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NONLINEAR PROPAGATION OF A WAVE PACKET IN A IN A HARD-WALLED CIRCULAR DUCT

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ABSTRACT

The method of multiple scales is used to derive a nonlinear Schrödinger equation for the temporal and spatial modulation of the amplitudes and the phases of waves propagating in a hard-walled circular duct. This equation is used to show that monochromatic waves are stable and to determine the amplitude dependance of the cut-off frequencies.

Introduction

Nonlinear wave propagation in hard-walled ducts was investigated by Coppens[1], Pestorius and Blackstock[2], Maslen and Moore[3], Burns[4], Keller and Millman[5], and Keller[6]. Coppens[1] and Pestorius and Blackstock[2] determined the viscous and thermal dissipative effects on the nonlinear propagation of plane waves. Maslen and Moore[3] used the method of strained parameters (e.g., Sec. 3.1 of Ref.7) to analyze strong transverse waves in a circular cylinder. Burns[4] obtained a straightforward expansion limited to small axial distances because it contains secular terms. Keller and Millman[5] determined the nonlinear wavenumber shift of the symmetric dispersive modes by using the method of strained parameters. Keller[6] determined the amplitude dependence of the cut-off frequencies for the case studied by Keller and Millman[5].

The nonlinear effects of acoustic materials were investigated experimentally by Zorumski and Parrott[8] and Kurze and Allen[9] and analytically by Kurze and Allen[9], Ingard[10], Isakovich[11], and Nayfeh and Tsai[12,13]. Nayfeh and Tsai[14,15] analyzed the combined nonlinear effects of the gas and the acoustic material.

The purpose of the present paper is to analyze the nonlinear propagation of a wave packet by deriving a nonlinear Schrödinger equation governing the temporal and spatial modulations of the amplitudes and the phases.

I. PROBLEM FORMULATION

We consider finite-amplitude waves propagating in a hard-walled cylindrical duct. Although the analysis is valid for any non-rectangular duct cross section, we treat the case of a circular cross section so that we would be able to give an explicit solution. The fluid is assumed to be inviscid, irrotational, and initially quiescent with a uniform pressure p_0^* and a uniform density ρ_0^* so that its subsequent motion can be represented by a potential function.

We introduce a cylindrical coordinate system (r, θ, x) whose x axis coincides with the duct axis. Dimensionless variables are introduced by using the radius of the duct R^*, the ambient speed of sound c_0^*, and the ambient density ρ_0^* as reference quantities. Thus, we let $\vec{r} = \vec{r}^*/R^*$, $\vec{v} = \vec{v}^*/c_0^*$, $\rho = \rho^*/\rho_0^*$, $p = p^*/\rho_0^* c_0^{*2}$, and $t = t^* c_0^*/R^*$, where starred and unstarred quantities denote dimensional and dimensionless quantities, respectively, $\vec{r}$ is the position vector, $\vec{v}$ is the velocity vector, ρ is the gas density, p is the gas pressure, and t is the time. In terms of these dimensionless quantities, the equations describing the conservation of mass and momentum are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = - \nabla p \tag{2}$$

The pressure is related to the density by the isentropic relationship

$$p^*/p_0^* = (\rho^*/\rho_0^*)^{\gamma}$$

or in dimensionless quantities by

$$\gamma p = \rho^{\gamma} \tag{3}$$

where γ is the gas specific heat ratio. Since the duct walls are assumed to be rigid, the appropriate boundary condition is the vanishing of the normal velocity at the duct walls; that is,

$$v = 0 \text{ at } r = 1 \tag{4}$$

where v is the radial component of velocity.

Since the flow is assumed to be inviscid and irrotational, the velocity $\vec{v}$ is derivable from a potential function $\phi(\vec{r},t)$ according to

$$\vec{v} = \nabla\phi \tag{5}$$

Substituting for p and $\vec{v}$ from Eqs. 3 and 5 into Eq. 2, using the irrotationality of the gas, and integrating the resulting equation, we obtain

$$\rho^{\gamma-1} = 1 + (1-\gamma)\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right] \tag{6}$$

Eliminating ρ from Eqs. 3 and 6 gives

$$\gamma p = \left\{1 + (1-\gamma)\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]\right\}^{\gamma/(\gamma-1)} \tag{7}$$

