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SOLUTION OF THE EXACT EQUATIONS FOR THREE-DIMENSIONAL ATMOSPHERIC ENTRY USING DIRECTLY MATCHED ASYMPTOTIC EXPANSIONS

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16. Abstract

The problem of determining the trajectories, partially or wholly contained in the atmosphere of a spherical, nonrotating planet, is considered. The exact equations of motion for three-dimensional, aerodynamically affected flight are derived. Modified Chapman variables are introduced and the equations are transformed into a set suitable for analytic integration using asymptotic expansions.

The trajectory is solved in two regions: the outer region, where the force may be considered a gravitational field with aerodynamic perturbations, and the inner region, where the force is predominantly aerodynamic, with gravity as a perturbation.

The two solutions are matched directly. A composite solution, valid everywhere, is constructed by additive composition.

This approach of directly matched asymptotic expansions applied to the exact equations of motion couched in terms of modified Chapman variables yields an analytical solution which should prove to be a powerful tool for aerodynamic orbit calculations.

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1. INTRODUCTION

With advancements in space flight technology, a new generation of aerospace vehicles will soon come into service. Such a vehicle, designed to have lifting capability, can operate as a satellite for an extended period outside a planetary atmosphere, and upon accomplishing its mission can enter the atmosphere and use its aerodynamic maneuverability to reach a prescribed region before performing an approach and landing on an airfield like an ordinary airplane.

The portion of the trajectory, starting at the top of the sensible atmosphere and ending at a point at low altitude and low speed where approach and landing procedures can be initiated--the entry trajectory--is the most critical portion of the flight path. It is during this phase of flight that the speed is reduced from orbital speed at high altitude to subsonic speed at low altitude. During this rapid decrease in the kinetic energy, the deceleration, dynamic pressure and heating rate all vary markedly. It is important for preliminary design or mission planning to have accurate, yet simple formulas expressing the behavior of different trajectory variables and physical quantities associated with the trajectory.

For planar entry trajectories, there exist several analytical theories with various degrees of sophistication. Mathematically, one of the best theories is Chapman's theory for analyses of planetary entry (Ref. 1). In Chapman's formulation, the pair of equations for planar entry into a planetary atmosphere is reduced to a single, ordinary, nonlinear differential equation of the second order by disregarding two relatively small terms and by introducing a certain mathematical transformation. The equation is integrated numerically for different cases of entry and the functions obtained, called the Z-functions, are tabulated (Ref. 2). An outstanding feature in Chapman's theory is that the tabulated data of the Z-functions are universal in the sense that every solution determines the motion and heating of a vehicle of arbitrary weight, dimensions, and shape entering a planetary atmosphere.

A major deficiency in Chapman's theory is that, because of his two main assumptions, the equations derived are only approximate, and the applications are restricted to entry trajectories with small flight path angle or small lift-to-drag ratio.

Recently, these restrictive assumptions have been successfully removed. A set of exact equations for three-dimensional entry trajectories has been developed using modified Chapman variables to perform the mathematical transformation (Refs. 3 - 6).

In this report the method of directly matched asymptotic expansions is applied to the exact equations for three-dimensional entry in terms of the modified Chapman variables, resulting in an accurate analytical solution. The two-regime approach of directly matched asymptotic expansions has proved to be feasible and effective in previous, more restricted, applications (Refs. 7 - 10). Now, applied to these exact equations, a powerful, useful solution appears.

The equations of motion for three-dimensional flight about a nonrotating spherical planet and inside of its atmosphere, assumed to be at rest, are derived in Section 2. The exact dimensionless equations using a set of modified Chapman variables are derived in Section 3. In Section 4, these equations are transmuted into a form most suitable for an analytical integration using the method of matched asymptotic expansions. The two solutions, one valid in the outer region where the gravitational force is predominant, and the other valid in the inner region where the aerodynamic force is predominant, are obtained. The two solutions are matched directly and the composite solution, uniformly valid everywhere, is constructed.

2. BASIC EQUATIONS OF MOTION

The vehicle is considered as a mass point, with constant mass m, moving about a nonrotating spherical planet. The atmosphere surrounding the planet is assumed to be at rest and the central gravitational field is the usual inverse square force field.

$$\begin{aligned} \vec{r}(t) &= \text{position vector} \\ \vec{V}(t) &= \text{velocity vector} \end{aligned} \tag{2.1}$$

The initial reference frame $OXYZ$ is the planet-fixed system with O at the center of the gravitational field. The OXY plane is referred to as the equatorial plane (Fig. 1).

The position vector $\vec{r}$ is defined in this planetocentric system by its magnitude r, its longitude θ, measured from the X-axis, in the equatorial plane, positively eastward, and its latitude ϕ, measured from the equatorial plane, along a meridian, and positively northward.

The velocity vector $\vec{V}$ is expressed in terms of its components in a rotating coordinate system $Oxyz$ such that the x-axis is along the position vector, the y-axis in the equatorial plane positive toward the direction of motion and orthogonal to the x-axis, and the z-axis completing

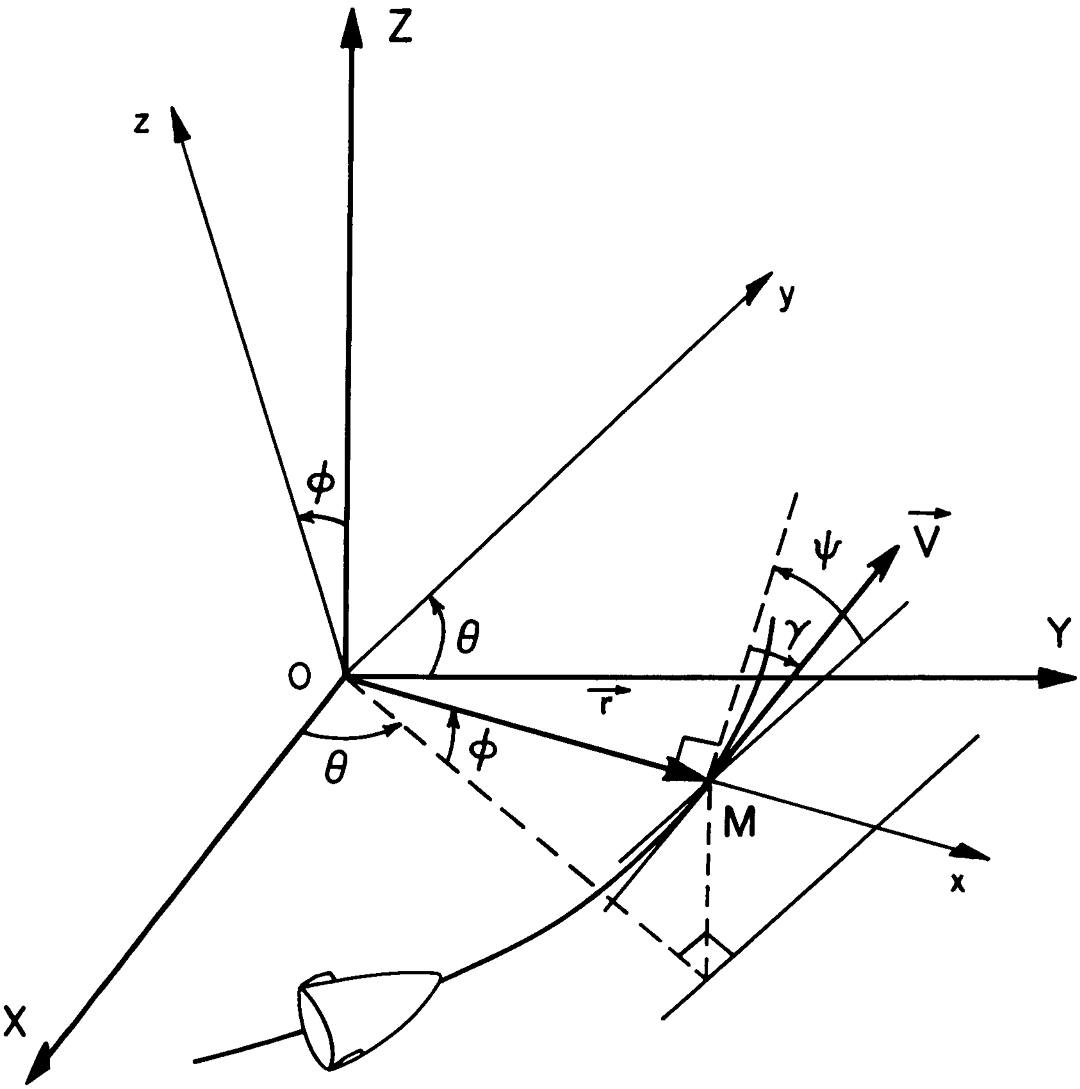

Fig. 1. Coordinate Systems.

