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PINES METHOD FOR CIRCULAR AND ELLIPTICAL ORBITS

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1. SUMMARY

The Pines method for trajectory integration is derived from the equations of two-body motion. The concept of the perturbation derivative is explained, and a formal definition is presented. An algorithm for applying the Pines method to numerical integration of trajectories is included.

2. INTRODUCTION

The Pines method is a variation-of-parameters method in which initial conditions on a two-body orbit are treated as elements of the orbit. This is possible because specification of position and velocity at any time uniquely determines a two-body orbit. If at time t_0, the position and velocity are $\underline{R}_0$ and $\underline{V}_0$, at some later time, t, the position and velocity are $\underline{R}$ and $\underline{V}$. In the absence of perturbations, the two-body orbit determined by the vectors $\underline{R}$ and $\underline{V}$ is the same as the two-body orbit determined by the vectors $\underline{R}_0$ and $\underline{V}_0$. If perturbations are present, however, $\underline{R}_0$ and $\underline{V}_0$ must gradually change so that the two-body trajectory on which they lie will intersect the position $\underline{R}$ with velocity $\underline{V}$. $\underline{R}_0$ and $\underline{V}_0$ are the elements of the osculating orbit, and the rate at which they change is called the "perturbation derivative" of $\underline{R}_0$ and $\underline{V}_0$. If the perturbation derivatives of $\underline{R}_0$ and $\underline{V}_0$ are known, $\underline{R}_0$ and $\underline{V}_0$ can be calculated for any time by numerical integration. Once $\underline{R}_0$ and $\underline{V}_0$ for a given time are known, the instantaneous two-body orbit is known and the actual position and velocity ($\underline{R}$, $\underline{V}$) can be calculated from the formulas for two-body motion. This is the essence of the Pines method.

3. THEORY OF TWO-BODY MOTION

The formulas for two-body motion are well known and may be found in a number of references, including reference 1 (chs. 1 and 2) and various astronomy texts (such as ref. 2 (ch. V, p. 107)). For the convenience of the reader, however, derivations of the formulas introduced in this section have been included in the appendix.

The position and velocity vectors for some point on a two-body orbit are given by eqs. (96) in the appendix

$$\underline{R} = \left[1 - \frac{a}{r_0}(1 - \cos q)\right]\underline{R}_0 \tag{1}$$

$$+ \left(t - t_0 - \frac{q - \sin q}{\sqrt{\mu/a^3}}\right)\underline{V}_0$$

$$\underline{V} = -\frac{\sqrt{\mu a}}{rr_0}\sin q\,\underline{R}_0 + \left[1 - \frac{a}{r}(1 - \cos q)\right]\underline{V}_0 \tag{2}$$

where quantities with subscript zero are evaluated at time t_0, q is the difference in eccentric anomaly $(E - E_0)$, μ is the gravitational constant times the mass of the central body, a is the semimajor axis, and r is the distance from the central body. Given an initial position and velocity, we may find position and velocity at some other time by using equations (1) and (2). We obtain a by using eq. (101) in the appendix

$$a = 1/\left(\frac{2}{r_0} - \frac{v_0^2}{\mu}\right) \tag{3}$$

where

$$v_0 = \left(\underline{V}_0 \cdot \underline{V}_0\right)^{1/2} \tag{4}$$

and

$$r_0 = \left(\underline{R}_0 \cdot \underline{R}_0\right)^{1/2} \tag{5}$$

The quantity $q = E - E_0$ may be obtained by solving Kepler's equation (eq. (93) in the appendix) in the form

$$\sqrt{\frac{\mu}{a^3}}\left(t - t_0\right) = q + \frac{\underline{R}_0 \cdot \underline{V}_0}{\sqrt{\mu a}}(1 - \cos q) - \left(1 - \frac{r_0}{a}\right)\sin q \tag{6}$$

Eqs. (1) and (2) are valid whether we go forward or backward in time from the initial point. We could express position and velocity at the earlier time in terms of position and velocity at time t and let this earlier time be t_0. The result is identical to eqs. (1) and (2) except that subscripted and unsubscripted quantities are interchanged and q is replaced by $-q$.

Before we do this, let us introduce the scalar functions f and g given by

$$f = 1 - \frac{a}{r_0}(1 - \cos q) \tag{7}$$

and

$$g = t - t_0 - \frac{q - \sin q}{\sqrt{\mu/a^3}} \tag{8}$$

Differentiating eq. (7), we find

$$\dot{f} = \frac{df}{dt} = -\frac{a}{r_0}\sin q\,\frac{dE}{dt} \tag{9}$$

We are confining ourselves to two-body motion for the moment; therefore, a and r_0 are constants. From eq. (84) in the appendix, we have

$$\frac{dE}{dt} = \frac{1}{r}\sqrt{\frac{\mu}{a}} \tag{10}$$

So eq. (9) may be written as

$$\dot{f} = -\frac{\sqrt{\mu a}}{r r_0} \sin q \tag{11}$$

Differentiating eq. (8) yields

$$\dot{g} = 1 - \sqrt{\frac{a^3}{\mu}}(1 - \cos q)\frac{dE}{dt}$$

And by applying eq. (10), we find that

$$\dot{g} = 1 - \frac{a}{r}(1 - \cos q) \tag{12}$$

Using eqs. (11) and (12), we may rewrite eqs. (1) and (2) in the form

$$\left.\begin{aligned} \underline{R} &= f\underline{R}_0 + g\underline{V}_0 \\ \underline{V} &= \dot{f}\underline{R}_0 + \dot{g}\underline{V}_0 \end{aligned}\right\} \tag{13}$$

where, after collecting our formulas together, we have

$$f = 1 - \frac{a}{r_0}(1 - \cos q) \qquad \text{(from eq. (7))}$$

$$g = t - t_0 - \frac{q - \sin q}{\sqrt{\mu/a^3}} \qquad \text{(from eq. (8))}$$

$$\dot{f} = -\frac{\sqrt{\mu a}}{rr_0} \sin q \qquad \text{(from eq. (11))}$$

$$\dot{g} = 1 - \frac{a}{r}(1 - \cos q) \qquad \text{(from eq. (12))}$$

If we invert eqs. (1) and (2) to express $\underline{R}_0$ and $\underline{V}_0$ in terms of $\underline{R}$ and $\underline{V}$, we have

