5} + 0$$

We derive now the term $d^2 I_5/dt^2$

$$\frac{d^2\bar{i}_5}{dt^2} = \frac{d}{dt}\left(\frac{d\bar{i}_5}{dt}\right)$$

$$= \frac{d}{dt}\left(\bar{\omega}_5 \times \bar{i}_5\right)$$

$$= \left(\frac{d\bar{\omega}_5}{dt}\right) \times \bar{i}_5 + \bar{\omega}_5 \times \frac{d\bar{i}_5}{dt}$$

(7.46)

This requires the evaluation first of

$$\frac{d\bar{\omega}_5}{dt} = \bar{i}_5\, \dot{\omega}_{x_5} + \bar{J}_5\, \dot{\omega}_{y_5} + \bar{K}_5\, \dot{\omega}_{z_5}$$

$$+ \left(\frac{d\bar{i}_5}{dt}\right) \omega_{x_5} + \left(\frac{d\bar{J}_5}{dt}\right) \omega_{y_5} + \left(\frac{d\bar{K}_5}{dt}\right) \omega_{z_5} \qquad (7.47)$$

$$= \bar{i}_5\, \dot{\omega}_{x_5} + \bar{J}_5\, \dot{\omega}_{y_5} + \bar{K}_5\, \dot{\omega}_{z_5}$$

$$+ \left(\bar{\omega}_5 \times \bar{i}_5\right) \omega_{x_5} + \left(\bar{\omega}_5 \times \bar{J}_5\right) \omega_{y_5} + \left(\bar{\omega}_5 \times \bar{K}_5\right) \omega_{z_5}$$

$$= \bar{i}_5\, \dot{\omega}_{x_5} + \bar{j}_5\, \dot{\omega}_{y_5} + \bar{K}_5\, \dot{\omega}_{z_5} + \bar{\omega}_5 \times \bar{\omega}_5$$

$$= \bar{i}_5\, \dot{\omega}_{x_5} + \bar{J}_5\, \dot{\omega}_{y_5} + \bar{K}_5\, \dot{\omega}_{z_5}$$

(7.48)

The first term in (7.46) becomes

$$\left(\frac{d\bar{\omega}_5}{dt}\right)\times\bar{i}_5 = \bar{i}_5\times\bar{i}_5\,\dot{\omega}_{x_5} + \bar{j}_5\times\bar{i}_5\,\dot{\omega}_{y_5} + \bar{k}_5\times\bar{i}_5\,\dot{\omega}_{z_5} \quad (7.49)$$

$$= \bar{i}_5\cdot 0 + \bar{j}_5\,\dot{\omega}_{z_5} - \bar{k}_5\,\dot{\omega}_{y_5} \quad (7.50)$$

after use of (7.44).

The second term in (7.46) is

$$\bar{\omega}_5 \times \frac{d\bar{i}_5}{dt} = \left(\bar{i}_5\,\omega_{x_5} + \bar{j}_5\,\omega_{y_5} + \bar{k}_5\,\omega_{z_5}\right)\times\left(\bar{j}_5\,\omega_{z_5} - \bar{k}_5\,\omega_{y_5}\right) \quad (7.51)$$

after substitution of (7.43) and (7.45), and reduces to

$$\bar{\omega}_5 \times \frac{d\bar{i}_5}{dt} = -\bar{i}_5\left(\omega_{y_5}^2 + \omega_{z_5}^2\right) + \bar{j}_5\,\omega_{x_5}\omega_{y_5} + \bar{k}_5\,\omega_{x_5}\omega_{z_5} \quad (7.52)$$

Substitution of (7.50) and (7.52) in (7.46) and similar treatments of $d^2\bar{j}_5/dt^2$ and $d^2\bar{k}_5/dt^2$ yields

$$\frac{d^2\bar{i}_5}{dt^2} = \bar{i}_5\left(0 - \omega_{y_5}^2 - \omega_{z_5}^2\right) + \bar{j}_5\left(\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5}\right) + \bar{k}_5\left(-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5}\right) \tag{7.53}$$

$$\frac{d^2\bar{j}_5}{dt^2} = \bar{i}_5\left(-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{x_5}\right) + \bar{j}_5\left(-\omega_{x_5}^2 + 0 - \omega_{z_5}^2\right) + \bar{k}_5\left(\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}\right)$$

$$\frac{d^2\bar{k}_5}{dt^2} = \bar{i}_5\left(\dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5}\right) + \bar{j}_5\left(-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5}\right) + \bar{k}_5\left(-\omega_{x_5}^2 - \omega_{y_5}^2 + 0\right)$$

Define

$$\frac{d^2\bar{P}}{dt^2} = \bar{i}_5 (a_p)_{x_5} + \bar{j}_5 (a_p)_{y_5} + \bar{k}_5 (a_p)_{z_5} \tag{7.54}$$

Substitute (7.45) and (7.53) in the expression for $d^2\bar{P}/dt^2$, (7.41). We obtain

$$(a_p)_{x_5} = \ddot{P}_{x_5} + 2\left(-\omega_{z_5}\dot{P}_{y_5} + \omega_{y_5}\dot{P}_{z_5}\right) + \left(-\omega_{y_5}^2 - \omega_{z_5}^2\right)P_{x_5} + \left(-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{x_5}\right)P_{y_5} + \left(\dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5}\right)P_{z_5} \tag{7.55}$$

$$(a_p)_{y_5} = \ddot{P}_{y_5} + 2(\omega_{z_5}\dot{P}_{x_5} - \omega_{x_5}\dot{P}_{z_5})$$
$$+ (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})P_{x_5}$$
$$+ (-\omega_{x_5}^2 - \omega_{z_5}^2)P_{y_5}$$
$$+ (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})P_{z_5}$$

$$(a_p)_{z_5} = \ddot{P}_{z_5} + 2(-\omega_{y_5}\dot{P}_{x_5} + \omega_{x_5}\dot{P}_{y_5})$$
$$+ (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5})P_{x_5}$$
$$+ (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})P_{y_5}$$
$$+ (-\omega_{x_5}^2 - \omega_{y_5}^2)P_{z_5}$$

These acceleration components are reduced further to useful forms by expressing $\bar{\omega}_5$ and $\bar{P}$ in terms of the generalized coordinates and the motion of the $\bar{X}_1$ axis.

The angular velocity of the $\bar{X}_5$ system is

$$\bar{\omega}_5 = \bar{\omega}_1 + \bar{k}_1\Omega + \bar{k}_4\dot{\delta} - \bar{j}_5\dot{\beta} \qquad (7.56)$$

which, with all contributions resolved to the $\bar{X}_5$ directions, is

$$\begin{Bmatrix} \omega_{x_5} \\ \omega_{y_5} \\ \omega_{z_5} \end{Bmatrix} = A_{-\beta}A_{-\delta}A_{-\psi} \times \begin{Bmatrix} \omega_{x_1} \\ \omega_{y_1} \\ \omega_{z_1} + \Omega \end{Bmatrix} + A_{-\beta} \times \begin{Bmatrix} 0 \\ 0 \\ \dot{\delta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dot{\beta} \\ 0 \end{Bmatrix} \qquad (7.57)$$

Taking account of the definition of the direction cosines a_{ij} in (7.29), the expression for A_β in (5.43), and defining

$$\omega_i^{(1)} = \omega_{x_1}, \omega_{y_1}, \omega_{z_1} + \Omega \qquad i = 1, 2, 3 \tag{7.58}$$

equation (7.57) reduces to

$$\begin{aligned} \omega_{x_5} &= a_{1j}\,\omega_j^{(1)} + s_\beta \dot{\delta} \\ \omega_{y_5} &= a_{2j}\,\omega_j^{(1)} - \dot{\beta} \\ \omega_{z_5} &= a_{3j}\,\omega_j^{(1)} + c_\beta \dot{\delta} \end{aligned} \tag{7.59}$$

where the repeated suffix j indicates summation from j = 1 to 3.

The angular accelerations in (7.55) are obtained by differentiating (7.59) with respect to time.

$$\begin{aligned} \dot{\omega}_{x_5} &= \dot{a}_{1j}\,\omega_j^{(1)} + a_{1j}\,\dot{\omega}_j^{(1)} + \dot{\beta} c_\beta \dot{\delta} + s_\beta \ddot{\delta} \\ \dot{\omega}_{y_5} &= \dot{a}_{2j}\,\omega_j^{(1)} + a_{2j}\,\dot{\omega}_j^{(1)} - \ddot{\beta} \\ \dot{\omega}_{z_5} &= \dot{a}_{3j}\,\omega_j^{(1)} + a_{3j}\,\dot{\omega}_j^{(1)} + \dot{\beta} s_\beta \dot{\delta} + c_\beta \ddot{\delta} \end{aligned} \tag{7.60}$$

Appendix (14.2) defines coefficients $\dot{a}_{ij}$, which are the derivatives of a_{ij} defined in (7.29). The angular accelerations $\dot{\omega}_i^{(1)}$ are

$$\dot{\omega}_i^{(1)} = \dot{\omega}_{x_1}, \dot{\omega}_{y_1}, \dot{\omega}_{z_1} \qquad i = 1, 2, 3 \tag{7.61}$$

obtained from (7.58), with $\dot{\Omega} = 0$.

Equations (7.59) and (7.60) involve the motion of the $\bar{X}_1$ axis, $\omega_{x1}\ \omega_{y1}\ \omega_{z1}$, $\dot{\omega}_{x1}\ \dot{\omega}_{y1}$, $\dot{\omega}_{z1}$, and the generalized coordinates β and δ, and are thus the required forms of angular velocities and accelerations.

We proceed now to find displacement vector $\bar{P}$. This is identical to $\bar{X}$, (6.6). Replace A $\bar{W}_e$ in (6.6) by W (Eq. (5.31)). We find

$$\bar{P} = A_\theta A_{\lambda_2} A_{\lambda_1} A_{\theta_e} \bar{X}_{10} + A_\theta A_{\lambda_2} A_{\lambda_1} \bar{W}_9 + \bar{W} + \bar{R} \tag{7.62}$$

Expansion of (7.62) yields

$$\begin{aligned} P_{x_5} &= r - \lambda_2 y_{10} - \lambda_1 z_{10} \\ P_{y_5} &= v + \Delta v + C_{\bar{\theta}} y_{10} - S_{\bar{\theta}} z_{10} \\ P_{z_5} &= w + \Delta w + S_{\bar{\theta}} y_{10} - C_{\bar{\theta}} z_{10} \end{aligned} \tag{7.63}$$

In the derivation of (7.63) from (7.62), we used (5.83) for v_9 and (5.84) for w_9 to express (7.63) as a function of Δv and Δw (See (5.75) and (5.76) for the expression for Δv and Δw).

To obtain (7.63) we neglected the elastic extension u* and third order products of small quantities involving λ_1, λ_2, θe, y_{10}, and z_{10}. Displacements v and w are second order products of elastic variables v, w, and θ_e (see (5.75) and (5.76)).

The retention of Δv and Δw in (7.63) is subsequently justified by showing that they yield terms in the generalized force Q_j comparable in magnitude to the terms contributed by the other terms in (7.63).

Differentiation of (7.63) yields

(7.64)

$$\dot P_{x_5} = -\dot\lambda_2 y_{10} \quad \dot\lambda_1 z_{10}$$
$$\dot P_{y_5} = \dot v + \Delta\dot v + \dot C_{\bar\theta}\, y_{10} - \dot S_{\bar\theta}\, z_{10}$$
$$\dot P_{z_5} = \dot w + \Delta\dot w + \dot S_{\bar\theta}\, y_{10} + \dot C_{\bar\theta}\, z_{10}$$

$$\ddot P_{x_5} = -\ddot\lambda_2 y_{10} - \ddot\lambda_1 z_{10} \qquad (7.65)$$
$$\ddot P_{y_5} = \ddot v + \Delta\ddot v + \ddot C_{\bar\theta}\, y_{10} - \ddot S_{\bar\theta}\, z_{10}$$
$$\ddot P_{z_5} = \ddot w + \Delta\ddot w + \ddot S_{\bar\theta}\, y_{10} + \ddot C_{\bar\theta}\, z_{10}$$

To complete the reduction of P_{x5}, P_{y5}, and P_{z5} and their derivatives to useful forms we require expressions for λ_1, λ_2, and $\bar\theta$ and their time derivatives in terms of the coordinates w, v, and θ_e and the non-elastic pitch angles. Before doing this, we state the following definitions for pitch angles.

$$\theta = \theta_c + \theta_t \qquad (7.66)$$

$$\bar\theta = \theta + \theta_e \qquad (7.67)$$

$$= \theta_c + \theta_{e_1}$$

(7.68)

where

$$\theta_{e_1} = \theta_t + \theta_e \qquad (7.69)$$

Expressions for λ_1 and λ_2 derive from (5.60) and (5.63), respectively.

$$\lambda_1 = -s_\theta v' + c_\theta w' \qquad (7.70)$$

$$\lambda_2 = c_\theta v' + s_\theta w'$$

The trigonomentric functions required in (7.63) to (7.65) and (7.70) are

$$s_\theta = s_{\theta_c} + \theta_t c_{\theta_c} \qquad (7.71)$$

$$c_\theta = c_{\theta_c} - \theta_t s_{\theta_c}$$

$$s_{\bar\theta} = s_{\theta_c} + \theta_{e_1} c_{\theta_c}$$

$$c_{\bar\theta} = c_{\theta_c} - \theta_{e_1} s_{\theta_c}$$

where θ_t and θ_{e1} are considered small enough to justify the expansions in (7.71).

Substitution of (7.71) in (7.70), differentiation of (7.70), and neglect of products of quantities involving v', w', θ_t and θ_{e1} and assembly with the corresponding approximation to (7.70) yields

$$\lambda_1 = -s_{\theta_c} v' + c_{\theta_c} w' \qquad (7.72)$$

$$\lambda_2 = c_{\theta_c} v' + s_{\theta_c} w'$$

$$\dot\lambda_1 = -s_{\theta_c} \dot v' + c_{\theta_c} \dot w'$$

$$\dot\lambda_2 = c_{\theta_c} \dot v' + s_{\theta_c} \dot w'$$

$$\ddot\lambda_1 = -s_{\theta_c} \ddot v' + c_{\theta_c} \ddot w'$$

$$\ddot\lambda_2 = c_{\theta_c} \ddot v' + s_{\theta_c} \ddot w'$$

Derivatives of trigonometric functions in (7.64) and (7.65) are

$$
\begin{aligned}
\dot{s}_{\bar{\theta}} &= \dot{\bar{\theta}}\, c_{\theta_c} \\
\dot{c}_{\bar{\theta}} &= -\dot{\bar{\theta}}\, s_{\theta_c} \\
\ddot{s}_{\bar{\theta}} &= \ddot{\bar{\theta}}\, c_{\theta_c} \\
\ddot{c}_{\bar{\theta}} &= -\ddot{\bar{\theta}}\, s_{\theta_c}
\end{aligned} \tag{7.73}
$$

where

$$
\begin{aligned}
\dot{\bar{\theta}} &= \dot{\theta}_{e1} \\
\ddot{\bar{\theta}} &= \ddot{\theta}_{e1}
\end{aligned} \tag{7.74}
$$

To obtain (7.73), we neglected products involving $\dot{\bar{\theta}}$, $\ddot{\bar{\theta}}$, and θ_{el}.

Equations (7.72) to (7.74) are the expressions which are substituted in the expressions for P_{x5}, P_{y5}, and P_{z5} and their derivatives, (7.63) to (7.65). The approximations (7.72) to (7.74) will be justified when generalized force Q_j is formed by showing that higher order approximations would contribute negligible terms to Q_j.

At this point we have derived all the information needed for a working form for the acceleration contribution, $d^2\bar{P}/dt^2$, and with this done we are in a position to present in the next section working forms for the total acceleration of the blade particle, defined by (7.4).

7.1 Versions of Blade Acceleration Expression

Four different versions are used for the blade particle acceleration expressions with a mind to easing and systematizing the subsequent derivations. The first version eases the evaluation of section integrals required in the expressions for the inertial loads p_{x5}^D, p_{y5}^D, p_{z5}^D, q_{x5}^D, q_{y5}^D, and q_{z5}^D. The second version provides an explicit notation defining the elastic displacement and pitch angle, θ_t, orders of magnitudes of the terms, to aid the derivation of approximations. The third version separates terms into radially independent and radially dependent members to aid the evaluation of modal integrals in Q_j^D. The fourth version facilitates the distinguishing of blade modal acceleration terms from modal displacement and velocity terms to assist the formation of the coefficient, s_{jk}, of the modal acceleration $\ddot{q}_k$ in (4.14)

a) Section Property Version of Acceleration

To facilitate the evaluation of section properties, we express the acceleration components in the X_5 system in terms of coefficients multiplying the section coordinates y_{10} and z_{10}

$$
\begin{aligned}
a_x &= A_x + B_x y_{10} + C_x z_{10} \\
a_y &= A_y + B_y y_{10} + C_y z_{10} \\
a_z &= A_z + B_z y_{10} + C_z z_{10}
\end{aligned}
\tag{7.75}
$$

To derive the coefficients we utilize (7.4) which reduces to

$$
\begin{aligned}
a_x &= (a_{o5})_{x_5} + g_{x_5} + (a_p)_{x_5} \\
a_y &= (a_{o5})_{y_5} + g_{y_5} + (a_p)_{y_5} \\
a_z &= (a_{o5})_{z_5} + g_{z_5} + (a_p)_{z_5}
\end{aligned}
\tag{7.76}
$$

following substitution of (7.54). Replacement of P_{x5}, P_{y5}, and P_{z5}, and their derivatives in (7.55) by (7.63) to (7.65), substitution of the resulting expressions for

$(a_p)_{x5}$, $(a_p)_{y5}$, and $(a_p)_{z5}$ in (7.76), and comparison of (7.75) and (7.76) yields the following expressions for the coefficients in (7.75).

(7.77)

$$
\begin{aligned}
A_x = {} & (a_{o_5})_{x_5} + g_{x_5} \\
& - 2\omega_{z_5}\dot{v} + 2\omega_{y_5}\dot{w} - (\omega_{y_5}^2 + \omega_{z_5}^2) r \\
& + (-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{x_5}) v + (\dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5}) w
\end{aligned}
$$

$$
\begin{aligned}
B_x = {} & -\ddot{\lambda}_2 - 2\omega_{z_5}\dot{C}_{\bar{\theta}} + 2\omega_{y_5}\dot{S}_{\bar{\theta}} \\
& + (\omega_{y_5}^2 + \omega_{z_5}^2)\lambda_2 \\
& + (-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{z_5}) C_{\bar{\theta}} + (\dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5}) S_{\bar{\theta}}
\end{aligned}
$$

$$
\begin{aligned}
C_x = {} & -\dot{\lambda}_1 + 2\omega_{z_5}\dot{S}_{\bar{\theta}} + 2\omega_{y_5}\dot{C}_{\bar{\theta}} \\
& + (\omega_{y_5}^2 + \omega_{z_5}^2)\lambda_1 \\
& - (-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{z_5}) S_{\bar{\theta}} + (\dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5}) C_{\bar{\theta}}
\end{aligned}
$$

$$
\begin{aligned}
A_y = {} & (a_{o_5})_{y_5} + g_{y_5} \\
& + \dot{v} + \Delta\dot{v} - 2\omega_{x_5}(\dot{w} + \Delta\dot{w}) + (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5}) r \\
& - (\omega_{x_5}^2 + \omega_{z_5}^2)(v + \Delta v) \\
& + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})(w + \Delta w)
\end{aligned}
$$

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$$
\begin{aligned}
B_y = {} & \ddot{c}_{\bar{\Theta}} - 2\omega_{z_5}\dot{\lambda}_2 - 2\omega_{x_5}\dot{s}_{\bar{\Theta}} \\
& - (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})\lambda_2 \\
& - (\omega_{x_5}^2 + \omega_{z_5}^2)\, c_{\bar{\Theta}} + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})\, s_{\bar{\Theta}}
\end{aligned}
$$

$$
\begin{aligned}
C_y = {} & -\ddot{s}_{\bar{\Theta}} - 2\omega_{z_5}\dot{\lambda}_1 - 2\omega_{x_5}\dot{c}_{\bar{\Theta}} \\
& - (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})\lambda_1 \\
& + (\omega_{x_5}^2 + \omega_{z_5}^2)\, s_{\bar{\Theta}} + (-\omega_{x_5} + \omega_{z_5}\omega_{y_5})\, c_{\bar{\Theta}}
\end{aligned}
$$

$$
\begin{aligned}
A_z = {} & (a_{0_5})_{z_5} + g_{z_5} \\
& + \ddot{w} + \Delta\ddot{w} + 2\omega_{x_5}(\dot{v} + \Delta\dot{v}) + (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5})\, r \\
& + (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})(v + \Delta v) - (\omega_{x_5}^2 + \omega_{y_5}^2)(w + \Delta w)
\end{aligned}
$$

$$
\begin{aligned}
B_z = {} & \ddot{s}_{\bar{\Theta}} + 2\omega_{y_5}\dot{\lambda}_2 + 2\omega_{x_5}\dot{c}_{\bar{\Theta}} \\
& - (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5})\lambda_2 \\
& + (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})\, c_{\bar{\Theta}} - (\omega_{x_5}^2 + \omega_{y_5}^2)\, s_{\bar{\Theta}}
\end{aligned}
$$

$$
\begin{aligned}
C_z = {} & \ddot{c}_{\bar{\Theta}} + 2\omega_{y_5}\dot{\lambda}_1 - 2\omega_{x_5}\dot{s}_{\bar{\Theta}} \\
& - (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5})\lambda_1 \\
& - (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})\, s_{\bar{\Theta}} - (\omega_{x_5}^2 + \omega_{y_5}^2)\, c_{\bar{\Theta}}
\end{aligned}
$$

We neglect the higher order displacements v and w and their derivatives in (7.77) in all coefficients except A_y and A_z. This approximation is justified in the next chapter by showing that in the inertia loads P_{x5}^D, P_{y5}^D, P_{z5}^D, q_{x5}^D, q_{y5}^D, q_{z5}^D, only A_y and A_z need to be approximated to this order tc comply with our approximations to the loading equilibrium or modal equations.

b) Order of Magnitude Version for Acceleration

To present a version of acceleration specifying the orders of magnitudes of members in the acceleration expressions in terms of elastic displacements or pitch angle, θ_t, we define the coefficients in (7.75) as

(7.78)

$$A_x = A_x^{(0)} + A_x^{(1)}$$

$$B_x = B_x^{(0)} + B_x^{(1)}$$

$$C_x = C_x^{(0)} + C_x^{(1)}$$

$$A_y = A_y^{(0)} + A_y^{(1)} + A_y^{(2)}$$

$$B_y = B_y^{(0)} + B_y^{(1)}$$

$$C_y = C_y^{(0)} + C_y^{(1)}$$

$$A_z = A_z^{(0)} + A_z^{(1)} + A_z^{(2)}$$

$$B_z = B_z^{(0)} + B_z^{(1)}$$

$$C_z = C_z^{(0)} + C_z^{(1)}$$

Superscript (0) indicates that the coefficient is of zeroeth order in terms of elastic displacements or angle θ_t - that is, such a coefficient contains no elastic displacements w, v, θ_e, or θ_t. Superscript (1) indicates a coefficient with a first order dependence on w, v, θ_e, and θ_t. Superscript (2) indicates a term which contains products of elastic displacements w and v, and θ_e.

Substitution in (7.77) of the approximations to $s_{\bar\theta}$, and $c_{\bar\theta}$ and their time derivatives defined in (7.71) and (7.73), and comparison of the resulting coefficients for A_x, B_x etc with the definition (7.78) yields the desired expressions for $A_x^{(0)}$, $A_x^{(1)}$ etc in (7.78). In these expressions λ_1, λ_2 and their derivatives are to be approximated by (7.72), and (7.69) defines θ_{e1} (= θ_t + θ_e). We obtain

(7.79)

$$A_x^{(0)} = (a_{c_5})_{x_5} + g_{x_5} - (\omega_{y_5}^2 + \omega_{z_5}^2)\, r$$

$$A_x^{(1)} = -2\omega_{z_5}\dot v + 2\omega_{y_5}\dot w + (-\dot\omega_{z_5} + \omega_{y_5}\omega_{x_5})\, v + (\dot\omega_{y_5} + \omega_{z_5}\omega_{x_5})\, w$$

$$B_x^{(0)} = (-\dot\omega_{z_5} + \omega_{y_5}\omega_{x_5})\, c_{\theta_c} + (\dot\omega_{y_5} + \omega_{z_5}\omega_{x_5})\, s_{\theta_c}$$

$$B_x^{(1)} = -\ddot\lambda_2 + 2\omega_{z_5}\dot\theta_{e1} s_{\theta_c} + 2\omega_{y_5}\dot\theta_{e1} c_{\theta_c} + (\omega_{y_5}^2 + \omega_{z_5}^2)\,\lambda_2 - (-\dot\omega_{z_5} + \omega_{y_5}\omega_{x_5})\,\theta_{e1} s_{\theta_c} + (\dot\omega_{y_5} + \omega_{z_5}\omega_{x_5})\,\theta_{e1} c_{\theta_c}$$

$$C_x^{(0)} = -(-\dot\omega_{z_5} + \omega_{y_5}\omega_{x_5})\, s_{\theta_c} + (\dot\omega_{y_5} + \omega_{z_5}\omega_{x_5})\, c_{\theta_c}$$

$$C_x^{(1)} = -\ddot\lambda_1 + 2\omega_{z_5}\dot\theta_{e1} c_{\theta_c} - 2\omega_{y_5}\dot\theta_{e1} s_{\theta_c} + (\omega_{y_5}^2 + \omega_{z_5}^2)\,\lambda_1 - (-\dot\omega_{z_5} + \omega_{y_5}\omega_{x_5})\,\theta_{e1} c_{\theta_c} - (\dot\omega_{y_5} + \omega_{z_5}\omega_{x_5})\,\theta_{e1} s_{\theta_c}$$

$$A_y^{(0)} = (a_{o_5})_{y_5} + g_{y_5} + (\dot\omega_{z_5} + \omega_{x_5}\omega_{y_5})r$$

$$A_y^{(1)} = \dot v - 2\omega_{x_5}\dot w - (\omega_{x_5}^2 + \omega_{z_5}^2)v + (-\dot\omega_{x_5} + \omega_{z_5}\omega_{y_5})w$$

$$A_y^{(2)} = \Delta\dot v - 2\omega_{x_5}\Delta\dot w - (\omega_{x_5}^2 + \omega_{z_5}^2)\Delta v$$
$$+ (-\dot\omega_{x_5} + \omega_{z_5}\omega_{y_5})\Delta w$$

$$B_y^{(0)} = -(\omega_{x_5}^2 + \omega_{z_5}^2)C_{\theta_c} + (-\dot\omega_{x_5} + \omega_{z_5}\omega_{y_5})S_{\theta_c}$$

$$B_y^{(1)} = -\ddot\theta_{e_1} S_{\theta_c} - 2\omega_{z_5}\dot\lambda_2 - 2\omega_{x_5}\dot\theta_{e_1} C_{\theta_c}$$
$$-(\dot\omega_{z_5} + \omega_{x_5}\omega_{y_5})\lambda_2$$
$$+(\omega_{x_5}^2 + \omega_{z_5}^2)\theta_{e_1} S_{\theta_c} + (-\dot\omega_{x_5} + \omega_{z_5}\omega_{y_5})\theta_{e_1} C_{\theta_c}$$

$$C_y^{(0)} = (\omega_{x_5}^2 + \omega_{z_5}^2)S_{\theta_c} + (-\dot\omega_{x_5} + \dot\omega_{z_5}\omega_{y_5})C_{\theta_c}$$

$$C_y^{(1)} = -\ddot\theta_{e_1} C_{\theta_c} - 2\omega_{z_5}\dot\lambda_1 + 2\omega_{x_5}\dot\theta_{e_1} S_{\theta_c}$$
$$-(\dot\omega_{z_5} + \omega_{x_5}\omega_{y_5})\lambda_1$$
$$+(\omega_{x_5}^2 + \omega_{z_5}^2)\theta_{e_1} C_{\theta_c} - (-\dot\omega_{x_5} + \omega_{z_5}\omega_{y_5})\theta_{e_1} S_{\theta_c}$$

$$A_z^{(0)} = (a_{0_5})_{z_5} + g_{z_5} + (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5}) r$$

$$A_z^{(1)} = \ddot{w} + 2\omega_{x_5} \dot{v} + (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) v - (\omega_{x_5}^2 + \omega_{y_5}^2) w$$

$$A_z^{(2)} = \Delta \ddot{w} + 2\omega_{x_5} \Delta \dot{v} + (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) \Delta v - (\omega_{x_5}^2 + \omega_{y_5}^2) \Delta w$$

$$B_z^{(0)} = (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) C_{\theta_c} - (\omega_{x_5}^2 + \omega_{y_5}^2) S_{\theta_c}$$

$$B_z^{(1)} = \ddot{\theta}_{e_1} C_{\theta_c} + 2\omega_{y_5} \dot{\lambda}_2 - 2\omega_{x_5} \dot{\theta}_{e_1} S_{\theta_c} - (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5}) \dot{\lambda}_2 - (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) \theta_{e_1} S_{\theta_c} - (\omega_{x_5}^2 + \omega_{y_5}^2) \theta_{e_1} C_{\theta_c}$$

$$C_z^{(0)} = -(\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) S_{\theta_c} - (\omega_{x_5}^2 + \omega_{y_5}^2) C_{\theta_c}$$

$$C_z^{(1)} = -\ddot{\theta}_{e_1} S_{\theta_c} + 2\omega_{y_5} \dot{\lambda}_1 - 2\omega_{x_5} \dot{\theta}_{e_1} C_{\theta_c} - (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5}) \lambda_1 - (\omega_{x_5} + \omega_{y_5}\omega_{z_5}) \theta_{e_1} C_{\theta_c} + (\omega_{x_5}^2 + \omega_{y_5}^2) \theta_{e_1} S_{\theta_c}$$

The second order character of $Ay^{(2)}$ and $A_z^{(2)}$ is due to the second order character of Δv and Δw which involve products of w, v, and θ_e (see 5.75 and 5.76).

c) Modal Integral Version of Acceleration

To introduce a version of acceleration appropriate to the evaluation of modal integrals we express the generalized coordinates in (7.79) as modal sums, and then employ a convention distinguishing the radial dependences of the terms.

The modal transformations substituted in (7.79) are

$$\beta = \beta_i \; q_i \qquad \delta = \delta_i \, q_i \tag{7.80}$$

$$\theta_e = \theta_i \, q_i \qquad \theta_{e_1} = \theta_t + \theta_i \, q_i \qquad w = w_i \, q_i \qquad v = v_i \, q_i \tag{7.81}$$

with the repeated suffix i indicating summation on the number of modes, M.

Derivatives of these modal sums required in (7.79) are

$$\dot{\beta}, \ddot{\beta} = \beta_i \dot{q}_i , \beta_i \ddot{q}_i \qquad \dot{\delta}, \ddot{\delta} = \delta_i \dot{q}_i , \delta_i \ddot{q}_i \tag{7.82}$$

$$\dot{\theta}_{e_1}, \ddot{\theta}_{e_1} = \dot{\theta}_t + \theta_i \dot{q}_i , \ddot{\theta}_t + \theta_i \ddot{q}_i \qquad \dot{w}, \ddot{w} = w_i \dot{q}_i , w_i \ddot{q}_i \qquad \dot{v}, \ddot{v} = v_i \dot{q}_i , v_i \ddot{q}_i \tag{7.83}$$

Substitution of (7.81) and (7.83) in the expressions for λ_1, λ_2, $s_{\bar\theta}$, $c_{\bar\theta}$ and their derivatives, (7.71) to (7.74), yields

(7.84)

$$\lambda_1 = -s_{\theta_c} v_i' q_i + c_{\theta_c} w_i' q_i$$
$$\lambda_2 = c_{\theta_c} v_i' q_i + s_{\theta_c} w_i' q_i$$
$$\dot\lambda_1 = -s_{\theta_c} v_i' \dot q_i + c_{\theta_c} w_i' \dot q_i$$
$$\dot\lambda_2 = c_{\theta_c} v_i' \dot q_i + s_{\theta_c} w_i' \dot q_i$$
$$\ddot\lambda_1 = -s_{\theta_c} v_i' \ddot q_i + c_{\theta_c} w_i' \ddot q_i$$
$$\ddot\lambda_2 = c_{\theta_c} v_i' \ddot q_i + s_{\theta_c} w_i' \ddot q_i$$
$$s_{\bar\theta} = s_{\theta_c} + (\theta_t + \theta_i q_i)\, c_{\theta_c}$$
$$c_{\bar\theta} = c_{\theta_c} - (\theta_t + \theta_i q_i)\, s_{\theta_c}$$
$$\dot s_{\bar\theta} = (\dot\theta_t + \theta_i \dot q_i)\, c_{\theta_c}$$
$$\dot c_{\bar\theta} = -(\dot\theta_t + \theta_i \dot q_i)\, s_{\theta_c}$$
$$\ddot s_{\bar\theta} = (\ddot\theta_t + \theta_i \ddot q_i)\, c_{\theta_c}$$
$$\ddot c_{\bar\theta} = -(\ddot\theta_t + \theta_i \ddot q_i)\, s_{\theta_c}$$

We also require the time derivatives of Δv and Δw for substitution in $A_y(2)$ and $A_z(2)$ in (7.79). From (5.75) and (5.76)

$$\Delta v = -\int_0^r \left[w - w(\xi) - w'(\xi)(r-\xi) \right] \theta_e'(\xi)\, d\xi$$

$$= -\int_0^r \left[w_i - w_i(\xi) - w_i'(\xi)(r-\xi) \right] q_i \, \theta_K'(\xi) q_K \, d\xi$$

$$\equiv \Delta v_{iK} q_i q_K \tag{7.85}$$

$$\Delta v_{iK} = -\int_0^r \left[w_i - w_i(\xi) - w_i'(\xi)(r-\xi) \right] \theta_K'(\xi)\, d\xi \tag{7.86}$$

$$\Delta w = \Delta w_{iK} q_i q_K$$

(7.87)

$$\Delta w_{iK} = \int_0^r \left[v_i - v_i(\xi) - v_i'(r-\xi) \right] \theta_K'(\xi)\, d\xi$$

(7.88)

Differentiation of (7.85) and (7.87) yields

$$\Delta \dot{v} = \Delta v_{iK} (\dot{q}_i q_K + q_i \dot{q}_K) \tag{7.89}$$

$$\Delta \ddot{v} = \Delta v_{iK} (\ddot{q}_i q_K + 2\dot{q}_i \dot{q}_K + q_i \ddot{q}_K)$$

$$\Delta \dot{w} = \Delta w_{iK} (\dot{q}_i q_K + q_i \dot{q}_K)$$

$$\Delta \ddot{w} = \Delta w_{iK} (\ddot{q}_i q_K + 2\dot{q}_i \dot{q}_K + q_i \ddot{q}_K)$$

The occurrence of two repeated suffixes, i and k, in (7.85), (7.87), and (7.89) indicates a double summation on the modes.

We illustrate now the expansion of typical terms in (7.79), to exemplify the derivation a version of the accelerations separating radially dependent terms from radially independent terms in the acceleration coefficients (7.79).

