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LIQUID JET IMPINGEMENT NORMAL TO A DISK IN ZERO GRAVITY

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Advisor

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An Abstract of

LIQUID JET IMPINGEMENT NORMAL TO A DISK IN ZERO GRAVITY

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An experimental and analytical investigation was conducted to determine the free surface shapes of circular liquid jets impinging normal to sharp-edged disks under both normal and zero gravity conditions. An order of magnitude analysis was conducted indicating regions where viscous forces were not significant when computing free surface shapes. The demarcation between the viscous and inviscid region was found to depend upon the flow Reynolds number and the ratio between the jet and disk radius.

Experiments conducted under zero gravity conditions yielded three distinct flow patterns. These flow patterns were defined as surface tension flow, transition flow, and inertia flow. The flow regions were classified in terms of the relative effects of surface tension and inertial forces. The transition between regions was correlated with the system Weber number and the ratio of the jet to the disk radius. The normal gravity plume shapes were observed to jump from one apparently stable flow pattern to another until steady-state was reached.

A zero gravity inviscid analysis was performed in which the governing equations and boundary conditions in the physical plane were transformed into an inverse plane. In the inverse plane, the stream function and velocity potential became the coordinates thus removing the prime difficulty in free surface problems, that of having to guess at the true position of the free surface. The governing equations were nonlinear in the inverse plane thus requiring a numerical solution in which sets of nonlinear algebraic equations were solved simultaneously. Comparisons between experiment and numerical computations were made for the infinite and finite plate cases with the result that good agreement for the free surface shapes were obtained.

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NOMENCLATURE

A	numerical constant
a	stream function increment
B	numerical constant
Bo	Bond number, $\rho g R_o^2/\sigma$
b	numerical constant
C	numerical constant
C_0, C_1, C_2	numerical constants
c	numerical constant
D	$= 1/\alpha^2$
d	designated reference point
f_o	unknown scale factor, m
f, f_1, f_2	fictitious points
g	acceleration due to gravity, m/sec^2
g	gas
H	distance between plate and nozzle, m
i	any point on free surface
$\hat{i}, \hat{j}$	unit vectors
J	Jacobian
$K\phi$	numerical constant
L	disk radius, m
$\hat{n}$	unit normal on free surface
O	stagnation point
p	pressure, N/m^2

p_o	unknown scale factor, N/m^2
Q	$= \frac{(r_1 - r_3)^2}{(r_f - r_4)^2} \cdot \frac{1}{\alpha^2}$
Q^*	$= \alpha^2 \frac{(f - z_4)^2}{(z_1 - z_3)^2}$
R^2	statistical variable
RD	$= L/R_o$
Re	Reynolds number, $\rho V R_o/\mu$
R_{max}	maximum radius for surface tension model, m
R_o	radius of nozzle, m
R_p	radius of plume, m
R_1, R_2	radii of curvature, m
r	radial coordinate, m
r_o	unknown scale factor, m
T^2	$= \alpha^2 r_o^2 \frac{(f - z_4)^2}{(z_1 - z_3)^2} + 1$
U_o	unknown scale factor, m/sec
u	radial velocity component, m/sec
V	jet velocity, m/sec
$\vec{V}$	vector velocity along free surface, m/sec
v	axial velocity component, m/sec
We	Weber number, $\rho V^2 R_o/\sigma$
y	dimensionless parameter
z	axial coordinate, m
z_1, z_2, z_3, z_4, z_5	dimensionless parameters

α	$\Delta\phi/\Delta\psi$
$\delta/\delta r$	finite difference analogy of $\partial/\partial r$
μ	liquid viscosity, g/m^2-sec
ρ	liquid density, kg/m^3
σ	liquid surface tension, N/m
ϕ	velocity potential, m^2/sec
χ	$= r_f - r_4$, m
ψ	Stokes' stream function, m^3/sec

Superscripts and subscripts:

Cr	critical
'	differentiation with respect to r
*	denotes dimensionless quantities
1,2,3,4	nodal point locations
s	surface

I. INTRODUCTION

A knowledge of the dynamics of free liquid jets is required for the solution of a variety of problems associated with fluid flow within propellant tanks under low gravitational conditions. In particular, an understanding of the liquid jet - impact process, as occurs when liquid impinges upon baffles or tank walls during an inflow or reorientation maneuver, will be required in order to predict liquid-propellant location, heat transfer rates and pressure distributions. The area of liquid jet impingement also has direct applicability to the spacecraft fire safety problem, in which water jets are employed as extinguishant agents under low gravity conditions. In order to be able to predict required delivery flow rates, the accurate prediction of flow surface coverage as a function of jet momentum is needed.

There generally appears to be three chief obstacles which have in the past prohibited the attainment of solutions to steady-state liquid jet-solid interaction problems. The major obstacle is the presence of the free surface. In order to apply numerical techniques to the solution of free-jet problems it is necessary to define the area over which the computations are made by boundaries defined by the free liquid surface. Unfortunately, the location of the free-surface is one of the items sought from the solution so that various techniques must be devised to circumvent this situation. Furthermore, analytical techniques are restricted solely to two-dimensional problems, whether a free surface exists or not. The second obstacle is gravity. Liquid jets in air (free jets), unlike liquid-into-liquid jets and gas-into-gas jets (submerged jets), are affected significantly by gravitational forces. The free-surface shape and velocity profiles are dependent on both the magnitude and the orientation of gravity. The addition of gravity necessarily complicates a model either through the governing equation or through the boundary conditions. Neglecting gravity in the model makes questionable the comparison of the theory with normal-gravity experimental data. The final obstacle is surface tension, an effect which has generally been neglected in almost all studies on free jets. The addition of surface tension into a model leads to non-linear free surface boundary conditions.

The purpose of this report is to present the results of an experimental and analytical study conducted at the NASA Lewis Research Center concerning zero gravity isothermal liquid jet impingement. An axisymmetric liquid jet was impinged normally onto a sharp-edged disk under conditions in which both inertial and surface tension forces are of importance. The experimental free surface shapes were correlated with known system parameters. An analytical model was formulated and the free surface shapes and streamlines were calculated for a number of discrete cases.

II. LITERATURE SURVEY

A. Experimental Studies

Very few experimental studies have been conducted to examine free jets impinging on solid surfaces. No work has been conducted where the major concern was either the shape of the free surface or the measurement of velocity profiles within the jet. Also, only one experiment has been conducted using a two-dimensional jet. A two-dimensional jet is one in which the flow emanates from a rectangular slot in which the width of the jet is very large relative to the thickness of the jet. Schach (38) measured the pressure distribution and analytically calculated the free surface shape and velocity distributions for jets impinging onto flat panels at various impingement inclinations relative to the direction of flow. The jet employed had dimensions of 21 by 115 mm. According to Schach, the jet diverged spatially after impinging upon the panel and thus it can only be considered as truly two-dimensional close to the centerline. An excellent account of an elaborate experimental apparatus for obtaining a quiescent circular water jet in normal gravity is given by Donnelly, et al. (11). Their major concern was jet stability under imposed audio frequency disturbances and, therefore, the impingement phenomenon was not directly observed. Rupe (37) and Stephens (41) experimentally measured the pressure distribution caused by circular jets striking solid surfaces in normal gravity. However, neither Rupe or Stephens measured the free-surface shape or discussed any instabilities which occurred.

In nearly all flows where a circular liquid jet strikes a large flat surface, typically what happens in normal gravity is that the liquid jet impinges on the surface and moves radially outward from the stagnation point until a certain radial distance is reached whereupon an instability known as a circular hydraulic jump occurs. The jump is characterized by an abrupt increase in the liquid depth and turbulent fluid motion. Koloseus, et al. (22) were concerned solely with predicting the behavior of the circular hydraulic jump. A water jet impinging on a flat plate of epoxy material was employed in these experiments. The circular hydraulic jump was the subject for a very complete study conducted by Nirapathdongporn (33), whose report contains an excellent description of various devices for measuring jet shapes and jet diameters.

All of the above mentioned studies deal with normal-gravity, liquid jet-solid impingement. There have been no experimental studies on the impingement of liquid jets under zero-gravity conditions.

B. Analytical Studies

1. Steady two-dimensional potential flow. - A number of papers and books have discussed steady-state two-dimensional free jets impinging on a variety of surfaces using analytical techniques. The majority of these studies were concerned with irrotational, incompressible, inviscid flow, in which the effects of gravity and surface tension were neglected. One of the major attractions of this type of problem is that it can be handled using complex potential theory and, therefore, can be treated analytically.

A two-dimensional jet striking an infinitely flat surface at various angles was examined by Batchelor (4), who solved for the limiting stream thickness as a function of flow impingement angle and jet diameter. However, no attempt was made to predict free-streamline shape. Schach (38) treated the impingement as a function of angle using Prandtl's hodograph method, and obtained the equations for the free surface shapes, flow distribution, and pressure distribution for the case of impingement on an infinitely wide plate. Kochin, et al. (21) also examined the impingement of a two-dimensional jet obliquely to an infinite flat plate, and discussed the case of impingement on a plate of finite length. The equation of the free-surface for the case of a two-dimensional jet striking a flat surface at right angles is presented by Milne-Thomson (30) who also solved for the velocity components within the jet. An excellent discussion of the techniques for handling two-dimensional free jet problems is presented by Gurevich (14). Some of the two-dimensional flows examined by Gurevich include flow around a finite wedge, perpendicular to a finite plate, obliquely to an infinite flat plate and flows where a variety of solid objects are positioned adjacent to one wall or between two walls. Chang, et al. (9) analyzed the two-dimensional flow of a jet interacting with a number of flat segments at angles to one another. The results include flow turning angles but not free surface shapes or velocity profiles. The irrotational flow pattern of a free jet discharging from a slot and flowing past a wedge was analyzed by Arbhabhirama (3). All of the above texts and articles were concerned with analytical techniques for obtaining solutions. The area of steady two-dimensional potential flow represents the most complete area of research in the field of jet impingement.

2. Steady axisymmetric viscous flow. - Watson (43) has analytically investigated free jet-impingement for the case of large Reynolds numbers where the viscous forces are confined to a thin boundary layer adjacent to the plate. A similarity solution was obtained for both the two-dimensional and axisymmetric velocity profiles and free surface shapes for the case of normal impingement. As mentioned by Watson, the similarity solution can only be expected to be valid when the radial distance is sufficiently large for the incident jet to have lost its influence. The effects of gravity and surface tension were neglected in the analysis. Watson solved for the radial position of the circular hydraulic jump.

C. Numerical Studies

1. Steady two-dimensional potential flow. - When the shape of the solid upon which the jet impinges becomes complex, numerical techniques for the solution of free jet problems have to be applied. Jeppson (19) presents an excellent article in this regard. Jeppson employed the stream function and the velocity potential as the independent variables and the coordinates as the dependent variables. A similar inversion approach has been previously used to solve a variety of fluid dynamics problems as shown in references 5, 20, 31, 42, and 44 and is mainly attributable to Thom and Apelt (42). Using this technique, Jeppson was able to circumvent the problem of working in the physical plane and having to guess at the true position of the free surface. The latter iterative approach was used in references 1, 8, 13, 27, 32, 36, and 40 with limited success. Jeppson solved the problem of the two-dimensional flow over a wedge, and, as such, is

the only one to have attempted numerical solutions of this problem. Lastly, Chan (7) applied the finite element method to a number of free-surface flow problems, including the flow from a circular orifice.

2. Steady axisymmetric potential flow. - The solution of axisymmetric flow problems cannot utilize the powerful tool of complex analysis. For this reason, only numerical solutions can be attempted for problems of this nature.

LeClerc (25) studied the impingement of an axially symmetric liquid jet perpendicular to a flat surface. The shape of the free surface was found using an electrical analogy. This method thus fixed the position of the free surface and enabled the author to apply standard finite-differencing methods and employ Southwell's relaxation technique to solve Laplace's equation. Jeppson (19) applied his inversion technique to find the flow pattern and free-surface shape for the case of axisymmetric flow past a variety of bodies of revolution, including cones. Jeppson also applied his technique to the solution of a jet of inviscid, incompressible fluid issuing from a nozzle into the free atmosphere. He indicates how his method may be extended to a variety of other problems. Schach (39) used a semi-analytical technique based on Trefftz's approximate method to find the shape of an axisymmetric free jet impinging normally on a plate. Also presented in Schach's article was the pressure distribution on the plate which was calculated from the velocity distribution using Bernoulli's equation. Young, et al. (45) and Brunauer (6) determined the flow pattern past two disks immersed in axisymmetric flow. Both Young and Brunauer solved Laplace's equation in the physical plane.

No analyses have been conducted for the case in which an inviscid free jet impinges upon a plate of finite thickness.

The articles mentioned above encompass all the known solutions with regard to axisymmetric jet impingement. References 7, 8, 20, 27, and 40 deal specifically with numerical methods applied to free surface problems in which no impingement occurs. Jeppson (20) employed an inverse formulation while the others worked in the physical plane.

3. Unsteady two-dimensional and axisymmetric potential flow. - Huang (17, 18) has investigated unsteady flows and considered the impact phenomena for both two-dimensional and axisymmetric jets. The major interest in these articles was in obtaining the initial pressure distribution due to liquid impact.

4. Steady potential flow including gravitational effects. - The addition of gravity in analyses for potential flows causes no serious formulation problem for either the two-dimensional or axisymmetric case. The reason for this is because its effect enters only through the free-surface boundary conditions and not the governing equations. Jeppson (19) included gravity in his analysis of the impingement on a two-dimensional wedge. Moayeri, et al. (31) Southwell, et al. (40) and Chan (8), all considered the effect of gravity in dealing with steady, potential, free-surface problems in which no impingement occurs.

5. Steady potential flow including surface tension. - Zhukovskii (46) has indicated how to include the effects of surface tension. He examined a two-dimensional problem using complex analysis, but his method is not extendable to either axisymmetric or three-dimensional

flows.

6. Steady three-dimensional potential flow. - Until very recently, very little had been accomplished in the area of three-dimensional potential flow with a free surface, much less including impingement. Davis and Jeppson (10) developed a computer program to solve free-surface problems of this type using the inverse method. Michelson (28, 29) also examined jets under these conditions. He treated the case of an axisymmetric jet impinging obliquely on a flat surface, and analytically showed the occurrence of wedge-shaped dry zones when the impingement angle was less than a critical value. Free-surface shapes are not obtainable using Michelson's method.

III. ORDER OF MAGNITUDE ANALYSIS

A. Formulation

The problem under consideration is the viscous flow of a circular liquid jet as it impinges normally to an infinite flat plate, as shown in Figure 1. The objective is to determine the free surface shape of the impinging liquid and the velocity profiles within the jet. In general, flows of the type described will depend on viscous, surface tension, inertial and body forces. Physical intuition tells us that if the velocity is large and the diameter of the plate is sufficiently small, there will be regions wherein viscous forces are not of prime importance in determining the resulting flow behavior, particularly the free surface shape. The viscous forces, in this case, will be confined to a thin boundary layer on the plate which originates from the stagnation point. The location of the stagnation point is shown in Figure 1. The jet or nozzle radius is R_o and the distance between the plate and nozzle is given as H. A cylindrical coordinate system (r,z) emanating from the stagnation point is chosen. An order of magnitude analysis will permit the governing equations to be simplified so that an analytical solution can be attempted. For axisymmetric, isothermal, incompressible steady flow under weightless conditions, the governing equations in cylindrical coordinates can be written:

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0 \qquad (1)$$

Momentum:

r Component:

$$\rho\left(u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right] + \frac{\partial^2 u}{\partial z^2}\right\} \tag{2}$$

z Component:

$$\rho\left(u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2}\right] \tag{3}$$

Boundary conditions are required on the flat plate, along the axis of symmetry, at the nozzle exit, on the free surface, and at $r = L$.

On plate

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\} \quad \text{on} \quad z = 0, \text{ all } r \tag{4}$$

Along axis of symmetry

$$\left.\begin{aligned} u &= 0 \\ \frac{\partial v}{\partial r} &= 0 \end{aligned}\right\} \quad \text{on} \quad r = 0, \text{ all } z \tag{5}$$

At the nozzle

$$\left.\begin{aligned} v &= -V \\ u &= 0 \end{aligned}\right\} \quad \text{on} \quad 0 \le r \le R_o,\ z = H \tag{6}$$

At $r = L$

$$\left.\begin{aligned} u &= u(z) \\ v &= v(z) \end{aligned}\right\} \quad \text{on} \quad r = L,\ 0 < z \le f(L) \tag{7}$$

On the free surface, denoted by $z_s = f(r)$, two boundary conditions are required since the free surface position is an unknown to be determined as part of the solution. The details of the calculation for the boundary conditions along the free surface can be found in Appendix A. (See eqs. (A.10) and (A.14).)

On the free surface

$$\frac{1}{2}\,(u^2 + v^2) - \frac{\sigma}{\rho r}\,\frac{d}{dr}\left[\frac{r\,\frac{df}{dr}}{\sqrt{1 + \left(\frac{df}{dr}\right)^2}}\right] = \frac{1}{2}\,V^2 - \frac{\sigma}{\rho R_o} \qquad \text{on} \quad z_s = f(r) \tag{8}$$

and

$$-u\,\frac{df}{dr} + v = 0 \qquad \text{on} \quad z_s = f(r) \tag{9}$$

In equation (4), the no-flow and no-slip boundary conditions are applied at the wall. Equation (5) is a statement involving the known geometrical symmetry of the problem, while equation (6) imposes an initially uniform velocity profile on the incoming jet. Equation (7) simply states the velocity distribution as the liquid leaves the control volume. Equation (8) is a statement of conservation of mechanical energy along a streamline, while equation (9) states that the normal velocity component on a streamline is zero. The second terms on the left and right sides of equation (8) are the contribution of surface tension to the mechanical energy balance.

The solution of the problem can be greatly facilitated by simplifying equations (1) to (8). Specifically, the method of obtaining the minimum parametric representation of a problem will be employed in order to simplify the governing equations. This method is described in detail by Krantz (23) and is the most systematic approach for scaling the governing equations. The initial step in the minimum parametric representation method is to form dimensionless variables by introducing characteristic scale factors for all dependent and independent variables. The unknown scale factors are defined as U_o, V_o, r_o, z_o, p_o, and f_o.

Dimensionless variables are now defined as:

$$u^* = \frac{u}{U_o},\ v^* = \frac{v}{V_o},\ r^* = \frac{r}{r_o},\ z^* = \frac{z}{z_o},\ p^* = \frac{p}{p_o},\ f^* = \frac{f}{f_o} \qquad (10)$$

Introducing these dimensionless variables into the differential equations and boundary conditions, and arbitrarily making the coefficient of one term in each differential equation and boundary condition equal to unity, results in:

Continuity:

$$\frac{\partial v^*}{\partial z^*} + \frac{U_o z_o}{V_o r_o} \frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* u^*) = 0 \qquad (11)$$

Momentum:

r Component:

$$\frac{\rho U_o z_o^2}{r_o \mu} u^* \frac{\partial u^*}{\partial r^*} + \frac{\rho V_o z_o}{\mu} v^* \frac{\partial u^*}{\partial z^*} = - \frac{p_o z_o^2}{r_o \mu U_o} \frac{\partial p^*}{\partial r^*}$$

$$+ \frac{z_o^2}{r_o^2} \frac{\partial}{\partial r^*} \left[\frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* u^*)\right] + \frac{\partial^2 v^*}{\partial z^{*2}} \qquad (12)$$

z Component:

$$\frac{U_o z_o}{r_o V_o} u^* \frac{\partial v^*}{\partial r^*} + v^* \frac{\partial v^*}{\partial z^*} = - \frac{p_o}{\rho V_o^2} \frac{\partial p^*}{\partial z^*}$$

$$+ \frac{\mu z_o}{r_o^2 \rho V_o} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial v^*}{\partial r^*}\right) + \frac{\mu}{z_o \rho V_o} \frac{\partial^2 v^*}{\partial z^{*2}} \qquad (13)$$

Boundary conditions:

On wall

$$u^* = 0 \quad \text{on} \quad z^* = 0, \text{ all } r^* \qquad (14)$$
$$v^* = 0$$

Along axis of symmetry

$$u^* = 0 \quad \text{on} \quad r^* = 0, \text{ all } z^* \qquad (15)$$
$$\frac{\partial v^*}{\partial r^*} = 0$$

At the nozzle

$$v^* = -\frac{V}{V_o} \qquad 0 \leq r^* \leq \frac{R_o}{r_o} \qquad (16)$$
$$u^* = 0 \qquad z^* = \frac{H}{R_o}$$

At $r = L/R_o$

$$u^* = u^*(z^*) \qquad \text{on} \quad r^* = \frac{L}{r_o} \qquad (17)$$
$$v^* = v^*(z)$$

On the free surface

$$\frac{1}{2}\left(u^{*2}\frac{U_o^2}{V_o^2} + v^{*2}\right) - \frac{\sigma}{\rho r^* V_o^2}\frac{f_o}{r_o^2}\frac{d}{dr^*}\left[\frac{r^*\frac{df^*}{dr^*}}{\sqrt{1 + \frac{f_o^2}{r_o^2}\left(\frac{df^*}{dr^*}\right)^2}}\right]$$

$$= \frac{1}{2}\frac{V^2}{V_o^2} - \frac{\sigma}{\rho R_o V_o^2} \qquad \text{on} \quad z_s^* = \frac{f^* f_o}{z_o} \qquad (18)$$

and

$$v^* - \frac{U_o f_o}{V_o r_o} u^* \frac{df^*}{dr^*} = 0 \qquad \text{on} \quad z_s^* = \frac{f^* f_o}{z_o} \qquad (19)$$

The scale factors must now be determined. This is done by setting some of the resulting dimensionless groups in the equations and boundary conditions equal to zero or unity, the groups chosen depending upon the physical conditions for which the equations are being scaled (23). Characteristic lengths are usually determined from the dimensionless groups generated by the boundary conditions, while characteristic times, velocities, etc., are determined from dimensionless groups generated by the differential equations. The guidelines in determining the unknown scale factors are:

(1) Do not introduce any mathematical contradictions

(2) Do not violate physical intuition.

Boundary conditions. - Examining the boundary conditions. it is apparent that the following dimensionless groups are introduced:

$$\frac{V}{V_o}, \quad \frac{R_o}{r_o}, \quad \frac{H}{z_o}, \quad \frac{L}{r_o}, \quad \frac{f(L)}{z_o}, \quad \frac{U_o^2}{V_o^2}, \quad \frac{\sigma f_o}{\rho V_o^2 r_o^2}, \quad \frac{f_o^2}{r_o^2}, \quad \frac{\sigma}{\rho R_o V_o^2}, \quad \frac{f_o}{z_o}, \quad \frac{U_o f_c}{V_o r_o}$$

It is known that v has the range 0 to $-V$, r has the range 0 to L, z has the range 0 to H and $f(r)$ also has the range 0 to H. Therefore, setting,

$$\frac{V}{V_o} = 1 \qquad \text{and} \qquad \frac{L}{r_o} = 1 \tag{20}$$

implies that

$$V_o = V \qquad \text{and} \qquad r_o = L \tag{21}$$

and yields two of the six unknown scale factors.

Some of the above remaining dimensionless groups cannot be set equal to one or zero without introducing contradictions. Setting $V_o = V$ and $r_o = L$ into the above ratios, and since it would be expected that

$$f_o = z_o \qquad (22)$$

the following meaningful ratios remain:

$$\frac{H}{z_o}, \quad \frac{U_o^2}{V^2}, \quad \frac{\sigma z_o}{\rho V^2 L^2}, \quad \frac{z_o^2}{L^2}, \quad \frac{U_o z_o}{VL}$$

Setting the second or fourth ratio equal to one or zero would violate physical intuition. Therefore, the ratios to be considered are

$$\frac{H}{z_o}, \quad \frac{\sigma z_o}{\rho V^2 L^2}, \quad \frac{U_o z_o}{VL}$$

At this point an attempt was made to set $H/z_o = 1$ such that z_o would equal H. This seemed logical because 0 to H was the range of z. However, this leads to some confusing results in terms of the physics. For a given flow condition, it is argued that for a certain (minimum) value of H up to $H = \infty$, the flow pattern in the vicinity of the plate is not expected to change. This is shown schematically in Figure 2. This argument has been experimentally verified and will be discussed at some length in Section IV, Experimentation. The major point is that H cannot be a characteristic length in the problem either with reference to z_o or f_o. This leaves two remaining ratios from consideration of the boundary conditions.

$$\frac{\sigma z_o}{\rho V^2 L^2} \quad \text{and} \quad \frac{U_o z_o}{VL}$$

Accordingly, all possible information from the boundary conditions has been obtained. Two of the six scale factors and a relationship between two others has been determined. The governing equations must now be examined.

B. Continuity Equation

From the physics of the problem, it is known that mass must be conserved. Hence, the continuity equation must be valid in its dimensionless form (eq. (11)). If the dimensionless derivatives $\partial v^*/\partial z^*$ and $(1/r^*)(\partial/\partial r^*)(r^*u^*)$ are to be of the same order of magnitude, it is required that

$$\frac{U_o z_o}{V_o r_o} = 1 \tag{23}$$

With $V_o = V$ and $r_o = L$, it is found that

$$\frac{U_o z_o}{VL} = 1 \tag{24}$$

Solving for U_o,

$$U_o = \frac{VL}{z_o} \tag{25}$$

This, of course, is an equation relating two unknowns, U_o and z_o. It is noted that the same information could have been obtained by setting the second of the two ratios remaining from the consideration of the boundary conditions equal to one.

C. Momentum Equations

It is the objective of this analysis to define that portion of the flow for which an inviscid solution is valid. For this case, the pressure forces are balanced by the inertia forces. This fact allows us to determine the scale factor for the pressure, P_o. If the dimensionless pressure gradient in equation (13) is to be the same order of magnitude as the dimensionless inertia term,

$$\frac{P_o}{\rho V_o^2} = 1 \tag{26}$$

This implies that

$$P_o = \rho V_o^2 = \rho V^2 \tag{27}$$

thus, the characteristic pressure is the stagnation value.

D. Physics of z_o

The remaining unknown to be determined is z_o. At this point some physical arguments are necessary. Recall that it has been shown that H cannot be considered as a characteristic scale factor for the problem at hand for reasonably large values of H. However, the free surface shape is expected to change as R_o varies (see Fig. 3). Therefore, from physical considerations this suggest that characteristic values of z_o vary as R_o. Defining

$$z_o = R_o \tag{28}$$

then U_o can be found from equation (25).

$$U_o = \frac{VL}{R_o} \tag{29}$$

Summarizing, the following has been determined

$$V_o = V$$

$$r_o = L$$

$$z_o = R_o$$

$$f_o = R_o$$

$$U_o = \frac{VL}{R_o}$$

$$P_o = \rho V^2$$

These scale factors can now be substituted into the governing equations to obtain the minimum parametric representation of the problem. The results are as follows:

$$\text{Re}\ u^* \frac{\partial u^*}{\partial r^*} + \text{Re}\ v^* \frac{\partial u^*}{\partial z^*} = -\text{Re}\left(\frac{R_o}{L}\right)^2 \frac{\partial p^*}{\partial r^*}$$

$$+ \left(\frac{R_o}{L}\right)^2 \frac{\partial}{\partial r^*}\left[\frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* u^*)\right] + \frac{\partial^2 v^*}{\partial z^{*2}} \tag{30}$$

$$\text{Re}\ u^* \frac{\partial v^*}{\partial r^*} + \text{Re}\ v^* \frac{\partial v^*}{\partial z^*} = -\text{Re} \frac{\partial p^*}{\partial z^*}$$

$$+ \left(\frac{R_o}{L}\right)^2 \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial v^*}{\partial r^*}\right) + \frac{\partial^2 v^*}{\partial z^{*2}} \tag{31}$$

where Re is the Reynolds number defined as

$$\text{Re} = \frac{\rho V R_o}{\mu} \tag{32}$$

The equations are now in the form in which it can be determined what the conditions must be such that viscous forces are not significant. The major parameters in this problem are (R_o/L) and Re. Since all the starred or dimensionless terms in equations (30) and (31) are of unit order, only the coefficients of the individual terms need be considered in order to make statements regarding simplifications. Considering equation (30), it can be seen that since $(R_o/L)^2 \ll 1$, both the inertia and pressure forces will be an order of magnitude greater than the viscous forces provided that,

$$\text{Re} \gg 1$$

$$\text{Re}\left(\frac{R_o}{L}\right)^2 \gg 1 \tag{33}$$

From equation (31), since $(R_o/L)^2 \ll 1$, no new information is obtained. The governing equations will be reduced to a simplified form of Euler's equations of motion.

$$u^* \frac{\partial u^*}{\partial r^*} + v^* \frac{\partial u^*}{\partial z^*} = - \left(\frac{R_o}{L}\right)^2 \frac{\partial p^*}{\partial r^*} \tag{34}$$

$$u^* \frac{\partial v^*}{\partial r^*} + v^* \frac{\partial v^*}{\partial z^*} = - \frac{\partial p^*}{\partial z^*} \tag{35}$$

E. Results

Restricting the viscous forces to be at least two orders of magnitude smaller than the inertial or pressure forces, there results

$$Re \geq 100$$

$$Re \left(\frac{R_o}{L}\right)^2 \geq 100 \tag{36}$$

Since (R_o/L) is less than one by definition, the coefficient to consider is the second one in equation (36), since this will be the limiting one. Under the following restrictions,

(1) $Re \geq 100$

(2) $\frac{R_o}{L} < 1$ (37)

the Euler's equation of motion are obtained. The equation $Re(R_o/L)^2 = 100$ is shown graphically in Figure 4. The line shown in Figure 4 separates the inviscid from the viscous region. A physical understanding of the problem is made clearer by reference to this plot. The higher the incoming jet Reynolds number becomes, the thinner will be the boundary layer at some fixed radial position from the stagnation point. As seen from Figure 4, at lower values of the ratio R_o/L (perhaps obtained by increasing the disk radius L), higher Reynolds numbers are required in order to avoid viscous influence. Finally, within the viscous region, the boundary layer will grow until a radial length is reached where it becomes equal to the free surface height. The effective design of an experiment is now possible so that the flow can be considered essentially inviscid.

IV. ZERO GRAVITY EXPERIMENTATION

A. Apparatus and Procedure

1. Test facility. - The experimental investigation was conducted in the 2.2-second drop tower. The exact specifications of the facility, the mode of operation, and release and recovery systems are described in detail in Appendix B. The drop tower provides us with a 2.2 second weightless environment in which to conduct the tests.

2. Experiment. - The experiment package used to obtain the data for this study is shown in Figure 5. It consists of an aluminum frame in which were mounted the jet reservoir, disk, a 16 millimeter high speed motion picture camera, supply tank, backlighting scheme, and batteries. The major functions were controlled by on-board sequence timers.

A diagram indicating the manner in which the flow system operates is shown in Figure 6. This is a pressure controlled system in which the flow was initiated by opening the solenoid valve. Prior to the drop, liquid was contained in the line between the liquid supply tank and the jet reservoir. In addition, the jet reservoir was completely filled with the test liquid.

Initially, the object was to employ two-dimensional jets during the experiments. Schach (38), was the only author who investigated two-dimensional jets, although it is impossible to determine how rectangular his jets really were. Several attempts were made to fabricate a two-dimensional nozzle (slot) for use in the zero gravity studies.

Initially, the slots built were quite crude; a rectangular hole cut in a section of plexiglass; a balsa wood jet sealed with epoxy cement. Later, slots were accurately designed in order to achieve the desired flow. The slot length was arbitrarily chosen as 5 centimeters and its width as 0.25 centimeters. Approximately 45° tapers were made to the opening along both the wide and narrow sides. The results of the zero gravity testing were as follows: The jet contracted in the long direction (5 cm) and expanded in the narrow direction (0.25 cm). A tendency to become cylindrical probably due to the effects of surface tension, was observed. A redesigned version of the slot was tested, which employed absolutely no taper in the narrow direction. This also failed to yield a rectangular jet. When these approximate slot jets impinged on various flat plates, the jet diverged radially. In other words, it was impossible to prevent spreading in the lateral direction for an unconstrained surface. Various methods were tried to eliminate this lateral spreading and to force the impingement flow to be completely two-dimensional. The flow was impinged on rectangular plates having the same width as that of the slot, 5 centimeters. The result of these tests was that the liquid simply fell off or went around the plate. A large rectangular plate was used and the region greater in width than 5 centimeters was sprayed with fluorocarbon since distilled water does not wet a fluorocarbon surface. This also failed. All attempts at using a two-dimensional slot jet were subsequently abandoned and axisymmetric jets were pursued.

A schematic drawing of the jet reservoir, which was fabricated out of acrylic plastic, is shown in Figures 7(a) and (b). The critical feature of the jet reservoir is the 45° taper to a circular hole of diameter D. The range of diameters studied was from 0.5 to 1.5 centimeters. In addition, the transition between the circular hole and conical taper was rounded smooth. The taper prohibits boundary layer buildup and allows the liquid jet to exit from the reservoir with a nearly uniform velocity profile. Based on the experimental results of reference 24, a total angle of 90° is more than sufficient to insure a uniform exiting velocity profile over the range of Reynolds numbers studied.

Sharp-edge disks, also fabricated from acrylic plastic, were mounted above the jet reservoir by means of a threaded rod such that the flat surface of the disk was at right angles to the impinging liquid jet. Sharp-edged disks were employed in the study rather than finite thickness disks in order to develop an analytical model for the flow. As was learned later, the thickness of the disk edge is only of importance for those flows dominated by the effects of surface tension. The diameters, 2L, of the disks were 2.0 and 3.0 centimeters. A schematic drawing of the disks is shown in Figure 8.

3. Test liquids. - Two test liquids were employed, anhydrous ethanol and trichlorotrifluoroethane. Their properties at 20° C are listed in Table 1. No attempt was made to correct the fluid properties for temperature changes. It is noted that both of these test fluids possess a nearly 0° contact angle on an acrylic plastic surface. However, this was not the reason why they were chosen as test fluids. They were chosen because of their relatively low viscosity and availability.

4. Test procedure. - Prior to a test run, the jet reservoir, disk and supply tanks were cleaned ultrasonically with a mild detergent.

After these parts were rinsed with methanol, they were dried in a warm air dryer. The supply tank was subsequently filled with the test liquid and the jet reservoir was filled by pressurizing the supply tank. This procedure eliminated air bubbles from the lines ensuring accurate flow rates. After the jet reservoir was completely full, the supply tank was sealed and two accumulator bottles (not shown in Fig. 5) were pressurized with gaseous nitrogen to a predetermined value. The accumulator bottles were designed to be of such a volume that no appreciable pressure drop occurred during the drop.

Electrical timers on the experiment package were set to control the initiation and duration of all functions programmed during the drop. The experiment package was then balanced and positioned within the prebalanced drag shield. The wire support was attached to the experiment package through an access hole in the shield (see Fig. B3(a) in Appendix B). Properly sized spikes tips were installed on the drag shield. Then the drag shield, with the experiment package inside, was hoisted to the predrop position at the top of the facility (Fig. B1) and connected to an external electrical power source. The wire support was attached to the release system, and the entire assembly was suspended from the wire. After final electrical checks were made and the experiment package was switched to internal power, the system was released. After completion of the test, the experiment package and drag shield were returned to the preparation area.

B. Experimental Results

1. General considerations. - In addition to the measurement and observation of steady state liquid flow patterns, two separate phenomena were observed. First, no circular hydraulic jump occurred during any

of the tests even though they are a common occurrence under normal gravity conditions. Secondly, the initial impact of the jet upon the solid surface provided another unusual phenomena, that of the rebounding liquid droplet. At high flow rates, the jet broke up prior to impinging upon the disk. The first droplet tended to impinge upon and stick to the disk, spreading as it did. However, the second droplet impinged and rebounded off this wetted surface sometimes into the incoming liquid jet. This provided no serious problem with the attainment of a steady state flow pattern since this all occurred during the transient phase.

The jet generally appeared to go through three phases during the impingement process. The initial phase, including the droplet pinch-off and subsequent impingement, was termed the transient phase. After a certain period of time, a steady state flow pattern was achieved from which the free surface shapes were measured and observations were made. A third phase was reached shortly after the jet had reached its equilibrium configuration. The flow pattern developed an instability. Initially, the instability started from the jet which began to oscillate, and then spread to the plume. Since the time over which the jet and flow pattern appeared stable and smooth was finite, steady state data could be cbtained. It was observed that the time before breakdown was inversely proportional to the back pressure during the flow. At 1 psia, for example, the jet remained stable for 1.5 seconds while at 10 psi, it was stable for approximately 0.4 seconds. An attempt was made to lengthen the time over which stability occurred by packing the nozzle chamber with steel wool. This appeared to be effective in improving the overall stability but had the negative effect of introducing a low level perturbation throughout the test and, thus, was not employed during any of the experiments.

2. Effect of nozzle height. - Several tests were conducted initially to determine the effect of H, the distance between the nozzle and the disk, on the experimental flow pattern. It was determined that there is no effect on the liquid flow provided that the ratio of the nozzle height H to the jet diameter is greater than 3. As a result, all zero gravity experiments were conducted in order to eliminate this effect. This fact, which was determined experimentally, supports the argument made in Section III, Order of Magnitude Analysis, concerning the assertion that H could not be a scale factor for the axial coordinate.

3. Steady state flow patterns. - The approximate steady state flows for three different jet velocities are shown photographically in Figure 9. The direction of flow of the liquid jet is vertically upwards. The threaded rod and bolt observed in the film clips is the disk holder which connects the sharp-edged disk assembly to the rig frame. The jet velocity increases from left to right in the figure. Three distinct classifications of flow patterns were observed to occur and are shown labeled in Figure 9 as surface tension flow, transition flow, and inertia flow. Surface tension flow (Fig. 9(a)) is defined as that flow in which the liquid flows completely around the disk with no separation occurring from the disk edge. In transition flow (Fig. 9(b)), surface tension and inertia forces are both important. Transition flow is defined as flow in which separation occurs from the disk and the resulting liquid sheet either collects upon itself forming an envelope or has the tendency to do so. Inertia flow (Fig. 9(c))

is defined as that flow in which the liquid separates from the disk with no liquid turning towards the jet centerline attempting to form an envelope. The flow pattern shown for transition flow is not really the steady state flow pattern one would expect if steady state conditions could have been reached. The reason for this is as follows: Some liquid is always traveling toward the disk from the point at which the envelope meets. This liquid flow strikes the back of the baffle and subsequently disrupts the initially formed envelope. The recirculation flow then is a strong function of the geometry of the disk holder, an uncontrollable parameter. This important distinction means that an analysis for free surface shapes would have to account for the geometry of the disk holder in the transition region. The surface tension flow is generally slow and, as a result, does not quite reach a steady state configuration on the back side of the disk. The inertia flow tests (represented by Fig. 9(c)) always reached steady state.

The experimental tests were conducted in the inviscid region of Figure 4. Depending on the particular ratio of (R_o/L), there exists a minimum Reynolds number below which the runs can no longer be considered as viscous-free. The experimental results are listed in tabular form in Table 2 in which all the important parameters as well as the flow classifications are contained. One additional parameter, the Weber number, is listed in Table 1. As will be shown in the next section, when the Reynolds number is no longer a parameter to consider for flow classifications, the Weber number and ratio (R_o/L) remain. The Weber number $\rho V^2 R_o/\sigma$, is basically the ratio of inertia to surface tension forces. In the flow category column, S indicates surface tension flow, T indicates transition flow, and I is inertia flow. Finally, it is noted that the designated flow classification for some cases, particularly those bordering transition or inertia flow, could easily fit into either category.

4. Zero gravity results. - The data contained in Table 2 is shown graphically in Figure 10. The lines indicated in the figure were faired in by hand and separate the various flow classifications. It is observed that at any particular value of the ratio (R_o/L), the flow classification is dependent only on the system Weber number. Two critical Weber numbers occur at a constant value of (R_o/L). The lowest critical Weber number separates the surface tension flow from the transition flow while the higher critical Weber number separates the transition flow from the inertia flow. In addition, the critical Weber number between regimes was found to decrease as (R_o/L) was increased.

V. POTENTIAL FORMULATION

A. Governing Equations and Boundary Conditions in Physical Plane Including Surface Tension

In Section III, Order of Magnitude Analysis, it was shown that at any particular value of R_o/L if $Re > Re_{cr}$ the flow in the region of the disk can be considered as viscous free. It is further assumed that the jet will continue to remain viscous free after leaving the disk. There will be no shear stress between the exiting radial jet and the ambient air surrounding it.