$$\underline{R}_0 = \left[1 - \frac{a}{r}(1 - \cos q)\right] \underline{R} - \left[\left(t - t_0\right) - \frac{q - \sin q}{\sqrt{\mu/a^3}}\right] \underline{V}$$

$$\underline{V}_0 = \frac{\sqrt{\mu a}}{rr_0} \sin q \underline{R} + \left[1 - \frac{a}{r_0}(1 - \cos q)\right] \underline{V}$$

or, comparing with eqs. (7), (8), (11), and (12), we find that

$$\left.\begin{aligned} \underline{R}_0 &= \dot{g}\underline{R} - g\underline{V} \\ \underline{V}_0 &= -\dot{f}\underline{R} + f\underline{V} \end{aligned}\right\} \qquad (14)$$

If we rewrite Kepler's equation (eq. (6)) in the same way, we have

$$\sqrt{\frac{\mu}{a^3}}\left(t - t_0\right) = q - \frac{\underline{R} \cdot \underline{V}}{\sqrt{\mu a}}(1 - \cos q) - \left(1 - \frac{r}{a}\right)\sin q \qquad (15)$$

Note that instead of $\underline{R}_0 \cdot \underline{V}_0$, we now have $\underline{R} \cdot V$; and instead of r_0, we have r. We can use either eq. (6) or eq. (15) to eliminate the time-dependent term in g. If we use eq. (6), we have

$$g = \frac{1}{\mu}\left[\underline{R}_0 \cdot \underline{V}_0 \, a(1 - \cos q) + r_0\sqrt{a\mu} \sin q\right] \tag{16}$$

and if we use eq. (15), we have

$$g = -\frac{1}{\mu}\left[\underline{R} \cdot \underline{V} a(1 - \cos q) - r\sqrt{a\mu} \sin q\right] \tag{17}$$

Either expression for g may be used - whichever is the most convenient.

From the equation of the orbit (eq. (83) in the appendix),

$$\begin{aligned} r &= a(1 - e \cos E) \\ &= a\left[1 - e \cos\left(E - E_0 + E_0\right)\right] \\ &= a\left(1 - e \cos E_0 \cos q + e \sin E_0 \sin q\right) \end{aligned} \tag{18}$$

From the appendix (eqs. (90) and (92)),

$$e \cos E_0 = 1 - \frac{r_0}{a} \tag{19}$$

$$e \sin E_0 = \left(\underline{R}_0 \cdot \underline{V}_0\right)/\sqrt{\mu a} \tag{20}$$

Substituting into eq. (18) yields

$$r = a(1 - \cos q) + r_0 \cos q + \underline{R}_0 \cdot \underline{V}_0 \sqrt{\frac{a}{\mu}} \sin q \tag{21}$$

Eq. (21) allows us to calculate r if we know only the initial conditions and q.

In the next section, we will need an expression for $\underline{R} \cdot \underline{V}$ in terms of the initial conditions. This is found by taking eq. (90) from the appendix in the form

$$\underline{R} \cdot \underline{V} = \sqrt{\mu a}\ e \sin E$$

and using the following expansion:

$$e \sin E = e \sin\left(E - E_0 + E_0\right)$$

$$= e \cos E_0 \sin q + e \sin E_0 \cos q$$

$$= \left(1 - \frac{r_0}{a}\right) \sin q + \frac{\underline{R}_0 \cdot \underline{V}_0}{\sqrt{\mu a}} \cos q$$

where we have used eqs. (19) and (20). With this result, we may write

$$\underline{R} \cdot \underline{V} = \sqrt{\mu a}\ \left(1 - \frac{r_0}{a}\right) \sin q + \underline{R}_0 \cdot \underline{V}_0 \cos q \tag{22}$$

The formulas presented in this section allow us to obtain the position and velocity of an object moving along an elliptical orbit provided that we have the initial conditions. We would proceed as follows: From the initial conditions $\underline{R}_0$ and $\underline{V}_0$, we can calcualte r_0, v_0, and $\underline{R}_0 \cdot \underline{V}_0$. Knowing r_0 and v_0, we can calculate a by using eq. (3). We can then solve Kepler's equation in the form of eq. (6) to find the value of q. Eq. (21)

can be used to find r, and we can then substitute the values of r_0, a, $\underline{R}_0 \cdot \underline{V}_0$, r, and q into the formulas for f and g and evaluate $\underline{R}$ and $\underline{V}$ by using eqs. (13).

4. THE PERTURBATION DERIVATIVE

In the presence of perturbations, the vectors $\underline{R}_0$ and $\underline{V}_0$ will change slowly in time. In order to calculate the rate of this change, we shall define a "perturbation derivative," which is the time rate of change of a quantity with respect to what it would have been in the absence of perturbations.

Let Q be some function of time along a real trajectory. If we expand Q in Taylor series about the time t, we have

$$Q(t + \Delta t) = Q(t) + \frac{dQ}{dt}\Delta t + \frac{1}{2}\frac{d^2Q}{dt^2}(\Delta t)^2 + \ldots$$

If the rate of change of Q along a two-body orbit at time t is written as Q_t, then the perturbation derivative is $Q_T = \frac{dQ}{dt} - Q_t$, or

$$\frac{dQ}{dt} = Q_t + Q_T$$

The T subscript specifies the perturbation derivative, and the t subscript specifies the derivative for the instantaneous two-body orbit with the same state vector.

To first order, then, we may write

$$Q(t + \Delta t) = Q(t) + Q_t\Delta t + Q_T\Delta t \tag{23}$$

The sum of the first two terms on the right is just the value that Q would have at $t + \Delta t$ if it were evaluated along the two-body orbit. We abbreviate this by

$$Q^*(t + \Delta t) = Q(t) + Q_t \Delta t \tag{24}$$

Solving eq. (23) for Q_T and taking the limit as Δt approaches zero, we have

$$Q_T = \lim_{\Delta t \to 0} \frac{Q(t + \Delta t) - Q^*(t + \Delta t)}{\Delta t} \tag{25}$$

Eq. (25) will be our formal definition of the perturbation derivative.