Substitute as required in (7.79) the modal sums and their derivatives, (7.81) and (7.83), and λ_1, λ_2, $s_{\bar{\beta}}$, $c_{\bar{\beta}}$ and their derivatives from (7.84), and introduce below a set of radially independent coefficients. We obtain, for example.

$$A_y^{(0)} = (a_{c_5})_{y_5} + g_{y_5} + (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5}) r \tag{7.90}$$

$$\equiv a_{0_1}^{y} + a_{0_2}^{y} r \tag{7.91}$$

$$A_y^{(1)} = v_i \ddot{q}_i - 2\omega_{x_5} w_i \dot{q}_i - (\omega_{x_5}^2 + \omega_{z_5}^2) v_i q_i \tag{7.92}$$

$$+ (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5}) w_i q_i$$

$$\equiv a_{11}^{y} w_i + a_{12}^{y} v_i \tag{7.93}$$

$$
\begin{aligned}
A_y^{(2)} = {} & \Delta v_{iK}\left(\ddot{q}_i q_K + 2\dot{q}_i \dot{q}_K + q_i \ddot{q}_K\right) \\
& - 2\omega_{x_5} \Delta w_{iK}\left(\dot{q}_i q_K + q_i \dot{q}_K\right) \\
& - \left(\omega_{x_5}^2 + \omega_{z_5}^2\right) \Delta v_{iK}\, q_i q_K \\
& + \left(-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5}\right) \Delta w_{iK}\, q_i q_K
\end{aligned}
\qquad (7.94)
$$

$$
\equiv a_{21}^y \Delta w_{iK} + a_{22}^y \Delta v_{iK} \qquad (7.95)
$$

$$
\begin{aligned}
B_y^{(1)} = {} & -\left(\ddot{\theta}_t + \theta_i \ddot{q}_i\right) S_{\theta_c} - 2\omega_{z_5}\left(C_{\theta_c} v_i' + S_{\theta_c} w_i'\right) \dot{q}_i \\
& - 2\omega_{x_5}\left(\dot{\theta}_t + \theta_i \dot{q}_i\right) C_{\theta_c} \\
& - \left(\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5}\right)\left(C_{\theta_c} v_i' + S_{\theta_c} w_i'\right) q_i \\
& + \left(\omega_{x_5}^2 + \omega_{z_5}^2\right)\left(\theta_t + \theta_i q_i\right) S_{\theta_c} \\
& + \left(-\dot{\omega}_{z_5} + \omega_{z_5}\omega_{y_5}\right)\left(\theta_t + \theta_i q_i\right) C_{\theta_c}
\end{aligned}
\qquad (7.96)
$$

$$
\begin{aligned}
\equiv {} & b_{11}^y S_{\theta_c} + b_{12}^y C_{\theta_c} + b_{13}^y \theta_i S_{\theta_c} + b_{14}^y \theta_i C_{\theta_c} \\
& + b_{15}^y S_{\theta_c} w_i' + b_{16}^y C_{\theta_c} v_i'
\end{aligned}
\qquad (7.97)
$$

Comparison of (7.90) and (7.91) yields the radially independent coefficients a_{o1}^{y} and a_{o2}^{y}. Similar comparisons yields the other coefficients. We obtain

(7.98)

$$a_{o_1}^{y} = (a_{c_5})_{y_5} + g_{y_5}$$

$$a_{o_2}^{y} = \dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5}$$

$$a_{11}^{y} = -2\omega_{x_5}\dot{q}_i + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})q_i$$

$$a_{12}^{y} = \ddot{q}_i - (\omega_{x_5}^2 + \omega_{z_5}^2)q_i$$

$$a_{21}^{y} = -2\omega_{x_5}(\dot{q}_i q_k + q_i\dot{q}_k) + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})q_i q_k$$

$$a_{22}^{y} = \ddot{q}_i q_k + 2\dot{q}_i\dot{q}_k + q_i\ddot{q}_k - (\omega_{x_5}^2 + \omega_{z_5}^2)q_i q_k$$

$$\vdots$$

$$b_{11}^{y} = -\ddot{\theta}_t + (\omega_{x_5}^2 + \omega_{z_5}^2)\theta_t$$

$$b_{12}^{y} = -2\omega_{x_5}\dot{\theta}_t + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})\theta_t$$

$$b_{13}^{y} = -\ddot{q}_i + (\omega_{x_5}^2 + \omega_{z_5}^2)q_i$$

$$\vdots$$

$$b_{16}^{y} = -2\omega_{z_5}\dot{q}_i - (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})q_i$$

$$\vdots$$

The equations given below list all terms derived in the above manner from (7.79) showing a separation of radially independent coefficients from radially dependent functions involving $r, w_i, v_i, \theta_1, \theta_c, \Delta v_{ik}$, and Δw_{ik}.

$$A_x^{(0)} = a_{01}^x + a_{02}^x r$$

$$A_x^{(1)} = a_{11}^x w_i + a_{12}^x v_i$$

$$B_x^{(0)} = b_{01}^x S_{\theta_c} + b_{02}^x C_{\theta_c}$$

$$B_x^{(1)} = b_{11}^x S_{\theta_c} + b_{12}^x C_{\theta_c} + b_{13}^x \theta_i S_{\theta_c} + b_{14}^x \theta_i C_{\theta_c} + b_{15}^x S_{\theta_c} w_i' + b_{16}^x C_{\theta_c} v_i'$$

$$C_x^{(0)} = C_{01}^x S_{\theta_c} + C_{02}^x C_{\theta_c}$$

$$C_x^{(1)} = C_{11}^x S_{\theta_c} + C_{12}^x C_{\theta_c} + C_{13}^x \theta_i S_{\theta_c} + C_{14}^x \theta_i C_{\theta_c} + C_{15}^x C_{\theta_c} w_i' + C_{16}^x S_{\theta_c} v_i'$$

$$A_y^{(0)} = a_{01}^y + a_{02}^y r$$

$$A_y^{(1)} = a_{11}^y w_i + a_{12}^y v_i$$

$$A_y^{(2)} = a_{21}^y \Delta w_{iK} + a_{22}^y \Delta v_{iK}$$

$$B_y^{(0)} = b_{01}^y S_{\theta_c} + b_{02}^y C_{\theta_c}$$

$$B_y^{(1)} = b_{11}^y S_{\theta_c} + b_{12}^y C_{\theta_c} + b_{13}^y \theta_i S_{\theta_c} + b_{14}^y \theta_i C_{\theta_c} + b_{15}^y S_{\theta_c} w_i' + b_{16}^y C_{\theta_c} v_i'$$

$$C_y^{(0)} = C_{01}^y S_{\theta_c} + C_{02}^y C_{\theta_c}$$

$$C_y^{(1)} = C_{11}^y S_{\theta_c} + C_{12}^y C_{\theta_c} + C_{13}^y \theta_i S_{\theta_c} + C_{14}^y \theta_i C_{\theta_c} + C_{15}^y C_{\theta_c} w_i' + C_{16}^y S_{\theta_c} v_i'$$

$$A_z^{(0)} = a_{01}^z + a_{02}^z r$$

$$A_z^{(1)} = a_{11}^z w_i + a_{12}^z v_i$$

$$A_z^{(2)} = a_{21}^z \Delta w_{iK} + a_{22}^z \Delta v_{iK}$$

$$B_z^{(0)} = b_{01}^z S_{\theta_c} + b_{02}^z C_{\theta_c}$$

$$B_z^{(1)} = b_{11}^z S_{\theta_c} + b_{12}^z C_{\theta_c} + b_{13}^z \theta_i S_{\theta_c} + b_{14}^z \theta_i C_{\theta_c} + b_{15}^z S_{\theta_c} w_i' + b_{16}^z C_{\theta_c} v_i'$$

$$C_z^{(0)} = C_{01}^z S_{\theta_c} + C_{02}^z C_{\theta_c}$$

$$C_z^{(1)} = C_{11}^z S_{\theta_c} + C_{12}^z C_{\theta_c} + C_{13}^z \theta_i S_{\theta_c} + C_{14}^z \theta_i C_{\theta_c} + C_{15}^z C_{\theta_c} w_i' + C_{16}^z S_{\theta_c} v_i'$$

(7.99)

$$a_{01}^x = (a_{05})_{x_5} + g_{x_5}$$

$$a_{02}^x = -(\omega_{y_5}^2 + \omega_{z_5}^2)$$

$$a_{11}^x = 2\omega_{y_5} \dot{q}_i + (\dot{\omega}_{y_5} + \omega_{z_5} \omega_{x_5}) q_i$$

$$a_{12}^x = -2\omega_{z_5} \dot{q}_i + (-\dot{\omega}_{z_5} + \omega_{y_5} \omega_{z_5}) q_i$$

$$b_{01}^x = \dot{\omega}_{y_5} + \omega_{z_5} \omega_{x_5}$$

$$b_{02}^x = -(\dot{\omega}_{z_5} + \omega_{y_5} \omega_{x_5})$$

$$b_{11}^x = 2\omega_{z_5} \dot{\theta}_t - (-\dot{\omega}_{z_5} + \omega_{y_5} \omega_{x_5}) \theta_t$$

$$b_{12}^x = 2\omega_{y_5} \dot{\theta}_t + (\dot{\omega}_{y_5} + \omega_{z_5} \omega_{x_5}) \theta_t$$

$$b_{13}^x = 2\omega_{z_5} \dot{q}_i - (-\dot{\omega}_{z_5} + \omega_{y_5} \omega_{x_5}) q_i$$

$$b_{14}^x = 2\omega_{y_5} \dot{q}_i + (\dot{\omega}_{y_5} + \omega_{z_5} \omega_{x_5}) q_i$$

$$b_{15}^x = -\ddot{q}_i + (\omega_{y_5}^2 + \omega_{z_5}^2) q_i$$

$$b_{16}^x = -\ddot{q}_i + (\omega_{y_5}^2 + \omega_{z_5}^2) q_i$$

$$C_{01}^{x} = -(-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{x_5})$$

$$C_{02}^{x} = \dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5}$$

$$C_{11}^{x} = -2\omega_{y_5}\dot{\theta}_t - (\dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5})\theta_t$$

$$C_{12}^{x} = 2\omega_{z_5}\dot{\theta}_t - (-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{x_5})\theta_t$$

$$C_{13}^{x} = -2\omega_{y_5}\dot{q}_i - (\dot{\omega}_{y_5} + \omega_{z_5}\omega_{x_5})q_i$$

$$C_{14}^{x} = 2\omega_{z_5}\dot{q}_i - (-\dot{\omega}_{z_5} + \omega_{y_5}\omega_{x_5})q_i$$

$$C_{15}^{x} = -\ddot{q}_i + (\omega_{y_5}^2 + \omega_{z_5}^2)q_i$$

$$C_{16}^{x} = \ddot{q}_i - (\omega_{y_5}^2 + \omega_{z_5}^2)q_i$$

$$a_{01}^{y} = (a_{05})_{y_5} + g_{y_5}$$

$$a_{02}^{y} = \dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5}$$

$$a_{11}^{y} = -2\omega_{x_5}\dot{q}_i + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})q_i$$

$$a_{12}^{y} = \ddot{q}_i - (\omega_{x_5}^2 + \omega_{y_5}^2)q_i$$

$$a_{21}^{y} \quad -2\omega_{x_5}(\dot{q}_i q_k + q_i\dot{q}_k) + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})q_i q_k$$

$$a_{22}^{y} = \ddot{q}_i q_k + 2\dot{q}_i\dot{q}_k + q_i\ddot{q}_k - (\omega_{x_5}^2 + \omega_{y_5}^2)q_i q_k$$

$$b_{01}^{y} = -\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5}$$

$$b_{02}^{y} = -(\omega_{x_5}^2 + \omega_{z_5}^2)$$

$$b_{11}^{y} = -\ddot{\theta}_t + (\omega_{x_5}^2 + \omega_{z_5}^2)\theta_t$$

$$b_{12}^{y} = -2\omega_{x_5}\dot{\theta}_t + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})\theta_t$$

$$b_{13}^{y} = -\ddot{q}_i + (\omega_{x_5}^2 + \omega_{z_5}^2)q_i$$

$$b_{14}^{y} = -2\omega_{x_5}\dot{q}_i + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})q_i$$

$$b_{15}^{y} = -2\omega_{z_5}\dot{q}_i - (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})q_i$$

$$b_{16}^{y} = -2\omega_{z_5}\dot{q}_i - (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})q_i$$

$$c_{01}^{y} = \omega_{x_5}^2 + \omega_{z_5}^2$$

$$c_{02}^{y} = -\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5}$$

$$c_{11}^{y} = 2\omega_{x_5}\dot{\theta}_t - (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})\theta_t$$

$$c_{12}^{y} = -\dot{\theta}_t + (\omega_{x_5}^2 + \omega_{z_5}^2)\theta_t$$

$$c_{13}^{y} = 2\omega_{x_5}\dot{q}_i - (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})q_i$$

$$c_{14}^{y} = -\dot{q}_i + (\omega_{x_5}^2 + \omega_{z_5}^2)q_i$$

$$c_{15}^{y} = -2\omega_{z_5}\dot{q}_i - (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})q_i$$

$$c_{16}^{y} = 2\omega_{z_5}\dot{q}_i + (\dot{\omega}_{z_5} + \omega_{x_5}\omega_{y_5})q_i$$

$$a_{0_1}^{z} = (a_{0_5})_{z_5} + g_{z_5}$$

$$a_{0_2}^{z} = -\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5}$$

$$a_{11}^{z} = \ddot{q}_i - (\omega_{x_5}^2 + \omega_{y_5}^2) q_i$$

$$a_{12}^{z} = 2\omega_{x_5}\dot{q}_i + (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) q_i$$

$$a_{21}^{z} = \ddot{q}_i q_k + 2\dot{q}_i \dot{q}_k + q_i \ddot{q}_k - (\omega_{x_5}^2 + \omega_{y_5}^2) q_i q_k$$

$$a_{22}^{z} = 2\omega_{x_5}(\dot{q}_i q_k + q_i \dot{q}_k) + (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) q_i q_k$$

$$b_{01}^{z} = -(\omega_{x_5}^2 + \omega_{y_5}^2)$$

$$b_{02}^{z} = \dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}$$

$$b_{11}^{z} = -2\omega_{x_5}\dot{\theta}_t - (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})\theta_t$$

$$b_{12}^{z} = \ddot{\theta}_t - (\omega_{x_5}^2 + \omega_{y_5}^2)\theta_t$$

$$b_{13}^{z} = -2\omega_{x_5}\dot{q}_i - (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5}) q_i$$

$$b_{14}^{z} = \ddot{q}_i - (\omega_{x_5}^2 + \omega_{y_5}^2) q_i$$

$$b_{15}^{z} = 2\omega_{y_5}\dot{q}_i - (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5}) q_i$$

$$b_{16}^{z} = 2\omega_{y_5}\dot{q}_i - (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5}) q_i$$

$$c_{0_1}^{z} = -(\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})$$

$$c_{0_2}^{z} = -(\omega_{x_5}^2 + \omega_{y_5}^2)$$

$$c_{11}^{z} = -\ddot{\theta}_t + (\omega_{x_5}^2 + \omega_{y_5}^2)\theta_t$$

$$c_{12}^{z} = -2\omega_{x_5}\dot{\theta}_t - (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})\theta_t$$

$$c_{13}^{z} = -\ddot{q}_c + (\omega_{x_5}^2 + \omega_{y_5}^2)q_c$$

$$c_{14}^{z} = -2\omega_{x_5}\dot{q}_c - (\dot{\omega}_{x_5} + \omega_{y_5}\omega_{z_5})q_c$$

$$c_{15}^{z} = 2\omega_{y_5}\dot{q}_c - (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5})q_c$$

$$c_{16}^{z} = -2\omega_{y_5}\dot{q}_c + (-\dot{\omega}_{y_5} + \omega_{x_5}\omega_{z_5})q_c$$

(7.100)

The symbolism for the coefficients a_{01}^x etc in (7.99) indicates the origin of these coefficients, and may be used to trace the source of a term and to attach a physical intepretation to it. A superscript x such as in a_{01}^x indicates a derivation from the x component of acceleration, a_x, in the X_5 system. Likewise, superscripts y and z indicate a_y and a_z derivations. A base 'a' indicates an A term origin in the accelerations, (7.76). A base 'b' would indicate a B term origin, - and a base 'c' - a C term origin. The first subscript in a_{01}^x, etc in (7.100) indicates the order of magnitude origin of the term. Thus a_{01}^x derives from $A_x^{(0)}$, a_{11}^x derives from $A_x^{(1)}$, and a_{21}^y derives from $A_y^{(2)}$.

The occurrence of integer subscripts i or k in any term in (7.99) defines the summation treatment required. Generally, no i or k subscript indicates no summation. Thus the terms $c_{01}^y\, s_{\theta_C}$, $c_{02}^y\, c_{\theta_C}$ in the expression for $C_y^{(0)}$ in (7.99) are not sums. The occurrence of a single i subscript requires a single summation. Thus the term $a_{11}^x\, w_i$ in $A_x^{(1)}$ in (7.99) is a single sum (Note that a a_{11}^x contains a suffix i and thus the product $a_{11}^x\, w_i$ complies with the repeated suffix summation convention). Similarly, $a_{21}^y\, \Delta w_{ik}$ in the expression for $Ay^{(2)}$ is a double sum with i and k repeated in a_{21}^y (See 7.100).

d) <u>Version of Acceleration Expression Separating Modal Acceleration $\ddot{q}$ from q and $\dot{q}$ Terms</u>

In this version of the acceleration expressions we separate the blade modal acceleration terms involving $\ddot{q}$, from the blade modal displacement and velocity terms involving q and $\dot{q}$ in acceleration coefficients a_{01}^x etc in (7.100). This version facilitates the formation of the coefficient s_{jk} of the blade modal acceleration, $\ddot{q}$, in (4.14).

The only sources of blade modal acceleration terms in coefficients a_{01}^x etc, (7.100), are $\dot{\omega}_{x5}$, $\dot{\omega}_{y5}$ $\dot{\omega}_{z5}$, $\dot{q}_i$ and $\dot{\theta}_t$. Employ (7.60) for $\dot{\omega}_{x5}$, $\dot{\omega}_{y5}$, and $\dot{\omega}_{z5}$, substitute the modal transformations $\ddot{\beta}$, $\ddot{\delta}$, (7.82) in (7.60), introduce the expression for θ_t, and underline the modal acceleration terms. We obtain

$$\dot{\omega}_{x_5} = \dot{a}_{1K}\,\omega_K^{(1)} + a_{1K}\,\dot{\omega}_K^{(1)} + \dot{\beta} c_\beta\, \dot{\delta} + \underline{S_\beta\, \delta_K\, \ddot{q}_K} \tag{7.101}$$

$$\dot{\omega}_{y_5} = \dot{a}_{2k}\,\omega_k^{(1)} + a_{2k}\,\dot{\omega}_k^{(1)} - \underline{\beta_k \ddot{q}_k}$$

$$\dot{\omega}_{z_5} = \dot{a}_{3k}\,\omega_k^{(1)} + a_{3k}\,\dot{\omega}_k^{(1)} + \dot{\beta} s_\beta \dot{\delta} + \underline{C_\beta \delta_k \ddot{q}_k}$$

$$\ddot{q}_i = \underline{\delta_{ik}\,\ddot{q}_k}$$

$$\ddot{\theta}_t = \Omega^2 (A_{1s} C_\psi + B_{1s} S_\psi) - \underline{(\beta_k \tan \delta_3' + \delta_k \tan \alpha}$$
$$\underline{+ (w'_k)_A \tan \delta_3)\,\ddot{q}_k}$$

The symbol δ_{ik} is the Kronecker delta and should not be confused with the lead angle component of the mode shape, δ_k. The expression for $\ddot{\theta}_t$ derives from the expression for θ_t established subsequently in Chapter (12).

Remove the modal acceleration terms from (7.101). We obtain the following residues, denoted with asterisks.

$$\dot{\omega}_{x_5}^* = \dot{a}_{1k}\,\omega_k^{(1)} + a_{1k}\,\dot{\omega}_k^{(1)} + \dot{\beta} c_\beta \dot{\delta} \tag{7.102}$$

$$\dot{\omega}_{y_5}^* = \dot{a}_{2k}\,\omega_k^{(1)} + a_{2k}\,\dot{\omega}_k^{(1)}$$

$$\dot{\omega}_{z_5}^* = \dot{a}_{3k}\,\omega_k^{(1)} + a_{3k}\,\dot{\omega}_k^{(1)} + \dot{\beta} s_\beta \dot{\delta}$$

$$\ddot{q}_i^* = 0$$

$$\ddot{\theta}_t^* = \Omega^2 (A_{1s} C_\psi + B_{1s} S_\psi)$$

Equation (7.101) may be expressed as

(7.103)

$$\dot{\omega}_{x_5} = \dot{\omega}_{x_5}^{*} + S_\beta \delta_K \ddot{q}_K$$

$$\dot{\omega}_{y_5} = \dot{\omega}_{y_5}^{*} - \beta_K \ddot{q}_K$$

$$\dot{\omega}_{z_5} = \dot{\omega}_{z_5}^{*} + C_\beta \delta_K \ddot{q}_K$$

$$\ddot{q}_i = \ddot{q}_i^{*} + \delta_{iK} \ddot{q}_K$$

$$\ddot{\theta}_t = \ddot{\theta}_t^{*} - (\beta_K \tan\delta_3' + \delta_K \tan\alpha + (w_K')_A \tan\delta_3)\ddot{q}_K$$

To develop an explicit notation for the residues of the acceleration coefficients deprived of $\ddot{q}$ terms, and for the removed terms themselves which are $\ddot{q}$ dependent, we re-define the acceleration coefficients (7.100), in terms of residues and $\ddot{q}$ - dependent terms.

We express the coefficients in (7.100) as

(7.104)

$$a_{01}^{x} = \alpha_{01}^{x} - \rho_{01}^{x} \ddot{q}_K$$

$$a_{02}^{x} = \alpha_{02}^{x} - \rho_{02}^{x} \ddot{q}_K$$

$$\vdots$$

$$b_{16}^{x} = \beta_{16}^{x} - \sigma_{16}^{x} \ddot{q}_K$$

$$c_{01}^{x} = \delta_{01}^{x} - \nu_{01}^{x} \ddot{q}_K$$

$$\vdots$$

$$a_{11}^{y} = \alpha_{11}^{y} - \rho_{11}^{y} \ddot{q}_K$$

$$\vdots$$

$$b_{01}^{z} = \beta_{01}^{z} - \sigma_{16}^{z} \ddot{q}_K$$

$$\vdots$$

$$a_{21}^{y} = \alpha_{21}^{y} - \rho_{21}^{y}(i, l, k)\ddot{q}_{K} \qquad (7.105)$$

$$a_{22}^{y} = \alpha_{22}^{y} + \ddot{q}_{i}\, q_{K} + q_{i}\,\ddot{q}_{K}$$

$$a_{21}^{z} = \alpha_{22}^{z} + \dot{q}_{i}\, q_{K} + q_{i}\,\ddot{q}_{K}$$

$$a_{22}^{z} = \alpha_{21}^{z} - \rho_{21}^{z}(i, l, k)\ddot{q}_{K}$$

All the coefficients in (7.100), except the four listed in (7.105) are like (7.104), and may be generalized to

$$(7.106)$$

$$a_{mn}^{X} = \alpha_{mn}^{X} - \rho_{mn}^{X}\ddot{q}_{K}$$

$$b_{mn}^{X} = \beta_{mn}^{X} - \sigma_{mn}^{Y}\ddot{q}_{K}$$

$$c_{mn}^{X} = \gamma_{mn}^{X} - \nu_{mn}^{X}\ddot{q}_{K}$$

In (7.106) x, y, z in (7.100) substitute for X to yield from (7.106) any required element in (7.104). The integer subscripts m and n are taken from the subscripts in (7.100) to generate the elements in (7.104), with exceptions not complying with this rule listed in (7.105).

We illustrate now the derivation of typical terms α_{mn}^{x}, β_{mn}^{x}, γ_{mn}^{x}, ρ_{mn}^{x}, σ_{mn}^{x}, and ν_{mn}^{x} in (7.104) and we also illustrate the derivation of terms in (7.105).

We substitute in the coefficients a_{01}^{x} etc in (7.100), the terms ω_{x5}, $\dot{\omega}_{y5}$ etc from (7.103), as required. Typically, we obtain from (7.100)

$$(7.107)$$

$$\begin{aligned} a_{11}^{X} &= 2\omega_{y5}\dot{q}_{i} + (\dot{\omega}_{y5} + \omega_{z5}\,\omega_{x5})q_{i} \\ &= 2\omega_{y5}\dot{q}_{i} + (\dot{\omega}_{x5}^{*} - \beta_{K}\ddot{q}_{K} + \omega_{z5}\,\omega_{x5})q_{i} \\ &\equiv \alpha_{11}^{X} - \rho_{11}^{X}\ddot{q}_{K} \end{aligned}$$

$$(7.108)$$

Comparison of terms in (7.107) and (7.108) yields

$$\alpha_{11}^{x} = 2\omega_{y_5}\dot{q}_i + (\dot{\omega}_{y_5}^{*} + \omega_{z_5}\omega_{x_5})\, q_i \tag{7.109}$$

$$\rho_{11}^{x} = \beta_K q_i \tag{7.110}$$

Similarly, we find

$$b_{13}^{y} = -\ddot{q}_i + (\omega_{x_5}^2 + \omega_{z_5}^2)\, q_i \tag{7.111}$$

$$\equiv \beta_{13}^{y} - \sigma_{13}^{y}\ddot{q}_K \tag{7.112}$$

yielding

$$\beta_{13}^{y} = (\omega_{x_5}^2 + \omega_{z_5}^2)\, q_i \tag{7.113}$$

The second coefficient, σ_{13}^{y}, comes from

$$\sigma_{13}^{y}\ddot{q}_K = \ddot{q}_i$$

$$= \delta_{iK}\ddot{q}_K$$

yielding

$$\sigma_{13}^{y} = \delta_{iK} \tag{7.114}$$

Determination of the coefficients in (7.105) is exemplified by

$$a_{21}^{y} = -2\omega_{x_5}(\dot{q}_i q_k + q_i \dot{q}_k) + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5}) q_i q_k$$
$$= -2\omega_{x_5}(\dot{q}_i q_k + q_i \dot{q}_k) +$$
$$\left[\dot{a}_{1k}\omega_k^{(1)} + a_{1k}\dot{\omega}_k^{(1)} + \dot{\beta} c_\beta \dot{S} + S_\beta \delta_k \ddot{q}_k\right] q_i q_k \leftarrow \ell$$

(7.115)

$$\equiv \alpha_{21}^{y} - \rho_{21}^{y}(i,\ell,k)\ddot{q}_k$$

(7.116)

Comparisons of terms in (7.115) and (7.116) yields

$$\alpha_{21}^{y} = -2\omega_{x_5}(\dot{q}_i q_k + q_i \dot{q}_k) + (-\dot{\omega}^{*}_{x_5} + \omega_{z_5}\omega_{y_5}) q_i q_k$$

(7.117)

(7.118)

$$\rho_{21}^{y}(i, l, k) = -S_\beta \delta_K q_i q_K \leftarrow l$$

The notation $k \leftarrow 1$ in (7.118) means that k must be replaced by the new dummy suffix 1 in q_k, and in multiplication of $q_k \leftarrow 1$ with terms like Δw_{ik}, subsequently performed, reminds us to change the k in Δw_{ik} to l. The reason for the replacement $k \leftarrow 1$ in (7.118) is to prevent the occurrence of a triple subscript k in (7.118), which has no meaning in our repeated suffix convention.

Equations (7.119) and (7.120) given below list all coefficients $\alpha_{mn}{}^{x}$, $\beta_{mn}{}^{x}$, $\delta_{mn}{}^{x}$, $\rho_{mn}{}^{x}$, $\sigma_{mn}{}^{x}$, and $\gamma_{mn}{}^{x}$ derived from the acceleration coefficients in the above manner.

$\alpha^X_{mn}, \beta^X_{mn}, \gamma^X_{mn}$

$$\alpha^X_{01} = (a_{0_5})_{x_5} + g_{x_5}$$

$$\alpha^X_{02} = -(\omega^2_{y_5} + \omega^2_{z_5})$$

$$\alpha^X_{11} = 2\omega_{y_5}\dot{q}_c + (\dot{\omega}^*_{y_5} + \omega_{z_5}\omega_{x_5})q_c$$

$$\alpha^X_{12} = -2\omega_{z_5}\dot{q}_c + (-\dot{\omega}^*_{z_5} + \omega_{y_5}\omega_{x_5})q_c$$

$$\beta^X_{01} = \dot{\omega}^*_{y_5} + \omega_{z_5}\omega_{x_5}$$

$$\beta^X_{02} = -(\dot{\omega}^*_{z_5} + \omega_{y_5}\omega_{x_5})$$

$$\beta^X_{11} = 2\omega_{z_5}\dot{\theta}_t - (-\dot{\omega}^*_{z_5} + \omega_{y_5}\omega_{x_5})\theta_t$$

$$\beta^X_{12} = 2\omega_{y_5}\dot{\theta}_t + (\dot{\omega}^*_{y_5} + \omega_{z_5}\omega_{x_5})\theta_t$$

$$\beta^X_{13} = 2\omega_{z_5}\dot{q}_c - (-\dot{\omega}^*_{z_5} + \omega_{y_5}\omega_{x_5})q_c$$

$$\beta^X_{14} = 2\omega_{y_5}\dot{q}_c + (\dot{\omega}^*_{y_5} + \omega_{z_5}\omega_{x_5})q_c$$

$$\beta^X_{15} = (\omega^2_{y_5} + \omega^2_{z_5})q_c$$

$$\beta^X_{16} = (\omega^2_{y_5} + \omega^2_{z_5})q_c$$

$$\gamma^X_{01} = -(-\dot{\omega}^*_{z_5} + \omega_{y_5}\omega_{x_5})$$

$$\gamma^X_{02} = \dot{\omega}^*_{y_5} + \omega_{z_5}\omega_{x_5}$$

$$\gamma^X_{11} = -2\omega_{y_5}\dot{\theta}_t - (+\dot{\omega}^*_{y_5} + \omega_{z_5}\omega_{x_5})\theta_t$$

$$\gamma^X_{12} = 2\omega_{z_5}\dot{\theta}_t - (-\dot{\omega}^*_{z_5} + \omega_{y_5}\omega_{x_5})\theta_t$$

$\alpha_{mn}^{x}, \beta_{mn}^{x}, \gamma_{mn}^{x}$

$$\gamma_{13}^{x} = -2\omega_{y_5}\dot{q}_i - (\dot{\omega}_{y_5}^{*} + \omega_{z_5}\omega_{x_5})q_i$$

$$\gamma_{14}^{x} = 2\omega_{z_5}\dot{q}_i - (-\dot{\omega}_{z_5}^{*} + \omega_{y_5}\omega_{x_5})q_i$$

$$\gamma_{15}^{x} = (\omega_{y_5}^{2} + \omega_{z_5}^{2})q_i$$

$$\gamma_{16}^{x} = -(\omega_{y_5}^{2} + \omega_{z_5}^{2})q_i$$

$$\alpha_{01}^{y} = (a_{o5})_{y_5} + g_{y_5}$$

$$\alpha_{02}^{y} = \dot{\omega}_{z_5}^{*} + \omega_{x_5}\omega_{y_5}$$

$$\alpha_{11}^{y} = -2\omega_{x_5}\dot{q}_i + (-\dot{\omega}_{x_5}^{*} + \omega_{z_5}\omega_{y_5})q_i$$

$$\alpha_{12}^{y} = -(\omega_{x_5}^{2} + \omega_{y_5}^{2})q_i$$

$$\alpha_{21}^{y} = -2\omega_{x_5}(\dot{q}_i q_K + q_i \dot{q}_K) + (-\dot{\omega}_{x_5} + \omega_{z_5}\omega_{y_5})q_i q_K$$

$$\alpha_{22}^{y} = 2\dot{q}_i \dot{q}_K - (\omega_{x_5}^{2} + \omega_{y_5}^{2})q_i q_K$$

$$\beta_{01}^{y} = -\dot{\omega}_{x_5}^{*} + \omega_{z_5}\omega_{y_5}$$

$$\beta_{02}^{y} = -(\omega_{x_5}^{2} + \omega_{z_5}^{2})$$

$$\beta_{11}^{y} = -\ddot{\theta}_t^{*} + (\omega_{x_5}^{2} + \omega_{z_5}^{2})\theta_t$$

$$\beta_{12}^{y} = -2\omega_{x_5}\dot{\theta}_t + (-\dot{\omega}_{x_5}^{*} + \omega_{z_5}\omega_{y_5})\theta_t$$

$$\beta_{13}^{y} = (\omega_{x_5}^{2} + \omega_{z_5}^{2})q_i$$

$$\beta_{14}^{y} = -2\omega_{x_5}\dot{q}_i + (-\dot{\omega}_{x_5}^{*} + \omega_{z_5}\omega_{y_5})q_i$$

$$\beta_{15}^{y} = -2\omega_{z_5}\dot{q}_i - (\dot{\omega}_{z_5}^{*} + \omega_{x_5}\omega_{y_5})q_i$$

$$\beta_{16}^{y} = -2\omega_{z_5}\dot{q}_i - (\dot{\omega}_{z_5}^{*} + \omega_{x_5}\omega_{y_5})q_i$$

$\alpha_{mn}^{x},\ \beta_{mn}^{x},\ \gamma_{mn}^{x}$

$$\gamma_{01}^{y} = \omega_{x_5}^2 + \omega_{z_5}^2$$

$$\gamma_{02}^{y} = -\dot{\omega}_{x_5}^{*} + \omega_{z_5}\omega_{y_5}$$

$$\gamma_{11}^{y} = 2\omega_{x_5}\dot{\theta}_t - (-\dot{\omega}_{x_5}^{*} + \omega_{z_5}\omega_{y_5})\theta_t$$

$$\gamma_{12}^{y} = -\ddot{\theta}_t^{*} + (\omega_{x_5}^2 + \omega_{z_5}^2)\theta_t$$

$$\gamma_{13}^{y} = 2\omega_{x_5}\dot{q}_i - (-\dot{\omega}_{x_5}^{*} + \omega_{z_5}\omega_{y_5})q_i$$

$$\gamma_{14}^{y} = (\omega_{x_5}^2 + \omega_{z_5}^2)q_i$$

$$\gamma_{15}^{y} = -2\omega_{z_5}\dot{q}_i - (\dot{\omega}_{z_5}^{*} + \omega_{x_5}\omega_{y_5})q_i$$

$$\gamma_{16}^{y} = 2\omega_{z_5}\dot{q}_i + (\dot{\omega}_{z_5}^{*} + \omega_{x_5}\omega_{y_5})q_i$$

$$\alpha_{01}^{z} = (a_{0_5})_{z_5} + g_{z_5}$$

$$\alpha_{02}^{z} = -\dot{\omega}_{y_5}^{*} + \omega_{x_5}\omega_{z_5}$$

$$\alpha_{11}^{z} = -(\omega_{x_5}^2 + \omega_{y_5}^2)q_i$$

$$\alpha_{12}^{z} = 2\omega_{x_5}\dot{q}_i + (\dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5})q_i$$

$$\alpha_{21}^{z} = 2\dot{q}_i\dot{q}_K - (\omega_{x_5}^2 + \omega_{y_5}^2)q_i q_K$$

$$\alpha_{22}^{z} = 2\omega_{x_5}(\dot{q}_i q_K + q_i\dot{q}_K) + (\dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5})q_i q_K$$

$$\beta_{01}^{z} = -(\omega_{x_5}^2 + \omega_{y_5}^2)$$

$$\beta_{02}^{z} = \dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5}$$

$$\beta_{11}^{z} = -2\omega_{x_5}\dot{\theta}_t - (\dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5})\theta_t$$

$$\beta_{12}^{z} = \ddot{\theta}_t^{*} - (\omega_{x_5}^2 + \omega_{y_5}^2)\theta_t$$

$$\overset{x}{\alpha}_{mn},\ \overset{x}{\beta}_{mn},\ \overset{x}{\gamma}_{mn}$$

$$\beta_{13}^{z} = -2\omega_{x_5}\dot{q}_i - (\dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5})q_i$$

$$\beta_{14}^{z} = -(\omega_{x_5}^{2} + \omega_{y_5}^{2})q_i$$

$$\beta_{15}^{z} = 2\omega_{y_5}\dot{q}_i - (-\dot{\omega}_{y_5}^{*} + \omega_{x_5}\omega_{z_5})q_i$$

$$\beta_{16}^{z} = 2\omega_{y_5}\dot{q}_i - (-\dot{\omega}_{y_5}^{*} + \omega_{x_5}\omega_{z_5})q_i$$

$$\gamma_{01}^{z} = -(\dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5})$$

$$\gamma_{02}^{z} = -(\omega_{x_5}^{2} + \omega_{y_5}^{2})$$

$$\gamma_{11}^{z} = -\ddot{\theta}_t^{*} + (\omega_{x_5}^{2} + \omega_{y_5}^{2})\theta_t$$

$$\gamma_{12}^{z} = -2\omega_{x_5}\dot{\theta}_t - (\dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5})\theta_t$$

$$\gamma_{13}^{z} = (\omega_{x_5}^{2} + \omega_{y_5}^{2})q_i$$

$$\gamma_{14}^{z} = -2\omega_{x_5}\dot{q}_i - (\dot{\omega}_{x_5}^{*} + \omega_{y_5}\omega_{z_5})q_i$$

$$\gamma_{15}^{z} = 2\omega_{y_5}\dot{q}_i - (-\dot{\omega}_{y_5}^{*} + \omega_{x_5}\omega_{z_5})q_i$$

$$\gamma_{16}^{z} = -2\omega_{y_5}\dot{q}_i + (-\dot{\omega}_{y_5}^{*} + \omega_{x_5}\omega_{z_5})q_i \tag{7.119}$$

ρ_{mn}^{x}, σ_{mn}^{x}, ν_{mn}^{x}

$\rho_{01}^{x} = 0$

$\rho_{02}^{x} = 0$

$\rho_{11}^{x} = \beta_K q_i$

$\rho_{12}^{x} = + c_\beta \delta_K q_i$

$\sigma_{01}^{x} = \beta_K$

$\sigma_{02}^{x} = c_\beta \delta_K$

$\sigma_{11}^{x} = - c_\beta \delta_K \theta_t$

$\sigma_{12}^{x} = \beta_K \theta_t$

$\sigma_{13}^{x} = - c_\beta \delta_K q_i$

$\sigma_{14}^{x} = \beta_K q_i$

$\sigma_{15}^{x} = \delta_{iK}$

$\sigma_{16}^{x} = \delta_{iK}$

$\nu_{01}^{x} = - c_\beta \delta_K$

$\nu_{02}^{x} = \beta_K$

$\nu_{11}^{x} = -\beta_K \theta_t$

$\nu_{12}^{x} = - c_\beta \delta_K \theta_t$

$\rho_{mn}^{x}, \sigma_{mn}^{x}, \nu_{mn}^{x}$

$$\nu_{13}^{x} = -\beta_K q_i$$

$$\nu_{14}^{x} = -c_\beta S_K q_i$$

$$\nu_{15}^{x} = \delta_{iK}$$

$$\nu_{16}^{x} = -\delta_{iK}$$

$$\rho_{01}^{y} = 0$$

$$\rho_{02}^{y} = -c_\beta S_K$$

$$\rho_{11}^{y} = s_\beta S_K q_i$$

$$\rho_{12}^{y} = -\delta_{iK}$$

$$\rho_{21}^{y} = s_\beta \delta_K q_i \; q_\ell$$

$$\sigma_{01}^{y} = s_\beta \delta_K$$

$$\sigma_{02}^{y} = 0$$

$$\sigma_{11}^{y} = -(\beta_K \tan \delta_3' + S_K \tan \alpha + (W_K')_A \tan \delta_3)$$

$$\sigma_{12}^{y} = s_\beta \delta_K \theta_t$$

$$\sigma_{13}^{y} = \delta_{iK}$$

$$\sigma_{14}^{y} = s_\beta S_K q_i$$

$$\sigma_{15}^{y} = c_\beta S_K q_i$$

$$\sigma_{16}^{y} = c_\beta \delta_K q_i$$

ρ^x_{mn}, σ^x_{mn}, ν^x_{mn}

$\nu^y_{01} = 0$

$\nu^y_{02} = S_\beta \delta_K$

$\nu^y_{11} = -S_\beta \delta_K \theta_i$

$\nu^y_{12} = -(\beta_K \tan \delta_3' + \delta_K \tan \alpha + (w_K')_A \tan \delta_3)$

$\nu^y_{13} = -S_\beta \delta_K q_i$

$\nu^y_{14} = \delta_{iK}$

$\nu^y_{15} \cdot C_\beta \delta_K q_i$

$\nu^y_{16} = -C_\beta \delta_K q_i$

$\rho^z_{01} = 0$

$\rho^z_{02} = -\beta_K$

$\rho^z_{11} = -\delta_{iK}$

$\rho^z_{12} = -S_\beta \delta_K q_i$

$\rho^z_{22} = -S_\beta \delta_K q_i q_l$

$\sigma^z_{01} = 0$

$\sigma^z_{02} = -S_\beta \delta_K$

$\sigma^z_{11} = S_\beta \delta_K \theta_i$

$\sigma^z_{12} = \beta_K \tan \delta_3' + \delta_K \tan \alpha + (w_K')_A \tan \delta_3$

$\rho^{x}_{mn}, \sigma_{mn}; \nu_{mn}$

$\sigma_{13}^{z} = S_{\beta} \delta_{K} q_{i}$

$\sigma_{14}^{z} = -\delta_{iK}$

$\sigma_{15}^{z} = \beta_{K} q_{i}$

$\sigma_{16}^{z} = \beta_{K} q_{i}$

$\nu_{01}^{z} = S_{\beta} \delta_{K}$

$\nu_{02}^{z} = 0$

$\nu_{11}^{z} = -(\beta_{K} \tan \delta_{3}' + \delta_{K} \tan \alpha + (w_{K}')_{H} \tan \delta_{3})$

$\nu_{12}^{z} = S_{\beta} \delta_{K} \theta_{i}$

$\nu_{13}^{z} = \delta_{iK}$

$\nu_{14}^{z} = S_{\beta} \delta_{K} q_{i}$

$\nu_{15}^{z} = \beta_{K} q_{i}$

$\nu_{16}^{z} = -\beta_{K} q_{i}$

(7.120)

We recall that α_{mn}^x, β_{mn}^x, and γ_{mn}^x, in (7.119) are the residues of a_{mn}^x, b_{mn}^x, and c_{mn}^x, (7.100), after removal of the blade modal acceleration terms $\ddot{q}$. Consequently, it is reasonable to expect that the substitution

$$\begin{aligned} a_{mn}^x &\leftarrow \alpha_{mn}^x \\ b_{mn}^x &\leftarrow \beta_{mn}^x \\ c_{mn}^x &\leftarrow \gamma_{mn}^x \end{aligned} \qquad (7.121)$$

in Q_j^D, formed in the next section, will yield Q_j^{D*}, which is the residue of Q_j^D after removal of the $\ddot{q}$ terms.