1. General formulation. - Consider the flow of a circular liquid jet impinging normally on a circular disk as shown in Figure 11. R_o is the jet radius, L is the disk radius, and H is the distance between the jet reservoir and the disk. The incoming jet velocity is given as V and the initial velocity profile is assumed to be uniform. There will be two free surfaces involved. The upper free surface is defined as $z_s = f_1(r)$ and the lower surface as $z_s = f_2(r)$. In addition, a third surface is required for the complete formulation of the boundary value problem. Initially, this surface was chosen as the straight line FE shown in Figure 11. However, this proved to be inconvenient since FE is arbitrary and, thus, has no known boundary condition. It was found convenient to instead choose the surface $z = f_3(r)$ to be a surface along which the velocity potential is constant. Various points in the physical plane have been designated with letters ranging from A to G for convenience. In cylindrical coordinates, the governing

equations and boundary conditions in terms of primary variables (u,v) are given as follows:

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0 \qquad (38)$$

Momentum:

r Component:

$$\rho\left(u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} \qquad (39)$$

z Component:

$$\rho\left(u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial z} \qquad (40)$$

The following boundary conditions are applied,

On DC

$$v = 0 \quad \text{on} \quad z = 0,\ 0 \le r \le L \qquad (41)$$

On AB

$$u = 0 \quad 0 \le r \le R_o,\ z = H \qquad (42)$$

On BC

$$\frac{\partial v}{\partial r} = 0 \quad r = 0,\ 0 < z \le H \qquad (43)$$

On AG

$$\frac{1}{2}(u^2 + v^2) - \frac{\sigma}{\rho r}\frac{d}{dr}\left(\frac{rf_1'}{\sqrt{1 + f_1'^2}}\right) = \frac{1}{2}V^2 - \frac{\sigma}{\rho R_o} \quad \text{on} \quad z_s = f_1(r) \qquad (44)$$

And

$$-uf_1' + v = 0 \quad \text{on} \quad z_s = f_1(r) \qquad (45)$$

On DE

$$\frac{1}{2}(u^2 + v^2) - \frac{\sigma}{\rho r}\frac{d}{dr}\left(\frac{rf_2'}{\sqrt{1 + f_2'^2}}\right) = \frac{1}{2}V^2 - \frac{\sigma}{\rho R_o} \quad \text{on} \quad z_s = f_2(r) \qquad (46)$$

And

$$-uf_2' + v = 0 \qquad \text{on} \quad z_s = f_2(r) \tag{47}$$

On GE

$$u + vf_3'(r) = 0 \qquad \text{on} \quad z = f_3(r) \tag{48}$$

The derivation for the boundary condition along GE is as follows. Along GE, the velocity potential ϕ is constant. By definition, the velocity vector $\vec{V}$ is normal to an equipotential surface. This means that

$$\vec{V} \times \hat{n} = 0 \qquad \text{on} \quad z = f_3(r) \tag{49}$$

The unit normal to $f_3(r)$ is

$$\hat{n} = \frac{-f_3'\hat{i} + \hat{j}}{\sqrt{f_3'^2 + 1}} \tag{50}$$

where $\hat{i}$ is the unit vector in the radial direction and $\hat{j}$ is the unit vector in the axial direction, and

$$\vec{V} = u\hat{i} + v\hat{j} \tag{51}$$

Application of equation (49) results in

$$\frac{u + vf_3'(r)}{\sqrt{f_3'^2 \div 1}} = 0 \tag{52}$$

From which equation (48) follows.

2. <u>Introduction of Stokes Stream Function</u>. - The primary variables contained in the governing equations are the scaler velocity components u and v. The fact that two functions are required to describe one vector field is cumbersome. As shown in the theory of hydrodynamics, the number of functions can be reduced for several important cases, one of these being axisymmetric flow. A function ψ, defined as Stokes stream function, can be introduced which automatically satisfies

the continuity equation. With the additional requirement of irrotationality, the governing equation in terms of Stokes stream function will result. According to Chan (7), Stokes stream function is a mathematical device used to describe the flow and has the following properties. First, when the stream function is set equal to a constant, it results in different annular stream surfaces in axisymmetric flow. Secondly, when it is differentiated properly, it yields the velocity components. Thirdly, taking the difference between the values at two adjacent stream surfaces yields the flow rate.

Starting with equation (38), the continuity equation for axisymmetric flow, a guess is made at what u and v are in order to satisfy continuity identically. Assume

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z} \tag{53}$$

And

$$v = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{54}$$

Substitution of equations (53) and (54) into equation (38) shows that continuity is identically satisfied by these two guesses. In addition, the assumption is made that the flow is also irrotational. As a result

$$\text{Curl}\ \vec{V} = 0 \tag{55}$$

For axisymmetric flow, this can be written,

$$\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} = 0 \tag{56}$$

Replacing u and v in the above by their relationships to ψ results in the governing equation for axisymmetric flow in terms of Stokes stream function

$$\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} = 0 \tag{57}$$

3. Derivation of boundary conditions in terms of ψ. - Since BC, CD, and DE are all a part of the same streamline, they must all have the same value for Stokes stream function. Let us arbitrarily set that value equal to zero. The value of ψ along AB and AG can be calculated from equation (54).

On AB

$$-V = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{58}$$

Integrating this yields

$$\psi = -\frac{Vr^2}{2} \tag{59}$$

On AG since AG is a line of constant ψ,

$$\psi = -\frac{VR_o^2}{2} \tag{60}$$

In addition, equation (44) applies. Substitution of equation (53) and (54) into (44) yields

$$\frac{1}{2r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] - \frac{\sigma}{\rho r}\frac{d}{dr}\left(\frac{rf_1'}{\sqrt{1 + f_1'^2}}\right) = \frac{1}{2}V^2 - \frac{\sigma}{\rho R_o} \quad \text{on} \quad z_s = f_1(r) \tag{61}$$

On DE

$$\frac{1}{2r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] - \frac{\sigma}{\rho r}\frac{d}{dr}\left(\frac{rf_2'}{\sqrt{1 + f_2'^2}}\right) = \frac{1}{2}V^2 - \frac{\sigma}{\rho R_o} \quad \text{on} \quad z_s = f_2(r) \tag{62}$$

It is noted that the second boundary conditions on the free surfaces, namely $-uf_1' + v = 0$ and $-uf_2' + v = 0$, are fully equivalent to the specification of the value of Stokes stream function along that surface. Since $\psi = \psi(r,z)$, it can be expanded and $d\psi$ set equal to zero along AG and ED.

On GE, equation (48) becomes

$$-\frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial r} f_3'(r) = 0 \qquad \text{on} \quad z = f_3(r) \tag{63}$$

4. Nondimensionalization of governing equations and boundary conditions in physical plane. - The governing equation in terms of Stokes stream function (eq. (57)) and the boundary conditions (eqs. (58) to (63)) are now put into dimensionless form by introducing arbitrary scale factors. Let the scale factor for the stream function be $-VR_o^2$. Let dimensionless quantities be represented by stars, i.e., ψ^* is dimensionless. The results of this manipulation are shown in Figure 12. Three parameters appear in the specification of boundary conditions. They include the Weber number, We, and the dimensionless length ratios L/R_o and H/R_o. Recalling the arguments in the Order of Magnitude Analysis section, H/R_o is not really a parameter provided it is larger than some minimum value. The dimensionless velocity components can be calculated from the following expressions

$$\frac{u}{V} = \frac{1}{r^*}\frac{\partial\psi^*}{\partial z^*} \tag{64}$$

$$\frac{v}{V} = -\frac{1}{r^*}\frac{\partial\psi^*}{\partial r^*} \tag{65}$$

The procedure for solution of the boundary value problem as set up in dimensionless form in the physical plane (Fig. 12) would be as follows: Initially, realistic variations for $f_1^*(r)$, $f_2^*(r)$, and $f_3^*(r)$ are assumed. Using only one of the two given boundary conditions on AG and GE (i.e., $\psi^* = 1/2$ and $\psi^* = 0$) solve for ψ^* using finite difference methods. With the initial solution for ψ^* check the validity of the second of the two boundary conditions on AG and GE. If the boundary conditions are not satisfied, new variations in f_1^*, f_2^*, and f_3^* must be assumed. A second iteration to ψ^* must be obtained and so on. One of the serious drawbacks of this outlined iteration scheme is the lack of knowledge concerning how to update assumed values of f_1^*, f_2^*, and f_3^* based on previous solutions. In other words, there is no logical way in which to make changes to the shape of the initially assumed control volume.

5. Surface tension dominated model. - For Weber numbers between 5 and 30 (depending on the ratio R_0/L), experimental data shows that the resulting steady-state flow pattern is surface tension dominated (see Fig. 9(a)). By previous definition, surface tension flow is defined as that flow in which the liquid flows completely around the disk with no separation from the edge. It is the intent to model this flow in order to solve for the theoretical free surface shapes and velocity profiles. Assuming axisymmetry, the physical plane model is shown in Figure 13. In the model, at some cross-section far downstream, the flow is assumed to approach the initially uniform flow it possessed at AB. The exiting plane is denoted by GE in the model. C and C' are both located at $r = 0$, $z = 0$. C is located on top of the plate while C' is on the bottom. The free surface $f_1(r)$, is not assumed to possess mirror symmetry about the $z = 0$ position. In cylindrical coordinates, the governing equations and boundary conditions in terms of the primary variables (u,v) are given as follows:

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0 \tag{66}$$

Momentum:

r Component

$$\rho\left(u\,\frac{\partial u}{\partial r} + v\,\frac{\partial u}{\partial z}\right) = -\,\frac{\partial p}{\partial r} \tag{67}$$

z Component

$$\rho\left(u\,\frac{\partial v}{\partial r} + v\,\frac{\partial v}{\partial z}\right) = -\,\frac{\partial p}{\partial z} \tag{68}$$

The following boundary conditions are applied

On DC

$$v = 0 \qquad \text{on} \quad z = 0,\ 0 \le r \le L \tag{69}$$

On DC'

$$v = 0 \qquad \text{on} \quad z = 0,\ 0 \le r \le L \tag{70}$$

On AB

$$u = 0 \qquad 0 \le r \le R_o,\ z = H \tag{71}$$

On BC

$$\frac{\partial v}{\partial r} = 0 \qquad \text{on} \quad r = 0,\ 0 < z \le H \tag{72}$$

On C'E

$$\frac{\partial v}{\partial r} = 0 \qquad \text{on} \quad r = 0,\ -H \le z < 0 \tag{73}$$

On GE

$$u = 0 \qquad 0 \le r \le R_o,\ z = -H \tag{74}$$

On AG

$$\frac{1}{2}\,(u^2 + v^2) - \frac{\sigma}{\rho r}\,\frac{d}{dr}\left(\frac{r f_1'}{\sqrt{1 + f_1'^2}}\right) = \frac{1}{2}\,V^2 - \frac{\sigma}{\rho R_o} \qquad \text{on} \quad z_s = f_1(r) \tag{75}$$

and

$$-u f_1' + v = 0 \qquad \text{on} \quad z_s = f_1(r) \tag{76}$$

Equations (75) and (76) represent two distinct pieces of information concerning $z_s = f_1(r)$. It is noted that f_1' can be eliminated from the set of equations by means of simple substitution. This fact will have implications later in the development leading to a major simplification. The following expression would result,

$$\frac{1}{2}\left(u^2 + v^2\right) - \frac{\sigma}{\rho r}\frac{d}{dr}\left(\frac{r}{\sqrt{\frac{u^2}{v^2} + 1}}\right) = \frac{1}{2}V^2 - \frac{\sigma}{\rho R_o} \qquad \text{on} \quad z_s = f_1(r) \qquad (77)$$

Direct substitution of equations (53) and (54), the relations between the velocity components and Stokes stream function, into the governing equation and boundary conditions for the surface tension model results in the formulation shown in Figure 14 after nondimensionalization. Similar to the general formulation in the last section, the various lengths in the problem are scaled with respect to R_o and the scale factor for Stokes stream function is $-VR_o^2$.

B. Inverse Plane Formulations

The procedure for solving for the free surface shapes in the physical plane has been outlined previously. The difficulties encountered when making adjustments to the free surfaces between iterations and the lack of a logical manner in which to make the adjustments have been cited. This would be a time consuming task even in the absence of surface tension. A computerized scheme is sought which offers the possibility of achieving the free-surface results with a minimum of computer iteration time and user interaction.

1. Transformation formulas. - An alternate approach to the physical plane solution is discussed in detail by Jeppson (19). In his article, Jeppson discusses a transformation technique into what is defined as the inverse plane. The coordinate axes in the inverse plane are the velocity potential ϕ and Stokes stream function ψ. The ad-

vantage to using the $\phi\psi$ space, or inverse plane, is that the free surface lies along a line of constant ψ and is, therefore, at a known position. Of course, one must pay the price for knowing the position of the free surface. As will be shown shortly, the governing equation is no longer linear as it was in the physical plane.

2. Introduction of velocity potential. - The use of a scaler function defined as the velocity potential has been used previously in the specification of the boundary GE (the exiting plane). The continuity equation for steady incompressible axisymmetric flow is given by equation (38). The flow is also assumed to be irrotational such that the condition given by equation (56) is also valid. Equation (56) implies that there exists a scaler potential function ϕ such that

$$\vec{V} = \text{grad } \phi \tag{78}$$

from which it follows

$$u = \frac{\partial \phi}{\partial r} \tag{79}$$

and

$$v = \frac{\partial \phi}{\partial z} \tag{80}$$

Substitution of equations (79) and (80) yields the governing equation for steady axisymmetric flow in terms of the velocity potential

$$\frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} = 0 \tag{81}$$

3. Relationship between ψ and ϕ. - It can now be stated that the relationships between the velocity potential and Stokes's stream function are given as:

$$u = \frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{82}$$

and

$$v = \frac{\partial\phi}{\partial z} = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{83}$$

4. Inverse plane formulation. - In order to formulate the problem in the physical plane, r and z must be known as functions of ψ and ϕ. In other words, we are attempting to reverse the roles of the dependent and independent variables in the inverse plane. The required relationships are obtained noting that if $\psi = \psi(r,z)$ and $\phi = \phi(r,z)$, then there exist inverse functions $r = r(\phi,\psi)$ and $z = z(\phi,\psi)$ as shown in reference 13 such that

$$\frac{\partial r}{\partial\phi} = -\frac{1}{J}\frac{\partial\psi}{\partial z} \tag{84}$$

$$\frac{\partial z}{\partial\psi} = -\frac{1}{J}\frac{\partial\phi}{\partial r} \tag{85}$$

$$\frac{\partial z}{\partial\phi} = \frac{1}{J}\frac{\partial\psi}{\partial r} \tag{86}$$

$$\frac{\partial r}{\partial\psi} = \frac{1}{J}\frac{\partial\phi}{\partial z} \tag{87}$$

and where the Jacobian J is defined as

$$J = \begin{vmatrix} \frac{\partial\phi}{\partial r} & \frac{\partial\phi}{\partial z} \\ \frac{\partial\psi}{\partial r} & \frac{\partial\psi}{\partial z} \end{vmatrix} = \left(\frac{\partial\phi}{\partial r}\right)\left(\frac{\partial\psi}{\partial z}\right) - \left(\frac{\partial\phi}{\partial z}\right)\left(\frac{\partial\psi}{\partial r}\right) \tag{88}$$

From the above set of equations follow two important relations. Substituting into equation (82) the values of $\partial\phi/\partial r$ and $\partial\psi/\partial z$ obtained from equations (84) and (85) yields

$$\frac{\partial z}{\partial\psi} = -\frac{1}{r}\frac{\partial r}{\partial\phi} \tag{89}$$

Similarly, substitution of the values of $\partial\phi/\partial z$ and $\partial\psi/\partial r$ from equations (86) and (87) into equation (83) yields

$$\frac{\partial r}{\partial\psi} = \frac{1}{r}\frac{\partial z}{\partial\phi} \tag{90}$$

For subsequent relations, it is necessary to express the Jacobian in terms of the inverse function $r(\psi,\phi)$ and $z(\psi,\phi)$. Substituting equations (84) to (87) into equation (88) results in

$$J = J^2\left[\left(\frac{\partial z}{\partial\psi}\right)\left(\frac{\partial r}{\partial\phi}\right) - \left(\frac{\partial r}{\partial\psi}\right)\left(\frac{\partial z}{\partial\phi}\right)\right] \tag{91}$$

Now, using equations (89) and (90) in the above yields,

$$J = -\frac{1}{r\left[\left(\frac{\partial z}{\partial\psi}\right)^2 + \left(\frac{\partial r}{\partial\psi}\right)^2\right]} = -\frac{r}{\left(\frac{\partial r}{\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial\phi}\right)^2} \tag{92}$$

5. <u>Governing equations for inverse functions</u>. - Differentiating equation (89) with respect to ψ yields

$$\frac{\partial^2 z}{\partial\psi^2} = -\frac{1}{r}\frac{\partial^2 r}{\partial\psi\,\partial\phi} + \frac{1}{r^2}\frac{\partial r}{\partial\psi}\frac{\partial r}{\partial\phi} \tag{93}$$

and combining this with the derivatives of equation (90) with respect to ϕ, which is

$$\frac{\partial^2 r}{\partial\phi\,\partial\psi} = -\frac{1}{r^2}\left(\frac{\partial r}{\partial\phi}\right)\left(\frac{\partial z}{\partial\phi}\right) + \frac{1}{r}\frac{\partial^2 z}{\partial\phi^2} \tag{94}$$

it is found that

$$\frac{\partial^2 z}{\partial\psi^2} = \frac{1}{r^3}\frac{\partial r}{\partial\phi}\frac{\partial z}{\partial\phi} - \frac{1}{r^2}\frac{\partial^2 z}{\partial\phi^2} + \frac{1}{r^2}\frac{\partial r}{\partial\phi}\frac{\partial r}{\partial\psi} \tag{95}$$

By using equations (89) and (90), the terms involving derivatives in equation (95) can be expressed entirely in terms of z, giving the important equation,

$$r^2\frac{\partial^2 z}{\partial\psi^2} + \frac{\partial^2 z}{\partial\phi^2} = -2\frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\phi} \tag{96}$$

On the other hand, differentiating equation (89) with respect to ϕ and equation (90) with respect to ψ and combining the result leads to the equation

$$\frac{\partial^2 r}{\partial \psi^2} = -\frac{1}{r^2}\frac{\partial^2 r}{\partial \phi^2} + \frac{1}{r^3}\left(\frac{\partial r}{\partial \phi}\right)^2 - \frac{1}{r^2}\frac{\partial r}{\partial \psi}\frac{\partial z}{\partial \phi} \tag{97}$$

By substitution for $\partial z/\partial \phi$ from equation (90), the following equation for $r(\phi,\psi)$ is obtained:

$$\frac{\partial^2 r}{\partial \psi^2} + \frac{1}{r^2}\frac{\partial^2 r}{\partial \phi^2} - \frac{1}{r^3}\left(\frac{\partial r}{\partial \phi}\right)^2 + \frac{1}{r}\left(\frac{\partial r}{\partial \psi}\right)^2 = 0 \tag{98}$$

For further discussion, equation (96) will be referred to as the z equation while equation (98) will be called the r equation. Both of these equations are nonlinear with the nonlinearity in the z equation also involving r. It is further noted that both of these equations are of the elliptic type, as shown in reference 19. This means that boundary conditions are required for all boundaries in the flow region. Since the r equation (eq. (98)) only involves that variable, a solution to the problem begins with its solution.

One final point involves the fact that the variables discussed in the above inverse transformations are all dimensional. It is noted that nondimensionalization of the above equations similar to the nondimensionalization done in the physical plane allows us to recover the same equation.

Let

$$\left.\begin{aligned}
z^* &= \frac{z}{z_o} && \text{where } z_o = R_o \\
r^* &= \frac{r}{r_o} && \text{where } r_o = R_o \\
\psi^* &= \frac{\psi}{\psi_o} && \text{where } \psi_o = -VR_o^2 \\
\phi^* &= \frac{\phi}{\phi_o} && \text{where } \phi_o = -VR_o
\end{aligned}\right\} \tag{99}$$

The results of this substitution yields exactly the same governing equations and transformation formulas in starred notation. In other words,

$$r^{*2} \frac{\partial^2 z^*}{\partial \psi^{*2}} + \frac{\partial^2 z^*}{\partial \phi^{*2}} = -2 \frac{\partial z^*}{\partial \psi^*} \frac{\partial z^*}{\partial \phi^*} \tag{100a}$$

and

$$\frac{\partial^2 r^*}{\partial \psi^{*2}} + \frac{1}{r^{*2}} \frac{\partial^2 r^*}{\partial \phi^{*2}} - \frac{1}{r^{*3}} \left(\frac{\partial r^*}{\partial \phi^*}\right)^2 + \frac{1}{r^*} \left(\frac{\partial r^*}{\partial \psi^*}\right)^2 = 0 \tag{100b}$$

6. Infinite flat plate excluding surface tension. - The first problem to be examined in which we employ the inverse transformation is the flow of a circular liquid jet normal to a flat plate. At this point, we drop the starred notation to designate dimensionless quantities. It is assumed that all quantities appearing henceforth are dimensionless unless otherwise stated (i.e., r, z, ψ, ϕ, are now dimensionless). For the case in which surface tension is excluded, the physical plane model depicted in Figure 12 must necessarily change. The model in Figure 12 is for the general case of a finite disk in which surface tension effects are important. For the case of an infinite flat plate, since the free surface ED no longer exists, the designation D does not appear in the model. Furthermore, for the case in which surface tension forces are unimportant, the boundary condition along the upper surface (referring to Fig. 12) can be written as follows:

On AG

$$\frac{1}{r^2} \left[\left(\frac{\partial \psi}{\partial z}\right)^2 + \left(\frac{\partial \psi}{\partial r}\right)^2\right] = 1 \tag{101}$$

This expression is obtained by allowing the Weber number to approach infinity. The reasoning for letting the Weber number become large is because it is the ratio of inertial to surface tension forces. The case we are examining is one in which the surface tension forces become vanishingly small or the inertial forces becoming large. Finally, the boundary GE will be nearly vertical implying that the velocity there is purely radial.

7. Derivation of boundary conditions in inverse plane for r formulation. -

On AB

$$\left.\begin{array}{l}\text{In the physical plane, } \psi = \frac{1}{2} r^2 \\ \text{In the inverse plane, } r = \sqrt{2\psi}\end{array}\right\} \quad (102)$$

On BC

$$\left.\begin{array}{l}\text{In the physical plane, } \psi = 0 \\ \text{In the inverse plane, } r = 0\end{array}\right\} \quad (103)$$

On EC

$$\left.\begin{array}{l}\text{In the physical plane, } \psi = 0 \\ \text{Also } v = 0\text{, by equation (83) this implies that } \frac{\partial\phi}{\partial z} \text{ and} \\ \quad \frac{\partial\psi}{\partial r} = 0. \\ \text{In the inverse plane (via eq. (87)), } \frac{\partial r}{\partial\psi} = 0\end{array}\right\} \quad (104)$$

On GE

$$\left.\begin{array}{l}\text{In the physical plane, } \frac{\partial\psi}{\partial r} = 0 \\ \text{In the inverse plane, } \frac{\partial r}{\partial\psi} = 0\end{array}\right\} \quad (105)$$

On AG

In the physical plane, $\frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] = 1$

In the inverse plane, using equations (82) and (83), this becomes

$$\left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 = 1$$

and upon using relations (85) and (87)

$$J^2 \left[\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial r}{\partial \psi}\right)^2\right] = 1$$

Substituting in for J^2 from equation (92) and then using (89) and (90) yields the final form

$$\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = 1 \tag{106}$$

The results of the above calculations are shown in Figure 15(a) in the inverse plane. For the case of flow normal to an infinite flat plate, the r values can now be completely found once we solve the nonlinear governing equation. The variable z does not appear anywhere in the formulation. In addition, the boundary conditions for the z formulation were also derived and are indicated in Figure 15(b). It follows that the z solution cannot be obtained until after r values are known since the r values are required on the free surface boundary and within the interior flow. The boundary conditions for the z formulation were derived as follows:

8. Derivation of boundary conditions in inverse plane for z formulation. -

On AB

In the physical plane, $u = 0$ which implies $\frac{\partial \phi}{\partial r} = 0$

From equation (85), $\frac{\partial z}{\partial \psi} = 0$

This implies that $z = z(\phi)$ alone and since AB is a line of constant ϕ,

in the inverse plane z = constant   (107)

On BC

In physical plane, $u = 0$

From which it follows, in the inverse plane $\frac{\partial z}{\partial \psi} = 0$ (108)

On EC

In physical plane $\psi = 0$

In inverse plane $z = 0$ (109)

On GE

In physical plane, $v = 0$

From equation (83), it follows that $\frac{\partial \psi}{\partial r} = 0$ (110)

and from equation (86), we have in the inverse plane $\frac{\partial z}{\partial \phi} = 0$

On AG

In the physical plane $\frac{1}{r^2}\left[\left(\frac{\partial \psi}{\partial z}\right)^2 + \left(\frac{\partial \psi}{\partial r}\right)^2\right] = 1$

In the inverse plane we can take over the equivalent expression given by equation (106) $\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = 1$ (111)

Using equations (89) and (90) to eliminate the derivative involving r results in $r^2\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = 1$

9. Finite plate excluding surface tension. - A second problem to be formulated is concerned with the impingement of a circular liquid jet normal to a disk of finite width. For the case in which surface tension effects are neglected, the model depicted in Figure 12 will vary only slightly. The only difference being that the boundary conditions along the top and bottom free surfaces are now expressed as

$$\frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] = 1$$

The r formulation in the inverse plane is shown in Figure 16(a). The only boundary condition that bears some explanation is that along GE, the exiting plane in the physical plane, from Figure 12,

$$\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial r} f_3' = 0 \tag{112}$$

Using equations (84) and (86) this can be written

$$\frac{\partial r}{\partial\phi} + \frac{\partial z}{\partial\phi} f_3' = 0 \tag{113}$$

Finally, using equation (90) to eliminate $\partial z/\partial\phi$,

$$\frac{\partial r}{\partial\phi} + r \frac{\partial r}{\partial\psi} f_3' = 0 \tag{114}$$

or

$$\frac{\partial r}{\partial\phi} + r\left(\frac{\partial r}{\partial\psi}\right)\left(\frac{dz}{dr}\right) = 0 \tag{115}$$

The z formulation in the inverse plane is shown in Figure 16(b). Here again, only the boundary condition along GE bears some explanation. Continuing with equation (115) and using equations (89) and (90) we obtain

$$r\left(\frac{\partial z}{\partial\psi}\right) - \left(\frac{\partial z}{\partial\phi}\right)\left(\frac{dz}{dr}\right) = 0 \tag{116}$$

The formulation of the problem of flow of a circular liquid jet normal to a finite disk in the inverse plane is now complete. The r solution (Fig. 16(a)) can no longer be obtained independent of the z solution due to the exiting jet boundary condition. Thus, the problem will necessitate a simultaneous solution of two partial differential equations of the nonlinear elliptic type.

10. Finite plate including surface tension. - A third problem that is formulated but not solved due to the complexity of applying it to a

real system involves the normal impingement to a finite disk in which both surface tension and inertial forces are important. These would include all the flows experimentally labeled as transition flows. This represents a complete inverse formulation for the exact physical model represented in Figure 12. The results of the inverse formulation for the r solution are shown in Figure 17. The way in which the inverse boundary conditions along the free surface were derived will be made more clear in the next section.

11. Surface tension dominated model - finite plate. - The final problem to be examined involves the surface tension dominated flow, described in Figure 14. The inverse plane formulation for the r solution is indicated in Figure 18(a). The boundary conditions are the same as derived previously with the exception of the free surface boundary. Along AG in the physical plane,

$$\frac{1}{2r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] - \frac{1}{We}\frac{1}{r}\frac{d}{dr}\left[\frac{r}{\sqrt{\left(\frac{\partial\psi/\partial z}{\partial\psi/\partial r}\right)^2 + 1}}\right] = \frac{1}{2} - \frac{1}{We} \tag{117}$$

In the derivation of equation (106), it was shown that

$$\frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] = \frac{1}{\left(\frac{\partial r}{\partial\phi}\right)^2 + r^2\left(\frac{\partial r}{\partial\psi}\right)^2} \tag{118}$$

In addition, using equations (84) and (86),

$$\left(\frac{\frac{\partial\psi}{\partial z}}{\frac{\partial\psi}{\partial r}}\right)^2 = \frac{1}{r^2}\left(\frac{\frac{\partial r}{\partial\phi}}{\frac{\partial r}{\partial\psi}}\right)^2 \tag{119}$$

Therefore, the inverse boundary condition along AG becomes

$$\frac{1}{2}\frac{1}{\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2} - \frac{1}{We}\frac{1}{r}\frac{d}{dr}\left[\frac{r}{\sqrt{\frac{1}{r^2}\left(\frac{\partial r/\partial \phi}{\partial r/\partial \psi}\right)^2 + 1}}\right] = \frac{1}{2} - \frac{1}{We} \qquad (120)$$

which can be written

$$\frac{1}{2}\frac{1}{\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2} - \frac{1}{We}\frac{1}{r}\frac{d}{dr}\left[\frac{r^2}{\sqrt{\left(\frac{\partial r/\partial \phi}{\partial r/\partial \psi}\right)^2 + r^2}}\right] = \frac{1}{2} - \frac{1}{We} \qquad (121)$$

Multiplying through by 2, and rearranging yields,

$$\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = \frac{1}{\left\{1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r}\frac{d}{dr}\left[\frac{r^2}{\sqrt{\left(\frac{\partial r/\partial \phi}{\partial r/\partial \psi}\right)^2 + r^2}}\right]\right\}} \qquad (122)$$

There has been no symmetry assumed for the r formulation, only axisymmetry. It is noted that the r formulation does not involve the variable z. This means that the r and z solutions can be obtained independently. The z formulation is indicated in Figure 18(b).

VI. CENTRAL FINITE DIFFERENCE REPRESENTATION

The finite difference operators for the nonlinear r and z elliptic partial differential equations resulting from the inversion were put into difference form. It was found from experience that considerable flexibility resulted if the difference equations were derived employing rectangular mesh. The reason for this becomes clearer as we progress into the numerical solution. Suffice to say that this allowed us to control the size of the flow region. Square meshes were originally attempted, but only led to solutions in the cases where the mesh sizes became vanishingly small.

A. Formulation

1. General considerations. - The partial derivatives appearing in the r and z governing equations are replaced by algebraic central finite difference operators. A complete derivation of the various operators is contained in reference 35. The notation used in this report is that shown in Figure 19. If the finite difference analogy of $\partial/\partial r$ is $\delta/\delta r$, the r derivatives can be written at point 0 as follows,

$$\frac{\delta r}{\delta \phi} = \frac{r_1 - r_3}{2\ \Delta\phi} \tag{123}$$

$$\frac{\delta r}{\delta \psi} = \frac{r_2 - r_4}{2\ \Delta\psi} \tag{124}$$

$$\frac{\delta^2 r}{\delta \phi^2} = \frac{r_1 + r_3 - 2r_o}{\Delta\phi^2} \tag{125}$$

$$\frac{\delta^2 r}{\delta\psi^2} = \frac{r_2 + r_4 - 2r_o}{\Delta\psi^2} \tag{126}$$

Similarly, the central finite difference representations for the z derivatives are,

$$\frac{\delta z}{\delta\phi} = \frac{z_1 - z_3}{2\ \Delta\phi} \tag{127}$$

$$\frac{\delta z}{\delta\psi} = \frac{z_2 - z_4}{2\ \Delta\psi} \tag{128}$$

$$\frac{\delta^2 z}{\delta\phi^2} = \frac{z_1 + z_3 - 2z_o}{\Delta\phi^2} \tag{129}$$

$$\frac{\delta^2 z}{\delta\psi^2} = \frac{z_2 + z_4 - 2z_o}{\Delta\psi^2} \tag{130}$$

2. Interior nodal points. - If equations (123) to (126) are substituted into the r governing equation (eq. (98)) with $\alpha = \Delta\phi/\Delta\psi$, the finite difference expression for all interior nodal points is obtained,

$$r_o^4 - \frac{r_o^3}{2}(r_2 + r_4) + r_o^2\left[\frac{1}{\alpha^2} - \frac{1}{8}(r_2 - r_4)^2\right] - \frac{r_o}{2\alpha^2}(r_1 + r_3) + \frac{1}{8\alpha^2}(r_1 - r_3)^2 = 0 \tag{131}$$

In addition, if equations (127) to (130) are substituted into the z governing equation (eq. (96)), the finite difference expression for all interior nodal points is obtained

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)}(z_2 + z_4) + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)}(z_1 + z_3) + \frac{1}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)}(z_2 - z_4)(z_1 - z_3) \tag{132}$$

where, as in the derivation of equation (131),

$$\alpha = \frac{\Delta\phi}{\Delta\psi} \tag{133}$$

B. Excluding Surface Tension

1. Infinite flat plate. - The finite difference representation for the infinite flat plate is shown in Figures 20(a) and (b). The algebraic expressions for the boundaries is derived by simultaneous application of the governing equation and boundary conditions at a fixed point. This application involves a fictitious point, f, outside the boundary which is subsequently eliminated. Point G represents a special point in the finite difference representation for the z solution since it is a part of two separate boundaries. The governing z equation (eq. (132)) was applied at point G which resulted in two fictitious points. Then the boundary conditions along both AG and GE were applied at point G. This allowed the elimination of the two fictitious points from the resulting finite difference expression. Detailed calculations for the boundaries are contained in Appendix C.

2. Finite plate. - The inverse formulation for the finite plate problem (Fig. 16), is shown in difference form in Figure 21. For this case, the difference operator along GE is more complicated than in the infinite plate case. In fact, both G and E represent special points in the formulation. However, one of these, point E, is specified as a known position (r = constant). At point G the equation to be satisfied in the r formulation is shown at the top of Figure 21(a). In the formulation (Fig. 21(b), both points G and E are special points. The following equations hold there.

At point G

$$z_o = \frac{r_o}{\left(r_o^2 + \frac{1}{\alpha^2}\right)}\left\{\frac{1}{r_o}\sqrt{\left[a^2 - r_o^2(r_o - r_4)^2\right]} + z_4\right\} + \frac{z_1 - r_o\alpha(r_o - r_4)}{r_o^2\alpha^2 + 1}$$

$$+ \frac{r_o - r_4}{r_o^2 + \frac{1}{\alpha^2}}\sqrt{\left[a^2 - r_o^2(r_o - r_4)^2\right]} \qquad (134)$$

At point E

$$z_o = \frac{r_o^2}{r_o^2 + \frac{1}{\alpha^2}}\left\{z_2 - \frac{1}{r_o}\sqrt{\left[a^2 - r_o^2(r_2 - r_o)^2\right]}\right\} + \frac{z_1 - r_o\alpha(r_2 - r_o)}{r_o^2\alpha^2 + 1}$$

$$+ \frac{r_2 - r_o}{r_o^2 + \frac{1}{\alpha^2}}\sqrt{\left[a^2 - r_o^2(r_2 - r_o)^2\right]} \qquad (135)$$

Detailed calculations for all the additional boundaries encountered for the finite plate case are contained in Appendix C.

C. Surface Tension

The difference formulation corresponding to the Surface Tension model (shown in Fig. 18) is indicated in Figure 22(a) and (b). The only boundary condition that must be explained is the one along the free surface AG. Details of this calculation are contained in Appendix C. It is noted that (see Fig. 22(a)) the r solution can be obtained independent of the z solution.

VII. DISCUSSION OF NUMERICAL TECHNIQUES

Since the governing equations for the r and z formulations are nonlinear in the inverse plane, in general, the finite difference operations at the interior and boundary points will be nonlinear (see eqs. (131) and (132)). For the infinite flat plate case, as described in Figure 20, the solution begins with $r = r(\psi,\phi)$ from Figure 20(a). Secondly, with a knowledge of the r solution, the z formulation, shown in Figure 20(b), is solved for $z(\psi,\phi)$. However, in the case where the plate is finite, the r formulation also contains z along the exiting plane GE (see Fig. 21(a)). As a result, the r and z formulations must be solved simultaneously. The surface tension model, described in Figure 22, also allows the solution of the r equation independent of the z equations since z appears nowhere in the formulation.

In any case, when solving the r equation, the finite difference representation of the problem results in N nonlinear algebraic equations in N unknowns. A variety of methods were applied in order to obtain a solution to the simultaneous nonlinear equations. These included Lieberstein's extension of Youngs' work on over-relaxation to nonlinear elliptic partial differential equations (26), and the familiar Newton-Raphson method. None of the above methods were successful in obtaining a convergent solution. The technique suggested by Powell (34) resulted in the method used in this paper to obtain solutions. Basically, Powell developed a subroutine which was essentially a "compromise between the Newton-Raphson algorithm and the methods of steepest descents." In his paper, a Fortran subroutine is described for solving the nonlinear set of equations,

$$f_K(x_1,x_2, \ldots, x_N) = 0 \qquad K = 1,2, \ldots, N \tag{138}$$

The objective is to minimize the function

$$F(x_1,x_2, \ldots, x_N) = \sum_{K=1}^{N} [f_K(x_1,x_2, \ldots, x_N)]^2 \tag{139}$$

As with many iteration schemes, initial guesses are required for Xi. This particular algorithm has an advantage in that the initial guessed values do not have to be that close to the exact solution. The computer program for the r solution contains the main program and three subroutines. The subroutine EQNS is the one supplied by Powell. The user supplies the subroutine MATINV which inverts the matrices and the subroutine CALFUN which contains the nonlinear functions F(Xi). A knowledge of the r solution makes the z formulation explicit in the unknown z_o at each nodal point. As a result, a Gauss-Siedel linear iteration scheme was employed to obtain the solution.

For details of the subroutines, the reader is referred to reference 34. As we get into the computer results in the next section, a complete printout of the subroutine will be presented.

VIII. NUMERICAL RESULTS

A. Exclusion of Surface Tension

1. Infinite flat plate (Weber number = ∞). - Initially, the finite flat plate problem was numerically solved employing a very coarse mesh. Referring to Figure 20(a), the total stream functio was divided into five equal parts. Recall that $\psi_G = 1/2$ and ψ which meant that the parameter $a = \Delta\psi$ was set equal to 0.1. In dition, the ϕ axis was divided up into eight equal parts. This vision was purely arbitrary. One consideration was that there wou be at least two vertical lines of constant ϕ between points C and B. This resulted in a total of 39 unknown nodal points for t r formulation indicated in Figure 20(a), and 41 unknown nodal poi for the z formulation shown in Figure 20(b). The value of $D =$ where $\alpha = \Delta\phi/\Delta\psi$, was set equal to 0.0204. In Figure 20(a), the va of C along GE was arbitrarily chosen as 4.0 while the value of for z along AB was chosen as 3.315. The major reason for the assignment of these two values for r and z was to be able to co pare our numerical solution with the semi-analytical results for th infinite flat plate case with Schach (39). Actually, the major var able choice in the entire program is $D = 1/\alpha^2$. Basically, D is a measure of how large the flow system is since rectangular mesh is be used, i.e., a measure of $\Delta\phi$ and ϕ total. The only concern is th $\Delta\phi$ is not chosen too small. In that event, ψ total would be too small to satisfy the incoming and exiting flow boundary conditions.

$\Delta\phi$ is chosen to large, i.e., α chosen too large or D chosen too small, all that is lost is accuracy due to larger mesh spacings. For the coarse mesh infinite flat plate problem, the results of the numerical solution as well as the computer listing and final output can be found in Appendix D - "Computer Solutions/Listings." The solution to the r formulation (Fig. 20(a)) required 1098 calls of the subroutine used to solve the simultaneous nonlinear equation. The sum of the squares of the error to the exact solution was reduced to 0.00357 at the last iteration.

The final values for the coarse mesh solution (r,z) were used to make initial guesses for the values for the fine mesh solution. The r solution required the simultaneous solution of 159 nonlinear algebraic equations. The z equation, again being explicit in z_o, resulted in 164 unknown values of z_o. The plot of the results from the computer program (details shown in Appendix D) are presented in Figure 24 in a print plot. The computer connected the nodal points with straight line segments. In general, there is very little difference between the fine mesh and coarse mesh solution. The fine mesh solution required the extended storage space option on the 1106 machine. The sum of the squares for the final r solution was reduced to 0.024 after approximately 1000 calls of the subroutine CALFUN used to solve the simultaneous nonlinear equations. For this case 1000 iterations were required to obtain a satisfactory z solution.

2. Finite plate. - As mentioned in Section VII, Discussions of Numerical Techniques, the finite plate formulation also contains z on the exiting jet surface (see Fig. 21(a)). As a result, the method of solution consisted of the following steps:

Initially, assumed values of z along GE were chosen. These values were used to compute an r solution. The computed r solution used 161 calls of the subroutine CALFUN to reduce the sum of the squares of the residuals at the nodal points to 0.00066. With this r solution, the linear iteration technique was used to calculate the complete z solution. Since the computed z solution resulted in refined approximations to the values of z along GE, changes could then be reflected in a new r computation.

For the finite disk case in which the ratio of the radius of the liquid jet to the radius of the disk was one-half, a second r calculation did not change when the z values were updated. A curve was faired through the calculated nodal points and is shown in Figure 25. Only two of the four available streamlines are shown in the figure. No attempt was made to refine the solution by completely doubling the number of vertical and horizontal grid lines. For this particular case, since no comparison with any existing analytical techniques existed, a more convenient value of 3.3 was chosen for $K\phi$. The value of D employed in the solution was 0.0138. The only specification along GE was that Z was set equal to 0.25. The computer listing as well as the calculated r and z values at each nodal point can be found in Appendix D.