Differentiating Eq. 6 with respect to t, eliminating ρ by using Eqs. 1, 5, and 6, and arranging, we obtain

$$\phi_{tt} - \nabla^2\phi = (1-\gamma)\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]\nabla^2\phi$$

$$- \frac{\partial}{\partial t}(\nabla\phi)^2 - \frac{1}{2}\nabla\phi\cdot\nabla(\nabla\phi)^2 \tag{8}$$

In terms of the potential function ϕ, the boundary condition 4 becomes

$$\phi_r = 0 \text{ at } r = 1 \tag{9}$$

To determine an approximate solution to Eq. 8 subject to the boundary condition 9, we use the method of multiple scales (e.g., chapter 6 of Ref. 7) and let

$$\phi(r,\theta,x,t) = \sum_{n=1}^{3} \varepsilon^n \phi_n(r,\theta,X_0,X_1,X_2, T_0,T_1,T_2) + O(\varepsilon^4) \tag{10}$$

where ε is a small but finite dimensionless parameter characterizing the amplitude of the wave and

$$X_n = \varepsilon^n x, \; T_n = \varepsilon^n t \tag{11}$$

Here, X_0 is a short scale characterizing the wavelength, X_1 and X_2 are long scales characterizing the amplitude and phase modulations with axial distance, T_0 is a short scale characterizing the frequency of the wave, and T_1 and T_2 are long scales characterizing the temporal amplitude and phase modulations. Using Eq. 11 and the chain rule, we express the temporal and axial derivatives as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X_0} + \varepsilon \frac{\partial}{\partial X_1} + \varepsilon^2 \frac{\partial}{\partial X_2} + \ldots \tag{12a}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \ldots \tag{12b}$$

Substituting Eqs. 10-12 into Eqs. 8 and 9 and equating coefficients of like powers of ε, we obtain

Order ε

$$\mathcal{L}(\phi_1) \equiv \left(\frac{\partial^2}{\partial T_0^2} - \nabla_0^2\right)\phi_1 = 0 \tag{13a}$$

$$\nabla_0^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial X_0^2} \tag{13b}$$

$$\partial\phi_1/\partial r = 0 \text{ at } r = 1 \tag{14}$$

Order ε^2

$$\mathcal{L}(\phi_2) = -2\frac{\partial^2\phi_1}{\partial T_0 \partial T_1} + 2\frac{\partial^2\phi_1}{\partial X_0 \partial X_1} + (1-\gamma)\frac{\partial\phi_1}{\partial T_0}\nabla_0^2\phi_1 - \frac{\partial}{\partial T_0}(\nabla_0\phi_1)^2 \tag{15}$$

$$\partial\phi_2/\partial r = 0 \text{ at } r = 1 \tag{16}$$

Order ε^3

$$\mathcal{L}(\phi_3) = -2\frac{\partial^2\phi_1}{\partial T_0\partial T_2} - \frac{\partial^2\phi_1}{\partial T_1^2} - 2\frac{\partial^2\phi_2}{\partial T_0\partial T_1} + 2\frac{\partial^2\phi_1}{\partial X_0\partial X_2}$$

$$+ \frac{\partial^2\phi_1}{\partial X_1^2} + 2\frac{\partial^2\phi_2}{\partial X_0\partial X_1} + (1-\gamma)\frac{\partial\phi_1}{\partial T_1}\nabla_0^2\phi_1$$

$$+ 2(1-\gamma)\frac{\partial\phi_1}{\partial T_0}\frac{\partial^2\phi_1}{\partial X_0\partial X_1} + (1-\gamma)\frac{\partial\phi_1}{\partial T_0}\nabla_0^2\phi_2$$

$$+ (1-\gamma)\frac{\partial\phi_2}{\partial T_0}\nabla_0^2\phi_1 + \frac{1}{2}(1-\gamma)(\nabla_0\phi_1)^2\nabla_0^2\phi_1 - \frac{\partial}{\partial T_1}(\nabla_0\phi_1)^2$$

$$-2\frac{\partial}{\partial T_0}\left(\frac{\partial\phi_1}{\partial X_0}\frac{\partial\phi_1}{\partial X_1}\right) - 2\frac{\partial}{\partial T_0}(\nabla_0\phi_1\cdot\nabla_0\phi_2) - \frac{1}{2}\nabla_0\phi_1\cdot\nabla_0(\nabla_0\phi_1)^2 \quad (17)$$

$$\partial\phi_3/\partial r = 0 \text{ at } r = 1 \quad (18)$$

II. SOLUTION

We take the solution of Eqs. 13 that is bounded at the axis in the form of a traveling wave packet centered at the frequency ω and the wavenumber k; that is, we let