Similarly, the substitution

$$\begin{aligned} a_{mn}^x &\leftarrow \rho_{mn}^x \\ b_{mn}^x &\leftarrow \sigma_{mn}^x \\ c_{mn}^x &\leftarrow \nu_{mn} \end{aligned} \qquad (7.122)$$

in the expression for Q_j^D should yield s_{jk}.

The validities of substitutions (7.121) and (7.122) are demonstrated in the next chapter, and these substitutions are thereafter employed to facilitate the derivations of Q_j^{D*} and s_{jk}.

7.2 Orders of Magnitudes of Acceleration Coefficients for Small Hub Accelerations

In this section, we illustrate, for small hub accelerations, the procedure used to determine the magnitudes of the acceleration coefficients in (7.119) and (7.120). This is followed in Section 8.4 with a description of the procedure for neglecting small terms contributing to Q_j^D for small hub accelerations. The elimination of these terms effects a substantial reduction in the number of significant terms retained in Q_j^D, considerably simplifying the algebra.

As a preliminary we state that ϵ will designate any small quantity while unity designates a non-small quantity. To make sure that the symbol ϵ, which uniformly designates the orders of the different parameters, assumes comparable values among the parameters, we non-dimensionalize the modal equations. In the following section, we assume that all quantitites have been non-dimensionalized, according to the method of Chapter (12), without introducing a new explicit notation for non-dimensional quantities.

The assumption of small hub accelerations requires the following $\overline{X}_1$ axis quantities to be small.

$$\begin{matrix} \dot{V}_{0x_1}, \dot{V}_{0y_1}, \dot{V}_{0z_1}, \\ \omega_{x_1}, \omega_{y_1}, \omega_{z_1} \\ \dot{\omega}_{x_1}, \dot{\omega}_{y_1}, \dot{\omega}_{z_1} \end{matrix} = O(\epsilon) \tag{7.123}$$

In addition, it is convenient to recall here that the following are also small.

$$\beta, \delta, w_1, v_1 = O(\epsilon) \tag{7.124}$$

$$\theta_e, w, v = O(\theta_e), O(w), O(v) \tag{7.125}$$

$$\begin{matrix} e, y_{10_{cg}}, K_{z_{10}}, K_{y_{10}} \\ K_A, e_A, g, \theta_t \end{matrix} = O(\epsilon) \tag{7.126}$$

We note that since $w_1 = w + \beta r$, and $v_1 = v + \delta r$, and β and δ are of $0(\epsilon)$, then w_1 and v_1 are of $0(\epsilon)$ in (7.124)

Displacements θ_e, w, and v in (7.125) are expressed as elastic displacement orders in contrast to an ϵ -order to achieve the specificness required by our assumptions distinguishing the orders of magnitudes of θ_e, w, and v from ϵ. These assumptions are invoked in the next chapter to derive Q_j^D.

We illustrate now the derivation of the orders of magnitudes of typical acceleration coefficients in (7.119) and (7.123).

From (7.119)

$$\alpha_{01}^{x} = (a_{05})_{x_5} + g_{x_5} \tag{7.127}$$

$$\alpha_{02}^{x} = -(\omega_{y_5}^2 + \omega_{z_5}^2) \tag{7.128}$$

Employ (7.27) to obtain $(a_{05})_{x5}$, (7.39) to obtain g_{x5}, and (7.59) to obtain ω_{y5} and ω_{z5}. We find

$$(a_{05})_{x_5} = a_{11}(a_{05})_{x_1} + a_{12}(a_{05})_{y_1} + (a_{13})(a_{05})_{z_1}$$

$$g_{x_5} = a_{11} g_{x_1} + a_{12} g_{y_1} + a_{13} g_{z_1} \tag{7.129}$$

(7.130)

$$\omega_{y_5} = a_{21}\omega_{x_1} + a_{22}\omega_{y_1} + a_{23}(\omega_{z_1} + \Omega) - \beta' \tag{7.131}$$

$$\omega_{z_5} = a_{31}\omega_{x_1} + a_{32}\omega_{y_1} + a_{33}(\omega_{z_1} + \Omega) + C_\beta \delta' \tag{7.132}$$

Equation (7.19) with (7.24) yields $(a_{05})_{x1}$, $(a_{05})_{y1}$ and $(a_{05})_{z1}$. Invoking the order assumptions (7.123) and (7.126), we find

$$(a_{05})_{x_1} = O(\epsilon) \tag{7.133}$$
$$(a_{05})_{y_1} = O(\epsilon)$$
$$(a_{05})_{z_1} = O(\epsilon)$$

Similarly from (7.37), (7.39), and assumption (7.126), we find

$$g_{x_1} = O(\epsilon) \tag{7.134}$$
$$g_{y_1} = O(\epsilon)$$
$$g_{z_1} = O(\epsilon)$$

Employ assumption (7.124) and the expressions for the direction cosines (7.29) to obtain the orders of magnitudes of these direction cosines. We find

$$\left[a_{ij}\right] = O\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix} \tag{7.135}$$

Insert (7.133) to (7.135) in (7.129) to (7.132). We find

$$(a_{05})_{x_5} = O(\epsilon) \tag{7.136}$$
$$g_{x_5} = O(\epsilon)$$
$$\omega_{y_5} = O(\epsilon)$$
$$\omega_{z_5} = O(1)$$

Substitution of (7.136) in (7.127) and (7.128) yields

$$\alpha_{01}^{x} = O(\epsilon) \tag{7.137}$$

$$\alpha_{02}^{x} = O(1) \tag{7.138}$$

Similarly, we find

$$\alpha_{02}^{y} = O(\epsilon) \tag{7.139}$$

$$\alpha_{11}^{z} = O(\epsilon^2 q_i) \tag{7.140}$$

and so on.

From (7.120) we find

$$\rho_{02}^{y}\ddot{q}_K = -c_{,\beta}\,\delta_K\ddot{q}_K = -c_{,\beta}\ddot{\delta} = O(\epsilon) \tag{7.141}$$

$$\rho_{11}^{z}\ddot{q}_K = -\delta_{iK}\ddot{q}_i = -\ddot{q}_i = O(q_i) \tag{7.142}$$

The orders of all terms in (7.119) and (7.120) are similarly derived.

It is necessary to keep the modal displacement q_i in the order statement to help us distinguish between terms of $0(\epsilon)$ and terms of $0(\theta_e)$, $0(w)$, and $0(v)$ when Q_j^D is formed. For example, when a term $0(q_i)$ combines with a term involving θ_i, the resulting order of the product is θ_e because $\theta_e = \theta_i q_i$, while a product $q_i \delta_i = \delta$ is of order ϵ. Because terms of $0(\theta_e)$, $0(w)$, and $0(v)$ are approximated differently from terms of $0(\epsilon)$, we have to retain q_i in the order expressions until a point where products like $\theta_i q_i$, $w_i q_i$, $\delta_i q_i$, etc are formed in the Q_j^D expression.

To emphasize the physical implications of our order statements we list a few examples from those given above, and revert to the interpretation that all quantities presented below are dimensional. Thus we have

$$
\begin{aligned}
\dot{V}_{0x_1}/\Omega^2 R &= 0(\epsilon) \\
\omega_{x_1}/\Omega &= 0(\epsilon) \\
\dot{\omega}_{x_r}/\Omega^2 &= 0(\epsilon) \\
e/R &= 0(\epsilon) \\
g/\Omega^2 R &= 0(\epsilon) \\
\omega_{z_5}/\Omega &= 0(1) \\
\alpha_{01}^{x}/\Omega^2 R &= 0(\epsilon) \\
\alpha_{02}^{x}/\Omega^2 &= 0(\epsilon)
\end{aligned}
\tag{7.143}
$$

8. Inertial Generalized Force Q_j^D

We derive in this chapter a working form for the inertia contribution to generalized force, Q_j^D.

We begin by deriving expressions for inertial loads p_{x5}^D, p_{y5}^D, p_{z5}^D, q_{x5}^D, q_{y5}^D, and q_{z5}^D in terms of accelerations in the X_5 system. (7.75). The resulting expressions involve the coefficients A_x, A_y, A_z etc in (7.75) and section properties like c.g. offset and radii of gyrations. The inertial loads are then substituted in the generalized force expressions F_1, F_2, and F_3 and integrated to form Q_j^{D1}, Q_j^{D2}, and Q_j^{D3}, (4.35). The different acceleration conventions of the previous chapter participate to effect approximations, develop modal integrals, and to separate $\ddot{q}$ dependent and $\ddot{q}$ independent terms to form Q_j^{D*} and s_{jk}.

8.1 Expressions for Inertia Loads

Here we derive expressions for the inertia loads per unit length of blade, p_{x5}^D, p_{y5}^D, p_{z5}^D, q_{x5}^D, q_{y5}^D, and q_{z5}^D, in terms of acceleration coefficients A_x A_y A_z etc of the X_5 system and section properties, comprising c.g. offset and section radii of gyrations.

Figure 17 illustrates the positive conventions for the inertia loads resolved to the $\bar{X}_{10}$ directions. In terms of accelerations resolved to the $\bar{X}_{10}$ directions, these $\bar{X}_{10}$ inertia loads are

$$p_{x_{10}} = \int_A (-a_{x_{10}}) \rho \, dA \qquad (8.1)$$

$$p_{y_{10}} = \int_A (-a_{y_{10}}) \rho \, dA$$

$$p_{z_{10}} = \int_A (-a_{z_{10}}) \rho \, dA$$

$$(8.2)$$

$$q_{x_{10}} = \int_A \left[-(-a_{y_{10}} \rho \, dA) z_{10} + (-a_{z_{10}} \rho \, dA) y_{10} \right]$$

$$q_{y_{10}} = \int_A -(a_{x_{10}} \rho \, dA) z_{10}$$

$$q_{z_{10}} = -\int_A -(a_{x_{10}} \rho \, dA) y_{10}$$

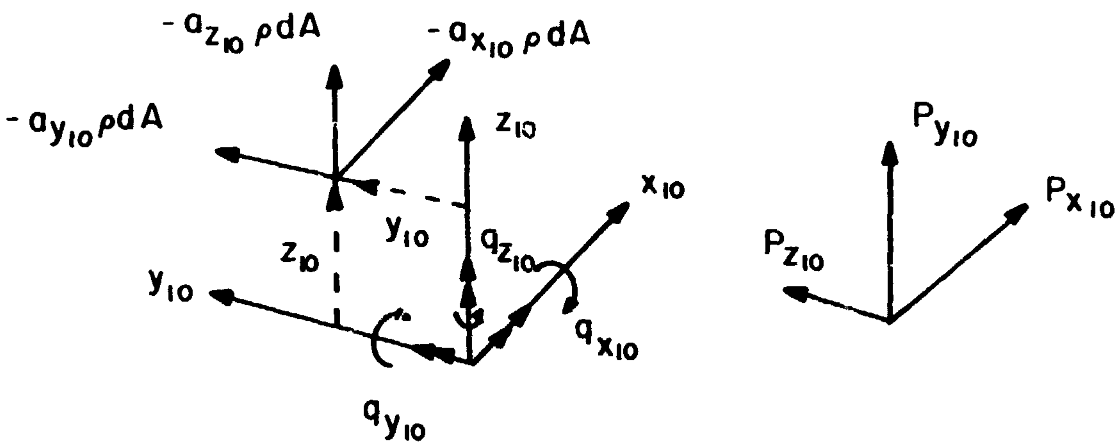

Figure 17. Positive Conventions for Inertia Loads Employed in Multi-Blade Rotor Analysis.

To derive $\bar{X}_5$ system loads in terms of $\bar{X}_5$ system accelerations, we employ the following rotation transformations relating free vectors $\bar{X}_5$ and $\bar{X}_{10}$.

$$\bar{X}_5 = A_\theta A_{\lambda_1} A_{\lambda_2} A_{\theta_e} \bar{X}_{10} \tag{8.3}$$

$$\bar{X}_{10} = A_{-\theta_e} A_{-\lambda_2} A_{-\lambda_1} A_{-\theta} \bar{X}_5 \tag{8.4}$$

Free vector $\bar{X}_5$ is selected from among $\bar{p}_5$, $\bar{q}_5$, and $\bar{a}_5$, and corresponding $\bar{X}_{10}$ vectors are $\bar{p}_{10}$, $\bar{q}_{10}$, and $\bar{a}_{10}$, with components

$$\bar{p}_5^{\,T} = p_{x_5}, p_{y_5}, p_{z_5} \tag{8.5}$$

$$\bar{q}_5^{\,T} = q_{x_5}, q_{y_5}, q_{z_5}$$

$$\bar{a}_5^{\,T} = a_x, a_y, a_z$$

$$\bar{p}_{10}^{\,T} = p_{x_{10}}, p_{y_{10}}, p_{z_{10}}$$

$$\bar{q}_{10}^{\,T} = q_{x_{10}}, q_{y_{10}}, q_{z_{10}}$$

$$\bar{a}_{10}^{\,T} = a_{x_{10}}, a_{y_{10}}, a_{z_{10}}$$

Application of (8.3) to $\bar{p}_5$ and $\bar{p}_{10}$, substitution of (8.1) for $\bar{p}_{10}$, and substitution of (8.4) to express $\bar{a}_{10}$ in terms of $\bar{a}_5$ yields the following set of results.

$$\begin{aligned}
\bar{p}_5 &= A_\theta A_{\lambda_1} A_{\lambda_2} A_{\theta_e} \bar{p}_{10} \\
&= A_\theta A_\lambda A_{\lambda_2} A_{\theta_e} \int_A (-\bar{a}_{10}) \rho \, dA \\
&= A_\theta A_{\lambda_1} A_{\lambda_2} A_{\theta_e} \int_A (-A_{\theta_e} A_{-\lambda_2} A_{-\lambda_1} A_{-\theta_e}) \bar{a}_5 \rho \, dA \\
&= \int_A -\bar{a}_5 \rho \, dA
\end{aligned} \tag{8.6}$$

The expanded form of (8.6) is

$$\begin{pmatrix} P_{x_5} \\ P_{y_5} \\ P_{z_5} \end{pmatrix} = -\int_A \rho\, dA \cdot \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \tag{8.7}$$

To obtain $\bar{q}_5$ in terms of $\bar{a}_5$, we first write (8.2) in the form

$$\bar{q}_{10} = \int_A \rho\, dz_{10}\, dy_{10} \begin{pmatrix} 0 & z_{10} & -y_{10} \\ -z_{10} & 0 & 0 \\ y_{10} & 0 & 0 \end{pmatrix} \bar{a}_{10} \tag{8.8}$$

Following a similar procedure to that used to derive (8.7), we obtain

$$\bar{q}_5 = A_\Theta A_{\lambda_1} A_{\lambda_2} A_{\theta_e} \bar{q}_{10}$$

$$= \int_A \rho\, dz_{10}\, dy_{10} \left(A_\Theta A_{\lambda_1} A_{\lambda_2} A_{\theta_e}\right) \begin{pmatrix} 0 & z_{10} & -y_{10} \\ -z_{10} & 0 & 0 \\ y_{10} & 0 & 0 \end{pmatrix} \left(A_{-\theta_e} A_{-\lambda_2} A_{-\lambda_2} A_{-\Theta}\right) \bar{a}_5$$

(8.9)

This reduces to

$$q_{x_5} = \int_A \rho\, dz_{10}\, dy_{10} \Big\{ \left[(c_{\bar{\Theta}} + \lambda_2 v') z_{10} + (s_{\bar{\Theta}} - \lambda_1 v') y_{10} \right] a_y + \left[(s_{\bar{\Theta}} + \lambda_2 w') z_{10} - (c_{\bar{\Theta}} + \lambda_1 w') y_{10} \right] a_z$$

(8.10)

$$q_{y_5} = \int_A \rho\, dz_{10}\, dy_{10} \left[(-c_{\bar{\Theta}}\, z_{10} - s_{\bar{\Theta}}\, y_{10}) a_x - (\lambda_1 z_{10} + \lambda_2 y_{10}) a_z \right]$$

(8.11)

$$q_{z_5} = \int_A \rho \, dz_{10} \, dy_{10} \left[\left(-s_{\bar{\theta}} z_{10} + c_{\bar{\theta}} y_{10} \right) a_x + \left(\lambda_1 z_{10} + \lambda_2 y_{10} \right) a_y \right] \tag{8.12}$$

We neglected third order products of elastic variables and fourth order products of small quantities to derive $q_{x_5}^D$, (8.10), and we neglected second order products of elastic variables and third order products of small quantities to obtain $q_{y_5}^D$, (8.11), and $q_{z_5}^D$, (8.12). These approximations are discussed in more detail later in this chapter.

To reduce (8.7) and (8.10) to (8.12) to forms involving the acceleration coefficients A_x, A_y, etc, we substitute (7.75). Assume inertial symmetry of each section and introduce the following definitions.

$$m = \int_A \rho \, dA \tag{8.13}$$

$$y_{10_{cg}} = \frac{1}{m} \int_A y_{10} \rho \, dA$$

$$K_{y_{10}}^2 = \frac{1}{m} \int_A z_{10}^2 \rho \, dA$$

$$K_{z_{10}}^2 = \frac{1}{m} \int_A y_{10}^2 \rho \, dA$$

We obtain

$$p_{x_5}^D = -m \left(A_x + B_x y_{10_{cg}} \right) \tag{8.14}$$

$$p_{y_5}^D = -m \left(A_y + B_y y_{10_{cg}} \right)$$

$$p_{z_5}^D = -m \left(A_z + B_z y_{10_{cg}} \right)$$

$$q_{x_5}^D = m \Big\{ \left[A_y \left(s_{\bar{\theta}} - \lambda_1 v' \right) - A_z \left(c_{\bar{\theta}} + \lambda_1 v' \right) \right] y_{10_{cg}} + \left(B_y s_{\bar{\theta}} - B_z c_{\bar{\theta}} \right) K_{z_{10}}^2 + \left(C_y c_{\bar{\theta}} + C_z s_{\bar{\theta}} \right) K_{y_{10}}^2 \Big\} \tag{8.15}$$

$$q_{y_5}^D = -m\Big[(A_x S_{\bar\theta} + A_z \lambda_2) y_{10cg} + (B_x S_{\bar\theta} + B_z \lambda_2) K_{z_{10}}^2 + (C_x C_{\bar\theta} + C_z \lambda_1) K_{y_{10}}^2\Big] \tag{8.16}$$

$$q_{z_5}^D = m\Big[(A_x C_{\bar\theta} + A_y \lambda_2) y_{10cg} + (B_x C_{\bar\theta} + B_y \lambda_2) K_{z_{10}}^2 + (-C_x S_{\bar\theta} + C_y \lambda_1) K_{y_{10}}^2 \tag{8.17}$$

Shears derive from (6.71)

$$F_{x_5}^D = \int_r^{r_T} p_{x_5}^D \, d\xi = -\int_r^{r_T} m(A_x + B_x y_{10cg})\, d\xi \tag{8.18}$$

$$F_{y_5}^D = \int_r^{r_T} p_{y_5}^D \, d\xi = -\int_r^{r_T} m(A_y + B_y y_{10cg}\, d\xi$$

$$F_{z_5}^D = \int_r^{r_T} p_{z_5}^D \, d\xi = -\int_r^{r_T} m(A_z + B_z y_{10cg})\, d\xi$$

Loads and tensions in the $\bar{X}_{10}$ system derive from (6.60) and (6.61).

$$P_{x_{10}}^D = P_{x_5}^D + v' P_{y_5}^D + w' P_{z_5}^D \tag{8.19}$$

$$F_{x_{10}}^D = F_{x_5}^D + v' F_{y_5}^D + w' F_{z_5}^D \tag{8.20}$$

Substitute in (8.14) to (8.17) the expansions for A_x, A_y, A_z etc defined by (7.78), and employ

$$S_{\bar{\theta}} = S_{\theta_c} + \theta_{e_1} C_{\theta_c} - \frac{1}{2}\theta_{e_1}^2 S_{\theta_c}$$ (8.21)

$$C_{\bar{\theta}} = C_{\theta_c} - \theta_{e_1} S_{\theta_c} - \frac{1}{2}\theta_{e_1}^2 C_{\theta_c}$$

$$\theta_{e_1} = \theta_t + \theta_e$$

We obtain the approximations

$$P_{x_5}^D = -m\left[A_x^{(0)} + A_x^{(1)} + (B_x^{(0)} + B_x^{(1)})y_{10_{cg}}\right]$$ (8.22)

$$P_{y_5}^D = -m\left[A_y^{(0)} + A_y^{(1)} + (B_y^{(0)} + B_y^{(1)})y_{10_{cg}}\right]$$

$$P_{z_5}^D = -m\left[A_z^{(0)} + A_z^{(1)} + (B_z^{(0)} + B_z^{(1)})y_{10_{cg}}\right]$$

$$\begin{aligned} q_{x_5}^D = m\Big\{ &\Big\langle A_y^{(0)}(S_{\theta_c} + C_{\theta_c}\theta_{e_1} - \tfrac{1}{2}\theta_{e_1}^2 S_{\theta_c} - \lambda_1 v') + A_y^{(1)}(S_{\theta_c} + \theta_{e_1}C_{\theta_c}) + A_y^{(2)} S_{\theta_c} \\ &- \big[A_z^{(0)}(C_{\theta_c} - S_{\theta_c}\theta_{e_1} - \tfrac{1}{2}\theta_{e_1}^2 C_{\theta_c} + \lambda_1 w') + A_z^{(1)}(C_{\theta_c} - \theta_{e_1} S_{\theta_c}) + A_z^{(2)} C_{\theta_c}\big]\Big\rangle y_{10_{cg}} \\ &+ \Big\langle B_y^{(0)}(S_{\theta_c} + \theta_{e_1} C_{\theta_c}) + B_y^{(1)} S_{\theta_c} - B_z^{(0)}(C_{\theta_c} - \theta_{e_1} S_{\theta_c}) - B_z^{(1)} C_{\theta_c}\Big\rangle k_{z_{10}}^2 \\ &+ \Big\langle C_y^{(0)}(C_{\theta_c} - \theta_{e_1} S_{\theta_c}) + C_y^{(1)} C_{\theta_c} + C_z^{(0)}(S_{\theta_c} + \theta_{e_1} C_{\theta_c}) + C_z^{(1)} S_{\theta_c}\Big\rangle k_{y_{10}}^2 \end{aligned}$$ (8.23)

$$\begin{aligned} q_{y_5}^D = -m\Big\{ &\big[A_x^{(0)}(S_{\theta_c} + \theta_{e_1} C_{\theta_c}) + A_x^{(1)} S_{\theta_c} + A_z^{(0)}\lambda_2\big] y_{10_{cg}} \\ &+ B_x^{(0)} S_{\theta_c} k_{z_{10}}^2 + C_x^{(0)} C_{\theta_c} k_{y_{10}}^2 \Big\} \end{aligned}$$ (8.24)

$$\begin{aligned} q_{z_5}^D = m\Big\{ &\big[A_x^{(0)}(C_{\theta_c} - \theta_{e_1} S_{\theta_c}) + A_x^{(1)} C_{\theta_c} + A_y^{(0)}\lambda_2\big] y_{10_{cg}} \\ &+ B_x^{(0)} C_{\theta_c} k_{z_{10}}^2 - C_x^{(0)} S_{\theta_c} k_{y_{10}}^2 \Big\} \end{aligned}$$ (8.25)

with λ_1 and λ_2 approximated by (7.72)

$$\lambda_1 = -S_{\theta_c} v' + C_{\theta_c} w' \tag{8.26}$$

$$\lambda_2 = C_{\theta_c} v' + S_{\theta_c} w'$$

Corresponding approximations to F_{x5}^D, F_{y5}^D, F_{z5}^D, (8.18), and P_{x10}^D, F_{x10}^D, (8.19) and (8.20) are

$$F_{x_5}^D = -\int_r^{r_T} m\left[A_x^{(0)} + A_x^{(1)} + \left(B_x^{(0)} + B_x^{(1)}\right) y_{10cg}\right] d\xi \tag{8.27}$$

$$F_{y_5}^D = -\int_r^{r_T} m\left[A_y^{(0)} + A_y^{(1)} + \left(B_y^{(0)} + B_y^{(1)}\right) y_{10cg}\right] d\xi$$

$$F_{z_5}^D = -\int_r^{r_T} m\left[A_z^{(0)} + A_z^{(1)} + \left(B_z^{(0)} + B_z^{(1)}\right) y_{10cg}\right] d\xi$$

and

$$P_{r_{10}}^D = -m\left[A_x^{(0)} + A_x^{(1)} + \left(B_x^{(0)} + B_x^{(1)}\right) y_{10cg} + v' A_y^{(0)} + w' A_z^{(0)}\right] \tag{8.28}$$

$$F_{x_{10}}^D = -\int_r^{r_T} m\left[A_x^{(0)} + A_x^{(1)} + \left(B_x^{(0)} + B_x^{(1)}\right) y_{10cg} + v' A_y^{(0)} + w' A_z^{(0)}\right] d\xi \tag{8.29}$$

In (8.22) to (8.29) our approximation to all loadings except q_{x5}^D are accurate to first order elastic terms w, v, and θ_e, and second order products of small quantities involving w, v, θ_e, θ_t, y_{10cg}, k_{y10} and k_{z10}. Loading q_{x5}^D is approximated to a higher order than the other loadings and is accurate to second order products of elastic terms and third order products of small quantities. We show in Section 8.4 that these approximations when substituted in the modal equations are sufficient to satisfy our approximations to the modal equations.

8.2 Derivation of Typical Terms in Q_j^D

In this section we illustrate the derivation of typical terms contributing to the generalized force, Q_j^D, to exemplify the procedure used to derive a working form for Q_j^D, and to introduce the modal integral notation.

We base the formation of Q_j^D on (4.36) and (4.35).

$$Q_j^D = Q_j^{D_1} + Q_j^{D_2} + Q_j^{D_3} \tag{8.30}$$

$$Q_j^{D_1} = \int_0^{r_T} \theta_j F_1^D dr \tag{8.31}$$

$$Q^{D_2} = \int_0^{r_T} w_{1_j} F_2^D dr$$

$$Q_j^{D_3} = \int_0^{r_T} v_{1_j} F_3^D dr$$

Substitute in (8.31) the expressions (6.63) to (6.65) for F_1^D to F_3^D. We obtain

$$Q_j^{D_1} = \int_0^{r_T} \theta_j \Big\{ q_{x_5}^D + m\,\Omega^2 (k_{z_{10}}^2 - k_{y_{10}}^2) C_{2\theta}\, \theta_e + m (k_{z_{10}}^2 + k_{y_{10}}^2) \ddot{\theta}_e$$

$$+ v' q_{y_5}^D + w' q_{z_5}^D$$

$$- \left(\left(\hat{T} - F_{x_5}^D \right) k_A^2 \theta_e' \right)'$$

$$+ \left(\theta' k_A^2 F_{x_{10}}^D \right)'$$

$$
\begin{aligned}
&+ c_\theta e_A F_{x_{10}}^D w'' \\
&- S_\theta e_A F_{x_{10}}^D v'' \\
&+ v'' S_\theta \Delta e_{A_{cw}} \int_{r_{ocw}}^{r} p_{x_{10}}^D \frac{m_{cw}}{m} d\xi \\
&- w'' C_\theta \Delta e_{A_{cw}} \int_{r_{ocw}}^{r} p_{x_{10}}^D \frac{m_{cw}}{m} d\xi \\
&+ \theta_e (v'' C_\theta + w'' S_\theta)\left(-e_A F_{x_5}^D + \Delta e_{A_{cw}} \int_{r_{ocw}}^{r} p_{x_5}^D \frac{m_{cw}}{m} d\xi\right) \\
&-(EI_z - EI_y)\left(v'' w'' C_{2\theta} + \tfrac{1}{2}(v''^2 - w''^2) S_{2\theta}\right)\Big\} dr
\end{aligned}
$$

(8.32)

$$
\begin{aligned}
Q_j^{D_2} = \int_0^{r_T} w_{1_j} \Big\{ & q_{y_5}^{D\,\prime} + (w' F_{x_5}^D)' + p_{z_5}^D + (-\hat{T} w_1')' + m \ddot{w}_1 \\
& - \Big[S_\theta \left(-e_A F_{x_{10}}^D + \Delta e_{A_{cw}} \int_{r_{ocw}}^{r} p_{x_{10}}^D \frac{m_{cw}}{m} d\xi\right) \\
& + \theta_e c_\theta \left(-e_A F_{x_5}^D + \Delta e_{A_{cw}} \int_{r_{ocw}}^{r} p_{x_5}^D \frac{m_{cw}}{m} d\xi\right)\Big]'' \Big\} dr
\end{aligned}
$$

(8.33)

$$
\begin{aligned}
Q_j^{D_3} = \int_0^{r_T} v_{1_j} \Big\{ & -q_{z_5}^{D\,\prime} + (v' F_{x_5}^D)' + p_{y_5}^D + (-\hat{T} v_1')' - m \Omega^2 v_1 + m \ddot{v}_1 \\
& - \Big[C_\theta \left(-e_A F_{x_{10}}^D + \Delta e_{A_{cw}} \int_{r_{ocw}}^{r} p_{x_{10}}^D \frac{m_{cw}}{m} d\xi\right) \\
& - \theta_e S_\theta \left(-e_A F_{x_5}^D + \Delta e_{A_{cw}} \int_{r_{ocw}}^{r} p_{x_5}^D \frac{m_{cw}}{m} d\xi\right)\Big]'' \Big\} dr
\end{aligned}
$$

(8.34)

We illustrate now the expansions of typical terms contributing to $Q_j^{D}{}_1$ to $Q_j^{D}{}_3$, (8.32) to (8.34). Use (8.23) for q_{x5}^{D} in (8.32), employ $\theta = \theta_c + \theta_t$ to expand the pitch angle, and neglect third order products of elastic variables, w, v, and θ_e, and fourth order products of small quantities. We find

$$\int_0^{r_T} \theta_j \left\{ q_{x5}^{D} + m\Omega^2 (K_{z_{10}}^2 - K_{y_{10}}^2) C_{2\theta}\, \theta_e + m (K_{z_{10}}^2 + K_{y_{10}}^2) \ddot{\theta}_e \right\} dr =$$

$$\int_0^{r_T} \theta_j\, m \Big\{$$

$$\Big\langle A_y^{(0)} (S_{\theta_c} + C_{\theta_c} \theta_{e_1} - \tfrac{1}{2} \theta_{e_1}^2 S_{\theta_c} - \lambda_1 v') + A_y^{(1)} (S_{\theta_c} + \theta_{e_1} C_{\theta_c}) + A_y^{(2)} S_{\theta_c}$$

$$- \Big[A_z^{(0)} (C_{\theta_c} - S_{\theta_c} \theta_{e_1} - \tfrac{1}{2} \theta_{e_1}^2 C_{\theta_c} + \lambda_1 w') + A_z^{(1)} (C_{\theta_c} - S_{\theta_c} \theta_{e_1}) + A_z^{(2)} C_{\theta_c} \Big] \Big\rangle y_{10_{CG}}$$

$$+ \Big\langle B_y^{(0)} (S_{\theta_c} + \theta_{e_1} C_{\theta_c}) + B_y^{(1)} S_{\theta_c} - B_z^{(0)} (C_{\theta_c} - \theta_{e_1} S_{\theta_c}) - B_z^{(1)} C_{\theta_c}$$

$$+ \Omega^2 C_{2\theta_c} + \ddot{\theta}_e \Big\rangle K_{z_{10}}^2$$

$$\Big\langle C_y^{(0)} (C_{\theta_c} - \theta_{e_1} S_{\theta_c}) + C_y^{(1)} C_{\theta_c} + C_z^{(0)} (S_{\theta_c} + \theta_{e_1} C_{\theta_c}) + C_z^{(1)} S_{\theta_c}$$

$$- \Omega^2 C_{2\theta_c} + \ddot{\theta}_e \Big\rangle K_{y_{10}}^2 \Big\} dr \qquad (8.35)$$

In (8.35) λ_1 and λ_2 are approximated by (7.72) (also duplicated as 8.26).

