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THE GENERATION OF GRAVITATIONAL WAVES

II. THE POST-LINEAR FORMALISM REVISITED*

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ABSTRACT

Two different versions of the Green's function for the scalar wave equation in weakly curved spacetime (one due to DeWitt and DeWitt, the other to Thorne and Kovács) are compared and contrasted; and their mathematical equivalence is demonstrated. Then the DeWitt-DeWitt Green's function is used to construct several alternative versions of the Thorne-Kovács post-linear formalism for gravitational-wave generation. Finally it is shown that, in calculations of gravitational bremsstrahlung radiation, some of our versions of the post-linear formalism allow one to treat the interacting bodies as point masses, while others do not.

I. INTRODUCTION

In the first paper in this series Thorne and Kovács (1975) [cited henceforth as TK] developed a "plug-in-and-grind", "post-linear" formalism for calculating the gravitational radiation from accelerated systems with weak internal gravitational fields but arbitrarily large internal velocities. Their method is applicable to a class of problems which heretofore have not been amenable to analysis. The application to the gravitational bremsstrahlung problem is of particular importance. Central to the development of their formalism is the determination of an appropriate approximation to the well-known exact Green's function for the scalar wave equation in curved spacetime developed by DeWitt and Brehme (1960).

The purpose of this note is threefold: First, to show that an approximate form of the exact Green's function developed previously by DeWitt and DeWitt (1964) [cited henceforth as DD] has exactly the same mathematical content (within the constraints of the approximations used) as that constructed by TK, although the two representations differ significantly in mathematical form and physical interpretation, and have slightly different realms of validity. Second, to use the Green's function of DD in the formulas of TK to establish several alternative representations of the TK post-linear gravitational-wave-generation formalism. Third, to determine, in the case of the gravitational bremsstrahlung problem, which representations of the formalism permit one to treat the interacting objects as point masses rather than as extended bodies, and which do not.

In §II we review and contrast the DD method of approximation with that of TK. The equivalence of the two approximations is demonstrated in §III. In §IV the alternative representations of the "plug-in-and-grind" formalism are given. Section V discusses the point-mass approximation to the bremsstrahlung problem.

In the following we shall deal with three sets of 2-tensors, namely, those associated with the event pairs (x,x'), (x,x'') and (x'',x'). To avoid confusion we shall distinguish the various 2-tensors by bars and tildes as follows (cf. fig. 1):

$$A = A(x,x'), \quad \bar{A} = A(x,x'') \quad \text{and} \quad \tilde{A} = A(x'',x') \quad . \tag{1}$$

Further, we shall let the indices on 2-tensors designate the events to which they refer (e.g.: $A_{,\mu} \equiv \partial A/\partial x^{\mu}$, $A_{,\mu'} \equiv \partial A/\partial x^{\mu'}$ etc.). Finally, we shall use geometrized units ($c = G = 1$).

II. WEAK-FIELD GREEN'S FUNCTIONS

a) Foundations for the Analysis

The defining equation for the Green's function, corresponding to the scalar wave equation in curved spacetime, is given by

$$(-g)^{1/2}\, g^{\mu\nu}\, G_{;\mu\nu} = -\delta^4(x,x') \equiv -\delta^4 \ , \tag{2}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, semicolons indicate covariant differentiation, and $\delta^4(x,x')$ is the four-dimensional bi-density Dirac delta function

$$\delta^4 \equiv \delta^4(x,x') = \delta(x^0 - x^{0'})\ \delta(x^1 - x^{1'})\ \delta(x^2 - x^{2'})\ \delta(x^3 - x^{3'}) = \delta^4(x',x). \tag{3}$$

In the absence of the crossing of light-cone geodesics, the exact solution of equation (2) (due to DeWitt and Brehme [1960]) consists of a "direct part" and a "tail"; i.e.,

$$G(x,x') = G^{DIRECT} + G^{TAIL} \quad , \tag{4}$$

where G^{DIRECT} is nonvanishing on the future light cone of the event x' alone, and is given by

$$G^{DIRECT} = (4\pi)^{-1} \Delta^{1/2}(x,x') \, \delta[\Omega(x,x')] \quad . \tag{5}$$

Here $\Delta^{1/2}$ is the "scalarized Van Vleck determinant" (see DeWitt and Brehme), $\Omega(x,x')$ is the "world-function" of Synge (1960), and $\delta[\Omega]$ is the retarded Dirac delta function and is nonzero only when x' lies on the past light cone of the event x . (Throughout this paper δ will represent the <u>retarded</u> delta function. The appropriate temporal order of the events in Ω will always be $x' \prec x'' \prec x$ where $\prec$ means "in the causal past of"; cf. Figure 1.) G^{TAIL} is a nonlocal term which arises from back-scattering of the direct field by the curvature of spacetime. The necessity to construct approximate Green's functions is chiefly a consequence of the complexity of the tail term.

For physical situations in which the gravitational field is weak, it is possible to choose coordinate systems in which the metric can be written $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ and $|h_{\mu\nu}| << 1$. Under such circumstances, to first order, the Green's function can be written

$$_1G(x,x') = {_0G}(x,x') + \Delta G \ , \tag{6a}$$

where

$$_0G(x,x') = (4\pi)^{-1} \delta[{_0\Omega}(x,x')] \equiv (4\pi)^{-1}\delta \quad , \tag{6b}$$

$$\begin{aligned} {_0\Omega}(x,x') &= \frac{1}{2} \eta_{\mu\nu}(x^\mu - x^{\mu'})(x^\nu - x^{\nu'}) \\ &= \frac{1}{2} \eta_{\mu\nu} \, {_0\Omega}^{,\mu} \, {_0\Omega}^{,\nu} \quad , \end{aligned} \tag{6c}$$

and where ΔG is the lowest-order contribution to the flat-spacetime Green's function ${}_0G(x,x')$ due to the nonzero values of $h_{\mu\nu}$.

Throughout this paper the symbol δ , standing by itself, will denote the retarded Dirac delta function of ${}_0\Omega(x,x')$, and similarly for $\bar{\delta}$ and $\tilde{\delta}$:

$$\delta \equiv \delta({}_0\Omega) \ , \quad \bar{\delta} \equiv \delta({}_0\bar{\Omega}) \ , \quad \tilde{\delta} \equiv \delta({}_0\tilde{\Omega}) \quad , \tag{7a}$$

$${}_0\Omega \equiv {}_0\Omega(x,x') \ , \quad {}_0\bar{\Omega} \equiv {}_0\Omega(x,x'') \ , \quad {}_0\tilde{\Omega} \equiv {}_0\Omega(x'',x') \ . \tag{7b}$$

Note that δ , $\bar{\delta}$, and $\tilde{\delta}$ are essentially flat-space propagators, while δ^4, $\bar{\delta}^4$, and $\tilde{\delta}^4$ are 4-dimensional Dirac delta functions of position (eq. [3]). A prime on a δ , $\bar{\delta}$, or $\tilde{\delta}$ will always mean derivative with respect to its argument

$$\delta' \equiv \partial\delta/\partial_0\Omega \ , \quad \bar{\delta}' \equiv \partial\bar{\delta}/\partial_0\bar{\Omega} \ , \quad \tilde{\delta}' \equiv \partial\tilde{\delta}/\partial_0\tilde{\Omega} \quad . \tag{7c}$$

The spatial gradients of ${}_0\Omega$, ${}_0\bar{\Omega}$, ${}_0\tilde{\Omega}$ are flat-space vectors connecting x',x'', and x ; we shall denote them by capital X's:

$$\begin{aligned}
X^{\mu} &\equiv -X^{\mu'} \equiv {}_0\Omega^{,\mu} = -{}_0\Omega^{,\mu'} = (x^{\mu} - x^{\mu'}) \ , \\
\bar{X}^{\mu} &\equiv -\bar{X}^{\mu''} \equiv {}_0\bar{\Omega}^{,\mu} = -{}_0\bar{\Omega}^{,\mu''} = (x^{\mu} - x^{\mu''}) \ , \\
\tilde{X}^{\mu''} &\equiv -\tilde{X}^{\mu'} \equiv {}_0\tilde{\Omega}^{,\mu''} = -{}_0\tilde{\Omega}^{,\mu'} = (x^{\mu''} - x^{\mu'}) \quad .
\end{aligned} \tag{7d}$$