a right handed system (Fig. 1). Let γ be the angle between the local horizontal plane, that is the plane passing through the vehicle and parallel to the Oyz plane, and the velocity vector $\vec{V}$. The angle γ is termed the flight path angle and is positive when $\vec{V}$ is above the horizontal plane. Let ψ be the angle between the local parallel of latitude and the projection of $\vec{V}$ on the horizontal plane. The angle ψ is termed the heading and is measured positively in the right-handed direction about the x-axis. Let $\vec{i}$, $\vec{j}$, and $\vec{k}$ be the unit vectors along the axes of the rotating system Oxyz . We have

$$\vec{r} = r\vec{i} \tag{2.2}$$

and

$$\vec{V} = (V\sin\gamma)\ \vec{i} + (V\cos\gamma\cos\psi)\ \vec{j} + (V\cos\gamma\sin\psi)\ \vec{k} \tag{2.3}$$

The system Oxyz is obtained from the system OXYZ by a rotation θ about the positive Z-axis, followed by a rotation ϕ about the negative y-axis. Hence the angular velocity $\vec{\Omega}$ of the rotating system Oxyz is

$$\vec{\Omega} = (\sin\phi\ \frac{d\theta}{dt})\ \vec{i} - (\frac{d\phi}{dt})\ \vec{j} + (\cos\phi\ \frac{d\theta}{dt})\ \vec{k} \tag{2.4}$$

We deduce the time derivative of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ with respect to the inertial system OXYZ

$$\begin{aligned} \frac{d\vec{i}}{dt} &= \vec{\Omega}X\vec{i} = (\cos\phi\ \frac{d\theta}{dt})\ \vec{j} + (\frac{d\phi}{dt})\ \vec{k} \\ \frac{d\vec{j}}{dt} &= \vec{\Omega}X\vec{j} = -\ (\cos\phi\ \frac{d\theta}{dt})\ \vec{i} + (\sin\phi\ \frac{d\theta}{dt})\ \vec{k} \\ \frac{d\vec{k}}{dt} &= \vec{\Omega}X\vec{k} = -\ (\frac{d\phi}{dt})\ \vec{i} - (\sin\phi\ \frac{d\theta}{dt})\ \vec{j} \end{aligned} \tag{2.5}$$

The equations of motion of the vehicle are

$$\frac{d\vec{r}}{dt} = \vec{V} \tag{2.6}$$

and

$$\frac{d\vec{V}}{dt} = \frac{1}{m}(\vec{L} + \vec{D}) + \vec{g} \tag{2.7}$$

where $\vec{g}$ is the acceleration of gravity and the forces $\vec{L}$ and $\vec{D}$ are the lift and the drag. Expressed in components along the rotating axes, $\vec{g}$ is simply

$$\vec{g} = -g(r)\,\vec{i} \tag{2.8}$$

The drag $\vec{D}$ is always opposite to the velocity vector, while the lift $\vec{L}$ is orthogonal to it. Hence, based on Eq. (2.3), we have immediately for the vector $\vec{D}$

$$\vec{D} = -(D\sin\gamma)\,\vec{i} - (D\cos\gamma\cos\psi)\,\vec{j} - (D\cos\gamma\sin\psi)\,\vec{k} \tag{2.9}$$

In planar flight, the vector $\vec{L}$ is in the $(\vec{r}\,,\vec{V})$ plane and there is no lateral aerodynamic force. By control action, if we rotate the vector $\vec{L}$ about the velocity vector $\vec{V}$ we create a lateral component of the lift force that has the effect of changing the orbital plane. To resolve the lift $\vec{L}$ into components along the rotating axes, we refer to Fig. 2. The vertical plane considered is the $(\vec{r}\,,\vec{V})$ plane. Assume the vector $\vec{L}$ is rotated out of this plane through an angle σ . The angle σ which is the angle between the vector $\vec{L}$ and the $(\vec{r}\,,\vec{V})$ plane will be referred to as the roll, or the bank, angle. The lift is decomposed into a component $\overrightarrow{L\cos\sigma}$ in the vertical plane and orthogonal to $\vec{V}$ and a component $\overrightarrow{L\sin\sigma}$ orthogonal to the vertical plane. Let x' , y' , and z' be the axes from the position M of the vehicle, parallel to