Employing (7.99) to expand the acceleration coefficients $Ay^{(0)}$, $Ay^{(1)}$, $Ay^{(2)}$, etc. in terms of radially dependent and radially independent terms, and employing

$$\theta_{e_1} = \theta_t + \theta_e$$
$$= \theta_t + \theta_i\, q_i \qquad (8.36)$$

where required, we find from (8.35) the typical development

$$\int_0^{r_T} \theta_j\, m A_y^{(0)} \left(S_{\theta_c} + C_{\theta_c} \theta_{e_1} - \frac{1}{2}\theta_{e_1}^2\right) y_{10_{cg}}\, dr =$$
$$\int_0^{r_T} \theta_j\, m (a_{01}^y + a_{02}^y r)\Big(S_{\theta_c} + C_{\theta_c}(\theta_t + \theta_i q_i)$$
$$- \frac{1}{2}(\theta_t^2 + 2\theta_t \theta_i q_i + \theta_i \theta_K q_i q_K) S_{\theta_c}\Big] y_{10_{cg}}\, dr$$

(8.37)

Denote modal integrals originating from Q_j^{D1} by $Q_1^{(1)}$, $Q_2^{(1)}$, etc. We find that a_{01}^y in (8.37) yields

$$\int_0^{r_T} \theta_j\, m\, a_{01}^y \Big(S_{\theta_c} + C_{\theta_c}(\theta_t + \theta_i q_i)$$
$$- \frac{1}{2}(\theta_t^2 + 2\theta_t\theta_i q_i + \theta_i\theta_K q_i q_K) S_{\theta_c}\Big] y_{10_{cg}}\, dr$$
$$= a_{01}^y Q_1^{(1)} + a_{01}^y \theta_t Q_2^{(1)} + a_{01}^y q_i Q_3^{(1)}$$
$$+ a_{01}^y \theta_t^2 Q_4^{(1)} + a_{01}^y q_i \theta_t Q_5^{(1)} + a_{01}^y q_i q_K Q_6^{(1)}$$

(8.38)

$$Q_1^{(1)} = \int_0^{r_T} m\,\theta_j\, S_{\theta_c}\, y_{10_{cg}}\, dr \qquad (8.39)$$
$$Q_2^{(1)} = \int_0^{r_T} m\,\theta_j\, C_{\theta_c}\, y_{10_{cg}}\, dr$$
$$Q_3^{(1)} = \int_0^{r_T} m\,\theta_j\,\theta_i\, C_{\theta_c}\, y_{10_{cg}}\, dr$$
$$Q_4^{(1)} = \int_0^{r_T} -\frac{1}{2} m\,\theta_j\, S_{\theta_c}\, y_{10_{cg}}\, dr$$
$$Q_5^{(1)} = \int_0^{r_T} -m\,\theta_j\,\theta_i\, S_{\theta_c}\, y_{10_{cg}}\, dr$$
$$Q_6^{(1)} = \int_0^{r_T} -\frac{1}{2} m\,\theta_j\,\theta_i\,\theta_K\, S_{\theta_c}\, y_{10_{cg}}\, dr$$

The contribution to (8.37) from a_{02}^{y} is

$$\int_0^{r_T} \theta_j\, m\, a_{02}^{y}\, r \Big(S\theta_c + C\theta_c(\theta_t + \theta_i q_i) - \tfrac{1}{2}(\theta_t^2 + 2\theta_t\theta_i q_i + \theta_i\theta_K q_i q_K) S\theta_c\Big] y_{10_{cg}}\, dr$$

$$= a_{02}^{y} Q_7^{(1)} + a_{02}^{y}\theta_t Q_8^{(1)} + a_{02}^{y} q_i Q_9^{(1)} + a_{02}^{y}\theta_t^2 Q_{10}^{(1)} + a_{02}^{y}\theta_t q_i Q_{11}^{(1)} + a_{02}^{y} q_i q_K Q_{12}^{(1)} \quad (8.40)$$

(8.41)

$$Q_7^{(1)} = \int_0^{r_T} m\,\theta_j\, r\, S\theta_c\, y_{10_{cg}}\, dr$$

$$Q_8^{(1)} = \int_0^{r_T} m\,\theta_j\, r\, C\theta_c\, y_{10_{cg}}\, dr$$

$$Q_9^{(1)} = \int_0^{r_T} m\,\theta_j\, r\,\theta_i\, C\theta_c\, y_{10_{cg}}\, dr$$

$$Q_{10}^{(1)} = \int_0^{r_T} -\tfrac{1}{2} m\,\theta_j\, r\, S\theta_c\, y_{10_{cg}}\, dr$$

$$Q_{11}^{(1)} = \int_0^{r_T} - m\,\theta_j\, r\,\theta_i\, S\theta_c\, y_{10_{cg}}\, dr$$

$$Q_{12}^{(1)} = \int_0^{r_T} -\tfrac{1}{2} m\,\theta_j\, r\,\theta_i\,\theta_K\, S\theta_c\, y_{10_{cg}}\, dr$$

Similarly, we find

$$\int_0^{r_T} \theta_j\, m\, A_y^{(1)} (S\theta_c + C\theta_c\,\theta_{e_1}) y_{10_{cg}}\, dr =$$

$$\int_0^{r_T} \theta_j\, m (a_{11}^{y} w_i + a_{12}^{y} v_i)(S\theta_c + C\theta_c(\theta_t + \theta_K q_K)) y_{10_{cg}}\, dr$$

(8.42)

The contribution to (8.42) from a_{11}^y is

$$\int_0^{r_T} \theta_j \, m \, a_{11}^y w_i (S\theta_c + C\theta_c(\theta_t + \theta_K q_K)) y_{10cg} \, dr =$$

$$a_{11}^y Q_{13}^{(1)} + a_{11}^y \theta_t Q_{14}^{(1)} + a_{11}^y q_K Q_{15}^{(1)} \quad (8.43)$$

$$Q_{13}^{(1)} = \int_0^{r_T} \theta_j m w_i S\theta_c y_{10cg} \, dr \quad (8.44)$$

$$Q_{14}^{(1)} = \int_0^{r_T} \theta_j m w_i C\theta_c y_{10cg} \, dr$$

$$Q_{15}^{(1)} = \int_0^{r_T} \theta_j m w_i C\theta_c \theta_K y_{10cg} \, dr$$

The contribution to (8.42) from a_{12}^y is

$$\int_0^{r_T} \theta_j \, m a_{12}^y v_i (S\theta_c + C\theta_c(\theta_t + \theta_K q_K)) y_{10cg} \, dr =$$

$$a_{12}^y Q_{16}^{(1)} + a_{12}^y \theta_t Q_{17}^{(1)} + a_{12}^y q_K Q_{18}^{(1)} \quad (8.45)$$

(8.46)

$$Q_{16}^{(1)} = \int_0^{r_T} \theta_j m v_i S\theta_c y_{10cg} \, dr$$

$$Q_{17}^{(1)} = \int_0^{r_T} \theta_j m v_i C\theta_c y_{10cg} \, dr$$

$$Q_{18}^{(1)} = \int_0^{r_T} \theta_j m v_i C\theta_c \theta_K y_{10cg} \, dr$$

Another characteristic term is

$$\int_0^{r_T} \theta_j m A_y^{(2)} S\theta_c y_{10cg} \, dr =$$

$$\int_0^{r_T} \theta_j m (a_{21}^y \Delta w_{iK} + a_{22}^y \Delta v_{iK}) S\theta_c y_{10cg} \, dr =$$

$$= a_{21}^{y} Q_{17}^{(1)} + a_{22}^{y} Q_{20}^{(1)} \tag{8.47}$$

$$Q_{19}^{(1)} = \int_0^{r_T} \theta_j \, m \Delta w_{\iota k} S\theta_c \, y_{10cg} \, dr \tag{8.48}$$

$$Q_{20}^{(1)} = \int_0^{r_T} \theta_j \, m \Delta v_{\iota k} S\theta_c \, y_{10cg} \, dr$$

We evaluate now the contribution to Q_j^{D1} from $-((\hat{T}-F_{x5}^{D}) k_A^2 \theta_e')'$ in (8.32), to illustrate a typical application of integration by parts to collectively differentiated terms. The contribution to Q_j^{D1} from this term is

$$-\int_0^{r_T} \theta_j ((\hat{T} - F_{x_5}^{D}) k_A^2 \theta_e')' dr =$$

$$-\left[\theta_j (\hat{T} - F_{x_5}^{D}) k_A^2 \theta_e'\right]_0^{r_T} + \int_0^{r_T} \theta_j' (\hat{T} - F_{x_5}^{D}) k_A^2 \theta_e' \, dr =$$

$$\left(\theta_j (\hat{T} - F_{x_5}^{D}) k_A^2 \theta_e'\right)_0 + \int_0^{r_T} \theta_j' (\hat{T} - F_{x_5}^{D}) k_A^2 \theta_e' \, dr \tag{8.49}$$

The condition

$$\hat{T} = F_{x_5}^{D} = 0 \quad , r = r_T \tag{8.50}$$

was used to obtain (8.49). Replace in (8.49) F_{x5}^{D} by (8.27), substitute $\theta_e' = \theta_i' q_i$, and remove negligible terms. We find

$$-\int_0^{r_T} \theta_j \left(\left(\hat{T} - F_{x_5}^D \right) k_A^2 \theta_e' \right)' dr =$$

$$(\theta_j \theta_i' k_A^2)_0 \, q_i \int_0^{r_T} m \left[\Omega^2 (e + \xi) + A_x^{(0)} \right] d\xi$$

$$+ \int_0^{r_T} \theta_j' \theta_i' k_A^2 q_i \int_r^{r_T} m \left[\Omega^2 (e + \xi) + A_x^{(0)} \right] d\xi \, dr \qquad (8.51)$$

Inserting (7.99) for $A_x^{(0)}$ in (8.51), we obtain

$$-\int_0^{r_T} \theta_j \left(\left(\hat{T} - F_{x_5}^D \right) k_A^2 \theta_e' \right)' dr =$$

$$(\Omega^2 e + a_{01}^x) q_i Q_{125}^{(1)} + (\Omega^2 + a_{02}^x) q_i Q_{126}^{(1)} \qquad (8.52)$$

$$Q_{125}^{(1)} = (\theta_j \theta_i' k_A^2 R_1)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta_i' R_1 \, dr \qquad (8.53)$$

$$Q_{126}^{(1)} = (\theta_j \theta_i' k_A^2 R_2)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta_i' R_2 \, dr$$

$$R_1 = \int_r^{r_T} m \, d\xi$$

$$R_2 = \int_r^{r_T} m \xi \, d\xi$$

The derivation of (8.52) exemplifies the production of terms involving end conditions like $(\theta_j \theta_i' k_A^2 R_1)_0$. In general, root or tip end terms occur from integrations by parts.

C 3

Examples of terms contributing to Q_j^{D2} and Q_j^{D3}, (8.33) and (8.34), are considered below to illustrate the use of superscripts to denote the origin of modal integrals, and to prepare the ground for the examples in the next section.

The contribution to Q_j^{D2} from p_{z5}^{D} in (8.33) follows from a similar treatment to that used for the Q_j^{D1} contributions.

$$\int_0^{r_T} w_{1_j} P_{z_5}^{D} dr =$$

$$-\int_0^{r_T} w_{1_j} \cdot m \left(A_z^{(0)} + A_z^{(1)} + \left(B_z^{(0)} + B_z^{(1)}\right) y_{10_{cg}} \right) dr =$$

$$-\int_0^{r_T} w_{1_j} \cdot m \Big\{ a_{01}^{z} + a_{02}^{z} r + a_{11}^{z} w_i + a_{12}^{z} v_i + \Big[b_{01}^{z} s_{\theta_c} + b_{02}^{z} c_{\theta_c} + b_{11}^{z} s_{\theta_c} + b_{12}^{z} c_{\theta_c} + b_{13}^{z} \theta_i s_{\theta_c} + b_{14}^{z} \theta_i c_{\theta_c} + b_{15}^{z} s_{\theta_c} w_i' + b_{16}^{z} c_{\theta_c} v_i' \Big] y_{10_{cg}} \Big\} dr \qquad (8.54)$$

Contributions from a_{02}^{z} and b_{14}^{z}, for example, are

$$\int_0^{r_T} - a_{02}^{z} w_{1_j} m r\, dr = a_{02}^{z} Q_{38}^{(2)} \qquad (8.55)$$

$$\int_0^{r_T} - b_{14}^{z} w_{1_j} m \theta_i c_{\theta_c} y_{10_{cg}} dr = b_{14}^{z} Q_{46}^{(2)}$$

$$Q_{38}^{(2)} = \int_0^{r_T} - w_{1_j} m r\, dr \qquad (8.56)$$

$$Q_{46}^{(2)} = \int_0^{r_T} - w_{1_j} m \theta_i c_{\theta_c} y_{10_{cg}} dr$$

The contribution to Q_j^{D3} from $m\ddot{v}_1$ in (8.34) is

$$\int_0^{r_T} v_{1_j} m \ddot{v}_1 = \ddot{q}_i Q_{51}^{(3)} \qquad (8.57)$$

$$Q_{51}^{(3)} = \int_0^{r_T} v_{1_j} \, m \, v_{1_i} \, dr \tag{8.58}$$

The examples (8.55) to (8.58) show the use of superscripts 2 and 3 to denote integrals originating from Q_j^{D2} and Q_j^{D3}.

To aid the identification of the physical origins of terms, subscripts attached to modal integrals increase progressively from a value of one upwards as one progresses among the terms composing Q_j^{D1} in (8.32). The first terms in the integrand of (8.32) contribute the lowest subscripts in the modal integrals, terms succeeding these contribute higher subscripts, and the final terms yield the highest subscripts. Equations (8.35) to (8.48) illustrate this convention. To preserve this sequence, modal integrals reducing to the same expressions are not given the same subscript if they originate from different places in (8.32). Exactly the same conventions apply to modal integrals derived from Q_j^{D2} and Q_j^{D3}, (8.33) and (8.34).

Table (1) relates the physical sources of modal integrals in the integrands of (8.32) to (8.34) to the subscripts attached to the modal integrals, and may be used to identify the sources of terms contributing to Q_j^D.

As always in this report, the interpretation of terms involving repeated suffixes in (8.38), (8.40), (8.43), (8.45), and (8.47) must comply with the repeated suffix convention for summations. It is necessary to keep in mind that suffixes may be buried in terms a_{11}^Y, a_{21}^Y, $Q_3^{(1)}$, $Q_{20}^{(1)}$, etc.

TERM IN INTEGRAND OF $G_j^{D'}$ (EQ. 8.32)	SUBSCRIPT i IN $Q_i^{(i)}$
$\theta_j (q_{x_5}^D + m\Omega^2(k_{z_{10}}^2 - k_{y_{10}}^2) c_{2\theta}\theta_e + m(k_{z_{10}}^2 + k_{y_{10}}^2)\ddot{\theta}_e)$	1 - 92
$\theta_j v' q_{y_5}^D$	93-108
$\theta_j w' q_{z_5}^D$	109-124
$-\theta_j ((\hat{T} - F_{x_5}^D) k_A^2 \theta_e')'$	125-126
$\theta_j (F_{x_{10}}^D k_A^2 \theta')'$	127-136
$\theta_j c_\theta e_A w'' F_{x_{10}}^D$	13 -148
$-\theta_j s_\theta e_A v'' F_{x_{10}}^D$	149-160
$\theta_j v'' s_\theta \Delta e_{A_{CW}} \int_{r_{0_{CW}}}^{r} \frac{m_{CW}}{m} p_{x_{10}}^D \, d\xi$	161-172
$-\theta_j w'' c_\theta \Delta e_{A_{CW}} \int_{r_{0_{CW}}}^{r} \frac{m_{CW}}{m} p_{x_{10}}^D \, d\xi$	173-184
$\theta_j \theta_e (v'' c_\theta + w'' s_\theta)(-e_A F_{x_5}^D + \Delta e_{A_{CW}} \int_{r_{0_{CW}}}^{r} \frac{m_{CW}}{m} p_{x_5}^D \, d\xi)$	185-192
$-\theta_j (EI_z - EI_y)(v'' w'' c_{2\theta} + \frac{1}{2}(w''^2 - v''^2) s_{2\theta})$	192-195

TABLE I. RELATIONSHIP BETWEEN PHYSICAL TERMS AND MODAL INTEGRAL SUBSCRIPTS.

TERM IN INTEGRAND OF $Q_j^{D_2}$ (EQ 8.33)	SUBSCRIPT i IN $Q_i^{(2)}$
$w_{1_j}\, q_{y_5}^{D\,\prime}$	1-32
$w_{1_j}\, (w' F_{x_5}^{D})'$	33-36
$w_{1_j}\, p_{z_5}^{D}$	37-48
$w_{1_j}\, (-\hat{T} w_1')'$	49
$w_{1_j}\, m \ddot{w}_1$	50
$-w_{1_j}\left[s_\theta \left(-e_A F_{x_{10}}^{D} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} \frac{m_{cw}}{\bar{m}} p_{x_{10}}^{D}\, d\xi\right)\right]''$	51-134
$-w_{1_j}\left[\theta_e c_\theta \left(-e_A F_{x_5}^{D} + \Delta e_{A_{cw}} \int_{r_{0_{cw}}}^{r} \frac{m_{cw}}{\bar{m}} p_{x_5}^{D}\, d\xi\right)\right]''$	135-150

TABLE I. CONTINUED.

TERM IN INTEGRAND OF Q_j^{D3} (EQ. 8.34)	SUBSCRIPT i IN $Q_i^{(3)}$
$-v_{1j}\, q_{x5}^{D\,\prime}$	1-32
$v_{1j}(v' F_{x5}^{D})'$	33-36
$v_{1j}\, P_{y5}^{D}$	37-48
$v_{1j}(-\hat{T} v_1')'$	49
$-v_{1j}\, m \Omega^2 v_1$	50
$v_{1j}\, m \ddot{v}_1$	51
$-v_{1j}\left[c_\theta\left(-e_A F_{x10}^{D} + \Delta e_{A_{CW}} \int_{r_{CW}'}^{r} \frac{m_{CW}}{m} P_{x10}^{D}\, d\xi\right)\right]''$	52-135
$v_{1j}\left[\theta_e s_\theta\left(-e_A F_{x5}^{D} + \Delta e_{A_{CW}} \int_{r_{CW}}^{r} \frac{m_{CW}}{m} P_{x5}^{D}\, d\xi\right)\right]''$	136-151

TABLE I CONCLUDED.

8.3 Inertia Force Components Q_j^{D*} and s_{jk}

In this section, we separate the terms composing Q_j^D into a component Q_j^{D*}, idependent of modal acceleration, and a component $s_{jk}\ddot{q}_k$, isolating the modal acceleration, and we list here all such terms in Q_j^{D*} and s_{jk}. The derivation is specialized for small hub accelerations and this assumption effects a substantial reduction in the number of significant terms retained in our equations. We present also a classification dividing components of Q_j^{D*} and s_{jk} into terms not involving modal sums, terms involving single modal sums, and terms involving double modal sums, to facilitate the programming of Q_j^{D*} and s_{jk}.

Our aim is to form the elements (4.42) and (4.15) in the modal equation, which are

$$Q_j^{D*} = Q_j^{D_1*} + Q_j^{D_2*} + Q_j^{D_3*} \tag{8.59}$$

$$S_{jK} = a_{jK} + S_{jK}^{(1)} + S_{jK}^{(2)} + S_{jK}^{(3)} \tag{8.60}$$

with

$$a_{jK} = \int_0^{r_T} m\left[\left(k_{z_{10}}^2 + k_{y_{10}}^2\right)\theta_j\,\theta_K + w_{1_j} w_{1_K} + v_{1_j} v_{1_K}\right] dr \tag{8.61}$$

The generalized mass a_{jk} is known from the solution for the normal modes, and the other elements in (8.59) and (8.60) are derived here.

We proceed now to illustrate the separation of Q_j^{D*} and s_{jk}, drawing on the examples of Section (8.2) for the derivation of typical contributions to Q_j^{D*} and s_{jk}. The basic step is the substitution of acceleration coefficients (7.104) to (7.106) in the terms composing Q_j^D, to effect this separation into Q_j^{D*} and s_{jk}.

Denote with an arrow a contribution to Q_j^D or s_{jk} from a subset of the elements making up Q_j^D or s_{jk}. We may write the contribution to Q_j^{D1} from the terms in (8.40) as

$$Q_j^{D_1} \longleftarrow a_{02}^y Q_7^{(1)} + a_{02}^y \theta_t Q_8^{(1)} + a_{02}^y q_i Q_9^{(1)} + a_{02}^y \theta_t^2 Q_{10}^{(1)} + a_{02}^y \theta_t q_i Q_{11}^{(1)} + a_{02}^y q_i q_k Q_{12}^{(1)} \tag{8.62}$$

Employ (7.104) or (7.106) to separate the acceleration coefficient a_{02}^y into a term independent of $\ddot{q}_k$ and a term dependent on $\ddot{q}_k$, that is,

$$a_{02}^y = \alpha_{02}^y - \rho_{02}^y \ddot{q}_k \tag{8.63}$$

Insert (8.63) in (8.62). We obtain

$$\begin{aligned} Q_j^{D_1} \longleftarrow\ & \alpha_{02}^y Q_7^{(1)} + \alpha_{02}^y \theta_t Q_8^{(1)} + \alpha_{02}^y q_i Q_9^{(1)} \\ & + \alpha_{02}^y \theta_t^{(2)} Q_{10}^{(1)} + \alpha_{02}^y \theta_t q_i Q_{11}^{(1)} + \alpha_{02}^y q_i q_k Q_{12}^{(1)} \\ & - \rho_{02}^y \ddot{q}_k Q_7^{(1)} - \rho_{02}^y \ddot{q}_k \theta_t Q_8^{(1)} - \rho_{02}^y \ddot{q}_k q_i Q_9^{(1)} \\ & - \rho_{02}^y \ddot{q}_k \theta_t^2 Q_{10}^{(1)} - \rho_{02}^y \ddot{q}_k \theta_t q_i Q_{11}^{(1)} - \rho_{02}^y \ddot{q}_k q_i q_\ell Q_{12}^{(1)} (k \leftarrow \ell) \end{aligned} \tag{8.64}$$

The notation $k \leftarrow l$ is designed to prevent the occurrences of three k subscripts in (8.64), to preserve the summation convention, as explained in the remarks following (7.118).

Comparison of (8.64) and (4.39), which is

$$Q_j^{D_1} = Q_j^{D_1*} - s_{jk}^{(1)} \ddot{q}_k \tag{8.65}$$

yields

$$Q_j^{D_1*} \leftarrow \alpha_{02}^y Q_7^{(1)} + \alpha_{02}^y \theta_t Q_8^{(1)} + \alpha_{02}^y q_i Q_9^{(1)} + \alpha_{02}^y \theta_t^2 Q_{10}^{(1)} + \underline{\alpha_{02}^y \theta_t q_i Q_{11}^{(1)}} + \underline{\alpha_{02}^y q_i q_k Q_{12}^{(1)}} \tag{8.66}$$

$$s_{jk}^{(1)} \leftarrow \rho_{02}^y Q_7^{(1)} + \rho_{02}^y \theta_t Q_8^{(1)} + \rho_{02}^y q_i Q_9^{(1)} + \rho_{02}^y \theta_t^2 Q_{10}^{(1)} + \underline{\rho_{02}^y \theta_t q_i Q_{11}^{(1)}} + \underline{\rho_{02}^y q_i q_\ell Q_{12}^{(1)} (k \leftarrow \ell)} \tag{8.67}$$

Equations (8.66) and (8.67) are illustrations confirming the use of substitutions (7.121) and (7.122) to obtain Q_j^{D*} and s_{jk} from Q_j^D. We observe that $a_{02}^y \leftarrow \alpha_{02}^y$ in the contribution to Q_j^{D1}, (8.62), will yield Q_j^{D1*}, (8.66). Similarly, $a_{02}^y \leftarrow \rho_{02}^y$ in the contribution to Q_j^{D1}, (8.62), will yield the contribution to $s_{jk}^{(1)}$, (8.67). In the subsequent illustrations, it is to be understood that substitutions (7.121) and (7.122) have been applied, without further preliminaries.

We underline in (8.66) and (8.67), and in the subsequent derivations of this section, negligible terms for small hub accelerations to remind us that such terms make no contributions to our set of terms composing Q_j^{D*}, presented below. Section (8.4) illustrates the procedure for establishing the magnitudes of terms contributing to Q_j^{D*} and s_{jk} for small hub accelerations.

Employing similar procedures, we find the contributions from terms in (8.45) to be

$$Q_j^{D_1*} \leftarrow \alpha_{12}^y Q_{16}^{(1)} + \alpha_{12}^y \theta_t Q_{17}^{(1)} + \alpha_{12}^y q_k Q_{18}^{(1)} \tag{8.68}$$

$$s_{jk}^{(1)} \leftarrow \rho_{12}^y Q_{16}^{(1)} + \rho_{12}^y \theta_t Q_{17}^{(1)} + \rho_{12}^y q_\ell Q_{18}^{(1)} (k \leftarrow \ell) \tag{8.69}$$

A simplification of (8.69) is achieved from the substitution $\rho_{13}^{Y} = -\delta_{ik}$ (Eq. 7.120). Rewrite the modal integrals in (8.46) as

$$Q_{16}^{(1)}(j,i) = \int_0^{r_T} \theta_j \, m v_i \, s_{\theta_c} \, y_{10cg} \, dr \qquad (8.70)$$

$$Q_{17}^{(1)}(j,i) = \int_0^{r_T} \theta_j \, m v_i \, c_{\theta_c} \, y_{10cg} \, dr$$

$$Q_{18}^{(1)}(j,i,k \leftarrow \ell) = \int_0^{r_T} \theta_j \cdot m v_i \, c_{\theta_c} \, \theta_{k \leftarrow \ell} \, y_{10cg} \, dr$$

$$Q_{18}^{(1)}(j,i,\ell) = \int_0^{r_T} \theta_j \, m v_i \, c_{\theta_c} \, \theta_\ell \, y_{10cg} \, dr$$

Substitute in (8.69) the result $\rho_{12}^{Y} = -\delta_{ik}$. We obtain

$$S_{jk}^{(1)} \leftarrow -\delta_{ik} Q_{16}^{(1)}(j,i) - \delta_{ik} \theta_t Q_{17}^{(1)}(j,i) - \delta_{ik} q_\ell Q_{18}^{(1)}(j,i,\ell)$$

(8.71)

$$= Q_{16}^{(1)}(j,k) - \theta_t Q_{17}^{(1)}(j,k) - q_\ell Q_{18}^{(1)}(j,k,\ell)$$

(8.72)

$$= -Q_{16}^{(1)}(j,k) - \theta_t Q_{17}^{(1)}(j,k) - q_i Q_{18}^{(1)}(j,k,i)$$

(8.73)

The properties of the Kronecker delta reduce the single and double sums in (8.71) to single terms and a single sum, respectively, in (8.73). The program exploits these properties of the Kronecker delta, wherever it occurs, by using forms like (8.73) instead of (8.71). Both forms are listed in the s_{jk} tables.

A third example is the contribution from (8.47), which requires a slightly different treatment because the expansions of a_{21}^{y} and a_{22}^{y}, (7.105), are exceptions to the usual expansions (7.104) or (7.106). Substitute (7.105) in (8.47). We find

(8.74)

$$Q_j^{0,*} \leftarrow \underline{\alpha_{21}^{y} Q_{19}^{(1)}} + \alpha_{22}^{y} Q_{20}^{(1)}$$

$$S_{jk}^{(1)} \leftarrow \rho_{21}^{y}(i,l,k)\, Q_{19}^{(1)}(j,i,k \leftarrow l) - (\ddot{q}_i q_k + q_i \ddot{q}_k)\, Q_{20}^{(1)}(j,i,k) \qquad (8.75)$$

With

(8.76)

$$Q_{19}^{(1)}(j,i,k) = \int_0^{r_T} \theta_j \; m \,\Delta w_{ik}\, S_{\theta_c}\, y_{10_{cg}}\, dr$$

$$Q_{20}^{(1)}(j,i,k) = \int_0^{r_T} \theta_j \; m \,\Delta v_{ik}\, S_{\theta_c}\, y_{10_{cg}}\, dr$$

we obtain from (8.75)

$$S_{jk}^{(1)} \leftarrow \underline{\beta_{21}^{y}(i,l,k)\, Q_{19}^{(1)}(j,i,l)} - q_i \dot{q}_k \left(Q_{20}^{(1)}(j,k,i) + Q_{20}^{(1)}(i,j,k) \right)$$

(8.77)

As indicated above, the underlined terms are neglected for small hub accelerations.

Proceeding now to the remaining examples considered in Section 8.2, we obtain the following contributions to Q_j^{D*} and s_{jk}.

a) Contribution from (8.52)

$$Q_j^{D,*} \leftarrow \underline{(\Omega^2 e + \alpha_{01}^x) q_i Q_{,25}^{(1)}} + (\Omega^2 + \alpha_{02}^x) q_i Q_{,26}^{(1)} \tag{8.78}$$

$$S_{jk}^{(1)} \leftarrow \rho_{01}^x q_i Q_{,25}^{(1)} + \rho_{02}^x q_i Q_{,26}^{(1)} = 0 \tag{8.79}$$

since $\rho_{01}^x = \rho_{02}^x = 0$.

b) Contribution from (8.55)

$$Q_j^{D_2 *} \leftarrow \alpha_{02}^z Q_{38}^{(2)} \tag{8.80}$$

$$S_{jk}^{(2)} \leftarrow \rho_{02}^z Q_{38}^{(2)} \tag{8.81}$$

$$Q_j^{D_2 *} \leftarrow \underline{\beta_{,4}^z Q_{46}^{(2)}} \tag{8.82}$$

$$\begin{aligned} S_{jk}^{(2)} &\leftarrow \sigma_{14}^z Q_{46}^{(2)} \\ &= -\delta_{ik} Q_{46}^{(2)}(j,i) \\ &= -Q_{46}^{(2)}(j,k) \end{aligned} \tag{8.83}$$

c) Contribution from (8.57)

$$Q_j^{D_3*} \leftarrow 0 \tag{8.84}$$

$$s_{jk}^{(3)} \leftarrow -\delta_{ik} Q_{51}^{(3)}(j,i) = -Q_{51}^{(3)}(j,k) \tag{8.85}$$

These examples complete our illustrations of the formation of elements composing Q_j^{D*} and s_{jk}.

We present now the division of components of Q_j^{D*} and s_{jk} into terms not involving modal sums, terms involving single modal sums, and terms involving double modal sums, to facilitate the programming of Q_j^{D*} and s_{jk}. At the same time, we introduce a notation indicating explicitly the subscript dependences of the terms composing Q_j^{D*} and s_{jk} to indicate clearly to the programmer where subscripted variables are required.

Recall that repeated suffixes indicate summations. We express Q_j^{D*} and s_{jk} as the sums

$$Q_j^{D*} = Q_j^{D_1*} + Q_j^{D_2*} + Q_j^{D_3*} \tag{8.86}$$

$$\begin{aligned} Q_j^{D_1*} &= (Q_j^{D_1*})_0 + (Q_j^{D_1*})_1 + (Q_j^{D_1*})_2 \\ Q_j^{D_2*} &= (Q_j^{D_2*})_0 + (Q_j^{D_2*})_1 \\ Q_j^{D_3*} &= (Q_j^{D_3*})_0 + (Q_j^{D_3*})_1 \end{aligned} \tag{8.87}$$

$$\left(Q_j^{D_1^*}\right)_0 = \alpha_m^{(1)} A_{m_j}^{(1)} \tag{8.88}$$

$$\left(Q_j^{D_1^*}\right)_1 = \alpha_{m_i}^{(1)} A_{i m_j}^{(1)}$$

$$\left(Q_j^{D_1^*}\right)_2 = \alpha_{m i k}^{(1)} A_{k i m_j}^{(1)}$$

$$\left(Q_j^{D_2^*}\right)_0 = \alpha_m^{(2)} A_{m_j}^{(2)}$$

$$\left(Q_j^{D_2^*}\right)_1 = \alpha_{m i}^{(2)} A_{i m_j}^{(2)}$$

$$\left(Q_j^{D_3^*}\right)_0 = \alpha_m^{(3)} A_{m_j}^{(3)}$$

$$\left(Q_j^{D_3^*}\right)_1 = \alpha_{m i}^{(3)} A_{i m_j}^{(3)}$$

$$S_{jK} = \alpha_{jK} + S_{jK}^{(1)} + S_{jK}^{(2)} + S_{jK}^{(3)} \tag{8.89}$$

$$S_{jK}^{(1)} = \left(S_{jK}^{(1)}\right)_0 + \left(S_{jK}^{(1)}\right)_1 + \left(S_{jK}^{(1)}\right)_2 \tag{8.90}$$

$$S_{jK}^{(2)} = \left(S_{jK}^{(2)}\right)_0 + \left(S_{jK}^{(2)}\right)_1$$

$$S_{jK}^{(3)} = \left(S_{jK}^{(3)}\right)_0 + \left(S_{jK}^{(3)}\right)_1$$

$$\left(S_{jK}^{(1)}\right)_0 = \beta_{Km}^{(1)} B_{m_j}^{(1)} \tag{8.91}$$

$$\left(S_{jK}^{(1)}\right)_1 = \beta_{Kmi}^{(1)} B_{i m_j}^{(1)}$$

$$\left(S_{jK}^{(1)}\right)_2 = \beta_{Kmil}^{(1)} B_{l i m_j}^{(1)}$$

$$\left(S_{jK}^{(2)}\right)_0 = \beta_{Km}^{(2)} B_{m_j}^{(2)}$$

$$\left(S_{jK}^{(2)}\right)_1 = \beta_{Kmi}^{(2)} B_{i m_j}^{(2)}$$

$$\left(S_{JK}^{(3)}\right)_0 = \beta_{Km}^{(3)} B_{mJ}^{(3)}$$

$$\left(S_{JK}^{(3)}\right)_1 = \beta_{Kmi}^{(3)} B_{imJ}^{(3)}$$

Table (2) lists all the elements derived in the manner of the examples given above, and tabulated according to definitions (8.88) and (8.91).

Referring to these tables, we see that index m ranges over the number of elements in a column of Table (2) and since m is repeated always, we must sum on m in a column to evaluate any term in (8.88) and (8.91). The other indexes i, j, k range on the number of modes, M. Appendix 14.3 defines the modal integrals required in Table (2).

We select now a few examples from these tables to illustrate the correspondences of elements $\alpha_m^{(1)}$, $A_{mj}^{(1)}$ $\beta_{km}^{(1)}$, $B_{kmj}^{(1)}$, etc., in (8.88) and (8.91) with the elements derived in the illustrations, given above.

a) Terms not involving modal sums

$$\left(Q_J^{D_1^*}\right)_0 \leftarrow \alpha_4^{(1)} A_{4J}^{(1)} \equiv \alpha_{02}^{9} Q_7^{(1)} \tag{8.92}$$

$$\left(S_{JK}^{(1)}\right)_0 \leftarrow \beta_{K1}^{(1)} B_{1J}^{(1)} \equiv \beta_{02}^{9} Q_7^{(1)} \tag{8.93}$$

b) Terms involving single modal sums

$$\left(Q_J^{D_1^*}\right)_1 \leftarrow \alpha_{2i}^{(1)} A_{i2J}^{(1)} \equiv \alpha_{02}^{9} q_i Q_9^{(1)} \tag{8.94}$$

$$\left(S_{JK}^{(1)}\right)_1 \leftarrow \beta_{K1i}^{(1)} B_{i1J}^{(1)} \equiv \beta_{02}^{9} q_i Q_9^{(1)} \tag{8.95}$$

b) Terms involving double modal sums

$$\left(Q_j^{D,*}\right)_2 \leftarrow \alpha_{1ik}^{(1)} A_{kij}^{(1)} \equiv \alpha_{22}^{y} Q_{20}^{(1)} \quad (8.96)$$

$$\left(s_{jk}^{(1)}\right)_2 = \beta_{kmil}^{(1)} B_{limj}^{(1)} \quad (8.97)$$

$$\begin{aligned} = -g_i \Big\{ & Q_{18}^{(1)}(j,k,i) \\ & + Q_{20}^{(1)}(j,i,k) + Q_{20}^{(1)}(j,k,i) \\ & + Q_{35}^{(1)}(j,k,i) \\ & + Q_{39}^{(1)}(j,i,k) + Q_{39}^{(1)}(j,k,i) \Big\} \end{aligned} \quad (8.98)$$

Because $(s_{jk}^{(1)})_2$ for small hub accelerations is expressible as the more convenient single modal sum, (8.98), we do not require the form (8.97), and its components are not listed in Table (2).