Note that

$$\delta_{,\mu} = \delta' X_{\mu} \ , \quad \delta_{,\mu'} = \delta' X_{\mu'} \ , \quad \tilde{\delta}_{,\mu'} = \tilde{\delta}' \tilde{X}_{\mu'} \ , \ \text{etc.}; \tag{7e}$$

$${}_0\Omega = \frac{1}{2}\eta^{\mu\nu} X_{\mu} X_{\nu} \ , \quad {}_0\bar{\Omega} = \frac{1}{2}\eta^{\mu\nu} \bar{X}_{\mu''} \bar{X}_{\nu''} \ , \ \text{etc.} \tag{7f}$$

Figure 1 may help one to remember the above conventions.

b) ${}_1G(x,x')$ of DeWitt and DeWitt

DeWitt and DeWitt (1964) derive a weak-field Green's function for the vector wave equation of curved-space electrodynamics. Here we sketch the obvious specialization of their derivation to the scalar wave equation.

The DeWitt-DeWitt analysis makes use of gauge invariance techniques developed by Schwinger (1951) to deal with Green's functions in quantum electrodynamics. Specifically, DD take $G(x,x')$ to be the matrix element of an abstract operator G in a fictitious Hilbert space:

$$G(x,x') = \langle x|G|x'\rangle \quad , \tag{8}$$

and they consider the defining differential equation (2) for G to be a matrix element of the operator equation

$$FG = -1 \text{ , with } F \equiv -p_\mu(-g)^{1/2}g^{\mu\nu}\, p_\nu \quad , \tag{9}$$

where the 1 is the identity operator in the Hilbert space, and the p_μ are Hermitian operators characterized by the commutation relations

$$[x^\mu,p_\nu] = i\delta^\mu{}_\nu \quad , \quad [p_\mu,p_\nu] = 0 \quad . \tag{10}$$

Taking the variation of equation (9) with respect to the metric tensor, one finds that:

$$\Delta G = G\,\Delta F\,G \tag{11}$$

where

$$\Delta F = \frac{1}{2}\, p_\mu(-g)^{1/2}[g^{\mu\sigma}g^{\nu\tau} + g^{\mu\tau}g^{\nu\sigma} - g^{\mu\nu}g^{\sigma\tau}]\,\Delta g_{\sigma\tau}p_\nu \quad . \tag{12}$$

By substituting equation (12) for ΔF into equation (11) and evaluating the matrix element, one obtains

$$\Delta G(x,x') = \int \frac{1}{2} G_{,\mu''}(x,x'')[g^{\mu''\sigma''}g^{\nu''\tau''} + g^{\mu''\tau''}g^{\nu''\sigma''} - g^{\mu''\nu''}g^{\sigma''\tau''}] \times$$

$$\times \quad \Delta g_{\sigma''\tau''} G_{,\nu''}(x'',x')(-g'')^{1/2} d^4x'' \quad . \tag{13}$$

The Green's function, to first order in $h_{\mu\nu}$, can now be obtained by making the substitutions

$$g^{\mu\nu} = \eta^{\mu\nu} \;, \quad \Delta g_{\mu\nu} = h_{\mu\nu} \quad \text{and} \quad G(x,x'') = {}_0G(x,x'')$$

in equation (13), and inserting the resulting ΔG into equation (6a). The result is:

$${}_1G_{DD}(x,x') = (4\pi)^{-1}\,\delta + (4\pi)^{-2}\int \bar{\delta}_{,\mu''}\,\bar{h}^{\mu''\nu''}\,\tilde{\delta}_{,\nu''}\,d^4x'' \quad , \tag{14}$$

where $\bar{h}^{\mu''\nu''}$ is the trace-reversed metric perturbation (a single-point function; <u>not</u> a bi-tensor)

$$\bar{h}^{\mu''\nu''} \equiv h^{\mu''\nu''} - \frac{1}{2}\eta^{\mu\nu}h'' \quad , \tag{15}$$

and where $\bar{\delta}$ and $\tilde{\delta}$ are defined by equations (7). This form of the first-order scalar Green's function is stated by DeWitt and DeWitt (1964) without proof.

c) ${}_1G(x,x')$ of Thorne and Kovács

The starting point of the Thorne-Kovács (1975) development of an approximate Green's function is equation (4). Using the explicit expression for G^{DIRECT}, TK insert equation (4) into the defining equation (2) to obtain a differential equation for the tail term:

$$g^{\mu\nu}G^{TAIL}{}_{;\mu\nu} = -(4\pi)^{-1}\, g^{\mu\nu}\Delta^{1/2}{}_{;\mu\nu}\,\delta(\Omega)\ . \tag{16}$$

Inverting equation (16) and combining with G^{DIRECT}, they obtain an alternative expression for the exact Green's function:

$$G(x,x') = (4\pi)^{-1}\Delta^{1/2}\,\delta(\Omega) + (4\pi)^{-1}\int \tilde{\Delta}^{1/2}{}_{;\mu''}{}^{;\mu''}\,\tilde{\delta}(\tilde{\Omega})G(x,x'')(-g'')^{1/2}d^4x''. \tag{17}$$

It is now possible to make a power series expansion of this equation to first order in $h_{\mu\nu}$. To this end, an expansion for the world function $\Omega(x,x')$ can be obtained by approximating the geodesic between the points x and x' by the "straight line"

$$\mathcal{C}_o(\lambda):\quad \xi^{\mu} = x^{\mu'} + \lambda X^{\mu}\ ,\quad \text{with}\quad 0 \le \lambda \le 1\ \ . \tag{18}$$

The errors introduced by such an approximation are of second order and thus do not affect first-order results. One finds (without imposing the de Donder gauge condition, or any other gauge condition, on $\bar{h}^{\mu\nu}$)[1]

[1] TK imposed the de Donder gauge condition, but the result is the same without it.

$$\Omega(x,x') = {}_0\Omega(x,x') + \gamma(x,x')\qquad , \tag{19a}$$

$$\gamma(x,x') \equiv \frac{1}{2}\ X^{\alpha}\ X^{\beta}\int_{\mathcal{C}_o} h_{\alpha\beta}\ d\lambda\qquad , \tag{19b}$$

$$\Delta(x,x')\ \equiv\ \frac{-\det\|\Omega_{,\alpha\beta'}\|}{|gg'|^{1/2}} = 1 + 2\alpha(x,x')\quad , \tag{19c}$$

$$\alpha(x,x') = \frac{1}{2}\ X^{\alpha}\, X^{\beta} \int_{\mathcal{C}_0} R_{\alpha\beta}\, \lambda(1-\lambda)\ d\lambda \quad . \tag{19d}$$

In equation (19d) $R_{\alpha\beta}$ is the Ricci tensor, accurate to first order in $h_{\mu\nu}$. Inserting equations (19) into equation (17) one obtains

$$ {}_1G_{TK}(x,x') = (4\pi)^{-1}(1+\alpha)\ \delta({}_0\Omega) + (4\pi)^{-1}\,\gamma\,\delta'({}_0\Omega)$$

$$+ (4\pi)^{-2} \int \tilde{\alpha}_{,\mu''}{}^{\mu''}\ \delta({}_0\overline{\Omega})\ \delta({}_0\tilde{\Omega})\ d^4x'' \quad . \tag{20}$$

This expression for ${}_1G(x,x')$ actually differs from that of Thorne and Kovåcs in two ways: First, to simplify computations when applying the formalism, TK have used a different but equivalent version of the tail term (the integral in equation [20]). However, in their Appendix C they prove that their different version is equivalent to the one given above.

Second, TK perform a renormalization (truncation) on the bi-scalar γ so as to make their Green's function and gravitational radiation formulas valid for field points very far away from the source. (The method of DD cannot reveal the breakdown in the formulas at large distances.) To effect a comparison of the two Green's functions we shall deal with points x and x' near enough to each other so that both Green's functions are valid, and that, consequently, the TK truncation is not needed; cf. Appendix C of TK.