The perturbation derivatives of $\underline{R}$ and $\underline{V}$ can readily be calculated from our definition and the equation of motion for two-body motion, which is

$$\dot{\underline{V}} + \frac{\mu \underline{R}}{r^3} = \underline{F} \tag{26}$$

Expanding $\underline{R}$ in Taylor series yields

$$\underline{R}(t + \Delta t) = \underline{R}(t) + \underline{V}(t)\Delta t + \frac{1}{2}\dot{\underline{V}}(t)(\Delta t)^2 + \ldots \tag{27}$$

$V(t)$ is the instantaneous two-body velocity; therefore,

$$\underline{R}(t) + \underline{V}(t)\Delta t = \underline{R}^*(t + \Delta t) \tag{28}$$

To first order,

$$\underline{R}(t + \Delta t) = \underline{R}^*(t + \Delta t) \tag{29}$$

Therefore, applying eq. (25), we find that

$$\underline{R}_T = 0 \tag{30}$$

The equation of motion, eq. (26), is just the statement that

$$\frac{d\underline{V}}{dt} = -\frac{\mu \underline{R}}{r^3} + \underline{F} \tag{31}$$

The first term on the right is the two-body acceleration. Expanding $\underline{V}(t + \Delta t)$ about t yields, to first order,

$$\underline{V}(t + \Delta t) = \underline{V}(t) - \frac{\mu \underline{R}}{r^3} \Delta t + \underline{F} \Delta t \tag{32}$$

but the first two terms on the right are just $V^*(t + \Delta t)$. We have

$$V(t + \Delta t) = V^*(t + \Delta t) + \underline{F} \Delta t \tag{33}$$

Therefore, by our formal definition,

$$\underline{V}_T = \underline{F} \tag{34}$$

We have now obtained the important results

$$\underline{R}_T = 0 \quad \text{(from eq. (30))}$$

and

$$\underline{V}_T = \underline{F} \quad \text{(from eq. (34))}$$

These are valuable results because any quantity for which we might need to calculate the perturbation derivative can be expressed as a function of $\underline{R}$ and $\underline{V}$. Taking the perturbation derivative of eqs. (14) and using eqs. (30) and (34) results in

$$\left.\begin{aligned} (\underline{R}_0)_T &= \dot{g}_T\underline{R} - g_T\underline{V} - g\underline{F} \\ (\underline{V}_0)_T &= -\dot{f}_T\underline{R} + f_T\underline{V} + f\underline{F} \end{aligned}\right\} \tag{35}$$

Expressions for f_T, g_T, $\dot{f}_T$, and $\dot{g}_T$ are needed; but before they can be derived, it is necessary to resolve a certain ambiguity about what is meant by initial conditions.

One way of defining our initial conditions would be to say that t_0 is a fixed time and that $\underline{R}_0$ and $\underline{V}_0$ are the position and velocity at t_0 for a two-body orbit determined by $\underline{R}$ and $\underline{V}$ at time t. Under this assumption, $t - t_0$ is just the elapsed time for a two-body orbit and

$$(t - t_0)_T = 0 \tag{36}$$

This choice leads to the set of equations originally presented by Pines (ref. 3).

Another way of defining initial conditions would be in terms of the eccentric anomaly. If E_0 is fixed and $\underline{R}_0$ and $\underline{V}_0$ are the position and velocity on a two-body orbit determined by $\underline{R}$ and $\underline{V}$ when the eccentric anomaly is E, then $q = E - E_0$ is just the change in eccentric anomaly along a two-body orbit and

$$q_T = \left(E - E_0\right)_T = 0 \tag{37}$$

This choice yields the set of equations presented by Pines in ref. 4 and is the choice used in the following work.

5. EQUATIONS OF MOTION

To save writing, let

$$d_0 = \underline{R}_0 \cdot \underline{V}_0$$

$$d = \underline{R} \cdot \underline{V}$$

Eq. (22) may be written as

$$d = \sqrt{\mu a} \sin q\left(1 - \frac{r_0}{a}\right) + d_0 \cos q \tag{38}$$

since $d = \underline{R} \cdot \underline{V}$, $d_T = \underline{R}_T \cdot \underline{V} + \underline{R} \cdot \underline{V}_T$; or by using eqs. (30) and (34),

$$d_T = \underline{R} \cdot \underline{F} \tag{39}$$

The perturbation derivative of a can be calculated by using

$$a = \frac{1}{\frac{2}{r} - \frac{v^2}{\mu}} = \frac{1}{\frac{2}{(\underline{R}\cdot\underline{R})^{\frac{1}{2}}} - \frac{\underline{V}\cdot\underline{V}}{\mu}} \tag{40}$$

Taking the perturbation derivative of eq. (40) and using eqs. (30) and (34) yields

$$a_T = 2a^2\underline{V}\cdot\underline{F}/\mu \tag{41}$$

In a similar manner, all the perturbation derivatives of f, g, $\dot{f}$, and $\dot{g}$ can be calculated by using the assumption $q_T = 0$. Applying the perturbation derivative to eq. (7), we have

$$f_T = \left\{a(1 - \cos q)\left[\frac{(r_0)_T}{r_0} - \frac{a_T}{a}\right]\right\}/r_0 \tag{42}$$

Taking the perturbation derivative of eq. (17) yields

$$g_T = \left[\frac{r \sin q}{2\sqrt{a\mu}} - \frac{d(1 - \cos q)}{\mu}\right]a_T - \frac{a(1 - \cos q)}{\mu}d_T \tag{43}$$

Direct application of the perturbation derivative to eqs. (11) and (12) gives

$$\dot{f}_T = \dot{f}\left[\frac{a_T}{2a} - \frac{(r_0)_T}{r_0}\right] \tag{44}$$

and

$$\dot{g}_T = -\frac{1 - \cos q}{r} a_T \tag{45}$$

We now have expressions for everything in eqs. (42) through (45) except $\left(r_0\right)_T$. If we invert eq. (21), we have

$$r_0 = a(1 - \cos q) + r \cos q - d\sqrt{\frac{a}{\mu}} \sin q$$

and direct application of the perturbation derivative yields

$$\left(r_0\right)_T = a_T(1 - \cos q) - \sqrt{\frac{a}{\mu}}\left(d_T + d\,\frac{a_T}{2a}\right)\sin q \tag{46}$$

where we have used the fact that

$$r_T = \left[(\underline{R}\cdot\underline{R})^{\frac{1}{2}}\right]_T = \left(\underline{R}\cdot\underline{R}_T\right)/r = 0$$

We now have everything we need to calculate the derivatives of $\underline{R}_0$ and $\underline{V}_0$ for any value of q. We could obtain q by solving Kepler's equation; however, it is convenient to introduce a seventh variable of integration from which q may be calculated.