The same method was used to numerically compute the finite plate case in which the ratio of the jet diameter to the disk diameter was three-fourths. There were 38 nodal points required for the r solution and 45 for the z solution. A complete print-plot of the results is shown in Figure 26. Two iterations were required for the z solution as well as for the r solution. The sum of the squares for the r solution at the final iteration was 0.001. The computer listing and printout can be found in Appendix D. Again, the value of $K\phi$ was arbitrarily chosen as 3.3.

The z solution corresponding to the r solution was obtained and is also presented in Appendix D along with its complete computer listing. The method applied to obtain the solution was a simple linear iteration technique since equations explicit in Z_o can be derived both in the interior and along the boundary points. There were 249 iterations required for the Z solution. Changes between the 248th and 249th iteration occurred in the fifth decimal point.

A physical plane description of the infinite flat plate solution is presented in Figure 23. Only two of the four internal streamlines are shown in the figure. Curves were faired through the available calculated nodal points (r,z) by a best fit process. There appeared, at the onset, the question of whether or not the coarse mesh employed could sufficiently describe the flow. When applying boundary conditions at the free surface, a larger number of nodal points are desirable. As a result, the existing mesh was doubled. The total ψ was divided up into ten equal parts such that $a = \Delta\psi = 0.05$, and the total ϕ was divided up into 16 equal parts.

B. Surface Tension Dominated Model

As previously mentioned, the r formulation for the surface tension dominated flow can be solved independent of the z formulation (refer to Figs. 22(a) and (b)). The initial case examined had a Weber number of 4 and the ratio of the radius of the jet to the radius of the disk was one-half. Also, the first solution to this problem assumed only axisymmetry. The results of the numerical solution, employing a coarse mesh, resulted in a symmetrical r solution; symmetrical about the equipotential line emmanating from point D. This allowed us to make a simplification to the problem in that not only could axisymmetry be assumed, but also mirror image symmetry (i.e., symmetry about the $z = 0$ plane). This is significant for problems in which the surface tension effects are to be taken into account since the surface tension forces are highly dependent on the curvature of the free surface. By taking advantage of the symmetry involved, additional nodal points can be placed on the free surface without using extended computer storage.

Referring now to Figure 22(a), a vertical line was drawn (equipotential line) emmanating from point D, where $r = RD$. RD is the dimensionless disk radius. The intersection of this equipotential line with the free surface AG was defined as point M. Along DM, it is known that $z = 0$. In addition, the variation of r with ψ can be computed. Let us now refer to Figure 27 in which the equipotential line DM is depicted. Since DM is an equipotential line, the velocity along this line must be equal to V, the incoming jet velocity with the exception of the point $r = L$ in the physical plane. At point D, a velocity discontinuity will exist. However, we can specify how r varies with ψ along this line as follows:

In the physical plane,

On DM

$$u = 0 \qquad \text{on} \quad z = 0, \; L \leq r \leq R_{max} \tag{140}$$

$$v = -V$$

where R_{max} is the maximum radius of the liquid flow pattern.

An expression for the stream function along DM can now be derived since the radial velocity component along DM is 0, we have

$$0 = -\frac{1}{r}\frac{\partial\psi}{\partial z} \qquad (141)$$

This implies that $\psi = \psi(r)$ alone. In order to find what the function is, the definition of the axial velocity is employed, namely, $v = (1/r)(\partial\psi/\partial r)$, on DM

$$-V = \frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (142)$$

Integrating this

$$\psi = -\frac{Vr^2}{2} + \text{Constant} \qquad (143)$$

Applying the boundary condition that $\psi = 0$ at $r = L$, the constant in (143) can be calculated. The following expression results,

$$\psi = -\frac{Vr^2}{2} + \frac{VL^2}{2} \qquad (144)$$

If this equation is nondimensionalized,

$$\psi^* = \frac{r^{*2}}{2} - \frac{1}{2}\left(\frac{L}{R_o}\right)^2 \qquad (145)$$

As we have done in the past derivations, the starred notation is dropped.

$$\psi = \frac{r^2}{2} - \frac{1}{2}\left(\frac{L}{R_o}\right)^2 \qquad (146)$$

Solving for r,

$$r = +\sqrt{2\psi + (RD)^2} \qquad (147)$$

where

$$\frac{L}{R_o} = RD \qquad (148)$$

Equation (147) represents the boundary condition employed along DM in the inverse plane for the r formulation. The condition $z = 0$ was used for the z formulation.

In the course of finding the solution to the z formulation, depicted in Figure 22(b), it was necessary to solve a cubic equation for the parameter T along the free surface. Physical as well as mathematical interpretation was required when choosing the proper root of the cubic since a possibility of three real roots existed. Referring to equation (C-78) in Appendix C, the only mathematically meaningful roots are those in which the absolute value of the parameter T was greater than or equal to one. However, the possibility still existed that all three roots would be real and in addition satisfy the requirement that their absolute values were greater than one. As a result, some physical insight was required when deciding upon which roots to employ in the equation relating T to the fictitious point f, (eq. (C-84)). For example, it is known that as the nozzle exit is approached along the free surface, $z = f(r)$ becomes steeper (i.e., $f'(r)$ approaches infinity). This would coincide with the curvature terms dropping out of the boundary condition in the physical plane formulation. An alternate approach to viewing this is that the fictitious point f approaches the image point z_4. As a result, the value of the parameter T approaches unity. The algorithm selected for choosing the proper root of the cubic was to select the value of T closest to unity but ensuring that its absolute value was greater than or equal to one. During the course of finding the solution, problems in implementing this algorithm occurred, particularly when close to the nozzle, since the T values closest to one were slightly less than one and were automatically discarded by the algorithm. The resulting potential lines and streamlines appeared inaccurate when plotted up. The only way found to circumvent this problem was to set up an additional algorithm which set T identically equal to one for several free surface nodal points in the vicinity of the nozzle (i.e., those nodal points in which the computed r value was ≤ 1.00 N). Physically, this reasoning is justified since it is known that f'(r) must approach infinity there.

The results of the numerical solution are indicated in Figure 28. The Weber number for this solution was 4 and the ratio of the radius of the liquid jet to the radius of the disk was 1/2. A value of 3.5 was chosen for $K\phi$, and D was set equal to 0.0204. The computer listing can be found in Appendix D along with the computed r and z values for the nodal points. The sum of the squares for the r solution was 0.77×10^{-6} and since the r solution was independent of the z solution, a second iteration was unnecessary.

C. Discussion of Zero Gravity Results

The numerical solutions were compared with the available semi-analytical results of Schach (39) for the case of the infinite flat plate. The method employed by Schach is attributed to Trefftz. The method used by Schach did not appear readily extendable to more complicated geometrical flows and could not be employed to account for the effects of surface tension. In making the comparison between reference 39 and the numerical results, the fine mesh solution presented in Figure 24 was used. The results of the comparison for the infinite flat plate are shown in Figure 29. The symbols indicated in the figure were obtained from Figure 11 of Schach's paper by using an expandable scale. As a result there is some unknown error associated with the process of taking the results from the reference. In any case, the agreement looks particularly well with the sole exception of the first r coordinate greater than unity. One final point with respect to the infinite flat plate solution concerns the extreme left coordinate in Figure 29, namely, $r = 4$, $z = 0.125$. These two values are fixed by continuity, both in our numerical program and in the semi-analytical results of Schach. This result was obtained as follows; assuming constant density, the volumetric flow rate into the control volume at AB must balance the flow out of the control volume at GE. In physical coordinates, the flow is given as $\pi R_o^2 V$ and the flow out by $2\pi R_{jet} Z_G V$. Equating these and cancelling leads to the fact that Z_G must equal $R_o^2/(2R_{jet})$. Nondimensionalizing with respect to R_o yields

$$z_G^* = \frac{1}{2R_{jet}^*} \tag{149}$$

Dropping the starred notation and observing that $R_{jet} = 4$ at the left boundary of the control volume shows that Z_o must equal 1/8.

As far as the finite plate is concerned, there was no available comparisons with past experimental or analytical work. As a result, comparisons were made with respect to our own zero gravity experimental data. The results are shown in dimensionless coordinates in Figures 30 and 31. Figure 30 is for the case where the ratio of the jet radius to the disk radius is one-half and Figure 31 indicates the comparison when the ratio of the jet radius to the disk radius is three-fourths. The comparisons were made with respect to the outer or top free surface since it was impossible to view the lower free surface because of the way in which the flow occurred. The results were generally good for both ratios compared. Experimental data points were obtained from both sides of the axisymmetric sheet as it flowed around the disk. An averaging procedure was subsequently used to plot the continuous lines indicated in Figures 30 and 31. The analysis corroborated the experiments in the sense that as the ratio (R_o/L) becomes smaller, the jet appears to leave the disk more tangentially.

IX. NORMAL GRAVITY EXPERIMENT SECTION

A. Apparatus and Procedure

1. Experiment. - The experimental test rig used to obtain the normal gravity data is shown in Figures 32(a) and (b). The rig consisted of an angle-iron frame in which was mounted a 10 gallon supply tank, a settling chamber, a 56 gallon catch basin, a return pump, a control box, a clock, sequence timers, a regulator, and a supply tank. The major functions were controlled through the control box. A high-speed Mitchell Monitor motion picture camera (nominal speed 400 frames/sec) was located directly in front of the experimental test rig. The camera was mounted on a Wollensak camera stand which was fastened to a concrete floor by means of conduit clamps.

The experiment could be conducted in either a pressurized or non-pressurized mode (gravity-feed). To operate in the pressurized mode, two vent valves located above the supply tank were closed and the system was pressurized through the regulator. The pressure level was recorded on the gage located immediately to the right of the regulator as shown in Figure 32(a). For both modes, the return pump was used to resupply liquid to the supply tank in order to maintain a nearly constant level of liquid. Since the supply tank was fabricated from stainless steel and provided no visible means for monitoring liquid level, attempts were made to connect a plastic hose between the needle valve located just upstream of the solenoid valve and the side of the supply tank. This system was generally inaccurate and as a rule the return pump was normally activated after completing two or three test runs.

In order to maintain a circular liquid jet having an initially uniform velocity profile, a settling chamber was designed to quiet the incoming flow. A similar technique was employed by Donnelly et al. (11) in their study of liquid jet stability. The settling chamber, Fig. 33, was fabricated from stainless-steel and had a 30° tapered approach to a circular hole in order to prevent boundary layer buildup. A total angle of 60° was employed in order to ensure a nearly uniform velocity profile at the exit to the settling chamber. Three settling chambers were employed having outlet diameters of 0.25, 0.50, and 1.0 centimeters. Problems developed when using an unbaffled settling chamber in that the entire flow developed a swirling action between 2 and 10 seconds after flow was initiated. In order to circumvent the swirling problem, the cylindrical section of the settling chamber was fitted with a $1\frac{3}{4}$ inch honeycomb spacer. A sixteen mesh stainless steel screen was mounted below the first screen and both screens were butted up against the bottom of the spacer. This technique seemed to eliminate any noticeable swirl in the flow.

Sharp-edged disks, fabricated from stainless steel, were mounted below the settling chamber and positioned at right angles to the impinging liquid jet. A photograph of the disks employed in the study is shown in Figure 34. Their diameters ranged from 1 to 4 centimeters. The disks were mounted onto a stainless steel device capable of being adjusted at any angle to the incoming flow. However, in this study the impingement was restricted to 90°.

2. <u>Test liquids</u>. - Two test liquids were employed, anhydrous

ethanol and distilled water. Their properties at 20° C are listed in Table 1. No attempt was made to correct these fluid properties for temperature changes.

3. Test procedure. - Prior to a test run the settling chamber was filled with the test liquid by opening the solenoid valve while holding the bottom of the settling chamber closed. The liquid completely filled the settling chamber until it flowed out of the relief screw. At that point, the relief screw was closed and the outlet could be opened without loss of liquid from the settling chamber since it was a stable pressure supported system. This method worked when using distilled water for all the nozzle openings. It did not work for anhydrous ethanol in conjunction with the largest or 1 centimeter diameter opening. As a result, for those tests, the nozzle opening was kept sealed until the solenoid was opened at the start of each test.

Calibration tests were made prior to every series of test runs. Electrical sequence times in the control box were used along with graduated cylinders to determine the volumetric flow rate. At least three calibration tests were made before each series of runs, and an average value for the flow rate was thus determined. The impingement velocity (velocity at the disk) was calculated by correcting for the effect of gravitational forces.

B. Results

1. Steady-state flow patterns. - Basically, two types of flow patterns were observed in the course of normal gravity liquid jet impingement. The first type occurred when a very high speed jet impinged upon a solid surface and spread out radially as a thin sheet. The sheet became thinner until it became unstable and broke up into

small liquid droplets. A second pattern is shown schematically in Figure 35. No breakup was observed in this pattern; the jet curved upon itself to form a surface of revolution or plume. The incoming jet velocity is V and the radius of the circular liquid jet is R_o. The disk radius is denoted as L while H is the distance between the nozzle and the disk. The maximum radius that the plume possessed is denoted as R_p. The maximum plume radius was the primary experimental variable for the normal gravity study since it was easily measurable and characteristic of the entire flow pattern. For the case when the plume was not visibly disturbed, the plume would slowly move to a final R_p, with visible surface ripples.

2. Unstable jumps. - In attempting to obtain steady-state impingement profiles, an unusual phenomenon was discovered. Instead of a single steady-state flow pattern, a number of unstable flow patterns were observed prior to the attaining of a stable steady state. Typically, the jet impinged upon the solid and formed a plume of some given radius with surface ripples present. If disturbed, either upstream or downstream of the plume, the jet would jump to another apparently stable configuration of larger plume radius. Normally, only one jump would occur in a single test run, but occasionally two jumps or three apparently stable configurations were observed. A series of tests indicating the flow patterns before and after a jump are shown in Figure 36. Figures 36(a), (c), and (e) illustrate the behavior before the respective jumps, while Figures 36(b), (d), and (f) occur after the jumps. The plume size for the final configurations appear to be nearly double what they were initially. However, there does not appear to be any correlation between initial and final size. The jumps

are natural in occurrence, such as when a small liquid droplet rebounds off the disk holder and impinges upon this film, but can be initiated by a disturbance, such as a pencil penetrating the flow. The final steady-state configuration is stable to any additional disturbances, i.e., it remains at some fixed plume shape. Under no circumstances was the plume observed to jump to a smaller plume configuration; all jumps were to larger plume sizes.

The actual cause of these jumps remains unanswered at this writing. Some possible causes have been eliminated, such as swirling flow. If the flow were swirling, the steady-state flow patterns would be significantly affected if half the flow over the disk was obstructed. However, the pattern was not affected at all when this was done. It is tentatively concluded that the jumps are natural in occurrence. The initial states and all transition states are unstable to small disturbances. For the purpose of the experimental investigation, then, the steady-state was chosen as the state in which further disturbances caused no change in the liquid flow pattern.

3. Experimental data. - The experimental runs were conducted such that the viscous dependence would be small. As shown in section III, depending upon the ratio (R_o/L), there exists a critical Reynolds number above which the flow can be considered as essentially inviscid. The Reynolds number was calculated at the point of impingement on the disk (i.e., the approach velocity to the disk was corrected for gravitational effects). From analytical considerations, numerous dimensionless parameters arise and thus form the basis for correlating the primary variable, the plume size. These parameters include the Weber number, which is the ratio of inertial to surface tension forces, and

the Bond number, which is the ratio between gravitational and surface tension forces. In addition, several geometrical ratios appear, such as R_o/L, the ratio of jet to the disk radius, and H/R_o, the ratio of the nozzle height to the jet radius. The experimental results are listed in tabular form in Table 3. The last column contains the plume radius, nondimensionalized with respect to the jet radius. In examining the difference between zero and normal gravity liquid jet impingement, it can be seen that two additional parameters are required in order to completely define this phenomena in normal gravity. They are the Bond number, Bo, and the dimensionless nozzle height, H/R_o. The Bond number relates the relative contribution of gravitational forces, while H/R_o has an effect in that jets flowing downward under the effect of gravity accelerate and thus shrink in size, thereby having an effect on the resulting flow.

4. Data analysis. - A linear regression analysis was employed in order to correlate the independent variable, R_p/R_o, with the remaining system parameters. Let us define the following variables,

$$Y = \frac{R_p}{R_o} \tag{150}$$

$$Z_1 = \frac{R_o}{L} \tag{151}$$

$$Z_2 = \frac{H}{R_o} \tag{152}$$

$$Z_3 = We \tag{153}$$

$$Z_4 = Re \tag{154}$$

$$Z_5 = Bo \tag{155}$$

The parameter Z_4, the Reynolds number, was included to see what the viscous influence was. The experimental data was fitted to the following complex model

$$\frac{R_p}{R_o} = 22.8241 + 7.67129\, X_1 - 41.2001\, X_1^2 + 42.9330\, X_1^3 + 1.91085\, X_2$$

$$+ 8.90368\, X_3 + 1.46319\, X_4 - 2.07823\, X_5 + 0.0696939\, X_2 X_5 \qquad (156)$$

where

$$X_1 = \frac{\frac{R_o}{L} - 0.371839}{0.154239} \qquad (157)$$

$$X_2 = \frac{\frac{H}{R_o} - 9.22727}{7.19571} \qquad (158)$$

$$X_3 = \frac{We - 129.227}{75.7108} \qquad (159)$$

$$X_4 = \frac{Re - 4663.89}{1897.57} \qquad (160)$$

$$X_5 = \frac{Bo - 2.15602}{1.12058} \qquad (161)$$

The value of R^2 for the above statistical model was 0.981. This means that 98.1 percent of the total variance in R_p/R_o is accounted for by the model. An examination of the signs in the equation for the complex model tells us how R_p/R_o varies with each of the parameters; R_p/R_o increases slightly as H/R_o and Re are increased, and increases significantly with X_3, the Weber number. Since X_5 is involved in two terms in the correlation, its variation depends on the strongest term (i.e., the one with the largest coefficient). As a result, R_p/R_o decreases with increasing Bo. The variation with X_1 is more confusing since it is involved in three terms. The actual data bears this out, R_p/R_o sometimes increasing, or sometimes decreasing. The coefficient for X_3 is essentially the largest of all (X_3 is related to We). As a result, a simpler model was pursued using only We as the independent variable. The following equation resulted;

$$\frac{R_p}{R_o} = 21.7059 + 10.1410\ \frac{We - 129.227}{75.7108} \tag{162}$$

For this simple model $R^2 = 0.916$, which means that the correlation accounts for 91.6 percent of the variance in the data. A plot of the data is displayed in Figure 37.

The Weber number turns out to be the most statistically significant variable. This is not to say that the other variables are not significant, but that the Weber number is the most dominant variable in determining R_p/R_o.

X. CONCLUSIONS

Zero gravity. - An experimental and analytical investigation was conducted to determine the free surface shape of circular liquid jets impinging normal to sharp-edged disks in zero gravity. The test liquids employed were anhydrous ethanol and trichlorotrifluoroethane. Jet radii were varied from 0.25 to 0.75 centimeter and disk radii of 1.0 and 1.5 centimeters were employed. The jet velocity was varied between 12 and 365 centimeters per second. Under the stipulation that the nozzle was located at least 5 centimeters from the disk, the investigation yielded the following results:

1. It was analytically determined that there exist flow regions where viscous forces are not significant when computing free surface shapes. It was shown that the Reynolds number $\rho V R_0/\mu$ and the ratio of jet to disk radius R_0/L uniquely define the flow regions. It was further shown that the Reynolds number specifying the transition between viscous and nonviscous flow decreased with increasing jet to disk radius ratio.

2. Within the inviscid region, three distinct flow regimes were experimentally found which depend uniquely on the Weber number $\rho V^2 R_0/\sigma$ and the ratio of the jet to disk radius R_0/L. These flows were defined as Surface Tension Flow, Transition Flow, and Inertia Flow. The critical Weber number between regimes was found to decrease with increasing jet to disk radius ratio.

3. A numerical solution yielding free surface shapes and stream-

lines was obtained for the case of impingement normal to an infinite flat plate and compared favorably with semi-analytical techniques in the literature.

4. A numerical solution yielding free surface shapes and streamlines was obtained for inertially dominated flows at ratios of jet to disk radius of one-half and three-fourths. The comparison with experiments showed good agreement for the upper free surface.

5. A surface tension dominated flow was formulated and solved numerically. The system Weber number was 4.0 and the ratio of the jet to the disk radius was one-half.

Normal gravity. - An experimental investigation was conducted to determine the characteristics of circular liquid jets impinging normal to a sharp-edged disk in normal gravity. The test liquids employed were distilled water and anhydrous ethanol. Jet radii between 0.125 and 0.50 centimeter were employed and the disk radii were varied between 0.5 and 2.0 centimeters. The jet velocity had the range of 75.5 to 484 centimeters per second. The distance between the nozzle and disk was varied between 0.25 and 5.0 centimeters. Under the stipulation that the Reynolds numbers were such that they exceeded the minimum value required to avoid viscous influence, the investigation yielded the following results:

1. The liquid flow pattern was observed to jump from one apparently stable flow pattern to another until a completely stable configuration was reached. The jumps were triggered by disturbances both upstream and downstream of the disk and were apparently natural in occurrence.

2. The dimensionless plume radius R_p/R_o was correlated by means

of a linear regression analysis. A simple model, employing only the Weber number, accounted for nearly 92 percent of the experimental data. The following empirical formula resulted

$$\frac{R_p}{R_o} = 21.7059 + 10.140 \frac{We - 129.227}{75.7108}$$

where R_p is the plume radius, R_o the nozzle radius, and We the system Weber number.

XI. APPENDIXES

Appendix A - Detailed Calculations of Free Surface Boundary Conditions

There are two boundary conditions required for the case of a free surface in a fluid dynamics problem. This is in comparison to known boundaries in which only one boundary condition is required. The two conditions to be satisfied are:

(1) Conservation of energy along a streamline

(2) The velocity normal to the streamline is zero.

Consider the following geometry:

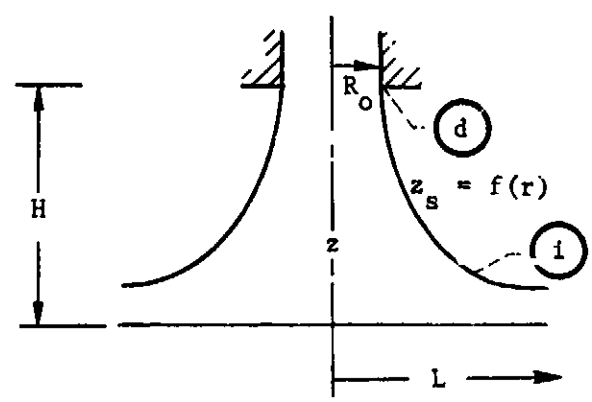


Figure A.1. - Schematic of Liquid Jet Impingement

where "i" represents any point on the free surface $z_s = f(r)$ and "d" designates the reference point which is chosen as the point where the liquid jet exits from the nozzle.

(1) Conservation of energy along a streamline

Bernoulli's equation written between points "i" and "d" becomes

$$\frac{1}{2}(u^2 + v^2) + \frac{P_i}{\rho} = \frac{1}{2}V^2 + \frac{P_d}{\rho} \qquad \text{on } z_s = f(r) \tag{A.1}$$

The pressure at point i, p_i, is not the same as it is at point d, P_d, due to the effects of surface tension. In general,

$$Pg - P = \sigma J \tag{A-2}$$

where Pg = known gas pressure, and

$$J = \frac{1}{R_1} + \frac{1}{R_2} \tag{A-3}$$

where R_1, R_2 are radii of curvature where

$$\frac{1}{R_1} = \frac{f''}{\left(1 + f'^2\right)^{3/2}} \tag{A-4}$$

and

$$\frac{1}{R_2} = \frac{f'}{r\left(1 + f'^2\right)^{1/2}} \tag{A-5}$$

Combining (A-4) and (A-5) with (A-2) we find

$$Pg - P = \sigma\left[\frac{f''}{\left(1 + f'^2\right)^{3/2}} + \frac{f'}{r\left(1 + f'^2\right)^{1/2}}\right] \tag{A-6}$$

which can be combined to yield,

$$Pg - P = \frac{\sigma}{r}\frac{d}{dr}\left(\frac{rf'}{\sqrt{1 + f'^2}}\right) \tag{A-7}$$

Applying at point "d"

$z = H, \qquad r = R_o, \qquad$ and $\qquad \frac{df}{dr} = f' = \infty$

For large f',

$$\sqrt{1 + f'^2} \doteq \sqrt{f'^2} = f'$$

Therefore, substitution into (A-7) yields

$$Pg - Pd = \frac{\sigma}{R_o} \tag{A-8}$$

Applying at point "i"

$$P_g - P_i = \frac{\sigma}{r}\frac{d}{dr}\left(\frac{rf'}{\sqrt{1+f'^2}}\right) \qquad \text{(A-9)}$$

Now (A-8) and (A-9) can be substituted into (A-1) to obtain

$$\frac{1}{2}(u^2+v^2) - \frac{\sigma}{\rho r}\frac{d}{dr}\left(\frac{rf'}{\sqrt{1+f'^2}}\right) = \frac{1}{2}V^2 - \frac{\sigma}{\rho R_o} \qquad \text{on } z_s = f(r) \qquad \text{(A-10)}$$

(2) The velocity normal to the free surface is zero

A portion of the free surface $z_s = f(r)$ is shown in Figure A.2

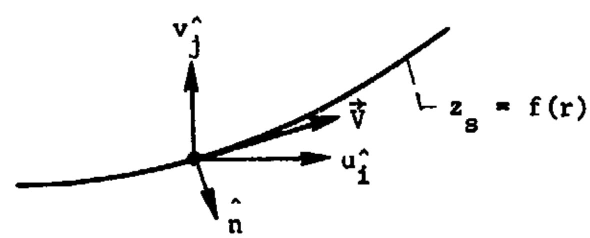


Figure A.2. - Velocity Vector at Free Surface

$\hat{n}$ is the unit outward normal to the surface at some point and u and v are the velocity components such that $\vec{V} = u\hat{i} + v\hat{j}$. Since the velocity normal to the surface is zero

$$\vec{V} \cdot \hat{n} = 0 \qquad \text{(A-11)}$$

With the surface given by $z_s = f(r)$, the unit normal is given as

$$\hat{n} = \frac{-f'\hat{i} + \hat{j}}{\sqrt{1+f'^2}} \qquad \text{(A-12)}$$

Hence

$$\vec{V} \cdot \hat{n} = -\frac{uf'}{\sqrt{1+f'^2}} + \frac{v}{\sqrt{1+f'^2}} = 0 \qquad \text{(A-13)}$$

which simplifies to

$$-uf' + v = 0 \qquad \text{on } z_s = f(r) \qquad \text{(A-14)}$$

Appendix B - Zero Gravity Drop Tower Test Facility

The experimental data for this study were obtained in the Lewis Research Center's 2.2-Second Zero Gravity Facility. A schematic diagram of this facility is shown in Figure B.1. The facility consists of a building 6.4 meters square by 30.5 meters tall. Contained within the building is a drop area 27 meters long with a cross section 1.5 by 2.75 meters.

The service building has a shop and service area, a calibration room, and a controlled environment room. Those components of the experiment that required special handling were prepared in the controlled environment room of the facility. This air-conditioned and filtered room (shown in Fig. B.2) contains an ultrasonic cleaning system and the laboratory equipment necessary for handling test liquids.

Mode of operation - A 2.2-second period of weightlessness is obtained by allowing the experiment package to free fall from the top of the drop area. In order to minimize drag on the experiment package, it is enclosed in a drag shield designed with a high ratio of weight to frontal area and a low drag coefficient. The relative motion of the experiment package with respect to the drag shield during a test is shown in Figure B.3. Throughout the test, the experiment package and drag shield fall freely and independently of each other; that is, no guide wires, electrical lines, etc., are connected to either. Therefore, the only force acting on the freely falling experiment package is the air drag associated with the relative motion of the package within the enclosure of the drag shield. This air drag results in an equivalent gravitational acceleration acting on the experiment, which is estimated to be below 10^{-5} g's.

Release system. - The experiment package, installed within the drag shield, is suspended at the top of the drop area by means of a highly stressed music wire attached to the release system. This release system consists of a double-acting air cylinder with a hard-steel knife edge attached to the piston. Pressurization of the air cylinder drives the knife edge against the wire, which is backed by an anvil. The resulting notch causes the wire to fail, smoothly releasing the experiment. No measurable disturbances are imparted to the package by this release procedure.

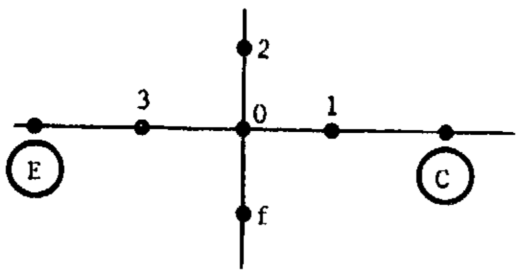
Recovery system. - After the experiment package and drag shield have traversed the total length of the drop area, they are recovered by deceleration in a 2.2-meter-deep container filled with sand. The deceleration rate (averaging 15 g's) is controlled by selectively varying the tips of the deceleration spikes mounted on the bottom of the drag shield (Fig. B.1). At the time of impact of the drag shield in the decelerator container, the experiment package has traversed the vertical distance within the drag shield (compare Figs. B3(a) and (c)).

Appendix C - Detailed Derivations of Finite Difference Operators Along Boundaries

A. Infinite Flat Plate

The finite difference representations for the derivatives are those shown in the text (eqs. (123) to (130)). Refer to Figures 15(a) and (b).

$\frac{\partial r}{\partial \psi} = 0$ on EC



Applying the r difference equation (eq. (131)) at point 0, where f is a fictitious point outside boundary

$$r_o^4 - \frac{r_o^3}{2}(r_2 + r_f) + r_o^2\left[\frac{1}{\alpha^2} - \frac{1}{8}(r_2 - r_f)^2\right] - \frac{r_o}{2\alpha^2}(r_1 + r_3)$$

$$+ \frac{1}{8\alpha^2}(r_1 - r_3)^2 = 0 \qquad \text{(C-1)}$$

Along EC, $\frac{\partial r}{\partial \psi} = 0$. This implies

$$\frac{r_2 - r_f}{2\,\Delta\psi} = 0 \qquad \text{(C-2)}$$

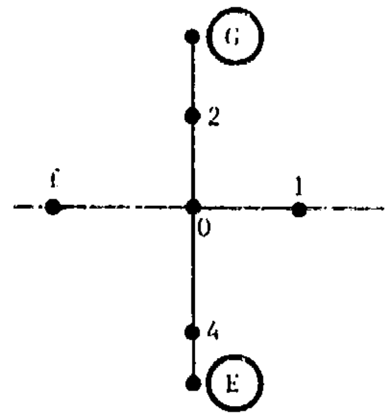
Therefore,

$$r_2 = r_f \qquad \text{(C-3)}$$

Equation (C-1) becomes

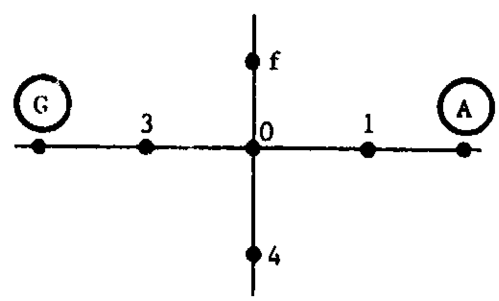
$$r_o^4 - r_o^3 r_2 + r_o^2 \frac{1}{\alpha^2} - \frac{r_o}{2\alpha^2}(r_1 + r_3) + \frac{1}{8\alpha^2}(r_1 - r_3)^2 = 0 \qquad (C-4)$$

<u>$\frac{\partial r}{\partial \psi} = 0$ on GE</u>



Application of $\partial r/\partial\psi$ on GE yields the fact that $r_2 = r_4$ and does not involve the unknown fictitious point f. However, $\partial r/\partial\psi = 0$ implies $r \neq r(\psi)$. Therefore, r must only be a function of ϕ, $r = r(\phi)$. But, along GE, ϕ = constant, ($\phi = 0$). Hence, r = constant along GE.

$$\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = 1 \quad \text{on AG}$$



The r difference equation can be written

$$r_o^4 - \frac{r_o^3}{2}(r_f + r_4) + r_o^2\left[\frac{1}{\alpha^2} - \frac{1}{8}(r_f - r_4)^2\right] - \frac{r_o}{2\alpha^2}(r_1 + r_3) + \frac{1}{8\alpha^2}(r_1 - r_3)^2 = 0 \qquad \text{(C-5)}$$

Along AC

$$\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = 1$$

This implies

$$\left(\frac{r_1 - r_3}{2\,\Delta\phi}\right)^2 + r_o^2\left(\frac{r_f - r_4}{2\,\Delta\psi}\right)^2 = 1 \qquad \text{(C-6)}$$

Rearranging and letting $\Delta\psi = a$ (recalling $\alpha = \Delta\phi/\Delta\psi$)

$$(r_f - r_4)^2 = \frac{1}{r_o^2}\left[4a^2 - \frac{1}{\alpha^2}(r_1 - r_3)^2\right] \qquad \text{(C-7)}$$

Whereupon we can calculate the two expressions

$$r_f - r_4 = \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(r_1 - r_3)^2\right]} \qquad \text{(C-8)}$$

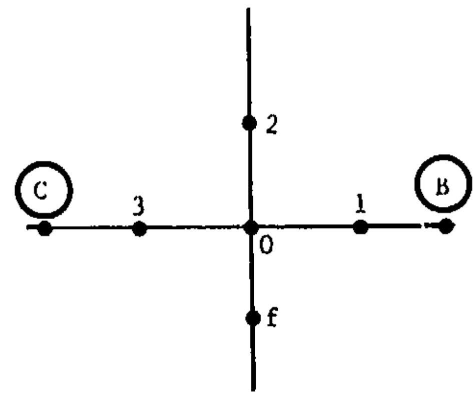
$$r_f + r_4 = \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(r_1 - r_3)^2\right]} + 2r_4 \qquad \text{(C-9)}$$

Inserting (C-8) and (C-9) into (C-5) yields the desired relation

$$r_o^4 - r_o^3 r_4 + r_o^2\left\{\frac{1}{\alpha^2} - \frac{1}{2}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(r_1 - r_3)^2\right]}\right\} - \frac{r_o}{2\alpha^2}(r_1 + r_3) + \frac{1}{8}\left[\frac{2}{\alpha^2}(r_1 - r_3)^2 - 4a^2\right] = 0 \qquad \text{(C-10)}$$

z = Constant on AB, let $z = K\phi$.

$\frac{\partial z}{\partial \phi} = 0$ on BC



Applying the z difference equation (eq. (132)) at point 0, where f is a fictitious point outside of boundary

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} (z_2 + z_f) + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)} (z_1 + z_3)$$

$$+ \frac{1}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} (z_2 - z_f)(z_1 - z_3) \qquad \text{(C-11)}$$

Along BC, $\frac{\partial z}{\partial \psi} = 0$. This implies

$$z_2 - z_f = 0$$

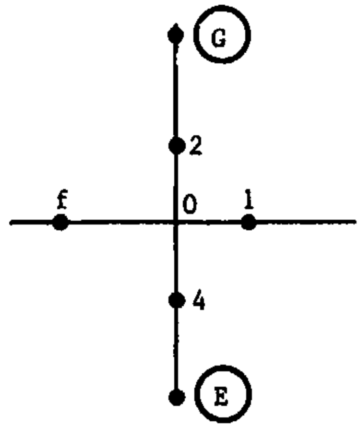
or

$$z_f = z_2 \qquad \text{(C-12)}$$

Therefore, equation (C-11) becomes

$$z_o = \frac{r_o^2 z_2}{r_o^2 + \frac{1}{\alpha^2}} + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)} (z_1 + z_3) \qquad \text{(C-13)}$$

$\frac{\partial z}{\partial \phi} = 0$ on GE



The z difference equation can be written

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} (z_2 + z_4) + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)} (z_1 + z_f)$$

$$+ \frac{1}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} (z_2 - z_4)(z_1 - z_f) \qquad \text{(C-14)}$$

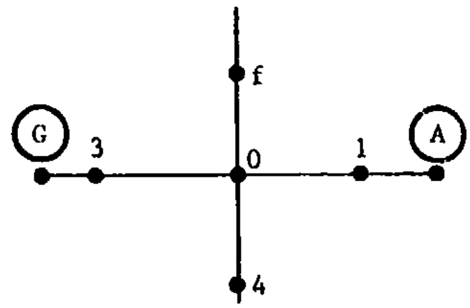
Along GE, $\frac{\partial z}{\partial \phi} = 0$ this implies $z_1 - z_f = 0$ or

$$z_f \equiv z_1 \qquad \text{(C-15)}$$

Therefore, equation (C-14) becomes

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} (z_2 + z_4) + \frac{z_1}{r_o^2\alpha^2 + 1} \qquad \text{(C-16)}$$

$r^2\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = 1$ on GA



The z difference equation can be written

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} (z_f - z_4) + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)} (z_1 + z_3)$$

$$+ \frac{1}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} (z_f - z_4)(z_1 - z_3) \qquad \text{(C-17)}$$

Along AG, $r^2\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = 1$. Hence,

$$r_o^2\left(\frac{z_f - z_4}{2\ \Delta\psi}\right)^2 + \left(\frac{z_1 - z_3}{2\ \Delta\phi}\right)^2 = 1 \qquad \text{(C-18)}$$

which can be expressed as

$$(z_f - z_4)^2 = \frac{1}{r_o^2}\left[4a^2 - \frac{1}{\alpha^2}\ (z_1 - z_3)^2\right] \qquad \text{(C-19)}$$

This yields the two relations,

$$z_f - z_4 = \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}\ (z_1 - z_3)^2\right]} \qquad \text{(C-20)}$$

$$z_f + z_4 = \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}\ (z_1 - z_3)^2\right]} + 2z_4 \qquad \text{(C-21)}$$

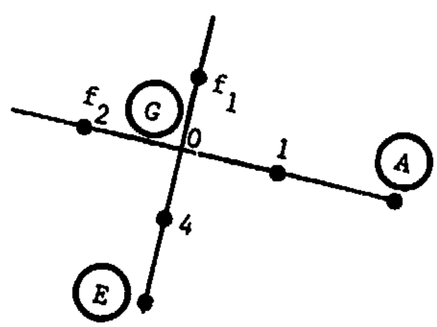
Inserting these last two equations in (C-17) eliminates the fictitious point z_f, yielding

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} \left\{ \frac{1}{r_o} \sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]} + 2z_4 \right\}$$

$$+ \frac{1}{2\left(r_o^2\alpha^2 + 1\right)} (z_1 + z_3)$$

$$+ \frac{z_1 - z_3}{4r_o\left(r_o^2\alpha + \frac{1}{\alpha}\right)} \sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]} \qquad \text{(C-22)}$$

A special boundary condition is required for point G since it is a part of two separate boundaries



Applying the z difference equation,

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} (f_1 + z_4) + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)} (z_1 + f_2)$$

$$+ \frac{1}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} (f_1 - z_4)(z_1 - f_2) \qquad \text{(C-23)}$$

Applying the boundary condition along GE, namely $\frac{\partial z}{\partial \phi} = 0$, it is found that $f_2 = z_1$. Therefore, we write

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} (f_1 + z_4) + \frac{z_1}{r_o^2\alpha^2 + 1} \tag{C-24}$$

Applying the boundary condition along AG (see eq. (C-20))

$$f_1 = z_4 + \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - f_2)^2\right]} \tag{C-25}$$

but $f_2 = z_1$, hence we write

$$f_1 = z_4 + \frac{2a}{r_o} \tag{C-26}$$

Equation (C-23) becomes

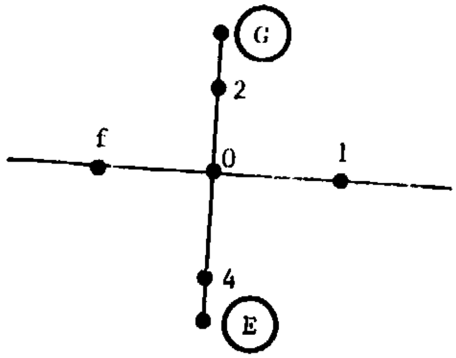
$$z_o = \frac{r_o\left(z_4 + \frac{a}{r_o}\right)}{r_o^2 + \frac{1}{\alpha^2}} + \frac{z_1}{r_o^2\alpha^2 + 1} \tag{C-27}$$

such that $z_o = z_o(z_1, z_4)$ at point G.