It should be pointed out that the Green's functions of DD and TK have both been developed in a gauge-invariant manner (cf. footnote 1). Of the two representations, that of DD has the advantage of formal simplicity, wh'`e that of TK is superior heuristically in that it is a sum of four ter. , each of which is easily understood physically. Also, it appears to us (cf. Kovàcs and Thorne 1976) that the DD form is superior for proving theorems, but the TK form is superior for practical computations involving bodies separated by distances large compared to their size.

III. EQUIVALENCE OF ${}_1G_{DD}$ AND ${}_1G_{TK}$

Assuming, of course, that there are no errors in the derivations sketched above, then in their common domain of validity ${}_1G_{DD}$ and ${}_1G_{TK}$ must be equivalent. In this section we shall present a direct proof of their equivalence. This is important because it demonstrates explicitly the relationship between ${}_1G_{DD}$ and ${}_1G_{TK}$, and thereby helps explain why the two formalisms are powerful for very different aspects of bremsstrahlung calculations (Kovàcs and Thorne 1976).

By inserting into expression (14) for ${}_1G_{DD}$ the identity (see Appendix)

$$\bar{\delta}_{,\mu''}\bar{h}^{\mu''\nu''}\tilde{\delta}_{,\nu''} = \bar{\delta}\tilde{\alpha}_{,\mu''}{}^{\mu''}\,\tilde{\delta} + 4\pi(\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta})\,\bar{\delta}^4$$

$$+ [\bar{\delta}\bar{h}^{\mu''\nu''}\tilde{\delta}_{,\nu''} + \eta^{\mu\nu}\bar{\delta}_{,\nu''}(\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta}) - \eta^{\mu\nu}\bar{\delta}(\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta})_{,\nu''}]_{,\mu''} \, , \qquad (21)$$

by using the divergence theorem to convert the volume integral of the second line into a surface integral, and by performing the integration over x" in the second term which involves $\bar{\delta}^4 \equiv \delta^4(x,x'')$, one obtains:

$${}_1G_{DD}(x,x') = {}_1G_{TK}(x,x') + (4\pi)^{-2}\int \{\bar{\delta}\bar{h}^{\mu''\nu''}\,\tilde{\delta}_{,\nu''}$$

$$+ \eta^{\mu\nu}\,\bar{\delta}_{,\nu''}[\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta}] - \eta^{\mu\nu}\,\bar{\delta}[\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta}]_{,\nu''}\}\, d^3\Sigma_{\mu''} \, . \qquad (22)$$

The three-surface over which the integral is evaluated is at spatial, null, and temporal infinity. The surface integral can be broken into five separate terms. Each term contains the delta function $\bar{\delta}$, or one of its derivatives, which are nonzero except on the past light-cone of the event x . Similarly, each of the five terms contains the delta function $\tilde{\delta}$, or one of its derivatives, which are nonzero only on the future light cone of the event x' . Therefore the integrands are zero everywhere except on the intersection of the two light cones. In particular, the integrands are zero at infinity. Thus the surface integral term in equation (22) vanishes and we obtain

$$ {}_1G_{DD}(x,x') = {}_1G_{TK}(x,x') \quad . \tag{23} $$

IV. ALTERNATIVE VERSIONS OF THE POST-LINEAR GRAVITATIONAL-WAVE-GENERATION FORMALISM

We now present several alternative representations of the TK formalism for calculating the gravitational radiation emitted by fast-motion, self-gravitating, weak-field sources.

In the TK formalism the gravitational radiation is the time-dependent, transverse-traceless part of a metric perturbation ${}_2\bar{h}^{\mu\nu}$, which is calculated at field points x in the radiation zone with accuracy of second order in the internal gravitational field of the source. This second-order field is expressed as a retarded integral over the source region (points x'):

$$ {}_2\bar{h}^{\mu\nu}(x) = 16\pi \int [(1-\bar{h}')T^{\mu'\nu'} + t_{LL}^{\mu'\nu'} + (16\pi)^{-1}\, \bar{h}^{\mu'\rho'}{}_{,\sigma'}\, \bar{h}^{\nu'\sigma'}{}_{,\rho'}]\, {}_1G(x,x')d^4x' \quad . \tag{24} $$

Here $T^{\mu\nu}$, $t_{LL}^{\mu\nu}$, and $\bar{h}^{\mu\nu}$ are the nongravitational stress-energy tensor (accurate to second order), the Landau-Lifshitz pseudotensor (accurate to second order), and a trace-reversed metric perturbation (accurate to first order)[2] which satisfy the following coupled equations

[2]In TK $T^{\mu\nu}$ is denoted ${}_2T^{\mu\nu}$, $t^{\mu\nu}_{LL}$ is denoted ${}_1t^{\mu\nu}_{LL}$, and $\bar{h}^{\mu\nu}$ is denoted ${}_1\bar{h}^{\mu\nu}$. We drop the prefixes to simplify the notation.

$$T^{\mu\nu}{}_{,\nu} = -\Gamma^{\mu}{}_{\alpha\nu}\, T^{\alpha\nu} - \Gamma^{\nu}{}_{\alpha\nu}\, T^{\mu\alpha} \quad , \tag{25a}$$

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}(h^{\mu}{}_{\alpha,\beta} + h^{\mu}{}_{\beta,\alpha} - h_{\alpha\beta}{}^{,\mu}) \quad , \tag{25b}$$

$$\begin{aligned} t^{\alpha\beta}_{LL} = (16\pi)^{-1} \{ & \tfrac{1}{2}\, \eta^{\alpha\beta}\eta_{\lambda\mu}\, \bar{h}^{\lambda\nu}{}_{,\rho}\, \bar{h}^{\rho\mu}{}_{,\nu} + \eta_{\lambda\mu}\eta^{\nu\rho}\, \bar{h}^{\alpha\lambda}{}_{,\nu}\, \bar{h}^{\beta\mu}{}_{,\rho} \\ & - (\eta^{\alpha\lambda}\eta_{\mu\nu}\, \bar{h}^{\beta\nu}{}_{,\rho}\, \bar{h}^{\mu\rho}{}_{,\lambda} + \eta^{\beta\lambda}\eta_{\mu\nu}\, \bar{h}^{\alpha\nu}{}_{,\rho}\, \bar{h}^{\mu\rho}{}_{,\lambda}) \\ & + \tfrac{1}{8}(2\eta^{\alpha\lambda}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\lambda\mu})(2\eta_{\nu\rho}\eta_{\sigma\tau} - \eta_{\rho\sigma}\eta_{\nu\tau})\, \bar{h}^{\nu\tau}{}_{,\lambda}\, \bar{h}^{\rho\sigma}{}_{,\mu} \} \quad , \end{aligned} \tag{25c}$$

$$\eta^{\alpha\beta}\bar{h}^{\mu\nu}{}_{,\alpha\beta} = -16\pi\, T^{\mu\nu} \quad . \tag{25d}$$

In their version of the formalism TK insert into equation (24) their own first-order Green's function ${}_1G_{TK}$ (eq. 20), with their modified tail term. If instead we insert ${}_1G_{DD}$, as given by equation (14), we obtain an alternative representation for the second-order field:

$$\begin{aligned} {}_2\bar{h}^{\mu\nu}(x) = 4 \int & [(1 - \bar{h}')\, T^{\mu'\nu'} + t^{\mu'\nu'}_{LL} + (16\pi)^{-1}\, \bar{h}^{\mu'\rho'}{}_{,\sigma'}\, \bar{h}^{\nu'\sigma'}{}_{,\rho'}]\delta d^4x' \\ & + \frac{1}{\pi} \iint \bar{\delta}_{,\alpha''}\, \bar{h}^{\alpha''\beta''}\, T^{\mu'\nu'}\, \tilde{\delta}_{,\beta''}\, d^4x'\, d^4x'' \quad . \end{aligned} \tag{26}$$

Equation (26) is, within the constraints of the weak-field approximation, completely equivalent to the ${}_2\bar{h}^{\mu\nu}$ developed by TK (their eq. [58]) so long as the field point x is near enough to the source that gravitational-redshift-induced phase shifts can be ignored (see the discussion of the truncation of γ in §II.c of this paper, and see §IV.c.ii of TK). Throughout this paper we restrict ourselves to such field points.