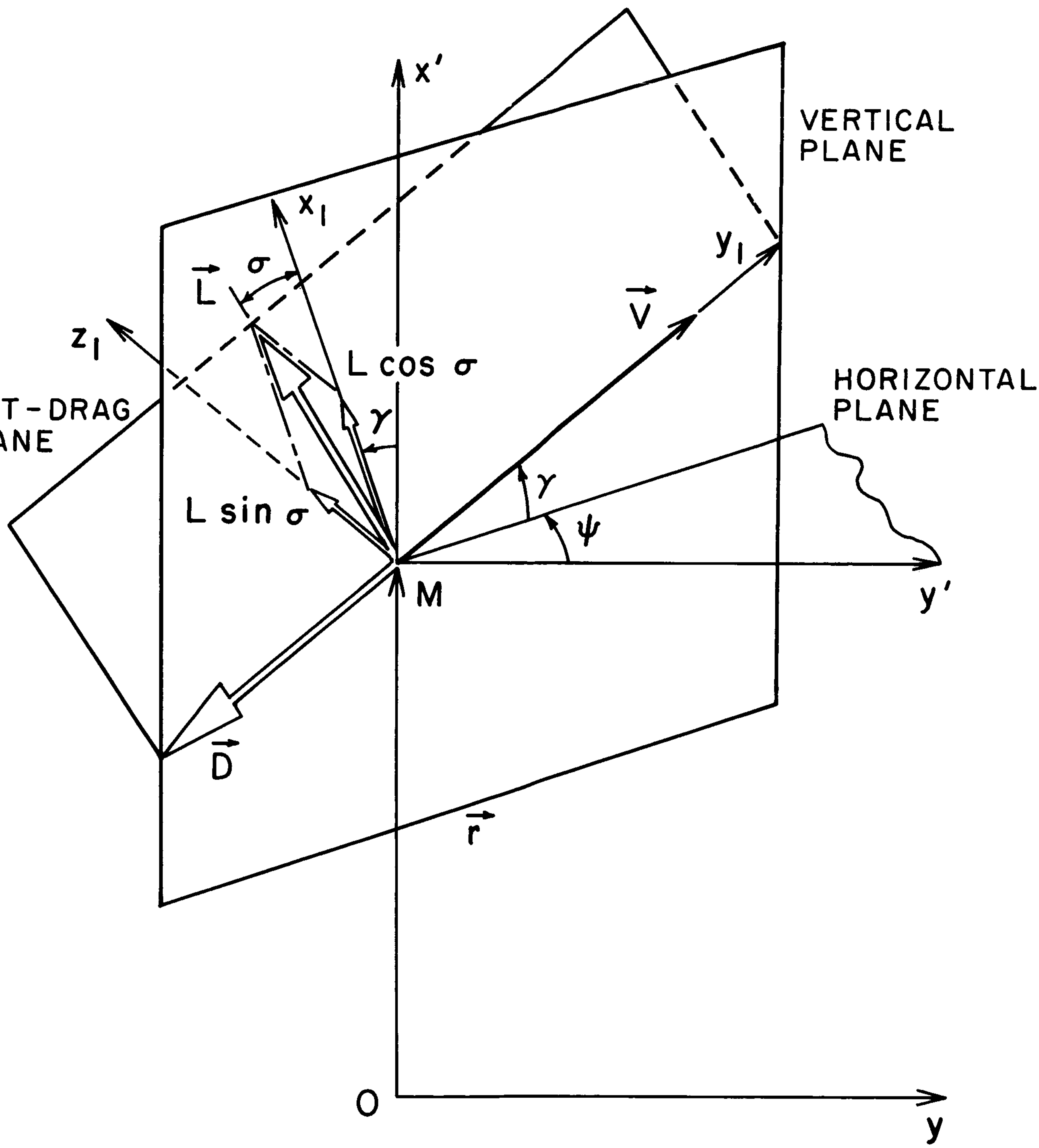

Fig. 2. Aerodynamic Forces

the rotating axes x , y , and z . Let x_1 , y_1 , and z_1 be the axes from the point M , along the direction of $\overrightarrow{L\cos\sigma}$, $\vec{V}$, and $\overrightarrow{L\sin\sigma}$, respectively. The system $Mx_1y_1z_1$ is deduced from the system $Mx'y'z'$ by a rotation ψ in the horizontal plane, followed by a rotation γ in the vertical plane. Hence, we have the transformation matrix equation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma\cos\psi & \cos\gamma\cos\psi & -\sin\psi \\ -\sin\gamma\sin\psi & \cos\gamma\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \tag{2.10}$$

Since the components of $\vec{L}$ in the $Mx_1y_1z_1$ system are $x_1 = L\cos\sigma$, $y_1 = 0$, $z_1 = L\sin\sigma$, we deduce the components of $\vec{L}$ along the system $Mx'y'z'$, or what is the same, along the rotating system Oxyz

$$\begin{aligned} \vec{L} = (L\cos\sigma\cos\gamma)\,\vec{i} - (L\cos\sigma\sin\gamma\cos\psi + L\sin\sigma\sin\psi)\,\vec{j} - \\ - (L\cos\sigma\sin\gamma\sin\psi - L\sin\sigma\cos\psi)\,\vec{k} \end{aligned} \tag{2.11}$$

Now, if we take the derivative of $\vec{r}$, as given by Eq. (2.2), using Eq. (2.5) for the derivative of $\vec{i}$, we have

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\vec{i} + (r\cos\phi\,\frac{d\theta}{dt})\,\vec{j} + (r\,\frac{d\phi}{dt})\,\vec{k} \tag{2.12}$$

By substituting into Eq. (2.6) and using Eq. (2.3) for the components of $\vec{V}$, we have three scalar equations

$$\frac{dr}{dt} = V\sin\gamma \tag{2.13}$$

$$\frac{d\theta}{dt} = \frac{V\cos\gamma\cos\psi}{r\cos\phi} \tag{2.14}$$

$$\frac{d\phi}{dt} = \frac{V\cos\gamma\sin\psi}{r} \tag{2.15}$$

These equations are the kinematic equations.

On the other hand, if we take the derivatives of the velocity vector $\vec{V}$, as given by Eq. (2.3), using the Eqs. (2.5) for the derivatives of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$, and subsequently the Eqs. (2.14) and (2.15) for $d\theta/dt$ and $d\phi/dt$, we have

$$\begin{aligned}\frac{d\vec{V}}{dt} &= [\sin\gamma \frac{dV}{dt} + V\cos\gamma \frac{d\gamma}{dt} - \frac{V^2}{r}\cos^2\gamma]\,\vec{i} \\ &+ [\cos\gamma\cos\psi \frac{dV}{dt} - V\sin\gamma\cos\psi \frac{d\gamma}{dt} - V\cos\gamma\sin\psi \frac{d\psi}{dt} \\ &+ \frac{V^2\cos\gamma\cos\psi}{r}(\sin\gamma - \cos\gamma\sin\psi\tan\phi)]\,\vec{j} \\ &+ [\cos\gamma\sin\psi \frac{dV}{dt} - V\sin\gamma\sin\psi \frac{d\gamma}{dt} + V\cos\gamma\cos\psi \frac{d\psi}{dt} \\ &+ \frac{V^2\cos\gamma}{r}(\sin\gamma\sin\psi + \cos\gamma\cos^2\psi\tan\phi)]\,\vec{k}\end{aligned} \tag{2.16}$$

By substituting into Eq. (2.7), and using the Eqs. (2.8), (2.9), and (2.11) for the components of $\vec{g}$, $\vec{D}$, and $\vec{L}$, we have three scalar equations

$$\sin\gamma \frac{dV}{dt} + V\cos\gamma \frac{d\gamma}{dt} - \frac{V^2\cos^2\gamma}{r} = \frac{L}{m}\cos\sigma\cos\gamma - \frac{D}{m}\sin\gamma - g \tag{2.17}$$

$$\begin{aligned}&\cos\gamma \frac{dV}{dt} - V\sin\gamma \frac{d\gamma}{dt} - V\cos\gamma\tan\psi \frac{d\psi}{dt} + \frac{V^2\cos\gamma}{r}(\sin\gamma - \cos\gamma\sin\psi\tan\phi) \\ &= -\frac{L}{m}\cos\sigma\sin\gamma - \frac{L}{m}\sin\sigma\tan\psi - \frac{D}{m}\cos\gamma\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\cos\gamma \frac{dV}{dt} - V\sin\gamma \frac{d\gamma}{dt} + \frac{V\cos\gamma}{\tan\psi}\frac{d\psi}{dt} + \frac{V^2\cos\gamma}{r}(\sin\gamma + \frac{\cos\gamma\cos^2\psi\tan\phi}{\sin\psi}) \\ &= -\frac{L}{m}\cos\sigma\sin\gamma + \frac{L\sin\sigma}{m\tan\psi} - \frac{D}{m}\cos\gamma\end{aligned} \tag{2.19}$$

Solving for the derivatives dV/dt , $d\gamma/dt$, and $d\psi/dt$, we have three scalar equations, the force equations,

$$\frac{dV}{dt} = -\frac{D}{m} - g\sin\gamma \tag{2.20}$$

$$V\frac{d\gamma}{dt} = \frac{L}{m}\cos\sigma - g\cos\gamma + \frac{V^2}{r}\cos\gamma \tag{2.21}$$

$$V\frac{d\psi}{dt} = \frac{L\sin\sigma}{m\cos\gamma} - \frac{V^2}{r}\cos\gamma\cos\psi\tan\phi \tag{2.22}$$

The six equations, Eqs. (2.13) - (2.15), and (2.20) - (2.22) are the exact equations of motion for flight over a spherical, nonrotating planet with its atmosphere at rest.

3. DIMENSIONLESS EQUATIONS USING MODIFIED CHAPMAN VARIABLES

In the equations of motion derived in Section 2, the aerodynamic lift and drag are now assumed to be

$$\begin{aligned} L &= \frac{1}{2}\rho S C_L V^2 \\ D &= \frac{1}{2}\rho S C_D V^2 \end{aligned} \tag{3.1}$$

where C_L and C_D are the lift and the drag coefficients, assumed independent of the Mach number and the Reynolds number in the hypervelocity regime. The density of the atmosphere, ρ , is assumed to be a known function of the radial distance r . For each flight program, the control functions C_L , C_D and σ are prescribed functions of the time, and the integration of the system of six equations of motion requires prescribing the six initial values for the state variables. In addition, for a given vehicle, the parameter S/m must also be given.