$$\phi_1 = A(X_1,X_2,T_1,T_2)J_m(\kappa r)\exp(i\zeta) + cc \quad (19a)$$

$$\zeta = kX_0 - \omega T_0 + m\theta \quad (19b)$$

where J_m is Bessel's function of order m, cc stands for the complex conjugate of the preceeding terms, and ω and k satisfy the dispersion relationship

$$\omega^2 - k^2 = \kappa^2 \quad (20)$$

Substituting Eqs. 19 into Eq. 14, we have

$$J_m'(\kappa) = 0 \quad (21)$$

In what follows, we exclude the non-dispersive case corresponding to $\kappa = 0$ (i.e., plane waves). Note that the function A is still undetermined at this level of approximation; it is determined by invoking the so-called solvability condition in the second-and third-order problems.

Substituting for ϕ_1 from Eq. 19a into Eq. 15, we obtain

$$\mathcal{L}(\phi_2) = 2i(\omega \frac{\partial A}{\partial T_1} + k \frac{\partial A}{\partial X_1})J_m(\kappa r)\exp(i\zeta)$$

$$+ [2i\omega\kappa^2 J^2_{m+1}(\kappa r) - 4i\omega\kappa m \frac{1}{r} J_{m+1}(\kappa r)J_m(\kappa r)$$

$$- i\omega(\gamma\omega^2 - \kappa^2 + k^2) J^2_m(\kappa r)]A^2 \exp(2i\zeta) + cc \tag{22}$$

Since the homogeneous second-order problem consisting of Eqs. 16 and 22 is the same as the first-order problem and since the latter has a non-trivial solution, the inhomogeneous second-order problem has a solution if, and only if, a solvability condition is satisfied; this condition yields

$$\omega \frac{\partial A}{\partial T_1} + k \frac{\partial A}{\partial X_1} = 0 \tag{23}$$

With this solvability condition, the solution of the second-order problem is

$$\phi_2 = [\Gamma_1 J^2_m(\kappa r) + \Gamma_2 r J_m(\kappa r)J_{m+1}(\kappa r) + \Gamma_3 J_{2m}(2\kappa r)]A^2 \exp(2i\zeta) + cc \tag{24a}$$

where

$$\Gamma_1 = - i\omega[\frac{1}{2} m(\gamma + 1)\omega^2\kappa^{-2} + 1], \; \Gamma_2 = \frac{1}{2} i(\gamma + 1)\omega^3\kappa^{-1},$$

$$\Gamma_3 = - \frac{1}{2}\Gamma_2 J_m(\kappa) [J_{m+1}(\kappa) + J'_{m+1}(\kappa)] / \kappa J'_m(2\kappa) \tag{24b}$$

Substituting for ϕ_1 and ϕ_2 from Eqs. 19 and 24 into Eq. 17, we have

$$\mathcal{L}(\phi_3) = [2i(\omega \frac{\partial A}{\partial T_2} + k \frac{\partial A}{\partial X_2}) - \frac{\partial^2 A}{\partial T_1^2} + \frac{\partial^2 A}{\partial X_1^2}]J_m(\kappa r)\exp(i\zeta)$$

$$- A^2\bar{A}F(r)\exp(i\zeta) + cc + \text{harmonics other than } \exp(\pm i\zeta) \qquad (25)$$

where F(r) is given in Appendix A. Since the homogeneous third-order problem consisting of Eqs. 18 and 25 has a non-trivial solution, the corresponding inhomogeneous problem has a solution if, and only if, a solvability condition is satisfied. To determine this solvability condition, we seek a particular solution of the form

$$\phi_3 = \psi(r,X_1,X_2,T_1,T_2)\exp(i\zeta) \qquad (26)$$

Substituting this solution into Eqs. 18 and 25 and equating the coefficients of exn($i\zeta$) on both sides, we obtain