The first term of expression (26) is exactly the same as the "direct" plus "whump" fields of the TK formalism. It is the second term which distinguishes this formula from theirs. The second term is equivalent to the sum of their "focusing", "transition" and "tail" terms. Since this representation appears to be substantially simpler than that of TK--at least in mathematical form--it should be useful in the proof of theorems and in the computational details of some applications.

Let us now consider the second term of equation (26), which we designate $I^{\mu\nu}$, to see if it can be manipulated into a more tractable form. Reordering the integrations we obtain:

$$
\begin{aligned}
I^{\mu\nu}(x) &= \frac{1}{4\pi}\int \bar{\delta}_{,\alpha''}\,\bar{h}^{\alpha''\beta''}\left(4\int T^{\mu'\nu'}\tilde{\delta}_{,\beta''}\,d^4x'\right)d^4x'' \\
&= \frac{1}{4\pi}\int \bar{\delta}_{,\alpha''}\,\bar{h}^{\alpha''\beta''}\left(4\int T^{\mu'\nu'}\,\tilde{\delta}\,d^4x'\right)_{,\beta''}d^4x'' \quad . \qquad (27)
\end{aligned}
$$

If now we use the inverted form of equation (25d), i.e.,

$$
\bar{h}^{\mu''\nu''} = 4\int T^{\mu'\nu'}\,\tilde{\delta}\,d^4x' \quad , \qquad (28)
$$

we find

$$
I^{\mu\nu}(x) = \frac{1}{4\pi}\int \delta_{,\alpha'}\,\bar{h}^{\alpha'\beta'}\,\bar{h}^{\mu'\nu'}{}_{,\beta'}\,d^4x' \quad . \qquad (29)
$$

While this is a quite workable form, it is possible to obtain two additional representations by integrating by parts. Considering the β' differentiation first and making use of the de Donder gauge condition, $\bar{h}^{\mu'\nu'}{}_{,\nu'} = 0$ (which follows to first order from eqs. [25a,d]), we find:

$$I^{\mu\nu}(x) = \frac{1}{4\pi}\int \{\delta_{,\alpha'}\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}\}_{,\beta'}\, d^4x'$$

$$- \frac{1}{4\pi}\int \delta_{,\alpha'\beta'}\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}\, d^4x' \quad . \tag{30}$$

The first term, which we identify as $A^{\mu\nu}(x)$, by use of the divergence theorem can be converted to a surface integral at spatial, null, and temporal infinity:

$$A^{\mu\nu}(x) = \frac{1}{4\pi}\int \delta_{,\alpha'}\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'} d^3\Sigma_{\beta'} \quad . \tag{31}$$

The integrand is nonzero only at the intersection of past null infinity, $\mathcal{I}^-$ with the past light cone of x. At that intersection $\delta_{,\alpha'} d^3\Sigma_{\beta'}$ $\sim |\underset{\sim}{x}'|^{-1} \cdot |\underset{\sim}{x}'|^2 \cdot \partial/\partial x^{\alpha'}$, while $\bar{h}^{\alpha'\beta'} \sim |\underset{\sim}{x}'|^{-1}$. Thus, whatever may be the time dependence of $\bar{h}^{\alpha'\beta'}$ at $\mathcal{I}^-$, the surface integral vanishes at least as fast as $|\underset{\sim}{x}'|^{-1}$. Consequently, equation (30) becomes

$$I^{\mu\nu}(x) = -\frac{1}{4\pi}\int [\delta \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}]_{,\alpha\beta}\, d^4x' \quad ,$$

where we have made u · of the identity $\delta_{,\alpha'\beta'} = \delta_{,\alpha\beta}$. A nicer expression will result if we interchange the order of differentiation and integration. However, in doing so we produce a divergent integral since at large radii $|\underset{\sim}{x}'|$, $\delta d^4x' \sim |\underset{\sim}{x}'|^2\, d|\underset{\sim}{x}'|$, and $\bar{h}^{\alpha'\beta'} \sim |x'|^{-1}$. To avoid the divergence we must confine the integral to a finite 4-volume $\mathcal{V}_4$ surrounding the source, then interchange integration and differentiation, then take the limit as the boundary of $\mathcal{V}_4$ goes to "infinity" (i.e., as $\mathcal{V}_4$ covers all of space-time):

$$I^{\mu\nu} = \lim_{\partial\mathcal{V}_4 \to \infty} \left\{ -\frac{1}{4\pi} \left[\int_{\mathcal{V}_4} \delta \bar{h}^{\alpha'\beta'} \, \bar{h}^{\mu'\nu'} \, d^4x' \right]_{,\alpha\beta} \right\} \quad . \tag{32}$$

In a similar manner, integrating by parts with respect to α' in equation (29) and invoking the deDonder gauge condition we obtain

$$I^{\mu\nu}(x) = B^{\mu\nu}(x) - \frac{1}{4\pi} \int \delta \, \bar{h}^{\alpha'\beta'} \, \bar{h}^{\mu'\nu'}{}_{,\alpha'\beta'} \, d^4x' \tag{33}$$

where

$$B^{\mu\nu}(x) = \frac{1}{4\pi} \int \{\delta \, \bar{h}^{\alpha'\beta'} \, \bar{h}^{\mu'\nu'}{}_{,\beta'}\}_{,\alpha'} \, d^4x'$$

$$= \frac{1}{4\pi} \int \delta \, \bar{h}^{\alpha'\beta'} \, \bar{h}^{\mu'\nu'}{}_{,\beta'} \, d^3\Sigma_{\alpha'} \quad . \tag{34}$$

Again, the surface integral can be seen to vanish at least as fast as $|\underset{\sim}{x}'|^{-1}$. Thus

$$I^{\mu\nu}(x) = -\frac{1}{4\pi} \int \delta \, \bar{h}^{\alpha'\beta'} \, \bar{h}^{\mu'\nu'}{}_{,\alpha'\beta'} \, d^4x' \quad . \tag{35}$$

In summary, the second-order gravitational field, ${}_2\bar{h}^{\mu\nu}(x)$, can be written

$$\begin{aligned} {}_2\bar{h}^{\mu\nu}(x) = 4 \int \Big[(1-\bar{h}') \, T^{\mu'\nu'} + t_{LL}^{\mu'\nu'} + (16\pi)^{-1} \, \bar{h}^{\mu'\rho'}{}_{,\sigma'} \, \bar{h}^{\nu'\sigma'}{}_{,\rho'} \Big] \delta d^4x' \\ + I^{\mu\nu}(x) \quad , \end{aligned} \tag{36}$$

where $I^{\mu\nu}(x)$ can be represented in the following ways

$$I^{\mu\nu}(x) = \pi^{-1} \iint \bar{\delta}_{,\alpha''}\, \bar{h}^{\alpha''\beta''}\, \tilde{\delta}_{,\beta''}\, T^{\mu'\nu'}\, d^4x'\, d^4x'' \tag{37a}$$

$$= (4\pi)^{-1} \int \delta_{,\alpha'}\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}{}_{,\beta'}\, d^4x' \tag{37b}$$

$$= \left(- (4\pi)^{-1} \int \delta\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}{}_{,\beta'}\, d^4x'\right)_{,\alpha} \tag{37c}$$

$$= - (4\pi)^{-1} \int \delta_{,\alpha'\beta'}\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}\, d^4x' \tag{37d}$$

$$= \lim_{\partial\mathcal{V}_4 \to \infty} \left[\left(- \frac{1}{4\pi} \int_{\mathcal{V}_4} \delta\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}\, d^4x'\right)_{,\alpha\beta}\right] \quad ; \tag{37e}$$

$$I^{\mu\nu}(x) = - (4\pi)^{-1} \int \delta\, \bar{h}^{\alpha'\beta'}\, \bar{h}^{\mu'\nu'}{}_{,\alpha'\beta'}\, d^4x' \quad . \tag{38}$$

All integrations except that in equation (37e) extend over all of spacetime; in (37e) the differentiation must be performed before extending $\mathcal{V}_4$ to all of spacetime.