For a planar trajectory with constant angle of attack, with the initial point taken as the origin for measuring the range, that is with $\theta_i = 0$, it is seen, by inspection of the equations of motion, that the following quantities must be prescribed:

a/ For the vehicle and flight parameters

$$SC_L/m \quad \text{and} \quad SC_D/m \tag{3.2}$$

or equivalently

$$SC_D/m \quad \text{and} \quad C_L/C_D \tag{3.3}$$

b/ For the initial conditions

$$r_i \ , \ V_i \ \text{and} \ \gamma_i \tag{3.4}$$

By a very ingenious coordinate transformation, Chapman has introduced two dimensionless variables defined as (Ref. 1)

$$\begin{aligned} \bar{u} &= \frac{V\cos\gamma}{\sqrt{gr}} \\ \bar{Z} &= \frac{\rho SC_D}{2m}\sqrt{\frac{r}{\beta}}\,\bar{u} \end{aligned} \tag{3.5}$$

where β is the atmospheric density height scale defining the atmosphere through the differential relation

$$\frac{d\rho}{\rho} = -\beta(r)dr \tag{3.6}$$

Through the transformation (3.5) and through some simplifying assumptions, Chapman has reduced the basic equations of motion to a single nonlinear differential equation with $\bar{Z}$ as the dependent variable and $\bar{u}$ as the independent variable. Chapman's equation is

$$\bar{u}\frac{d}{d\bar{u}}\left(\frac{d\bar{Z}}{d\bar{u}} - \frac{\bar{Z}}{\bar{u}}\right) - \frac{1-\bar{u}^2}{\bar{u}\,\bar{Z}}\cos^4\gamma + \sqrt{\beta r}\,\frac{C_L}{C_D}\cos^3\gamma = 0 \tag{3.7}$$

with the flight path angle γ obtained from

$$\frac{d\bar{Z}}{d\bar{u}} - \frac{\bar{Z}}{\bar{u}} = \sqrt{\beta r}\,\sin\gamma \tag{3.8}$$

The varying quantity r enters the equations as the product βr. In the lower altitudes of planetary atmospheres, where aerodynamic forces are effective, the quantity βr oscillates about a mean value. The flight parameter appears in Chapman's equation, Eq. (3.7), as the lift-to-drag ratio C_L/C_D. Furthermore, for entry trajectories, since the initial value of $\bar{Z}$ is nearly zero, only the initial values $\bar{u}_i$ and γ_i need be specified for the integration of Chapman's equations. Hence, for a given atmosphere, with βr considered as constant (e.g., for the Earth $\beta r \simeq 900$), Chapman can integrate numerically his equations and tabulate the results for each set of values of C_L/C_D, $\bar{u}_i$, and γ_i. These Tables of the $\bar{Z}$ functions (Ref. 2) can be used for any vehicle of arbitrary weight, dimensions and shape entering the specified planetary atmosphere. Furthermore, all the physical quantities during entry, such as deceleration, dynamic pressure, heating rate, and heat transfer rate, can be easily obtained as simple functions of the variables $\bar{Z}$, $\bar{u}$ and γ.

The major deficiency of Chapman theory is that, because of his two main assumptions, namely that

a/ the percentage change in distance from the planet center is small compared to the percentage change in the horizontal component of the velocity, that is

$$\left|\frac{dr}{r}\right| << \left|\frac{d(V\cos\gamma)}{V\cos\gamma}\right| \tag{3.9}$$

and

b/ the quantity $(C_L/C_D)\tan\gamma$ is small, that is

$$\left|\frac{C_L}{C_D}\tan\gamma\right| \ll 1 \tag{3.10}$$

the validity of Chapman's numerical analysis is restricted to trajectories with small flight path angle, or trajectories with small lift-to-drag ratio.

These restrictions have been removed by deriving the exact equations for three-dimensional reentry using a set of modified Chapman variables (Refs. 3 - 6). Chapman's assumptions (a) and (b), Eqs. (3.9) and (3.10), imply that the independent variable $\bar{u}$ is monotonically decreasing. This is only true for the last portion of the trajectory since at high altitude $\bar{u}$ is oscillatory, and in the limit, for flight in the vacuum, $\bar{u}$ is purely periodic (Ref. 5). To avoid this difficulty the following dimensionless variable is introduced as the independent variable

$$s = \int_0^t \left(\frac{V}{r}\right) \cos\gamma \, dt \tag{3.11}$$

The variable is strictly increasing as long as $\cos\gamma > 0$, a condition which is always satisfied for entry at constant lift-to-drag ratio, and for all physically realistic entry trajectories. For the equations,

$$\begin{aligned} u &= \frac{V^2\cos^2\gamma}{gr} \\ Z &= \frac{\rho S C_D}{2m}\sqrt{\frac{r}{\beta}} \end{aligned} \tag{3.12}$$

Expressed in terms of the original Chapman variables, we simply have

$$Z = \frac{\bar{Z}}{\bar{u}} \quad , \quad u = \bar{u}^2 \tag{3.13}$$

The new variables lead to a set of differential equations in a simpler form allowing a complete qualitative discussion of the three-dimensional reentry trajectory.

We shall present below a slightly different derivation of the exact dimensionless equations from the one given in Ref. 5.

First, by eliminating the time, by dividing Eqs. (2.14), (2.15), and (2.20) - (2.22) by Eq. (2.13), we have

$$\begin{aligned}
\frac{d\theta}{dr} &= \frac{\cos\psi}{r\cos\phi\tan\gamma} \\
\frac{d\phi}{dr} &= \frac{\sin\psi}{r\tan\gamma} \\
\frac{dV^2}{dr} &= -\frac{\rho SC_D V^2}{m\sin\gamma} - 2g \qquad (3.14) \\
\frac{d\gamma}{dr} &= \frac{\rho SC_L\cos\sigma}{2m\sin\gamma} - \frac{g}{V^2\tan\gamma} + \frac{1}{r\tan\gamma} \\
\frac{d\psi}{dr} &= \frac{\rho SC_L\sin\sigma}{2m\sin\gamma\cos\gamma} - \frac{\cos\psi\tan\phi}{r\tan\gamma}
\end{aligned}$$

Next, from Eq. (3.11), using Eq. (2.13), we have

$$\frac{ds}{dr} = \frac{ds}{dt}\frac{dt}{dr} = \frac{1}{r\tan\gamma} \qquad (3.15)$$

Hence, using this equation, we can rewrite the Eqs. (3.14) with s as independent variable

$$\begin{aligned}
\frac{d\theta}{ds} &= \frac{\cos\psi}{\cos\phi} \\
\frac{d\phi}{ds} &= \sin\psi \\
\frac{dV^2}{ds} &= -\frac{r\rho SC_D V^2}{m\cos\gamma} - 2gr\tan\gamma \qquad (3.16) \\
\frac{d\gamma}{ds} &= \frac{r\rho SC_L\cos\sigma}{2m\cos\gamma} + \left(1 - \frac{gr}{V^2}\right) \\
\frac{d\psi}{ds} &= \frac{r\rho SC_L\sin\sigma}{2m\cos^2\gamma} - \cos\psi\tan\phi
\end{aligned}$$

Using the Chapman formulation, the variable V^2 is replaced by the variable u, while the radial distance r is replaced by the variable Z. By taking the derivative of u, as defined by the first equation (3.12), with respect to s, using an inverse-square law for the acceleration of the gravity, we have

$$\frac{du}{ds} = \frac{\cos^2\gamma}{gr}\frac{dV^2}{ds} - \frac{2V^2\sin\gamma\cos\gamma}{gr}\frac{d\gamma}{ds} + \frac{V^2\cos^2\gamma}{gr^2}\frac{dr}{ds} \tag{3.17}$$

Using the definition (3.12) for u and Z, with the appropriate derivatives from Eqs. (3.15) and (3.16), we have the differential equation for u

$$\frac{du}{ds} = -\frac{2\sqrt{\beta r}\,Zu}{\cos\gamma}\left[1 + \left(\frac{C_L}{C_D}\right)\cos\sigma\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\,Z}\right] \tag{3.18}$$

On the other hand, by taking the derivative of Z, as defined by the second Eq. (3.12), with respect to s, we have

$$\frac{dZ}{ds} = -\beta r\left(-\frac{1}{\rho\beta}\frac{d\rho}{dr} - \frac{1}{2\beta r} + \frac{\beta'}{2\beta^2}\right) Z\tan\gamma \tag{3.19}$$

where $\beta' = d\beta/dr$.