B. Finite Plate

Referring to Figures 16(a) and (b), the changes in the boundary conditions between the infinite and finite plate occur on GE and the addition of the free surface ED. In addition, G becomes a special point in the r formulation while both G and E become special points in the z formulation.

$$\underline{\left(\frac{\partial r}{\partial \phi}\right) + r\left(\frac{\partial r}{\partial \psi}\right)\left(\frac{dz}{dr}\right) = 0 \quad \text{on GE}}$$



Along GE, the r difference equation is

$$r_o^4 - \frac{r_o^3}{2}(r_2 + r_4) + r_o^2\left[\frac{1}{\alpha^2} - \frac{1}{8}(r_2 - r_4)^2\right] - \frac{r_o}{2\alpha^2}(r_1 + f)$$

$$+ \frac{1}{8\alpha^2}(r_1 - f)^2 = 0 \qquad \text{(C-28)}$$

in finite difference form, the boundary condition is,

$$\frac{r_1 - f}{\Delta\phi} + \frac{r_o}{\Delta\psi}(z_2 - z_4) = 0 \qquad \text{(C-29)}$$

solving for f,

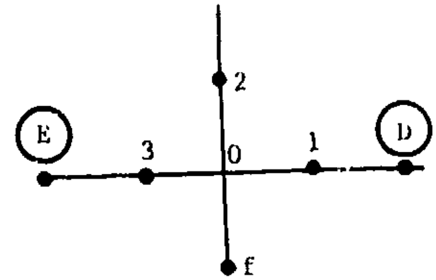
$$f = r_1 + \alpha r_o(z_2 - z_4) \qquad \text{(C-30)}$$

and, inserting this into the r difference equation yields the desired result.

$$r_o^4 - \frac{r_o^3}{2}(r_2 + r_4) + r_o^2\left[\frac{1}{\alpha^2} - \frac{1}{8}(r_2 - r_4)^2\right] - \frac{r_o}{2\alpha^2}\left[2r_1 + \alpha r_o(z_2 - z_4)\right]$$

$$+ \frac{r_o^2}{8}(z_2 - z_4)^2 = 0 \qquad \text{(C-31)}$$

$\left(\frac{\partial r}{\partial\phi}\right)^2 + r^2\left(\frac{\partial r}{\partial\psi}\right)^2 = 1$ on ED



The r difference equation can be written

$$r_o^4 - \frac{r_o^3}{2}(r_2 + r_f) + r_o^2\left[\frac{1}{\alpha^2} - \frac{1}{8}(r_2 - r_f)^2\right] - \frac{r_o}{2\alpha^2}(r_1 + r_3) + \frac{1}{8\alpha^2}(r_1 - r_3)^2 = 0 \qquad \text{(C-32)}$$

Using $\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = 1$ on ED,

$$\left(\frac{r_1 - r_3}{2\ \Delta\phi}\right)^2 + r_o^2\left(\frac{r_2 - r_f}{2\ \Delta\psi}\right)^2 = 1 \qquad \text{(C-33)}$$

Yields

$$(r_2 - r_f)^2 = \frac{1}{r_o^2}\left[4a^2 - \frac{1}{\alpha^2}(r_1 - r_3)^2\right] \qquad \text{(C-34)}$$

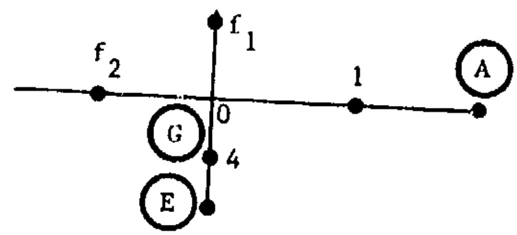
and

$$r_2 + r_f = 2r_2 - \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(r_1 - r_3)^2\right]} \qquad \text{(C-35)}$$

Substituting (C-34) and (C-35) into (C-32) we obtain

$$r_o^4 - r_o^3 r_2 + r_o^2\left\{-\frac{1}{\alpha^2} + \frac{1}{2}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(r_1 - r_3)^2\right]}\right\} - \frac{r_o}{2\alpha^2}(r_1 + r_3) + \frac{1}{8}\left[-4a^2 + \frac{2}{\alpha^2}(r_1 - r_3)^2\right] = 0 \qquad \text{(C-36)}$$

Now, points E and G will be special points since equation (C-31) cannot be directly applied there. One of these positions can be specified as known, r_E = constant. Let us examine special point G



Applying equation (C-10) at point G,

$$r_o^4 - r_o^3 r_4 + r_o^2\left\{\frac{1}{\alpha^2} - \frac{1}{2}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(r_1 - f_2)^2\right]}\right\} - \frac{r_o}{2\alpha^2}(r_1 + f_2)$$

$$+ \frac{1}{8}\left[\frac{2}{\alpha^2}(r_1 - f_2)^2 - 4a^2\right] = 0 \qquad \text{(C-37)}$$

Now, applying $\left(\frac{\partial r}{\partial \phi}\right) + r\left(\frac{\partial r}{\partial \psi}\right)\left(\frac{dz}{dr}\right) = 0$ along GE without involving f_1,

$$\left(\frac{r_1 - f_2}{2\,\Delta\phi}\right) + r_o\left(\frac{r_o - r_4}{\Delta\psi}\right)\left(\frac{z_o - z_4}{r_o - r_4}\right) = 0 \qquad \text{(C-38)}$$

Solving for f_2,

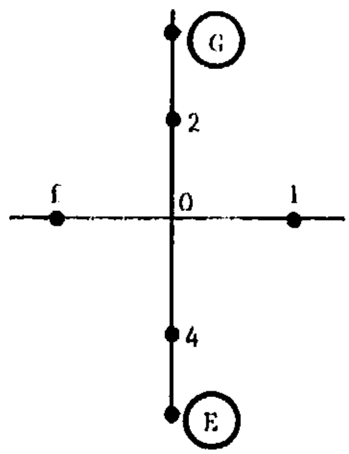
$$f_2 = r_1 + 2\alpha r_o(z_o - z_4) \qquad \text{(C-39)}$$

Therefore, equation (C-37) becomes

$$r_o^4 - r_o^3 r_4 + r_o^2\left\{\frac{1}{\alpha^2} - \sqrt{\left[a^2 - r_o^2(z_o - z_4)^2\right]}\right\} - \frac{r_o}{\alpha^2}\left[r_1 + \alpha r_o(z_o - z_4)\right]$$

$$+ \left[r_o^2(z_o - z_4)^2 - \frac{a^2}{2}\right] = 0 \qquad \text{(C-40)}$$

$r\left(\frac{\partial z}{\partial \psi}\right) - \left(\frac{\partial z}{\partial \phi}\right)\left(\frac{dz}{dr}\right) = 0$ on GE



Application of z difference equation yields,

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)}(z_2 + z_4) + \frac{z_1 + f}{2\left(r_o^2\alpha^2 + 1\right)} + \frac{(z_2 - z_4)(z_1 - f)}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} \qquad \text{(C-41)}$$

Now, along GE, $r\left(\frac{\partial z}{\partial \psi}\right) - \left(\frac{\partial z}{\partial \phi}\right)\left(\frac{dz}{dr}\right) = 0$, which in difference form can be written,

$$r_o\left(\frac{z_2 - z_4}{2\,\Delta\psi}\right) - \left(\frac{z_1 - f}{2\,\Delta\phi}\right)\left(\frac{z_2 - z_4}{r_2 - r_4}\right) = 0 \qquad \text{(C-42)}$$

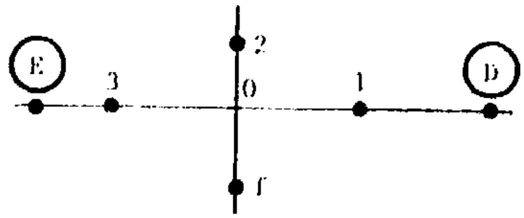
Solving for f,

$$f = z_1 - r_o\alpha(r_2 - r_4) \qquad \text{(C-43)}$$

Hence, equation (C-41) becomes

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)}(z_2 + z_4) + \frac{2z_1 - r_o\alpha(r_2 - r_4)}{2\left(r_o^2\alpha^2 + 1\right)} + \frac{(z_2 - z_4)[r_o\alpha(r_2 - r_4)]}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} \qquad \text{(C-44)}$$

$$r^2\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = 1 \quad \text{on ED}$$



The z difference equation can be written,

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} (z_2 + z_f) + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)} (z_1 + z_3)$$

$$+ \frac{1}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} (z_2 - z_f)(z_1 - z_3) \qquad \text{(C-45)}$$

Applying $r^2\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = 1$ on ED,

$$r_o^2\left(\frac{z_2 - z_f}{2\ \Delta\psi}\right)^2 + \left(\frac{z_1 - z_3}{2\ \Delta\phi}\right)^2 = 1 \qquad \text{(C-46)}$$

Solving for $(z_2 - z_f)$.

$$z_2 - z_f = \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]} \qquad \text{(C-47)}$$

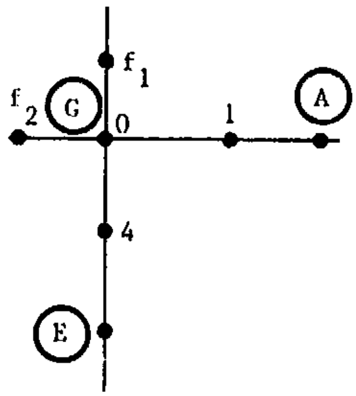
Also,

$$z_f + z_2 = 2z_2 - \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]} \qquad \text{(C-48)}$$

Substitution of these last two equations into equation (C-43) yields the desired expression

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} \left\{2z_2 - \frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]}\right\} + \frac{z_1 + z_3}{2\left(r_o^2\alpha^2 + 1\right)}$$

$$+ \frac{z_1 - z_3}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} \frac{1}{r_o} \sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]} \qquad \text{(C-49)}$$



Applying equation (C-22) at point G yields,

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} \left\{\frac{1}{r_o}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - f_2)^2\right]} + 2z_4\right\} + \frac{z_1 + f_2}{2\left(r_o^2\alpha^2 + 1\right)}$$

$$+ \frac{z_1 - f_2}{4r_o\left(r_o^2\alpha + \frac{1}{\alpha}\right)} \sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - f_2)^2\right]} \qquad \text{(C-50)}$$

To find f_2, we apply $r\left(\frac{\partial z}{\partial \psi}\right) - \left(\frac{\partial z}{\partial \phi}\right)\left(\frac{dz}{dr}\right) = 0$ without involving f_1,

$$r_o\left(\frac{z_o - z_4}{\Delta\psi}\right) - \left(\frac{z_1 - f_2}{2\,\Delta\psi}\right)\left(\frac{z_o - z_4}{r_o - r_4}\right) = 0 \qquad \text{(C-51)}$$

Solving for f_2,

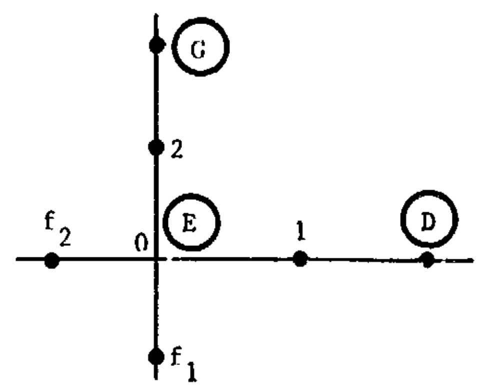
$$f_2 = z_1 - 2r_o\alpha(r_o - r_4) \qquad \text{(C-52)}$$

Substitution into (C-50) yields

$$z_o = \frac{r_o}{r_o^2 + \frac{1}{\alpha^2}} \left\{ \frac{1}{r_o} \sqrt{\left[a^2 - r_o^2 (r_o - r_4)^2\right]} + z_4 \right\} + \frac{z_1 - r_o \alpha (r_o - r_4)}{r_o^2 \alpha^2 + 1}$$

$$+ \frac{r_o - r_4}{r_o^2 + \frac{1}{\alpha^2}} \sqrt{\left[a^2 - r_o^2 (r_o - r_4)^2\right]} \qquad \text{(C-53)}$$

Let us examine the special point at E



Application of equation (C-49) at point E results in,

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)} \left\{ 2z_2 - \frac{1}{r_o} \sqrt{\left[4a^2 - \frac{1}{\alpha^2} (z_1 - f_2)^2\right]} \right\}$$

$$+ \frac{1}{2\left(r_o^2 \alpha^2 + 1\right)} (z_1 + f_2)$$

$$+ \frac{z_1 - f_2}{4r_o\left(r_o^2 \alpha + \frac{1}{\alpha}\right)} \sqrt{\left[4a^2 - \frac{1}{\alpha^2} (z_1 - f_2)^2\right]} \qquad \text{(C-54)}$$

In order to find the fictitious point, f_2, we apply

$r\left(\frac{\partial z}{\partial \psi}\right) - \left(\frac{\partial z}{\partial \phi}\right)\left(\frac{dz}{dr}\right) = 0$ on GE, without involving f_1

$$\frac{r_o(z_2 - z_o)}{\Delta\phi} - \frac{z_1 - f_2}{2\,\Delta\phi}\,\frac{z_2 - z_o}{r_2 - r_o} = 0 \qquad \text{(C-55)}$$

Solving for f_2,

$$f_2 = z_1 - 2r_o\alpha(r_2 - r_o) \qquad \text{(C-56)}$$

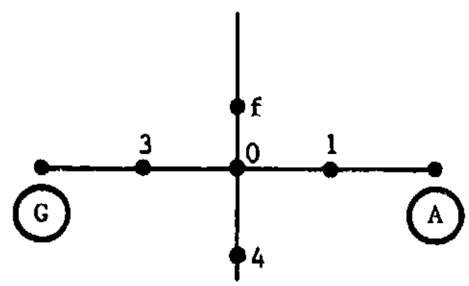
Finally, obtaining from equation (C-54),

$$z_o = \frac{r_o^2}{r_o^2 + \frac{1}{\alpha^2}}\left\{z_2 - \frac{1}{r_o}\sqrt{\left[a^2 - r_o^2(r_2 - r_o)^2\right]}\right\} + \frac{z_1 - r_o\alpha(r_2 - r_o)}{r_o^2\alpha^2 + 1}$$

$$+ \frac{r_2 - r_o}{r_o^2 + \frac{1}{\alpha^2}}\sqrt{\left[a^2 - r_o^2(r_2 - r_o)^2\right]} \qquad \text{(C-57)}$$

C. Surface Tension

$$\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = \frac{1}{\left\{1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r}\frac{d}{dr}\left[\frac{r^2}{\sqrt{\left(\frac{\partial r/\partial\phi}{\partial r/\partial\psi}\right)^2 + r^2}}\right]\right\}} \quad \text{on } AG$$



Applying r difference equation (eq. (131)) at point 0 where f is a fictitious point outside the boundary

$$r_o^4 - \frac{r_o^3}{2} (r_f + r_4) + r_o^2 \left[\frac{1}{\alpha^2} - \frac{1}{8} (r_f - r_4)^2\right] - \frac{r_o}{2\alpha^2} (r_1 + r_3) + \frac{1}{8\alpha^2} (r_1 - r_3)^2 = 0 \qquad \text{(C-58)}$$

in difference form, the free surface boundary condition becomes

$$\left(\frac{r_1 - r_3}{2\ \Delta\phi}\right)^2 + r_o^2\left(\frac{r_f - r_4}{2\ \Delta\psi}\right)^2 = \frac{1}{1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r_o}\frac{d}{dr_o}\left\{\frac{r_o^2}{\sqrt{\left[\frac{(r_1 - r_3)/2\ \Delta\phi}{(r_f - r_4)/2\ \Delta\psi}\right]^2 + r_o^2}}\right\}} \qquad \text{(C-59)}$$

Simplifying by using $\alpha = \Delta\phi/\Delta\psi$ and defining $a = \Delta\psi$ yields,

$$\frac{1}{\alpha^2} (r_1 - r_3)^2 + r_o^2(r_f - r_4)^2 = \frac{4a^2}{1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r_o}\frac{d}{dr_o}\left(\frac{r_o^2}{\sqrt{Q + r_o^2}}\right)} \qquad \text{(C-60)}$$

Now, examine

$$\frac{1}{r_o}\frac{d}{dr_o}\left(\frac{r_o^2}{\sqrt{Q + r_o^2}}\right)$$

As an approximation to this derivative Q is assumed as a constant. In actuality, $Q = f(r_f)$ and $r_f = f(r_o)$. Expanding

$$\frac{1}{r_o}\frac{d}{dr_o}\left(\frac{r_o^2}{\sqrt{Q + r_o^2}}\right)$$

yields

$$\frac{1}{r_c}\frac{d}{dr_o}\left(\frac{r_o^2}{\sqrt{Q + r_o^2}}\right) = \left[\frac{2Q + r_o^2}{\sqrt{\left(Q + r_o^2\right)}\left(Q + r_o^2\right)}\right] \tag{C-61}$$

Therefore, the finite difference representation for the free surface becomes

$$\frac{1}{\alpha^2}\left(r_1 - r_3\right)^2 + r_o^2\left(r_f - r_4\right)^2 = \frac{4a^2}{1 - \frac{2}{We} + \frac{2}{We}\left[\frac{2Q + r_o^2}{\sqrt{Q + r_o^2}\left(Q + r_o^2\right)}\right]} \tag{C-62}$$

r_f must be eliminated between equations (C-58) and (C-62). Let us define

$$\chi = r_f - r_4 \tag{C-63}$$

Equation (C-62) can be written

$$\frac{1}{\alpha^2}\left(r_1 - r_3\right)^2 + r_o^2\chi^2 = \frac{4a^2}{1 - \frac{2}{We} + \frac{2}{We}\left[\frac{2Q + r_o^2}{\sqrt{Q + r_o^2}\left(Q + r_o^2\right)}\right]} \tag{C-64}$$

Where

$$Q = \frac{\left(r_1 - r_3\right)^2}{\chi^2} \cdot \frac{1}{\alpha^2} \tag{C-65}$$

Inserting (C-63) into (C-58) yields

$$\frac{r_o^2\chi^2}{8} + \frac{r_o^3}{2}\chi - r_o^4 + r_o^3 r_4 - \frac{r_o^2}{\alpha^2} + \frac{r_o}{2\alpha^2}\left(r_1 + r_3\right) - \frac{1}{8\alpha^2}\left(r_1 - r_3\right)^2 = 0 \tag{C-66}$$

This is of the form

$$A'\chi^2 + B'\chi + C' = 0$$

Hence

$$\chi = \frac{-B' + \sqrt{B'^2 - 4A'C'}}{2A'} \tag{C-67}$$

where

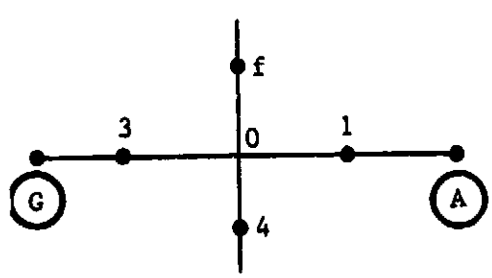
$$A' = \frac{r_o^2}{8} \tag{C-68}$$

$$B' = \frac{r_o^3}{2} \tag{C-69}$$

$$C' = -r_o^4 + r_o^3 r_4 - \frac{r_o^2}{\alpha^2} + \frac{r_o}{2\alpha^2}(r_1 + r_3) - \frac{1}{8\alpha^2}(r_1 - r_3)^2 \tag{C-70}$$

The sign in front of the square root in equation (C-67) was chosen as positive since χ must be greater than or equal to zero. In addition,

$$r^2\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = \frac{1}{1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r}\frac{d}{dr}\left[\frac{r}{\sqrt{r^2\left(\frac{\partial z/\partial\psi}{\partial z/\partial\phi}\right)^2 + 1}}\right]} \quad \text{on AG}$$



Applying z difference equation (132) at point 0 where f is a fictitious point outside the boundary

$$z_o = \frac{r_o^2}{2\left(r_o^2 + \frac{1}{\alpha^2}\right)}(f + z_4) + \frac{1}{2\left(r_o^2\alpha^2 + 1\right)}(z_1 + z_3) + \frac{(f - z_4)(z_1 - z_3)}{4\left(r_o^2\alpha + \frac{1}{\alpha}\right)} \tag{C-71}$$

In finite difference form, the free surface boundary condition can be written,

$$r_o^2(f - z_4)^2 + \frac{1}{\alpha^2}(z_1 - z_3)^2 = \frac{4a^2}{1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r_o}\frac{d}{dr_o}\left(\frac{r_o}{\sqrt{r_o^2 Q^* + 1}}\right)} \qquad \text{(C-72)}$$

where,

$$Q^* = \alpha^2 \frac{(f - z_4)^2}{(z_1 - z_3)^2} \qquad \text{(C-73)}$$

As an approximation, Q^* is assumed to be a constant. Expanding

$$\frac{1}{r_o}\frac{d}{dr_o}\left(\frac{r_o}{\sqrt{r_o^2 Q^* + 1}}\right)$$

obtaining

$$\frac{1}{r_o}\frac{d}{dr_o}\left(\frac{r_o}{\sqrt{r_o^2 Q^* + 1}}\right) = \frac{1}{r_o\left(r_o^2 Q^* + 1\right)\sqrt{r_o^2 Q^* + 1}} \qquad \text{(C-74)}$$

Therefore, the finite difference representation along the free surface becomes

$$r_o^2(f - z_4)^2 + \frac{1}{\alpha^2}(z_1 - z_3)^2 = \frac{4a^2}{\left[1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r_o}\frac{1}{\left(r_o^2 Q^* + 1\right)\sqrt{r_o^2 Q^* + 1}}\right]} \qquad \text{(C-75)}$$

f must now be eliminated between equation (C-71) and (C-75). The manner in which this is done is as follows: Equation (C-75) is solved for $f = F_{ct}(z_1, z_3, z_4)$. The results are then inserted into equation (C-71). In this way, equation (C-71) remains explicit in z_o. Actually, it will be more convenient to solve for the variable $(f - z_4)$ instead of f since equation (C-71) can be expressed as,

$$z_o = \frac{r_o^2 z_4}{r_o^2 + \frac{1}{\alpha^2}} + \frac{z_1 + z_3}{2\left(r_o^2\alpha^2 + 1\right)} + (f - z_4)\left[\frac{2\alpha r_o^2 + (z_1 - z_3)}{4\alpha\left(r_o^2 + \frac{1}{\alpha^2}\right)}\right] \qquad \text{(C-76)}$$

Turning our attention to equation (C-74), it is solved for $(f - z_4)$

$$\frac{\alpha^2 r_o^2 (f - z_4)^2}{(z_1 - z_3)^2} + 1 = \frac{\frac{4a^2\alpha^2}{(z_1 - z_3)^2}}{\left\{1 - \frac{2}{We} + \frac{2}{We}\frac{1}{r_o}\frac{1}{\left[\frac{r_o^2\alpha^2(f - z_4)^2}{(z_1 - z_3)^2} + 1\right]\sqrt{\frac{r_o^2\alpha^2(f - z_4)^2}{(z_1 - z_3)^2} + 1}}\right\}} \qquad \text{(C-77)}$$

A new variable T^2 is introduced

$$T^2 = \frac{\alpha^2 r_o^2 (f - z_4)^2}{(z_1 - z_3)^2} + 1 \qquad \text{(C-78)}$$

Also, let

$$C_0 = \frac{4a^2\alpha^2}{(z_1 - z_3)^2} \qquad \text{(C-79)}$$

$$C_1 = 1 - \frac{2}{We} \qquad \text{(C-80)}$$

$$C_2 = \frac{2}{We} \cdot \frac{1}{r_o} \qquad \text{(C-81)}$$

Making these substitutions into equation (C-77) results in a cubic equation for T

$$T^3 - \frac{C_0}{C_1} T + \frac{C_2}{C_1} = 0 \tag{C-82}$$

Rewriting this as follows:

$$T^3 + aT + b = 0 \tag{C-83}$$

Once T is found from (C-83), $(f - z_4)$ is calculated from equation (C-78) as follows:

$$f - z_4 = \pm \frac{z_1 - z_3}{\alpha r_o} \sqrt{T^2 - 1} \tag{C-84}$$

Appendix D - "Computer Solutions/Listings"

A. Infinite Flat Plate

1. Computer Listing

II. Coarse Mesh Solution for r and z

III. Computer Listing

IV. Fine Mesh Solution for r and z

B. Finite Plate

V. Computer Listing ($R_o/L = 1/2$)

VI. Coarse Mesh Solution

VII. Computer Listing ($R_o/L = 3/4$)

VIII. Coarse Mesh Solution

C. Surface Tension Model

IX. Computer Listing

X. r Solution

XI. z Solution

I. r AND z COMPUTER PROGRAM/INFINITE PLATE/COARSE SOLUTION

```
              INFINITE FLAT PLATE                                                                            DATE  070274     PAGE     1

@FOR,IS  MAIN
FOR 011A-07/02/74-15 29 09 (,0)

   MAIN PROGRAM

   STORAGE USED  CODE(1) 000050; DATA(0) 017271; BLANK COMMON(2) 000000

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003   ECNS
    0004   NINTR$
    0005   NRDU$
    0006   NIO1$
    0007   NIO2$
    0010   NSTOP$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0000    017261 100F       0001   000014 107G      0000 R 017256 ACC       0000 R 000144 AJINV     0000 R 017260 DMAX
    0000 R 000062 F           0000 I 017254 IPRINT    0000 I 017253 J         0000 I 017257 MAXFUN    0000 I 017252 N
    0000 R 017255 STEP        0000 R 005050 W         0000 R 000000 X


00101       1*          DIMENSION X(50),F(50),AJINV(50,50),W(5250)
00103       2*      100 FORMAT (8F10.2))
00104       3*          N=39
00105       4*          READ(5,100)(X(J),J=1,N)
00113       5*          IPRINT=1
00114       6*          STEP=.0001
00115       7*          ACC=.000001
00116       8*          MAXFUN=390
00117       9*          DMAX=1.0
00120      10*          CALL TONS(N,X,F,AJINV,STEP,DMAX,ACC,MAXFUN,IPRINT,W)
00121      11*          END

     END OF COMPILATION         NO  DIAGNOSTICS.
```

ORIGINAL PAGE IS
OF POOR QUALITY

INFINITE FLAT PLATE   DATE 070274   PAGE 2

@FOR,IS EQ
FOR 011A-07/02/74-15 29 13 (,0)

SUBROUTINE EQNS   ENTRY POINT 002561

STORAGE USED CODE(1) 002736; DATA(0) 005200; BLANK COMMON(2) 000000

EXTERNAL REFERENCES (BLOCK, NAME)

```
0003   CALFUN
0004   MATINV
0005   NWDU$
0006   N102$
0007   NIO1$
0010   NERR2$
0011   SORT
0012   NERR3$
```

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

```
0001     000102 1L       0001     000210 10L      0001     002164 1011G    0001     002250 1033G    0001     002304 1045G
0001     002330 1054G    0001     002366 1070G    0001     002372 1074G    0001     000300 11L      0001     002460 1117G
0001     002506 1126G    0001     000270 13L      0001     000120 140G     0001     000276 15L      0000     005020 16F
0001     000155 151G     0001     000234 17L      0000     005037 19F      0001     000244 215G     0001     000317 22L
0000     005054 23F      0001     000362 24L      0001     000344 257G     0000     005070 26F      0001     002024 27L
0001     000406 273G     0001     002226 28L      0001     000443 29L      0001     000130 3L       0001     000374 30L
0001     000455 312G     0001     000427 32L      0001     000432 33L      0001     000540 330G     0001     000542 333G
0001     000613 352G     0001     000614 [illegible]  0001 000660 364G     0001     000661 367G     0001     000673 38L
0001     000176 4L       0001     000725 400G     0001     000734 406G     0001     001052 41L      0001     001030 43L
0001     001037 435G     0001     001022 44L      0000     005101 45F      0001     001110 463G     0001     001114 467G
0001     001102 48L      0001     000173 5L       0001     001307 50L      0001     001246 524G     0001     001200 53L
0001     001325 545G     0001     001166 56L      0001     001364 560G     0001     001375 565G     0001     001421 58L
0001     001311 60L      0001     001434 603G     0001     001446 611G     0001     001526 64L      0001     001550 644G
0001     001561 651G     0001     001467 66L      0001     001614 664G     0001     001635 675G     0001     001512 68L
0001     001517 69L      0000     004765 7F       0001     001604 70L      0001     001662 704G     0001     001730 721G
0001     001752 752G     0001     002002 742G     0001     002063 770G     0000     005001 8F       0001     001741 80L
0001     002053 81L      0001     002152 83L      0000     004764 86F      0001     002146 87L      0000     005011 9F
0001     002422 95L      0001     001231 98L      0000 R 004747 ANMULT     0000 R 000000 BJINV      0000 R 004717 DD
0000 R 004721 DM         0000 R 004722 DMM        0000 R 004745 DMULT      0000 R 004742 DN         0000 R 004741 DS
0000 R 004720 DSS        0000 R 004707 DTEST      0000 R 004746 DW         0000 R 004716 FMIN       0000 R 004757 FNP
0000 R 004725 FSG        0000 I 004726 I          0000 I 004731 IC         0000   005131 INJP$      0000 I 004723 IS
0000 I 004734 J          0000 I 004732 K          0000 I 004752 KK         0000 I 004754 KMN        0000 I 004751 KPN
0000 I 004753 KS         0000 I 004740 KSIG       0000 I 004704 MAXC       0000 I 004713 MW         0000 I 004762 MWPI
0000 I 004715 ND         0000 I 004714 NDC        0000 I 004736 NDCPI      0000 I 004737 NDCPIP     0000 I 004750 NOPI
0000 I 004735 NDPN       0000 I 004711 NF         0000 I 004727 NFPI       0000 I 004744 NFPJ       0000 I 004705 NT
0000 I 004706 NTEST      0000 I 004712 NW         0000 I 004755 NWPT       0000 I 004756 NWPJ       0000 I 004710 NX
0000 I 004730 NXPT       0000 I 004733 NXPIC      0000 R 004763 PA         0000 R 004761 PJ         0000 R 004743 SP
0000 R 004760 SS         0000 R 004724 TINC
```

```
00101      1*      SUBROUTINE EQNS(N,X,F,AJINV,DSTEP,DMAX,ACC,MAXFUN,IPRINT,W)
```

INFINITE FLAT PLATE

```
      DIMENSION X(1),F(1),AJINV(N,N),W(1)
      DIMENSION BJINV(50,50)
C
C     SET VARIOUS PARAMETERS
C
      MAXC = 0
C
C     MAXC COUNTS THE NUMBER OF CALLS OF CALFUN
C
      NT= N+4
      NTEST= NT
C
C     NT AND NTEST CAUSE AN ERROR RETURN IF F(X) DOES NOT DECREASE
C
      DTEST = FLOAT(N+N) -0.5
C
C     DTEST IS USED TO MAINTAIN LINEAR INDEPENDENCE
C
      NX = N*N
      NF = NX+N
      NW = NF+N
      MW = NW+N
      NDC = MW+N
      ND = NDC+N
C
C     THESE PARAMETERS SEPARATE THE WORKING SPACE ARRAY W
C
      FMIN = 0.0
C
C     USUALLY FMIN IS THE LEAST CALCULATED VALUE OF F(X)
C     AND THE BEST X IS IN W(NX+1) TO W(NX+N)
C
      DD = 0.0
      DSS = DSTEP*DSTEP
      DM = DMAX*DMAX
      DMM = 4.0*DM
      IS = 5
C
C     IS CONTROLS A GO TO STATEMENT FOLLOWING A CALL OF CALFUN
      TINC = 1.0
C
C     TINC IS USED IN THE CRITERION TO INCREASE THE STEP LENGTH
C     START A NEW PAGE FOR PRINTING
      IF (IPRINT) 1,1,85
   85 WRITE(6,86)
   86 FORMAT(1H1)
C
C     CALL SUBROUTINE CALFUN
    1 MAXC = MAXC+1
      CALL CALFUN(N,X,F)
C
C     TEST FOR CONVERGENCE
      FSQ = 0.0
      DO 2 I=1,N
      FSQ = FSQ + F(I)*F(I)
    2 CONTINUE
      IF(FSQ-ACC) 3,3,4
```

```
00145   C
00145   C     PROVIDE PRINTING OF FINAL SOLUTION IF REQUESTED
00150       3 IF(IPRINT) 5,5,6
00153       6 WRITE(6,7) MAXC
00156       7 FORMAT   (1HK,5X 28HTHE FINAL SOLUTION REQUIRED I5,2X22HCALLS OF C
00156        1ALFUN AND IS)
00157         WRITE (6,8) (I,X(I),F(I),I=1,N)
00167       8 FORMAT(1HK,4X1HI,9X4HX(I),19X4HF(I)//(I5,2E17.8))
00170         WRITE (6,9) FSQ
00173       9 FORMAT (1HK,6X22HTHE SUM OF SQUARES IS E17.8)
00174       5 RETURN
00174   C
00174   C     TEST FOR ERROR RETURN BECAUSE F(X) DOES NOT DECREASE
00174   C
00175       4 GO TO (10,11,11,10,11),IS
00176      10 IF (FSQ - FMIN) 15,20,20
00201      20 IF (DD -DSS) 12,12,11
00204      12 NTEST = NTEST -1
00205         IF (NTEST) 13,14,11
00210      14 WRITE (6,16) NT
00213      16 FORMAT( 1HK,6X21HERROR RETURN BECAUSE I5,2X46HCALLS OF CALFUN FAIL
00213        1ED TO IMPROE THE RESIDUALS)
00214      17 DO 18 I=1,N
00217         NFPI = NF + I
00220         NXPI = NX + I
00221         X(I) = W(NXPI)
00222         F(I) = W(NFPI)
00223      18 CONTINUE
00225         FSQ = FMIN
00226         GO TO 3
00226   C
00226   C     ERROR RETURN BECAUSE A NEW JACOBIAN IS UNSUCESSFUL
00226   C
00227      13 WRITE(6,19)
00231      19 FORMAT(1HK,6X64HERROR RETURN BECAUSE F(X) FAILED TO DECREASE USING
00231        1A NEW JACOBEAN)
00232         GO TO 17
00233      15 NTEST = NT
00233   C
00233   C     TEST WHETHER THERE HAVE BEEN MAXFUN CALLS OF CALFUN
00233   C
00234      11 IF (MAXFUN - MAXC) 21,21,22
00237      21 WRITE(6,23) MAXC
00242      23 FORMAT (1HK,6X37HERROR RETURN BECAUSE THERE HAVE BEEN I5,2X 15HCAL
00242        1LS OF CALFUN)
00243         IF (FSQ - FMIN) 3,17,17
00243   C
00243   C     PROVIDE PRINTING IF REQUESTED
00243   C
00246      22 IF (IPRINT) 24,24,25
00251      25 WRITE(6,26) MAXC
00254      26 FORMAT(1HK,6X7HAT THE I5,1X25HTH CALL OF CALFUN WE HAVE)
00255         WRITE(6,8) (I,X(I),F(I),I=1,N)
00267         WRITE(6,9) FSQ
00270      24 GO TO (27,28,29,87,30),IS
00270   C
00270   C     STORE THE RESULT OF THE INITIAL CALL OF CALFUN
```

```
C
   30 FMIN = FSO
      DO 31 I=1,N
      NXPI = NX + I
      NFPI = NF + I
      W(NXPI) = X(I)
      W(NFPI) = F(I)
   31 CONTINUE
C
C     CALCULATE A NEW JACOBIAN APPROXIMATION
C
   32 IC = 0
      IS = 3
   33 IC = IC+1
      X(IC) = X(IC) + DSTEP
      GO TO 1
   29 K = IC
      DO 34 I=1,N
      NFPI = NF + I
      W(K) = (F(I)-W(NFPI))/DSTEP
      K= K+N
   34 CONTINUE
      NXPIC = NX + IC
      X(IC) = W(NXPIC)
      IF (IC-N) 33,35,35
C
C     CALCULATE THE INVERSE OF THE JACOBIAN AND SET THE DIRECTION MATRIX
C
   35 K=0
      DO 36 I=1,N
      DO 37 J=1,N
      K= K+1
      AJINV(I,J)=W(K)
      NDPK = ND + K
      W(NDPK) =0.0
   37 CONTINUE
      NDCPI = NDC + I
      NDCPIP  = NDCPI + N
      W(NDCPIP ) =1.0
      W(NDCPI) = 1.0+FLOAT(N-I)
   36 CONTINUE
      DO 1000 I = 1,N
      DO 1000 J = 1,N
 1000 BJINV(I,J) = AJINV(I,J)
      CALL MATINV(BJINV,N,KSIG)
      DO 1001 I = 1,N
      DO 1001 J = 1,N
 1001 AJINV(I,J) = BJINV(I,J)
C
C     START ITERATION BY PREDICTING THE DESCENT AND NEWTON MINIMA
C
   38 DS = 0.0
      DN = 0.0
      SP = 0.0
      DO 39 I=1,N
      X(I) = 0.0
      F(I) = 0.0
```

INFINITE FLAT PLATE

```
      K= I
      DO 40 J=1,N
      NFPJ = NF + J
      X(I) = X(I)-W(K)*W(NFPJ)
      F(I) = F(I)-AJINV(I,J)*W(NFPJ)
      K= K+N
   40 CONTINUE
      DS = DS + X(I)*X(I)
      DN= DN+F(I)*F(I)
      SP= SP+X(I)*F(I)
   39 CONTINUE
C
C     TEST WHETHER A NEARBY STATIONARY POINT IS PREDICTED
C
      IF(FMIN*FMIN-DMM*DS) 41,41,42
C
C     IF SO THEN RETURN OR REVISE JACOBIAN
C
   42 GO TO (43,43,44),IS
   44 WRITE (6,45)
   45 FORMAT (1HK, 6X68HERROR RETURN BECAUSE A NEARBY STATIONARY POINT
     1OF F(X) IS PREDICTED)
      GO TO 17
   43 NTEST = 0
      DO 46 I=1,N
      NXPI = NX + I
      X(I) = W(NXPI)
   46 CONTINUE
      GO TO 32
C
C     TEST WHETHER TO APPLY THE FULL NEWTON CORRECTION
C
   41 IS = 2
      IF(DN-DD) 47,47,48
   47 DD= AMAX1(DN,DSS)
      DS = .25*DN
      TINC = 1.0
      IF(DN-DSS) 49,58,58
   49 IS = 4
      GO TO 90
C
C     CALCULATE THE LENGTH OF THE STEEPEST DESCENT STEP
C
   48 K=0
      DMULT = 0.0
      DO 51 I=1,N
      DW = 0.0
      DO 52 J=1,N
      K= K+1
      DW = DW + W(K)*X(J)
   52 CONTINUE
      DMULT= DMULT+DW*DW
   51 CONTINUE
      DMULT= DS/DMULT
      DS = DS* DMULT*DMULT
C
C     TEST WHETHER TO USE THE STEEPEST DESCENT DIRECTION
```

ORIGINAL PAGE IS
OF POOR QUALITY


```
C
      IF(DS-DD) 53,54,54
C
C     TEST WHETHER INITIAL VALUE OF DD HAS BEEN SET
C
   54 IF(DD) 55,55,56
   55 DD = AMAX1(DSS,AMIN1(DN,DS))
      DS = DS/(DMULT*DMULT)
      GO TO 41
C
C     SET THE MULTIPLIER OF THE STEEPEST DESCENT DIRECTION
C
   56 ANMULT = 0.0
      DMULT = DMULT*SQRT(DD/DS)
      GO TO 98
C
C     INTERPOLATE BETWEEN STEEPEST DESCENT AND NEWTON DIRECTIONS
C
   53 SP = SP*DMULT
      ANMULT= (DD-DS)/((SP-DS)+SQRT((SP-DD)**2+(DN-DD)*(DD-DS)))
      DMULT = DMULT*(1.0-ANMULT)
C
C     CALCULATE THE CHANGE IN X AND ITS ANGLE WITH THE FIRST DIRECTION
C
   98 DN = 0.0
      SP = 0.0
      DO 57 I=1,N
      F(I)= DMULT*X(I)+ANMULT*F(I)
      DN= DN+F(I)*F(I)
      NDPI = ND + I
      SP= SP+F(I)*W(NDPI)
   57 CONTINUE
      DS = .25* DN
C
C     TEST WHETHER AN EXTRA STEP IS NEEDED FOR INDEPENDENCE
C
      IF(W(NDC+1)-DTEST) 58,58,59
   59 IF(SP*SP-DS) 60,53,58
C
C     TAKE THE EXTRA STEP AND UPDATE THE DIRECTION MATRIX
C
   50 IS = 2
   60 DO 61 I=1,N
      NXPI = NX + I
      NDCPI = NDC + I
      NDPI = ND + I
      X(I) = W(NXPI) + DSTEP*W(NDPI)
      W(NDCPI) = W(NDCPI+1) + 1.0
   61 CONTINUE
      W(ND) = 1.0
      DO 62 I=1,N
      K= ND+I
      SP= W(K)
      DO 63 J=2,N
      KPN = K + N
      W(K) = W(KPN)
      K= K+N
```