$(Q_J^{D*})_0$

m	$\alpha_m^{(1)}$	$A_{mj}^{(1)}$	m	$\alpha_m^{(1)}$	$A_{mj}^{(1)}$	m	$\alpha_m^{(1)}$	$A_{mj}^{(1)}$
1	α_{01}^y	$Q_1^{(1)}$	19	β_{02}^z	$Q_{56}^{(1)}$	37	β_{01}^x	$Q_{131}^{(1)}$
2	$\alpha_{01}^y\theta_t$	2	20	$\beta_{02}^z\theta_t$	57	38	β_{02}^x	132
3	$\alpha_{01}^y\theta_t^2$	4	21	β_{11}^z	59			
4	α_{02}^y	7	22	β_{12}^z	60			
5	$\alpha_{02}^y\theta_t$	8	23	γ_{01}^y	67			
6	$\alpha_{02}^y\theta_t^2$	10	24	$\gamma_{01}^y\theta_t$	68			
7	α_{01}^z	21	25	γ_{02}^y	70			
8	$\alpha_{01}^z\theta_t$	22	26	$\gamma_{02}^y\theta_t$	71			
9	$\alpha_{01}^z\theta_t^2$	24	27	γ_{11}^y	73			
10	α_{02}^z	27	28	γ_{12}^y	74			
11	$\alpha_{02}^z\theta_t$	28	29	γ_{01}^z	79			
12	$\alpha_{02}^z\theta_t^2$	30	30	$\gamma_{01}^z\theta_t$	80			
13	β_{01}^y	41	31	γ_{01}^z	82			
14	$\beta_{01}^y\theta_t$	42	32	$\gamma_{01}^z\theta_t$	83			
15	β_{02}^y	44	33	γ_{11}^z	85			
16	β_{11}^y	47	34	γ_{12}^z	86			
17	β_{12}^y	48	35	α_{01}^x	127			
18	β_{01}^z	53	36	α_{02}^x	129			

TABLE 2. ELEMENTS IN Q_j^{D*} AND S_{jk}

$(Q_J^{D_1 *})_1$

m	$\alpha_m^{(1)}$	$A_{imj}^{(1)}$	m	$\alpha_{mi}^{(1)}$	$A_{imj}^{(1)}$	m	$\alpha_{mi}^{(1)}$	$A_{imj}^{(1)}$
1	$\alpha_{01}^y q_i$	$Q_3^{(1)}$	19	$\alpha_{02}^x q_i$	$Q_{112}^{(1)}$	37		
2	$\alpha_{02}^y q_i$	9	20	$\alpha_{02}^x q_i \theta_t$	113	38		
3	α_{11}^y	13	21	$(\Omega^2+\alpha_{02}^x) q_i$	126	39		
4	$\alpha_{01}^z q_i$	23	22	α_{12}^x	130	40		
5	$\alpha_{02}^z q_i$	29	23	$\alpha_{01}^x q_i$	137	41		
6	α_{12}^z	36	24	$\alpha_{02}^x q_i$	138	42		
7	β_{15}^y	51	25	$\alpha_{01}^x q_i$	149	43		
8	β_{16}^y	52	26	$\alpha_{02}^x q_i$	150	44		
9	$q_i \Omega^2$	[illegible]	27	$\alpha_{02}^x \theta_t q_i$	160	45		
10	$\delta_{01}^y q_i$	69	28	$\alpha_{01}^x q_i$	161	46		
11	γ_{14}^y	76	29	$\alpha_{02}^x q_i$	162	47		
12	γ_{15}^y	77	30	$\alpha_{02}^x q_i \theta_t$	172	48		
13	γ_{16}^y	78	31	$\alpha_{01}^x q_i$	173	49		
14	$\Omega^2 q_i$	91	32	$\alpha_{02}^x q_i$	174	50		
15	$\alpha_{01}^x q_i$	93	33	$\alpha_{02}^x \theta_t q_i$	184	51		
[illegible]	$\alpha_{02}^x q_i$	96	34	α_{12}^y	16	52		
17	$\alpha_{02}^x q_i \theta_t$	97	35	$\alpha_{12}^y \theta_t$	17	53		
18	$\alpha_{0?}^x q_i$	109	36			54		

TABLE 2 CONTINUED

$(Q_j^{D_1 *})_2$

m	$\alpha_{mik}^{(1)}$	$A_{Kim_j}^{(1)}$
1	α_{22}^{y}	$Q_{20}^{(1)}$
2	α_{21}^{z}	39
3	$\alpha_{02}^{x} q_i q_k$	98
4	$\alpha_{12}^{x} q_i$	100
5	$\alpha_{02}^{x} q_i q_k$	114
6	$\alpha_{12}^{x} q_i$	116
7	$\alpha_{12}^{x} q_i$	140
8	$\alpha_{12}^{x} q_i$	152
9	$\alpha_{12}^{x} q_i$	164
10	$\alpha_{12}^{x} q_i$	176
11	$\alpha_{02}^{x} q_i q_k$	187
12	$\alpha_{02}^{y} q_i q_k$	188
13	$\alpha_{02}^{y} q_i q_k$	191
14	$\alpha_{02}^{y} q_i q_k$	192
15	$q_i q_k$	193
16	$q_i q_k$	194
17	$q_i q_k$	195
18	$\alpha_{12}^{y} q_k$	18

TABLE 2 CONTINUED.

$(Q_j^{D_2^*})_0$

m	$\alpha_m^{(2)}$	$A_{m_j}^{(2)}$	m	$\alpha_m^{(2)}$	$A_{m_j}^{(2)}$	m	$\alpha_m^{(2)}$	$A_{m_j}^{(2)}$
1	α_{01}^x	$Q_1^{(2)}$	19	β_{01}^z	$Q_{41}^{(2)}$	37	$\alpha_{01}^z\theta_t$	$Q_{83}^{(2)}$
2	α_{02}^x	2	20	β_{02}^z	42	38	$\alpha_{02}^x\theta_t$	84
3	$\alpha_{01}^x\theta_t$	5	21	β_{11}^z	43	39	$\alpha_{01}^x\theta_t$	85
4	$\alpha_{02}^x\theta_t$	7	22	β_{12}^z	44	40	$\alpha_{02}^x\theta_t$	86
5	β_{01}^x	13	23	α_{01}^x	51	41	α_{01}^x	87
6	β_{02}^x	14	24	α_{02}^x	52	42	α_{02}^x	88
7	γ_{01}^x	15	25	β_{01}^x	55	43	β_{01}^x	91
8	γ_{02}^x	16	26	β_{02}^x	56	44	β_{02}^x	92
9	$\alpha_{01}'^x$	17	27	$\alpha_{01}^x\theta_t$	61	45	$\alpha_{01}^x\theta_t$	97
10	$\alpha_{02}'^x$	18	28	$\alpha_{02}^x\theta_t$	62	46	$\alpha_{02}^x\theta_t$	98
11	$\alpha_{01}^x\theta_t$	21	29	α_{01}^x	63	47	α_{01}^x	99
12	$\alpha_{02}^x\theta_t$	23	30	α_{02}^x	64	48	α_{02}^x	100
13	β_{01}^x	29	31	β_{01}^x	67	49	β_{01}^x	103
14	β_{02}^x	30	32	β_{02}^x	68	50	β_{02}^x	104
15	γ_{01}^x	31	33	α_{01}^x	73	51	$\alpha_{01}^x\theta_t$	109
16	γ_{02}^x	32	34	α_{02}^x	74	52	$\alpha_{02}^x\theta_t$	110
17	α_{01}^z	37	35	β_{01}^x	77	53	α_{01}^x	111
18	α_{02}^z	38	36	β_{02}^x	78	54	α_{02}^x	112

TABLE 2 CONTINUED

$(Q_j^{D_2^*})_0$

m	$\alpha_m^{(2)}$	$A_{m_j}^{(2)}$	m	$\alpha_m^{(2)}$	$A_{m_j}^{(2)}$	m	$\alpha_m^{(2)}$	$A_{m_j}^{(2)}$
55	β_{01}^x	$Q_{115}^{(2)}$						
56	β_{02}^x	116						
57	$\alpha_{01}^x\theta_t$	121						
58	$\alpha_{02}^x\theta_t$	122						
59	α_{01}^x	123						
60	α_{02}^x	124						
61	β_{01}^x	127						
62	β_{02}^x	128						
63	$\alpha_{01}^x\theta_t$	133						
64	$\alpha_{02}^x\theta_t$	134						
65								
66								
67								
68								
69								
70								
71								
72								

TABLE 2. CONTINUED

$(Q_j^{D_2^*})_1$

m	$\alpha_{mi}^{(2)}$	$A_{imj}^{(2)}$	m	$\alpha_{mi}^{(2)}$	$A_{imj}^{(2)}$	m	$\alpha_{mi}^{(2)}$	$A_{imj}^{(2)}$
1	α_{12}^{x}	$Q_4^{(2)}$	19	$\alpha_{02}^{x} q_i$	142			
2	$\alpha_{02}^{x} q_i$	8	20	$\alpha_{02}^{x} q_i$	144			
3	α_{12}^{x}	20	21	$\alpha_{02}^{x} q_i$	146			
4	$\alpha_{02}^{x} q_i$	24	22	$\alpha_{02}^{x} q_i$	148			
5	$\alpha_{01}^{x} q_i$	33	23	$\alpha_{02}^{x} q_i$	150			
6	$\alpha_{02}^{x} q_i$	34						
7	α_{12}^{z}	40						
8	$\Omega^2 q_i$	49						
9	α_{12}^{x}	54						
10	α_{12}^{x}	66						
11	α_{12}^{x}	76						
12	α_{12}^{x}	90						
13	α_{12}^{x}	102						
14	α_{12}^{x}	114						
15	α_{12}^{x}	126						
16	$\alpha_{02}^{x} q_i$	136						
17	$\alpha_{02}^{x} q_i$	140						
18	$\alpha_{02}^{x} q_i$	141						

TABLE 2. CONTINUED.

$(Q_j^{D_3^*})_0$

m	$\alpha_m^{(3)}$	$A_{mj}^{(3)}$	m	$\alpha_m^{(3)}$	$A_{mj}^{(3)}$	m	$\alpha_m^{(3)}$	$A_{mj}^{(3)}$
1	α_{01}^x	$Q_1^{(3)}$	19	β_{01}^y	$Q_{41}^{(3)}$	37	$\alpha_{01}^x\theta_t$	$Q_{84}^{(3)}$
2	α_{02}^x	2	20	β_{02}^y	42	38	$\alpha_{02}^x\theta_t$	85
3	$\alpha_{01}^x\theta_t$	5	21	β_{11}^y	43	39	$\alpha_{01}^x\theta_t$	86
4	$\alpha_{02}^x\theta_t$	7	22	β_{12}^y	44	40	$\alpha_{02}^x\theta_t$	87
5	β_{01}^x	13	23	α_{01}^x	52	41	α_{01}^x	88
6	β_{02}^x	14	24	α_{02}^x	53	42	α_{02}^x	89
7	γ_{01}^x	15	25	β_{01}^x	56	43	β_{01}^x	92
8	γ_{02}^x	16	26	β_{02}^x	57	44	β_{02}^x	93
9	α_{01}^x	17	27	$\alpha_{01}^x\theta_t$	62	45	$\alpha_{01}^x\theta_t$	98
10	α_{02}^x	18	28	$\alpha_{02}^x\theta_t$	63	46	$\alpha_{02}^x\theta_t$	99
11	$\alpha_{01}^x\theta_t$	21	29	α_{01}^x	64	47	α_{01}^x	100
12	$\alpha_{02}^x\theta_t$	23	30	α_{02}^x	65	48	α_{02}^x	101
13	β_{01}^x	29	31	β_{01}^x	68	49	β_{01}^x	104
14	β_{02}^x	30	32	β_{02}^x	69	50	β_{02}^x	105
15	γ_{01}^x	31	33	α_{01}^x	74	51	$\alpha_{01}^x\theta_t$	110
16	γ_{02}^x	32	34	α_{02}^x	75	52	$\alpha_{02}^x\theta_t$	111
17	α_{01}^y	37	35	β_{01}^x	78	53	α_{01}^x	112
18	α_{02}^y	38	36	β_{02}^x	79	54	α_{02}^x	113

TABLE 2. CONTINUED.

$(Q_j^{D_3^*})_o$

m	$\alpha_m^{(3)}$	$A_{m_j}^{(3)}$	m	$\alpha_m^{(3)}$	$A_{m_j}^{(3)}$	m	$\alpha_m^{(3)}$	$A_{m_j}^{(3)}$
55	β_{01}^x	$Q_{116}^{(3)}$						
56	β_{02}^x	117						
57	$\alpha_{01}^x \theta_t$	122						
58	$\alpha_{02}^x \theta_t$	133						
59	α_{01}^x	124						
60	α_{02}^x	125						
61	β_{01}^x	128						
62	β_{02}^x	129						
63	$\alpha_{01}^x \theta_t$	134						
64	$\alpha_{02}^x \theta_t$	135						
65								
66								
67								
68								
69								
70								
71								
72								

TABLE 2. CONTINUED.

$(Q_J^{D_3^*})_1$

m	$\alpha_{mi}^{(3)}$	$A_{mJ}^{(3)}$	m	$\alpha_{mi}^{(3)}$	$A_{mJ}^{(3)}$
1	α_{12}^x	$Q_4^{(3)}$	19	$\alpha_{02}^x q_i$	$Q_{137}^{(3)}$
2	$\alpha_{02}^x q_i$	8	20	$\alpha_{02}^x q_i$	141
3	α_{12}^x	20	21	$\alpha_{02}^x q_i$	142
4	$\alpha_{02}^x q_i$	24	22	$\alpha_{02}^x q_i$	143
5	$\alpha_{01}^x q_i$	33	23	$\alpha_{02}^x q_i$	145
6	$\alpha_{02}^x q_i$	34	24	$\alpha_{02}^x q_i$	147
7	α_{11}^y	39	25	$\alpha_{02}^x q_i$	149
8	β_{15}^y	47	26	$\alpha_{02}^x q_i$	151
9	β_{16}^y	48	27	α_{12}^y	40
10	$\Omega^2 q_i$	49			
11	$\Omega^2 q_i$	50			
12	α_{12}^x	55			
13	α_{12}^x	67			
14	α_{12}^x	77			
15	α_{12}^x	91			
16	α_{12}^x	103			
17	α_{12}^x	115			
18	α_{12}^x	127			

TABLE 2. CONTINUED.

$(S_{jk}^{(1)})_0$

m	$\beta_{Km}^{(1)}$	$\beta_{mj}^{(1)}$
1	ρ_{02}^{y}	$Q_7^{(1)}$
2	$\rho_{02}^{y}\,\theta_t$	8
3	$\rho_{02}^{y}\,\theta_t^2$	10
4	ρ_{02}^{z}	27
5	$\rho_{02}^{z}\,\theta_t$	28
6	$\rho_{02}^{z}\,\theta_t^2$	30
7	σ_{01}^{y}	41
8	σ_{11}^{y}	47
9	σ_{12}^{y}	48
10	σ_{02}^{z}	56
11	σ_{12}^{z}	60
12	γ_{02}^{y}	70
13	γ_{12}^{y}	74
14	γ_{01}^{z}	79
15	γ_{01}^{z}	82
16	γ_{11}^{z}	85
17	σ_{01}^{x}	131
18	σ_{02}^{x}	132

TABLE 2 CONTINUED

$(S_{jk}^{(1)})$,

m	$\beta_{kmi}^{(1)}$	$B_{imj}^{(1)}$	$\beta_{kmi}^{(1)} B_{imj}^{(1)}$
1	$\rho_{02}^{y} q_i$	$Q_9^{(1)}$	
2	ρ_{12}^{y}	16	$-Q_{16}^{(1)}(j,k)$
3	$\rho_{12}^{y}\theta_t$	17	$-\theta_t Q_{16}^{(1)}(j,k)$
4	$\rho_{02}^{y} q_i$	29	
5	ρ_{11}^{z}	33	$-Q_{33}^{(1)}(j,k)$
6	$\rho_{11}^{z}\theta_t$	34	$-\theta_t Q_{34}^{(1)}(j,k)$
7	σ_{13}^{y}	49	$Q_{49}^{(1)}(j,k)$
8	σ_{14}^{z}	62	$-Q_{62}^{(1)}(j,k)$
9	$-\delta_{ik}$	66	$-Q_{66}^{(1)}(j,k)$
10	ν_{14}^{y}	76	$Q_{76}^{(1)}(j,k)$
11	ν_{13}^{z}	87	$Q_{87}^{(1)}(j,k)$
12	$-\delta_{ik}$	92	$-Q_{92}^{(1)}(j,k)$

TABLE 2. CONTINUED.

$$
\begin{aligned}
(S_{jk}^{(1)})_2 = & -q_i\, Q_{18}^{(1)}(j,k,i) \\
& -q_i \left\{ Q_{20}^{(1)}(j,i,k) + Q_{20}^{(1)}(j,k,i) \right\} \\
& -q_i\, Q_{35}^{(1)}(j,k,i) \\
& -q_i \left\{ Q_{39}^{(1)}(j,i,k) + Q_{39}^{(1)}(j,k,i) \right\}
\end{aligned}
$$

TABLE 2. CONTINUED.

$(S_{JK}^{(2)})_0$

m	$\beta_{Km}^{(2)}$	$\beta_{mJ}^{(2)}$	m	$\beta_{Km}^{(2)}$	$\beta_{mJ}^{(2)}$
1	σ_{01}^{x}	$a_{13}^{(2)}$	19	σ_{02}^{x}	$a_{92}^{(2)}$
2	σ_{02}^{x}	14	20	σ_{01}^{x}	103
3	V_{01}^{x}	15	21	σ_{02}^{x}	104
4	V_{02}^{x}	16	22	σ_{01}^{x}	115
5	σ_{01}^{x}	29	23	σ_{02}^{x}	116
6	σ_{02}^{x}	30	24	σ_{01}^{x}	127
7	V_{01}^{x}	31	25	σ_{02}^{x}	128
8	V_{02}^{x}	32			
9	ρ_{02}^{z}	38			
10	σ_{02}^{z}	42			
11	σ_{12}^{z}	44			
12	σ_{01}^{x}	55			
13	σ_{02}^{x}	56			
14	σ_{01}^{x}	67			
15	σ_{02}^{x}	68			
16	σ_{01}^{x}	77			
17	σ_{02}^{x}	78			
18	σ_{01}^{x}	91			

TABLE 2. CONTINUED.

$(S_{jk}^{(2)})_1$

m	$\beta_{kmi}^{(2)}$	$\beta_{imj}^{(2)}$	$\beta_{kmi}^{(2)}\beta_{imj}^{(2)}$
1	ρ_{11}^{2}	$Q_{39}^{(2)}$	$-Q_{39}^{(2)}(j,k)$
2	σ_{14}^{2}	46	$-Q_{46}^{(2)}(j,k)$
3	$-\delta_{ik}$	50	$-Q_{50}^{(2)}(j,k)$

TABLE 2 CONTINUED.

			$(\zeta_{jk}^{(3)})_0$		
m	$\beta^{(3)}_{km}$	$\beta^{(3)}_{mj}$	m	$\beta^{(3)}_{km}$	$\beta^{(3)}_{mj}$
1	σ_{01}^{y}	$a^{(3)}_{13}$	19	σ_{02}^{y}	$a^{(3)}_{93}$
2	σ_{02}^{x}	14	20	σ_{01}^{y}	104
3	ν_{01}^{x}	15	21	σ_{02}^{x}	105
4	ν_{02}^{x}	16	22	σ_{01}^{x}	116
5	σ_{01}^{y}	29	23	σ_{02}^{x}	117
6	σ_{02}^{x}	30	24	σ_{01}^{x}	128
7	ν_{01}^{x}	31	25	σ_{02}^{x}	129
8	ν_{02}^{x}	32			
9	ρ_{02}^{y}	38			
10	σ_{01}^{y}	41			
11	σ_{11}^{y}	43			
12	σ_{01}^{x}	56			
13	σ_{02}^{x}	57			
14	σ_{01}^{x}	68			
15	σ_{02}^{x}	69			
16	σ_{01}^{y}	78			
17	σ_{02}^{y}	79			
18	σ_{01}^{x}	92			

TABLE 2 CONTINUED.

$(S_{jk}^{(3)})_1$

m	$\beta_{kmi}^{(3)}$	$B_{imj}^{(3)}$	$\beta_{kmi}^{(3)} B_{imj}^{(3)}$
1	ρ_{12}^{y}	$Q_{40}^{(3)}$	$-Q_{40}^{(3)}(j,k)$
2	σ_{13}^{y}	45	$Q_{45}^{(3)}(j,k)$
3	$-\delta_{ik}$	51	$-Q_{51}^{(3)}(j,k)$

TABLE 2 CONCLUDED.

8.4 Approximations to Terms Composing Q_j^D

In this section we

1) Verify the adequacy of our approximations to section loads p_{x5}^D, p_{y5}^D, p_{z5}^D, q_{x5}^D, q_{y5}^D, q_{z5}^D, cited earlier in this chapter.

2) Illustrate the procedure for neglecting small terms contributing to Q_j^{D*} and s_{jk} for small hub accelerations.

We consider first the approximations (8.22) to (8.29), to the inertia loads. We recall from Section 4.2 that the torsion modal equation (4.61) excludes third order products of elastic variables and fourth order products of small quantities. The flatwise-edgewise modal equation, (4.65), excludes second order products of elastic variables, and third order products of small quantities. We verify the approximations to p_{x5}^D, p_{y5}^D, etc. by inspection of (8.32) to (8.34). We see that q_{x5}^D in (8.32) must be approximated as (8.23) which includes second order products of elastic variables and third order products of small quantities. The occurrences of $v'q_{y5}^D$ and $w'q_{z5}^D$ in (8.32) indicate that q_{y5}^D and q_{z5}^D need to include only first order elastic terms and second order products of small quantities, such as in (8.24) and (8.25). Similar inspections of the other terms in (8.32) to (8.34) justify the adequacies of all the approximations, (8.22) to (8.29), for p_{x5}^D, p_{y5}^D, p_{z5}^D, q_{x5}^D, q_{y5}^D, q_{z5}^D, F_{x5}^D, F_{y5}^D, F_{z5}^D, p_{x10}^D. and F_{x10}^D. Inspection of (8.22) to (8.29) verifies also the adequacy of approximations (7.72) for λ_1 and λ_2. Finally, approximation (8.23) for q_{x5}^D justifies the assertion of Section 7.1(b) that only A_y and A_z require expansions to second orders $(A_y^{(2)}$ and $A_z^{(2)})$ in (7.78). All other coefficients $A_x^{(0)}$, $A_x^{(1)}$, etc., in (7.78) occur as zeroeth and first order terms in p_{x5}^D, p_{y5}^D, etc., (8.22) to (8.29). We recall that $A_y^{(2)}$ and $A_z^{(2)}$ are functions of Δv and Δw (see 7.79), which are second order translations accompanying the rotation θ_e (see Section 5.3). Consequently, the presence of these terms in q_{x5}^D, (8.23), shows that these second order displacements are comparable to other terms contributing to q_{x5}^D. This justifies the inclusion of Δv and Δw in our calculation of blade accelerations.

We illustrate now the procedure for neglecting small terms contributing to Q_j^{D*} and s_{jk} for small hub accelerations. We consider first $\alpha_{02}^Y \theta_t q_i Q_{11}^{(1)}$ in (8.66). From Section (7.2) we obtain the magnitudes of the components in $\alpha_{02}^Y \theta_t q_i Q_{11}^{(1)}$.

$$\alpha_{02}^{y} = O(\epsilon) \tag{8.99}$$

$$\theta_t = O(\epsilon) \tag{8.100}$$

$$\begin{aligned} q_i Q_{11}^{(1)} &= -q_i \int_0^{r_T} m\, \theta_j\, r\, \theta_i\, S_{\theta_c}\, y_{10_{cg}}\, dr \\ &= -\int_0^{r_T} m\, \theta_j\, r\, \theta_e\, S_{\theta_c}\, y_{10_{cg}}\, dr \\ &= O(\theta_e \epsilon) \end{aligned} \tag{8.101}$$

Combining (8.99) to (8.101), we find

$$\alpha_{02}^{y}\, \theta_i\, q_i\, Q_{11}^{(1)} = \underline{O(\epsilon^3 \theta_e)} \tag{8.102}$$

which according to the assumptions listed in Chapter 2 is negligible. Similarly, we find the magnitudes of the other terms in (8.66), and we deduce that all terms must be retained except those involving $Q_{11}^{(1)}$ and $Q_{12}^{(1)}$.

To approximate the terms making up s_{jk}, we employ only the assumptions that blade flap and lead angles, β and δ are small. We may restrict our approximations to these angles because these are the only motion parameters present in s_{jk}. By following a similar procedure to that used to establish the magnitudes of (8.102), we deduce from the product $s_{jk}^{(1)}\ddot{q}_k$ that all terms in (8.67) should be retained except those involving $Q_{11}^{(1)}$ and $Q_{12}^{(1)}$.

When hub induced accelerations, blade offset, and rigid blade angular deflections are among the small quantities considered, we approximated the terms in $(Q_j^{D1*})_o$ to $O(\epsilon^4)$, and the terms in $(Q_j^{D2*})_o$ and $(Q_j^{D3*})_o$ each to $O(\epsilon^3)$. Similarly, we approximated the terms in $(s_{jk}^{(1)})_o\ddot{q}_k$ to $O(\epsilon^4)$, and the terms in $(s_{jk}^{(2)})_o\ddot{q}_k$ and $(s_{jk}^{(3)})_o\ddot{q}_k$ each to $O(\epsilon^3)$. These approximations may be inferred to allow somewhat larger hub and blade motions for systems modeled with rigid body flap and lead motions without pitch freedom. These orders, however, are not consistent with our approximations excluding from the general blade model other terms of comparable order, like those from cyclic pitch inputs. For this general model, the policy of retaining such high orders in the working equations is sound provided applications are restricted to problems for which the high order terms are insignificant. It is well to note also that because these additional terms do not require summation on the modes, no significant penalty in program execution time is incurred by them.

9. Aerodynamic Generalized Force Q_j^A

The calculation of the aerodynamic excitation, Q_j^A is organized to yield Q_j^A from the calculated motions and tables of non-dimensional C_l, C_D, and $C_{m\alpha}$ characteristics as functions of section angle of attack, and section Mach number. The table look-up procedure is a consequence of blade element theory, which is assumed to apply to our model.

The first section of this chapter contains a derivation of Q_j^A as an integral of section lifts, drags, and pitching moments. The second section of this chapter completes the information needed to evaluate numerically the integral Q_j^A. We derive there expressions for flow velocity, section angle of attack, and Mach number which then yield the sections lifts, drags, and pitching moments in Q_j^A from tables of these characteristics.

Unsteady aerodynamic states are represented by tables of section aerodynamic characteristics as functions of angle of attack α, and the parameters A and B of Ref. (8), proportional to $\dot{\alpha}$ and $\ddot{\alpha}$, respectively. The time derivatives $\dot{\alpha}$ and $\ddot{\alpha}$ are formed by numerical differentiation in the program, and are not discussed further in this chapter.

In addition to the restrictions imposed by blade element assumptions, the analysis does not include a radial flow model.

9.1 Expression for Q_j^A in Terms of Section Loading Characteristics

The starting points for the derivation of Q_j^A are equations (4.37) and (4.38) which are

$$Q_j^A = Q_j^{A_1} + Q_j^{A_2} + Q_j^{A_3} \tag{9.1}$$

$$Q_j^{A_1} = \int_0^{r_T} \Theta_j \, F_1^A \, dr \tag{9.2}$$

$$Q_j^{A_2} = \int_0^{r_T} w_{1_j} \, F_2^A \, dr$$

$$Q_j^{A_3} = \int_0^{r_2} v_{1_j} \, F_3^A \, dr$$

Equations (6.63) to (6.65) define F_1 to F_3 in terms of section loads. To obtain F_1^A to F_3^A from (6.63) to (6.65) we employ the following results for the section loads.

$$p_{x_{10}}^{A} = 0 \tag{9.3}$$

$$p_{y_{10}}^{A} = \ell\, s_{\alpha_r} - d\, c_{\alpha_r} \tag{9.4}$$

$$p_{z_{10}}^{A} = \ell\, c_{\alpha_r} + d\, s_{\alpha_r} \tag{9.5}$$

$$q_{x_{10}}^{A} = m_{c/4} + p_{z_{10}}^{A}\, y_{10_{c/4}} + m_d \tag{9.6}$$

$$q_{y_{10}}^{A} = 0 \tag{9.7}$$

$$q_{z_{10}}^{A} = 0 \tag{9.8}$$

$$\ell = c_\ell \tfrac{1}{2}\rho U^2 c \tag{9.10}$$

$$d = c_d \tfrac{1}{2}\rho U^2 c \tag{9.11}$$

$$m_{c/4} = c_{m_{c/4}} \tfrac{1}{2}\rho U^2 c^2 \tag{9.12}$$

$$m_d = -\frac{\pi}{8}\rho c^3 U\left(\tfrac{1}{2} - a_o\right)\dot{\bar{\theta}} \tag{9.13}$$

$$a_o = \frac{2}{c} y_{10_{c/4}} - \frac{1}{2} \qquad U_T > 0 \tag{9.14}$$

$$= \frac{1}{2} - \frac{2}{c} y_{10_{c/4}} \qquad U_T < 0 \tag{9.15}$$

$$\alpha_r = \bar{\theta} + \phi \tag{9.16}$$

$$\bar{\theta} = \theta + \theta_e \tag{9.17}$$

$$\theta = \theta_c + \theta_\epsilon \tag{9.18}$$

$$\dot{\theta} = \dot{\theta}_t + \dot{\theta}_c \tag{9.19}$$

$$\phi = \tan^{-1}(U_P/U_T) \tag{9.20}$$

$$U = (U_P^2 + U_T^2)^{\frac{1}{2}} \tag{9.21}$$

$$M = U/a_\infty \tag{9.22}$$

In the above equations, l, d, and $m_{c/4}$ define aerodynamic lift, drag, and pitching moments. Vectors l and d are perpendicular and parallel to the relative flow vector, Fig. 18, which subtends an angle α_{r_A} to the local chord. The corresponding direct loads $p_{y10}{}^A$ and $p_{z10}{}^A$ are expressed in terms of l and d by means of a rotation transformation defined in (9.4) and (9.5). Equations (9.3), (9.7) and (9.8) are assumptions that $p_{x10}{}^A$, $q_{y10}{}^A$ and $q_{z10}{}^A$ are negligible. These assumptions are justified by the smallness of blade surface viscons tractions and components of normal surface forces in the x_{10} direction, as contributions to $Q_j{}^A$, for largely cylindrical blades. When steady state section aerodynamic data are used, we add to the other terms in the pitching moment expression, (9.6), the quasi-static approximation to damping in pitch, m_d (p 279, Ref (9)). Conditions (9.14) and (9.15) simulate forward and reverse flows, respectively.

Equations (9.16) to (9.22) complete the information required to obtain numerical values of the loads $p_{y10}{}^A$, $p_{z10}{}^A$, etc., subsequently used to obtain an expression for $Q_j{}^A$ which is numerically integrated. In particular, parameters α_r and M are known when the rotor state of motion is known, and these in turn yield from tables the C_l, C_d, and $C_{mc/4}$ occurring in the loading expressions.

The flow velocity components are referred to the $\bar{x}_{10}'$ axis, defined in the next section by (9.47), and these velocity components are evaluated at the three-quarter chord ($(y_{10})3c/4$). The radial component U_{x10}' is disregarded in our model, and does not participate in the determination of the aerodynamic loads. The magnitude of $\bar{U}_p$ and $\bar{U}_T$ is denoted U, and the inclination of $\bar{U}_p + \bar{U}_T$ to $\bar{y}_{10}'$ is denoted by ϕ, Fig. 18. Expressions relating these velocity components to the rotor motions are derived in the next section.

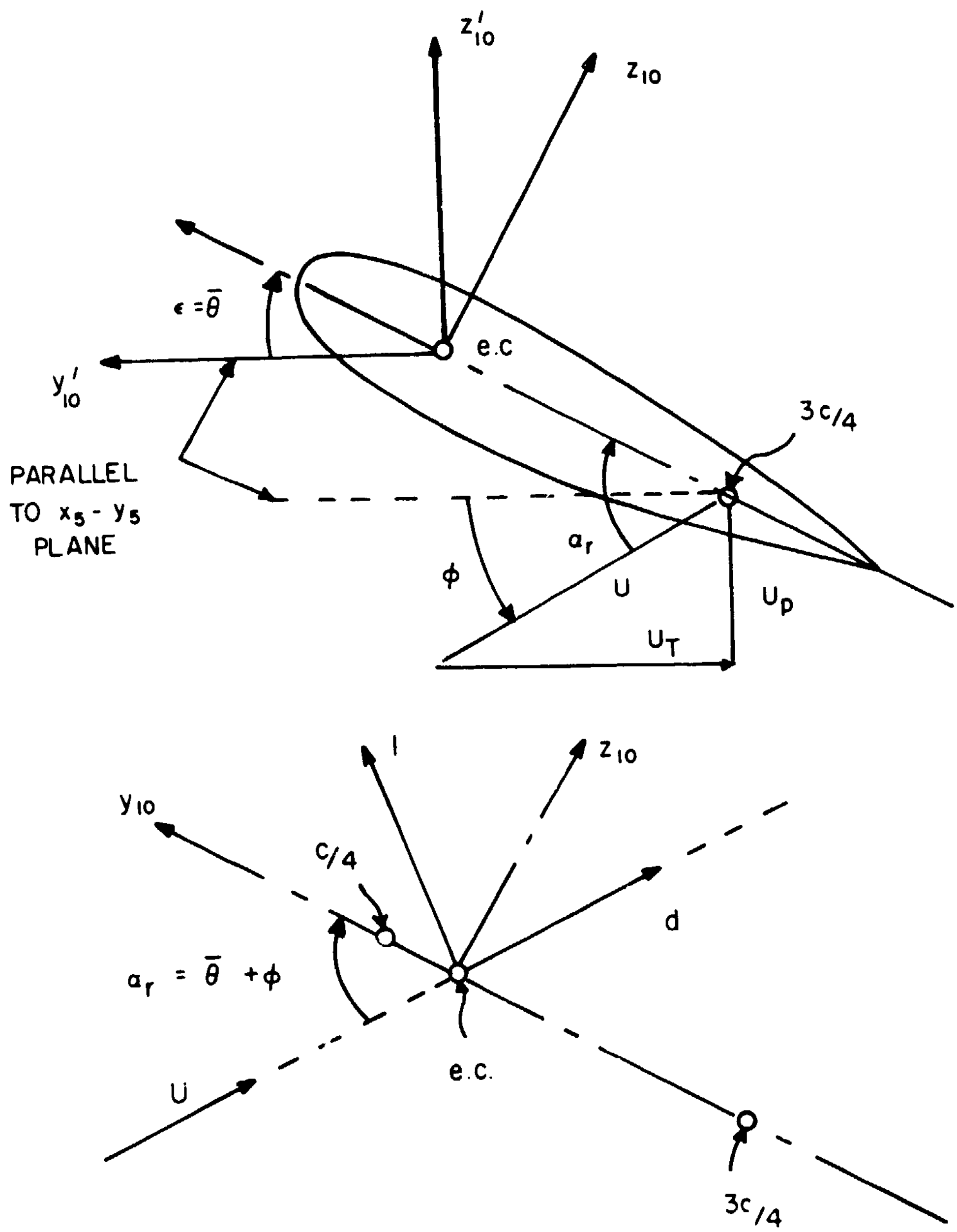

Figure 18. Blade Element Force and Velocity Vectors.