Let $W = q\sqrt{a}$. The total derivative of W is

$$\frac{dW}{dt} = \sqrt{a}\,\frac{dq}{dt} + \frac{q}{2\sqrt{a}}\frac{da}{dt}$$

From eq. (10),

$$\frac{dq}{dt} = \frac{dE}{dt} = \frac{1}{r}\sqrt{\frac{\mu}{a}}$$

Eq. (10) applies because the perturbation derivative of q is zero. The rate of change of q is the same as it would be for a two-body orbit; therefore, we can use eq. (10) to express it.

Since a is constant for a two-body orbit, the total derivative of a equals the perturbation derivative $\frac{da}{dt} = a_T$ and we have, using eq. (41),

$$\frac{dW}{dt} = \frac{\sqrt{\mu}}{r} + \frac{aW}{\mu}\,\underline{V}\cdot\underline{F} \tag{47}$$

We can integrate $\frac{dW}{dt}$, together with $\left(\underline{R}_0\right)_T$ and $\left(\underline{V}_0\right)_T$, at each step and calculate q by using

$$q = W/\sqrt{a} \tag{48}$$

Collecting together the differential equations which are to be integrated, we obtain the following set of equations of motion:

$$\frac{d}{dt}\,\underline{R}_0 = \dot{g}_T\underline{R} - g_T\underline{V} - g\underline{F}$$

$$\frac{d}{dt}\,\underline{V}_0 = -\dot{f}_T\,\underline{R} + f_T\underline{V} + f\underline{F} \qquad \text{(from eq. (35))}$$

$$\frac{dW}{dt} = \frac{\sqrt{\mu}}{r} + \frac{aW}{\mu}\,\underline{V}\cdot\underline{F} \tag{47}$$

where we have used the fact that the total derivative of the initial conditions is equal to the perturbation derivative.

We now have everything we need to integrate along the trajectory.

6. INTEGRATION ALGORITHM

STEP 1: Initialize variables

At the beginning of the integration, set

$$t_0 = t$$
$$W = 0$$
$$\underline{R}_0 = \underline{R}$$
$$\underline{V}_0 = \underline{V}$$

STEP 2: Function evaluation

The following order of evaluation of the variables may be used to calculate the derivatives of the initial conditions and the auxiliary variable W.

$$r_0 = \left(\underline{R}_0 \cdot \underline{R}_0\right)^{\frac{1}{2}}$$

$$v_0 = \left(\underline{V}_0 \cdot \underline{V}_0\right)^{\frac{1}{2}}$$

$$a = 1/\left(\frac{2}{r_0} - \frac{v_0^2}{\mu}\right)$$

$$q = W/\sqrt{a}$$

$$d_0 = \underline{R}_0 \cdot \underline{V}_0$$

$$f = 1 - a(1 - \cos q)/r_0$$

$$g = \frac{1}{\mu}\left[r_0\sqrt{\mu a}\,\sin q + d_0 a(1 - \cos q)\right]$$

$$r = a(1 - \cos q) + r_0 \cos q + d_0\sqrt{\frac{a}{\mu}}\sin q$$

$$\dot{f} = -\frac{\sqrt{\mu a}}{rr_0}\sin q$$

$$\dot{g} = 1 - \frac{a}{r}(1 - \cos q)$$

$$\underline{R} = f\underline{R}_0 + g\underline{V}_0$$

$$\underline{V} = \dot{f}\underline{R}_0 + \dot{g}\underline{V}_0$$

$\underline{F} = \underline{F}(\underline{R}, \underline{V})$ (Evaluate perturbations.)

$$d_T = \underline{R}\cdot\underline{F}$$

$$d = \sqrt{\mu a}\left(1 - \frac{r_0}{a}\right)\sin q + d_0 \cos q$$

$$a_T = \left(2a^2\underline{V}\cdot\underline{F}\right)/\mu$$

$$\left(r_0\right)_T = (1 - \cos q)a_T - \sqrt{\frac{a}{\mu}}\sin q\left(d_T + \frac{da_T}{2a}\right)$$

$$f_T = \frac{a}{r_0}(1 - \cos q)\left[\frac{\left(r_0\right)_T}{r_0} - \frac{a_T}{a}\right]$$

$$g_T = \left[\frac{r \sin q}{2\sqrt{a\mu}} - \frac{d(1 - \cos q)}{\mu}\right]a_T - \frac{a(1 - \cos q)}{\mu}d_T$$

$$\dot{f}_T = \dot{f}\left[\frac{a_T}{2a} - \frac{\left(r_0\right)_T}{r_0}\right]$$

$$\dot{g}_T = -\frac{(1 - \cos q)}{r}a_T$$

$$\frac{d}{dt}\underline{R}_0 = \dot{g}_T\underline{R} - g_T\underline{V} - g\underline{F}$$

$$\frac{d}{dt}\underline{V}_0 = -\dot{f}_T\underline{R} + f_T\underline{V} + f\underline{F}$$

$$\frac{dW}{dt} = \frac{\sqrt{\mu}}{r} + \frac{aW}{\mu}\,\underline{V}\cdot\underline{F}$$

Note that the total derivatives of $\underline{R}_0$ and $\underline{V}_0$ are just the perturbation derivatives because they are constants along the two-body orbit, whereas W does change along a two-body orbit, it being a function of eccentric anomaly.