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```
   63 CONTINUE
      W(K) = SP
   62 CONTINUE
      GO TO 1
C
C     EXPRESS THE NEW DIRECTION IN TERMS OF THOSE OF THE DIRECTION
C     MATRIX, AND UPDATE THE COUNTS IN W(NDC+I) ETC.
C
   58 SP= 0.
      K= ND
      DO 64  I=1,N
      NDCPI = NDC + I
      X(I) = DW
      DW = D.
      DO 65 J=1,N
      K= K+1
      DW = DW+F(J)*W(K)
   65 CONTINUE
      GO TO (68,66),IS
   66 W(NDCPI) = W(NDCPI)+1.D
      SP= SP+DW*DW
      IF (SP-DS) 64,64,67
   67 IS=1
      KK =I
      X(1) = DW
      GO TO 69
   68 X(I) = DW
      NDCPI = NDC + I
   69 W(NDCPI) = W(NDCPI+1)+1.D
   64 CONTINUE
      W(ND) = 1.D
C
C     REORDER THE DIRECTIONS SO THAT KK IS FIRST
C
      IF(KK-1) 70,70,71
   71 KS= NDC+KK*N
      DO 72 I=1,N
      K= KS+I
      SP = W(K)
      DO 73 J=2,KK
      KMN = K - N
      W(K)= W(KMN)
      K= K-N
   73 CONTINUE
      W(K)= SP
   72 CONTINUE
C
C     GENERATE THE NEW ORTHOGOBAL DIRECTION MATRIX
C
   70 DO 74 I=1,N
      NWPI = NW + I
      W(NWPI) = 0.D
   74 CONTINUE
      SP= X(1)*X(1)
      K= ND
      DO 75 I=2,N
      DS = SQRT(SP*(SP+X(I)*X(I)))
```

```
      DW = SP/DS
      DS = X(I)/DS
      SP = SP+X(I)*X(I)
      DO 76 J=1,N
      K= K+1
      NWPJ = NW + J
      KPN = K + N
      W(NWPJ)=W(NWPJ)+X(I-1)*W(K)
      W(K) = DW*W(KPN)-DS*W(NWPJ)
   76 CONTINUE
   75 CONTINUE
      SP= 1.0/SQRT(DN)
      DO 77 I=1,N
      K= K+1
      W(K) = SP*F(I)
   77 CONTINUE
C
C     CALCULATE THE NEXT VECTOR X, AND PREDICT THE RIGHT HAND SIDES
C
   80 FNP= 0.0
      K=0
      DO 78 I=1,N
      NXPI = NX + I
      NWPI = NW + I
      NFPI = NF + I
      X(I) = W(NXPI)+F(I)
      W(NWPI) = W(NFPI)
      DO 79 J=1,N
      K= K+1
      W(NWPI) = W(NWPI)+W(K)*F(J)
   79 CONTINUE
      FNP= FNP+W(NWPI)**2
   78 CONTINUE
C
C     CALL CALFUN USING THE NEW VECTOR OF VARIABLES
C
      GO TO 1
C
C     UPDATE THE STEP SIZE
C
   27 DMULT = .9*FMIN+.1*FNP-FSQ
      IF (DMULT) 82,81,81
   82 DD = AMAX1(DSS,0.25*DD)
      TINC = 1.0
      IF(FSQ-FMIN) 83,28,28
C
C     TRY THE TEST TO DECIDE WHETHER TO INCREASE THE STEP LENGTH
C
   81 SP= 0.0
      SS = 0.0
      DO 84 I=1,N
      NWPI = NW + I
      SP= SP+ABS(F(I)*(F(I)-W(NWPI)))
      SS = SS + (F(I)-W(NWPI))**2
   84 CONTINUE
      PJ = 1.0+ DMULT/(SP+SQRT(SP*SP+DMULT*SS))
      SP= AMIN1(4.0,TINC,PJ)
```

```
INFINITE FLAT PLATE

      TINC = PJ/SP
      DD = AMIN1(DM,SP*DD)
      GO TO 83
C
C     IF F(X) IMPROVES STORE THE NEW VALUE OF X
C
   87 IF (FSO - FMIN) 83,50,50
   83 FMIN = FSO
      DO 88 I=1,N
      SP = X(I)
      NXPI = NX + I
      NWPI = NW + I
      NFPI = NF + I
      X(I) = W(NXPI)
      W(NXPI) = SP
      SP= F(I)
      F(I) = W(NFPI)
      W(NFPI) = SP
      W(NWPI) = -W(NWPI)
   88 CONTINUE
      IF(IS-1) 28,28,50
C
C     CALCULATE THE CHANGES IN F AND X
C
   28 DO 89 I=1,N
      NXPI = NX + I
      NFPI = NF + I
      X(I) = X(I) - W(NXPI)
      F(I) = F(I)-W(NFPI)
   89 CONTINUE
C
C     UPDATE THE APPROXIMATIONS TO J AND AJINV
C
      K = 0
      DO 90 I=1,N
      NWPI = NW + I
      MWPI = MW + I
      W(MWPI) = X(I)
      W(NWPI) = F(I)
      DO 91 J=1,N
      W(MWPI) = W(MWPI)-AJINV(I,J)*F(J)
      K=K+1
      W(NWPI) = W(NWPI)- W(K)*X(J)
   91 CONTINUE
   90 CONTINUE
      SP = 0.0
      SS = 0.0
      DO 92 I=1,N
      DS = 0.0
      DO 93 J=1,N
      DS = DS + AJINV(J,I)*X(J)
   93 CONTINUE
      SP = SP+ DS*F(I)
      SS = SS + X(I)*X(I)
      F(I) = DS
   92 CONTINUE
      DMULT = 1.0
```

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```
01107            IF (ABS(SP)-.1*SS) 94,95,95
01112         94 DMULT = 0.8
01113         95 PJ = DMULT/SS
01114            PA = DMULT/(DMULT*SP+(1.0-DMULT)*SS)
01115            K= 0
01116            DO 96 I=1,N
01121            NWPI = NW + I
01122            MWPI = MW + I
01123            SP = PJ*W(NWPI)
01124            SS = PA*W(MWPI)
01125            DO 97 J=1,N
01130            K= K+1
01131            W(K) = W(K)+SP*X(J)
01132            AJINV(I,J)= AJINV(I,J)+SS*F(J)
01133         97 CONTINUE
01135         96 CONTINUE
01137            GO TO 38
01140            END
```

END OF COMPILATION   NO DIAGNOSTICS.

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                                                                                  DATE  070274    PAGE  12

                    INFINITE FLAT PLATE

@FOR,IS  MATIN
FOR 011A-07/02/74-15 29 31 (,0)

  SUBROUTINE MATINV    ENTRY POINT 000655

  STORAGE USED  CODE(1) 000715; DATA(0) 012003; BLANK COMMON(2) 000000

  EXTERNAL REFERENCES (BLOCK, NAME)

   0003    NEPR3$

  STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

   0001     000007 1L          0001     000327 11L         0001     000376 13L         0001     000127 137G        0001     000133 143G
   0001     000423 15L         0001     000156 151G        0001     000174 155G        0001     000231 173G        0001     000040 2L
   0001     000250 205G        0001     000562 21L         0001     000264 213G        0001     000302 221G        0001     000614 23L
   0001     000352 234G        0001     000366 242G        0001     000412 251G        0001     000427 263G        0001     000435 267G
   0001     000444 272G        0001     000450 276G        0001     000045 3L          0001     000520 311G        0001     000522 314G
   0001     000525 320G        0001     000601 335G        0001     000602 340G        0001     000053 4L          0001     000060 5L
   0001     000206 7L          0000 D 000062 A         0000 R 000072 DELT        0000 R 000104 DET         0000 R 000073 EPS
   0000 I 000077 I          0000    011726 INJP$      0000 I 000100 J           0000 I 000000 K           0000 I 000101 L
   0000 I 000074 LOOPS      0000 I 000075 M           0000 L 000070 MATIN       0000 I 000076 MM          0000 I 000071 MMAX
   0000 I 000102 N          0000 D 000066 PROD        0000 D 000064 R           0000 D 000105 T           0000 R 000103 TEST
   0000 D 000105 Z1         0000 D 000105 Z2

                                                                                        MINV   2
 00101        1*               SUBROUTINE MATINV(S,MX,KSIG)
 00101        2*       C.....MATRIX INVERSION BY GAUSS-JORDAN ELIMINATION
 00103        3*               DIMENSION S(50,50),T(50,50),Z1(50,50),Z2(50,50),K(50)        MINV   5
 00104        4*               DOUBLE PRECISION A,R,PROD,T,Z1,Z2
 00105        5*               EQUIVALENCE (T,Z1,Z2)                                        MINV   8
 00106        6*               LOGICAL MATIN                                                MINV   9
 00107        7*               DATA MMAX, DELT, EPS, LOOPS/ 5,0.0001,1.0E-8,1/              MINV  13
 00114        8*               MATIN=.TRUE.                                                 MINV  14
 00115        9*               GO TO 1                                                      MINV  15
 00115       10*       C.....INITIALIZE ROUTINE AND TEST MX (=ORDER OF MATRIX)              MINV  16
 00116       11*            1 M=MX                                                          MINV  17
 00117       12*               IF(M.GT.1.AND.M.LE.MMAX) GO TO 5                             MINV  18
 00121       13*               IF(M.EQ.1) GO TO 2
 00123       14*               KSIG=KSIG+1
 00124       15*               RETURN
 00125       16*            2 PROD=S(1,1)                                                   MINV  20
 *DIAGNOSTIC*  THE TEST FOR EQUALITY BETWEEN NON-INTEGERS MAY NOT BE MEANINGFUL.           MINV  21
 00126       17*               IF(PROD.NE.0.0) GO TO 4                                      MINV  22
 00130       18*            3 KSIG=KSIG+2
 00131       19*               RETURN                                                       MINV  24
 00132       20*            4 S(1,1)=1.0/PROD                                               MINV  25
 00133       21*               GO TO 23                                                     MINV  26
 00134       22*            5 PROD=1.0                                                      MINV  27
 00135       23*               MM=M-1
 00136       24*               DO 6 I=1,M
```

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```
00141           K(I)=I                                                  MINV  28
00142           DO 6 J=1,M                                              MINV  29
00145         6 T(I,J)=S(I,J)
00145     C.....BEGIN BY FINDING LARGEST PIVOTAL ELEMENT
00150           DO 11 I=1,M                                             MINV  31
00153           A=0.0                                                   MINV  32
00154           DO 7  J=I,M                                             MINV  33
00157           IF(ABS(T(J,I)).LE.A) GO TO 7                            MINV  34
00161           A=ABS(T(J,I))
00162           L=J                                                     MINV  37
00163         7 CONTINUE                                                MINV  38
00165 *DIAGNOSTIC*  THE TEST FOR EQUALITY BETWEEN NON-INTEGERS MAY NOT BE MEANINGFUL.
00165           IF(A.EQ.0.0) GO TO 3                                    MINV  39
00165     C.....REARRANGE ROWS AND ORDER ARRAY                          MINV  40
00167           N=K(L)                                                  MINV  41
00170           K(L)=K(I)                                               MINV  42
00171           K(I)=N                                                  MINV  43
00172           DO 8  J=1,M                                             MINV  44
00175           A=T(I,J)
00176           T(I,J)=T(L,J)
00177         8 T(L,J)=A
00177     C.....REDUCE PIVOTAL ROW                                      MINV  48
00201           A=T(I,I)
00202           IF(.NOT.MATIN) PROD=PROD*A                              MINV  50
00204           DO 9 J=1,MM                                             MINV  51
00207         9 T(I,J)=T(I,J+1)/A
00211           T(I,M)=1.0/A
00211     C.....REDUCE REMAINING ROWS                                   MINV  54
00212           DO 11 L=1,M                                             MINV  55
00215           IF(L.EQ.I) GO TO 11                                     MINV  56
00217           A=T(L,I)
00220           DO 10 N=1,MM                                            MINV  58
00223           T(L,N)=T(L,N+1)-A*T(I,N)
00224        10 IF(ABS(T(L,N)).LT.(ABS(T(L,N+1))*EPS))   T(L,N)=0.0
00227           T(L,M)=-A*T(I,M)
00230        11 CONTINUE                                                MINV  62
00230     C.....UNSCRAMBLE INVERTED MATRIX                              MINV  63
00233           DO 15 I=1,M                                             MINV  64
00236           IF(K(I).EQ.I) GO TO 15                                  MINV  65
00240           PROD=-PROD                                              MINV  66
00241           DO 12 J=I,M                                             MINV  67
00244        12 IF(K(J).EQ.I) GO TO 13                                  MINV  68
00247           GO TO 3                                                 MINV  69
00250        13 DO 14 L=1,M                                             MINV  70
00253           A=T(L,I)
00254           T(L,I)=T(L,J)
00255        14 T(L,J)=A
00257           K(J)=K(I)                                               MINV  74
00260        15 CONTINUE                                                MINV  75
00260     C.....OBTAIN ERROR MATRIX                                     MINV  76
00262           DO 20 N=1,LOOPS                                         MINV  77
00265           TEST=0.0                                                MINV  78
00266           DO 17 I=1,M                                             MINV  79
00271           DO 17 J=1,M                                             MINV  80
00274           R=0.0                                                   MINV  81
00275           DO 16 L=1,M                                             MINV  82
00300        16 R=R-Z(I(L,J)*S(I,L)
```

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INFINITE FLAT PLATE                                          DATE 070274 - - PAGE

00302    81*        IF(I.EQ.J) R=R+1.0                       MINV  84
00304    82*        TEST=AMAX1(TEST,ABS(R))                  MINV  85
00305    83*     17 Z2(I,L)=R                                MINV  87
00310    84*        DO 19 I=1,M                              MINV  88
00313    85*        DO 19 J=1,M                              MINV  89
00316    86*        A=0.0                                    MINV  90
00317    87*        DO 18 L=1,M
00322    88*     18 A=A+Z1(I,L)*Z2(L,J)
00324    89*     19 Z1(I,J)=Z1(I,J)+A                        MINV  93
00327    90*        IF(TEST.LE.DELT) GO TO 21                MINV  94
00331    91*     20 CONTINUE                                 MINV  95
00333    92*        KSIG=KSIG+3                              MINV  96
00333    93*  C.....TRANSFER FINAL INVERSE                   MINV  97
00334    94*     21 DO 22 I=1,M                              MINV  98
00337    95*        DO 22 J=1,M
00342    96*     22 S(I,J)=Z1(I,J)                           MINV 100
00345    97*     23 IF(MATIN) RETURN                         MINV 101
00347    98*        DET=PROD                                 MINV 102
00350    99*        RETURN                                   MINV 103
00351   100*        END

   END OF COMPILATION          2 DIAGNOSTICS.
```

```
@FOR,IS CAL
FOR 011A-07/02/74-15 29 39 (,0)

   SUBROUTINE CALFUN    ENTRY POINT 001224

   STORAGE USED  CODE(1) 001247; DATA(0) 000203; BLANK COMMON(2) 000000

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003   SQRT
    0004   NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001   000246 121G      0001   000414 136G      0001   000517 146G      0001   000600 161G      0001   000666 170G
    0001   000736 176G      0001   001001 204G      0001   001050 212G      0001   001112 220G      0001   001151 226G
    0001   000457 62L       0000 R 000066 AL        0000 R 000064 D         0000   000106 INJP$     0000 I 000070 IS
    0000 I 000071 IT        0000 I 000067 K         0000 R 000072 NR        0000 R 000062 NS        0000 R 000065 PSIG
    0000 R 000000 U         0000 R 000063 US


00101            SUBROUTINE CALFUN(N,Y,F)
00103            DIMENSION U(50)
00104            DIMENSION Y(50)
00105            DIMENSION F(50)
00106            REAL NS
00107             NS=1.
00110            US=NS**2
00111            D=.0204
00112            PSIG=NS/2.
00113            AL=PSIG/5.
00113      C ALL STATEMENT NUMBERS LOCAL TO CAL
00113      C DEFINE F(I) IN TERMS OF Y(J)
00114        50 U(1)=4.*AL**2/US-D*(Y(2)-4.)**2
00115            IF(U(1).LE.0.0)U(1)=0.0
00117            F(1)=Y(1)**4-Y(8)*Y(1)**3+
00117           1Y(1)**2*(D-.5*SQRT(U(1)))-Y(1)*D*(Y(2)+4.)/2.+
00117           2(2.*D*(Y(2)-4.)**2-4.*AL**2/US)/8.
00120            DO 60 K=2,6
00123            U(K)=4.*AL**2/US-D*(Y(K+1)-Y(K-1))**2
00124            IF(U(K).LE.0.0)U(K)=0.0
00126            F(K)=Y(K)**4-Y(K+7)*Y(K)**3+
00126           1Y(K)**2*(D-.5*SQRT(U(K)))-
00126           2Y(K)*D*(Y(K+1)+Y(K-1))/2.+(2.*D*(Y(K+1)-Y(K-1))**2-4.*AL**2/US)/8.
00127        60 CONTINUE
00131            U(7)= 4.*AL**2/US-(1.-Y(6))**2*D
00132            IF(U(7).LE.0.0)U(7)=0.0
00134            F(7)=Y(7)**4-Y(14)*Y(7)**3+
00134           1Y(7)**2*(D-.5*SQRT'U(7)))-
00134           2Y(7)*D*(1.+Y(6))/2.+(2.*D*(1.-Y(6))**2-4.*AL**2/US)/8.
00135            DO 61 K=8,29,7
```

```
00140          F(K)=Y(K)**4-Y(K)**3*(Y(K-7)+Y(K+7))/2.+
00140         1Y(K)**2*(D-.125*(Y(K-7)-Y(K+7))**2)-
00140         2Y(K)*D*(Y(K+1)+4.)/2.+D*(Y(K+1)-4.)**2/8.
00141       61 CONTINUE
00143          IS=9
00144       62 IT=IS+4
00145          DO 63 K=IS,IT
00150          F(K)=Y(K)**4-Y(K)**3*(Y(K-7)+Y(K+7))/2.+
00150         1Y(K)**2*(D-.125*(Y(K-7)-Y(K+7))**2)-D*Y(K)*(Y(K+1)+Y(K-1))/2.
00150         2+D*(Y(K+1)-Y(K-1,)**2/8.
00151       63 CONTINUE
00153          IS=IS+7
00154          IF(IS.LE.23)GO TO 62
00156          REAL MR
00157          MR=1.
00160          DO 64 K=14,28,7
00163          F(K)=Y(K)**4-Y(K)**3*(Y(K-7)+Y(K+7))/2.+Y(K)**2*(D-.125*(Y(K-7)-
00163         1Y(K+7))**2)-Y(K)*D*(SQRT(2.*(5.-MR)*AL/NS)+Y(K-1))/2.+
00163         2D*(SQRT(2.*(5.-MR)*AL/NS)-Y(K-1))**2/8.
00164          MR=MR+1.
00165       64 CONTINUE
00167          DO 65 K=30,32
00172          F(K)=Y(K)**4-Y(K)**3*(Y(K-7)+Y(K+7))/2.+
00172         1Y(K)**2*(D-.125*(Y(K-7)-Y(K+7))**2)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
00172         2D*(Y(K+1)-Y(K-1))**2/8.
00173       65 CONTINUE
00175          DO 67 K=33,34
00200          F(K)=Y(K)**4-Y(K)**3*(Y(K-7))/2.+
00200         1Y(K)**2*(D-.125*(Y(K-7))**2)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
00200         2D*(Y(K+1)-Y(K-1))**2/8.
00201       67 CONTINUE
00203          DO 68 K=35,35
00206          F(K)=Y(K)**4-Y(K)**3*Y(K-7)/2.+Y(K)**2*(D-.125*Y(K-7)**2)-
00206         1D*Y(K)*(SQRT(2.*AL/NS)+Y(K-1))/2.
00206         2+D*(SQRT(2.*AL/NS)-Y(K-1))**2/8.
00207       68 CONTINUE
00211          DO 21 K=36,36
00214          F(K)= Y(K)**4-Y(K)**3*Y(K-7)+D*Y(K)**2
00214         1-D*Y(K)*(Y(K+1)+4.)/2.+D*(Y(K+1)-4.)**2/8.
00215       21 CONTINUE
00217          DO 22 K=37,38
00222          F(K)=Y(K)**4-Y(K-7)*Y(K)**3+D*Y(K)**2
00222         1-D*Y(K)*(Y(K+1)+Y(K-1))/2.+D*(Y(K+1)-Y(K-1))**2/8.
00223       22 CONTINUE
00225          DO 130 K=39,39
00230          F(K)= Y(K)**4-Y(K)**3*Y(K-7)+D*Y(K)**2
00230         1-D*Y(K)*(Y(K-1))/2.+D*Y(K-1)**2/8.
00231      130 CONTINUE
00233          RETURN
00234          END
```

END OF COMPILATION NO DIAGNOSTICS.

@XQT

AT THE 318 TH CALL OF CALFUN WE HAVE

I	X(I)	F(I)
1	.36591012+01	-.50244624-02
2	.26165584+01	-.63966683-02
3	.22588039+01	-.10192208-01
4	.15253005+01	-.19077991-02
5	.10540853+01	-.77230524-02
6	.10241688+01	.36837752-02
7	.10074260+01	-.12102634-02
8	.36554049+01	.10294695-01
9	.26161744+01	-.14457410-01
10	.22327907+01	-.21219265-01
11	.14919475+01	-.35230514-02
12	.96486716+00	.34360909-02
13	.91838167+00	-.92781783-03
14	.90438066+00	.50412014-03
15	.36525446+01	.10723169-01
16	.26160268+01	-.13809766-01
17	.22123906+01	-.20670719-01
18	.14605228+01	-.57012009-02
19	.84837027+00	-.10842451-02
20	.80062543+00	.13186946-02
21	.78636679+00	.86075103-03
22	.36504966+01	.11411039-01
23	.26161263+01	-.12569365-01
24	.21977497+01	-.20706564-01
25	.14338906+01	-.38107553-02
26	.70283066+00	-.91968715-04
27	.65504960+00	.88849347-03
28	.64302110+00	.99666334-03
29	.36492280+01	.11610316-01
30	.26163948+01	-.11036446-01
31	.21890788+01	-.22334901-01
32	.14129902+01	-.74147554-02
33	.49379254+00	-.19728177-04
34	.45625705+00	.93313894-03
35	.44746030+00	.89681619-03
36	.36487271+01	.58850507-02
37	.26166984+01	-.51837265-02
38	.21868414+01	-.11770966-01
39	.14038445+01	-.42180943-02

THE SUM OF SQUARES IS .35713191-02
ERROR RETURN BECAUSE F(X) FAILED TO DECREASE USINGA NEW JACOBEAN

ORIGINAL PAGE IS
OF POOR QUALITY

```
                 INFINITE PLATE Z SOLUTION                                              DATE  091774     PAGE      1
 @FOR,IS MAIN
 FOR 011A-09/17/74-10 26 18 (,0)

    MAIN PROGRAM

    STORAGE USED  CODE(1) 000752; DATA(0) 000315; BLANK COMMON(2) 000000

    EXTERNAL REFERENCES (BLOCK, NAME)

     0003   NINTRS
     0004   NRDUS
     0005   NIO1$
     0006   NIO2$
     0007   NWDUS
     0010   SORT
     0011   NSTOP$

    STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

     0000   000244 100F       0001   000032 1206       0001   000044 1266       0001   000125 1506       0001   000211 1626
     0001   000372 2005       0001   000450 2106       0001   000517 2216       0001   000575 2306       0000   000246 26F
     0001   000415 3L         0001   000053 50L        0000   000257 8F         0001   000750 99L        0000 R 000234 AL
     0000 I 000237 COUNT      0000 R 000232 D          0000 I 000240 I          0000 I 000242 IS         0000 I 000243 IT
     0000 I 000236 J          0000 I 000241 N          0000 R 000227 KO         0000 I 000230 MM         0000 I 000235 N
     0000 R 000226 NS         0000 R 000233 PSIG       0000 R 000000 U          0000 R 000231 US         0000 R 000062 Y
     0000 R 000144 Z

 00101      1*           DIMENSION U(50)
 00103      2*           DIMENSION Y(50)
 00104      3*           DIMENSION Z(50)
 00105      4*           REAL NS
 00106      5*           REAL KO
 00107      6*           MM=250
 00110      7*           NS=1.
 00111      8*           US=NS**2
 00112      9*           D=.0204
 00113     10*           PSIG=NS/2.
 00114     11*           AL=PSIG/5.
 00115     12*           N=41
 00116     13*           READ(5,100) (Y(J),J=1,N)
 00124     14*           READ(5,100)(Z(J),J=1,N)
 00132     15*       100 FORMAT(8(F10.2))
 00133     16*           KO=3.315
 00134     17*           INTEGER COUNT
 00135     18*           COUNT=0
 00136     19*        50 COUNT=COUNT+1
 00137     20*           IF(COUNT.GE.MM) GO TO 99
 00141  *DIAGNOSTIC*   THIS STATEMENT HAS TOO MANY LEFT PARENTHESES.
 00141     21*        26 FORMAT(1HK,6X7HAT THE I5,1X25HTH ITERATION WE HAVE)
 00142     22*           WRITE(6,26) COUNT
 00145     23*         8 FORMAT(1HK,4X1HI,9X4HZ(I)//(I5,E17.8))
 00146     24*           WRITE(6,8) (I,Z(I),I=1,N)
```

INFINITE PLATE Z SOLUTION DATE 091774 PAGE 2

```
00155         U(1)= 4.*AL**2/US-D*(Z(2)-.125)**2
00156         IF(U(1).LE.0.0)  U(1)= 0.0
00160         Z(1)=Y(1)**2*(SQRT(U(1))/Y(1)+2.*Z(9))/(2.*(Y(1)**2+D)) +
00160        1(Z(2)+.125)/(2.*(Y(1)**2/D+1.))+
00160        2(Z(2)-.125)*SQRT(U(1))/(4.*Y(1)*(Y(1)**2/SQRT(D)+SQRT(D)))
00161         DO 7 K=2,6
00164         U(K)= 4.*AL**2/US-D*(Z(K+1)-Z(K-1))**2
00165         IF(U(K).LE.0.0)  U(K)=0.0
00167         Z(K)=Y(K)**2*(SQRT(U(K))/Y(K)+2.*Z(K+8))/(2.*(Y(K)**2+D))+
00167        1(Z(K+1)+Z(K-) )/(2.*(Y(K)**2/D+1.))+
00167        2(Z(K+1)-Z(K-. )*SQRT(U(K))/(4.*Y(K)*(Y(K)**2/SQRT(D)+SQRT(D)))
00170       7 CONTINUE
00172         U(7)=4.*AL**2/US-D*(KO-Z(6))**2
00173         IF(U(7).LE.0.0)  U(7)=0.0
00175         Z(7)=Y(7)**2*(SQRT(U(7))/Y(7)+2.*Z(15))/(2.*(Y(7)**2+D))+
00175        1(KO+Z(6))/(2.*(Y(7)**2/D+1.))+
00175        2(KO-Z(6))*SQRT(U(7))/(4.*Y(7)*(Y(7)**2/SQRT(D)+SQRT(D)))
00176         Z(8)=Y(8)**2*(.125+Z(16))/(2.*(Y(8)**2+D))+
00176        1Z(9)/ (Y(8)**2/D+1.)
00177         DO 2  K=16,24,8
00202         Z(K)= Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D)) +
00202        1Z(K+1)/(Y(K)**2/D+1.)
00203       2 CONTINUE
00205         IS=9
00206       3 IT= IS + 5
00207         DO 4 K=IS,IT
00212         Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
00212        1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00212        2(Z(K-8)-Z(K+8))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00213       4 CONTINUE
00215         IS=IS+8
00216         IF(IS.LE.25)GO TO 3
00220         DO 5 K=IS,31,8
00223         Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
00223        1(KO+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00223        2(Z(K-8)-Z(K+8))*(KO-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00224       5 CONTINUE
00226         Z(32)=Y(32)**2*Z(24)/(2.*(Y(32)**2+D))+Z(33)/(Y(32)**2/D+1.)
00227         DO 6 K=33,37
00232         Z(K)=Y(K)**2*Z(K-8)/(2.*(Y(K)**2+D))+
00232        1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00232        2Z(K-8)*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00233       6 CONTINUE
00235         Z(38)=Y(38)**2*(Z(30)+Z(40))/(2.*(Y(38)**2+D))+
00235        1(Z(39)+Z(37))/(2.*(Y(38)**2/D+1.))+
00235        2(Z(30)-Z(40))*(Z(39)-Z(37))/(4.*(Y(38)**2/SQRT(D)+SQRT(D)))
00236         Z(39)=Y(39)**2*(Z(31)+Z(41))/(2.*(Y(39)**2+D))+
00236        1(KO+Z(38))/(2.*(Y(39)**2/D+1.))+
00236        2(Z(31)-Z(41))*(KO-Z(38))/(4.*(Y(39)**2/SQRT(D)+SQRT(D)))
00237         Z(40)=Y(40)**2*Z(38)/(Y(40)**2+D)+
00237        1Z(41)/(2.*(Y(40)**2/D+1.))
00240         Z(41)=Y(41)**2*Z(39)/(Y(41)**2+D)+
00240        1(KO+Z(40))/(2.*(Y(41)**2/D+1.))
00241         GO TO 50
00242      99 END
```

AT THE 249 TH ITERATION WE HAVE)

I	Z(I)
1	.13719856+00
2	.19091894+00
3	.22606329+00
4	.36715700+00
5	.87868027+00
6	.14528295+01
7	.23498205+01
8	.10003767+00
9	.10985776+00
10	.15276201+00
11	.18180807+00
12	.30635310+00
13	.91400262+00
14	.14496883+01
15	.23491339+01
16	.75050307-01
17	.82451005-01
19	.11458908+00
19	.13696562+00
20	.23945656+00
21	.73381828+00
22	.14382671+01
23	.23461986+01
24	.50044060-01
25	.54994802-01
26	.76401815-01
27	.91624808-01
28	.16603324+00
29	.62519554+00
30	.14139485+01
31	.23390695+01

INFINITE PLATE Z SOLUTION

32	.25025193-01
33	.27505639-01
34	.38203740-01
35	.45913611-01
36	.85877217-01
37	.45195695+00
38	.13611427+01
39	.23212366+01
40	.11050000+01
41	.22100000+01

@FIN

II. INFINITE FLAT PLATE/COARSE MESH

ψ

	G								A
$\psi=\frac{1}{2}$	4 .125	3.659 .137	2.616 .191	2.258 .226	1.525 .367	1.054 .878	1.024 1.45	1.007 2.35	Kφ=3.315 1.0
		z_1	z_2	z_3	z_4	z_5	z_6	z_7	
	4 .100	3.655 .109	2.616 .152	2.232 .181	1.492 .306	.965 .814	.918 1.44	.904 2.35	Kφ=3.315 .895
	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}	z_{15}	
	4 .075	3.652 .082	2.616 .114	2.212 .136	1.460 .239	.848 .733	.8006 1.43	.986 2.34	Kφ=3.315 .775
	z_{16}	z_{17}	z_{18}	z_{19}	z_{20}	z_{21}	z_{22}	z_{23}	
	4 .05	3.650 .055	2.616 .076	2.198 .091	1.433 .166	.702 .625	.655 1.41	.643 2.34	Kφ=3.315 .632
	z_{24}	z_{25}	z_{26}	z_{27}	z_{28}	z_{29}	z_{30}	z_{31}	
	4 .025	3.649 .027	2.616 .038	2.189 .045	1.412 .085	.494 .45	.456 1.36	.4474 2.32	Kφ=3.315 .447
	z_{32}	z_{33}	z_{34}	z_{35}	z_{36}	z_{37}	z_{38}	z_{39}	
	4 0	3.648 0	2.616 0	2.187 0	1.404 0	0 0	0 1.10	0 2.21	Kφ=3.315 .447
							z_{40}	z_{41}	
ψ=0	E					C			B φ

III. r AND z COMPUTER PROGRAM/INFINITE PLATE/FINE SOLUTION

```
                                                                  DATE  082774     PAGE      1
         INFINITE FLAT PLATE

@FOR,IS  MAIN
FOR 011A-06/27/74-17 12 12 (,0)

  MAIN PROGRAM

  STORAGE USED  CODE(1) 000051; DATA(0) 000516; BLANK COMMON(2) 000000

   COMMON BLOCKS

    0003    A       061301
    0004    B       061301
    0005    C       165140
    0006    D       061301

   EXTERNAL REFERENCES (BLOCK, NAME)

    0007   EQNS
    0010   NINTR$
    0011   NROU$
    0012   NIO1$
    0013   NIO2$
    0014   NSTOP$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0000   000505 100F      0001   000017 1156       0000 R 000502 ACC       0004   000000 AJINV      0006   000000 BJINV
    0000 R 000504 DMAX      0000 R 000237 F          0000 I 000500 IPRINT    0000 I 000477 J          0000 I 000503 MAXFUN
    0000 I 000476 N         0000 R 000501 STEP       0003   000000 T         0005   000000 W          0000 R 000000 X


  00101      1*          COMPILER(XM=1)
  00103      2*          DIMENSION X(159),F(159)
  00105      3*          COMMON/A/T(159,159)
  00106      4*          COMMON/B/AJINV(159,159)
  00107      5*          COMMON/C/W(60000)
  00110      6*          COMMON/D/BJINV(159,159)
  00111      7*      100 FORMAT (8(F10.2))
  00112      8*          N=159
  00113      9*          READ(5,100)(X(J),J=1,N)
  00121     10*          IPRINT=1
  00122     11*          STEP=.0001
  00123     12*          ACC=.000001
  00124     13*          MAXFUN=700
  00125     14*          DMAX=1.0
  00126     15*          CALL EQNS(N,X,F,STEP,DMAX,ACC,MAXFUN,IPRINT)
  00127     16*          END

     END OF COMPILATION           NO  DIAGNOSTICS.
```

```
@FOR,IS EG
FOR 011A-08/27/74-17 12 21 (,0)

  SUBROUTINE EQNS       ENTRY POINT 002525

  STORAGE USED  CODE(1) 002652; DATA(0) 000276; BLANK COMMON(2) 000000

   COMMON BLOCKS

   0003   A        061301
   0004   B        061301
   0005   C        165140
   0006   D        061301

  EXTERNAL REFERENCES (BLOCK, NAME)

   0007   CALFUN
   0010   MATINV
   0011   NWDUS
   0012   NIO2$
   0013   NIO1$
   0014   NERR2$
   0015   SQRT
   0016   NERR3$

  STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

   0001   000110 1L       0001   000216 10L      0001   002137 1026G    0001   002221 1040G    0001   002253 1052G
   0001   002277 1061G    0001   002335 1075G    0001   000306 11L      0001   002341 1101G    0001   002424 1124G
   0001   002452 1133G    0001   000276 13L      0001   000126 145G     0001   000304 15L      0000   000114 16F
   0001   000163 166G     0001   000242 17L      0000   000133 19F      0001   000325 22L      0001   000252 222G
   0000   000150 23F      0001   000370 24L      0000   000164 26F      0001   000352 264G     0001   001777 27L
   0001   002201 28L      0001   000451 29L      0001   000136 3L       0001   000402 30L      0001   000414 306G
   0001   000463 317G     0001   000435 32L      0001   000440 33L      0001   000533 335G     0001   000535 340G
   0001   000605 357G     0001   000606 362G     0001   000641 371G     0001   000642 374G     0001   000654 38L
   0001   000204 4L       0001   000700 405G     0001   001025 41L      0001   000707 413G     0001   001003 43L
   0001   000775 44L      0001   001012 442G     0000   000175 45F      0001   001063 470G     0001   001067 474G
   0001   001055 48L      0001   000201 5L       0001   001262 50L      0001   001153 53L      0001   001221 531G
   0001   001300 552G     0001   001141 56L      0001   001337 565G     0001   001350 572G     0001   001374 58L
   0001   001264 60L      0001   001407 610G     0001   001421 616G     0001   001501 64L      0001   001523 651G
   0001   001534 656G     0001   001442 66L      0001   001567 671G     0001   001465 68L      0001   001472 69L
   0000   000061 7F       0001   001557 70L      0001   001610 702G     0001   001635 711G     0001   001703 726G
   0001   001725 737G     0001   001755 747G     0001   002036 775G     0000   000075 8F       0001   001714 80L
   0001   002026 81L      0001   002125 83L      0000   000060 86F      0001   002121 87L      0000   000105 9F
   0001   002574 95L      0001   001204 98L      0004 R 000000 AJINV    0000 R 000043 ANMULT   0006 R 000000 BJINV
   0000 R 000013 DD       0000 R 000015 DM       0000 R 000016 DMM      0000 R 000041 DMULT    0000 R 000036 DN
   0000 R 000035 DS       0000 R 000014 DSS      0000 R 000003 DTEST    0000 R 000042 DW       0000 R 000012 FMIN
   0000 R 000053 FNP      0000 R 000021 FSQ      0000 I 000022 I        0000 I 000025 IC       0000   000225 INJP$
   0000 I 000017 I[illegible]   0000 I 000030 J        0000 I 000026 K        0000 I 000046 KK       0000 I 000050 KMN
   0000 I 000045 KPN      0000 I 000047 KS       0000 I 000034 KSIG     0000 I 000000 MAXC     0000 I 000007 MW
   0000 I 000056 MWPI     0000 I 000011 ND       0000 I 000010 NDC      0000 I 000032 NDCPI    0000 I 000033 NDCPIP
   0000 I 000044 NDPI     0000 I 000031 NDPM     0000 I 000005 NF       0000 I 000023 NFPI     0000 I 000040 NFPJ
   0000 I 000001 NT       0000 I 000002 NTEST    0000 I 000006 NW       0000 I 000051 NWPI     0000 I 000052 NWFJ
```

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                 INFINITE FLAT PLATE                                                              DATE  082774     PAGE    3

  0000 I 000004 NX      0000 I 000024 NXPI     0000 I 000027 NXPIC    0000 R 000057 PA      0000 R 000055 PJ
  0000 R 000037 SP      0000 R 000054 SS       0003   000000 T        0000 R 000020 TINC    0005 R 000000 W

 00101            COMPILER(XM=1)
 00103            SUBROUTINE CONSIN,X,F,DSTEP,DMAX,ACC,MAXFUN,IPRINT)
 00105            DIMENSION X(1),F(1)
 00106            COMMON/A/T(159,159)
 00107            COMMON/B/AJINW(159,159)
 00110            COMMON/C/W(60000)
 00111            COMMON/D/BJINV(159,159)
 00111      C
 00111      C     SET VARIOUS PARAMETERS
 00111      C
 00112            MAXC = 0
 00112      C
 00112      C     MAXC COUNTS THE NUMBER OF CALLS OF CALFUN
 00112      C
 00113            NT= N+4
 00114            NTEST= NT
 00114      C
 00114      C     NT AND NTEST CAUSE AN ERROR RETURN IF F(X) DOES NOT DECREASE
 00114      C
 00115            DTEST = FLOAT(N+N) -0.5
 00115      C
 00115      C     DTEST IS USED TO MAINTAIN LINEAR INDEPENDENCE
 00115      C
 00116            NX = N+N
 00117            NF = NX+N
 00120            NW = NF+N
 00121            NW = NW+N
 00122            NDC = NW+N
 00123            ND = NDC+N
 00123      C
 00123      C     THESE PARAMETERS SEPARATE THE WORKING SPACE ARRAY W
 00123      C
 00124            FMIN = 0.0
 00124      C
 00124      C     USUALLY FMIN IS THE LEAST CALCULATED VALUE OF F(X)
 00124      C     AND THE BEST X IS IN W(NX+1) TO W(NX+N)
 00124      C
 00125            DD = 0.0
 00126            DSS = DSTEP*DSTEP
 00127            DM = DMAX*DMAX
 00130            DMM = 4.0*DM
 00131            IS = 5
 00131      C
 00131      C     IS CONTROLS A GO TO STATEMENT FOLLOWING A CALL OF CALFUN
 00132            TINC = 1.0
 00132      C
 00132      C     TINC IS USED IN THE CRITERION TO INCREASE THE STEP LENGTH
 00132      C     START A NEW PAGE FOR PRINTING
 00133            IF (IPRINT) 1,1,85
 00136         85 WRITE(6,86)
 00140         86 FORMAT(1H1)
```

```
00140   C
00140   C     CALL SUBROUTINE CALFUN
00141       1 MAXC = MAXC+1
00142         CALL CALFUN(N,X,F)
00142   C
00142   C     TEST FOR CONVERGENCE
00143         FSQ = 0.0
00144         DO 2 I=1,N
00147         FSQ = FSQ + F(I)*F(I)
00150       2 CONTINUE
00152         IF(FSQ-ACC) 3,3,4
00152   C
00152   C     PROVIDE PRINTING OF FINAL SOLUTION IF REQUESTED
00155       3 IF(IPRINT) 5,5,6
00160       6 WRITE(6,7) MAXC
00163       7 FORMAT   (1HK,5X 28HTHE FINAL SOLUTION REQUIRED I5,2X22HCALLS OF C
00163        1ALFUN AND IS)
00164         WRITE (6,8) (I,X(I),F(I),I=1,N)
00174       8 FORMAT(1HK,4X1HI,9X4HX(I),14X4HF(I)//(I5,2E17.8))
00175         WRITE (6,9) FSQ
00200       9 FORMAT (1HK,6X22HTHE SUM OF SQUARES IS E17.8)
00201       5 RETURN
00201   C
00201   C     TEST FOR ERROR RETURN BECAUSE F(X) DOES NOT DECREASE
00201   C
00202       4 GO TO (10,11,11,10,11),IS
00203      10 IF (FSQ - FMIN) 15,20,20
00206      20 IF (DD -DSS) 12,12,11
00211      12 NTEST = NTEST -1
00212         IF (NTEST) 13,14,11
00215      14 WRITE (6,16) NT
00220      16 FORMAT( 1HK,6X21HERROR RETURN BECAUSE I5,2X46HCALLS OF CALFUN FAIL
00220        1ED TO IMPROE THE RESIDUALS)
00221      17 DO 18 I=1,N
00224         NFPI = NF + I
00225         NXPI = NX + I
00226         X(I) = W(NXPI)
00227         F(I) = W(NFPI)
00230      18 CONTINUE
00232         FSQ = FMIN
00233         GO TO 3
00233   C
00233   C     ERROR RETURN BECAUSE A NEW JACOBIAN IS UNSUCESSFUL
00233   C
00234      13 WRITE(6,19)
00236      19 FORMAT(1HK,6X64HERROR RETURN BECAUSE F(X) FAILED TO DECREASE USING
00236        1A NEW JACOBEAN)
00237         GO TO 17
00240      15 NTEST = NT
00240   C
00240   C     TEST WHETHER THERE HAVE BEEN MAXFUN CALLS OF CALFUN
00240   C
00241      11 IF (MAXFUN - MAXC) 21,21,22
00244      21 WRITE(6,23) MAXC
00247      23 FORMAT (1HK,6X37HERROR RETURN BECAUSE THERE HAVE BEEN I5,2X 15HCAL
00247        1LS OF CALFUN)
00250         IF (FSQ - FMIN) 3,17,17
```

```
C
C     PROVIDE PRINTING IF REQUESTED
C
   22 IF (IPRINT) 24,24,25
   25 WRITE(6,26) MAXC
   26 FORMAT(1HK,6X7HAT THE 15,1X25HTH CALL OF CALFUN WE HAVE)
      WRITE(6,8) (I,X(I),F(I),I=1,N)
      WRITE(6,9) FSQ
   24 GO TO (27,28,29,87,30),IS
C
C     STORE THE RESULT OF THE INITIAL CALL OF CALFUN
C
   30 FMIN = FSQ
      DO 31 I=1,N
      NXPI = NX + I
      NFPI = NF + I
      W(NXPI) = X(I)
      W(NFPI) = F(I)
   31 CONTINUE
C
C     CALCULATE A NEW JACOBIAN APPROXIMATION
C
   32 IC = 0
      IS = 3
   33 IC = IC+1
      X(IC) = X(IC) + DSTEP
      GO TO 1
   29 K = IC
      DO 34 I=1,N
      NFPI = NF + I
      WTK) = (F(I)-W(NFPI))/DSTEP
      K= K+N
   34 CONTINUE
      NXPIC = NX + IC
      X(IC) = W(NXPIC)
      IF (IC-N) 33,35,35
C
C     CALCULATE THE INVERSE OF THE JACOBIAN AND SET THE DIRECTION MATRIX
C
   35 K=0
      DO 36 I=1,N
      DO 37 J=1,N
      K= K+1
      AJINV(I,J)=W(K)
      NDPK = ND + K
      W(NDPK) =0.0
   37 CONTINUE
      NDCPI = NDC + I
      NOCPIP = NOCPI + K
      W(NDCPIP ) =1.0
      W(NDCPI) = 1.0-FLOAT(N-I)
   36 CONTINUE
      DO 1000 I = 1,N
      DO 1000 J = 1,N
 1000 PJINV(I,J) = AJINV(I,J)
      CALL MATINV(PJINV,N,KSIG)
      DO 1001 I = 1,N
```

```
      DO 1001 J = 1,N
 1001 AJINV(I,J) = BJINV(I,J)
C
C     START ITERATION BY PREDICTING THE DESCENT AND NEWTON MINIMA
C
   38 DS = 0.0
      DN = 0.0
      SP = 0.0
      DO 39 I=1,N
      X(I) = 0.0
      F(I) = 0.0
      K= 1
      DO 40 J=1,N
      NFPJ = NF + J
      X(I) = X(I)-W(K)*W(NFPJ)
      F(I) = F(I)-AJINV(I,J)*W(NFPJ)
      K= K+N
   40 CONTINUE
      DS = DS + X(I)*X(I)
      DN= DN+F(I)*F(I)
      SP= SP+X(I)*F(I)
   39 CONTINUE
C
C     TEST WHETHER A NEARBY STATIONARY POINT IS PREDICTED
C
      IF(FMIN*FMIN-DMM*DS) 41,41,42
C
C     IF SO THEN RETURN OR REVISE JACOBIAN
C
   42 GO TO (43,43,44),IS
   44 WRITE (6,45)
   45 FORMAT (1HK, 6X68HERROR RETURN BECAUSE A NEARBY STATIONARY POINT
     1OF F(X) IS PREDICTED)
      GO TO 17
   43 NTEST = 0
      DO 46 I=1,N
      NXPI = NX + I
      X(I) = W(NXPI)
   46 CONTINUE
      GO TO 32
C
C     TEST WHETHER TO APPLY THE FULL NEWTON CORRECTION
C
   41 IS = 2
      IF(DN-DD) 47,47,48
   47 DD= AMAX1(DN,DSS)
      DS = .25*DN
      TINC = 1.0
      IF(DN-DSS) 49,58,58
   49 IS = 4
      GO TO 80
C
C     CALCULATE THE LENGTH OF THE STEEPEST DESCENT STEP
C
   48 K=0
      DMULT = 0.0
      DO 51 I=1,N
```