The $\bar{X}_5$ system loads required in the F_1 to F_3 expressions (6.63) to (6.65), are obtained from the $\bar{X}_{10}$ system loads by means of the transformation

$$\bar{X}_5 = A_\theta A_{\lambda_2} A_{\lambda_1} A_{\theta_e} \bar{X}_{10} \tag{9.23}$$

with (6.33) defining the transformation

(9.24)

$$A_\theta A_{\lambda_2} A_{\lambda_1} A_{\theta_e} = \begin{matrix} 1 & -\lambda_2 - \lambda_1 \theta_e & \lambda_2 \theta_e - \lambda_1 \\ v' & C\bar{\theta} & -S\bar{\theta} \\ w' & S\bar{\theta} & C\bar{\theta} \end{matrix}$$

Identify the free vector $\bar{X}_5$ with $\bar{p}_5$ or $\bar{q}_5$. Identify $\bar{X}_{10}$ with $\bar{p}_{10}$ or $\bar{q}_{10}$, and substitute (9.3) to (9.8) as convenience suggests. We obtain from (9.23) the $\bar{X}_5$ system aerodynamic loads. These are

$$p_{x_5}^A = (-\lambda_2 - \lambda_1 \theta_e) p_{y_{10}}^A + (\lambda_2 \theta_e - \lambda_1) p_{z_{10}}^A \tag{9.25}$$

$$p_{y_5}^A = C\bar{\theta}\, p_{y_{10}}^A - S\bar{\theta}\, p_{z_{10}}^A \tag{9.26}$$

$$= \ell s_\phi - d c_\phi \tag{9.27}$$

$$p_{z_5}^A = S\bar{\theta}\, p_{y_{10}}^A + C\bar{\theta}\, p_{z_{10}}^A \tag{9.28}$$

$$= \ell c_\phi + d s_\phi \tag{9.29}$$

$$q_{x_5}^A = q_{x_{10}}^A \tag{9.30}$$

$$q_{y_5}^A = v' q_{x_{10}}^A \tag{9.31}$$

$$q_{z_5}^A = w' q_{x_{10}}^A \tag{9.32}$$

The tensions required in the F_1 to F_3 expressions, (6.63) to (6.65), are

$$F_{x_{10}}^{A} = \int_r^{r_T} p_{x_{10}}^{A}\, dx = 0 \tag{9.33}$$

$$F_{x_5}^{A} = \int_r^{r_T} p_{x_5}^{A}\, dx \tag{9.34}$$

$$= \int_r^{r_T} \left[(-\lambda_2 - \lambda_1 \theta_e) P_{y_{10}}^{A} + (\lambda_2 \theta_e - \lambda_1) P_{z_{10}}^{A} \right] dx \tag{9.35}$$

Substitute (9.25) to (9.32), and (9.3) in the F_1 to F_3 expressions, (6.63) to (6.65). We obtain

$$F_1^{A} = \left[1 + (v')^2 + (w')^2\right] q_{x_{10}}^{A}$$
$$+ \underline{\left\{ \theta_e' K_A^2 \int_r^{r_T} \left[(-\lambda_2 - \lambda_1 \theta_e) P_{y_{10}}^{A} + (\lambda_2 \theta_e - \lambda_1) P_{z_{10}}^{A} \right] dx \right\}'}$$
$$\underline{- \theta_e (v'' c_\theta + w'' s_\theta) e_A \int_r^{r_T} \Big[(-\lambda_2 - \lambda_1 \theta_e) P_{y_{10}}^{A}}$$
$$\underline{+ (\lambda_2 \theta_e - \lambda_1) P_{z_{10}}^{A} \Big] dx}$$

(9.36)

$$F_2^{A} = \left(v' q_{x_{10}}^{A}\right)'$$
$$+ \underline{\left\{ w' \int_r^{r_T} \left[(-\lambda_2 - \lambda_1 \theta_e) P_{y_{10}}^{A} + (\lambda_2 \theta_e - \lambda_1) P_{z_{10}}^{A} \right] dx \right\}'}$$
$$+ P_{z_5}^{A}$$
$$\underline{- \left\{ \theta_e c_\theta e_A \int_r^{r_T} \left[(-\lambda_2 - \lambda_1 \theta_e) P_{y_{10}}^{A} + (\lambda_2 \theta_e - \lambda_1) P_{z_{10}}^{A} \right] dx \right\}''}$$

(9.37)

$$F_3^A = -\left(w' q x_{10}^A\right)' + \underline{\left\{ v' \int_r^{r_T} \left[\left(-\lambda_2 - \lambda_1 \theta_c\right) P y_{10}^A + \left(\lambda_2 \theta_e - \lambda_1\right) P z_{10}^A \right] dx \right\}'}$$

$$+ P y_5^A$$

$$\underline{- \left\{ \theta_e s_\theta e_A \int_r^{r_T} \left[\left(-\lambda_2 - \lambda_1 \theta_e\right) P y_{10}^A + \left(\lambda_2 \theta_e - \lambda_1\right) P z_{10}^A \right] dx \right\}''}$$

(9.38)

with λ_1 and λ_2 derived from (5.60) and (5.63).

Neglect the underlined terms in (9.36) to (9.38). We obtain

$$F_1^A = \left(1 + (v')^2 + (w')^2\right) q x_{10}^A \tag{9.39}$$

$$F_2^A = P z_5^A + \left(v' q x_{10}^A\right)'$$

$$F_3^A = P y_5^A - \left(w' q x_{10}^A\right)'$$

Substitute (9.39) in (9.2), and integrate by parts, as required. We obtain the aerodynamic generalized forces

$$Q_j^{A_1} = \int_0^{r_T} \theta_j \left[1 + (v')^2 + (w')^2\right] q x_{10}^A \, dr \tag{9.40}$$

$$Q_j^{A_2} = \int_0^{r_T} \left(w_{i_j} P z_5^A - w_{i_j}' v' q x_{10}^A\right) dr + \left(w_{i_j} v' q x_{10}^A\right)_T$$

$$Q_j^{A_3} = \int_0^{r_T} \left(v_{i_j} P y_5^A + v_{i_j}' w' q x_{10}^A\right) dr - \left(v_{i_j} w' q x_{10}^A\right)_T$$

$$p_{y_{10}}^A = p_{y_5}^A c_{\bar\theta} + p_{z_5}^A s_{\bar\theta}$$

$$p_{z_{10}}^A = -p_{y_5}^A s_{\bar\theta} + p_{z_5}^A c_{\bar\theta}$$

$$q_{x_{10}}^A = m_{c/4} + m_d + p_{z_{10}}^A y_{10_{c/4}} \qquad (9.41)$$

$$p_{y_5}^A = \ell s_\phi - d c_\phi$$

$$p_{z_5}^A = \ell c_\phi + d s_\phi \qquad (9.42)$$

In the above, subscript T denotes a quantity at the tip, $r = r_T$. Substitution of (9.40) in (9.1) yields the aerodynamic generalized force, Q_j^A. In the computer program, the integrals in (9.40) are evaluated numerically by the trapezium rule.

As a convenience, we list in (9.41) and (9.42) the loads required in (9.40), as well as loads employed to display the aerodynamic loads in the blade-oriented $\bar{X}_{10}$ system. The $\bar{X}_{10}$ system loads p_{y10}^A and p_{z10}^A derive from (9.4) and (9.5), or from the transformation inverse to (9.23). The additional variables required in (9.42) are listed in (9.10) to (9.22).

A slight variation expresses the loads in (9.42) in terms of C_l, C_d, U, U_p and U_T and may be a convenience. Substituting in (9.42) the results from (9.10) to (9.21), as required, we obtain

$$p_{y_5}^A = \tfrac{1}{2}\rho c U (c_\ell U_P - c_d U_T) \qquad (9.43)$$

$$p_{z_5}^A = \tfrac{1}{2}\rho c U (c_\ell U_T + c_d U_P)$$

9.2 Expressions for Relative Flow Velocity

In this section, we derive expressions for the relative flow velocity at a blade section in terms of the generalized coordinates in physical space and their time-derivatives, and the velocities induced by the rotor hub motion. These quantities are defined when the modal displacements and velocities are found from the integration of the blade modal differential equations and the solution to the support or body equation. With the relative flow velocity known, we are able to define the effective angle of attack α_r of a section, (9.16), and local Mach number, M, (9.22), which yields the section load from tables.

For the sake of completeness and as a convenience, we include in this Section and Section 9.4, the description of a gust model which is preferable to that supplied previously under contract to NASA and described in Ref (2). This theoretical gust model was not incorporated in the present program.

We express the velocity of the flow relative to a point on the blade as

$$\bar{U} = -\bar{v} + \bar{v}_{AIR} \tag{9.44}$$

Here $\bar{v}$ is the velocity of a point on the blade induced by the motion of the rotor blade and hub relative to the stationary axis $\bar{X}_I$. The term $\bar{v}_{AIR}$ isolates all contributions to $\bar{u}$ whose sources are independent of the rotor state during the calculation. Typically, $\bar{v}_{AIR}$ may derive from a wind tunnel, gust, or rotor variable inflow source. In our scheme, the rotor inflow is independent of the rotor state during the calculation and the inflow velocity is conveniently located in $\bar{v}_{AIR}$. Fig. 19 illustrates the construction of the flow velocity $\bar{u}$ for a typical case comprising a wind tunnel source V_{xI}, a rotor induced inflow, $\bar{y}$, and a flow component $-\bar{v}$ induced by the rotor motion.

As a preliminary to finding $\bar{U}$, we define the $\bar{X}_{10}'$ axis (Fig. 18) by the rotation transformation

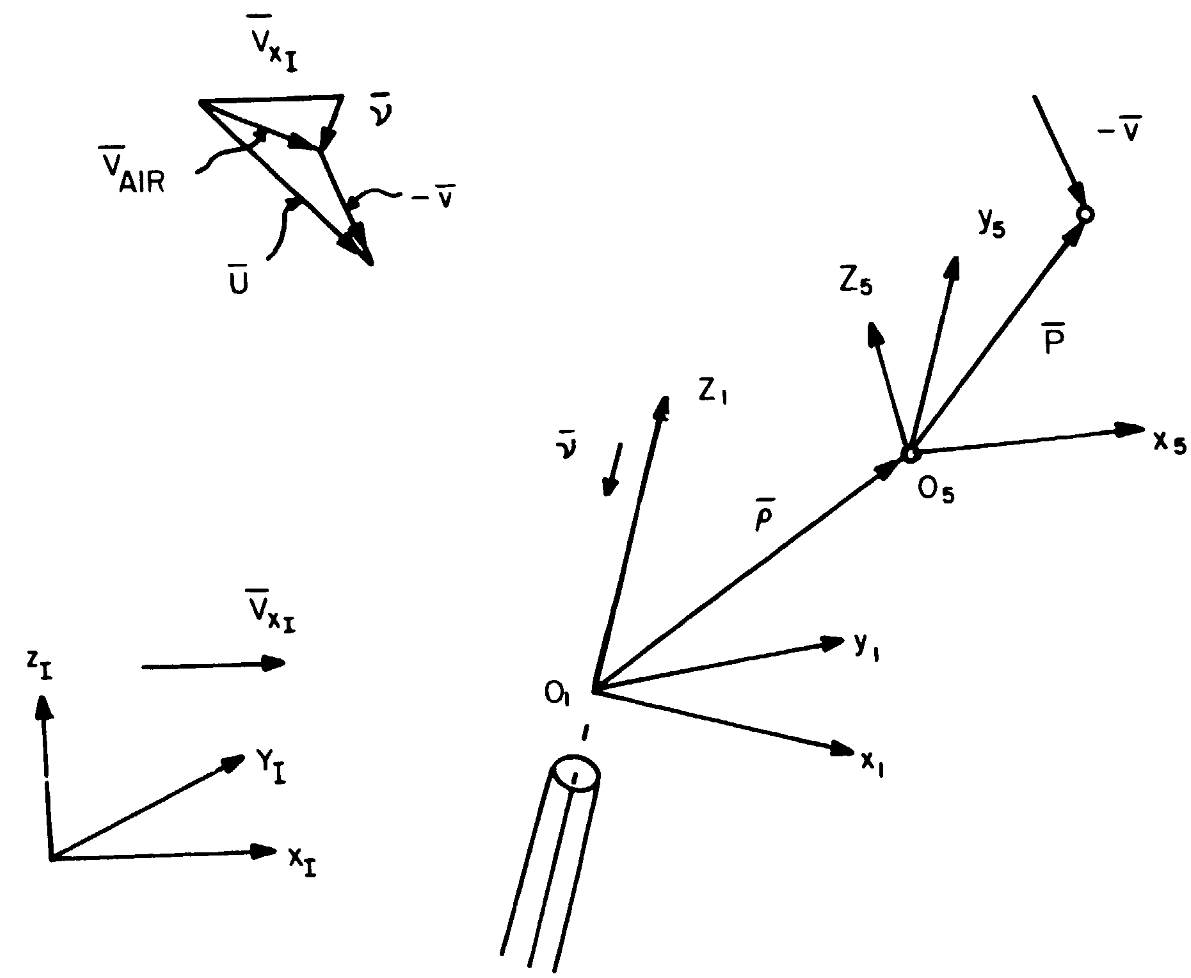

Figure 19. Example of Construction of Relative Airflow Velocity, $\overline{U}$.

$$\bar{X}_{10} = A_{\epsilon} \bar{X}_{10}' \tag{9.45}$$

with
$$A_{\epsilon} = \begin{matrix} 1 & 0 & 0 \\ 0 & c_{\epsilon} & -s_{\epsilon} \\ 0 & s_{\epsilon} & c_{\epsilon} \end{matrix} \tag{9.46}$$

Here ϵ is the angle subtended by the chord and the line y_{10}' parallel to the x_5-y_5 plane, and lying in the y_{10}-z_{10} plane (Fig. 18). A study of Figure 8 shows that ϵ is approximated by the total angle $\bar{\theta}$, defined in (9.17). To preserve the rigor of our derivations, however, we do not assume that $\epsilon = \bar{\theta}$, and prove in the next section that our approximations consistently yield this result.

Defining the component of relative airflow velocity in the $\bar{X}_{10}'$ axis as

$$\bar{U}^T = U_{x_{10}'}, -U_T, U_P \tag{9.47}$$

with U_P and U_T illustrated in Fig. 18, our objective reduces to finding U_T and U_P in terms of our generalized coordinates, and the hub motions.

Our approach will be to refer to the $\bar{X}_5$ axis all the contributions to the relative air flow appearing on the right-side of (9.44), and to resolve these components to the $\bar{X}_{10}'$ axis by the transformation

$$\bar{X}_{10}' = A_{-\epsilon} A_{-\theta_e} A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} \bar{X}_5 \tag{9.48}$$

Substituting in the transformation $A_{-\epsilon}$ the result of the next section that

$$\epsilon = \bar{\theta} \tag{9.49}$$

and multiplying the transformation matrices in (9.48), we find

$$A_{-\epsilon} A_{-\theta_e} A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} =$$

$$A_{-\bar{\theta}} A_{\theta_e} A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} = \begin{bmatrix} 1 & v' & w' \\ -v' & 1 & 0 \\ -w' & 0 & 1 \end{bmatrix} \quad (9.50)$$

after neglecting second and higher order products of elastic variables.

Considering first the component, $\bar{V}_{AIR}$ in (9.44), we express this as components v_{xI}, v_{yI}, v_{zI} referred to the $\bar{X}_I$ axis, and a rotor inflow component $v_{z1} = -\nu$ referred to the $\bar{X}_1$ axis. The components v_{xI}, v_{yI}, and v_{zI} are known functions of the spatial variables x_I, y_I and z_I, and the time, t, and represent a known wind flow or assumed three-dimensional gust. The variable inflow is modeled as a known function of blade radius and azimuth.

For the grounded support shaft axis system of Chapter 5, our $\bar{X}_1$ system velocities are

$$\begin{pmatrix} (V_{AIR})_{x_1} \\ (V_{AIR})_{y_1} \\ (V_{AIR})_{z_1} \end{pmatrix} = A_{-\phi_S} A_{-\theta_S} A_{-\psi_S} \begin{pmatrix} V_{x_I} \\ V_{y_I} \\ V_{z_I} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\nu \end{pmatrix} \quad (9.51)$$

For the rigid body in free flight shaft axes of Chapter 5, our $\bar{X}_1$ system velocities are

$$\begin{pmatrix} (V_{AIR})_{x_1} \\ (V_{AIR})_{y_1} \\ (V_{AIR})_{z_1} \end{pmatrix} = \left(A_{-\theta'}\right)_{\theta'=\pi} A_{-\phi} A_{-\theta'} A_{-\psi'} \left(A_{\theta'}\right)_{\theta'=\pi} \begin{pmatrix} V_{x_I} \\ V_{y_I} \\ V_{z_I} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\nu \end{pmatrix} \quad (9.52)$$

The transformation matrices in (9.51) and (9.52) are

$$A_{-\phi_S} \, A_{-\theta_S} \, A_{-\psi_S} =$$

$$\begin{matrix} C_{\theta_S} C_{\psi_S} & , \; C_{\theta_S} S_{\psi_S} & , \; -S_{\theta_S} \\ S_{\phi_S} S_{\theta_S} C_{\psi_S} - C_{\theta_S} S_{\psi_S} & , \; S_{\phi_S} S_{\theta_S} S_{\psi_S} + C_{\phi_S} C_{\psi_S}, & S_{\theta_S} C_{\theta_S} \\ C_{\phi_S} S_{\theta_S} C_{\psi_S} + S_{\phi_S} S_{\psi_S}, & C_{\phi_S} S_{\theta_S} S_{\psi_S} - S_{\phi_S} C_{\psi_S}, & C_{\phi_S} C_{\theta_S} \end{matrix}$$

(9.53)

$$(A_{-\theta'})_{TT} \, A_{-\phi'} A_{-\theta'} \, A_{-\psi'} \, (A_{\theta'})_{TT} =$$

$$\begin{matrix} C_{\theta'} C_{\psi'} & , \; -C_{\theta'} S_{\psi'} & , \; -S_{\theta'} \\ -S_{\phi'} S_{\theta'} C_{\psi'} + C_{\theta'} S_{\psi'} & , \; S_{\phi'} S_{\theta'} S_{\psi'} + C_{\phi'} C_{\psi'} & , \; -S_{\phi'} C_{\theta'} \\ C_{\phi'} S_{\theta'} C_{\psi'} + S_{\phi'} S_{\psi'} & , \; -C_{\phi'} S_{\theta'} S_{\psi'} + S_{\phi'} C_{\psi'} & , \; C_{\phi'} C_{\theta'} \end{matrix}$$

(9.54)

Substitute (9.53) in (9.51) and represent the airspeed in a wind tunnel as $v_{xI} = V_{AIR}$. We obtain for the grounded support

$$
\begin{aligned}
(v_{AIR})_{x_1} &= C_{\theta_S} C_{\psi_S} v_{AIR} \\
(v_{AIR})_{y_1} &= (S_{\phi_S} S_{\theta_S} C_{\psi_S} - C_{\theta_S} S_{\psi_S}) v_{AIR} \\
(v_{AIR})_{z_1} &= (C_{\phi_S} S_{\theta_S} C_{\psi_S} + S_{\phi_S} S_{\psi_S}) v_{AIR} - \nu
\end{aligned}
\tag{9.55}
$$

Substitute (9.54) in (9.52) and restrict the gust model to vertical gusts. We obtain for the rotor coupled to a rigid body in free flight

$$
\begin{aligned}
(v_{AIR})_{x_1} &= -S_{\theta'} v_{z_I} \\
(v_{AIR})_{y_1} &= -S_{\phi'} C_{\theta'} v_{z_I} \\
(v_{AIR})_{z_1} &= C_{\phi'} C_{\theta'} v_{z_I} - \nu
\end{aligned}
\tag{9.56}
$$

This model allows arbitrary cylindrical gusts in steady horizontal translation to be represented. The gust velocity profile v_{zI} is defined as a function of distance perpendicular to the moving wave front. Section 9.4 describes the procedure for obtaining from this gust function the velocity v_{zI} induced by the gust at any point on an arbitrary blade of the multi-blade rotor.

The program supplied under contract does not represent the gust in the manner described here and in Section 9.4. Details on the type of gust used are given in Ref. (2).

To find the $\bar{X}_5$ components of flow velocity induced by wind tunnel or gust sources, we define

$$(V_{AIR}^{(1)})_i \equiv (V_{AIR})_{x_1},\ (V_{AIR})_{y_1},\ (V_{AIR})_{z_1} \qquad i = 1, 2, 3 \tag{9.57}$$

with the right-side components derived from (9.55) for the grounded support mode and from (9.56) for the rigid body in free flight mode, respectively. We obtain the $\overline{X}_5$ axis components of $\overline{V}_{AIR}$ from

$$\begin{pmatrix} (V_{AIR})_{x_5} \\ (V_{AIR})_{y_5} \\ (V_{AIR})_{z_5} \end{pmatrix} = A_{-\beta} A_{-\delta} A_{-\psi} \times \begin{pmatrix} (V_{AIR})_{x_1} \\ (V_{AIR})_{y_1} \\ (V_{AIR})_{z_1} \end{pmatrix} = \begin{pmatrix} a_{1j} (V_{AIR}^{(1)})_j \\ a_{2j} (V_{AIR}^{(1)})_j \\ a_{3j} (V_{AIR}^{(1)})_j \end{pmatrix} \tag{9.58}$$

with Equation (7.29) defining the direction cosines a_{ij} in (9.58). This completes the derivation of $\overline{V}_{AIR}$.

Consider now the velocity of a point on the blade induced by the motion of the blade, we see that this follows from (7.2), which is

$$\overline{V} = \overline{V}_{05} + \frac{d\overline{P}}{dt} \tag{9.59}$$

Equation (7.18) yields the components of the velocity of the origin of $\overline{X}_5$ system resolved to the $\overline{X}_1$ system. These components are

$$(v_{05})_{x_1} = v_{0x_1} + \dot{\rho}_{x_1} + \omega_{y_1}\rho_{z_1} - \omega_{z_1}\rho_{y_1} \tag{9.60}$$

$$(v_{05})_{y_1} = v_{0y_1} + \dot{\rho}_{y_1} - \omega_{x_1}\rho_{z_1} + \omega_{z_1}\rho_{x_1}$$

$$(v_{05})_{z_1} = v_{0z_1} + \dot{\rho}_{z_1} + \omega_{x_1}\rho_{y_1} - \omega_{y_1}\rho_{x_1}$$

with ρ_{x1}, ρ_{y1}, ρ_{z1} and their time derivatives defined in (7.24). Figure 15 may be used to recall the physical meanings of the other parameters in (9.60).

We define

$$(v_{05}^{(1)})_i \equiv (v_{05})_{x_1},\ (v_{05})_{y_1},\ (v_{05})_{z_1} \qquad i = 1, 2, 3 \tag{9.61}$$

Then by a similar derivation to that used to obtain (9.58), we find

$$(v_{05})_{x_5} = a_{1j}\,(v_{05}^{(1)})_j \tag{9.62}$$

$$(v_{05})_{y_5} = a_{2j}\,(v_{05}^{(1)})_j$$

$$(v_{05})_{z_5} = a_{3j}\,(v_{05}^{(1)})_j$$

From (7.40) we have

$$\bar{P} = \bar{i}_5\, P_{x_5} + \bar{j}_5\, P_{y_5} + \bar{k}_5\, P_{z_5} \tag{9.63}$$

and differentiating this we obtain

$$\frac{d\bar{P}}{dt} = \bar{i}_5 \frac{dP_{x_5}}{dt} + \bar{j}_5 \frac{dP_{y_5}}{dt} + \bar{k}_5 \frac{dP_{z_5}}{dt}$$

$$+ \frac{d\bar{i}_5}{dt} P_{x_5} + \left(\frac{d\bar{j}_5}{dt}\right) P_{y_5} + \left(\frac{d\bar{k}_5}{dt}\right) P_{z_5} \qquad (9.64)$$

Employ in (9.64), equations (7.42) to (7.44) for d_{i5}/dt, d_{j5}/dt, and d_{k5}/dt. We obtain

$$\frac{d\bar{P}}{dt} = \bar{i}_5 \left(\dot{P}_{x_5} + \omega_{y_5} P_{z_5} - \omega_{z_5} P_{y_5} \right)$$

$$+ \bar{j}_5 \left(\dot{P}_{y_5} - \omega_{x_5} P_{z_5} + \omega_{z_5} P_{x_5} \right)$$

$$+ \bar{k}_5 \left(\dot{P}_{z_5} + \omega_{x_5} P_{y_5} - \omega_{y_5} P_{x_5} \right) \qquad (9.65)$$

with Equation (7.59) yielding ω_{x5}, ω_{y5}, and ω_{z5} required in (9.65). Neglect second and higher order products of elastic deflections in (7.63) and (7.64) and specify the blade section three-quarter point as the point at which the relative air velocity is to be evaluated, that is

$$y_{10} = (y_{10})_{3c/4} \qquad (9.66)$$

$$z_{10} = 0$$

We obtain from (7.63) and (7.64)

$$P_{x_5} = r - \lambda_2 (y_{10})_{3c/4} \qquad (9.67)$$

$$P_{y_5} = v + C_{\bar{\Theta}} (y_{10})_{3c/4}$$

$$P_{z_5} = w + S_{\bar{\Theta}} (y_{10})_{3c/4}$$

$$\dot{P}_{x_5} = -\dot{\lambda}_2 (y_{10})_{c/4} \qquad (9.68)$$

$$\dot{P}_{y_5} = \dot{v} + \dot{C}_{\bar{\Theta}} (y_{10})_{3c/4}$$

$$\dot{P}_{z_5} = \dot{w} + \dot{S}_{\bar{\Theta}} (y_{10})_{3c/4}$$

The neglect of second and higher order products of elastic variables in (9.67) and (9.68) requires the approximations defined by (7.72) to (7.74) to quantities appearing in (9.67) and (9.68). The required elements are

$$\lambda_2 = C_{\theta_c} v' + S_{\theta_c} w' \tag{9.69}$$

$$\dot{\lambda}_2 = C_{\theta_c} \dot{v}' + S_{\theta_c} \dot{w}'$$

$$S_{\bar{\theta}} = S_{\theta_c} + \theta_{e_1} C_{\theta_c}$$

$$\dot{S}_{\bar{\theta}} = \dot{\theta}_{e_1} C_{\theta_c}$$

$$C_{\bar{\theta}} = C_{\theta_c} - \theta_{e_1} S_{\theta_c}$$

$$\dot{C}_{\bar{\theta}} = -\dot{\theta}_{e_1} S_{\theta_c}$$

$$\theta_{e_1} = \theta_t + \theta_e$$

Substitution of (9.59) and (9.65) in (9.44) yields the airflow velocity referred to the $\bar{X}_5$ axis. This is

$$U_{x_5} = -(V_{05})_{x_5} - (\dot{P}_{x_5} + \omega_{y_5} P_{z_5} - \omega_{z_5} P_{y_5}) + (V_{AIR})_{x_5}$$

$$U_{y_5} = -(V_{05})_{y_5} - (\dot{P}_{y_5} - \omega_{x_5} P_{z_5} + \omega_{z_5} P_{x_5}) + (V_{AIR})_{y_5}$$

$$U_{z_5} = -(V_{05})_{z_5} - (\dot{P}_{z_5} + \omega_{x_5} P_{y_5} - \omega_{y_5} P_{x_5}) + (V_{AIR})_{z_5} \tag{9.70}$$

Employ (9.47) to (9.50) to obtain from U_{x5}, U_{y5}, and U_{z5} the $\bar{X}_{10}'$ components of airflow velocity. We find that

$$\begin{Bmatrix} U_{x_{10}} \\ -U_T \\ U_P \end{Bmatrix} = \begin{pmatrix} 1 & v' & w' \\ -v' & 1 & 0 \\ -w' & 0 & 1 \end{pmatrix} \times \begin{Bmatrix} U_{x_5} \\ U_{y_5} \\ U_{z_5} \end{Bmatrix} \tag{9.71}$$

Substitute (9.70) in (9.71) and neglect second and higher order products of elastic variables. We obtain

$$U_T = -v'\left[(V_{05})_{x_5} - (V_{AIR})_{x_5} + \dot{P}_{x_5} + \omega_{y_5} P_{z_5}^{(0)} - \omega_{z_5} P_{y_5}^{(0)}\right]$$
$$+ \left[(V_{05})_{y_5} - (V_{AIR})_{y_5} + \dot{P}_{y_5} - \omega_{x_5} P_{z_5} + \omega_{z_5} P_{x_5}\right]$$

$$U_P = w'\left[(V_{05})_{x_5} - (V_{AIR})_{x_5} + \omega_{y_5} P_{z_5}^{(0)} - \omega_{z_5} P_{y_5}^{(0)}\right]$$
$$- \left[(V_{05})_{z_5} - (V_{AIR})_{z_5} + \dot{P}_{z_5} + \omega_{x_5} P_{y_5} - \omega_{y_5} P_{x_5}\right]$$

$$P_{x_5}^{(0)} = r \tag{9.72}$$

$$P_{y_5}^{(0)} = C_{\theta_c} (y_{10})_{3c/4} \tag{9.73}$$

$$P_{z_5}^{(0)} = S_{\theta_c} (y_{10})_{3c/4}$$

The magnitude of the flow velocity, airfoil angle of attack, and flow Mach number derive from

$$U = \left(U_P^2 + U_T^2\right)^{\frac{1}{2}} \tag{9.74}$$

$$\alpha_r = \tan^{-1} U_P / U_T \tag{9.75}$$

$$M = U / a_\infty \tag{9.76}$$

Equations (9.72), and the other equations defining the elements in (9.72) given in this section and Section 9.4 for the gust, provide the working forms for the evaluation of U_T and U_P. It is to be understood that blade modal displacements, q_i, and velocities, $\dot{q}_i$ are known. These are employed in modal sums, like (7.80) and (7.81), to obtain blade physical displacements and the required spatial and temporal derivatives of these displacements. The substitution in (9.72) of hub motion velocities and shaft angle displacements deriving from the grounded support or rigid body support system of equations completes the determination of U_T and U_P, (9.72). The parameters, U, α_r, and M required to calculate the aerodynamic loads in Q_j^A follow from (9.74) to (9.76).

9.3 Proof That Rotation Angle $\epsilon = \bar{\theta}$

We prove that the rotation angle ϵ relating $\bar{X}_{10}'$ and $\bar{X}_{10}$ is equal to the total angle $\bar{\theta}$ defined in (9.17). Multiplying the rotation matrices in (9.48) and neglecting second and higher order product of elastic variables, we find

$$A_{-\epsilon} A_{-\theta_e} A_{-\lambda_1} A_{-\lambda_2} A_{-\theta} = \begin{bmatrix} 1 & v' & w' \\ -C_\epsilon \lambda_2 + S_\epsilon \lambda_1 & C_{\epsilon-\bar{\theta}} & -S_{\epsilon-\bar{\theta}} \\ -S_\epsilon \lambda_2 - C_\epsilon \lambda_1 & S_{\epsilon-\bar{\theta}} & C_{\epsilon-\bar{\theta}} \end{bmatrix} \tag{9.77}$$

The scalar product of the unit vectors $\bar{j}_{10}'$ and $\bar{k}_5$ is the direction cosine $-s_{\epsilon-\bar{\theta}}$, that is

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$$\bar{j}_{10}' \cdot \bar{K}_5 = -S_{\epsilon - \bar{\theta}} \qquad (9.78)$$

Since $\bar{j}_{10}'$ is perpendicular to $\bar{k}_5$, by construction, their scalar product is zero, and, hence, from (9.78)

$$\epsilon = \bar{\theta} \qquad (9.79)$$

9.4 Derivation of Gust Induced Flow Velocity v_{zI} From the Prescribed Gust Function

We describe here the coordinate transformations for finding from the input gust function the velocity v_{zI} induced by the gust at any point on an arbitrary blade of the multi-blade rotor.

Our procedure consists of obtaining the stationary axis coordinates x_I, y_I corresponding to an arbitrary point on any blade of the rotor in terms of blade radial and azimuthal coordinates. Using these values of x_I and y_I we interpolate the gust function to obtain the velocity v_{zI} at the point r, ψ on the blade. We neglect in the determination of v_{zI} the effect on x_I and y_I of blade elastic displacements and blade pitch. The velocity transformations of Section 9.2 then yield the contributions to U_T and U_P from v_{zI}, thereby completing the numerical definitions of U_T and U_P.

Fig. 20 illustrates the geometry of the gust model. The gust is cylindrical vertical gust moving with a steady velocity V_G perpendicular to the wave front. The gust vertical velocity profile in the direction z_I is f.(ξ_{I1}) in a coordinate system, ξ_{I1}, shown in Fig. 20, which moves with the gust, and with ξ_{I1} perpendicular to the gust front. The length of the gust is ξ_{I1} = L. At the instant t = t_o that the gust first impinges on the rotor disk, the origin O of the moving axis $\bar{X}_I'$ attached to the rotor is assumed to coincide with the origin of the stationary axis $\bar{X}_I$. At this instant, the gust is inclined at an angle α to the rotor axis $\bar{X}_I'$ as shown in Fig. 20, and the rotor configuration is such that the azimuth angle ψ_R of a reference blade used as a time parameter, is $\psi_R = (\psi_R)_o$. The helicopter is assumed to be moving always with a velocity V in the negative x_I' direction. We wish to find x_I and y_I for a point ψ, r on any blade of the rotor, and thereby to obtain v_{zI} from f(ξ_{I1}).

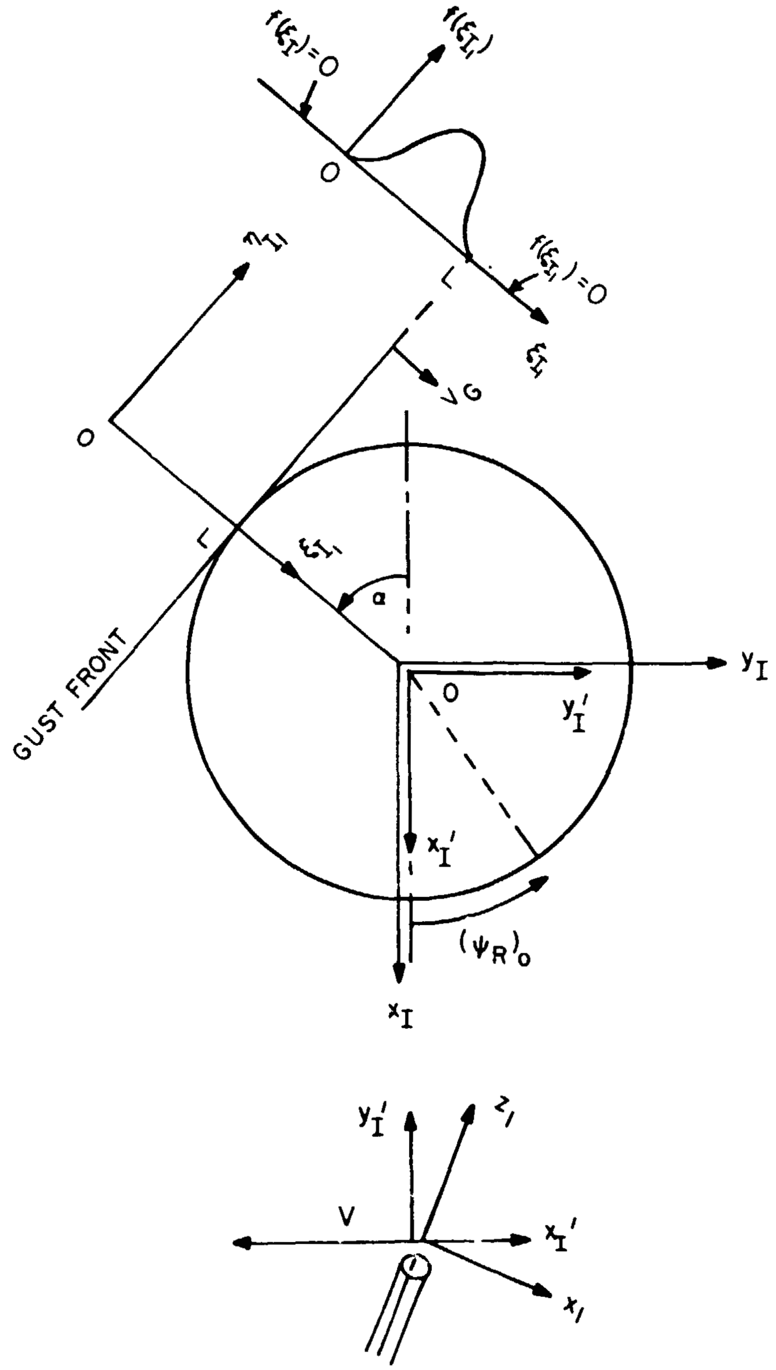

Figure 20. Geometry of Gust Model.

Fig. 20 yields the transformation between the moving $\bar{\xi}_{I1}$ axis and the stationary $\bar{X}_I$ axis. We require only ξ_{I1} and this is

$$\xi_{I1} = x_I C_\alpha + y_I S_\alpha + R + L - V_c(t - t_o) \tag{9.80}$$

The transformations of Section 5 yield

$$\bar{X}_I = \bar{X}_{I_o} + (A_{\theta}')_{\pi} A_{\psi}' A_{\theta}' A_{\phi}' (A_{-\theta}')_{\pi} \bar{X}_1$$

$$\bar{X}_1 = A_\psi \bar{E} + A_\psi A_\delta A_\beta \bar{X}_5 \tag{9.81}$$

with $\bar{X}_{Io}$ representing the $\bar{X}_I$ coordinates of the origin O of the $\bar{X}_I$ system. Since $\bar{X}_{Io} = 0$ at $t = t_o$ we have

$$x_{I_o} = \int_{t_o}^{t} V_{x_o}\, dt \tag{9.82}$$

$$= -V(t - t_o) \tag{9.83}$$

$$y_{I_o} = \int_{t_o}^{t} v_{y_o}\, dt \tag{9.84}$$

$$= 0 \tag{9.85}$$

Symbols v_{xo}, v_{yo} (and v_{zo}, not required here) are components of hub velocity, referred to the $\bar{X}_I$ system.