As discussed in Ref. 3, the term in parentheses in equation (3.19) is, for the locally exponential atmosphere of equation (3.6), $1 - \frac{1}{2\beta r} + \frac{\beta'}{2\beta^2}$. For an isothermal atmosphere βr^2 is constant and this term becomes $-\frac{1}{\rho\beta}\frac{d\rho}{dr} - \frac{3}{2\beta r}$. For a strictly exponential atmosphere this term is $1 - \frac{1}{2\beta r}$. Finally, for an atmosphere with βr constant, this term is $-\frac{1}{\rho\beta}\frac{d\rho}{dr} - \frac{1}{\beta r}$. In all these cases the direct dependency on r is of the form βr. In fact, for Earth, where βr is about 900, the term in question will be unity to a high degree of accuracy. As analyzed in

Refs. 3 and 5, the equations with βr as a constant may still justifiably be considered exact.

Finally, the differential equations for γ and ψ, written in terms of the dimensionless variables are, respectively

$$\frac{d\gamma}{ds} = \frac{\sqrt{\beta r}\, Z}{\cos\gamma}\left[\left(\frac{C_L}{C_D}\right)\cos\sigma + \frac{\cos\gamma}{\sqrt{\beta r}\, Z}\left(1 - \frac{\cos^2\gamma}{u}\right)\right] \tag{3.20}$$

and

$$\frac{d\psi}{ds} = \frac{\sqrt{\beta r}\, Z}{\cos^2\gamma}\left[\left(\frac{C_L}{C_D}\right)\sin\sigma - \frac{\cos^2\gamma\cos\psi\tan\phi}{\sqrt{\beta r}\, Z}\right] \tag{3.21}$$

In summary, the exact equations of motion for three dimensional flight in an isothermal atmosphere, using modified Chapman variables are

$$\begin{aligned}
\frac{dZ}{ds} &= -\beta r\left(1 - \frac{3}{2\beta r}\right) Z\tan\gamma \\
\frac{du}{ds} &= -\frac{2\sqrt{\beta r}\, Zu}{\cos\gamma}\left(1 + \lambda\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\, Z}\right) \\
\frac{d\gamma}{ds} &= \frac{\sqrt{\beta r}\, Z}{\cos\gamma}\left[\lambda + \frac{\cos\gamma}{\sqrt{\beta r}\, Z}\left(1 - \frac{\cos^2\gamma}{u}\right)\right] \\
\frac{d\theta}{ds} &= \frac{\cos\psi}{\cos\phi} \\
\frac{d\phi}{ds} &= \sin\psi \\
\frac{d\psi}{ds} &= \frac{\sqrt{\beta r}\, Z}{\cos^2\gamma}\left(\delta - \frac{\cos^2\gamma\cos\psi\tan\phi}{\sqrt{\beta r}\, Z}\right)
\end{aligned} \tag{3.22}$$

where

$$\lambda = \frac{C_L}{C_D}\cos\sigma \quad , \quad \delta = \frac{C_L}{C_D}\sin\sigma \tag{3.23}$$

It may be observed that, although the equations are derived for three-dimensional flight, the first three equations are decoupled from the last three equations. Hence, by using only the assumption of constant value for βr, for constant lift-to-drag ratio and constant bank angle, λ is the only flight parameter that needs to be specified. For each λ, with specified initial conditions u_i , γ_i , and $Z_i \simeq 0$, the first three equations (3.22) can be integrated and Tables of the Z functions can be generated. Since Chapman's two restrictive assumptions have been removed, the new Tables are valid for three-dimensional flight without any restriction concerning the flight path angle and the lift-to-drag ratio.

Reference 5 presents a qualitative and quantitative analysis of these equations. It is the purpose of the present work to obtain an analytic solution to the complete set of the exact equations of motion. These equations are integrated by the method of matched asymptotic expansions. The results of investigation corroborate the assessment in Ref. 5 that the assumption of a constant βr is very accurate. Hence, the equations (3.22) with βr = constant can be considered as the exact equations for reentry, and Tables of the Z functions based on the new equations should provide accurate data for analyses of planetary entry.

4. SOLUTIONS BY DIRECTLY MATCHED ASYMPTOTIC EXPANSIONS

In this application of the method of matched asymptotic expansions, the solutions are obtained separately for an outer region, near the vacuum, where the gravity force is predominant, and for an inner region, near the planetary surface, where the aerodynamic force is predominant. Hence, the altitude is the appropriate independent variable selected for the integration.

Let y be the altitude and let subscript $\dagger$ denote the reference altitude, taken at sea level. Then

$$r = r_\dagger + y = r_\dagger(1 + h) \tag{4.1}$$

where the dimensionless altitude h is defined as

$$h = \frac{y}{r_\dagger} \tag{4.2}$$

The differential relation between s and h is

$$ds = \frac{dh}{(1 + h)\ \tan\gamma} \tag{4.3}$$

For the integration, we adopt a strictly exponential atmosphere, but the general method can be applied to any more realistic atmosphere such as, for example, the one proposed in Ref. 7. For an exponential atmosphere

$$\rho = \rho_\dagger\, e^{-\beta y} = \rho_\dagger\, e^{-\frac{h}{\varepsilon}} \tag{4.4}$$

where

$$\varepsilon = \frac{1}{\beta r_\dagger} \tag{4.5}$$

Since the constant $\beta r_\dagger$ is large, e.g., for the Earth atmosphere $\beta r_\dagger \simeq 900$, the parameter ε is a small quantity. By the definition (3.12) of Z

$$Z = \frac{\rho_\dagger S C_D}{2m\beta} \sqrt{\frac{(1 + h)}{\varepsilon}}\ e^{-\frac{h}{\varepsilon}} \tag{4.6}$$

We define the ballistic coefficient

$$B = \frac{S C_D \rho_\dagger}{2m\beta} \tag{4.7}$$

For each vehicle, B is specified and the variable Z is obtained from

$$Z = B\sqrt{\frac{(1 + h)}{\varepsilon}}\ e^{-\frac{h}{\varepsilon}} \tag{4.8}$$

By this relation, the first Eq. (3.22) can be deleted, and we write the other equations with the dimensionless altitude h as independent variable

$$\frac{du}{dh} = -\frac{u}{(1 + h)} - \frac{2Bu(1 + \lambda\tan\gamma)}{\varepsilon\sin\gamma} e^{-\frac{h}{\varepsilon}}$$

$$\frac{dq}{dh} = -\frac{q}{(1 + h)}(1 - \frac{q^2}{u}) - \frac{\lambda B}{\varepsilon} e^{-\frac{h}{\varepsilon}}$$

$$\frac{d\theta}{dh} = \frac{\cos\psi}{(1 + h)\ \cos\phi\tan\gamma} \tag{4.9}$$

$$\frac{d\phi}{dh} = \frac{\sin\psi}{(1 + h)\ \tan\gamma}$$

$$\frac{d\psi}{dh} = -\frac{\cos\psi\tan\phi}{(1 + h)\ \tan\gamma} + \frac{B\delta e^{-\frac{h}{\varepsilon}}}{\varepsilon\sin\gamma\cos\gamma}$$

We have defined

$$q = \cos\gamma \tag{4.10}$$

Also, we recall the definition of the flight parameters

$$\lambda = \frac{C_L}{C_D}\cos\sigma \quad , \quad \delta = \frac{C_L}{C_D}\sin\sigma \tag{4.11}$$

The Eqs. (4.9) are in a suitable form for numerical integration for flight inside an atmosphere. For an analytical solution of the re-entry trajectory using the method of matched asymptotic expansions, we shall use a more convenient form using some elements of the orbit as introduced in celestial mechanics, since these elements are constants of the motion for flight in a vacuum.