We have given expression (38) for $I^{\mu\nu}$ an equation number of its own for two reasons: (i) it is badly behaved in point-mass bremsstrahlung calculations (see §V below), and (ii) it is most naturally thought of, not as arising from the DD Green's function (which is the way we derived it), but rather from the flat-space Green's function.

The second point (flat-space Green's function as origin of expression [38]) can be seen as follows: Expand the Einstein field equations to second-order in the metric perturbation in the manner of TK, and impose the de Donder gauge condition to obtain

$$\eta^{\alpha\beta}\, {}_2\bar{h}^{\mu\nu}{}_{,\alpha\beta} = -16\pi(1 - \bar{h})T^{\mu\nu} - 16\pi\, t^{\mu\nu}_{LL} - \bar{h}^{\mu\alpha}{}_{,\beta}\, \bar{h}^{\nu\beta}{}_{,\alpha} + \bar{h}^{\alpha\beta}\, \bar{h}^{\mu\nu}{}_{,\alpha\beta} \tag{39}$$

(cf. eqs. [9f] and [10b] of TK). This expression ıeduces to the first-order field equations (25d) if one neglects the quadratic terms on the right hand side. The quadratic terms provide the next-higher-order, nonlinear correction to the gravitational field. Equation (39) is written in terms of a flat-space wave operator, whereas in TK the same equation was written in terms of the wave operator for weakly curved space (TK eq. [10b]). If we invert it using the flat-space Green's function ${}_0G(x,x') = (4\pi)^{-1}\delta$ (rather than using the weakly curved Green's function ${}_1G_{TK}$ or ${}_1G_{DD}$) we obtain expressions (36), (38) for ${}_2\bar{h}^{\mu\nu}$ and $I^{\mu\nu}$.

We now see that there are at least three distinct ways by which one can obtain expressions for the second-order field, corresponding to three distinct Green's functions: ${}_1G_{DD}$, ${}_1G_{TK}$, and ${}_0G$. When one deals with extended sources with truly weak fields in their interiors, the three methods and their resulting formulas are all equivalent (aside from the delicate issue of validity for field points far far from the source). The DD formulas developed here (eqs. [36] and [37]) and the TK formulas (their eqs. [58] and [59]), however, have an advantage over the flat-space formulas (eqs. [36] and [38]) in their ability to approximate widely separated bodies as point masses.

V. THE POINT-MASS APPROXIMATION TO THE BREMSSTRAHLUNG PROBLEM

The most important application of the post-linear formalism is the calculation of the gravitational radiation from a collection of gravitationally interacting stars--the gravitational bremsstrahlung problem. To this problem we now turn our attention.

It is reasonable to expect that in stellar encounters with impact parameters large compared to stellar radii, the monopole fields of the stars should produce the major contribution to the time-dependent part of the second-order field ${}_2\bar{h}^{\mu\nu}$ far from the source, and thence to the gravitational radiation. Moreover, in calculating the effects of the monopole fields, it is tempting to idealize the stars as point masses. Of course, such an idealization violates the weak-field assumption of post-linear theory since the field of a point mass diverges at the position of the mass. But that violation is not important. The important question (formulated so as to mesh with the following analysis) is this:

Consider a near encounter between two stars A and B in which (i) to avoid issues of gravitational waves from stellar pulsations, the stars are assumed to not pulsate at all; (ii) the stars have weak internal gravity:

$$r_A \gg m_A \quad , \qquad r_B \gg m_B \tag{40a}$$

where r_J is the radius of star J as measured in its own rest frame and m_J is its mass; and (iii) the radii of the stars are small compared to their Lorentz-contracted distance of closest approach:

$$r_A + r_B \ll b/\gamma \quad , \qquad \gamma \equiv (1 - v^2)^{-1/2} \ . \tag{40b}$$

Here v is the relative velocity of the stars at their point of closest approach. Question: <u>Will a monopole, point-mass calculation with the formulas of post-linear theory give (very nearly) the same result for the gravitational radiation emitted as one would get from the same post-linear formulas, treating the stars correctly as finite bodies with realistic multipole structures?</u>

Kovàcs and Thorne (private communication) have proved that the answer is "yes" for their post-linear formulas. In fact, the demand for a "yes" answer was a guiding principle in the original development of their formalism. In this section of the paper we shall show that the answer to the above question is also "yes" for the DD forms of the post-linear equations (eqs. [36] and [37]), but "no" for the flat-space forms (eqs. [36] and [38]).

Begin with the DD equations. We presume that the equations of motion for the source (eqs.[25a,b,d]) are solved in such a manner that the monopole, point-mass calculation gives essentially the same stellar trajectories through the flat background spacetime as the finite-body calculation. We must then check whether expressions (36) and (37) for ${}_2\bar{h}^{\mu\nu}$ are sensitive to the monopole, point-mass idealization.

The fields in the regions exterior to each of the stars can be represented by multipole expansions. Near the surface of star A its monopole component is greater than or of the order of all other components; at a distance r' A's monopole component is larger than all others by a factor $(r'/r_A)^2$. Thus, there exists a certain radius R_A, measured from the center of star A in the rest frame of A, beyond which the field of A can accurately be considered a pure monopole field. Moreover, we can choose R_A such that

$$r_A \ll R_A \ll b/\gamma \quad . \tag{40c}$$

A similar situation holds for star B . In the volume integrals (36),(37) for ${}_2\bar{h}^{\alpha\beta}$ the only regions that can be sensitive to a monopole, point-mass approximation are the neighborhoods $\mathcal{N}(A)$ and $\mathcal{N}(B)$ of stars A and B with neighborhood radii R_A and R_B.

Because the stars always remain physically separated, and because the equations governing the first-order field $\bar{h}^{\alpha\beta}$ are linear, it is possible to split $T^{\mu\nu}$ and $\bar{h}^{\alpha\beta}$ into independent contributions from each of the stars

$$T^{\mu\nu} = T_A^{\mu\nu} + T_B^{\mu\nu} \quad , \qquad \bar{h}^{\alpha\beta} = \bar{h}_A^{\alpha\beta} + \bar{h}_B^{\alpha\beta} \quad . \tag{41}$$

When the split (41) is inserted into expressions (36) and (37) for ${}_2\bar{h}^{\mu\nu}$, three types of terms result: (i) the linearized field

$$_2\bar{h}_L^{\mu\nu} = 4 \int (T_A^{\mu'\nu'} + T_B^{\mu'\nu'})\, \delta\, d^4x' \quad , \tag{42}$$

(ii) "self-energy terms" which involve $\bar{h}_A\bar{h}_A$ or $\bar{h}_B\bar{h}_B$ or $\bar{h}_A T_A$ or $\bar{h}_B T_B$, and (iii) "interaction terms" which involve $\bar{h}_A\bar{h}_B$ or $\bar{h}_A T_B$ or $\bar{h}_B T_A$.

The linearized field (42) is obviously insensitive to a monopole, point-mass idealization since the field point x lies in the radiation zone, which is far outside $\mathcal{N}(A)$ and $\mathcal{N}(B)$.

When one ignores stellar pulsations (in keeping with our assumptions), the self-energy terms lead to a simple renormalization of the active gravitational mass of each star; they are not time-dependent at infinity, and they do not contribute to the gravitational radiation (transverse-traceless, time-dependent part of ${}_2\bar{h}^{\mu\nu}$). Therefore, we can drop them from our calculation and restrict attention to the interaction terms and the linearized field.