By the reversal law for transposed products, we can prove that

$$(A_{\theta}')_{\pi} A_\psi' A_\theta' A_\phi' (A_{-\theta}')_{\pi} = \left((A_{-\theta}')_{\pi} A_{-\phi}' A_{-\theta}' A_{-\psi}' (A_{\theta}')_{\pi}\right)^T \tag{9.86}$$

$$A_\psi A_\delta A_\beta = (A_{-\beta} A_{-\delta} A_{-\psi})^T \tag{9.87}$$

Equations (9.86) and (9.87) enable us to use the transposes of (9.54) and (7.29) to obtain the left-sides of (9.86) and (9.87) required in (9.81). Recall from (7.28) that direction cosines a_{ij} are given by

$$[a_{ij}] = (A_{-\beta} A_{-\delta} A_{-\psi}) \tag{9.88}$$

with i in a_{ij} denoting a row, and j denoting a column. Define the new set of direction cosines b_{ij} from

$$[b_{ij}] = (A_{-\theta'})_{T} A_{-\phi'} A_{-\theta'} A_{-\psi'} (A_{\theta'})_{T} \tag{9.89}$$

$$= \begin{matrix} C_{\theta'} C_{\psi'} , & -C_{\theta'} S_{\psi'} , & -S_{\phi'} \\ -S_{\phi'} S_{\theta'} C_{\psi'} + C_{\theta'} S_{\psi'} , & S_{\phi'} S_{\theta'} S_{\psi'} + C_{\phi'} S_{\psi'} , & -S_{\phi'} C_{\theta'} \\ C_{\phi'} S_{\theta'} C_{\psi'} + S_{\phi'} S_{\psi'} , & -C_{\phi'} S_{\theta'} S_{\psi'} + S_{\phi'} C_{\psi'} , & C_{\phi'} C_{\theta'} \end{matrix} \tag{9.90}$$

with i in b_{ij} denoting a row and j denoting a column. We may express the components of (9.81) as

$$\begin{aligned} x_I &= -V(t - t_0) + b_{11} x_1 + b_{21} y_1 + b_{31} z_1 \\ y_I &= b_{12} x_1 + b_{22} y_1 + b_{32} z_1 \end{aligned} \tag{9.91}$$

$$\begin{aligned} x_1 &= e C_\psi + a_{11} r \\ y_1 &= e S_\psi + a_{12} r \\ z_1 &= a_{13} r \end{aligned} \tag{9.92}$$

Replacing t by ψ_R/Ω and t_0 by $(\psi_R)_0/\Omega$, we summarize the formulas required to form v_{zI}.

$$v_{zI} = f\ (\xi_{I_1}) \tag{9.93}$$

$$\xi_{I_1} = x_I c_\alpha + y_I s_\alpha + R + L - \frac{V_G}{\Omega}\left(\psi_R - (\psi_R)_o\right)$$

$$x_I = -\frac{V}{\Omega}\left(\psi_R - (\psi_R)_o\right) + b_{11} x_1 + b_{21} y_1 + b_{31} z_1$$

$$y_I = b_{12} x_1 + b_{22} y_1 + b_{32} z_1$$

$$x_1 = e c_\psi + a_{11} r$$

$$y_1 = e s_\psi + a_{12} r$$

$$z_1 = a_{13} r$$

Direction cosines a_{ij} and b_{ij} derive from (7.29) and (9.89). The argument of the direction cosines a_{ij} and functions c_ψ and s_ψ in (9.93) is the angle to the point ψ, r on a blade at which v_{zI} is required. It is not to be replaced by ψ_R unless the v_{zI} at the reference blade is needed.

A restriction of the gust model that should be kept in mind is that the rotor hub is assumed to be translating with a steady velocity $v_{xo} = -V$ in the calculation of the gust-induced velocity, v_{zI}. This restriction is introduced in the expressions for the hub translations x_{Io}, and y_{Io}, (9.82) and (9.84). The effect of the restriction is an error in the calculation of v_{zI} when the hub motion is non-uniform, and v_{xo}, and v_{yo} exhibit time-dependence. It was thought that the restriction to steady hub translation in the calculation of the gust profile, v_{zI}, was worth the simplifications achieved in the x_{Io} and y_{Io} expressions, (9.83) and (9.85). Further, it was felt that our gust model is an adequate representation for problems in which the major translation is approximated by $v_{xo} = -V$. It should be noted that there are no hub motion restrictions in the velocity transformation yielding U_P and U_T from v_{zI}, which include hub angular displacements for general displacements without restriction. Also, the use of $v_{xo} = -V$ to calculate v_{zI} should not be taken to mean that $v_{xo} = -V$ applies to other parts of the theory, such as in the inertial and other aerodynamic elements of the model where no such assumption is made.

A minor point to note is that for substantial rotor disk inclinations our gust will not penetrate the rotor disk until some time after $t = t_0$, and this should be understood to interpret a response to this gust model.

10. Lag Damper Generalized Force Q_j^P

Equations (4.38) and (4.35) are the basis for the formation of the generalized force induced by concentrated loads.

The equations are

$$Q_j^P = Q_j^{P_1} + Q_j^{P_2} + Q_j^{P_3} \tag{10.1}$$

$$Q_j^{P_1} = \int_0^{r_T} \theta_j F_1^P dr \tag{10.2}$$

$$Q_j^{P_2} = \int_0^{r_T} w_{1j} F_2^P dr$$

$$Q_j^{P_3} = \int_0^{r_T} v_{1j} F_3^P dr$$

In the present model, we assume that the lag damper is the only contributor to Q_j^P. Other physically concentrated loads, like those induced by the pushrod, are embodied in the root springs in the calculation of normal modes (Chapter 3.). Fig. 21 illustrates the geometry assumed, which is such that the lag damper transmits to the blade only a moment M_{Z4} in the X_4 axis, that is

$$\bar{M}_4^T = M_{x_4}, M_{y_4}, M_{z_4} \tag{10.3}$$

$$= 0, 0, M_{z_4} \tag{10.4}$$

With

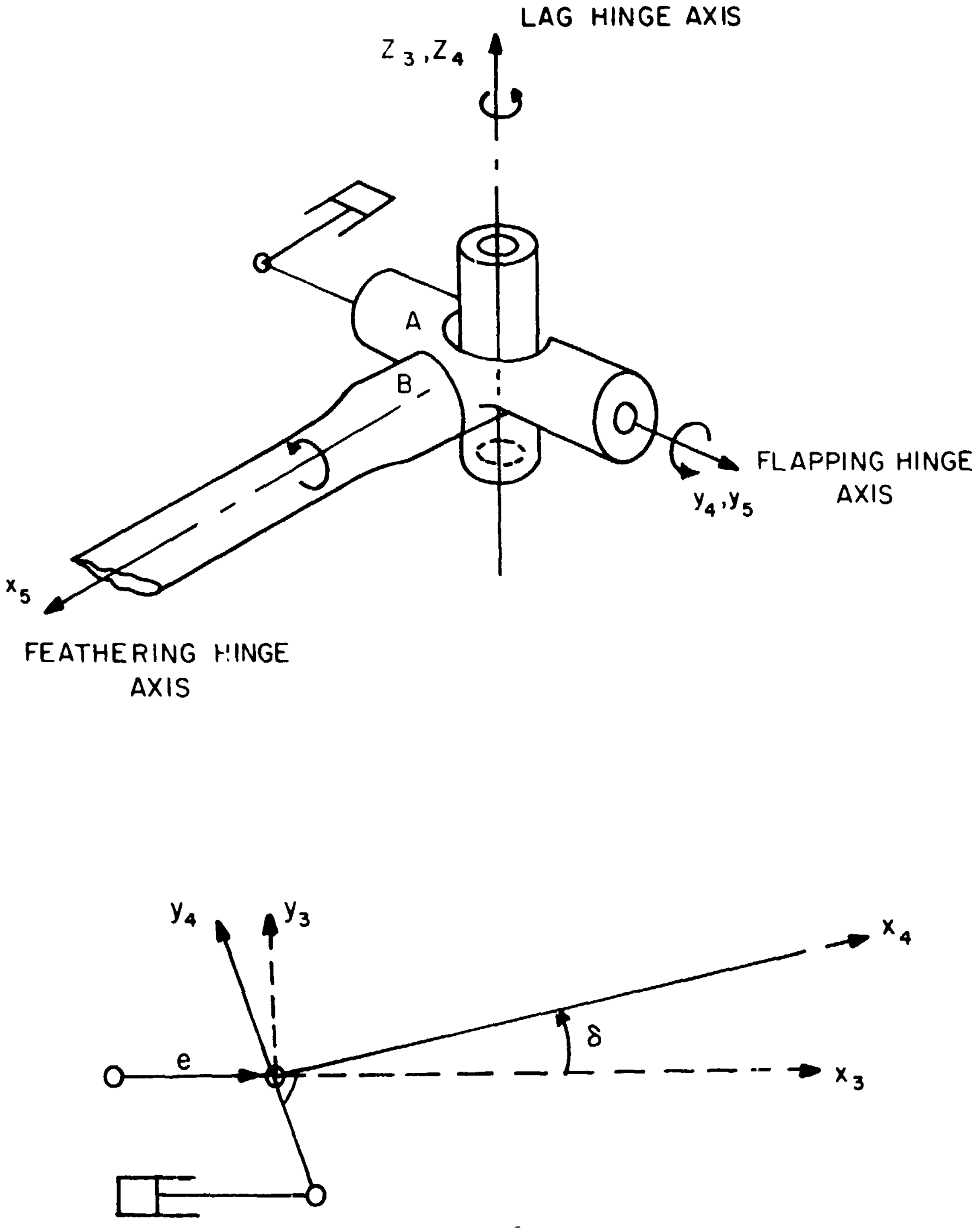

Figure 21. Lag Damper and Hinge Geometry.

$$\overline{M}_5 = M_{x_5}, M_{y_5}, M_{z_5} \tag{10.5}$$

The $\overline{M}_5$ moments applied to the yoke A derive from

$$\overline{M}_5 = A_{-\beta} \overline{M}_4 \tag{10.6}$$

from which we find

$$\overline{M}_5^T = S_\beta M_{z_4}, 0, C_\beta M_{z_4} \tag{10.7}$$

A torsion moment M_{x5} cannot transfer from yoke A to cuff B across the feathering bearing. It follows that $M_{x5} = 0$ at B and the only non-zero moment applied by the lag damper to the blade at B is

$$M_{z_5} = C_\beta M_{z_4} \tag{10.8}$$

The moment per unit length of blade follows from (6.68), which yields

$$q_{z_5} = -M_{z_5}' \tag{10.9}$$

To obtain this result we assumed the existence of the derivatives of M_{z5} on the grounds that the applied loads are distributed physically over a small but non-vanishing region of blade, in such a way as to make the derivatives of M_{z5} finite. Substituting (10.8) in (10.9) and assuming M_{z5}' to be predominantly larger than M_{z5}, we find

$$q_{z_5} \simeq -C_\beta M_{z_4}' \tag{10.10}$$

Substitution of (6.63) to (6.65) for F_1 to F_3 in (10.2) yields

$$Q_j^{P_1} = \int_0^{r_T} \theta_j w' q_{z_5} dr \tag{10.11}$$

$$Q_j^{P_2} = 0 \tag{10.12}$$

$$Q_j^{P_3} = \int_c^{r_T} -v_{1j} q_{z_5}' dr \tag{10.13}$$

Assume that all derivatives in (10.13) are finite, integrate (10.13) by parts, and set $v_{1j} = 0$ at $r = 0$ and $q_{z5}' = 0$ at $r = r_T$. We find

$$Q_j^{P_3} = \int_0^{\Delta} v_{1j}' q_{z_5} dr \tag{10.14}$$

where Δ is a small distance over which is applied the moment per unit span, q_{z5}, induced by the lag damper. We approximate this moment as

$$q_{z_5} = -c_\beta m_{z_4}' \tag{10.15}$$

$$= c_\beta \frac{m_{z_4}}{\Delta} \tag{10.16}$$

Approximating the blade deflections as

$$w = \frac{1}{2}(w'')_0 r^2 \tag{10.17}$$

$$v_{1j} = \frac{1}{2}(v_j'')_0 r^2 + \delta_j r \tag{10.18}$$

where subscript zero indicates a quantity at the lag hinge, substituting (10.17) and (10.18) in (10.11) and (10.14) and taking the limits as $\Delta \to 0$, we find

$$Q_j^{P_1} = 0 \tag{10.19}$$

$$Q_j^{P_2} = 0 \tag{10.20}$$

$$Q_j^{P_3} = \delta_j C_\beta M_{z_4} \tag{10.21}$$

Assume the lag damping moment representation to be

$$M_{z_4} = -C_{LD}(\dot{v}_1)_0' \tag{10.22}$$

$$= -C_{LD}\dot{\delta} \tag{10.23}$$

where C_{LD} is the lag damper constant (lb-ft-sec/rad units), and substitute (10.23) in (10.21). We obtain

$$Q_j^{P_3} = -C_{LD}\,\delta_j C_\beta \dot{\delta} \tag{10.24}$$

Substitution of (10.19), (10.20), and (10.24) in (10.1) yields

$$Q_j^P = -C_{LD}\,\delta_j\,C_\beta\,\dot{\delta} \tag{10.25}$$

which completes the definition of $Q_j P$ required in the element t_j, defined by (4.77).

11. Shears and Moments

We derive here the expressions used in the program to obtain the running moments along the blade and the force and moments applied by the rotor to the hub. The running moments are employed to display the blade response. The rotor forces and moments applied to the hub are employed to calculate the excitation of the support to find the response of the coupled rotor/support system. The coupled system includes either the grounded flexible support or the coupled fuselage in free flight.

Blade running moments for display of program responses are calculated from the following approximate internal moment expressions.

$$M_{x_{10}} = GJ\theta_e{}' + (\theta' + \theta_e{}')\,k_A^2\,\hat{T} \tag{11.1}$$

$$M_{y_{10}} = -EI_y(C_\theta w'' - S_\theta v'') \tag{11.2}$$

$$M_{z_{10}} = EI_z(S_\theta w'' + C_\theta v'') - e_A\hat{T} \tag{11.3}$$

$$\hat{T} = \int_r^{r_T} m\,\Omega^2(e+x)\,dx \tag{11.4}$$

$$= e\Omega^2 R_1 + \Omega^2 R_2 \tag{11.5}$$

Expressions (11.1) to (11.3) derive from (6.17), (6.30) and (6.31) with $\hat{T}$ substituting for F_{x10}. For certain conditions - for example, when hub accelerations are important - $\hat{T}$ should be replaced by F_{x10} (See (6.61) and (6.71) to obtain F_{x10}). Also, the expression for M_{z10} is not strictly correct when a Sikorsky-type counterweight is present, and preferably should be replaced by (6.31).

Blade root shears and moments resolved to the $\bar{X}_5$ axis derive from the external force and moment expressions, (6.71) and (6.74) to (6.76), specialized for x = 0. The resulting shears and moments are subscripted with the symbol e.

$$(F_{x_5})_e = \int_0^{r_T} \left(p_{x_5}^D + p_{x_5}^A \right) dx \qquad (11.6)$$

$$(F_{x_5})_e = \int_0^{r_T} \left(p_{y_5}^D + p_{y_5}^A \right) dx$$

$$(F_{z_5})_e = \int_0^{r_T} \left(p_{z_5}^D + p_{z_5}^A \right) dx$$

$$(11.7)$$

$$(M_{x_5})_e = \int_0^{r_T} \left(q_{x_5}^D + q_{x_5}^A \right) dx + \int_0^{r_T} (v - v(0)) \left(p_{z_5}^D + p_{z_5}^A \right) dx - \int_0^{r_T} (w - w(0)) \left(p_{y_5}^D + p_{y_5}^A \right) dx$$

$$(M_{y_5})_e = \int_0^{r_T} \left(q_{y_5}^D + q_{y_5}^A \right) dx + \int_0^{r_T} (w - w(0)) \left(p_{x_5}^D + p_{x_5}^A \right) dx - \int_0^{r_T} (x - 0) \left(p_{z_5}^D + p_{z_5}^A \right) dx$$

$$(M_{z_5})_e = \int_0^{r_T} \left(q_{z_5}^D + q_{z_5}^A \right) dx - \int_0^{r_T} (v - v(0)) \left(p_{x_5}^D + p_{x_5}^A \right) dx + \int_0^{r_T} (x - 0) \left(p_{z_5}^D + p_{z_5}^A \right) dx$$

The inertia loads p_{x5}^D to q_{z5}^D in (11.6) and (11.7) are obtained from (8.14) to (8.17) with (7.77) yielding A_x to C_z, and (5.63) and (5.60) yielding λ_1 and λ_2. Aerodynamic loads p_{x5}^D to q_{z5}^A are obtained from (9.25) to (9.32) with (9.4) to (9.6) yielding p_{y10}^A, p_{z10}^A, and q_{x10}^A required in these expressions. For articulated rotors the program sets the hinge moments to zero and loops around the calculation of (11.7).

The corresponding shears and moments resolved to the $\bar{X}_1$ axis are

$$\begin{pmatrix} (F_{x_1})_e \\ (F_{y_1})_e \\ (F_{z_1})_e \end{pmatrix} = A_\psi A_\delta A_\beta \cdot \begin{pmatrix} (F_{x_5})_e \\ (F_{y_5})_e \\ (F_{z_5})_e \end{pmatrix} \tag{11.8}$$

and this may be written

$$\begin{aligned} (F_{x_1})_e &= a_{11}(F_{x_5})_e + a_{21}(F_{y_5})_e + a_{31}(F_{z_5})_e \\ (F_{y_1})_e &= a_{12}(F_{x_5})_e + a_{22}(F_{y_5})_e + a_{32}(F_{z_5})_e \\ (F_{z_1})_e &= a_{13}(F_{x_5})_e + a_{23}(F_{y_5})_e + a_{33}(F_{z_5})_e \end{aligned} \tag{11.9}$$

with the a_{ij} obtained from (7.29). Similarly

$$\begin{aligned} (M_{x_1})_e &= a_{11}(M_{x_5})_e + a_{21}(M_{y_5})_e + a_{31}(M_{z_5})_e \\ (M_{y_1})_e &= a_{12}(M_{x_5})_e + a_{22}(M_{y_5})_e + a_{32}(M_{z_5})_e \\ (M_{z_1})_e &= a_{13}(M_{x_5})_e + a_{23}(M_{y_5})_e + a_{33}(M_{z_5})_e \end{aligned} \tag{11.10}$$

Fig. 22 may be used to derive hu' shears and mome.. s induced by a single blade. The result.s ,

$$
\begin{aligned}
(F_{x_1})_H &= (F_{x_1})_e \\
(F_{y_1})_H &= (F_{y_1})_e \\
(F_{z_1})_H &= (F_{z_1})_e
\end{aligned}
\tag{11.11}
$$

$$
\begin{aligned}
(M_{x_1})_H &= (M_{x_1})_e + e\, S_\psi (F_{z_1})_e \\
(M_{y_1})_H &= (M_{y_1})_e - e\, c_\psi (F_{z_1})_e \\
(M_{z_1})_H &= (M_{z_1})_e + e\, c_\psi (F_{y_1})_e - e\, s_\psi (F_{z_1})_e
\end{aligned}
\tag{11.12}
$$

Shears and moments from all the blades are summed to obtain forces and moments applied to the hub by the rotor.

The forces and moments in (11.11) and (11.12) are components in the direction of the shaft oriented axis $\bar{X}_1$ used with the grounded support (Fig. 4). To obtain components in the directions of the shaft oriented axis $\bar{\xi}_1$ coupled to the rigid body (Fig. 5) required for the determination of body response in free flight, we apply the following additional transformations.

$$
(F_{\xi_1}, F_{\eta_1}, F_{\zeta_1})_H = (-F_{x_1}, F_{y_1}, -F_{z_1})_H \tag{11.13}
$$

$$
(M_{\xi_1}, M_{\eta_1}, M_{\zeta_1})_H = (-M_{x_1}, M_{y_1}, -M_{\eta_1})_H \tag{11.14}
$$

These relations may be verified from (5.12) or Fig. 6.

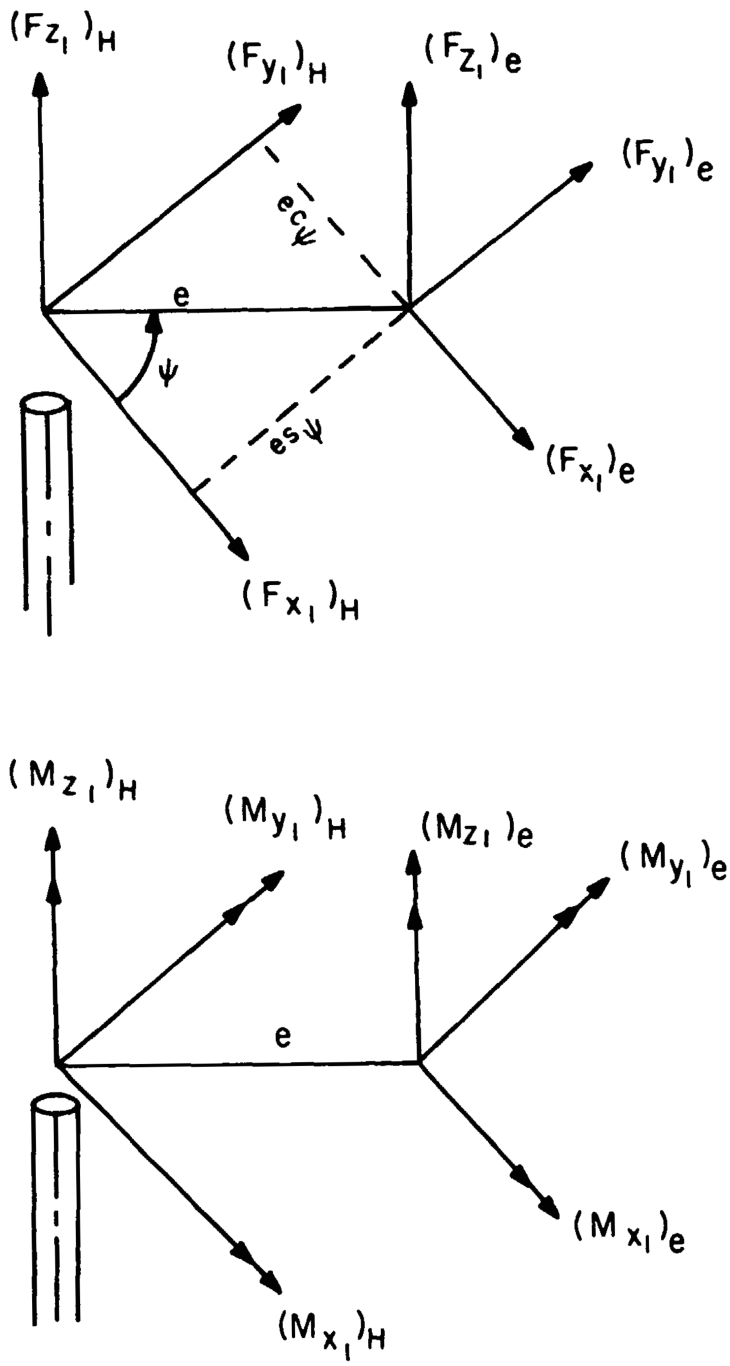

Figure 22. Blade Root and Fixed System Forces and Moments.

An alternative to (11.7) is to express the moments at the hinge as internal reactions consisting of the moments illustrated in Fig. (23). Resolve these to the $\bar{X}_5$ axis. We obtain for the moments applied by a blade to the rotor

$$\begin{Bmatrix} (M_{x_5})_e \\ (M_{y_5})_e \\ (M_{z_5})_e \end{Bmatrix} = \begin{Bmatrix} K_p(\theta_c)_0 \\ 0 \\ 0 \end{Bmatrix} + A_{-\beta} \begin{Bmatrix} 0 \\ -K_F \beta \\ -C_{LD}\dot{\delta} \end{Bmatrix} + A_{-\beta} A_{-\delta} \begin{Bmatrix} 0 \\ 0 \\ K_D \delta \end{Bmatrix} \tag{11.15}$$

which reduces to

$$\begin{aligned} (M_{x_5})_e &= K_p(\theta_e)_0 - C_{LD}\, s_\beta \dot{\delta} + K_D s_\beta \delta \\ (M_{y_5})_e &= -K_F \beta \\ (M_{z_5})_e &= -C_{LD} c_\beta \dot{\delta} + K_D c_\beta \delta \end{aligned} \tag{11.16}$$

The motivation for using the external forces to calculate the shears from (11.6) was a desire to increase the accuracy of the root shear calculation in comparison with the internal force method. The latter method would require third derivatives of mode shapes and for accurate results usually many more blade modes than are adequate for the response calculation would be needed. On the other hand, the use of (11.16) instead of (11.7) for the hinge moments would be more efficient computationally and should be accurate, except possibly for very stiff springs. Also, the more accurate external running moment expressions, (6.74) to (6.76), should replace the internal running moment expression, (11.1) to (11.3), now used in the program.

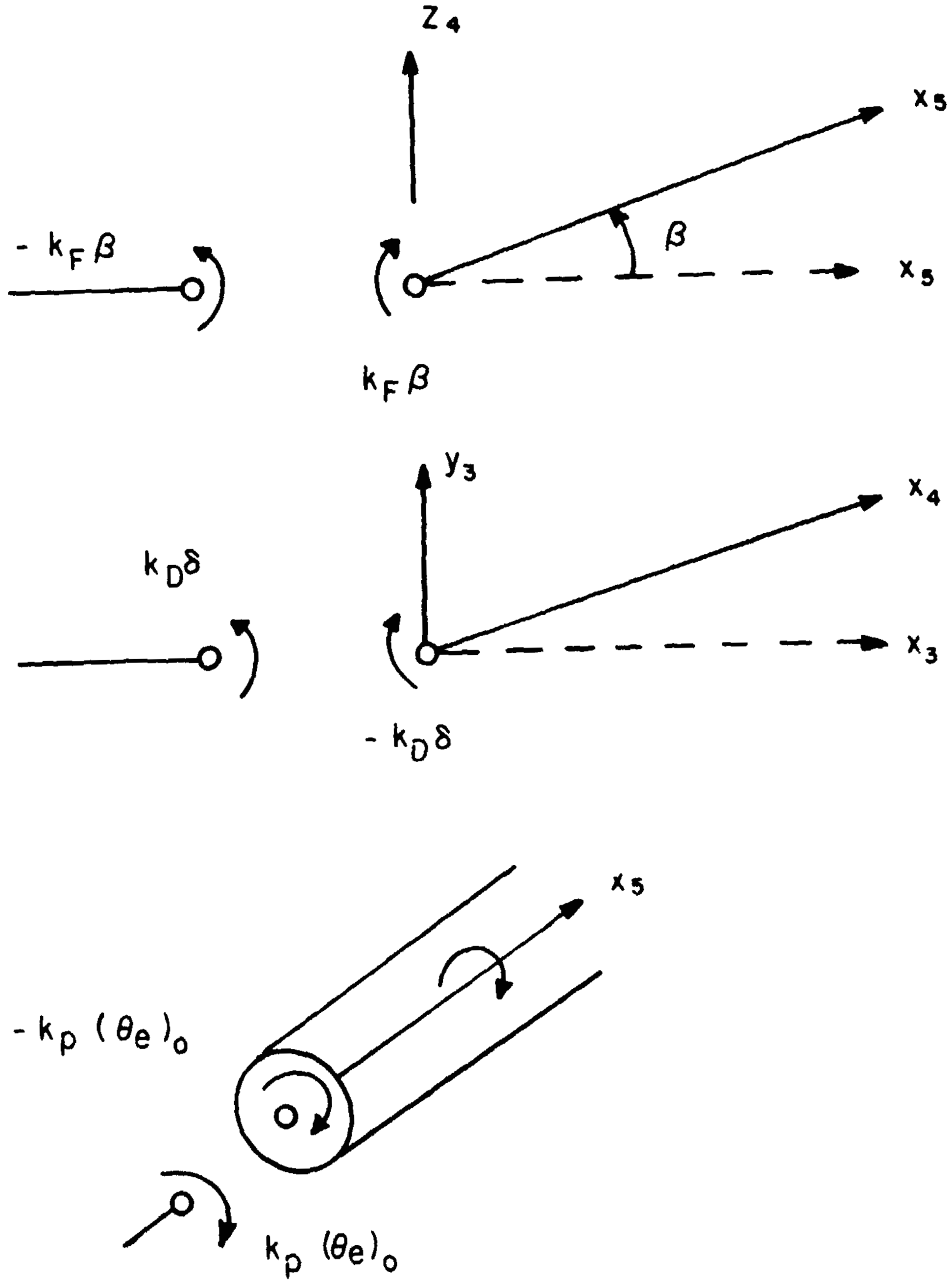

Figure 2 . Internal Reaction Moments at the Blade Root.

12. Expression for Time-Dependent Component of Blade Pitch Angle, and Non-Dimensionalization Rules

We derive here the expression used for the time-dependent component of blade pitch angle, θ_t, and the rules employed in the program to non-dimensionalize the modal equations.

12.1 Pitch Angle Component, θ_t

The expression for the time-dependent contribution to the pitch angle induced by cyclic control inputs and pitch-flap and pitch-lag couplings is

$$\theta_t = -A_{1S}\, C_\psi - B_{1S}\, S_\psi - \beta \tan \delta_3' - \delta \tan \alpha - w_A' \tan \delta_3 \qquad (12.1)$$

Angles A_{1S} and B_{1S} are cyclic inputs. The remaining terms are derived below, and the version of (12.1) used in the program is defined.

Fig. (24) illustrates the geometry assumed to obtain the pitch-flap coupling. Point A is the attachment of the pitch horn. Point P is the attachment of the pushrod. If the point P is unrestrained and free to move with the blade, it moves to P_1, following a flapping deflection and elastic bending deflection at A. A rigid pushrod restrains P such that it stays at P_2, and this geometric configuration is assumed.

The change in pitch angle is

$$\Delta\theta = -\frac{\Delta z_4}{y_P} \qquad (12.2)$$

$$\Delta z_4 = (z_4)_{P_1} - (z_4)_{P_2} \qquad (12.3)$$

$$(z_4)_{P_1} \simeq \beta r_P + w_A - w_A'(r_A - r_P) \qquad (12.4)$$

$$(z_4)_{P_2} = 0 \qquad (12.5)$$

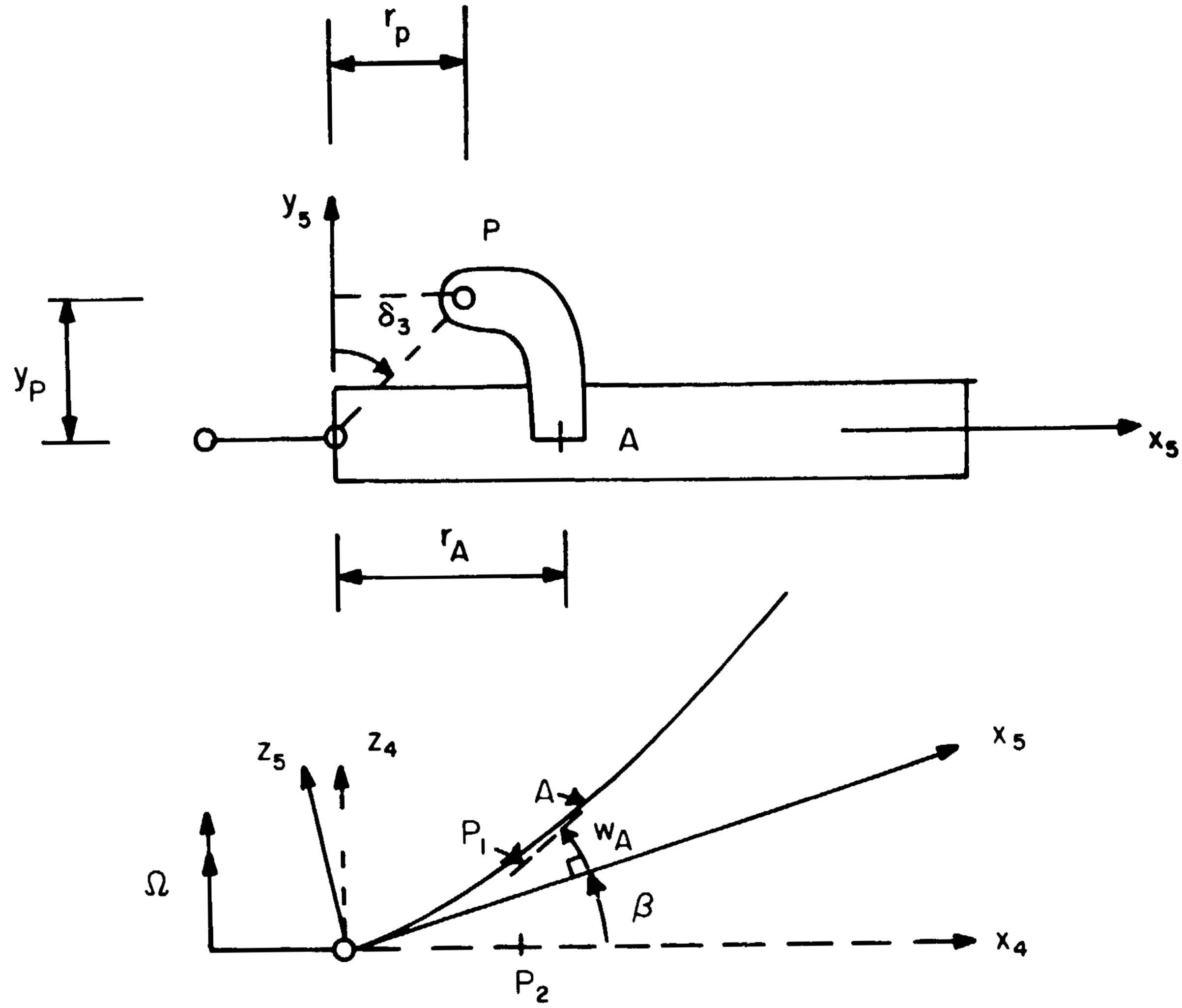

Figure 24. Geometry Assumed for Calculation of Pitch-Flap Coupling.

Substitute (12.3) to (12.5) in (12.2). We obtain

$$\Delta\theta = \frac{\beta r_p}{y_p} - \frac{1}{y_p}\left(w_A - w_A'(r_A - r_p)\right) \tag{12.6}$$

Expand the elastic deflection as a Taylor series from the blade hinge. We obtain

$$w_A = \frac{w_e''}{2} r_A^2 \tag{12.7}$$

$$w_A' = w_e'' r_A \tag{12.8}$$

From (3.12) and (3.14) we find

$$w_e'' = \frac{1}{EI_y EI_z}\left[(EI_y S_\theta^2 + EI_z C_\theta^2) K_F \beta - (EI_z - EI_y) S_\theta C_\theta K_D \delta\right] \tag{12.9}$$

Approximate θ by θ_c in (12.9) and substitute (12.7) to (12.9) in (12.6). Add to these expressions for $\Delta\theta$ the cyclic input contribution and the program input pitch lag coefficient $\tan\alpha_1$. We obtain

(12.10)

$$\theta_t = -A_{1S} C_\psi - B_{1S} S_\psi - \beta \tan\delta_3' - \delta \tan\alpha - w_A' \tan\delta_3$$

$$\tan\delta_3 = r_P / y_P$$

$$\tan\delta_3' = \tan\delta_3 - \frac{r_A^2}{2y_P}\left\{\frac{1}{EI_Z\,EI_y}\left(EI_y\,S_{\theta_c}^2 + EI_Z\,C_{\theta_c}^2\right)K_F\right\}_0 \qquad (12.11)$$

$$\tan\alpha = \tan\alpha_1 + \frac{r_A^2}{2y_P}\left\{\frac{1}{EI_Z\,EI_y}\left(EI_Z - EI_y\right)S_{\theta_c}\,C_{\theta_c}\,K_D\right\}_0$$

where $\tan\alpha_1$ is the input pitch-lag coefficient.

In the program the pitch horn is assumed to be located at the hinge ($r_A = 0$). The elastic deflection, w_A, is assumed to apply at the mid-point of the first segment of the blade. For these assumptions, (12.10) reduces to

$$\theta_t = -A_{1s}\,C_\psi - B_{1s}\,S_\psi - \beta\tan\delta_3 - \delta\tan\alpha_1 - w_A'\tan\delta_3 \qquad (12.12)$$

12.2 Non-Dimensionalization Rules

The purpose of this section is to define the rules for obtaining non-dimensional forms of the variables appearing in the blade modal equation. These variables are used in the coding of the modal equations. With these non-dimensional variables, the modal equation in the non-dimensional space have exactly the same forms as the dimensional modal equations, cited in the earlier text and no corrections reflecting changes in form are required.

To obtain a non-dimensional variable, we select a combination of terms from among m_o, Ω, and R which will non-dimensionalize the variable. Here m_o is a reference mass per unit length of the blade (slug/ft). We replace derivatives with respect to time, d/dt, by d/dψ. We replace radial derivative d/dr by d/d(r/R). Frequencies are replaced by frequency ratios, ω/Ω and rotor speed Ω is replaced by $\Omega/\Omega = 1$. We denote variables non-dimensionalized in this way by overbars, and use an arrow to denote a substitution. The following are examples of this transformation.