As seen in Fig. 3, if I is the inclination of the plane of the osculating orbit, that is the $(\vec{r}\ ,\ \vec{V})$ plane, Ω the longitude of the ascending node, and α the angle between the line of the ascending node and the position vector, we have the following pertinent relations from

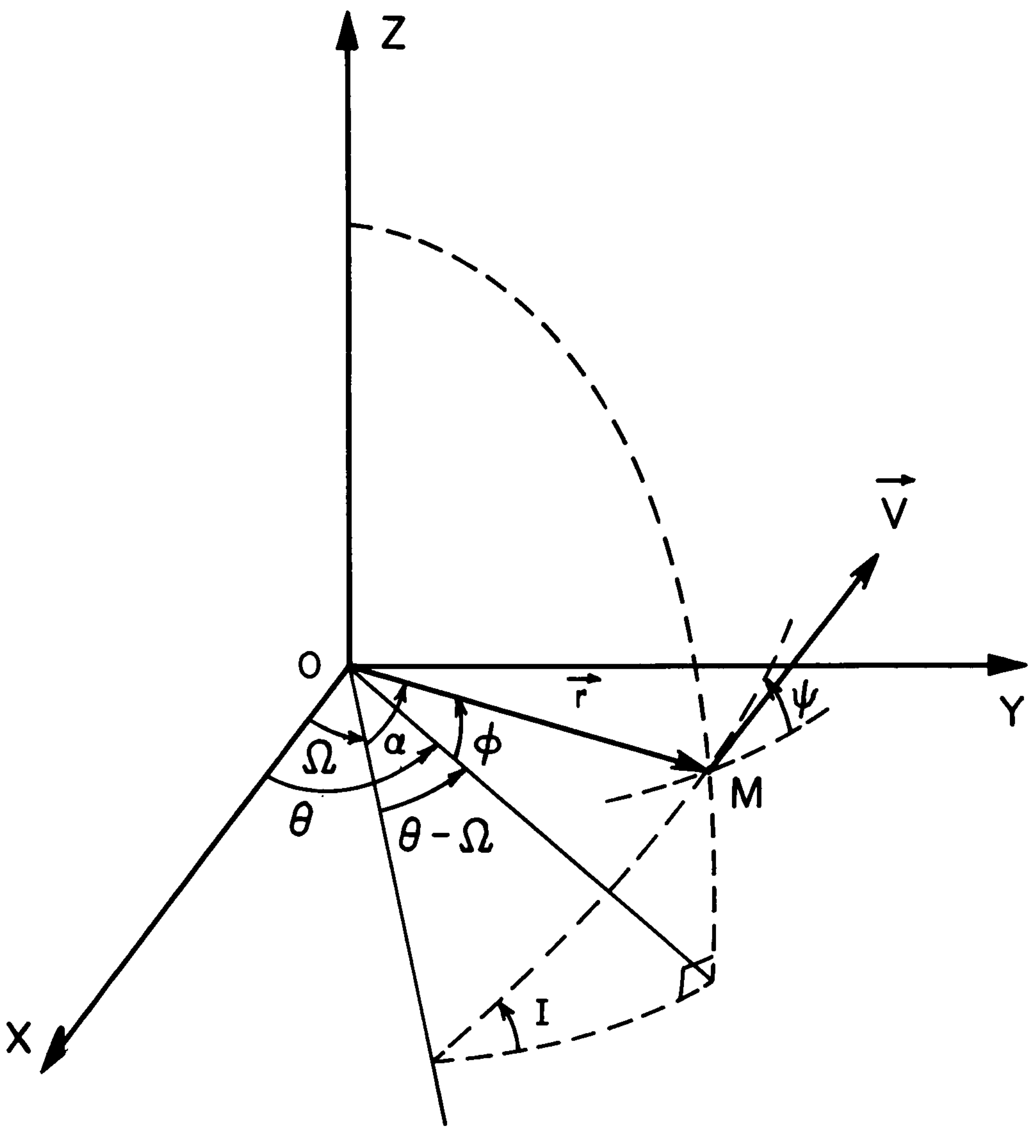

Fig. 3. The Osculating Plane and the Orbital Elements.

spherical trigonometry

$$\cos\phi\cos\psi = \cos I$$

$$\sin(\theta - \Omega) = \frac{\tan\phi}{\tan I} \tag{4.12}$$

$$\cos\alpha = \cos\phi\cos(\theta - \Omega)$$

These relations are independent. We can easily deduce

$$\sin\phi = \sin I\sin\alpha$$

$$\sin\psi = \frac{\tan\phi}{\tan\alpha} \tag{4.13}$$

$$\sin\psi = \sin I\cos(\theta - \Omega)$$

Using these relations, we replace the variables θ , ϕ , and ψ by the new variables α , Ω , and I . The Eqs. (4.9) now become

$$\frac{du}{dh} = -\frac{u}{(1+h)} - \frac{2Bu(1+\lambda\tan\gamma)}{\varepsilon\sin\gamma} e^{-\frac{h}{\varepsilon}}$$

$$\frac{dq}{dh} = -\frac{q}{(1+h)}(1 - \frac{q^2}{u}) - \frac{\lambda B}{\varepsilon} e^{-\frac{h}{\varepsilon}}$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{B\delta\sin\alpha}{\varepsilon\tan I\sin\gamma\cos\gamma} e^{-\frac{h}{\varepsilon}} \tag{4.14}$$

$$\frac{d\Omega}{dh} = \frac{B\delta\sin\alpha}{\varepsilon\sin I\sin\gamma\cos\gamma} e^{-\frac{h}{\varepsilon}}$$

$$\frac{dI}{dh} = \frac{B\delta\cos\alpha}{\varepsilon\sin\gamma\cos\gamma} e^{-\frac{h}{\varepsilon}}$$

The Eqs. (4.14) are most suitable for an integration using the method of matched asymptotic expansions. We notice that, once the elements α , Ω , and I are known, we obtain the original variables θ , ϕ , and ψ from

$$\tan(\theta - \Omega) = \cos I \tan\alpha$$
$$\sin\phi = \sin I \sin\alpha \tag{4.15}$$
$$\tan\psi = \cos\alpha \tan I$$

4.1 Outer Expansions (Keplerian Region)

The Eqs. (4.14) are expressed in terms of the outer variables. The outer expansions are introduced to study the limiting condition of the solution in the region near the vacuum where the gravitational force is predominant. They are obtained by repeated application of the outer limit, which is defined as the limit when $\varepsilon \to 0$ with the variable h and other dimensionless quantities held fixed.

We assume the following expansions

$$\begin{aligned}
u &= u_0(h) + \varepsilon u_1(h) + \ldots \\
q &= q_0(h) + \varepsilon q_1(h) + \ldots \\
\alpha &= \alpha_0(h) + \varepsilon \alpha_1(h) + \ldots \\
\Omega &= \Omega_0(h) + \varepsilon \Omega_1(h) + \ldots \\
I &= I_0(h) + \varepsilon I_1(h) + \ldots
\end{aligned} \tag{4.16}$$

By substituting into Eqs. (4.14) and equating coefficients of like power in ε , the differential equations with zero order of ε are

$$\begin{aligned}
\frac{du_0}{dh} &= -\frac{u_0}{(1+h)} \\
\frac{dq_0}{dh} &= -\frac{q_0}{(1+h)}\left(1 - \frac{q_0^2}{u_0}\right) \\
\frac{d\alpha_0}{dh} &= \frac{1}{(1+h)\tan\gamma_0} \\
\frac{d\Omega_0}{dh} &= 0 \\
\frac{dI_0}{dh} &= 0
\end{aligned} \tag{4.17}$$

The solution of this system is

$$
\begin{aligned}
u_0(1 + h) &= C_1 \\
\frac{1}{q_0^2} &= \frac{2(1 + h)}{C_1} - C_2(1 + h)^2 \\
u_0 &= 1 + \sqrt{1 - C_1^2 C_2}\,\cos(\alpha_0 - C_3) \qquad (4.18) \\
\Omega_0 &= C_4 \\
I_0 &= C_5
\end{aligned}
$$

where the C_i are constants of integration. The first and higher order solutions are all equal to zero because at high altitude, in the limit the atmospheric density is zero and the motion is Keplerian.