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```
00472           DW = 0.0
00473           DO 52 J=1,N
00476           K= K+1
00477           DW = DW + W(K)*X(J)
00500     52    CONTINUE
00502           DMULT= DMULT+DW*DW
00503     51    CONTINUE
00505           DMULT= DS/DMULT
00506           DS = DS* DMULT*DMULT
00506     C
00506     C     TEST WHETHER TO USE THE STEEPEST DESCENT DIRECTION
00506     C
00507           IF(DS-DD) 53,54,54
00507     C
00507     C     TEST WHETHER INITIAL VALUE OF DD HAS BEEN SET
00507     C
00512     54    IF(DD) 55,55,56
00515     55    DD = AMAX1(DSS,AMIN1(DM,DS))
00516           DS = DS/(DMULT*DMULT)
00517           GO TO 41
00517     C
00517     C     SET THE MULTIPLIER OF THE STEEPEST DESCENT DIRECTION
00517     C
00520     56    ANMULT = 0.0
00521           DMULT = DMULT*SQRT(DD/DS)
00522           GO TO 98
00522     C
00522     C     INTERPOLATE BETWEEN STEEPEST DESCENT AND NEWTON DIRECTIONS
00522     C
00523     53    SP = SP*DMULT
00524           ANMULT= (DD-DS)/((SP-DS)+SQRT((SP-DD)**2+(DN-DD)*(DD-DS)))
00525           DMULT = DMULT*(1.0-ANMULT)
00525     C
00525     C     CALCULATE THE CHANGE IN X AND ITS ANGLE WITH THE FIRST DIRECTION
00525     C
00526     98    DN = 0.0
00527           SP = 0.0
00530           DO 57 I=1,N
00533           F(I)= DMULT*X(I)+ANMULT*F(I)
00534           DN= DN+F(I)*F(I)
00535           NDPI = ND + I
00536           SP= SP+F(I)*W(NDPI)
00537     57    CONTINUE
00541           DS = .25* DN
00541     C
00541     C     TEST WHETHER AN EXTRA STEP IS NEEDED FOR INDEPENDENCE
00541     C
00542           IF(W(NDC+1)-DTEST) 58,58,59
00545     59    IF(SP*SP-DS) 60,58,58
00545     C
00545     C     TAKE THE EXTRA STEP AND UPDATE THE DIRECTION MATRIX
00545     C
00550     60    IS = 2
00551     10    DO 61 I=1,N
00554           NXPI = NX + I
00555           NDCPI = NDC + I
00555           NDPI = ND + I
```

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```
      X(I) = W(NXPI) + DST(P*W(NDPI))
      W(NDCPI) = W(NDCPI+1) + 1.0
   61 CONTINUE
      W(ND) = 1.0
      DO 62 I=1,N
      K= ND+I
      SP= W(K)
      DO 63 J=2,N
      KPN = K + N
      W(K) = W(KPN)
      K= K+N
   63 CONTINUE
      W(K) = SP
   62 CONTINUE
      GO TO 1
C
C     EXPRESS THE NEW DIRECTION IN TERMS OF THOSE OF THE DIRECTION
C     MATRIX, AND UPDATE THE COUNTS IN W(NDC+1) ETC.
C
   58 SP= 0.
      K= ND
      DO 64  I=1,N
      NDCPI = NDC + I
      X(I) = DW
      DW = 0.
      DO 65 J=1,N
      K= K+1
      DW = DW+F(J)*W(K)
   65 CONTINUE
      GO TO (68,66),IS
   66 W(NDCPI) = W(NDCPI)+1.0
      SP= SP+DW*DW
      IF (SP-DS) 64,64,67
   67 IS=1
      KK =I
      X(1) = DW
      GO TO 69
   68 X(I) = DW
      NDCPI = NDC + I
   69 W(NDCPI) = W(NDCPI+1)+1.0
   64 CONTINUE
      W(ND) = 1.0
C
C     REORDER THE DIRECTIONS SO THAT KK IS FIRST
C
      IF(KK-1) 70,70,71
   71 KS= NDC+KK*N
      DO 72 I=1,N
      K= KS+I
      SP = W(K)
      DO 73 J=2,KK
      KMN = K - N
      W(K)= W(KMN)
      K= K-N
   73 CONTINUE
      W(K)= SP
   72 CONTINUE
```

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```
C
C     GENERATE THE NEW ORTHOGONAL DIRECTION MATRIX
C
   70 DO 74 I=1,N
      NWPI = NW + I
      W(NWPI) = 0.0
   74 CONTINUE
      SP= X(1)*X(1)
      K= NO
      DO 75 I=2,N
      DS = SQRT(SP*(SP+X(I)*X(I)))
      DW = SP/DS
      DS = X(I)/DS
      SP = SP+X(I)*X(I)
      DO 75 J=1,N
      K= K+1
      NWPJ = NW + J
      KPN = K + N
      W(NWPJ)=W(NWPJ)+X(I-1)*W(K)
      W(K) = DW*W(KPN)-DS*W(NWPJ)
   76 CONTINUE
   75 CONTINUE
      SP  1.0/SQRT(DN)
      DO 77 I=1,N
      K= K+1
      W(K) = SP*F(I)
   77 CONTINUE
C
C     CALCULATE THE NEXT VECTOR X, AND PREDICT THE RIGHT HAND SIDES
C
   80 FNP= 0.0
      K=0
      DO 79 I=1,N
      NXPI = NX + I
      NWPI = NW + I
      NFPI = NF + I
      X(I) = W(NXPI)+F(I)
      W(NWPI) = W(NFPI)
      DO 79 J=1,N
      K= K+1
      W(NWPI) = W(NWPI)+W(K)*F(J)
   79 CONTINUE
      FNP= FNP+W(NWPI)**2
   78 CONTINUE
C
C     CALL CALFUN USING THE NEW VECTOR OF VARIABLES
C
      GO TO 1
C
C     UPDATE THE STEP SIZE
C
   27 DMULT   .5*FMIN+.1*FNP-FSW
      IF (DMULT) 82,81,81
   82 DD    AMAX1(DSS,0.25*DD)
      TINC   1.0
      IF(FSQ-FMIN) 53,28,28
C
```

INFINITE FLAT PLATE

```
C     TRY THE TEST TO DECIDE WHETHER TO INCREASE THE STEP LENGTH
C
   81 SP = 0.0
      SS = 0.0
      DO 84 I=1,N
      NWPI = NW + I
      SP= SP+ABS(F(I)*(F(I)-W(NWPI)))
      SS = SS + (F(I)-W(NWPI))**2
   84 CONTINUE
      PJ = 1.0+ DMULT/(SP+SQRT(SP*SP+DMULT*SS))
      SP= AMIN1(4.0,TINC,PJ)
      TINC = PJ/SP
      DD = AMIN1(DM,SP*DD)
      GO TO 53
C
C     IF F(X) IMPROVES STORE THE NEW VALUE OF X
C
   87 IF (FSQ - FMIN) 83,50,50
   83 FMIN = FSQ
      DO 88 I=1,N
      SP = X(I)
      NXPI = NX + I
      NWPI = NW + I
      NFPI = NF + I
      X(I) = W(NXPI)
      W(NXPI) = SP
      SP= F(I)
      F(I) = W(NFPI)
      W(NFPI) = SP
      W(NWPI) = -W(NWPI)
   88 CONTINUE
      IF(IS-1) 28,29,50
C
C     CALCULATE THE CHANGES IN F AND X
C
   28 DO 89 I=1,N
      NXPI = NX + I
      NFPI = NF + I
      X(I) = X(I) - W(NXPI)
      F(I) = F(I)-W(NFPI)
   89 CONTINUE
C
C     UPDATE THE APPROXIMATIONS TO J AND AJINV
C
      K = 0
      DO 91 I=1,N
      NWPI = NW + I
      NGPI = NG + I
      W(NGPI) = X(I)
      W(NWPI) = F(I)
      DO 91 J=1,N
      W(NWPI) = W(NWPI)-AJINV(I,J)*F(J)
      K = K+1
      W(NGPI) = W(NGPI)- W(K)*X(J)
   91 CONTINUE
   92 CONTINUE
```

```
01072          SS = 0.0
01074          DO 92 I=1,N
01077          DS = 0.0
01100          DO 93 J=1,N
01103          DS = DS + AJINV(J,I)*X(J)
01104       93 CONTINUE
01106          SP = SP+ DS*F(I)
01107          SS = SS + X(I)*X(I)
01110          F(I) = DS
01111       92 CONTINUE
01113          DMULT = 1.0
01114          IF (ABS(SP)-.1*SS) 94,95,95
01117       94 DMULT = 0.8
01120       95 PJ = DMULT/SS
01121          PA = DMULT/(DMULT*SP+(1.0-DMULT)*SS)
01122          K= 0
01123          DO 96 I=1,N
01126          NWPI = NW + 1
01127          MWPI = MW + 1
01130          SP = PJ*W(NWPI)
01131          SS = PA*W(MWPI)
01132          DO 97 J=1,N
01135          K= K+1
01136          W(K) = W(K)+SP*X(J)
01137          AJINV(I,J)= AJINV(I,J)+SS*F(J)
01140       97 CONTINUE
01142       96 CONTINUE
01144          GO TO 38
01145          END
```

END OF COMPILATION   NO DIAGNOSTICS.

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@FOR,IS  MATIN
FOR S11A-08/27/74-17 12 45 (,0)

   SUBROUTINE MATINV     ENTRY POINT 000741

   STORAGE USED  CODE(1) 001001; DATA(0) 000260; BLANK COMMON(2) 000000

    COMMON BLOCKS

    0003    A       142602
    0004    B       061301
    0005    C       165140
    0006    D       061301

   EXTERNAL REFERENCES (BLOCK, NAME)

    0007    NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001    000036 1L          0001   000373 111       0001    000451 13L       0001    000500 15L       0001    000161 151E
    0001    000165 155G        0001   000214 161G      0001    000236 167G      0001    000067 2L        0001    000275 205G
    0001    000645 21L         0001   000313 217G      0001    000327 225G      0001    000700 23L       0001    000346 233G
    0001    000422 246G        0001   000440 251G      0001    000470 263G      0001    000505 275G      0001    000074 3L
    0001    000515 303G        0001   000523 314G      0001    000527 310G      0001    000602 323G      0001    000605 324G
    0001    000610 332G        0001   000665 347G      0001    000666 352G      0001    000102 4L        0001    000107 5L
    0001    000252 7L          0003 D 000237 A         0004    000000 AJINV     0006    000000 BJINV     0000 R 000247 DELT
    0000 R 000261 EPP          0000 R 000250 EPS       0000 I 000254 I         0000    000271 INJP$     0000 I 000255 J
    0000 I 000000 K            0000 I 000256 L         0000 I 000251 LOOPS     0000 I 000252 M         0000 L 000245 MATIN
    0000 I 000253 MM           0000 I 000246 NMAX      0000 I 000257 N         0000 D 000243 PROD      0000 D 000241 R
    0003 D 000000 T            0000 R 000260 TEST      0005    000000 W         0003 D 000000 Z1        0003 D 000000 Z2


 00101      1*        COMPILER(XM=11),(EQUIV=CMN)
 00103      2*        SUBROUTINE MATINV(S,PX,NSIG)
 00104      3*        DIMENSION S(159,159)
 00105      4*     C.....MATRIX INVERSION BY GAUSS-JORDAN ELIMINATION              MINV   2
 00106      5*        DIMENSION M(159)
 00107      6*        DIMENSION Z1(159,159)
 00110      7*        DIMENSION Z2(159,159)
 00111      8*        COMMON/A/T(159,159)
 00112      9*        COMMON/B/AJINV(159,159)
 00113     10*        COMMON/C/W(6000)
 00114     11*        COMMON/D/BJINV(159,159)
 00115     12*        DOUBLE  PRECISION A,T,PROD,R,Z1,Z2
 00116     13*        EQUIVALENCE(Z1,T)
 00117     14*        EQUIVALENCE(Z2,T)
 00120     15*        LOGICAL MATIN                                               MINV   5
 00121     16*        DATA NMAX, DELT, EPS, LOOPS/ 5,0.0001,1.0E-6,1/
 00122     17*        MATIN =TRUE.                                                MINV   8
 00123     18*        GO TO 1                                                     MINV   9
```

```
00127   C.....INITIALIZE ROUTINE AND TEST MX (=ORDER OF MATRIX)               MINV  13
00130     1 M=MX                                                              MINV  14
00131       IF(M.GT.1.AND.M.LE.MPAX) GO TO 5                                  MINV  15
00132       IF(M.EQ.1) GO TO 2                                                MINV  16
00135       KSIG=KSIG+1                                                       MINV  17
00136       RETURN                                                            MINV  18
00137     2 PROD=S(1,1)
00140   *DIAGNOSTIC*  THE TEST FOR EQUALITY BETWEEN NON-INTEGERS MAY NOT BE MEANINGFUL.
00140       IF(PROD.NE.0.0) GO TO 4                                           MINV  20
00142     3 KSIG=KSIG+2                                                       MINV  21
00143       RETURN                                                            MINV  22
00144     4 S(1,1)=1.0/PROD
00145       GO TO 23                                                          MINV  24
00146     5 PROD=1.0                                                          MINV  25
00147       MM=M-1                                                            MINV  26
00150       DO 5 I=1,M                                                        MINV  27
00153       K(I)=I                                                            MINV  28
00154       DO 6 J=1,M                                                        MINV  29
00157     6 T(I,J)=S(I,J)
00157   C.....BEGIN BY FINDING LARGEST PIVOTAL ELEMENT                        MINV  31
00162       DO 11 I=1,M                                                       MINV  32
00165       A=0.0                                                             MINV  33
00166       DO 7 J=I,M                                                        MINV  34
00171       IF(ABS(T(J,I)).LE.A) GO TO 7
00173       A=ABS(T(J,I))
00174       L=J                                                               MINV  37
00175     7 CONTINUE                                                          MINV  38
00177   *DIAGNOSTIC*  THE TEST FOR EQUALITY BETWEEN NON-INTEGERS MAY NOT BE MEANINGFUL.
00177       IF(A.EQ.0.0) GO TO 3                                              MINV  39
00177   C.....REARRANGE ROWS AND ORDER ARRAY                                  MINV  40
00201       N=K(L)                                                            MINV  41
00202       K(L)=K(I)                                                         MINV  42
00207       K(I)=N                                                            MINV  43
00204       DO 8 J=1,M                                                        MINV  44
00207       A=T(I,J)
00211       T(I,J)=T(L,J)
00211     8 T(L,J)=A
00211   C.....REDUCE PIVOTAL ROW                                              MINV  49
00213       A=T(I,I)
00214       IF(.NOT.MATINV) PROD=PROD*A                                       MINV  50
00216       DO 9 J=1,MM                                                       MINV  51
00221     9 T(I,J)=T(I,J+1)/A
00223       T(I,M)=1.0/A
[illegible]   C.....REDUCE REMAINING ROWS                                     MINV  54
00224       DO 11 L=1,M                                                       MINV  55
[illegible]       IF(L.EQ.I) GO TO 11                                         MINV  56
00241       A=T(L,I)
[illegible]       DO 10 N=1,MM                                                MINV  59
00235       T(L,N)=T(L,N+1)-A*T(I,N)
[illegible]    10 IF(ABS(T(L,N)).LT.(ABS(T(L,N+1))*EPS))   T(L,N)=0.0
[illegible]       T(L,M)=-A*T(I,M)
[illegible]    11 CONTINUE                                                    MINV  62
[illegible]   C.....UNSCRAMBLE INVERTED MATRIX                                MINV  63
[illegible]       DO 1[illegible] I=1,M                                       MINV  64
[illegible]       IF(K(I).EQ.I) GO TO 1[illegible]                            MINV  65
[illegible]       [illegible]                                                 MINV  66
[illegible]       DO 1[illegible] J=1,M                                       MINV  67
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                                                                         DATE  082774    PAGE    18
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00256     74*         12 IF(K(J).EQ.I) GO TO 13                          MINV  68
00261     75*            GO TO 3                                         MINV  69
00262     76*         13 DO 14 L=1,M                                     MINV  70
00265     77*            A=T(L,I)
00266     78*            T(L,I)=T(L,J)
00267     79*         14 T(L,J)=A
00271     80*            K(J)=K(I)                                       MINV  74
00272     81*         15 CONTINUE                                        MINV  75
00272     82*   C.....OBTAIN ERROR MATRIX                                MINV  76
00274     83*            DO 23 N=1,LOOPS                                 MINV  77
00277     84*            TEST=0.0                                        MINV  78
00300     85*            DO 17 I=1,M                                     MINV  79
00303     86*            DO 17 J=1,M                                     MINV  80
00306     87*            R=0.0                                           MINV  81
00307     88*            DO 16 L=1,M                                     MINV  82
00312     89*         16 R=R-Z1(L,J)*S(I,L)
00314     90*            IF(I.EQ.J) R=R+1.0                              MINV  84
00316     91*            TEST=AMAX1(TEST,ABS(R))                         MINV  85
00317     92*         17 Z2(I,L)=R
00322     93*            DO 19 I=1,M                                     MINV  87
00325     94*            DO 19 J=1,M                                     MINV  88
00330     95*            A=0.0                                           MINV  89
00331     96*            DO 18 L=1,M                                     MINV  90
00334     97*         18 A=A+Z1(I,L)*Z2(L,J)
00336     98*         19 Z1(I,J)=Z1(I,J)+A
00341     99*            IF(TEST.LE.DELT) GO TO 21                       MINV  93
00343    100*         20 CONTINUE                                        MINV  94
00345    101*            NSIG=NSIG+3                                     MINV  95
00345    102*   C.....TRANSFER FINAL INVERSE                             MINV  96
00346    103*         21 DO 22 I=1,M                                     MINV  97
00351    104*            DO 22 J=1,M                                     MINV  98
00354    105*         22 S(I,J)=Z1(I,J)
00357    106*         23 IF(MATIN) RETURN                                MINV 100
00361    107*            DET=PROD                                        MINV 101
00362    108*            RETURN                                          MINV 102
00363    109*            END                                             MINV 103

       END OF COMPILATION          2 DIAGNOSTICS.
```

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@FOR,IS  CAL
FOR C11A-08/27/74-17 12 56 (,C)

   SUBROUTINE CALFUN     ENTRY POINT 001236

   STORAGE USED  CODE(1) 001262; DATA(0) 000361; BLANK COMMON(2) 000000

    COMMON BLOCKS

    0003   A      061301
    0004   B      061301
    0005   C      165140
    0006   D      061301

   EXTERNAL REFERENCES (BLOCK, NAME)

    0007   SQRT
    0010   NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001   000255 127G      0001    000425 144G      0001    000530 154G      0001    000611 167G      0001    000677 176G
    0001   000747 204G      0001    001012 212G      0001    001061 220G      0001    001123 226G      0001    001162 234G
    0001   000470 62L       0004    000000 AJINV     0000 R 000243 AL        0006    000000 BJINV     0000 R 000241 D
    0000   000262 INJP$     0000 I 000245 IS         0000 I 000246 IT        0000 I 000244 K         0000 R 000247 NR
    0000 R 000237 NS        0000 R 000242 PSIG       0003    000000 T        0000 R 000000 U         0000 R 000240 US
    0005   000000 W


 00101            COMPILER(XM=1)
 00103            SUBROUTINE CALFUN(N,Y,F)
 00105            DIMENSION U(159)
 00106            DIMENSION Y(159)
 00107            DIMENSION F(159)
 00110            COMMON/A/T(159,159)
 00111            COMMON/B/AJINV(159,159)
 00112            COMMON/C/W(60000)
 00113            COMMON/D/BJINV(159,159)
 00114            REAL NS
 00115            NS=1.
 00116            US=NS**2
 00117            D=.0002
 00120            PSIG=NS/2.
 00121            AL=PSIG/10.
 00121      C ALL STATEMENT NUMBERS LOCAL TO CAL
 00121      C DEFINE F(I) IN TERMS OF Y(J)
 00122         50 U(1)=4.*AL**2/US-D*(Y(2)-4.)**2
 00123            IF(U(1).LE.0.0)U(1)=0.0
 00125            F(1)=Y(1)**4-Y(3)*Y(1)**3+
 00125           1Y(1)**2*(D-.5*SQRT(U(1)))-Y(1)*D*(Y(2)+4.)/2.+
 00125           2(2.*D*(Y(2)-4.)**2-4.*AL**2/US)/8.
```

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```
      DO 60 K=2,14
      U(K)=4.*AL**2/US-D*(Y(K+1)-Y(K-1))**2
      IF(U(K).LE.0.0)U(K)=0.0
      F(K)=Y(K)**4-Y(K+15)*Y(K)**3+
     1Y(K)**2*(D-.5*SQRT(U(K)))-
     2Y(K)*D*(Y(K+1)+Y(K-1))/2.+(2.*D*(Y(K+1)-Y(K-1))**2-4.*AL**2/US)/8.
   60 CONTINUE
      U(15)=4.*AL**2/US-(1.-Y(14))**2*D
      IF(U(15).LE.0.0)U(15)=0.0
      F(15)=Y(15)**4-Y(30)*Y(15)**3+
     1Y(15)**2*(D-.5*SQRT(U(15)))-
     2Y(15)*D*(1.+Y(14))/2.+(2.*D*(1.-Y(14))**2-4.*AL**2/US)/8.
      DO 61 K=16,136,15
      F(K)=Y(K)**4-Y(K)**3*(Y(K-15)+Y(K+15))/2.+
     1Y(K)**2*(D-.125*(Y(K-15)-Y(K+15))**2)-
     2Y(K)*D*(Y(K+1)+4.)/2.+(  Y(K+1)-4.)**2/8.
   61 CONTINUE
      IS=17
   62 IT=IS+12
      DO 63 K=IS,IT
      F(K)=Y(K)**4-Y(K)**3*(Y(K-15)+Y(K+15))/2.+
     1Y(K)**2*(D-.125*(Y(K-15)-Y(K+15))**2)-D*Y(K)*(Y(K+1)+Y(K-1))/2.
     2+D*(Y(K+1)-Y(K-1))**2/8.
   63 CONTINUE
      IS=IS+15
      IF(IS.LE.122)GO TO 62
      REAL MR
      MR=1.
      DO 64 K=30,135,15
      F(K)=Y(K)**4-Y(K)**3*(Y(K-15)+Y(K+15))/2.+Y(K)**2*(D-.125*(Y
     8(K-15)-
     1Y(K+15))**2)-Y(K)*D*(SQRT(2.*(10.-MR)*AL/NS)+Y(K-1))/2.+
     2D*(SQRT(2.*(10.-MR)*AL/NS)-Y(K-1))**2/8.
      MR=MR+1.
   64 CONTINUE
      DO 65 K=137,144
      F(K)=Y(K)**4-Y(K)**3*(Y(K-15)+Y(K+15))/2.+
     1Y(K)**2*(D-.125*(Y(K-15)-Y(K+15))**2)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
     2D*(Y(K+1)-Y(K-1))**2/8.
   65 CONTINUE
      DO 67 K=145,149
      F(K)=Y(K)**4-Y(K)**3*(Y(K-15))/2.+
     1Y(K)**2*(D-.125*(Y(K-15))**2)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
     2D*(Y(K+1)-Y(K-1))**2/8.
   67 CONTINUE
      DO 68 K=150,150
      F(K)=Y(K)**4-Y(K)**3*Y(K-15)/2.+Y(K)**2*(D-.125*Y(K-15)**2)-
     1D*Y(K)*(SQRT(2.*AL/NS)+Y(K-1))/2.+
     2+D*(SQRT(2.*AL/NS)-Y(K-1))**2/8.
   68 CONTINUE
      DO 21 K=151,151
      F(K)=Y(K)**4-Y(K)**3*Y(K-15)+D*Y(K)**2
     1-D*Y(K)*(Y(K+1)+4.)/2.+D*(Y(K+1)-4.)**2/8.
   21 CONTINUE
      DO 22 K=152,158
      F(K)=Y(K)**4-Y(K-15)*Y(K)**3+D*Y(K)**2
     1-D*Y(K)*(Y(K+1)+Y(K-1))/2.+D*(Y(K+1)-Y(K-1))**2/8.
```

INFINITE FLAT PLATE DATE 082774 PAGE 17

```
00231        22 CONTINUE
00233           DO 130 K=159,159
00236           F(K)=Y(K)**4-Y(K)**3*Y(K-15)+D*Y(K)**2
00236          1-D*Y(K)*(Y(K-1))/2.+D*Y(K-1)**2/8.
00237       130 CONTINUE
00241           RETURN
00242           END
```

END OF COMPILATION NO DIAGNOSTICS.

ORIGINAL PAGE IS
OF POOR QUALITY

INFINITE FLAT PLATE

152	.35502090+01	-.10223294-01
153	.31082110+01	-.20614537-02
154	.25507895+01	-.99112723-02
155	.23763375+01	-.85941342-02
156	.21749562+01	-.55698398-02
157	.17930298+01	-.44024225-02
158	.13956247+01	-.27326857-02
159	.72633759+00	-.25541893-01

THE SUM OF SQUARES IS .24419923-01
AT THE 342 TH CALL OF CALFUN WE HAVE

I	X(I)	F(I)
1	.35706209+01	-.20943897-01
2	.30993267+01	-.11893887-01
3	.31331326+01	-.22705568-02
4	.26524323+01	-.12444856-01
5	.24702245+01	-.10237938-01
6	.22716574+01	-.60595024-02
7	.18992573+01	-.47812623-02
8	.15520390+01	-.55619102-02
9	.13039200+01	.26925764-02
10	.10592116+01	.32742196-03
11	.10405279+01	-.50723535-03
12	.10212715+01	-.40999346-03
13	.10153981+01	.14096924-02
14	.10085590+01	.15918617-02
15	.10034740+01	.76040100-03
16	.36585436+01	-.43987904-01
17	.36901187+01	-.22641685-01
18	.31281456+01	-.11711255-01
19	.26378301+01	-.23336743-01
20	.24518497+01	-.19027022-01
21	.22534223+01	-.10415060-01
22	.14799610+01	-.75625005-02
23	.15260547+01	-.51633511-02
24	.12660419+01	.13252547-02
25	.10113753+01	-.94485466-03
26	.99194919+00	.59441818-03
27	.97148565+00	.55613314-03
28	.96761350+00	-.73801924-03
29	.95589875+00	-.11324055-02
30	.95165323+00	-.49600205-03
31	.38479794+01	-.40374468-01
32	.36415480+01	-.20637518-01
33	.31239312+01	-.61907623-02
34	.26253943+01	-.20008824-01
35	.24359747+01	-.15187323-01
36	.22372173+01	-.77020774-02
37	.18628629+01	-.56927804-02
38	.15024925+01	-.24966497-02
39	.12295424+01	.27693803-02
40	.96115863+00	.42153371-03
41	.93916821+00	.40597743-03
42	.91761503+00	.27313177-04
43	.91057023+00	.51264857-03
44	.90310714+00	.65632428-03
45	.89835249+00	.30196631-03

INFINITE FLAT PLATE

46	.38388188+01	-.38706297-01
47	.36740452+01	-.20197499-01
48	.31201432+01	-.89057230-02
49	.26148053+01	-.16703597-01
50	.24221447+01	-.13402154-01
51	.22229895+01	-.82[illegible]79057-02
52	.18475634+01	-.4[illegible]987181-02
53	.14799536+01	-.46388752-02
54	.11874722+01	.38466343-02
55	.90510487+00	-.56023805-03
56	.88231392+00	.38250503-03
57	.86017717+00	.20055662-03
58	.85280239+00	.44289521-04
59	.84510340+00	-.10085039-03
60	.84014207+00	-.57532954-04
61	.38310136+01	-.3634[illegible]604-01
62	.36676014+01	-.20513883-01
63	.31169724+01	-.61903790-02
64	.26057286+01	-.15264948-01
65	.24101633+01	-.12482065-01
66	.22105391+01	-.77308029-02
67	.18336841+01	-.41287756-02
68	.14599760+01	-.38990654-02
69	.11394441+01	.24453780-02
70	.84445985+00	.19575331-03
71	.82034443+00	.56284290-03
72	.79777423+00	.35806775-03
73	.79068265+00	.39778119-03
74	.78315047+00	.27732580-03
75	.77812924+00	.21727554-03
76	.38244888+01	-.35651930-01
77	.36622369+01	-.14936249-01
78	.31142508+01	-.61575244-02
79	.25980272+01	-.15179482-01
80	.23999427+01	-.11578273-01
81	.21998200+01	-.76504316-02
82	.18212293+01	-.55705432-02
83	.14423754+01	-.25729887-02
84	.10862246+01	.32284358-02
85	.77541749+00	-.25580759-03
86	.75107636+00	.24852378-03
87	.72831628+00	.13083171-03
88	.72150990+00	.95527917-04
89	.71457072+00	.59446591-04
90	.70970825+00	.96217276-05
91	.38192264+01	-.35742138-01
92	.36577310+01	-.20363044-01
93	.31119849+01	-.75230513-02
94	.25917112+01	-.14878319-01
95	.23913869+01	-.17716955-01
96	.21908612+01	-.80675537-02
97	.18107419+01	-.73292132-02
98	.14265759+01	-.3943110[illegible]-02
99	.10253684+01	.33450835-02
100	.69705420+00	-.12706861-03
101	.67323023+00	.97315728-03
102	.65049320+00	.27966834-03

INFINITE FLAT PLATE

		.27023044-03
103	.64441901+00	.25288274-03
104	.63629708+00	.15301702-03
105	.63286577+00	-.31873726-01
106	.38152350+01	-.20015015-01
107	.36543093+01	-.62071178-02
108	.31102727+01	-.15182895-01
109	.25867629+01	-.12317716-01
110	.23846911+01	-.84554983-02
111	.21937790+01	-.61725041-02
112	.18028988+01	-.35246652-02
113	.14139543+01	.87481745-03
114	.95452069+00	-.80672418-03
115	.60458649+00	-.26506491-03
116	.58187600+00	-.38865030-03
117	.55968283+00	-.33286008-03
118	.55448864+00	-.30949535-03
119	.54930135+00	-.30219442-03
120	.54621821+00	-.34852651-01
121	.38123794+01	-.16660110-01
122	.36519618+01	-.67789138-02
123	.31090327+01	-.15830001-01
124	.25832305+01	-.13536847-01
125	.23798207+01	-.89943574-02
126	.21786797+01	-.54854267-02
127	.17973454+01	-.23707625-02
128	.14043501+01	-.21482936-02
129	.87640881+00	-.58371471-03
130	.49678179+00	-.49337271-04
131	.47661173+00	-.14605951-03
132	.45596735+00	-.86756361-04
133	.65172288+00	-.44401193-04
134	.44739817+00	.30522881-04
135	.44599961+00	-.37951167-01
136	.38107696+01	-.19526803-01
137	.36505533+01	-.84611035-02
138	.31083069+01	-.16796501-01
139	.25811989+01	-.14634226-01
140	.23769747+01	-.96733702-02
141	.21756909+01	-.65851419-02
142	.17939216+01	-.42003277-02
143	.13976639+01	-.26023505-02
144	.79864386+00	.77200018-03
145	.35883616+00	.83770940-03
146	.34214295+00	.53506868-03
147	.32432985+00	.49948927-03
148	.32055851+00	.40652101-03
149	.31593683+00	.40653087-03
150	.31502096+00	-.14179488-01
151	.38105192+01	-.10907302-01
152	.36502035+01	-.19307238-02
153	.31082098+01	-.99584091-02
154	.25807864+01	-.85449792-02
155	.23763276+01	-.55677543-02
156	.21749593+01	-.44125743-02
157	.17930288+01	-.27909659-02
158	.13956358+01	-.25539310-01
159	.72634292+00	