Of the interaction terms, those involving $\bar{h}_A T_B$ or $\bar{h}_B T_A$ (eqs. [36] and [37a]) are the easiest to analyze. The (37a) hT terms can be converted into the "hh" form (37b) by the manipulations (27)-(29), equally well in a monopole point-mass calculation or in an extended-body calculation. Since we will treat all hh terms below, we need not consider the (37a) hT terms. The $\bar{h}_B T_A$ term in (36) is best analyzed in the rest frame of star A, where, using

$$\delta \equiv \delta({}_0\Omega) \approx r^{-1}\,\delta(t' - t + r - \underset{\sim}{n}\cdot\underset{\sim}{x}') \;, \quad r \equiv |\underset{\sim}{x}|, \quad \underset{\sim}{n} \equiv \underset{\sim}{x}/r \tag{43}$$

we can bring it into the form

$$ {}_2\bar{h}^{\mu\nu}_{I1} = -4r^{-1} \int [T^{\mu'\nu'}_A \; \bar{h}'_B]_{t' = t - r + \underset{\sim}{n}\cdot\underset{\sim}{x}'} \, d^3x' \;. \tag{44}$$

The integrand is nonzero only inside star A , which is far enough from star B that only the monopole field of B contributes significantly to $\bar{h}'_B$. Also, because star A is small compared to the impact parameter ($r_A \ll b$), and because the field of B inside A changes on a timescale $\gtrsim b/(v\gamma) \gg r_A$, we can ignore the spatial variation of $\bar{h}'_B$ across star A, and we can also ignore its time retardation from point to point across A ; i.e., we can write

$$\begin{aligned} {}_2\bar{h}^{\mu\nu}_{I1} &= -4r^{-1}\, \bar{h}^{mono}_B(\underset{\sim}{x}' = 0,\; t' = t - r) \int T^{\mu'\nu'}_A \, d^3x' \\ &= -4r^{-1}\, \bar{h}^{mono}_B(\underset{\sim}{x}' = 0,\; t' = t - r)\; m_A\, \delta^{\mu}_0\, \delta^{\nu}_0 \;. \end{aligned} \tag{45}$$

This is precisely the same result as one would get from the monopole, point-mass approximation. The interaction term involving $\bar{h}_A T_B$ in expression (36), when analyzed in the rest frame of star B , gives a similar result. Thus, all the $\bar{h}T$ interaction terms are amenable to the monopole, point-mass approximation.

There are many interaction terms involving $\bar{h}_A\bar{h}_B$: A large number come from the $t^{\mu'\nu'}_{LL}$ term of expression (36) (cf. eq.[25c]); two come from the term following $t^{\mu'\nu'}_{LL}$ in (36); and two each come from expressions (37b,c,d,e). Each such term involves an integral over all spacetime (all x'). The only portions of the integrals which could possibly be sensitive to the monopole, point-mass idealization are the portions which come from the non-monopole regions $\mathcal{V}(A)$ and $\mathcal{V}(B)$; and because our analysis is insensitive to the change of names $A \leftrightarrow B$, it is adequate for us to consider contributions from $\mathcal{V}(A)$.

Each of the extended-body $\bar{h}_A \bar{h}_B$ integrals over $\mathcal{V}(A)$ can be split into two parts: the contribution from the interior of star A, $\mathcal{V}_I(A)$, and the contribution from its exterior, $\mathcal{V}_E(A)$:

$$\mathcal{V}_I(A): \quad 0 \leq r' \leq r_A , \qquad \mathcal{V}_E(A): \quad r_A \leq r' \leq R_A \quad . \tag{46}$$

Because we ignore pulsations of A, $T_A^{\mu'\nu'}$ is independent of t', and to first order the only nonzero component of $\bar{h}_A^{\alpha'\beta'}$ is $\bar{h}_A^{0'0'}$. In the interior of A, $\bar{h}_A^{0'0'}$ and its spatial derivatives have magnitude

$$\bar{h}_A^{0'0'} \sim 4m_A/r_A , \quad \bar{h}_A^{0'0'}{}_{,j'} \sim 4m_A/r_A^2 , \quad \bar{h}_A^{0'0'}{}_{,j'k'} \sim 4m_A/r_A^3 \quad \text{in} \quad \mathcal{V}_I(A). \tag{47a}$$

In the exterior of A we can expand $\bar{h}_A^{0'0'}$ in multipoles, obtaining

$$\bar{h}_A^{0'0'} = (4m_A/r') \left\{1 + \sum_{\ell=2}^{\infty} \alpha_\ell(\theta',\phi')(r_A/r')^\ell\right\} \quad \text{in} \quad \mathcal{V}_E(A) , \tag{47b}$$

where the α_ℓ are all of order unity. Because $b >> r_B$, the field of B inside $\mathcal{V}(A)$ is (very nearly) equal to B's monopole field at the center of A, plus a fractional correction of order $\gamma r'/b$:

$$\bar{h}_B^{\mu'\nu'} = \bar{h}_{B\ \mathrm{mono}}^{\mu'\nu'}(t',\underset{\sim}{x}'= 0)[1 + O(\gamma r'/b)] . \tag{48a}$$

The motion of star B past star A causes the field of B near A to vary on a timescale $\Delta t' \gtrsim b/v\gamma >> R_A$, leading to a time derivative of $\bar{h}_B^{\mu'\nu'}$ given by

$$\bar{h}_B^{\mu'\nu'}{}_{,0'} = (\partial/\partial t')\bar{h}_{B\,\mathrm{mono}}^{\mu'\nu'}(t',\underset{\sim}{x}'=0)[1+O(\frac{\gamma r'}{b})+O(\frac{r'}{vt'})], \tag{48b}$$

where $t'=0$ is the moment of closest approach, near which time

$$\frac{(\partial/\partial t')\bar{h}_{\mathrm{mono}}^{\mu'\nu'}(t',\underset{\sim}{x}'=0)}{(\partial/\partial t')\,\bar{h}_{\mathrm{mono}}^{\mu'\nu'}(t'\sim b/v\gamma,\underset{\sim}{x}'=0)} \sim \frac{\gamma v t'}{b} \quad \text{for } |t'| << \frac{b}{v\gamma} \quad . \tag{48c}$$

The error terms in (48b) are much smaller than the leading term at all times except $|t'| \sim r'/v\gamma$; and the error terms for $|t'| \lesssim r'/v$ are negligible compared to the leading term for nearby times $|t'| \sim b/v\gamma$. Because the field point x is very far outside the source region $\mathcal{V}(A)$, we can write the propagator δ in the form (43); and we can write

$$\delta_{,j'} = -n_j\delta_{,0'} \;; \quad \delta_{,j'k'} = n_j n_k\,\delta_{,0'0'} \quad . \tag{49}$$

Equations (43) and (46)-(49) are the foundation for evaluating the $\bar{h}_A\bar{h}_B$ interaction integrals. By inserting them into each interaction integral in turn, one can verify that all the interaction integrals in (36) and (37) are insensitive to the point-mass idealization. Consider, for example, the interaction integral

$$J^{\mu\nu\alpha}(x) \equiv \int_{\mathcal{V}(A)} \delta\,\bar{h}_A^{\alpha'\beta'}\,\bar{h}_B^{\mu'\nu'}{}_{,\beta'}\,d^4x' \quad , \tag{50}$$

which comes from expression (37c). By inserting expression (43) for δ and integrating over t' , and by using the fact that $\bar{h}_A^{\alpha'\beta'}$ is independent of t' and is zero unless $\alpha' = \beta' = 0$, we bring this into the form

$$J^{\mu\nu\alpha}(x) = r^{-1}\delta_0^\alpha \int_{\mathcal{V}(A)} \bar{h}_A^{0'0'}\,[\bar{h}_B^{\mu'\nu'}{}_{,0'}]_{t'=t-r+\underset{\sim}{n}\cdot\underset{\sim}{x}'}\,d^3x' \quad . \tag{51}$$