STEP 3: Integration

After having obtained the needed derivatives, new approximations of $\underline{R}_0$, $\underline{V}_0$, and W may be obtained according to the integration scheme being used. We then return to step 2 for the next function evaluation and iterate.

7. USE OF INITIAL TIME AS AN ELEMENT

The form of the equations of motion as presented in section 5 have one peculiarity, the fact that all the quantities being integrated are slowly varying except the seventh variable W which varies rapidly along a conic. The variable W was introduced to provide a way of calculating the difference in eccentric anomaly and to avoid having to solve Kepler's equation. For near-circular orbits, however, the solution of Kepler's equation is not time consuming, therefore the use of an alternate form of the Pines method should be considered.

The value of q may be found by solving eq. (6). The left-hand side of eq. (6) is the mean motion

$$M = \sqrt{\frac{\mu}{a_3}}(t - t_0) \tag{48}$$

and eq. (6) may be rewritten in the form

$$M = q + \frac{d_0}{\sqrt{\mu a}} (1 - \cos q) - (1 - \frac{r_0}{a}) \sin q \tag{49}$$

A Newton-Raphson technique may be used to solve eq. (49). The initial estimate of q is

$$q_0 = M \tag{50}$$

and successive approximations are found according to the scheme

$$q_{n+1} = q_n + \frac{M - M_n}{\frac{d}{dq} M_n} \tag{51}$$

where

$$M_n = q_n + \frac{d_0}{\sqrt{\mu a}} (1 - \cos q_n) - (1 - \frac{r_0}{a}) \sin q_n \tag{52}$$

and

$$\frac{d}{dq} M_n = 1 + \frac{d_0}{\sqrt{\mu a}} \sin q_n - (1 - \frac{r_0}{a}) \cos q_n \tag{53}$$

For elliptical orbits, the eccentricity lies between 0 and 1. The Newton-Raphson solution of eq. (4) has been found to converge within five iterations for eccentricities between 0 and 0.5. Once q has been found, the Pines method proceeds as previously outlined, except that the differential equation for W is removed and replaced by a differential equation for t_0.

The initial time t_0 is not a constant because of our definition of initial conditions. Because our initial conditions were defined in terms of eccentric anomaly so that eq. (37) would be satisfied, eq. (36) is not satisfied, therefore t_0 is not fixed, but varies slowly as the other initial conditions.

A differential equation for t_0 may be most easily found from eq. (15) which can be rewritten as

$$t - t_0 = \sqrt{\frac{a^3}{\mu}}\, q - \frac{a}{\mu}\, d(1 - \cos q) - \sqrt{\frac{a^3}{\mu}}\,(1 - \frac{r}{a}) \sin q \tag{54}$$

The total derivative of $t - t_0$ is obviously

$$\frac{d}{dt}(t - t_0) = 1 - \frac{dt_0}{dt} \tag{55}$$

but it is also true that

$$\frac{d}{dt}(t - t_0) = (t - t_0)_t + (t - t_0)_T \tag{56}$$

where, as before, the t subscript means the time rate of change of the quantity when initial conditions (in this case, t_0) are held constant and the T subscript indicates the perturbation derivative.

Now

$$(t - t_0)_t = 1$$

therefore eq. (56) is equivalent to

$$\frac{d}{dt}(t - t_0) = 1 + (t - t_0)_T \tag{57}$$

Using eqs. (57) and (55), we find that

$$\frac{dt_0}{dt} = - (t - t_0)_T \tag{58}$$

which means that the time rate of change of t_0 is the negative of the <u>perturbation</u> derivative of the right-hand side of eq. (54).

$$\frac{dt_0}{dt} = - \sqrt{\frac{a}{\mu}} \left\{ \frac{1}{2}\left[3(q - \sin q) + r \sin q/a - \frac{2d}{\sqrt{\mu a}} (1 - \cos q)\right] a_T - \sqrt{\frac{a}{\mu}} (1 - \cos q)\, d_T \right\} \tag{59}$$

where a_T and d_T are the perturbation derivatives of a and d (given by eqs. (41) and (39), respectively). Eq. (59) replaces eq. (47) and the seventh variable is now t_0 instead of W.

If the Pines method is set up in the form described in this section, the technique will provide an exact solution to the two-body problem. When no perturbations are present, all perturbation derivatives are zero and a state vector may be propagated ahead to any time to the limit of machine accuracy in a single step.

8. CONCLUSIONS

The Pines method for circular and elliptical orbits outlined in the previous sections provides a more stable and accurate method for integrating the equations of motion than does the Cowell method. The numerical accuracy of this method is not as great as that of the KS method (ref. 5), but this method is easier to program, requires less storage, and has a shorter cycle time. Only seven simultaneous differential equations must be integrated with the Pines method, whereas the KS method requires 10. Time is the independent variable in the Pines method, whereas time is an element in the KS method; thus, the Pines method is considerably simpler to program in typical applications. Test runs with this method using an eighth-order potential and including perturbations due to the sun, the moon, and drag have shown the error to be 11.5 meters after 2 days when a time step of 1/50 revolution for a Shuttle-type orbit is used.

APPENDIX

TWO-BODY MOTION

CONSERVATION OF ANGULAR MOMENTUM

The equations of motion for a point mass in an inverse square attractive force field are, in vector form,

$$\underline{\ddot{R}} + \frac{\mu}{r^3}\underline{R} = 0 \tag{60}$$

Forming the cross-product of $\underline{R}$ with eq. (60), we have

$$\underline{R} \times \underline{\ddot{R}} = 0$$

which is equivalent to

$$\frac{d}{dt}(\underline{R} \times \underline{\dot{R}}) = 0$$

from which we conclude that

$$\underline{R} \times \underline{\dot{R}} = \underline{h} \tag{61}$$

where $\underline{h}$ is a constant vector. It can be seen, by forming the dot product of $\underline{R}$ with eq. (61), that $\underline{R}\cdot\underline{h} = 0$; consequently, the motion of the orbiting body is confined to a plane that is perpendicular to $\underline{h}$ and that contains the center of attraction. Since this is true, it is convenient to choose a coordinate system in which the x and y axes are in the plane of motion and in which the z-axis is parallel to $\underline{h}$. Writing out the equations of motion in scalar form yields

$$\left.\begin{aligned} \ddot{x} + \frac{\mu}{r^3} x = 0 \\ \\ \ddot{y} + \frac{\mu}{r^3} y = 0 \end{aligned}\right\} \tag{62}$$