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + (\kappa^2 - \frac{m^2}{r^2})\psi = [2i(\omega \frac{\partial A}{\partial T_2} + k \frac{\partial A}{\partial X_2})$$

$$- \frac{\partial^2 A}{\partial T_1^2} + \frac{\partial^2 A}{\partial X_1^2}]J_m(\kappa r) - A^2\bar{A}\,F(r) \qquad (27)$$

$$\partial\psi/\partial r = 0 \text{ at } r = 1 \qquad (28)$$

Multiplying Eq. 27 by $rJ_m(\kappa r)$, integrating by parts from $r = 0$ to $r = 1$, and using Eq. 28, we obtain the solvability condition in the form

$$2i(\omega \frac{\partial A}{\partial T_2} + k \frac{\partial A}{\partial X_2}) - \frac{\partial^2 A}{\partial T_1^2} + \frac{\partial^2 A}{\partial X_1^2} = \Lambda\, A^2\bar{A} \qquad (29)$$

where

$$\Lambda = [\int_0^1 rF(r)J_m(\kappa r)dr]/[\int_0^1 rJ_m^2(\kappa r)dr] \qquad (30)$$

Eliminating $\partial^2 A/\partial X_1^2$ from Eq. 29 by using Eq. 23 gives

$$2i(\omega \frac{\partial A}{\partial T_2} + k \frac{\partial A}{\partial X_2}) + (\frac{\omega^2}{k^2} - 1)\frac{\partial^2 A}{\partial T_1^2} = \Lambda\, A^2\bar{A} \qquad (31)$$

To simplify Eq. 31, we differentiate the dispersion relationship 20 with respect to ω and obtain

$$kk' = \omega \tag{32}$$

where $k' = dk/d\omega$ the inverse of the group velocity. Differentiating Eq. 32 with respect to ω gives

$$kk'' = 1 - k'^2 = 1 - \omega^2/k^2 \tag{33}$$

Using Eqs. 32 and 33, letting $T_n = \varepsilon^n t$ and $X_2 = \varepsilon^2 x$, and arranging, we rewrite Eq. 31 as

$$\frac{\partial A}{\partial x} + k' \frac{\partial A}{\partial t} + \frac{1}{2} i k'' \frac{\partial^2 A}{\partial t^2} = - \frac{1}{2} i \varepsilon^2 \frac{\Lambda}{k} A^2 \bar{A} \tag{34}$$

Changing the independent variables from x and t to

$$\xi = t - k'x, \quad \eta = x \tag{35}$$

we express Eq. 34 in the form

$$\frac{\partial A}{\partial \eta} + \frac{1}{2} i k'' \frac{\partial^2 A}{\partial \xi^2} = - \frac{1}{2} i \varepsilon^2 \frac{\Lambda}{k} A^2 \bar{A} \tag{36}$$

which is a nonlinear Schrödinger equation. Letting $A = \frac{1}{2} a \exp(i\beta)$ with real a and β in Eq. 36 and separating real and imaginary parts, we obtain

$$\frac{\partial a}{\partial \eta} - k'' \left[\frac{\partial a}{\partial \xi} \frac{\partial \beta}{\partial \xi} + \frac{1}{2} a \frac{\partial^2 \beta}{\partial \xi^2}\right] = 0 \tag{37}$$

$$\frac{\partial \beta}{\partial \eta} + \frac{1}{2} k'' \left[\frac{1}{a} \frac{\partial^2 a}{\partial \xi^2} - \left(\frac{\partial \beta}{\partial \xi}\right)^2\right] = - \frac{1}{8} \varepsilon^2 \frac{\Lambda}{k} a^2 \tag{38}$$

The Case of Monochromatic Waves

For monochromatic waves, $\partial a/\partial \xi = \partial \beta/\partial \xi = 0$, and Eqs. 37 and 38 can be integrated to give

$$a = a_0 \text{ and } \beta = -\frac{1}{8}\varepsilon^2 \frac{\Lambda}{k} a_0^2 \eta + \beta_0 \tag{39}$$

where a_0 and β_0 are constants. Substituting for a and β from Eqs. 39 into Eqs. 19 and 24, we rewrite Eq. 10 in terms of the original variables as

$$\begin{aligned} \phi = {} & \varepsilon a J_m(\kappa r) \cos (\hat{k}x - \omega t + m\theta + \beta_0) \\ & + \frac{1}{2}\varepsilon^2 a^2 \left[\Gamma_1 J_m^2(\kappa r) + \Gamma_2 r J_m(\kappa r) J_{m+1}(\kappa r) + \Gamma_3 J_{2m}(2\kappa r) \right] \\ & \cos [\, 2(\hat{k}x - \omega t + m\theta + \beta_0) \,] + O(\varepsilon^3) \end{aligned} \tag{40}$$

where

$$\hat{k} = k - \frac{1}{8}\varepsilon^2 \Lambda k^{-1} a_0^2 \tag{41}$$

This result can be obtained as a special case from the solution of Nayfeh and Tsai[15] by letting the resistivity of the acoustic material to be infinite. Moreover, our solution reduces when $m = 0$ (i.e., symmetric modes) to that of Keller and Millman, which they obtained using the method of strained parameters.