We next split the integral into interior and exterior contributions; we use (47a) and (48b,c) in the interior, and (47b) and (48b, ·) in the exterior, thereby obtaining

$$J_I^{\mu\nu 0}(x) \sim \frac{16\pi m_A}{r} r_A{}^2 \frac{\partial}{\partial t} \bar{h}_{B\ \mathrm{mono}}^{\mu'\nu'}(t'= t-r, \underset{\sim}{x}'= 0)\left[1+O\left(\frac{\gamma r_A}{b}\right)+O\left(\frac{r_A}{v(t-r)}\right)\right], \quad (52a)$$

$$J_E^{\mu\nu 0}(x) = \frac{16\pi m_A}{r} R_A{}^2 \frac{\partial}{\partial t} \bar{h}_{B\ \mathrm{mono}}^{\mu'\nu'}(t'= t-r,\ \underset{\sim}{x}'= 0) \times$$

$$\times \left\{1 + O\left[\left(\frac{r_A}{R_A}\right)^2 \ell n\left(\frac{R_A}{r_A}\right)\right] + O\left[\frac{\gamma R_A}{b}\right] + O\left[\frac{R_A}{(t-r)}\right]\right\} \quad . \quad (52b)$$

Notice that (i) the exterior contribution $J_E^{\mu\nu 0}$ dominates over the interior contribution $J_I^{\mu\nu 0}$ by a factor $(R_A/r_A)^2$; (ii) except for the "error terms" the exterior contribution is precisely the result which one would obtain from a monopole, point-mass calculation; (iii) the error terms are negligible at all times except $|t-r| \lesssim R_A/v$ when the dominant, "point-mass contribution" is going through zero; (iv) even when they dominate (for $|t-r| \lesssim R_A/v$) the error terms are negligible, $O(\gamma R_A/b)$, compared to the point-mass contribution at nearby times ($|t-r| \sim b/v\gamma$). These facts show that, for all practical purposes, the interaction integral (50) is insensitive to a monopole, point-mass idealization.[3]

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[3] One can show that because of their angular dependences inside the d^3x' integral, the error terms shown explicitly in (52) actually vanish. Thus the dominant errors are of even smaller order than indicated in the O[ ] expressions of (52).

All other $\bar{h}_A\bar{h}_B$ terms in expressions (36) and (37) can be handled similarly, giving the same conclusion.

Thus, for the DD version of the post-linear formalism (eqs. [36] and [37]) the answer to our question (italicized sentence following eq. [40b]) is "yes". The monopole, point-mass approximation is valid.

Not so for the flat-space version of the post linear formalism (eqs. [36] and [38]): The second derivatives which occur in expression (38) wreak havoc with the monopole, point-mass idealization. To see this, consider the following interaction integral, which comes from expression (38):

$$K^{\mu\nu} = \int_{\eta(A)} \delta\, \bar{h}_A^{\mu'\nu'}{}_{,\alpha'\beta'}\, \bar{h}_B^{\alpha'\beta'}\, d^4x' \quad . \tag{53}$$

Evaluate the integral over t' in the rest frame of A, using expression (43) for δ and using the fact that $\bar{h}_A^{\mu'\nu'}$ is independent of t' and is zero at first order unless $\mu' = \nu' = 0'$; the result is

$$K^{\mu\nu} = r^{-1}\, \delta_0^\mu\, \delta_0^\nu \int \bar{h}_A^{0'0'}{}_{,j'k'} [\bar{h}_B^{j'k'}]_{t'=t-r+\underset{\sim}{n}\cdot\underset{\sim}{x}'}\, d^3x' \; . \tag{54}$$

The contribution to $K^{\mu\nu}$ from the interior of star A is obtained by inserting (47a) and (48a) into (54), and integrating over $r' \leq r_A$; the result is:

$$K_I^{\mu\nu} \sim (16\pi m_A/r)\delta_0^\mu\, \delta_0^\nu\, \bar{h}_{B\,\mathrm{mono}}^{j'k'}(t'=t-r,\, \underset{\sim}{x}'=0)\cdot[1+O(\gamma r_A/b)] \; . \tag{55}$$

This internal contribution does <u>not</u> go to zero as the size of star A, r_A, is shrunk to zero. Independently of r_A, it is comparable in magnitude to the total gravitational-wave amplitude--and for observers not in the rest frame of A it is an indispensible, non-negligible contributor to the gravitational waves. A monopole, point-mass calculation will miss this contribution.

VI. CONCLUSION

There are three different Green's functions which one can use to calculate the second-order gravitational field ${}_2\bar{h}^{\mu\nu}$ of post-linear theory: ${}_1G_{TK}$, ${}_1G_{DD}$, and ${}_0G$. The first-order Green's functions ${}_1G_{TK}$ and ${}_1G_{DD}$ are completely equivalent. ${}_1G_{TK}$ leads to the TK formulas for ${}_2\bar{h}^{\mu\nu}$ (their eqs. [58] and [59]), while ${}_1G_{DD}$ leads to the formulas of this paper (eqs. [36] and [37]). Both sets of formulas (TK and DD) remain valid if one idealizes the stars of a bremsstrahlung calculation as monopole point masses. However, the formulas for ${}_2\bar{h}^{\mu\nu}$ which are obtained from the zero-order Green's function ${}_0G$ (eqs. [36] and [38]), although valid for extended bodies, are <u>not</u> valid in the monopole, point-mass idealization of a bremsstrahlung calculation.

APPENDIX

The proof of the equivalence of the Green's functions of DD and TK is based upon the identity (eq. [21]):

$$\bar{\delta}_{,\mu''}\bar{h}^{\mu''\nu''}\,\tilde{\delta}_{,\nu''} = \bar{\delta}\tilde{\alpha}_{,\mu''}{}^{\mu''}\tilde{\delta} + 4\pi(\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta})\,\bar{\delta}^4$$

$$+ [\bar{\delta}\bar{h}^{\mu''\nu''}\tilde{\delta}_{,\nu''} + \eta^{\mu\nu}\bar{\delta}_{,\nu''}(\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta}) - \eta^{\mu\nu}\bar{\delta}(\tilde{\gamma}\tilde{\delta}' + \tilde{\alpha}\tilde{\delta})_{,\nu''}]_{,\mu''}\ . \qquad \text{(A1)}$$

(See eqs. [1] and [7] and fig. 1 for notation.) We shall now prove this identity.

If we carry out the differentiations of the round and square brackets, make use of the defining differential equation for the flat-spacetime Green's function

$$\eta^{\mu\nu}\,\bar{\delta}_{,\mu''\nu''} = 2\,{}_0\bar{\Omega}\,\bar{\delta}'' + 4\bar{\delta}' = -4\pi\bar{\delta}^4 \qquad \text{(A2)}$$

and the fact that

$$\tilde{\alpha}\,\eta^{\mu\nu}\,\tilde{\delta}_{,\mu''\nu''} = -4\pi\,\tilde{\alpha}\,\tilde{\delta}^4 = 0$$

which follows from equation (19d) since $\tilde{\alpha}(x',x') = 0$, the proof of the identity is reduced to showing that

$$K \equiv \tilde{X}_{\nu''}\,\bar{h}^{\mu''\nu''}{}_{,\mu''}\,\tilde{\delta}' + \bar{h}^{\mu''\nu''}\,\tilde{\delta}_{,\mu''\nu''} - \tilde{\gamma}_{,\mu''}{}^{\mu''}\tilde{\delta}'$$

$$-2\tilde{\gamma}^{,\mu''}\,X_{\mu''}\tilde{\delta}'' - \tilde{\gamma}\,\eta^{\mu\nu}\,\tilde{\delta}'_{,\mu''\nu''} - 2\tilde{\alpha}^{,\mu''}\,\tilde{X}_{\mu''}\,\tilde{\delta}' = 0\ . \qquad \text{(A3)}$$

Using the definitions of $\tilde{\alpha}$ and $\tilde{\gamma}$ given by equations (19b,d), it is relatively straightforward to show that

$$\tilde{\gamma}_{,\mu''}{}^{\mu''} = -2\tilde{\alpha}^{,\mu''}\tilde{x}_{\mu''} + h'' + \tilde{x}_{\mu''}\bar{h}^{\mu''\nu''}{}_{,\nu''} , \tag{A4}$$

and

$$\tilde{\gamma}^{,\mu''}\tilde{x}_{\mu''} = \tilde{\gamma} + \frac{1}{2}\tilde{x}^{\mu''}\tilde{x}^{\nu''}h_{\mu''\nu''} . \tag{A5}$$