Let r, θ be polar coordinates such that

$$\left.\begin{aligned} x = r \cos \theta \\ y = r \cos \theta \end{aligned}\right\} \tag{63}$$

Differentiating twice and substituting in eq. (62) yields

$$\ddot{r} \cos \theta - r\ddot{\theta} \sin \theta - r\dot{\theta}^2 \cos \theta - 2\dot{r}\dot{\theta} \sin \theta + \frac{\mu}{r^2} \cos \theta = 0 \tag{64a}$$

and

$$\ddot{r} \sin \theta - r\ddot{\theta} \cos \theta - r\dot{\theta}^2 \sin \theta + 2\dot{r}\dot{\theta} \cos \theta + \frac{\mu}{r^2} \sin \theta = 0 \tag{64b}$$

Multiplying eq. (64a) by $\cos \theta$, multiplying eq. (64b) by $\sin \theta$, and adding yields

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0 \tag{65a}$$

Multiplying eq. (64a) by $\sin \theta$, multiplying eq. (64b) by $\cos \theta$, and subtracting yields

$$-r\ddot{\theta} - 2\dot{r}\dot{\theta} = 0$$

or

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{65b}$$

Eq. (65b) implies that $r^2\dot{\theta}$ is a constant; and if we compare with eq. (61) written in polar coordinates, we see that the constant is the magnitude of $\underline{h}$. Therefore,

$$r^2\dot{\theta} = h \tag{66}$$

Using eqs. (65a) and (66), we find the equation of the orbit by making the change of variable $r = 1/u$ and using eq. (66) to eliminate the time derivative. Set $r = 1/u$. Then, differentiating with respect to time yields

$$\dot{r} = -\frac{1}{u^2}\dot{u} = \frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} \tag{67}$$

From eq. (66), however,

$$\dot{\theta} = h/r^2 = hu^2$$

Therefore,

$$\dot{r} = -h\frac{du}{d\theta} \tag{68}$$

Differentiating with respect to time again yields

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2} \tag{69}$$

Substituting for $\ddot{r}$ and $\dot{\theta}$ in eq. (65a) yields

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{70}$$

The solution of this differential equation is easily found and can be written in the form

$$u = \frac{\mu}{h^2}\left[1 + e\cos(\theta - \omega)\right] \tag{71}$$

where e and ω are the constants of integration. Replacing u by $1/r$ yields the equation of the orbit in polar coordinates:

$$r = \frac{h^2/\mu}{1 + e\cos(\theta - \omega)} \tag{72}$$

The constant e is called the eccentricity, and ω is the longitude (or argument) of periapsis. When $\theta = \omega$, r has a minimum value. Let $P = h^2/\mu$, where P is called the parameter (or semilatus rectum). If the coordinate axes are rotated so that the x-axis is along the line of periapsis (line of apsides) and we define $\nu = \theta - \omega$, where ν is called the true anomaly, then

$$\left.\begin{aligned} x &= r\cos\nu \\ y &= r\sin\nu \end{aligned}\right\} \tag{73}$$

Eq. (72) can be written as

$$r + er\cos\nu = P$$

or

$$\sqrt{x^2 + y^2} + ex = P$$

$$\sqrt{x^2 + y^2} = P - ex \tag{74}$$

Squaring eq. (74), we have

$$x^2 + y^2 = P^2 - 2ePx + e^2x^2 \tag{75}$$

If we collect the terms containing x together and complete the square, we have

$$\left(x + \frac{eP}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{P^2}{\left(1 - e^2\right)^2} \tag{76}$$

which can be put in the form

$$\frac{(x + c)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{77}$$

where

$$\left.\begin{aligned} a &= \frac{P}{1 - e^2} \\ b &= \frac{P}{\sqrt{1 - e^2}} \\ c &= \frac{eP}{1 - e^2} = ea \end{aligned}\right\} \tag{78}$$

provided that $e < 1$. If $e < 1$, eq. (77) is the equation of an ellipse with the right-hand focus at the origin of the coordinate system and the other focus on the $-x$ axis. The geometric center of the ellipse will be at the point $(-c, 0)$, as can be seen in figure 1. If we draw a circle of radius a about the center of the ellipse and then draw a line through the point (x, y) parallel to the y-axis to intersect the circle at point P, the angle subtended by the arc of the circle from point P to where the circle cuts the x-axis is called the eccentric anomaly, E. By inspection of figure 1, we see that

$$x = a \cos E - c = a(\cos E - e) \tag{79}$$

where we have used the fact that $c = ae$ from eqs. (78). By substituting with eq. (79) into eq. (76) and replacing c by ae and b by $P/\sqrt{1 - e^2}$, we obtain

$$y = \frac{P}{\sqrt{1 - e^2}} \sin E = \sqrt{aP} \sin E \tag{80}$$

So, we may write

$$\underline{R} = a(\cos E - e)\hat{x} + \sqrt{aP} \sin E \hat{y} \tag{81a}$$

and, by differentiating,

$$\underline{V} = -a\dot{E} \sin E \hat{x} + \sqrt{aP} \dot{E} \cos E \hat{y} \tag{81b}$$

where $\hat{x}$ and $\hat{y}$ are the unit vectors along the x and y axes.