(12.13)

$$\Omega \leftarrow 1$$
$$\omega \leftarrow \bar{\omega} = \omega/\Omega$$
$$(\ddot{\ }) \leftarrow (\overset{xx}{\ }) = d^2/d\psi^2$$
$$(\ )' \leftarrow d/d\bar{r} = d/d(r/R)$$
$$m \leftarrow \bar{m} = m/m_o$$
$$EI \leftarrow \overline{EI} = EI/m_o\Omega^2R^4$$
$$C_{FD} \leftarrow \bar{C}_{FD} = C_{FD}/m_o\Omega R$$
$$\rho \leftarrow \bar{\rho} = \rho R^2/m_o$$
$$U \leftarrow \bar{U} = U/\Omega R$$
$$c \leftarrow \bar{c} = c/R$$
$$r \leftarrow \bar{r} = r/R$$
$$K_A \leftarrow \bar{K}_A = K_A/R$$
$$K_{Z_{10}} \leftarrow \bar{K}_{Z_{10}} = K_{Z_{10}}/R$$
$$P_{X_5} \leftarrow \overline{P}_{X_5} = P_{X_5}/m_o\Omega^2R$$
$$A_X \leftarrow \bar{A}_X = A_X/\Omega^2R$$

13. REFERENCES

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14. - Appendices

Appendix 14.1 - Coefficients in Expression for Correction to Modal Stiffness.

$$(a_{11}^*)_0 = \left\{ \frac{1}{EI_y EI_z} \left[(EI_y c^2_{\theta_c} + EI_z s^2_{\theta_c})(EI_y s^2_{\theta_B^*} + EI_z c^2_{\theta_B^*}) - (EI_z - EI_y)^2 s_{\theta_c} c_{\theta_c} s_{\theta_B^*} c_{\theta_B^*} \right] - 1 \right\}_{r=0}$$

$$(a_{11}^*)_1 = \left\{ \frac{(EI_z - EI_y)}{EI_y EI_z} \left[(EI_y s^2_{\theta_B^*} + EI_z c^2_{\theta_B^*}) s_{2\theta_c} - (EI_z - EI_y) c_{2\theta_c} s_{\theta_B^*} c_{\theta_B^*} \right] \right\}_{r=0}$$

$$(a_{12}^*)_0 = \left\{ \frac{(EI_z - EI_y)}{EI_z EI_y} \left[(EI_y s^2_{\theta_B^*} + EI_z c^2_{\theta_B^*}) s_{\theta_c} c_{\theta_c} - (EI_y s^2_{\theta_c} + EI_z c^2_{\theta_c}) s_{\theta_B^*} c_{\theta_B^*} \right] \right\}_{r=0}$$

$$(a_{12}^*)_1 = \left\{ \frac{(EI_z - EI_y)}{EI_z EI_y} \left[(EI_y s^2_{\theta_B^*} + EI_z c^2_{\theta_B^*}) c_{2\theta_c} + (EI_z - EI_y) s_{2\theta_c} s_{\theta_B^*} c_{\theta_B^*} \right] \right\}_{r=0}$$

$$(a_{21}^*)_0 = \left\{ \frac{(EI_z - EI_y)}{EI_z EI_y} \left[-(EI_y c^2_{\theta_c} + EI_z s^2_{\theta_c}) s_{\theta_B^*} c_{\theta_B^*} + (EI_y c^2_{\theta_B^*} + EI_z s^2_{\theta_B^*}) s_{\theta_c} c_{\theta_c} \right] \right\}_{r=0}$$

$$(a_{21}^*)_1 = \left\{ \frac{(EI_z - EI_y)}{EI_z EI_y} \left[-(EI_z - EI_y) s_{2\theta_c} c_{\theta_B^*} s_{\theta_B^*} + (EI_y c^2_{\theta_B^*} + EI_z s^2_{\theta_B^*}) c_{2\theta_c} \right] \right\}_{r=0}$$

$$(a_{22}^*)_0 = \left\{ \frac{1}{EI_y EI_z} \left[-(EI_z - EI_y)^2 s_{\theta_c} c_{\theta_c} s_{\theta_B^*} c_{\theta_B^*} + (EI_y s^2_{\theta_c} + EI_z c^2_{\theta_c})(EI_y c^2_{\theta_B^*} + EI_z s^2_{\theta_B^*}) \right] - 1 \right\}_{r=0}$$

$$(a_{22}^*)_1 = \left\{ \frac{(EI_z - EI_y)}{EI_y EI_z} \left[-(EI_z - EI_y) c_{2\theta_c} s_{\theta_B^*} c_{\theta_B^*} - (EI_y c^2_{\theta_B^*} + EI_z s^2_{\theta_B^*}) s_{2\theta_c} \right] \right\}_{r=0}$$

Appendix 14.2 - Derivatives of Direction Cosines a_{ij}.

$$\dot{a}_{11} = (-S_\beta C_\delta C_\psi + S_\beta S_\delta S_\psi)\dot{\beta} + (-C_\beta S_\delta C_\psi - C_\beta C_\delta S_\psi)\dot{\delta} + (-C_\beta C_\delta S_\psi - C_\beta S_\delta C_\psi)\Omega$$

$$\dot{a}_{12} = (-S_\beta C_\delta S_\psi - S_\beta S_\delta C_\psi)\dot{\beta} + (C_\beta S_\delta S_\psi + C_\beta C_\delta C_\psi)\dot{\delta} + (C_\beta C_\delta C_\psi - C_\beta S_\delta S_\psi)\Omega$$

$$\dot{a}_{13} = C_\beta \dot{\beta}$$

$$\dot{a}_{21} = (-C_\delta C_\psi + S_\delta S_\psi)\dot{\delta} + (S_\delta S_\psi - C_\delta C_\psi)\Omega$$

$$\dot{a}_{22} = (-C_\delta S_\psi - S_\delta C_\psi)\dot{\delta} + (-S_\delta C_\psi - C_\delta S_\psi)\Omega$$

$$\dot{a}_{23} = 0$$

$$\dot{a}_{31} = (-C_\beta C_\delta C_\psi + C_\beta S_\delta S_\psi)\dot{\beta} + (S_\beta S_\delta C_\psi + S_\beta C_\delta S_\psi)\dot{\delta} + (S_\beta C_\delta S_\psi + S_\beta S_\delta C_\psi)\Omega$$

$$\dot{a}_{32} = (-C_\beta C_\delta S_\psi - C_\beta S_\delta C_\psi)\dot{\beta} + (S_\beta S_\delta S_\psi - S_\beta C_\delta C_\psi)\dot{\delta} + (-S_\beta C_\delta C_\psi + S_\beta S_\delta S_\psi)\Omega$$

$$\dot{a}_{33} = -S_\beta \dot{\beta}$$

Appendix 14.3 - Table of Modal Integrals.

i	$Q_i^{(1)}$ INTEGRANDS OR ROOT OR TIP VALUES YIELDING $Q_i^{(1)}$
1	$m\theta_j s_{\theta_c} y_{10cg}$
2	$m\theta_j c_{\theta_c} y_{10cg}$
3	$m\theta_j \theta_i c_{\theta_c} y_{10cg}$
4	$-\frac{1}{2} m\theta_j s_{\theta_c} y_{10cg}$
7	$m\theta_j r s_{\theta_c} y_{10cg}$
8	$m\theta_j r c_{\theta_c} y_{10cg}$
9	$m\theta_j r \theta_i c_{\theta_c} y_{10cg}$
10	$-\frac{1}{2} m\theta_j r s_{\theta_c} y_{10cg}$
13	$\theta_j m w_i s_{\theta_c} y_{10cg}$
16	$\theta_j m v_i s_{\theta_c} y_{10cg}$
17	$\theta_j m v_i c_{\theta_c} y_{10cg}$
18	$\theta_j m v_i c_{\theta_c} \theta_k y_{10cg} = Q_{18}^{(1)}(j,i,k)$
20	$\theta_j m \Delta v_{ik} s_{\theta_c} y_{10cg} = Q_{20}^{(1)}(j,i,k)$
21	$-\theta_j m c_{\theta_c} y_{10cg}$
22	$\theta_j m s_{\theta_c} y_{10cg}$

TABLE 3 TABLE OF MODAL INTEGRALS

i	$Q_i^{(1)}$	
23	$\theta_j m \theta_i s_{\theta_i} y_{10cg}$	
24	$\frac{1}{2}\theta_j m c_{\theta_i} y_{10cg}$	
27	$-\theta_j m r c_{\theta_i} y_{10cg}$	
28	$\theta_j m r s_{\theta_i} y_{10cg}$	
29	$\theta_j m r \theta_i s_{\theta_i} y_{10cg}$	
30	$\frac{1}{2}\theta_j m r c_{\theta_i} y_{10cg}$	
33	$-\theta_j m w_i c_{\theta_i} y_{10cg}$	
34	$\theta_j m w_i s_{\theta_i} y_{10cg}$	
35	$\theta_j m w_i \theta_k s_{\theta_i} y_{10cg}$	$= Q_{35}^{(1)}(j, i, k)$
36	$-\theta_j m v_i c_{\theta_i} y_{10cg}$	
39	$-\theta_j m \Delta w_{ik} c_{\theta_i} y_{10cg}$	$= Q_{39}^{(1)}(j, i, k)$
41	$\theta_j m s_{\theta_i}^2 k_{z_{10}}^2$	
42	$\theta_j m s_{\theta_i} c_{\theta_i} k_{z_{10}}^2$	
44	$\theta_j m c_{\theta_i} s_{\theta_i} k_{z_{10}}^2$	
47	$\theta_j m s_{\theta_i}^2 k_{z_{10}}^2$	
48	$\theta_j m c_{\theta_i} s_{\theta_i} k_{z_{10}}^2$	

TABLE 3 CONTINUED.

i	$Q_i^{(1)}$
49	$\theta_J m \theta_i s_{\theta_c}^2 k_{z_{10}}^2$
51	$\theta_J m s_{\theta_c} w_i' s_{\theta_c} k_{z_{10}}^2$
52	$\theta_J m c_{\theta_c} v_i' s_{\theta_c} k_{z_{10}}^2$
53	$-\theta_J m s_{\theta_c} c_{\theta_c} k_{z_{10}}^2$
56	$-\theta_J m c_{\theta_c}^2 k_{z_{10}}^2$
57	$\theta_J m c_{\theta_c} s_{\theta_c} k_{z_{10}}^2$
59	$-\theta_J m s_{\theta_c} c_{\theta_c} k_{z_{10}}^2$
60	$-\theta_J m c_{\theta_c}^2 k_{z_{10}}^2$
62	$-\theta_J m \theta_i c_{\theta_c}^2 k_{z_{10}}^2$
65	$\theta_J m c_{2\theta_c} \theta_i k_{z_{10}}^2$
66	$\theta_J m \theta_i k_{z_{10}}^2$
67	$\theta_J m s_{\theta_c} c_{\theta_c} k_{y_{10}}^2$
68	$-\theta_J m s_{\theta_c}^2 k_{y_{10}}^2$
69	$-\theta_J m s_{\theta_c} \theta_i s_{\theta_c} k_{y_{10}}^2$
70	$\theta_J m c_{\theta_c}^2 k_{y_{10}}^2$
71	$-\theta_J m c_{\theta_c} s_{\theta_c} k_{y_{10}}^2$

TABLE 3 CONTINUED

i	$Q_i^{(1)}$
73	$\theta_j\, m\, s_{\theta_c} c_{\theta_c} k_{y_{10}}^2$
74	$\theta_j\, m\, c_{\theta_c}^2 k_{y_{10}}^2$
76	$\theta_j\, m\, \theta_i\, c_{\theta_c}^2 k_{y_{10}}^2$
77	$\theta_j\, m\, c_{\theta_c}^2 w_i' k_{y_{10}}^2$
78	$\theta_j\, m\, s_{\theta_c} c_{\theta_c} v_i' k_{y_{10}}^2$
79	$\theta_j\, m\, s_{\theta_c}^2 k_{y_{10}}^2$
80	$\theta_j\, m\, s_{\theta_c} c_{\theta_c} k_{y_{10}}^2$
82	$\theta_j\, m\, c_{\theta_c} s_{\theta_c} k_{y_{10}}^2$
83	$\theta_j\, m\, c_{\theta_c}^2 k_{y_{10}}^2$
85	$\theta_j\, m\, s_{\theta_c}^2 k_{y_{10}}^2$
86	$\theta_j\, m\, c_{\theta_c} s_{\theta_c} k_{y_{10}}^2$
87	$\theta_j\, m\, \theta_i\, s_{\theta_c}^2 k_{y_{10}}^2$
91	$-\theta_j\, m\, c_{2\theta_c} \theta_i\, k_{y_{10}}^2$
92	$\theta_j\, m\, \theta_i\, k_{y_{10}}^2$
93	$-\theta_j\, v_i'\, m\, s_{\theta_c} y_{10cg}$
96	$-\theta_j\, v_i'\, m\, r\, s_{\theta_c} y_{10cg}$

TABLE 3 CONTINUED.

i	$Q_i^{(1)}$
97	$-\theta_j v_i' m r c_{\theta_c} y_{10cg}$
98	$-\theta_j v_i' m r \theta_k c_{\theta_c} y_{10cg}$
100	$-\theta_j v_i' m v_k s_{\theta_c} y_{10cg}$
109	$\theta_j w_i' m c_{\theta_c} y_{10cg}$
112	$\theta_j w_i' m r c_{\theta_c} y_{10cg}$
113	$-\theta_j w_i' m r s_{\theta_c} y_{10cg}$
114	$-\theta_j w_i' m r \theta_k s_{\theta_c} y_{10cg}$
116	$\theta_j w_i' m v_k c_{\theta_c} y_{10cg}$
126	$(\theta_j \theta_i' k_A^2 R_2)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta_i' R_2 \, d\xi = Q_{126}^{(1)}$
127	$(\theta_j k_A^2 \theta' R_1)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta' R_1 \, d\xi = Q_{127}^{(1)}$
128	$(\theta_j k_A^2 \theta' R_2)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta' R_2 \, d\xi = Q_{128}^{(1)}$
130	$(\theta_j k_A^2 \theta' R_4)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta' R_4 \, d\xi = Q_{130}^{(1)}$
131	$(\theta_j k_A^2 \theta' R_5)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta' R_5 \, d\xi = Q_{131}^{(1)}$
132	$(\theta_j k_A^2 \theta' R_6)_0 + \int_0^{r_T} \theta_j' k_A^2 \theta' R_6 \, d\xi = Q_{132}^{(1)}$
137	$-\theta_j c_{\theta_c} e_A w_i'' R_1$
138	$-\theta_j c_{\theta_c} e_A w_i'' R_2$

TABLE 3 CONTINUED

i	$Q_i^{(1)}$
140	$-\theta_j c_{\theta_i} e_A w_i'' R_4$
149	$\theta_j s_{\theta_c} e_A v_i'' R_1$
150	$\theta_j s_{\theta_c} e_A v_i'' R_2$
152	$\theta_j s_{\theta_c} e_A v_i'' R_4$
160	$\theta_j c_{\theta_c} e_A v_i'' R_2$
161	$-\theta_j v_i'' s_{\theta_c} \Delta e_{A_{cw}} R_7$
162	$-\theta_j v_i'' s_{\theta_c} \Delta e_{A_{cw}} R_8$
164	$-\theta_j v_i'' s_{\theta_c} \Delta e_{A_{cw}} R_{10}$
172	$-\theta_j v_i'' c_{\theta_c} \Delta e_{A_{cw}} R_2$
173	$\theta_j w_i'' c_{\theta_c} \Delta e_{A_{cw}} R_7$
174	$\theta_j w_i'' c_{\theta_c} \Delta e_{A_{cw}} R_8$
176	$\theta_j w_i'' c_{\theta_c} \Delta e_{A_{cw}} R_{10}$
184	$-\theta_j w_i'' s_{\theta_c} \Delta e_{A_{cw}} R_8$
187	$\theta_j \theta_i v_K'' c_{\theta_c} e_A R_2$
188	$\theta_j \theta_i w_K'' s_{\theta_c} e_A R_2$
191	$-\theta_j \theta_i v_K'' c_{\theta_c} \Delta e_{A_{cw}} R_8$

TABLE 3 CONTINUED

i	$Q_i^{(1)}$
192	$-\theta_j \theta_i w_k'' S_{\theta_i} \Delta e_{A_{cw}} R_8$
193	$-\theta_j (EI_z - EI_y) v_i'' w_k'' C_{2\theta_i}$
194	$-\frac{1}{2}\theta_j (EI_z - EI_y) w_i'' w_k'' S_{2\theta_i}$
195	$\frac{1}{2}\theta_j (EI_z - EI_y) v_i'' v_k'' S_{2\theta_i}$

TABLE 3 CONTINUED

i	$Q_i^{(2)}$ INTEGRANDS OR ROOT OR TIP VALUES YIELDING $Q_i^{(2)}$
1	$-(w_{1j}\, m s_{\theta_c} y_{10_{cg}})_T$
2	$-(w_{1j}\, m r s_{\theta_c} y_{10_{cg}})_T$
4	$-(w_{1j}\, m v_i s_{\theta_c} y_{10_{cg}})_T$
5	$-(w_{1j}\, m c_{\theta_c} y_{10_{cg}})_T$
7	$-(w_{1j}\, m r c_{\theta_c} y_{10_{cg}})_T$
8	$-(w_{1j}\, m r \theta_i c_{\theta_c} y_{10_{cg}})_T$
13	$-(w_{1j}\, m s^2_{\theta_c} k^2_{z_{10}})_T$
14	$-(w_{1j}\, m c_{\theta_c} s_{\theta_c} k^2_{z_{10}})_T$
15	$-(w_{1j}\, m s_{\theta_c} c_{\theta_c} k^2_{y_{10}})_T$
16	$-(w_{1j}\, m c^2_{\theta_c} k^2_{y_{10}})_T$
17	$w'_{1j}\, m s_{\theta_c} y_{10_{cg}}$
18	$w'_{1j}\, m r s_{\theta_c} y_{10_{cg}}$
20	$w'_{1j}\, m v_i s_{\theta_c} y_{10_{cg}}$
21	$w'_{1j}\, m c_{\theta_c} y_{10_{cg}}$
23	$w'_{1j}\, m r \quad c_{\theta_c} y_{10_{cg}}$

TABLE 3. CONTINUED

i	$Q_i^{(a)}$
24	$w_{1j}' m r \theta_i c_{\theta_c} y_{10cg}$
29	$w_{1j}' m s_{\theta_c}^2 k_{z_{10}}^2$
30	$w_{1j}' m s_{\theta_c} c_{\theta_c} k_{z_{10}}^2$
31	$w_{1j}' m s_{\theta_c} c_{\theta_c} k_{y_{10}}^2$
32	$w_{1j}' m c_{\theta_c}^2 k_{y_{10}}^2$
33	$w_{1j}' w_i' R_1$
34	$w_{1j}' w_i' R_2$
37	$-w_{1j} m$
38	$-w_{1j} m r$
39	$-w_{1j} m w_i$
40	$-w_{1j} m v_i$
41	$-w_{1j} m s_{\theta_c} y_{10cg}$
42	$-w_{1j} m c_{\theta_c} y_{10cg}$
43	$-w_{1j} m s_{\theta_c} y_{10cg}$
44	$-w_{1j} m c_{\theta_c} y_{10cg}$
46	$-w_{1j} m \theta_i c_{\theta_c} y_{10cg}$

TABLE 3 CONTINUED.

i	$Q_i^{(2)}$
49	$w_{1j}' \hat{T} w_{1i}' / \Omega^2$
50	$w_{1j}\, m\, w_{1i}$
51	$(w_{1j}\, m\, s_{\theta_c}\, e_A)_T$
52	$(w_{1j}\, m\, s_{\theta_c}\, e_A\, r)_T$
54	$(w_{1j}\, m\, s_{\theta_c}\, e_A\, v_i)_T$
55	$(w_{1j}\, m\, s_{\theta_c}^2\, e_A\, y_{10cg})_T$
56	$(w_{1j}\, m\, s_{\theta_c}\, c_{\theta_c}\, e_A\, y_{10cg})_T$
61	$(w_{1j}\, m\, c_{\theta_c}\, e_A)_T$
62	$(w_{1j}\, m\, c_{\theta_c}\, e_A\, r)_T$
63	$(w_{1j}\, \theta'\, c_{\theta_c}\, \Delta e_{A_{cw}}\, R_7)_T$
64	$(w_{1j}\, \theta'\, c_{\theta_c}\, \Delta e_{A_{cw}}\, R_8)_T$
66	$(w_{1j}\, \theta'\, c_{\theta_c}\, \Delta e_{A_{cw}}\, R_{10})_T$
67	$(w_{1j}\, \theta'\, c_{\theta_c}\, \Delta e_{A_{cw}}\, R_{11})_T$
68	$(w_{1j}\, \theta'\, c_{\theta_c}\, \Delta e_{A_{cw}}\, R_{12})_T$
73	$(w_{1j}\, s_{\theta_c}\, \Delta e_{A_{cw}}'\, R_7)_T$
74	$(w_{1j}\, s_{\theta_c}\, \Delta e_{A_{cw}}'\, R_8)_T$

TABLE 3 CONTINUED.

i	$Q_i^{(2)}$
76	$(w_{1_j} S_{\theta_c} \Delta e'_{A_{cw}} R_{10})_T$
77	$(w_{1_j} S_{\theta_c} \Delta e'_{A_{cw}} R_{11})_T$
78	$(w_{1_j} S_{\theta_c} \Delta e'_{A_{cw}} R_{12})_T$
83	$-(w_{1_j} \theta' S_{\theta_c} \Delta e_{A_{cw}} R_7)_T$
84	$-(w_{1_j} \theta' S_{\theta_c} \Delta e_{A_{cw}} R_8)_T$
85	$(w_{1_j} C_{\theta_c} \Delta e'_{A_{cw}} R_7)_T$
86	$(w_{1_j} C_{\theta_c} \Delta e'_{A_{cw}} R_8)_T$
87	$-(w'_{1_j} \Delta e_{A_{cw}} S_{\theta_c} R_7)_T$
88	$-(w'_{1_j} \Delta e_{A_{cw}} S_{\theta_c} R_8)_T$
90	$-(w'_{1_j} \Delta e_{A_{cw}} S_{\theta_c} R_{10})_T$
91	$-(w'_{1_j} \Delta e_{A_{cw}} S_{\theta_c} R_{11})_T$
92	$-(w'_{1_j} \Delta e_{A_{cw}} S_{\theta_c} R_{12})_T$
97	$-(w'_{1_j} \Delta e_{A_{cw}} C_{\theta_c} R_1)_T$
98	$-(w'_{1_j} \Delta e_{A_{cw}} C_{\theta_c} R_2)_T$
99	$-(w'_{1_j} e_A S_{\theta_c} R_1)_0$
100	$-(w'_{1_j} e_A S_{\theta_c} R_2)_0$

TABLE 3 CONTINUED

i	$Q_i^{(2)}$
102	$-(w_{1j}{}' e_A S_{\theta_c} R_4)_0$
103	$-(w_{1j}{}' e_A S_{\theta_c} R_5)_0$
104	$-(w_{1j}{}' e_A S_{\theta_c} R_6)_0$
114	$-w_{1j}{}'' e_A S_{\theta_c} R_4$
115	$-w_{1j}{}'' e_A S_{\theta_c} R_5$
116	$-w_{1j}{}'' e_A S_{\theta_c} R_6$
121	$-w_{1j}{}'' e_A C_{\theta_c} R_1$
122	$-w_{1j}{}'' e_A C_{\theta_c} R_2$
123	$w_{1j}{}'' \Delta e_{A_{cw}} S_{\theta_c} R_7$
124	$w_{1j}{}'' \Delta e_{A_{cw}} S_{\theta_c} R_8$
126	$w_{1j}{}'' \Delta e_{A_{cw}} S_{\theta_c} R_{10}$
127	$w_{1j}{}'' \Delta e_{A_{cw}} S_{\theta_c} R_{11}$
128	$w_{1j}{}'' \Delta e_{A_{cw}} S_{\theta_c} R_{12}$
133	$w_{1j}{}'' \Delta e_{A_{cw}} C_{\theta_c} R_7$
134	$w_{1j}{}'' \Delta e_{A_{cw}} C_{\theta_c} R_8$
136	$(w_{1j} \theta_i C_{\theta_c} e_A m r)_T$

TABLE 3 CONTINUED.

i	$Q_i^{(2)}$
140	$(w_{1j}\,\theta_i'\,c_{\theta_c}\,\Delta e_{A_{CW}}\,R_8)_T$
141	$-(w_{1j}\,\theta_i\,s_{\theta_c}\,\theta'\,\Delta e_{A_{CW}}\,R_8)_T$
142	$(w_{1j}\,\theta_i\,c_{\theta_c}\,\Delta e_{A_{CW}}'\,R_8)_T$
144	$-(w_{1j}'\,\theta_i\,c_{\theta_c}\,\Delta e_{A_{CW}}\,R_8)_T$
146	$-(w_{1j}'\,\theta_i\,c_{\theta_c}\,e_A\,R_2)_0$
148	$-\,w_{1j}''\,\theta_i\,c_{\theta_c}\,e_A\,R_2$
150	$w_{1j}''\,\theta_i\,c_{\theta_c}\,\Delta e_{A_{CW}}\,R_8$

TABLE 3 CONTINUED

i	$Q_i^{(3)}$ INTEGRANDS OR ROOT OR TIP VALUES YIELDING $Q_i^{(3)}$
1	$-(v_{1j}\, m c_{\theta_c}\, y_{10cg})_T$
2	$-(v_{1j}\, m r c_{\theta_c}\, y_{10cg})_T$
4	$-(v_{1j}\, m v_i\, c_{\theta_c}\, y_{10cg})_T$
5	$(v_{1j}\, m s_{\theta_c}\, y_{10cg})_T$
7	$(v_{1j}\, m r s_{\theta_c}\, y_{10cg})_T$
8	$(v_{1j}\, m r \theta_i\, s_{\theta_c}\, y_{10cg})_T$
13	$-(v_{1j}\, m s_{\theta_c}\, c_{\theta_c}\, k_{z_{10}}^2)_T$
14	$-(v_{1j}\, m c_{\theta_c}^2\, k_{z_{10}}^2)_T$
15	$(v_{1j}\, m s_{\theta_c}^2\, k_{y_{10}}^2)_T$
16	$(v_{1j}\, m s_{\theta_c}\, c_{\theta_c}\, k_{y_{10}}^2)_T$
17	$v_{1j}'\, m c_{\theta_c}\, y_{10cg}$
18	$v_{1j}'\, m r c_{\theta_c}\, y_{10cg}$
20	$v_{1j}'\, m v_i\, c_{\theta_c}\, y_{10cg}$
21	$-v_{1j}'\, m s_{\theta_c}\, y_{10cg}$
23	$-v_{1j}'\, m r s_{\theta_c}\, y_{10cg}$

TABLE 3 CONTINUED

i	$Q_i^{(3)}$
24	$-v_{1j}' m r \theta_i S_{\theta_c} y_{10cg}$
29	$v_{1j}' m S_{\theta_c} C_{\theta_c} k_{z_{10}}^2$
30	$v_{1j}' m C_{\theta_c}^2 k_{z_{10}}^2$
31	$-v_{1j}' m S_{\theta_c}^2 k_{y_{10}}^2$
32	$-v_{1j}' m C_{\theta_c} S_{\theta_c} k_{y_{10}}^2$
33	$v_{1j}' v_i' R_1$
34	$v_{1j}' v_i' R_2$
37	$-v_{1j} m$
38	$-v_{1j} m r$
39	$-v_{1j} m w_i$
40	$-v_{1j} m v_i$
41	$-v_{1j} m S_{\theta_c} y_{10cg}$
42	$-v_{1j} m C_{\theta_c} y_{10cg}$
43	$-v_{1j} m S_{\theta_c} y_{10cg}$
44	$-v_{1j} m C_{\theta_c} y_{10cg}$
45	$-v_{1j} m \theta_i S_{\theta_c} y_{10cg}$

TABLE 3. CONTINUED

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i	$Q_i^{(3)}$
47	$-v_{1j}\, m\, s_{\theta_c}\, w_i'\, y_{10cg}$
48	$-v_{1j}\, m\, c_{\theta_c}\, v_i'\, y_{10cg}$
49	$v_{1j}'\, \hat{T}\, v_{1i}' / \Omega^2$
50	$-v_{1j}\, m\, v_{1i}$
51	$v_{1j}\, m\, v_{1i}$
52	$(v_{1j}\, m\, c_{\theta_c}\, e_A)_T$
53	$(v_{1j}\, m\, c_{\theta_c}\, e_A\, r)_T$
55	$(v_{1j}\, m\, c_{\theta_c}\, e_A\, v_i)_T$
56	$(v_{1j}\, m\, c_{\theta_c}\, s_{\theta_c}\, e_A\, y_{10cg})_T$
57	$(v_{1j}\, m\, c_{\theta_c}^2\, e_A\, y_{10cg})_T$
62	$-(v_{1j}\, m\, s_{\theta_c}\, e_A)_T$
63	$-(v_{1j}\, m\, s_{\theta_c}\, e_A)_T$
64	$-(v_{1j}\, \theta'\, s_{\theta_c}\, \Delta e_{A_{cw}}\, R_7)_T$
65	$-(v_{1j}\, \theta'\, s_{\theta_c}\, \Delta e_{A_{cw}}\, R_8)_T$
67	$-(v_{1j}\, \theta'\, s_{\theta_c}\, \Delta e_{A_{cw}}\, R_{10})_T$
68	$-(v_{1j}\, \theta'\, s_{\theta_c}\, \Delta e_{A_{cw}}\, R_{11})_T$

TABLE 3. CONTINUED.

i	$Q_i^{(3)}$
69	$-(v_{1j}\,\theta'\,S_{\theta_c}\,\Delta e_{A_{CW}}\,R_{12})_T$
74	$(v_{1j}\,C_{\theta_c}\,\Delta e'_{A_{CW}}\,R_7)_T$
75	$(v_{1j}\,C_{\theta_c}\,\Delta e'_{A_{CW}}\,R_8)_T$
77	$(v_{1j}\,C_{\theta_c}\,\Delta e'_{A_{CW}}\,R_{10})_T$
78	$(v_{1j}\,C_{\theta_c}\,\Delta e'_{A_{CW}}\,R_{11})_T$
79	$(v_{1j}\,C_{\theta_c}\,\Delta e'_{A_{CW}}\,R_{12})_T$
84	$(v_{1j}\,\theta'\,C_{\theta_c}\,\Delta e_{A_{CW}}\,R_7)_T$
85	$(v_{1j}\,\theta'\,C_{\theta_c}\,\Delta e_{A_{CW}}\,R_8)_T$
86	$(v_{1j}\,S_{\theta_c}\,\Delta e'_{A_{CW}}\,R_7)_T$
87	$(v_{1j}\,S_{\theta_c}\,\Delta e'_{A_{CW}}\,R_8)_T$
88	$-(v_{1j}'\,\Delta e_{A_{CW}}\,C_{\theta_c}\,R_7)_T$
89	$-(v_{1j}'\,\Delta e_{A_{CW}}\,C_{\theta_c}\,R_8)_T$
91	$-(v_{1j}'\,\Delta e_{A_{CW}}\,C_{\theta_c}\,R_{10})_T$
92	$-(v_{1j}'\,\Delta e_{A_{CW}}\,C_{\theta_c}\,R_{11})_T$
93	$-(v_{1j}'\,\Delta e_{A_{CW}}\,C_{\theta_c}\,R_{12})_T$
98	$(v_{1j}'\,\Delta e_{A_{CW}}\,S_{\theta_c}\,R_1)_T$

TABLE 3. CONTINUED

i	$Q_i^{(3)}$
99	$(v_{1j}{}' \Delta e_{A_{cw}} s_{\theta_c} R_2)_T$
100	$-(v_{1j}{}' e_A c_{\theta_c} R_1)_0$
101	$-(v_{1j}{}' e_A c_{\theta_c} R_2)_0$
103	$-(v_{1j}{}' e_A c_{\theta_c} R_4)_0$
104	$-(v_{1j}{}' e_A c_{\theta_c} R_5)_0$
105	$-(v_{1j}{}' e_A c_{\theta_c} R_6)_0$
115	$-v_{1j}{}'' e_A c_{\theta_c} R_4$
116	$-v_{1j}{}'' e_A c_{\theta_c} R_5$
117	$-v_{1j}{}'' e_A c_{\theta_c} R_6$
122	$v_{1j}{}'' e_A s_{\theta_c} R_1$
123	$v_{1j}{}'' e_A s_{\theta_c} R_2$
124	$v_{1j}{}'' \Delta e_{A_{cw}} c_{\theta_c} R_7$
125	$v_{1j}{}'' \Delta e_{A_{cw}} c_{\theta_c} R_8$
127	$v_{1j}{}'' \Delta e_{A_{cw}} c_{\theta_c} R_{10}$
128	$v_{1j}{}'' \Delta e_{A_{cw}} c_{\theta_c} R_{11}$
129	$v_{1j}{}'' \Delta e_{A_{cw}} c_{\theta_c} R_{12}$

TABLE 3. CONTINUED.

i	$Q_i^{(3)}$
134	$-v_{1_j}'' \Delta e_{A_{CW}} s_{\theta_c} R_7$
135	$-v_{1_j}'' \Delta e_{A_{CW}} s_{\theta_c} R_8$
137	$-(v_{1_j} \theta_i s_{\theta_c} e_A mr)_T$
141	$-(v_{1_j} \theta_i' s_{\theta_c} \Delta e_{A_{CW}} R_8)_T$
142	$-(v_{1_j} \theta_i c_{\theta_c} \theta' \Delta e_{A_{CW}} R_8)_T$
143	$-(v_{1_j} \theta_i s_{\theta_c} \Delta e_{A_{CW}}' R_8)_T$
145	$(v_{1_j}' \theta_i s_{\theta_c} \Delta e_{A_{CW}} R_8)_T$
147	$(v_{1_j}' \theta_i s_{\theta_c} e_A R_2)_0$
149	$v_{1_j}'' \theta_i s_{\theta_c} e_A R_2$
151	$-v_{1_j}'' \theta_i s_{\theta_c} \Delta e_{A_{CW}} R_8$

TABLE 3 CONTINUED

$$R_1 = \int_r^{r_T} m \, d\xi$$

$$R_2 = \int_r^{r_T} m \xi \, d\xi$$

$$R_3 = \int_r^{r_T} m w_i \, d\xi$$

$$R_4 = \int_r^{r_T} m v_i \, d\xi$$

$$R_5 = \int_r^{r_T} m s_{\theta_c} y_{10_{cg}} \, d\xi$$

$$R_6 = \int_r^{r_T} m c_{\theta_c} y_{10_{cg}} \, d\xi$$

$$R_7 = \int_{r_{0_{cw}}}^{r} m_{cw} \, d\xi$$

$$R_8 = \int_{r_{0_{cw}}}^{r} m_{cw} \xi \, d\xi$$

$$R_9 = \int_{r_{0_{cw}}}^{r} m_{cw} w_i \, d\xi$$

$$R_{10} = \int_{r_{0_{cw}}}^{r} m_{cw} v_i \, d\xi$$

$$R_{11} = \int_{r_{0_{cw}}}^{r} m_{cw} s_{\theta_c} y_{10_{cg}} \, d\xi$$

$$R_{12} = \int_{r_{0_{cw}}}^{r} m_{cw} c_{\theta_c} y_{10_{cg}} \, d\xi$$

$$R_{13} = \int_{r_{0_{cw}}}^{r} m_{cw} v_i' \, d\xi$$

$$R_{14} = \int_{r_{0_{cw}}}^{r} m_{cw} v_i' \xi \, d\xi$$

$$R_{15} = \int_{r_{0_{cw}}}^{r} m_{cw} w_i' \, d\xi$$

$$R_{16} = \int_{r_{0_{cw}}}^{r} m_{cw} w_i' \xi \, d\xi$$

$$\hat{T} = \int_r^{r_T} m (e + \xi) \Omega^2 \, d\xi = e \Omega^2 R_1 + \Omega^2 R_2$$

TABLE 3 CONTINUED

$$\Delta v_{ik} = -\int_0^r \left[w_i - w_i(\xi) - w_i'(\xi)(r-\xi) \right] \theta_k'(\xi)\, d\xi$$

$$\Delta w_{ik} = -\int_0^r \left[v_i - v_i(\xi) - v_i'(\xi)(r-\xi) \right] \theta_k'(\xi)\, d\xi$$

NOTE: $\Delta e'_{A_{cw}} \equiv 0$ IN PROGRAM Y210

TABLE 3 CONCLUDED