4.2 Inner Expansions (Aerodynamic-Predominated Region)

The inner expansions are introduced to study the limiting condition of the solution near the planetary surface where the aerodynamic force is predominant. They are obtained by repeated application of the inner limit, which is defined as the limit when $\varepsilon \to 0$ with the new stretched altitude

$$\tilde{h} = \frac{h}{\varepsilon} \qquad (4.19)$$

and the other dimensionless quantities held fixed.

We assume the following expansions

$$
\begin{aligned}
u &= \tilde{u}_0(\tilde{h}) + \varepsilon \tilde{u}_1(\tilde{h}) + \ldots \\
q &= \tilde{q}_0(\tilde{h}) + \varepsilon \tilde{q}_1(\tilde{h}) + \ldots \\
\alpha &= \tilde{\alpha}_0(\tilde{h}) + \varepsilon \tilde{\alpha}_1(\tilde{h}) + \ldots \qquad (4.20) \\
\Omega &= \tilde{\Omega}_0(\tilde{h}) + \varepsilon \tilde{\Omega}_1(\tilde{h}) + \ldots \\
I &= \tilde{I}_0(\tilde{h}) + \varepsilon \tilde{I}_1(\tilde{h}) + \ldots
\end{aligned}
$$

By substituting into Eqs. (4.14) and equating coefficients of like powers in ε, the differential equations with zero order of ε are

$$
\begin{aligned}
\frac{d\tilde{u}_0}{d\tilde{h}} &= -\frac{2B\tilde{u}_0(1+\lambda\tan\tilde{\gamma}_0)}{\sin\tilde{\gamma}}\, e^{-\tilde{h}} \\
\frac{d\tilde{q}_0}{d\tilde{h}} &= -\lambda B e^{-\tilde{h}} \\
\frac{d\tilde{\alpha}_0}{d\tilde{h}} &= -\frac{B\delta\sin\tilde{\alpha}_0}{\tan\tilde{I}_0\sin\tilde{\gamma}_0\cos\tilde{\gamma}_0}\, e^{-\tilde{h}} \\
\frac{d\tilde{\Omega}_0}{d\tilde{h}} &= \frac{B\delta\sin\tilde{\alpha}_0}{\sin\tilde{I}_0\sin\tilde{\gamma}_0\cos\tilde{\gamma}_0}\, e^{-\tilde{h}} \\
\frac{d\tilde{I}_0}{d\tilde{h}} &= \frac{B\delta\cos\tilde{\alpha}_0}{\sin\tilde{\gamma}_0\cos\tilde{\gamma}_0}\, e^{-\tilde{h}}
\end{aligned}
\tag{4.21}
$$

The solution of this system is

$$
\begin{aligned}
\tilde{u}_0 &= \tilde{C}_1\tilde{q}_0^2 \exp\left[-\frac{2\tilde{\gamma}_0}{\lambda}\right] \\
\tilde{q}_0 &= \lambda B e^{-\tilde{h}} + \tilde{C}_2 \\
\sin\tilde{\alpha}_0\sin\tilde{I}_0 &= \sin\tilde{C}_3 \\
\cos\tilde{\alpha}_0 &= \cos\tilde{C}_3\cos(\tilde{C}_4 - \tilde{\Omega}_0) \\
\cos\tilde{I}_0 &= \cos\tilde{C}_3\cos\left\{\frac{\delta}{\lambda}\log\left[\tan\left(\frac{\Pi}{4}+\frac{\tilde{\gamma}_0}{2}\right)\right] + \tilde{C}_5\right\}
\end{aligned}
\tag{4.22}
$$

where the $\tilde{C}_j$ are constants of integration.

4.3 Asymptotic Matching and Composite Expansions

The constants of integration $\tilde{C}_j$ in the inner expansions will be determined by matching with the outer expansions. In this problem, matching is accomplished by expanding the inner solutions for large $\tilde{h}$, expressing the results in terms of the outer variables and matching with the outer solutions for small h.

The outer solutions, Eqs. (4.18), become for small h

$$
\begin{aligned}
u_0 &= C_1 \\
q_0 &= \sqrt{\frac{C_1}{2 - C_1C_2}} \\
\alpha_0 &= \cos^{-1}\left(\frac{C_1 - 1}{\sqrt{1 - C_1^2C_2}}\right) + C_3 \qquad (4.23) \\
\Omega_0 &= C_4 \\
I_0 &= C_5
\end{aligned}
$$

On the other hand, the inner solutions, Eqs. (4.22), become for large $\tilde{h}$

$$
\begin{aligned}
\tilde{u}_0 &= \tilde{C}_1\tilde{C}_2^2 \exp\left[-\frac{2}{\lambda}\cos^{-1}\tilde{C}_2\right] \\
\tilde{q}_0 &= \tilde{C}_2 \\
\sin\tilde{\alpha}_0 \sin\tilde{I}_0 &= \sin\tilde{C}_3 \qquad (4.24) \\
\cos\tilde{\alpha}_0 &= \cos\tilde{C}_3 \cos(\tilde{C}_4 - \Omega_0) \\
\cos\tilde{I}_0 &= \cos\tilde{C}_3 \cos\left\{\frac{\delta}{\lambda}\log\left[\tan\left(\frac{\Pi}{4} + \frac{1}{2}\cos^{-1}\tilde{C}_2\right)\right] + \tilde{C}_5\right\}
\end{aligned}
$$

Matching Eqs. (4.24) with Eqs. (4.23) provides the constants $\tilde{C}_j$ in terms of the constants C_j . We have

$$
\begin{aligned}
\tilde{C}_1 &= (2 - C_1C_2)\ \exp[\tfrac{2}{\lambda}\cos^{-1}\sqrt{\frac{C_1}{2 - C_1C_2}}\ ] \\
\tilde{C}_2 &= \sqrt{\frac{C_1}{2 - C_1C_2}} \\
\sin\tilde{C}_3 &= \sin C_5 \sin[\cos^{-1}(\frac{C_1 - 1}{\sqrt{1 - C_1^2C_2}}) + C_3] \qquad (4.25) \\
\tilde{C}_4 &= \cos^{-1}\{\cos[\cos^{-1}(\frac{C_1 - 1}{\sqrt{1 - C_1^2C_2}}) + C_3]/\cos\tilde{C}_3\} + C_4 \\
\tilde{C}_5 &= \cos^{-1}[\cos C_5/\cos\tilde{C}_3] - \frac{\delta}{\lambda}\log[\tan(\frac{\Pi}{4} + \frac{1}{2}\cos^{-1}\sqrt{\frac{C_1}{2 - C_1C_2}}\ )]
\end{aligned}
$$

Hence, the constants $\tilde{C}_j$ are expressed explicitly in terms of the constants C_j . Substitution into Eqs. (4.22) gives the inner solutions. It is convenient to use the following notation to write these solutions in a symmetric form. Let

$$
\begin{aligned}
u_* &= C_1 \\
\cos\gamma_* &= \sqrt{\frac{C_1}{2 - C_1C_2}} \\
\sin\phi_* &= \sin C_5 \sin[\cos^{-1}(\frac{C_1 - 1}{\sqrt{1 - C_1^2C_2}}) + C_3] \qquad (4.26) \\
\theta_* &= \cos^{-1}\{\cos[\cos^{-1}(\frac{C_1 - 1}{\sqrt{1 - C_1^2C_2}}) + C_3]/\cos\phi_*\} + C_4 \\
I_* &= C_5
\end{aligned}
$$