Using eqs. (81a), (81b), and (61), we find that

$$\underline{h} = h\hat{z} = (a^3P)^{\frac{1}{2}}(1 - e \cos E)\dot{E}\hat{z}$$

Therefore,

$$\dot{E} = \frac{h}{(a^3P)^{\frac{1}{2}}(1 - e \cos E)}$$

But

$$P = \frac{h^2}{\mu}$$

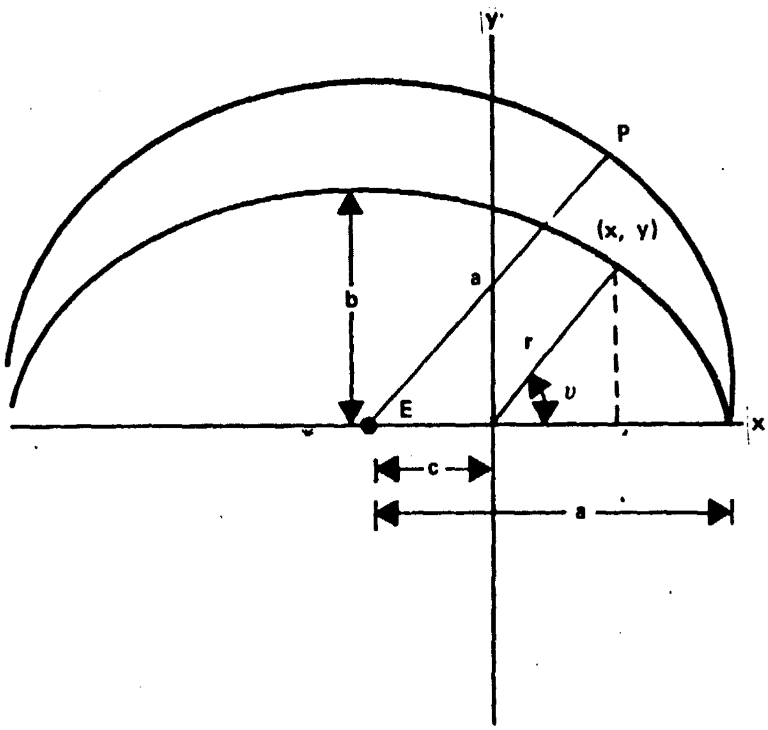

Figure 1.- Elements of the orbit.

So

$$\dot{E} = \sqrt{\frac{\mu}{a}}\ \frac{1}{a(1 - e \cos E)} \tag{82}$$

Now $r = \left(x^2 + y^2\right)^{\frac{1}{2}}$; so if we use eqs. (79), (80), and (78), we find that

$$r = a(1 - e \cos E) \tag{83}$$

Therefore,

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}} \tag{84}$$

Substituting back into eqs. (81a) and (81b), we have

$$\left.\begin{aligned} \underline{R} &= a(\cos E - e)\hat{x} + \sqrt{aP}\sin E\ \hat{y} \\ \underline{V} &= -\frac{1}{r}\sqrt{a\mu}\sin E\ \hat{x} + \frac{1}{r}\sqrt{\mu P}\cos E\ \hat{y} \end{aligned}\right\} \tag{85}$$

By solving eq. (85) straightforwardly with the use of Kramer's rule, we obtain

$$\left.\begin{aligned} \hat{x} &= \frac{\cos E}{r}\underline{R} - \sqrt{\frac{a}{\mu}}\sin E\ \underline{V} \\ \hat{y} &= \sqrt{\frac{a}{P}}\frac{\sin E}{r}\underline{R} + \frac{a}{\sqrt{\mu P}}(\cos E - e)\underline{V} \end{aligned}\right\} \tag{86}$$

KEPLER'S LAW

We need an equation for E, the eccentric anomaly. Using eqs. (84) and (83), we find that

$$(1 - e \cos E) \frac{dE}{dt} = \sqrt{\frac{\mu}{a^3}} \tag{87}$$

which is easily integrated and yields

$$\sqrt{\frac{\mu}{a^3}} \left(t - t_0\right) = E - E_0 - e \left(\sin E - \sin E_0\right) \tag{88}$$

which is one form of Kepler's equation, where E is the eccentric anomaly at time t and E_0 is the eccentric anomaly at t_0.

We would like to get eqs. (85) and (88) into a form such that eccentric anomaly only appears in the form $E - E_0$. We can write

$$\begin{aligned} \sin E &= \sin\left(E - E_0 + E_0\right) \\ &= \cos E_0 \sin\left(E - E_0\right) + \sin E_0 \cos\left(E - E_0\right) \end{aligned}$$

and put eq. (88) in the form

$$\sqrt{\frac{\mu}{a^3}} \left(t - t_0\right) = E - E_0 + e \sin E_0 \left[1 - \cos\left(E - E_0\right)\right] - e \cos E_0 \sin\left(E - E_0\right) \tag{78}$$

We can use eq. (85) to calculate $\underline{R} \cdot \underline{V}$ If we replace P by $a(1 - e^2)$ (eqs. (78)) and use eq. (83), we find that

$$\frac{\underline{R} \cdot \underline{V}}{\sqrt{\mu a}} = e \sin E \tag{90}$$

Evaluating eq. (90) at time $t = t_0$ gives

$$\frac{\underline{R}_0 \cdot \underline{V}_0}{\sqrt{\mu a}} = e \sin E_0 \tag{91}$$

Evaluating eq. (83) at t_0 and solving for $e \cos E_0$ yields

$$\left(1 - \frac{r_0}{a}\right) = e \cos E_0 \tag{92}$$

Eqs. (91) and (92) can be used to put Kepler's law in the desired form by substitution into eq. (89).

$$\sqrt{\frac{\mu}{a^3}}\left(t - t_0\right) = E - E_0 + \frac{\underline{R}_0 \cdot \underline{V}_0}{\sqrt{\mu a}}\left[1 - \cos\left(E - E_0\right)\right]$$
$$-\left(1 - \frac{r_0}{a}\right)\sin\left(E - E_0\right) \tag{93}$$

POSITION AND VELOCITY FROM
INITIAL POSITION AND VELOCITY VECTORS

Eqs. (86) expressed the unit vectors parallel to the semimajor and semiminor axes in terms of position and velocity vectors. Since the unit vectors are constant vectors, the relations (eqs. (86)) hold for any time, t_0.

$$\left.\begin{aligned} \hat{x} &= \frac{\cos E_0}{r_0}\underline{R}_0 - \sqrt{\frac{a}{\mu}}\sin E_0 \underline{V}_0 \\ \hat{y} &= \sqrt{\frac{a}{P}}\frac{\sin E_0}{r_0}\underline{R}_0 + \frac{a}{\sqrt{\mu P}}\left(\cos E_0 - e\right)\underline{V}_0 \end{aligned}\right\} \tag{94}$$