ORIGINAL PAGE IS
OF POOR QUALITY


```
FOR,IS  MAIN
FOR 011A-11/04/74-15 07 30 (,0)

 MAIN PROGRAM

 STORAGE USED  CODE(1) 001001  DATA(0) 001044  BLANK COMMON(2) 000000

 EXTERNAL REFERENCES (BLOCK, NAME)

  0003   NINTR$
  0004   NRDU$
  0005   NIO1$
  0006   NIO2$
  0007   SQRT
  0010   NWDU$
  0011   NSTOP$

 STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

  0000   000772 100F      0001   000032 120G      0001   000044 126G      0001   000162 146G      0001   000345 164G
  0001   000420 174G      0001   000467 205G      0001   000545 214G      0001   000602 222G      0001   000712 232G
  0001   000770 251G      0000   000774 26F       0001   000370 3L        0001   000053 50L       0000   001005 8F
  0001   000752 99L       0000 R 000762 AL        0000 I 000765 COUNT     0000 R 000760 D         0000 I 000771 I
  0000 I 000767 IS        0000 I 000770 IT        0000 I 000764 J         0000 I 000766 K         0000 R 000755 KO
  0000 I 000756 MM        0000 I 000763 N         0000 R 000754 NS        0000 R 000761 PSIG      0000 R 000000 U
  0000 R 000757 US        0000 R 000244 Y         0000 R 000510 Z


 00101      1*          DIMENSION U(164)
 00103      2*          DIMENSION Y(164)
 00104      3*          DIMENSION Z(164)
 00105      4*          REAL NS
 00106      5*          REAL KO
 00107      6*          MM=1000
 00110      7*          NS=1.
 00111      8*          US= NS**2
 00112      9*          D=.0082
 00113     10*          PSIG=NS/2.
 00114     11*          AL=PSIG/10.
 00115     12*          N=164
 00116     13*          READ(5,100) (Y(J),J=1,N)
 00124     14*          READ(5,100) (Z(J),J=1,N)
 00132     15*      100 FORMAT(8(F10.2))
 00133     16*          KO= 0.115
 00134     17*          INTEGER COUNT
 00135     18*          COUNT= 0
 00136     19*       50 COUNT= COUNT + 1
 [illegible] 20*        IF(COUNT.GT.MM) GO TO 99
 [illegible] 21*        [illegible]*(Z(2)-.125)**2
 [illegible] 22*        [illegible]
 [illegible] 23*        [illegible]/Y(I)+2.*Z(17))/(2.*(Y(I)**2+D))+
 [illegible] 24*        1[illegible]*(Y(I)**2/D+1.))+
 [illegible] 25*        [illegible](U(I))/(4.*Y(I)*(Y(I)**2/SQRT(D)+SQRT(D)))
```

```
      DO 1  K=2,14
      U(K)=4.*AL**2/US-D*(Z(K+1)-Z(K-1))**2
      IF(U(K).LE.0.0) U(K)= 0.0
      Z(K)= Y(K)**2*(SQRT(U(K))/Y(K)+2.*Z(K+16))/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K+1)-Z(K-1))*SQRT(U(K))/(4.*Y(K)*(Y(K)**2/SQRT(D)+SQRT(D)))
    1 CONTINUE
      U(15)=4.*AL**2/US-D*(KO-Z(14))**2
      IF(U(15).LE.0.0) U(15)= 0.0
      Z(15)= Y(15)**2*(SQRT(U(15))/Y(15)+2.*Z(31))/(2.*(Y(15)**2+D))+
     1(KO+Z(14))/(2.*(Y(15)**2/D+1.))+
     2(KO-Z(14))*SQRT(U(15))/(4.*Y(15)*(Y(15)**2/SQRT(D)+SQRT(D)))
      Z(16)= Y(16)**2*(.125+Z(32))/(2.*(Y(16)**2+D))+
     1Z(17)/(Y(16)**2/D+1.)
      DO 2  K=32,128,16
      Z(K)= Y(K)**2*(Z(K-16)+Z(K+16))/(2.*(Y(K)**2+D))+
     1Z(K+1)/(Y(K)**2/D+1.)
    2 CONTINUE
      IS= 17
    3 IT= IS + 13
      DO 4 K=IS,IT
      Z(K)=Y(K)**2*(Z(K-16)+Z(K+16))/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K-16)-Z(K+16))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    4 CONTINUE
      IS=IS+16
      IF(IS.LE.129)GO TO 3
      DO 5 K=31,143,16
      Z(K)=Y(K)**2*(Z(K-16)+Z(K+16))/(2.*(Y(K)**2+D))+
     1(KO+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K-16)-Z(K+16))*(KO-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    5 CONTINUE
      Z(144)=Y(144)**2*Z(128)/(2.*(Y(144)**2+D))+
     1Z(145)/(Y(144)**2/D+1.)
      DO 6 K=145,154
      Z(K)=Y(K)**2*Z(K-16)/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2Z(K-16)*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    6 CONTINUE
      DO 7 K=155,158
      Z(K)=Y(K)**2*(Z(K-16)+Z(K+5))/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K-16)-Z(K+5))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    7 CONTINUE
      Z(159)=Y(159)**2*(Z(143)+Z(164))/(2.*(Y(159)**2+D))+
     1(KO+Z(158))/(2.*(Y(159)**2/D+1.))+
     2(Z(143)-Z(164))*(KO-Z(158))/(4.*(Y(159)**2/SQRT(D)+SQRT(D)))
      Z(160)=Y(160)**2*Z(155)/(Y(160)**2+D)+
     1Z(161)/(2.*(Y(160)**2/D+1.))
      DO 9 K=161,163
      Z(K)=Y(K)**2*Z(K-5)/(Y(K)**2+D)+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))
    9 CONTINUE
      Z(164)=Y(164)**2*Z(159)/(Y(164)**2+D)+
     1(KO+Z(163))/(2.*(Y(164)**2/D+1.))
      GO TO 50
   99 CONTINUE
```

```
00242  *DIAGNOSTIC*  THIS STATEMENT HAS TOO MANY LEFT PARENTHESES.
00242     83*        75 FORMAT(1H*,6X7HAT THE I5,1X25HTH ITERATION WE HAVE)
00243     84*            WRITE(6,26) COUNT
00246     85*          5 FORMAT(1H*,4X1HI,9X4HZ(I)//(I5,E17.8))
00247     86*            WRITE(6,8) (I,Z(I),I=1,N)
00256     87*            END

      END OF COMPILATION          1 DIAGNOSTICS.


 XQT
MAP 0025-11/04-15 07
AFCM STATUS OF OUTPUT ELEMENT=UNKNOWN

ADDRESS LIMITS     001000 011534       4445 IBANK WORDS DECIMAL
                   040000 044661       2482 DBANK WORDS DECIMAL
STARTING ADDRESS  010534



            SEGMENT $MAIN$          001000 011534       040000 044661

NSWTC$/FOR69-6          $(1)   001000 001024
NPPLN$/FOR69            $(1)   001025 001047
NRWND$/FOR68            $(1)   001050 001131     $(2)   040000 040011
NWEF$/FOR69-6           $(1)   001132 001335     $(2)   040012 040031
NBOCV$/FOR69            $(1)   001336 001463     $(2)   040032 040074
NFTCH$/FOR68-6          $(1)   001464 001746     $(2)   040075 040110
NETW$/FOR               $(1)   001747 001771
NCNVT$/FOR69            $(1)   001772 002213     $(2)   040111 040205
NCLOS$/FOR69            $(1)   002214 002404     $(2)   040206 040236
NWBLK$/FOR69            $(1)   002405 002516
NRSBL$/FOR69            $(1)   002517 002557
NUFCA$/FOR69            $(1)   002560 002613
NEFCL$/FOR                                       $(2)   040237 042440
NCTIN$/FOR69            $(1)   002614 003110     $(2)   042441 042444
NOUT$/FOR68-10          $(1)   003111 004265     $(2)   042445 042503
NICER$/FOR68-6          $(1)   004266 004455     $(2)   042504 042642
NINTN$/FOR69            $(1)   004456 004646     $(2)   042643 042646
NINPT$/FOR68-6          $(1)   004647 005655     $(2)   042647 042677
NFMT$/FOR69-6           $(1)   005656 006532     $(2)   042700 042754
NFCHN$/FOR68-6          $(1)   006533 007520     $(2)   042755 043130
                                                 $(4)   043131 043202
NTAB$/FOR                                        $(2)   043203 043241
NERR$/FOR69-6           $(1)   007521 010121     $(2)   043242 043432
NSTOP$/FOR68-6          $(1)   010122 010146     $(2)   043433 043442
NCRUF$/FOR69            $(1)   010147 010207
SQRT$/FOR69             $(1)   010210 010250     $(2)   043443 043454
NIER$/FOR69-6           $(1)   010251 010432     $(2)   043455 043575
NIRUF$/FOR69            $(1)   010433 010472     $(2)   043576 043576
NINTR$/FOR68-6          $(1)   010473 010533     $(2)   043577 043615
BLANK$COMMON(COMMONBLOCK)
```

ORIGINAL PAGE IS
OF POOR QUALITY

```
MAIN                    $(1)  010534 011534      $(0)  043616 044661
                                                 $(2)  BLANK$COMMON

SYS$APLIBS. LEVEL 169-11
END OF COLLECTION - TIME 1.788 SECONDS

     AT THE  1000 TH ITERATION WE HAVE)
```

I	Z(I)
1	.12926090+00
2	.13557399+00
3	.15978673+00
4	.18621344+00
5	.20291018+00
6	.22212020+00
7	.26726576+00
8	.34090018+00
9	.49826545+00
10	.86261203+00
11	.13234586+01
12	.19167855+01
13	.23241685+01
14	.28176644+01
15	.33134465+01
16	.11251234+00
17	.11634062+00
18	.12205526+00
19	.14374010+00
20	.16917690+00
21	.18266869+00
22	.20010785+00
23	.24101676+00
24	.30940100+00
25	.45402895+00
26	.87037245+00
27	.10951458+01
28	.17951226+01
29	.27024924+01
30	.27446399+01
31	.32649556+01
32	.10022086+00
33	[illegible]
34	[illegible]
35	[illegible]
36	[illegible]
37	[illegible]
38	[illegible]
39	[illegible]
40	[illegible]
41	[illegible]
42	[illegible]
43	[illegible]
44	[illegible]
45	[illegible]
46	[illegible]
47	[illegible]
48	[illegible]

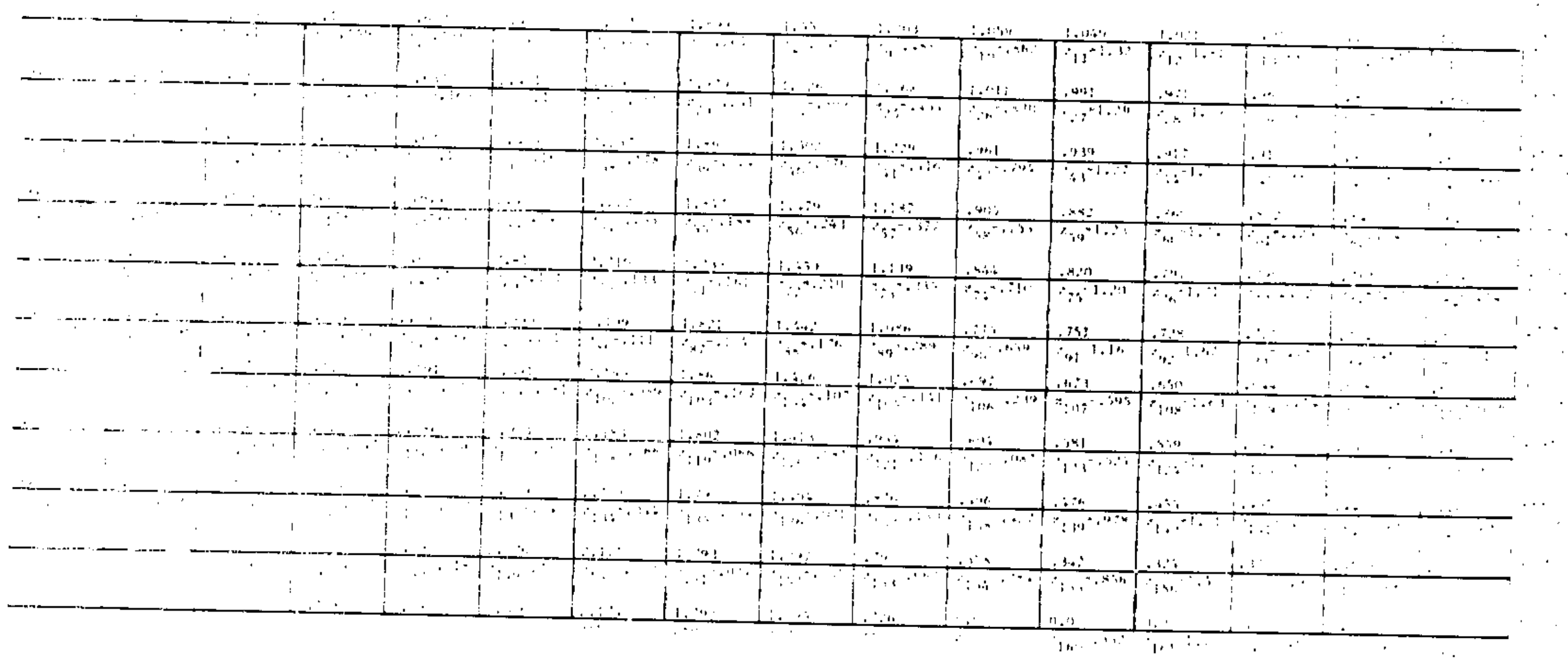
49	.90444881-01
50	.94492665-01
51	.11191759+00
52	.13172798+00
53	.14214194+00
54	.15590622+00
55	.19811943+00
56	.24362266+00
57	.37740524+00
58	.75542359+00
59	.12391459+01
60	.17443967+01
61	.22524386+01
62	.27417992+01
63	.31875903+01
64	.75027792-01
65	.77569769-01
66	.91431447-01
67	.95947965-01
68	.11069439+00
69	.12126948+00
70	.13372847+00
71	.16149657+00
72	.21036405+00
73	.33508680+00
74	.71059171+00
75	.12040106+01
76	.17140506+01
77	.22228151+01
78	.27111933+01
79	.31477911+01
80	.62527132-01
81	.69643410-01
82	.67572060-01
83	.79963597-01
84	.94073946-01
85	.10156534+00
86	.11150775+00
87	.13474805+00
88	.17636620+00
89	.28965115+00
90	.65910494+00
91	.11635657+01
92	.16789699+01
93	.21988950+01
94	.26768430+01
95	.31060276+01
96	.50024305-01
97	.51716003-01
98	.54305953-01
99	.63477616-01
100	.75256092-01
101	.81264149-01
102	.89250657-01
103	.10751665+00
104	.14145467+00
105	.24068281+00

106	.59681696+00
107	.11157328+01
108	.16372463+01
109	.21488974+01
110	.26373804+01
111	.30673063+01
112	.37519744-01
113	.38767750-01
114	.40734181-01
115	.47987140-01
116	.56440290-01
117	.60954619-01
118	.66964105-01
119	.81007968-01
120	.10687605+00
121	.19768257+00
122	.52466883+00
123	.10567261+01
124	.15853485+01
125	.20946093+01
126	.25942516+01
127	.30238705+01
128	.25013894-01
129	.25558857-01
130	.27158404-01
131	.31[illegible]93296-01
132	.37626016-01
133	.40539528-01
134	.44655236-01
135	.54039444-01
136	.71501604-01
137	.13027528+00
138	.42767770+00
139	.97224229+00
140	.15154195+01
141	.20539053+01
142	.25297273+01
143	.29740939+01
144	.12507158-01
145	.12429536-01
146	.13579887-01
147	.15497209-01
148	.1[illegible]12755-01
149	.20320[illegible]-01
150	.22331391-01
151	.27[illegible]-01
152	.3[illegible]-01
153	.6[illegible]5716-01
154	.28461449+00
155	[illegible]
156	.14[illegible]4577+01
157	[illegible]
158	.24[illegible]+01
159	[illegible]+01
160	[illegible]
161	[illegible]
162	.16[illegible]+01

163 .22599999+01
164 .27624999+01

FIN

IV. INFINITE PLATE/FINE MESH



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V. r AND z COMPUTER PROGRAM/FINITE PLATE/($R_o/L = 1/2$)

```
                 FINITE PLATE                                                              DATE  120374    PAGE    1
@FOR,IS  MAIN
FOR 011A-12/03/74-12 56 39 (,0)

 MAIN PROGRAM

 STORAGE USED  CODE(1) 000050; DATA(0) 017271; BLANK COMMON(2) 000000

 EXTERNAL REFERENCES (BLOCK, NAME)

  0003   EQNS
  0004   NINTRS
  0005   NRDUS
  0006   NIO1S
  0007   NIO2S
  0010   NSTOPS

 STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

  0000   017261 100F    0001   000014 107G     0000 R 017256 ACC     0000 R 000144 AJINV    0000 R 01726C DMAX
  0000 R 000062 F       0000 I 017254 IPRINT   0000 I 017253 J       0000 I 017257 MAXFUN   0000 I 017252 N
  0000 R 017255 STEP    0000 R 005050 W        0000 R 000000 X

 00101    1*         DIMENSION X(50),F(50),AJINV(50,50),W(5250)
 00103    2*     100 FORMAT (8(F10.2))
 00104    3*         N=44
 00105    4*         READ(5,100)(X(J),J=1,N)
 00113    5*         IPRINT=1
 00114    6*         STEP=.001
 00115    7*         ACC=.000001
 00116    8*         MAXFUN=200
 00117    9*         DMAX=1.0
 00120   10*         CALL EQNS(N,X,F,AJINV,STEP,DMAX,ACC,MAXFUN,IPRINT,W)
 00121   11*         END

   END OF COMPILATION         NO  DIAGNOSTICS.
```

FINITE PLATE

FOR,IS CAL
FOR [illegible]-12/03/74-12 57 22 (,0)

SUBROUTINE CALFUN ENTRY POINT 001455

STORAGE USED CODE(1) 001102; DATA(0) 000275; BLANK COMMON(2) 000000

EXTERNAL REFERENCES (BLOCK, NAME)

0003 SQRT
0004 NERR3$

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

0001 000233 131L	0001 000406 146G	0001 000577 156G	0001 000660 171G	0001 000744 200G
0001 001104 210L	0001 000456 3L	0000 R 000150 AL	0000 R 000146 D	0000 000200 INJP$
0000 I 000153 IS	0000 I 000154 IT	0000 I 000152 K	0000 R 000155 MR	0000 R 000144 NS
0000 R 000147 PSIG	0000 R 000151 RD	0000 R 000000 U	0000 R 000145 US	0000 R 000062 Z

```
      SUBROUTINE CALFUN(N,Y,F)
      DIMENSION U(50)
      DIMENSION Y(50)
      DIMENSION F(50)
      DIMENSION Z(50)
      Z(1)=-.250
      Z(9)=-.267
      Z(17)=-.284
      Z(25)=-.301
      Z(33)=-.317
      Z(41)=-.333
      REAL NS
      NS=1.
      US=NS**2
      D=.0178
      PSIG=NS/2.
      AL=PSIG/5.
      RD=2.0
C     RO/L=1/2
C ALL STATEMENT NUMBERS LOCAL TO CAL
C  DEFINE F(I) IN TERMS OF Y(J)
   50 U(1)=AL**2/US-Y(1)**2*(Z(1)-Z(9))**2
      IF(U(1).LE.0.0) U(1)=0.0
      F(1)=Y(1)**4-Y(1)**3*Y(9)+Y(1)**2*(D-SQRT(U(1)))-
     1Y(1)*D*(Y(2)+Y(1)*(Z(1)-Z(9))/SQRT(D))+
     2Y(1)**2*(Z(1)-Z(9))**2-AL**2/'2.*US)
      DO 1 K=2,7
      U(K)=4.*AL**2/US-D*(Y(K+1)-Y(K-1))**2
      IF(U(K).LE.0.0) U(K)=0.0
      F(K)=Y(K)**4-Y(K)**3*Y(K+8)+Y(K)**2*(D-.5*SQRT(U(K)))-
     1Y(K)*D*(Y(K+1)+Y(K-1))/2.+
```

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FINITE PLATE


```
     2(2.*D*(Y(K+1)-Y(K-1))**2-4.*AL**2/US)/8.
   1 CONTINUE
     U(8)=4.*AL**2/US-D*(1.-Y(7))**2
     IF(U(8).LE.0.0)U(8)=0.0
     F(8)=Y(8)**4-Y(8)**3*Y(16)+Y(8)**2*(D-.5*SQRT(U(8)))-
    1Y(8)*D*(1.+Y(7))/2.+
    2(2.*D*(1.-Y(7))**2-4.*AL**2/US)/8.
     DO 2 K=9,33,8
     F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
    1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-
    2.5*Y(K)*D*(2.*Y(K+1)+Y(K)*(Z(K-8)-Z(K+8))/SQRT(D))+
    3Y(K)**2*(Z(K-8)-Z(K+8))**2/8.
   2 CONTINUE
     IS=10
   3 IT=IS+5
     DO 4 K=IS,IT
     F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
    1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
    2D*(Y(K+1)-Y(K-1))**2/8.
   4 CONTINUE
     IS=IS+8
     IF(IS.LE.26) GO TO 3
     REAL MR
     MR=1.
     DO 5 K=16,32,8
     F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
    1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-
    2Y(K)*D*(SQRT(2.*(5.-MR)*AL/NS)+Y(K-1))/2.+
    3D*(SQRT(2.*(5.-MR)*AL/NS)-Y(K-1))**2/8.
     MR=MR+1.
   5 CONTINUE
     DO 6 K=34,35
     F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
    1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
    2D*(Y(K+1)-Y(K-1))**2/8.
   6 CONTINUE
     F(36)=Y(36)**4-Y(36)**3*(Y(28)+RD)/2.+
    1Y(36)**2*(D-(Y(28)-RD)**2/8.)-
    2Y(36)*D*(Y(35)+Y(37))/2.+D*(Y(37)-Y(35))**2/8.
     F(37)=Y(37)**4-Y(37)**3*(Y(29)+Y(44))/2.+
    1Y(37)**2*(D-(Y(29)-Y(44))**2/8.)-
    2Y(37)*D*(Y(38)+Y(36))/2.+D*(Y(38)-Y(36))**2/8.
     DO 7 K=38,39
     F(K)=Y(K)**4-Y(K)**3*Y(K-8)/2.+Y(K)**2*(D-.125*Y(K-8)**2)-
    1Y(K)*D*(Y(K+1)+Y(K-1))/2.+D*(Y(K+1)-Y(K-1))**2/8.
   7 CONTINUE
     F(40)=Y(40)**4-Y(40)**3*Y(32)/2.+Y(40)**2*(D-.125*Y(32)**2)-
    1Y(40)*D*.5*(SQRT(2.*AL/NS)+Y(39))+
    2D*(SQRT(2.*AL/NS)-Y(39))**2/8.
     U(41)=AL**2/US-Y(41)**2*(Z(33)-Z(41))**2
     IF(U(41).LE.0.0) U(41)=0.0
     F(41)=Y(41)**4-Y(41)**3*Y(33)+
    1Y(41)**2*(D+SQRT(U(41)))-
    2Y(41)*D*(Y(42)+Y(41)*(Z(33)-Z(41))/SQRT(D))+
    3(Y(41)**2*(Z(33)-Z(41))**2-AL**2/(2.*US))
     U(42)=4.*AL**2/US-D*(Y(43)-Y(41))**2
     IF(U(42).LE.0.0) U(42)=0.0
```

FINITE PLATE

```
      F(42)=Y(42)**4-Y(42)**3*Y(34)+
     1Y(42)**2*(D+.5*SQRT(U(42)))-
     2Y(42)*D*(Y(41)+Y(43))/2.+
     3(2.*D*(Y(43)-Y(41)))**2-4.*AL**2/US)/8.
      U(43)=4.*AL**2/US-D*(RD-Y(42))**2
      IF(U(43).LE.0.0) U(43)=0.0
      F(43)=Y(43)**4-Y(43)**3*Y(35)+
     1Y(43)**2*(D+SQRT(U(43))/2.)-
     2Y(43)*D*(RD+Y(42))/2.+
     3(2.*D*(RD-Y(42)))**2-4.*AL**2/US)/8.
      F(44)=Y(44)**4-Y(44)**3*Y(37)+Y(44)**2*D-
     1Y(44)*D*RD/2.+D*RD*2/8.
      RETURN
      END
```

END OF COMPILATION   NO DIAGNOSTICS.

```
MAP [illegible]
WCH STATUS OF OUTPUT ELEMENT [illegible]

HIGHEST LIMITS     001000 016140      6760 IBANK WORDS DECIMAL
                   040000 102606     17799 DBANK WORDS DECIMAL
STARTING ADDRESS   016111

SEGMENT MAINS         001000 016140      040000 102606

[illegible]   $(1)  001000 001024
[illegible]   $(1)  001025 001047
[illegible]   $(1)  001050 001131     $(2)  040000 040011
[illegible]   $(1)  001132 001335     $(2)  040012 040031
[illegible]   $(1)  001336 001620     $(2)  040032 040045
[illegible]   $(1)  001621 001746     $(2)  040046 040110
[illegible]   $(1)  001747 001771
[illegible]   $(1)  001772 002213     $(2)  040111 040205
[illegible]   $(1)  002214 002404     $(2)  040206 040236
[illegible]   $(1)  002405 002516
[illegible]   $(1)  002517 002557
[illegible]   $(1)  002560 002613
[illegible]                           $(2)  040237 042440
[illegible]   $(1)  002614 003004     $(2)  042441 042444
[illegible]   $(1)  003005 004013     $(2)  042445 042475
[illegible]   $(1)  004014 004310     $(2)  042476 042501
[illegible]   $(1)  004311 005465     $(2)  042502 042540
[illegible]   $(1)  005456 006342     $(2)  042541 042615
[illegible]   $(1)  006343 006532     $(2)  042616 042754
[illegible]   $(1)  006533 007520     $(2)  042755 043130
[illegible]                           $(4)  043131 043207
[illegible]                           $(2)  043210 043241
[illegible]
```

FINITE PLATE

42	.33039415+01	.37030713-02
43	.25128601+01	.79215138-03
44	.24445660+00	.30347845-04

THE SUM OF SQUARES IS .66547137-03
AT THE 159 TH CALL OF CALFUN WE HAVE

I	X(I)	F(I)
1	.39341670+01	-.23602791-01
2	.33991330+01	.49464226-02
3	.26338977+01	.16310143-02
4	.20499423+01	.38696657-02
5	.1318022[illegible]+01	-.16036470-02
6	.10024985+01	.19650487-03
7	.99442448+00	-.13980562-02
8	.99309520+00	-.16876350-02
9	.39156185+01	.39886636-01
10	.33901747+01	.43136225-02
11	.26105419+01	.10215487-02
12	.20336420+01	.20429420-02
13	.95617022+00	.10781657-02
14	.89784078+00	.53937199-04
15	.89015571+00	.10530984-02
16	.88896102+00	.11668912-02
17	.38956643+01	-.13990909-01
18	.33613225+01	.10635939-01
19	.25969206+01	-.25515474-04
20	.20195355+01	-.13577905-02
21	.45257745+00	-.10640944-03
22	.77723780+00	.20022892-03
23	.76945426+00	.84211205-04
24	.76701429+00	-.15459932-04
25	.39761078+01	.19496341-01
26	.33422521+01	.9785[illegible]014-02
27	.25679136+01	.1530[illegible]723-02
28	.[illegible]0369720+01	.67176441-03
29	.72[illegible]47598+[illegible]	.17472710-03
30	.[illegible]	.25467442-03
31	.[illegible]	.3[illegible]474984-03
32	.[illegible]	.2[illegible]04[illegible]216-03
33	.3[illegible]59223+01	-.24757713-03
34	.[illegible]721+01	.57041624-02
35	.[illegible]2652+01	.5[illegible]406-02
36	.[illegible]21691+01	.17407403-02
37	.[illegible]23142+01	-.4767[illegible]229-04
38	.[illegible]4997+[illegible]	.2098[illegible]417-03
39	.[illegible]	.[illegible]31-03
40	.[illegible]	.[illegible]429-03
41	.[illegible]+01	.15[illegible]42-01
42	.[illegible]+01	.4[illegible]7121-02
43	.[illegible]+01	-.353[illegible]07-02
44	.[illegible]	.[illegible]351104-04

THE SUM OF SQUARES IS .33711370-02
AT THE [illegible] TH CALL OF CALFUN WE HAVE

I	X(I)	F(I)
[illegible]	.[illegible]+01	.[illegible]1256-[illegible]
[illegible]	.[illegible]17+01	.[illegible]035-[illegible]

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FINITE PLATE DATE 120374 PAGE 153

3	.26339346+01	.89024083-03
4	.20496580+01	.26688096-02
5	.10386839+01	-.36344730-03
6	.10024565+01	.13316044-03
7	.99444565+00	-.13595246-02
8	.99317597+00	-.15506664-02
9	.39148330+01	.94817183-02
10	.33799877+01	.74407990-02
11	.26106225+01	.13415673-02
12	.20334443+01	.12815797-02
13	.95577931+00	.35519752-03
14	.89745270+00	.84258494-04
15	.89014006+00	.10337403-02
16	.88891198+00	.10970674-02
17	.38954268+01	.91589284-02
18	.33609975+01	.76580360-02
19	.25869103+01	.81492195-03
20	.20196086+01	-.21384814-03
21	.85271550+00	.12440665-03
22	.77723162+00	.19464283-03
23	.76845360+00	.89114238-04
24	.76702218+00	.25911929-05
25	.38756332+01	.91445014-02
26	.33419447+01	.77595818-02
27	.25628153+01	.18746395-02
28	.20090393+01	.12175383-02
29	.72095893+00	.13686364-03
30	.63205345+00	.25432670-03
31	.62298401+00	.37392579-03
32	.62162420+00	.22793029-03
33	.38554731+01	.88259769-02
34	.33228296+01	.75189451-02
35	.25381564+01	.18588411-02
36	.23020854+01	.99075942-04
37	.54023023+00	-.46073110-04
38	.43624417+00	.20977520-03
39	.42829492+00	.25629085-03
40	.42754654+00	.23420318-03
41	.38349783+01	.46051143-02
42	.33036727+01	.39352653-02
43	.25128599+01	.77474740-03
44	.24445911+00	.33299299-04

THE SUM OF SQUARES IS .66368585-03
AT THE 161 TH CALL OF CALFUN WE HAVE

I	X(I)	F(I)
1	.39334570+01	.46992741-02
2	.33986154+01	.35526160-02
3	.26539582+01	.97243446-03
4	.22495302+01	.26114930-02
5	.13386733+01	-.36563282-03
6	.10024502+01	.13255415-03
7	.99443902+00	-.13623016-02
8	.99716355+00	-.15613231-02
	.39144374+01	.92594462-02
10	.33796737+01	.74521928-02
11	.26105361+01	.1.167006-02

```
@FOR,IS  MAIN
FOR 1118-12/04/74-11 29 57 (,0)

   MAIN PROGRAM

   STORAGE USED  CODE(1) 001105; DATA(0) 000324; BLANK COMMON(2) 000000

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003   NINTR$
    0004   NRDU$
    0005   NIO1$
    0006   NIO2$
    0007   NWDU$
    0010   SQRT
    0011   NSTOP$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001     000246 100F      0001    000357 122G      0001    000051 130G      0001    000121 152G      0001    000134 151G
    0001     000303 176G      0001    000416 206G      0001    000465 217G      0001    000525 225G      0001    000566 233G
    0001     000250 26F       0001    000347 3L        0001    000060 50L       0000    000261 8F        0001    001103 99L
    0000 R 000234 AL          0000 I 000241 COUNT      0000 R 000232 D          0000 I 000242 I          0000 I 000244 IS
    0000 I 000245 IT          0000 I 000240 J          0000 I 000243 K          0000 I 000237 KE         0000 R 000227 KO
    0000 I 000230 MM          0000 I 000235 N          0000 R 000226 NS         0000 R 000233 PSIG       0000 R 000236 RD
    0000 R 000000 U           0000 R 000231 US         0000 R 000062 Y          0000 R 000144 Z


 00101      1*          DIMENSION U(50)
 00103      2*          DIMENSION Y(50)
 00104      3*          DIMENSION Z(50)
 00105      4*          REAL NS
 00106      5*          REAL KO
 00107      6*          MM=250
 00110      7*          NS=1.
 00111      8*          US=NS**2
 00112      9*          DT=.0178
 00113     10*          PSIG=NS/2.
 00114     11*          AL=PSIG/5.
 00115     12*          N=45
 00116     13*          RO=2.0
 00117     14*          KE=44
 00120     15*          READ(5,100) (Y(J),J=1,KE)
 00126     16*          READ(5,100) (Z(J),J=1,N)
 00134     17*      100 FORMAT(8(F10.2))
 00135     18*          KO=4.3
 00136     19*          INTEGER COUNT
 00137     20*          COUNT=0
 00140     21*       50 COUNT=COUNT+1
 00141     22*          IF(COUNT.GE.MM) GO TO 99
 00143  *DIAGNOSTIC*  THIS STATEMENT HAS TOO MANY LEFT PARENTHESES.
 00143     23*       26 FORMAT(1HK,6X7HAT THE I5,1X25HTH ITERATION WE HAVE)
 00144     24*          WRITE(6,26) COUNT
```

```
8 FORMAT(1HK,*XIHI,9X*HZ(I)//(I5,E17.8))
  WRITE(6,8) (I,Z(I),I=1,N)
  Z(1)=-.25
  DC 1 K=2,7
  U(K)=4.*AL**2/US-D*(Z(K+1)-Z(K-1))**2
  IF(U(K).LE.0.0) U(K)=0.0
  Z(K)=Y(K)**2*(SQRT(U(K))/Y(K)+2.*Z(K+8))/(2.*(Y(K)**2+D))+
 1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
 2(Z(K+1)-Z(K-1))*SQRT(U(K))/(4.*Y(K)*(Y(K)**2/SQRT(D)+SQRT(D)))
1 CONTINUE
  U(8)=4.*AL**2/US-D*(KO-Z(7))**2
  IF(U(8).LE.0.0)U(8)=0.0
  Z(8)=Y(8)**2*(SQRT(U(8))/Y(8)+2.*Z(16))/(2.*(Y(8)**2+D))+
 1(KO+Z(7))/(2.*(Y(8)**2/D+1.))+
 2(KO-Z(7))*SQRT(U(8))/(4.*Y(8)*(Y(8)**2/SQRT(D)+SQRT(D)))
  DO 2 K=9,33,8
  Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
 1(2.*Z(K+1)-Y(K)*(Y(K-8)-Y(K+8))/SQRT(D))/(2.*(Y(K)**2/D+1.))+
 2(Z(K-8)-Z(K+8))*Y(K)*(Y(K-8)-Y(K+8))/(4.*(Y(K)**2+D))
2 CONTINUE
  IS=10
3 IT=IS+5
  DO 4 K=IS,IT
  Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
 1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
 2(Z(K-8)-Z(K+8))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
4 CONTINUE
  IS=IS+8
  IF(IS.LE.26) GO TO 3
  DO 5 K=16,32,8
  Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
 1(KO+Z(K-1))/(2.*(Y(K)**2/D+1.))+
 2(Z(K-8)-Z(K+8))*(KO-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
5 CONTINUE
  DO 6 K=34,35
  Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
 1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
 2(Z(K-8)-Z(K+8))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
6 CONTINUE
  DO 7 K=36,38
  Z(K)=Y(K)**2*Z(K-8)/(2.*(Y(K)**2+D))+
 1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
 2Z(K-8)*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
7 CONTINUE
  Z(39)=Y(39)**2*(Z(31)+Z(44))/(2.*(Y(39)**2+D))+
 1(Z(40)+Z(38))/(2.*(Y(39)**2/D+1.))+
 2(Z(31)-Z(44))*(Z(40)-Z(38))/(4.*(Y(39)**2/SQRT(D)+SQRT(D)))
  Z(40)=Y(40)**2*(Z(32)+Z(45))/(2.*(Y(40)**2+D))+
 1(KO+Z(39))/(2.*(Y(40)**2/D+1.))+
 2(Z(32)-Z(45))*(KO-Z(39))/(4.*(Y(40)**2/SQRT(D)+SQRT(D)))
  U(41)=AL**2/U[illegible]-Y(41)**2*(Y(33)-Y(41))**2
  IF(U(41).LE.0.0) U(41)=0.0
  Z(41)=Y(41)**2*(Z(33)-SQRT(U(41))/Y(41))/(Y(41)**2+D)+
 1(Z(42)-Y(41)*(Y(33)-Y(41))/SQRT(D))/(Y(41)**2/D+1.)+
 2(Y(33)-Y(41))*SQRT(U(41))/(Y(41)**2+D)
  U(42)=4.*AL**2/US-D*(Z(43)-Z(41))**2
  IF(U(42).LE.0.0)U(42)=0.0
```

```
 251    [illegible]   Z(42)=Y(42)**2*(2.*Z(34)-SQRT(U(42))/Y(42))/(2.*(Y(42)**2+D))+
 251    [illegible]  1(Z(42)+Z(41))/(2.*(Y(42)**2/D+1.))+
 251    [illegible]  2(Z(42)-Z(41))*SQRT(U(42))/(4.*Y(42)*(Y(42)**2/SQRT(D)+SQRT(D)))
 252    [illegible]   U(43)=4.*AL**2/US-D*Z(42)**2
 253    [illegible]   IF(U(43).LE.0.0) U(43)=0.0
 255    [illegible]   Z(43)=Y(43)**2*(2.*Z(35)-SQRT(U(43))/Y(43))/(2.*(Y(43)**2+D))+
 255    [illegible]  1Z(42)/(2.*(Y(43)**2/D+1.))-
 255    [illegible]  2Z(42)*SQRT(U(43))/(4.*Y(43)*(Y(43)**2/SQRT(D)+SQRT(D)))
 256    [illegible]   Z(44)=Z(45)/2.
 257    [illegible]   Z(45)=(AC+Z(44))/2.
 [illegible]  [illegible]   GO TO 57
 261    [illegible]  49 END

 END OF COMPILATION          1 DIAGNOSTICS.


 [illegible]
 [illegible]
 [illegible] STATUS OF OUTPUT ELEMENT[illegible]UNKNOWN

 [illegible]ADDRESS LIMITS   001000 011640      4513 IBANK WORDS DECIMAL
                  040000 044141      2146 DBANK WORDS DECIMAL
 STARTING ADDRESS  010530


         SEGMENT [illegible]         001000 011640     040000 044141

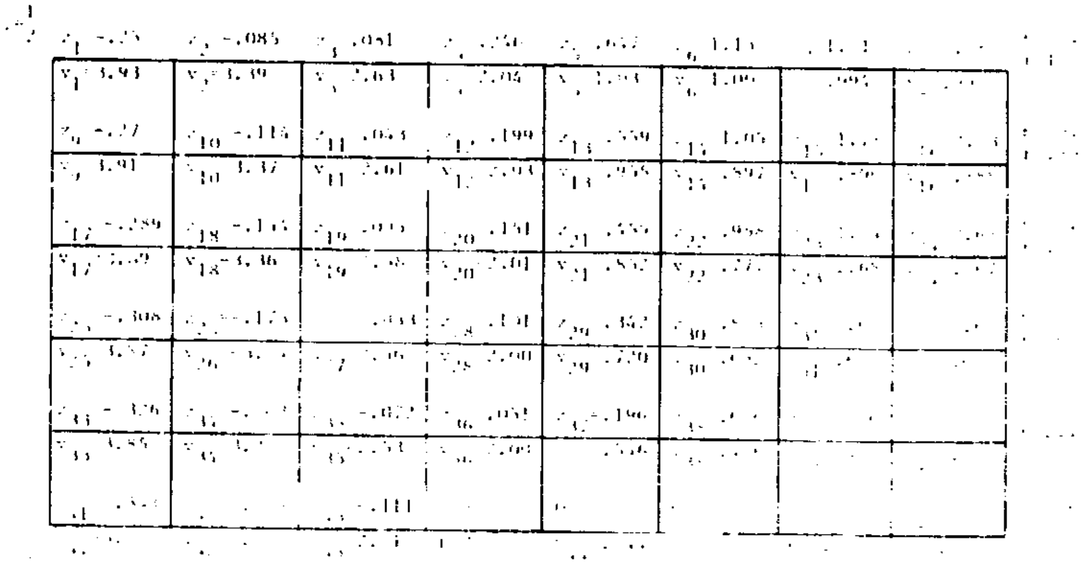
 [illegible]     $(1)  001000 001024
 [illegible]     $(1)  001025 001047
 [illegible]     $(1)  001050 001131   $(2)  040000 040011
 [illegible]     $(1)  001132 001335   $(2)  040012 040031
 [illegible]     $(1)  001336 001463   $(2)  040032 040074
 [illegible]     $(1)  001464 001746   $(2)  040075 040110
 [illegible]     $(1)  001747 001771
 [illegible]     $(1)  001772 002213   $(2)  040111 040205
 [illegible]     $(1)  002214 002404   $(2)  040206 040236
 [illegible]     $(1)  002405 002516
 [illegible]     $(1)  002517 002557
 [illegible]     $(1)  002560 002613
 [illegible]                           $(2)  040237 042440
 [illegible]     $(1)  002614 003110   $(2)  042441 042444
 [illegible]     $(1)  003111 004265   $(2)  042445 042503
 [illegible]     $(1)  004266 004455   $(2)  042504 042642
 [illegible]     $(1)  004456 004646   $(2)  042643 042646
 [illegible]     $(1)  004647 005655   $(2)  042647 042677
 [illegible]     $(1)  005656 006532   $(2)  042700 042754
 [illegible]     $(1)  006533 007520   $(2)  042755 043130
                                       $(4)  043131 043202
 [illegible]                           $(2)  043203 043241
 [illegible]     $(1)  007521 010121   $(2)  043242 043432
 [illegible]     $(1)  010122 010146   $(2)  043433 043442
```

43	-.1119[illegible]+00
44	.11000000+01
45	.22000000+01

AT THE 249 TH ITERATION WE HAVE)

I	Z(I)
1	-.75000000+00
2	-.45597292-01
3	.41004726-01
4	.24657763+00
5	.64709301+00
6	.11656670+01
7	.19145725+01
8	.27453169+01
9	-.27024435+00
10	-.11499717+00
11	.43756[illegible]1-01
12	.14457199+00
13	.55966750+00
14	.10574910+01
15	.17460419+01
16	.27369965+01
17	-.19977259+00
18	-.14401989+00
19	.50591786-02
20	.15116852+00
21	.45980714+00
22	.[illegible]51779+00
23	.17468780+01
24	.2[illegible]+01
25	-.30155760+00
26	-.17786217+00
27	-.73269978-01
28	.11155259+00
29	.49712013+00
30	.1[illegible]74572+01
31	.16722778+01
32	.25169160+01
33	-.32656971+00
34	-.20162134+00
35	-.72263801-01
36	.[illegible]16-01
37	.1[illegible]85559+00
38	.[illegible]76+00
39	.15[illegible]24+01
40	.[illegible]7+01
41	-.78577610+00
42	-.23749555+00
43	-.11165277+00
44	.11000000+01
45	.22000000+01