Making use of equations (A4) and (A5) in (A3) we find

$$K = \bar{h}^{\mu''\nu''}\tilde{\delta}_{,\mu''\nu''} - h''\tilde{\delta}' - \tilde{x}_{\mu''}\tilde{x}_{\nu''}h^{\mu''\nu''}\tilde{\delta}'' - \tilde{\gamma}(2\tilde{\delta}'' + \eta^{\mu\nu}\tilde{\delta}'_{,\mu''\nu''}) . \tag{A6}$$

If we now employ the identity

$$\tilde{\delta}_{,\mu''\nu''} = \tilde{x}_{\mu''}\tilde{x}_{\nu''}\tilde{\delta}'' + \eta_{\mu\nu}\tilde{\delta}' \tag{A7}$$

as well as the definition $\bar{h}^{\mu''\nu''} = h^{\mu''\nu''} - \frac{1}{2}\eta^{\mu\nu}h''$ and equation (A2), we find

$$K = 2\pi h''\tilde{\delta}^4 - \tilde{\gamma}(2\tilde{\delta}'' + \eta^{\mu\nu}\tilde{\delta}'_{,\mu''\nu''}) . \tag{A8}$$

Using equation (A5) for $\tilde{\gamma}$ and the identity

$$\eta^{\mu\nu}\tilde{\delta}'_{,\mu''\nu''} = 2\,{}_0\tilde{\Omega}\tilde{\delta}''' + 4\tilde{\delta}'' \tag{A9}$$

we can rewrite (A8) as

$$K = 2\pi h''\tilde{\delta}^4 - (\tilde{\gamma}^{,\mu''} - \frac{1}{2}\tilde{x}_{\nu''}h^{\mu''\nu''})\tilde{x}_{\mu''}(6\tilde{\delta}'' + 2\,{}_0\tilde{\Omega}\tilde{\delta}''') . \tag{A10}$$

By differentiating (A2)we obtain the identity

$$x_{\mu''}(6\tilde{\delta}'' + 2\,{}_0\tilde{\Omega}\tilde{\delta}''') = -4\pi\tilde{\delta}^4{}_{,\mu''} . \tag{A11}$$

Making use of this in (A10), we obtain

$$K = 2\pi \; h'' \, \tilde{\delta}^4 + 4\pi(\tilde{\gamma}^{,\mu''} - \frac{1}{2}\tilde{X}_{\nu''}\, h^{\mu''\nu''})\tilde{\delta}^4{}_{,\mu''}$$

$$= 2\pi \; h'' \, \tilde{\delta}^4 + 4\pi[(\tilde{\gamma}^{,\mu''} - \frac{1}{2}\tilde{X}_{\nu''}\, h^{\mu''\nu''})\tilde{\delta}^4]_{,\mu''}$$

$$- 4\pi(\tilde{\gamma}^{,\mu''}{}_{\mu''} - \frac{1}{2} h'' - \frac{1}{2}\tilde{X}_{\nu''}\, h^{\mu''\nu''}{}_{,\mu''})\tilde{\delta}^4 \quad . \qquad \text{(A12)}$$

By virtue of the properties of the Dirac delta function and the fact that $\tilde{\gamma}^{,\mu''}(x',x') = 0$, $\tilde{X}_{\nu''}(x',x') = 0$, and $\tilde{\gamma}^{,\mu''}{}_{\mu''}(x',x') = h''$, the second term vanishes and the third term reduces to $-2\pi h'' \tilde{\delta}^4$ which cancels the first term. Thus $K = 0$ and the identity (A1) is established.

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FIGURE CAPTIONS

Figure 1. A mnemonic diagram for helping one to remember: (i) the temporal order of the events x', x'', x; (ii) our notational conventions for 2-point functions (eq. [1]) and especially for ${}_0\Omega$, ${}_0\overline{\Omega}$, ${}_0\tilde{\Omega}$, δ, $\overline{\delta}$, $\tilde{\delta}$ (eqs. [7a,b]); (iii) our definitions of X, $\overline{X}$, and $\tilde{X}$ (eq. [7d]); and our formulas for the spatial gradients of the propagators δ, $\overline{\delta}$, $\tilde{\delta}$ (eq. [7e]).

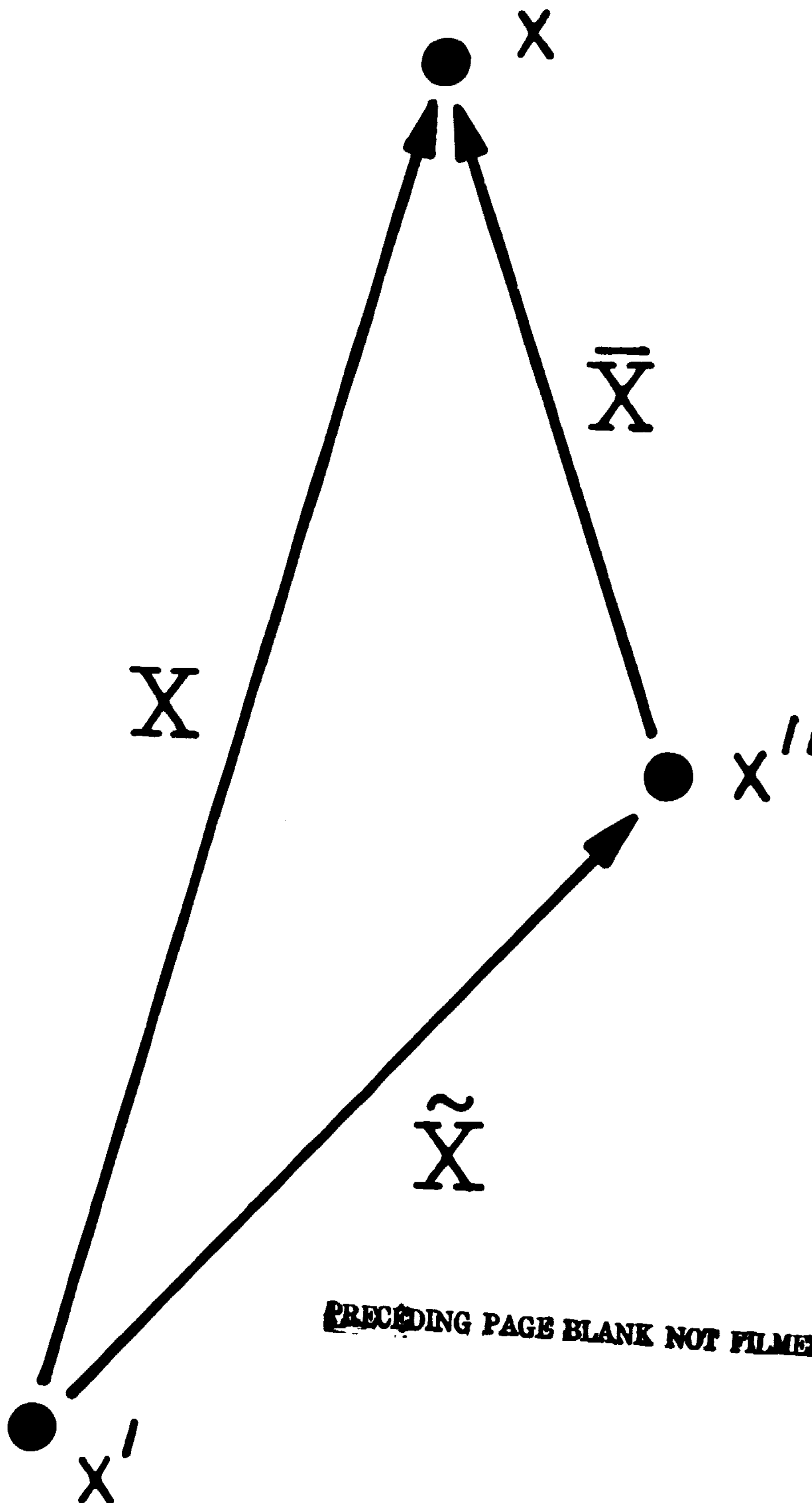

Fig. 1

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