The constants with subscript * are explicit functions of the constants C_j . Then the inner solutions are

$$\frac{\tilde{u}_0}{u_*} = \frac{\cos^2\tilde{\gamma}_0}{\cos^2\gamma_*} \exp[\frac{2}{\lambda}(\gamma_* - \tilde{\gamma}_0)]$$

$$\cos\tilde{\gamma}_0 = \cos\gamma_* + \lambda B e^{-\tilde{h}}$$

$$\sin\tilde{\alpha}_0 \sin\tilde{I}_0 = \sin\phi_* \qquad (4.27)$$

$$\cos\tilde{\alpha}_0 = \cos\phi_* \cos(\theta_* - \tilde{\Omega}_0)$$

$$\cos^{-1}(\frac{\cos\tilde{I}_0}{\cos\phi_*}) - \cos^{-1}(\frac{\cos I_*}{\cos\phi_*}) = \frac{\delta}{\lambda} \log[\tan(\frac{\Pi}{4} + \frac{\tilde{\gamma}_0}{2})/\tan(\frac{\Pi}{4} + \frac{\gamma_*}{2})]$$

The Eqs. (4.27) show that during the phase of aerodynamic force predominant turning, the latitude ϕ and the longitude θ remain constant. The last equation gives the change in the heading ψ during that phase.

The composite expansions, valid everywhere, can be constructed by the method of additive composition. The additive composition is obtained by taking the sum of the inner and the outer expansions, Eqs. (4.27) and 4.18), and subtracting the part they have in common (the inner limit of the outer expansions or the outer limit of the inner expansions), Eqs. (4.23) or (4.24). Thus, for the variables u and γ, using subscript c for the composite solution,

$$\frac{u_c}{u_*} = - \frac{h}{(1 + h)} + \frac{\cos^2\tilde{\gamma}_0}{\cos^2\gamma_*} \exp[\frac{2}{\lambda}(\gamma_* - \tilde{\gamma}_0)]$$

$$(4.28)$$

$$\cos\gamma_c = \cos\gamma_* \sqrt{\frac{u_*}{2\cos^2\gamma_*(1 + h) + (u_* - 2\cos^2\gamma_*)(1 + h)^2}} + \lambda B e^{-\frac{h}{\varepsilon}}$$

For the angular variables α , Ω and I , we have

$$\alpha_c = \alpha_0 + \tilde{\alpha}_0 - C_3 - \cos^{-1}\left(\frac{C_1 - 1}{\sqrt{1 - C_1^2 C_2}}\right)$$

$$\Omega_c = \Omega_0 + \tilde{\Omega}_0 - C_4 \tag{4.29}$$

$$I_c = I_0 + \tilde{I}_0 - C_5$$

Hence, from equation (4.23), we have immediately $I_c = \tilde{I}_0$, and

$$\cos I_c = \cos\phi_* \cos\{\cos^{-1}\left(\frac{\cos I_*}{\cos\phi_*}\right) + \frac{\delta}{\lambda}\log[\tan\left(\frac{\Pi}{4} + \frac{\tilde{\gamma}_0}{2}\right)/\tan\left(\frac{\Pi}{4} + \frac{\gamma_*}{2}\right)]\} \tag{4.30}$$

For the angle $\Omega_c = \tilde{\Omega}_0$, we can use the second Eq. (4.12) to have

$$\Omega_c = \theta_* - \sin^{-1}\left[\frac{\tan\phi_*}{\tan I_c}\right] \tag{4.31}$$

where I_c is given by Eq. (4.30).

Finally, the angle α_c is given by

$$\alpha_c = \sin^{-1}\left[\frac{\sin\phi_*}{\sin I_c}\right] + \cos^{-1}\left[\frac{\cos\gamma_*}{\sqrt{u_*^2 + (1 - 2u_*)\cos^2\gamma_*}}\left(\frac{u_*}{1 + h} - 1\right)\right]$$
$$- \cos^{-1}\left[\frac{\cos\gamma_*(u_* - 1)}{\sqrt{u_*^2 + (1 - 2u_*)\cos^2\gamma_*}}\right] \tag{4.32}$$

The composite solutions are expressed explicitly in terms of the constants of integration u_* , γ_* , ϕ_* , θ_* and I_* . For the computation in terms of the independent variable h , the angle $\tilde{\gamma}_0$ is first calculated from the second Eq. (4.27). Subsequently, we have u_c , γ_c and I_c , and then Ω_c and α_c .

4.4 The Composite Solutions In Terms of the Initial Conditions

For the initial conditions to be satisfied identically, the five constants of integration C_j, or equivalently the five constants with subscript $*$, as defined by Eqs. (4.26), are to be evaluated by using the composite solutions. Let the conditions at h_i be

$$u = u_i \ , \ \gamma = \gamma_i \ , \ \alpha = \alpha_i \ , \ \Omega = \Omega_i \ , \ I = I_i \qquad (4.33)$$

Using these conditions in the composite solutions, the constants u_*, γ_*, ϕ_*, θ_*, and I_* are obtained upon solving a set of transcendental equations which can only be done numerically. Another obstacle arises when, as is a common practice, in order to reduce the number of prescribed initial values, one takes the initial $(\vec{r}_i \ , \vec{V}_i)$ plane as the reference OXY plane. In doing so we have $\alpha_i = 0$, and $I_i = 0$, but when $I = 0$, the longitude of the ascending node Ω is not defined as evidenced by Eq. (4.31). This singularity can always be avoided by rotating the OXY plane through a fixed and arbitrary angle, say 45° about the $\vec{r}_i$ axis. Then the initial condition at h_i is

$$u = u_i \ , \ \gamma = \gamma_i \ , \ \alpha_i = 0 \ , \ \Omega_i = 0 \ , \ I_i = 45° \qquad (4.34)$$

The equivalent condition for the variables θ, ϕ and ψ is

$$\theta_i = 0 \quad , \quad \phi_i = 0 \quad , \quad \psi_i = 45° \qquad (4.35)$$

Finally, it should be noted that the composite solutions for u and γ, Eqs. (4.28) remain unchanged for the planar case. For the planar case, with the motion along the equatorial plane, the variable α is the same as the longitude θ. The composite solution for θ can be seen to be

$$\theta = \cos^{-1}\left[\frac{\cos\gamma_*}{\sqrt{u_*^2 + (1 - 2u_*)\cos^2\gamma_*}}\left(\frac{u_*}{1 + h} - 1\right)\right] + w_* \qquad (4.36)$$

where w_* is a constant of integration. The three constants of integration u_* , γ_* and w_* in Eqs. (4.28) and (4.36) for the planar case are evaluated using the initial conditions u_i , γ_i , and θ_i .

5. CONCLUSIONS

For a spherical, nonrotating planet with a spherically symmetric, but otherwise arbitrary, atmosphere the exact equations for three-dimensional, aerodynamically affected flight have been derived. The equations are transformed using modified Chapman variables into a set suitable for analytic integration using asymptotic expansions.

The solutions by asymptotic expansion are obtained as an inner expansion in the aerodynamically dominated region, and an outer expansion in the gravitationally dominated region.

The inner and outer expansions are matched directly, and a composite solution, valid everywhere, is constructed by additive composition.

This method of directly matched asymptotic expansions has provided highly accurate and useful solutions to less general atmospheric trajectory equations, Ref. 7. In that work extensive numerical calculations demonstrated the accuracy compared with exact numerical solutions.

The behavior of the exact entry equations was examined in Ref. 5. The method of directly matching the inner and outer expansions for atmospheric entry trajectories was proven valid for some restricted problems in Refs. 8 - 10.

This report has accomplished the wedding of the exact atmospheric trajectory equations, using the modified Chapman variables, with the method of directly matched asymptotic expansions and provides an analytical solution which should prove to be a powerful tool for aerodynamic orbit calculations.

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