Direct substitution with eqs. (94) into eqs. (85) yields

$$\left.\begin{aligned}
\underline{R} &= \left\{\frac{a}{r_0}\left[\cos\left(E - E_0\right) - e\cos E_0\right]\right\}\underline{R}_0 \\
&\quad + \left\{\sqrt{\frac{a^3}{\mu}}\left[\sin\left(E - E_0\right) + e\sin E_0 - e\sin E\right]\right\}\underline{V}_0 \\
\underline{V} &= -\frac{\sqrt{a\mu}}{rr_0}\sin\left(E - E_0\right)\underline{R}_0 + \left\{\frac{a}{r}\left[\cos\left(E - E_0\right) - e\cos E\right]\right\}\underline{V}_0
\end{aligned}\right\} \quad (95)$$

Eqs. (95) have two undesirable features: They contain terms in eccentric anomaly that do not depend only on $E - E_0$, and the eccentricity appears. Using eqs. (83) and (92), we can get rid of the terms $e\cos E_0$ and $e\cos E$. From Kepler's law (eq. (88)), we can write that

$$-\sqrt{\frac{a^3}{\mu}}\,e\left(\sin E - \sin E_0\right) = t - t_0 - \sqrt{\frac{a^3}{\mu}}\left(E - E_0\right)$$

and use this expression to eliminate the terms $e\sin E$ and $e\sin E_0$. The result is

$$\left.\begin{aligned}
\underline{R} &= \left\{1 - \frac{a}{r_0}\left[1 - \cos\left(E - E_0\right)\right]\right\}\underline{R}_0 \\
&\quad + \left\{t - t_0 - \sqrt{\frac{a^3}{\mu}}\left[\left(E - E_0\right) - \sin\left(E - E_0\right)\right]\right\}\underline{V}_0 \\
\underline{V} &= -\frac{\sqrt{a\mu}}{rr_0}\sin\left(E - E_0\right)\underline{R}_0 + \left\{1 - \frac{a}{r}\left[1 - \cos\left(E - E_0\right)\right]\right\}\underline{V}_0
\end{aligned}\right\} \quad (96)$$

Eqs. (96) are the usual form for the f and g expansion for two-body motion.

CONSERVATION OF ENERGY

Up to this point, we have everything we need to describe two-body motion except a formula for a. In order to obtain such a formula, we need to derive the conservation-of-energy formula. This is done by dotting equation (60) with $\underline{V}$ and using $\underline{\ddot{R}} = \underline{\dot{V}}$.

$$\underline{V}\cdot\underline{\dot{V}} + \frac{\mu}{r^3}\,\underline{R}\cdot\underline{V} = 0 \tag{97}$$

Noting that $r = (\underline{R}\cdot\underline{R})^{\frac{1}{2}}$, we can put eq. (97) in the form

$$\frac{d}{dt}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right) - \mu\,\frac{d}{dt}(\underline{R}\cdot\underline{R})^{-\frac{1}{2}} = 0$$

Therefore,

$$\frac{v^2}{2} - \frac{\mu}{r} = K \tag{98}$$

where K is some constant. We can determine the value of K by evaluating eq. (98) at any point on the orbit. Let $E = 0$ and use eqs. (85) to get v and r. We find that

$$v_0 = \frac{1}{r_0}\sqrt{\mu P}$$

$$r_0 = a(1 - e)$$

And substituting into eq. (98) yields

$$\frac{\mu P}{2a^2(1 - e)^2} - \frac{\mu}{a(1 - e)} = K \tag{99}$$

But $P = a(1 - e^2)$ (eqs. (78)), so we have

$$\frac{\mu}{a}\left[\frac{1 - e^2}{2(1 - e)^2} - \frac{1}{1 - e}\right] = K$$

which reduces to $K = -\frac{\mu}{2a}$. Eq. (98) can now be written as

$$\frac{v^2}{2} = \mu\left(\frac{1}{r} - \frac{1}{2a}\right) \tag{100}$$

which is the familiar <u>Vis</u> <u>Viva</u> integral. From eq. (100), an expression for a is obtained.

$$a = 1/\left(\frac{2}{r} - \frac{v^2}{\mu}\right) \tag{101}$$

COLLECTION OF FORMULAS

The results from this appendix that we shall need are as follows: eqs. (96),

$$\underline{R} = \left\{1 - \frac{a}{r_0}\left[1 - \cos\left(E - E_0\right)\right]\right\}\underline{R}_0$$
$$+ \left\{t - t_0 - \sqrt{\frac{a^3}{\mu}}\left[\left(E - E_0\right) - \sin\left(E - E_0\right)\right]\right\}\underline{V}_0$$

$$\underline{V} = -\frac{\sqrt{a\mu}}{rr_0}\sin\left(E - E_0\right)\underline{R}_0 + \left\{1 - \frac{a}{r}\left[1 - \cos\left(E - E_0\right)\right]\right\}\underline{V}_0$$

Kepler's Law in the form of eq. (93),

$$\sqrt{\frac{\mu}{a^3}}\left(t - t_0\right) = \left(E - E_0\right) + \frac{\underline{R}_0 \cdot \underline{V}_0}{\sqrt{\mu a}}\left[1 - \cos\left(E - E_0\right)\right]$$

$$- 1\left(- \frac{r_0}{a}\right)\sin\left(E - E_0\right)$$

and the relations

$$\frac{\underline{R} \cdot \underline{V}}{\sqrt{\mu a}} = e \sin E \qquad \text{(eq. (90))}$$

$$\left(1 - \frac{r_0}{a}\right) = e \cos E_0 \qquad \text{(eq. (92))}$$

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}} \qquad \text{(eq. (84))}$$

$$a = 1/\left(\frac{2}{r} - \frac{v^2}{\mu}\right) \qquad \text{(eq. (101))}$$

$$r = a(1 - e \cos E) \qquad \text{(eq. (83))}$$

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