Equation 41 shows that the nonlinearity of the gas results in a shift in the wavenumber from k to $\hat{k} = k - \frac{1}{8}\epsilon^2\Lambda k^{-1}a^2$. The numerical results of Nayfeh and Tsai[15] show that the nonlinearity shifts the wavenumber to lower values resulting in higher phase speeds. The wavenumber shift increases with increasing frequency and decreases with increasing azimuthal or radial mode number.

Equations 37 and 38 can be used to analyze the stability of the aforementioned monochromatic solution. To do this, we let

$$a = a_0 + a_1 \ , \ \beta = -\frac{1}{8}\epsilon^2\frac{\Lambda}{k}a_0^2\eta + \beta_0 + \beta_1 \tag{42}$$

where a_1 and β_1 are small compared with the preceeding terms. Substituting Eq. 42 into Eqs. 37 and 38 and neglecting the nonlinear terms in a_1 and β_1, we obtain

$$\frac{\partial a_1}{\partial \eta} - \frac{1}{2}k'' a_0 \frac{\partial^2 \beta_1}{\partial \xi^2} = 0 \tag{43}$$

$$\frac{\partial \beta_1}{\partial \eta} + \frac{1}{2}k'' \frac{1}{a_0}\frac{\partial^2 a_1}{\partial \xi^2} = -\frac{1}{4}\epsilon^2\frac{\Lambda}{k}a_0 a_1 \tag{44}$$

Since Eqs. 43 and 44 linear, we seek their solution in the form

$$a_1 = \tilde{a}_1 \exp[i(\tilde{k}\eta - \tilde{\omega}\xi)], \ \beta_1 = \tilde{\beta}_1 \exp[i(\tilde{k}\eta - \tilde{\omega}\xi)] \tag{45}$$

where $\tilde{a}_1$ and β_1 are constants. Substituting this solution into Eqs. 43 and 44 and eliminating $\tilde{a}_1$ and $\tilde{\beta}_1$, we obtain

$$\tilde{k}^2 = \frac{1}{4}k''^2\tilde{\omega}^2 \ (\tilde{\omega}^2 - \frac{1}{2}\epsilon^2\Lambda a_0^2/kk'') \tag{46}$$

which shows that, if $\Lambda/k''<0$, $\tilde{k}$ is always real for all values of $\tilde{\omega}$ so that the monochromatic waves given by Eqs. 40 and 41 are neutrally stable. On the other hand, if $\Lambda/k''>0$, $\tilde{k}^2$ is negative for all

$\tilde{\omega} < \epsilon a_0 \, (\Lambda/2kk'')^{1/2}$; consequently, disturbances grow exponentially with ξ and monochromatic waves are unstable. Since the numerical results of Nayfeh and Tsai[18] show that Λ is positive, monochromatic waves are stable only if $k'' < 0$. However, Eqs. 20 and 33 show that $k'' = -\kappa^2/k^3 < 0$. Therefore, monochromatic waves are stable.

Solution Near Cut-Off Frequencies

Although Eqs. 40 and 41 are valid for a wide range of frequencies, they break down as $k \to 0$ (i.e., near the linear cut-off frequencies) because the wavenumber shift approaches infinity. However, the basic equations 23 and 29 for the modulation of the amplitude and the phase with axial distance and time are valid for all frequencies. In this section, we specialize these equations to frequencies near the cut-off values. To do this, we use Eq. 23 to express $\partial A/\partial T_1$ as $-(k/\omega)\, \partial A/\partial X_1$, substitute this result into Eq. 29, and obtain

$$2i\omega \frac{\partial A}{\partial T_2} + 2ik \frac{\partial A}{\partial X_2} + \left(1 - \frac{k^2}{\omega^2}\right) \frac{\partial^2 A}{\partial X_1^2} = \Lambda A^2 \bar{A} \tag{47}$$

Letting $T_2 = \epsilon^2 t$ and $X_n = \epsilon^n x$ in Eq. 47, we rewrite it as

$$2i\omega \frac{\partial A}{\partial t} + 2ik \frac{\partial A}{\partial x} + \left(1 - \frac{k^2}{\omega^2}\right) \frac{\partial^2 A}{\partial x^2} = \epsilon^2 \Lambda A^2 \bar{A} \tag{48a}$$

$$\frac{\partial A}{\partial t} + \omega' \frac{\partial A}{\partial x} - \frac{1}{2} i\omega'' \frac{\partial^2 A}{\partial x^2} = -\frac{1}{2} i\Lambda\omega^{-1} A^2 \bar{A} \tag{48b}$$

For monochromatic waves, $\partial A/\partial t = 0$ and Eq. 48a becomes

$$2ik \frac{dA}{dx} + \left(1 - \frac{k^2}{\omega^2}\right) \frac{d^2 A}{dx^2} = \epsilon^2 \Lambda A^2 \bar{A} \tag{49}$$

which is valid for all frequencies away from zero.

Equation 49 has solutions of the form

$$A = \frac{1}{2}a \exp (i\beta) \tag{50}$$

where a is constant and

$$\frac{d\beta}{dx} = \{ - k + [k^2 - \frac{1}{4}\varepsilon^2 (1 - \frac{k^2}{\omega^2}) \Lambda a^2]^{1/2}\} \quad (1 - \frac{k^2}{\omega^2})^{-1} \tag{51}$$

Away from the cut-off frequencies, k is away from zero and the radical in Eq. 51 can be expanded for small ε yielding

$$\frac{d\beta}{dx} = - \frac{1}{8} \varepsilon^2\Lambda\ k^{-1}\ a^2 \tag{52}$$

in agreement with the monochromatic solution obtained above. On the other hand, when $k \to 0$ (i.e., near the cut-off frequencies), Eq. 51 tends to

$$\frac{d\beta}{dx} = - k + (k^2 - \frac{1}{4} \varepsilon^2\Lambda a^2)^{1/2} \tag{53}$$

Substituting for A from Eq. 50 into Eqs. 19 and 24, using Eq. 53, and letting $T_0 = t$ and $X_n = \varepsilon^n x$, we obtain Eq. 40; however, $\tilde{k}$ of Eq. 41 is modified to

$$\tilde{k} = (k^2 - \frac{1}{4} \varepsilon^2\Lambda a^2)^{1/2} \tag{54}$$

Therefore, the cut-off frequencies are solutions of

$$k^2 - \frac{1}{4} \varepsilon^2\Lambda a^2 = 0 \tag{55}$$

Since $k^2 = \omega^2 - \kappa^2$ according to Eq. 20, the cut-off frequencies are

$$\omega = \kappa + \frac{1}{8} \varepsilon^2 a^2 \Lambda\ (\kappa) + \ldots \tag{56}$$

where $\Lambda\ (\kappa)$ stands for the value of Λ when $\omega = \kappa$. These cut-off frequencies reduce when m = 0 to those obtained by Keller[6].

Using Eq. 48a, one can carry out a stability analysis and show that this modified solution is stable.

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Appendix A

$$F(r) = \kappa^2 J_m(\kappa r)\ J_m'^2(\kappa r) \left[\frac{1}{2}(\gamma-1)\ \omega^2 + \frac{m(5m-3)}{r^2} - 3\kappa^2 + 2k^2\right]$$

$$- \kappa\ J_m^2\ (\kappa r)\ J_m'\ (\kappa r) \left[\frac{m^2}{r^3} + 2i\omega\ \left(\frac{2m\Gamma_1}{r} + \Gamma_2\kappa r\right)\right]$$

$$+ \frac{3\kappa^2}{r} J_m'^2\ (\kappa r)\ J_{m+1}\ (\kappa r) - 2i\omega\kappa J_m'(\kappa r)$$

$$\left[- \Gamma_2\kappa r J_{m+1}^2\ (\kappa r) - 2\Gamma_1\kappa Jm(\kappa r)\ J_{m+1}(\kappa r) + \frac{2m}{r}\Gamma_3 J_{2m}(2\kappa r) - 2\Gamma_3\kappa J_{2m+1}(2\kappa r)\right]$$

$$- [\ (\gamma-1)\omega^2 + \frac{2m^2}{r^2} + 2k^2]\ \Big\{J_m^3(\kappa r)\ [\frac{m^2}{2r^2} + \frac{1}{2}k^2 + (\gamma-1)\omega^2 + 2i\omega\Gamma_1]$$

$$+ 2i\omega\ J_m(\kappa r)\ [\Gamma_2\gamma J_m(\kappa r)\ J_{m+1}(\kappa r) + \Gamma_3\ J_{2m}(2\kappa r)]\Big\}$$

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