VI. FINITE PLATE/(R_o/L) = 1/2



VII. r AND z COMPUTER PROGRAM/FINITE PLATE/($R_c/L = 3/4$)

```
FINITE PLATE                                                  DATE 120574     PAGE    1

 FOR,IS MAIN
FOR [illegible]-12/05/74-19 14 57 (1,0)

   MAIN PROGRAM

   STORAGE USED: CODE(1) [illegible]; DATA(0) 017271; BLANK COMMON(2) 000000

   EXTERNAL REFERENCES (BLOCK, NAME)

    [illegible]
    [illegible]
    [illegible]
    [illegible]
    [illegible]
    [illegible]

   STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    [illegible]            0001   000014 107L     0000 R 017256 ACC      0000 R 000144 AJINV    0000 R 017260 DMAX
    [illegible]            0000 I 017254 IPRINT   0000 I 017253 J        0000 I 017257 MAXFUN   0000 I 017252 N
    [illegible] STEP       0000 R [illegible] W   0000 R 000000 X


 [illegible]     1*          DIMENSION X(50),F(50),AJINV(50,50),W(2550)
 [illegible]     2*     100  FORMAT (5(F10.2))
 [illegible]     3*          N=50
 [illegible]     4*          READ(5,100)(X(J),J=1,N)
 [illegible]     5*          IPRINT=1
 [illegible]     6*          STEP=.001
 [illegible]     7*          ACC=.00001
 [illegible]     8*          MAXFUN=[illegible]
 [illegible]     9*          DMAX=[illegible]
 [illegible]    10*          CALL [illegible](N,X,F,AJINV,STEP,DMAX,ACC,MAXFUN,IPRINT,W)
 [illegible]    11*          END

      END OF COMPILATION:        NO DIAGNOSTICS.
```

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```
[illegible] CAL
FOR [illegible]

SUBROUTINE CALFU.  ENTRY POINT 001455

STORAGE USED  CODE(1) 001502  DATA(0) 000076  BLANK COMMON(2) 000000

EXTERNAL REFERENCES (BLOCK, NAME)

0003  SQRT
0004  NERR3$

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

[illegible]  0001 000475 1L   0001 000577 1560  0001 000660 1710  0001 000744 2001
[illegible]  0001 000475 3L   0000 R 000150 AL  0000 R 000142 D   0000 000273 INJP$
[illegible]  0000 I 000154 IT 0000 I 000152 K   0000 R 000155 RP  0000 R 000144 NS
[illegible]  0000 R 000151 RO 0000 R 000000 U   0000 R 000145 US  0000 R 000062 Z

00101     1*         SUBROUTINE CALFU(N,Y,F)
00103     2*         DIMENSION U(50)
00104     3*         DIMENSION Y(50)
00105     4*         DIMENSION F(50)
00106     5*         DIMENSION Z(50)
00107     6*         Z(1)=1.40
00110     7*         Z(9)=1.417
00111     8*         Z(17)=1.435
00112     9*         Z(25)=1.452
00113    10*         Z(33)=1.469
00114    11*         Z(41)=1.486
00115    12*         REAL [illegible]
00116    13*         [illegible]
00117    14*         US=NS**2
00120    15*         [illegible]
00121    16*         [illegible]
00122    17*         AL=RS[illegible]/5.
00123    18*         RO=1.[illegible]
00124    19*   C     [illegible]
00125    20*   C ALL STATEMENT NUMBERS LOCAL TO CAL
00126    21*   C  DEFINE F(I) IN TERMS OF Y(J)
00127    22*         [illegible]
00130    23*         IF(U(1).LE.0.0) U(1)=0.0
00131    24*         F(1)=Y(1)***-Y(1)**3*Y(9)+Y(1)**2*(D-SQRT(U(1))))-
00132    25*        1Y(1)*[illegible]*(Y(2)+Y(1)*(Z(2)-Z(9))/SQRT(D))+
00133    26*        2*Y(1)**2*(Z(1)-Z(9))**2-AL**2/(2.*US)
00134    27*         [illegible]
00135    28*         [illegible]
00136    29*         IF(U(K).LE.0.0) U(K)=0.0
00137    30*         F(K)=Y(K)***-Y(K)**3*Y(K+8)+Y(K)**2*(-.5*SQRT(U(K)))-
00140    31*        1Y(K)*[illegible]*Y(K+1)+Y(K-1))/2.+
```

```
      2(2.*D*(Y(K+1)-Y(K-1))**2-4.*AL**2/US)/8.
    1 CONTINUE
      U(8)=4.*AL**2/US-D*(1.-Y(7))**2
      IF(U(8).LE.0.0)U(8)=0.0
      F(8)=Y(8)**4-Y(8)**3*Y(16)+Y(9)**2*(D-.5*SQRT(U(8)))-
     1Y(8)*D*(1.+Y(7))/2.+
     2(2.*D*(1.-Y(7))**2-4.*AL**2/US)/8.
      DO 2 K=9,33,8
      F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
     1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-
     2.5*Y(K)*D*.2.*Y(K+1)+Y(K)*(Z(K-8)-Z(K+8))/SQRT(D))+
     3Y(K)**2*(Z(K-8)-Z(K+8))**2/8.
    2 CONTINUE
      IS=10
    3 IT=IS+5
      DO 4 K=IS,IT
      F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
     1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
     2D*(Y(K+1)-Y(K-1))**2/8.
    4 CONTINUE
      IS=IS+8
      IF(IS.LE.26) GO TO 3
      REAL MR
      MR=1.
      DO 5 K=16,32,8
      F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
     1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-
     2Y(K)*D*(SQRT(2.*(5.-MR)*AL/NS)+Y(K-1))/2.+
     3D*(SQRT(2.*(5.-MR)*AL/NS)-Y(K-1))**2/8.
      MR=MR+1.
    5 CONTINUE
      DO 6 K=34,35
      F(K)=Y(K)**4-Y(K)**3*(Y(K-8)+Y(K+8))/2.+
     1Y(K)**2*(D-(Y(K-8)-Y(K+8))**2/8.)-Y(K)*D*(Y(K+1)+Y(K-1))/2.+
     2D*(Y(K+1)-Y(K-1))**2/8.
    6 CONTINUE
      F(36)=Y(36)**4-Y(36)**3*(Y(28)+RD)/2.+
     1Y(36)**2*(D-(Y(28)-RD)**2/8.)-
     2Y(36)*D*(Y(35)+Y(37))/2.+D*(Y(37)-Y(35))**2/8.
      F(37)=Y(37)**4-Y(37)**3*(Y(29)+Y(44))/2.+
     1Y(37)**2*(D-(Y(29)-Y(44))**2/8.)-
     2Y(37)*D*(Y(38)+Y(36))/2.+D*(Y(38)-Y(36))**2/8.
      DO 7 K=38,39
      F(K)=Y(K)**4-Y(K)**3*Y(K-8)/2.+Y(K)**2*(D-.125*Y(K-8)**2)-
     1Y(K)*D*(Y(K+1)+Y(K-1))/2.+D*(Y(K+1)-Y(K-1))**2/8.
    7 CONTINUE
      F(40)=Y(40)**4-Y(40)**3*Y(32)/2.+Y(40)**2*(D-.125*Y(32)**2)-
     1Y(40)*D*(SQRT(2.*AL/NS)+Y(39))/2.+
     2D*(SQRT(2.*AL/NS)-Y(39))**2/8.
      U(41)=AL**2/US-Y(41)**2*(Z(33)-Z(41))**2
      IF(U(41).LE.0.0) U(41)=0.0
      F(41)=Y(41)**4-Y(41)**3*Y(33)+
     1Y(41)**2*(D-SQRT(U(41)))-
     2Y(41)*D*(Y(42)+Y(41)*(Z(33)-Z(41))/SQRT(D))+
     3(Y(41)**2*(Z(33)-Z(41))**2-AL**2/(2.*US))
      U(42)=4.*AL**2/US-D*(Y(42)-Y(41))**2
      IF(U(42).LE.0.0) U(42)=0.0
```

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OF POOR QUALITY

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                FINITE PLATE                                                DATE  120574    PAGE    17

 00225    89*        F(42)=Y(42)***4-Y(42)**3*Y(34)+
 00225    90*       1Y(42)**2*(D*.5*SQRT(U(42)))-
 00225    91*       2Y(42)*D*(Y(41)+Y(43))/2.+
 00225    92*       3T.5*D*(Y(35)-Y(41))**2-4.*AL**2/US)/8.
 00226    93*        U(43)=4.*AL**2/US-D*(RD-Y(42))**2
 00227    94*        IF(U(43).LE.0.0) U(43)=0.0
 00231    95*        F(43)=Y(43)***4-Y(43)**3*Y(35)+
 00231    96*       1Y(43)**2*(D*SQRT(U(43))/2.)-
 00231    97*       2Y(43)*D*(RD+Y(42))/2.+
 00231    98*       3(2.*D*(RD-Y(42))**2-4.*AL**2/US)/8.
 00232    99*        F(44)=Y(44)***4-Y(44)**3*Y(37)+Y(44)**2*D-
 00232   100*       1Y(44)*D*RD/2.+D*RD*2/8.
 00233   101*        RETURN
 00234   102*        END

      END OF COMPILATION         NO  DIAGNOSTICS.

 XQT
 MAP 0026-12/05-19 15
 AFCM STATUS OF OUTPUT ELEMENT=UNKNOWN

 ADDRESS LIMITS     001000 016160      6769 IBANK WORDS DECIMAL
                    040000 102607      .7600 DBANK WORDS DECIMAL
 STARTING ADDRESS   010211

           SEGMENT MAIN*          001000 016160       040000 102607

 [illegible]       $(1)  001000 001024
 [illegible]       $(1)  001025 001047
 [illegible]       $(1)  001050 001131     $(2)  040000 040011
 [illegible]       $(1)  001132 001335     $(2)  040012 040031
 [illegible]       $(1)  001336 001620     $(2)  040032 040045
 [illegible]       $(1)  001621 001746     $(2)  040046 040110
 [illegible]       $(1)  001747 001771
 [illegible]       $(1)  001772 002213     $(2)  040111 040205
 [illegible]       $(1)  002214 002404     $(2)  040206 040236
 [illegible]       $(1)  002405 002516
 [illegible]       $(1)  002517 002557
 [illegible]       $(1)  002560 002613
 [illegible]                               $(2)  040237 042440
 [illegible]       $(1)  002614 003004     $(2)  042441 042444
 [illegible]       $(1)  003005 004013     $(2)  042445 042475
 [illegible]       $(1)  004014 004310     $(2)  042476 042501
 [illegible]       $(1)  004311 004465     $(2)  042502 042540
 [illegible]       $(1)  004466 004542     $(2)  042541 042615
 [illegible]       $(1)  004543 004572     $(2)  042616 042754
 [illegible]       $(1)  004573 007520     $(2)  042755 043130
                                           $(4)  043131 043202
 [illegible]                               $(2)  043203 043241
```

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18	.77445259+01	-.39294415-02
19	.24447781+01	.56926779-02
20	.14665-75+01	.77283109-03
21	.94517175+00	-.90812791-04
22	.776-124e+00	.21471525-03
23	[illegible]	.26749460-03
24	.76746555+00	.22044677-03
25	.47705A08+01	-.31917183-01
26	.73273455+01	.12327062-01
27	.76247185+01	.19840744-02
28	.14224275+01	.34505674-03
29	[illegible]	.24747597-03
30	.47142151+00	.34456569-03
31	.47734960+00	.30597735-03
32	.47011277+00	.25676421-03
33	.79416370+01	.27793770-01
34	.77047771+01	-.91050789-02
35	.70127113+01	.30946679-03
36	.17777497+01	-.34447755-03
37	.57358814+00	.19945351-03
38	.47094376+00	.34972720-03
39	.47149712+00	.30612477-03
40	.47406556+00	.27476595-03
41	.79422271+01	-.57558792-02
42	[illegible]	.17791710-01
43	.79781752+01	.71928414-02
44	.77129659+00	-.14025923-02

THE SUM OF SQUARES IS .52099239-02

ERROR RETURN BECAUSE THERE HAVE BEEN 150 CALLS OF CALFUN

THE FINAL SOLUTION REQUIRED 150 CALLS OF CALFUN AND IS

I	X(I)	F(I)
1	.40559576+01	.17173493-02
2	.77947552+01	.17457861-02
3	[illegible]	.17146415-02
4	.75507758+01	-.95850783-03
5	.10517457+01	-.23764770-03
6	.17024776+01	.20597165-03
7	.99145572+00	.16721779-03
8	.99627771+00	.11151586-03
9	.70377234+01	.25491024-02
10	.77729573+01	.35775001-02
11	.75419254+01	.26197561-04
12	.75092701+01	-.25976995-03
13	.95047644+00	-.25310781-03
14	.99747202+00	.24957773-03
15	.94476944+00	.27971098-03
16	.95550776+00	.25576177-03
17	.40192191+01	.31427470-02
18	.77499986+01	.71967193-02
19	.26445941+01	.27741578-02
20	.19462579+01	-.44577706-03
21	.94537125+00	-.36745953-04
22	.77591878+00	.21593411-03
23	.76474254+00	.26742777-03
24	.76744575+00	.27782976-03
25	.47059376+01	.31196425-02

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[illegible]	.33473362+01	.32196984-02
[illegible]	.2625[illegible]+01	.29978565-02
[illegible]	.14242035+01	-.11345975-03
[illegible]	.7209[illegible]+00	.24[illegible]9776-03
[illegible]	.631[illegible]2169+00	.34755135-03
[illegible]	[illegible]	.3[illegible]76-03
[illegible]	.52211327+00	.25[illegible]3110-03
[illegible]	[illegible]+01	.3[illegible]14-02
[illegible]	.77[illegible]+01	.35179519-02
[illegible]	.2[illegible]127590+01	.34[illegible]76092-02
[illegible]	.13775276+01	.34272077-03
[illegible]	[illegible]	[illegible]
[illegible]	.4[illegible]+00	.3[illegible]118-03
[illegible]	.4[illegible]579[illegible]+00	.30[illegible]1928-03
[illegible]	.4[illegible]+00	.274[illegible]0273-03
[illegible]	.7[illegible]92+01	.19[illegible]52[illegible]5-02
[illegible]	.72[illegible]14197+01	.19[illegible]-02
[illegible]	[illegible]+01	[illegible]26-02
[illegible]	[illegible]195+00	-.19027914-02

THE SUM OF SQUARES IS .19037081-03

```
@FOR,IS  MAIN
FOR 311A-12/26/74-11 39 38 (,0)

  MAIN PROGRAM

  STORAGE USED  CODE(1) 001105; DATA(0) 000325; BLANK COMMON(2) 000000

  EXTERNAL REFERENCES (BLOCK, NAME)

   0003   NINTR$
   0004   NRDU$
   0005   NIO1$
   0006   NIO2$
   0007   NWDU$
   0010   SQRT
   0011   NSTOP$

  STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

   0001     000240 100F      0001    000037 122G     0001    000051 133G     0001    000121 152G     0001    000134 161G
   0001     000303 176G      0001    000416 206G     0001    000465 217G     0001    000525 225G     0001    000566 233G
   0000     000250 26F       0001    000347 3L       0001    000060 50L      0000    000261 8F       0001    001103 99L
   0000 R   000234 AL        0000 I  000241 COUNT    0000 R  000232 D        0000 I  000242 I        0000 I  000244 IS
   0000 I   000245 IT        0000 I  000240 J        0000 I  000243 M        0000 I  000237 ME       0000 R  000227 KO
   0000 I   000230 MM        0000 I  000235 N        0000 R  000226 NS       0000 R  000233 PSIG     0000 R  000236 RO
   0000 R   000000 U         0000 R  000231 US       0000 R  000062 Y        0000 R  000144 Z

 00101       1*          DIMENSION U(50)
 00103       2*          DIMENSION Y(50)
 00104       3*          DIMENSION Z(50)
 00105       4*          REAL NS
 00106       5*          REAL KO
 00107       6*          MM=250
 00110       7*          NS=1.
 00111       8*          US=NS**2
 00112       9*          D=.02*27
 00113      10*          PSIG=NS/2.
 00114      11*          AL=PSIG/D.
 00115      12*          N=45
 00116      13*          RO=1.733
 00117      14*          M=144
 00120      15*          READ(5,100) (Y(J),J=1,ME)
 00125      16*          READ(5,100) (Z(J),J=1,N)
 00134      17*     100 FORMAT(8F10.2)
 00135      18*          KO=7.5
 00136      19*          INTEGER COUNT
 00137      20*          COUNT=0
 00140      21*      50 COUNT=COUNT+1
 00141      22*          IF(COUNT.GE.MM) GO TO 99
 00142  *DIAGNOSTIC*  THIS STATEMENT HAS TOO MANY LEFT PARENTHESES.
 00143      23*          [illegible] FORMAT(1H-,4X,THAT THE IS,1X,[illegible]TH ITERATION WE HAVE)
 00144      24*          WRITE(6,[illegible]) COUNT
```

```
    8 FORMAT(1HK,4X1HI,9X4HZ(I)//(I5,E17.8))
      WRITE(6,3) (I,Z(I),I=1,N)
      Z(1)=-1.40
      DO 1 K=2,7
      U(K)=4.*AL**2/US-D*(Z(K+1)-Z(K-1))**2
      IF(U(K).LE.0.0) U(K)=0.0
      Z(K)=Y(K)**2*(SQRT(U(K))/Y(K)+2.*Z(K+8))/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K+1)-Z(K-1))*SQRT(U(K))/(4.*Y(K)*(Y(K)**2/SQRT(D)+SQRT(D)))
    1 CONTINUE
      U(8)=4.*AL**2/US-D*(HC-Z(7))**2
      IF(U(8).LE.0.)U(8)=0.0
      Z(8)=Y(8)**2*(SQRT(U(8))/Y(8)+2.*Z(16))/(2.*(Y(8)**2+D))+
     1(HC+Z(7))/(2.*(Y(8)**2/D+1.))+
     2(HC-Z(7))*SQRT(U(8))/(4.*Y(8)*(Y(8)**2/SQRT(D)+SQRT(D)))
      DO 2 K=9,73,8
      Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
     1(2.*Z(K+1)-Y(K)*(Y(K-8)-Y(K+8))/SQRT(D))/(2.*(Y(K)**2/D+1.))+
     2(Z(K-8)-Z(K+8))*Y(K)*(Y(K-8)-Y(K+8))/(4.*(Y(K)**2+D))
    2 CONTINUE
      IS=10
    3 IT=IS+5
      DO 4 K=IS,IT
      Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K-8)-Z(K+8))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    4 CONTINUE
      IS=IS+8
      IF(IS.LE.76) GO TO 3
      DO 5 K=16,72,8
      Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
     1(HC+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K-8)-Z(K+8))*(HC-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    5 CONTINUE
      DO 6 K=74,[illegible]
      Z(K)=Y(K)**2*(Z(K-8)+Z(K+8))/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2(Z(K-8)-Z(K+8))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    6 CONTINUE
      DO 7 K=[illegible]
      Z(K)=Y(K)**2*[illegible](K-8)/(2.*(Y(K)**2+D))+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2([illegible])*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
    7 CONTINUE
      Z([illegible])=Y([illegible])**2*(Z([illegible])+Z(44))/(2.*(Y([illegible])**2+D))+
     1([illegible])/(2.*(Y([illegible])**2/D+1.))+
     2([illegible])/(4.*(Y([illegible])**2/SQRT(D)+SQRT(D)))
      Z([illegible])=Y([illegible])**2*([illegible])/(2.*(Y([illegible])**2+D))+
     1([illegible])/(2.*(Y([illegible])**2/D+1.))+
     2([illegible])/(4.*(Y([illegible])**2/SQRT(D)+SQRT(D)))
      [illegible]
      [illegible]
      Z([illegible])=Y([illegible])**2*[illegible]/(Y([illegible])**2+D)+
     1[illegible]/(Y([illegible])**2/D+1.)+
     2[illegible]
      [illegible]
      [illegible]
```

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OF POOR QUALITY

```
      Z(42)=Y(42)**2*(2.*Z(34)-SQRT(U(42))/Y(42))/(2.*(Y(42)**2+D))+
     1(Z(42)+Z(41))/(2.*(Y(42)**2/D+1.))+
     2(Z(42)-Z(41))*SQRT(U(42))/(4.*Y(42)*(Y(42)**2/SQRT(D)+SQRT(D)))
      U(43)=4.*AL**2/US-U*Z(42)**2
      IF(U(43).LE.0.0) U(43)=0.0
      Z(43)=Y(43)**2*(2.*Z(35)-SQRT(U(43))/Y(43))/(2.*(Y(43)**2+D))+
     1(Z(43))/(2.*(Y(43)**2/D+1.))-
     2.(43)*SQRT(U(43))/(4.*Y(43)*(Y(43)**2/SQRT(D)+SQRT(D)))
      Z(44)=Z(43)/2.
      Z(45)=(Z(44)+Z(44))/2.
      GO TO 50
      END
```

END OF COMPILATION   [illegible] DIAGNOSTICS.

[illegible]

```
 43     -.49109963+00
 44      .11000000+01
 45      .22000000+01
    AT THE    249 TH ITERATION WE HAVE)
  I            Z(I)

  1     -.14000000+01
  2     -.86180836+00
  3     -.33984133+00
  4      .302453[illegible]+00
  5      .57056443+00
  6      .89035826+00
  7      .17222387+01
  8      .28459164+01
  9     -.14177022+01
 10     -.90654879+00
 11     -.36431603+00
 12      .24277105+00
 13      .47660011+00
 14      .80372937+00
 15      .16317861+01
 16      .27757206+01
 17     -.14352173+01
 18     -.93155911+00
 19     -.39831503+00
 20      .18244438+00
 21      .37593754+00
 22      .59901792+00
 23      .16286174+01
 24      .27026399+01
 25     -.14525366+01
 26     -.95684376+00
 27     -.42880629+00
 28      .12166324+00
 29      .26627696+00
 30      .56494156+00
 31      .15544137+01
 32      .26148431+01
 33     -.14696506+01
 34     -.98240872+00
 35     -.45975687+00
 36      .60706190-01
 37      .14376328+00
 38      .37423098+00
 39      .14354450+01
 40      .24927655+01
 41     -.14965488+01
 42     -.10082602+01
 43     -.49113024+00
 44      .11000000+01
 45      .22000000+01

#FIN
```

VIII. FINITE PLATE/(R_o/L) = 3/4

z_1=-1.40	z_2=-.887	z_3=-.341	z_4=.?96	z_5=.597	z_6=1.03	z_7=1.88	z_8=2.93	A K=3.3 r=1
y_1=?.05	y_2=3.39	y_3=2.67	y_4=1.55	y_5=1.05	y_6=1.00	y_7=.995	y_8=.99?	
z_9=-1.517	z_{10}=-.37?	z_{11}=-.??	z_{12}=.238	z_{13}=.505	z_{14}=.948	z_{15}=1.83	z_{16}=2.86	K=3.3 r=.895
y_9=5.03	y_{10}=3.37	y_{11}=2.66	y_{12}=1.50	y_{13}=.956	y_{14}=.897	y_{15}=.889	y_{16}=.888	
z_{17}=-1.534	z_{18}=-.938	z_{19}=-.?04	z_{20}=.179	z_{21}=.404	z_{22}=.842	z_{23}=1.77	z_{24}=2.77	K=3.3 r=.775
y_{17}=?.0?	y_{18}=3.34	y_{19}=2.64	y_{20}=1.46	y_{21}=.845	y_{22}=.776	y_{23}=.768	y_{24}=.767	
z_{25}=-1.??	z_{26}=-.96?	z_{27}=-.?47	z_{28}=.119	z_{29}=.29?	z_{30}=.70?	z_{31}=1.6?	z_{32}=2.6?	K=3.3 r=.63?
y_{25}=5.00	y_{26}=[illegible]	y_{27}=2.6?	y_{28}=1.42	y_{29}=.?10	y_{30}=.631	y_{31}=.6??	y_{32}=.6??	
z_{33}=-1.?6?	z_{34}=-.99?	z_{35}=-.??	z_{36}=.?59	z_{37}=.159	z_{38}=.491	z_{39}=1.55	z_{40}=[illegible]	[illegible]
y_{33}=?.9?	y_{34}=?.??	y_{35}=2.61	y_{36}=1.[illegible]	y_{37}=.533	y_{38}=.?36	y_{39}=.??	y_{40}=[illegible]	
z_{41}=-1.??	[illegible]	[illegible]		[illegible]	[illegible]	z_{44}?=1.1?	[illegible]	[illegible]
y_{41}=3.96	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	

IX. COMPUTER LISTING/SURFACE TENSION MODEL

```
                    FINITE PLATE                                              DATE  091875    PAGE   1

 @FOR,IS MAIN
 FOR 011A-09/10/75-17 35 58 (,0)

   MAIN PROGRAM

   STORAGE USED  CODE(1) 000050; DATA(0) 000510; BLANK COMMON(2) 000000

   COMMON BLOCKS

   0003   A      057720
   0004   B      057420
   0005   C      165140
   0006   D      057420

   EXTERNAL REFERENCES (BLOCK, NAME)

   0007   FUNS
   0010   NINTR$
   0011   NRDU$
   0012   NIO1$
   0013   NIO2$
   0014   NSTOP$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

   0000    000477 170F      0001    000017 115G     0000 R 000474 ACC      0004   000000 AJINV     0006   000000 BJINV
   0000 R  000476 HMAX      0000 R  000239 F        0000 I 000472 IPRINT   0000 I 000471 J         0000 I 000475 MAXFUN
   0000 I  000467 N         0000 R  000473 STEP     0003   000000 T        0005   000000 W         0000 R 000000 X

 00101      1*           COMPILER(XM=1)
 00103      2*           DIMENSION X(156),F(156)
 00103      3*           COMMON/A/T(156,156)
 00104      4*           COMMON/B/AJINV(156,156)
 00107      5*           COMMON/C/W(6,2000)
 00110      6*           COMMON/D/BJINV(156,156)
 00111      7*       100 FORMAT ( (F10.2))
 00112      8*           N=156
 00113      9*           READ(5,100)(X(J),J=1,N)
 00121     10*           IPRINT=0
 00122     11*           STEP=.0001
 00123     12*           ACC=.00001
 00124     13*           MAXFUN=0
 00125     14*           DMAX=1.0
 00126     15*           CALL FUNC(N,X,F,STEP,DMAX,ACC,MAXFUN,IPRINT)
 00127     16*           END

       END OF COMPILATION         NO  DIAGNOSTICS.
```

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```
                  FINITE PLATE                                                               DATE  061075     PAGE      15

FOR,IS  CAL
FOR 0111-04/1.275-17 76 87 1.01

   SUBROUTINE CALFUN     ENTRY POINT 002223

   STORAGE USED  CODE(1) 002246; DATA(0) 000675; BLANK COMMON(2) 000000

    COMMON BLOCKS

    0004    A       057420
    0005    B       057420
    0006    C       165140
    0007    D       057420

   EXTERNAL REFERENCES (BLOCK, NAME)

    0007    SQRT
    0010    NERR3S

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001    000324 124L       0001    000357 123G       0001    000517 146G       0001    000707 163G       0001    000657 2L
    0001    001201 220F       0001    001161 211F       0001    001265 276F       0001    001350 235G       0001    001415 243G
    0001    001     251G      0001    001532 257G       0001    001602 265G       0001    001645 273G       0001    001714 301G
    0001    001   2 377G      0001    002010 315F       0001    002052 323G       0001    002113 331G       0001    002151 337G
    0001    001124 5L         0001    001234 6L         0000 R  000477 A          0004    000000 AJINV      0000 R  000474 AL
    0000 R  000500 B          0006    000000 BJINV      0000 R  000511 C          0000 R  000475 COUNT      0000 R  000470 D
    0000    000521 INJPS      0000 I  000505 IPR        0000 I  000503 IS         0000 I  000504 IT         0000 I  000476 N
    0000 R  000502 PE         0000 R  000473 PSIC       0000 R  000000 C          0000 R  000472 RD         0003    000000 T
    0005    000000 W          0000 R  000471 WE         0000 R  000234 X

00101       1*            COMPILER(XM=1)
00103       2*            SUBROUTINE CALFUN(N,Y,F)
00105       3*            DIMENSION Z(156)
00106       4*            DIMENSION Y(156)
00107       5*            DIMENSION F(156)
00110       6*            DIMENSION X(156)
00111       7*            COMMON/A/T(156,156)
00112       8*            COMMON/B/AJINV(156,156)
00113       9*            COMMON/C/W(60000)
00114      10*            COMMON/D/BJINV(156,156)
00115      11*            DT=.0204
00116      12*            WE=8.
00117      13*            RD=2.0
00120      14*            PSIC=.5
00121      15*            AL= PSIC/10.
00121      16*      C  RO/L=1/2
00121      17*      C     SURFACE TENSION MODEL
00121      18*      C     DEFINE F(I) IN TERMS OF Y(J)
00121      19*      C
```

FINITE PLATE DATE 081075 PAGE 15

```
      COUNT=1.
      DO 22 K=1,1M3,15
      Y(K)=(COUNT+2.*(1.-COUNT)*AL*RC**2)
      COUNT=COUNT+1.
   22 CONTINUE
      DO 1 K=1,1
      A=Y(K)**4/8.
      B=Y(K)**3/2.
      C=-Y(K)**6*Y(K)**3*Y(K+15)-Y(K)**2*D+.5*Y(K)*F*(1.+Y(K+1))-
     1F*(Y(K+1)-1.)**2/8.
C   NO CHECK ON ARGUMENT
      X(K)=(-B+SQRT(B**2-4.*A*C))/(2.*A)
      D(K)=F*(Y(K+1)-1.)**2/X(K)**.
      F(K)=Y(K)**2*X(K)**2+F*(Y(K+1)-1.)**.-
     1H.*AL**2/(1.-[illegible]*2.*(2.*Q(K)*Y(K)**2)/(WE*(SQRT(Q(K)*
     2Y(K)**2))**3))
    1 CONTINUE
      DO 2 K=2,1M
      IF(K.EQ.[illegible]) GO TO 2
      A=Y(K)**4/8.
      B=Y(K)**3/2.
      C=-Y(K)**6*Y(K)**3*Y(K+15)-Y(K)**2*D+.5*Y(K)*D*(Y(K+1)+Y(K-1))-
     1F*(Y(K+1)-Y(K-1))**2/8.
      X(K)=(-B+SQRT(B**2-4.*A*C))/(2.*A)
      D(K)=F*(Y(K+1)-Y(K-1))**2/X(K)**2
      F(K)=Y(K)**2*X(K)**2+F*(Y(K+1)-Y(K-1))**2-
     1H.*AL**2/(1.-[illegible]*2.*(2.*G(K)*Y(K)**2)/(WE*(SQRT(G(K)*
     2Y(K)**2))**3))
    2 CONTINUE
      DO 3 K=15,15
      A=Y(K)**2/8.
      B=Y(K)**3/2.
      C=-Y(K)**6*Y(K)**3*Y(K+15)-Y(K)**2*D+.5*Y(K)*D*(1.+Y(K-1))-
     1F*(1.-Y(K-1))**2/8.
      X(K)=(-B+SQRT(B**2-4.*A*C))/(2.*A)
      D(K)=F*(1.-Y(K-1))**2/X(K)**2
      F(K)=Y(K)**2*X(K)**2+F*(1.-Y(K-1))**.-
     1H.*AL**2/(1.-2.*[illegible]*2.*(2.*Q(K)*Y(K)**2)/(WE*(SQRT(Q(K)*
     2Y(K)**2))**3))
    3 CONTINUE
      REAL MR
      MR=1.
      DO 4 K=16,121,15
      F(K)= Y(K)**4-Y(K)**3*(Y(K-15)+Y(K+15))/2.+
     1Y(K)**2*(D-(Y(K-15)-Y(K+15))**2/8.)-
     2.5*Y(K)*D*(Y(K+1)+SQRT(2.*(10.-MR)*AL))+
     3D*(Y(K+1)-SQRT(2.*(10.-MR)*AL))**2/8.
      MR= MR + 1.
    4 CONTINUE
      IS= 17
    5 IT= IS+12
      DO 6 K= IS,IT
      IWR=IS+6
      IF(K.EQ.IWR) GO TO 6
      F(K)= Y(K)**4-.5*Y(K)**3*(Y(K-15)+Y(K+15))+
     1Y(K)**2*(D-(Y(K-15)-Y(K+15))**2/8.)
     7-.5*Y(K)*D*(Y(K+1)+Y(K-1))+D*(Y(K+1)-Y(K-1))**2/8.
```

```
      6 CONTINUE
        IS= IS + 15
        IF(IS.LE.122) GO TO 5
        MP= 1.
        DO 7  M=30,135,15
        F(M)= Y(M)**4-.5*Y(M)**3* (Y(M-15)+Y(M+15))+
       1Y(M)**2*(D-(Y(M-15)-Y(M+15))**2/8.)-
       2.5*Y(M)*D*(SQRT(2.*(10.-MP)*AL)+Y(M-1))+
       3D*(SQRT(2.*(10.-MP)*AL)-Y(M-1))**2/8.
        MP= MP + 1.
      7 CONTINUE
        DO 8  M=136,136
        F(M)=Y(M)**4-.5*Y(M)**3*Y(M-15)+Y(M)**2*(D-Y(M-15)**2/ .)-
       1.5*Y(M)*D*(Y(M+1)+SQRT(2.*AL)) + D*(Y(M+1)-SQRT(2.*AL))**2/8.
      8 CONTINUE
        DO 9 M=137,139
        F(M)= Y(M)**4-Y(M)**3*Y(M-15)/2.+
       1Y(M)**2*(D-Y(M-15)**2/8.)-.5*Y(M)*D*(Y(M+1)+Y(M-1))+
       2D*(Y(M+1)-Y(M-1))**2/8.
      9 CONTINUE
        DO 10  M=140,142
        F(M)= Y(M)**4-Y(M)**3*(Y(M-15)+Y(M+1))/2.+
       1Y(M)**2*(D-(Y(M-15)-Y(M+1))**2/8.)-Y(M)*D*(Y(M+1)+Y(M-1))/2.+
       2D*(Y(M+1)-Y(M-1))**2/8.
     10 CONTINUE
        DO 11  M=144,146
        F(M)= Y(M)**4-Y(M)**3*(Y(M-15)+Y(M+1))/2.+
       1Y(M)**2*(D-(Y(M-15)-Y(M+1))**2/ .)-
       2.5*Y(M)*D*(Y(M+1)+Y(M-1))+
       3D*(Y(M+1)-Y(M-1))**2/8.
     11 CONTINUE
        DO 12  M=147,149
        F(M)= Y(M)**4-Y(M)**3*Y(M-15)/2.+
       1Y(M)**2*(D-Y(M-15)**2/8.)-
       2.5*Y(M)*D*(Y(M+1)+Y(M-1))+D*(Y(M+1)-Y(M-1))**2/8.
     12 CONTINUE
        DO 14  M=135,135
        F(M)= Y(M)**4-.5*Y(M)**3*Y(M-15)+
       1Y(M)**2*(D-Y(M-15)**2/8.)-
       2.5*Y(M)*D*(SQRT(2.*AL)) + Y(M-1)) +
       3D*(SQRT(2.*AL)-Y(M-1))**2/8.
     14 CONTINUE
        DO 15  M=151,151
        F(M)= Y(M)**4-Y(M)**3*Y(M-1)[illegible]
       1[illegible]
     15 CONTINUE
        DO 16  M=152,1[illegible]
        F(M)= Y(M)**4-Y(M)**3*Y(M-1)[illegible]
       1[illegible]*(Y(M+1)-Y(M-1))**2/8.
     16 CONTINUE
        DO 17  M=153,1[illegible]
        F(M)= Y(M)**4-Y(M)**3*Y(M-1)[illegible]
       1[illegible]*Y(M-1)[illegible]-Y(M-1))**2/8.
     17 CONTINUE
        DO 1[illegible]  M=154,154
        F(M)= Y(M)**4-Y(M)**3*Y(M-1)[illegible]
       1[illegible]
```

FINITE PLATE

```
   18 CONTINUE
      DO 19  K=155,155
      F(K)= Y(K)**4-Y(K)**3*Y(K-10)*C*Y(K)**2-
     1.5*C*Y(K)*(Y(K+1)*Y(K-1))*C*(Y(K+1)-Y(K-1))**2/E
   19 CONTINUE
      DO 20  K=156,156
      F(K)= Y(K)**4-Y(K)**3*Y(K-10)*D*Y(K)**2-
     1.5*D*Y(K)*Y(K-1)*D*Y(K-1)**2/R.
   20 CONTINUE
      RETURN
      END
```

END OF COMPILATION   NO DIAGNOSTICS.

112	.13749468+01	.34034509-02
113	.20736441+01	.00000000
114	.13776295+01	.38455729-02
115	.64696667+00	-.32924126-02
116	.58086179+00	.44755812-03
117	.55179533+00	-.32818407-03
118	.54954679+00	-.56954402-04
119	.54984693+00	-.83145680-05
120	.54810754+00	-.77902000-05
121	.44788775+00	.13638503-03
122	.44877667+00	.14531502-03
123	.44982877+00	.15184014-03
124	.45120259+00	-.14645468-03
125	.49412851+00	-.64453074-03
126	.56131039+00	-.36694615-02
127	.13634169+01	.30222110-02
128	.20493901+01	.00000000
129	.13667614+01	.33995816-02
130	.56141394+00	-.36786349-02
131	.49412665+00	-.64512353-03
132	.45120234+00	-.14645684-03
133	.44982857+00	.15185056-03
134	.44877643+00	.14531991-03
135	.44788770+00	.13638160-03
136	.31402625+00	.30660806-03
137	.31501478+00	.33294888-03
138	.31657759+00	.36752484-03
139	.31996438+00	.28722718-03
140	.38819164+00	-.14894305-03
141	.47588971+00	-.45549141-02
142	.13551607+01	.29539487-02
143	.20248457+01	.00000000
144	.13589291+01	.33374926-02
145	.47906323+00	-.45599598-02
146	.38818995+00	-.14956713-03
147	.31996447+00	.28724359-03
148	.31657759+00	.36752997-03
149	.31501478+00	.33295525-03
150	.31402625+00	.30663949-03
151	.24725760+00	-.15004302-02
152	.44601861+00	-.30220743-02
153	.13517796+01	.13546498-02
154	.13556335+01	.16085955-02
155	.44632958+00	-.30111935-02
156	.24726079+00	-.15006954-02

THE SUM OF SQUARES IS .63935093-03

ERROR RETURN BECAUSE THERE HAVE BEEN 400 CALLS OF CALFUN

THE FINAL SOLUTION REQUIRED 400 CALLS OF CALFUN AND IS

I	X(I)	F(I)
1	.10003373+01	.26559224-04
2	.10007266+01	.32765442-04
3	.10016510+01	.34883735-04
4	.10059315+01	.15685440-03
5	.10196154+01	.13530743-03
6	.10329327+01	-.15972726-02
7	.14389698+01	.10145338-01

FINITE PLATE

	.7723[illegible]+01	.[illegible]00
	.14772512+01	.95904946-02
10	.1[illegible]29951+01	-.20[illegible]799-02
11	.1[illegible]196951+01	.1421 771-03
12	.1[illegible]59[illegible]6+01	.1692[illegible]89-03
13	.1[illegible]16635+01	.36917976-04
14	.1[illegible]07297+01	.31364732-04
15	.1[illegible]03432+01	.2674[illegible]932-04
16	.94[illegible]3[illegible]69+00	.1[illegible]3[illegible]36-04
17	.94941455+00	.1457[illegible]19-04
18	.9[illegible]34447+00	.6976[illegible]412-05
19	.[illegible]9441[illegible]7+00	-.[illegible]911717-05
20	.[illegible]65947+00	.[illegible]172746-05
21	.[illegible]9764375+00	-.1453[illegible]237-02
22	.14369501+01	.5293 563-02
23	.2135943+01	.[illegible]0000
24	.1472401 5+01	.3027 247-02
25	.[illegible]743794+00	-.14273156-02
26	.96692347+00	.15771981-04
27	.95041[illegible]71+00	-.3117[illegible]498-04
28	.95 35223+00	.54914072-05
29	.[illegible]4421[illegible]3+01	.17943324-04
30	.[illegible]3[illegible]365[illegible]+00	.1[illegible]49[illegible]62-04
31	.[illegible]474755+00	.24224196-05
32	.[illegible]11526+00	-.77189932-05
33	.[illegible]01471+00	-.24[illegible]159-04
34	.[illegible]979469+00	-.1711[illegible]473-03
35	.91495596+00	-.2530733[illegible]-03
36	.95553152+00	-.16794260-02
37	.143191[illegible]+01	.30335904-02
38	.7190[illegible]2+01	.[illegible]00
39	.1428[illegible]+01	.3627[illegible]495-02
40	.95515113+00	-.1977[illegible]292-02
41	.91499968+00	-.2[illegible]22[illegible]703-03
42	.3979398+00	-.35057530-04
43	.[illegible]9[illegible]212[illegible]2+00	-.26441134-04
44	.[illegible]512734+00	-.5[illegible]3[illegible]415-05
45	.994[illegible]627+00	.26871594-05
46	.93653667+00	-.795[illegible]560-05
47	.43731192+00	-.27897315-04
48	.8331760[illegible]+00	-.56253573-04
49	.541670[illegible]4+00	-.24385776-03
57	.95774695+00	-.51722055-03
51	.9[illegible]09255+00	-.21153676-02
52	.1423[illegible]60+01	.41685[illegible]24-02
53	.21679483+01	.[illegible]000
54	.14220927+01	.43196935-02
55	.9[illegible]903219+00	-.15794282-02
55	.95776351+00	-.57107382-03
57	.84155912+00	-.26976874-03
58	.93918751+00	-.55766871-04
59	.93731136+00	-.2595[illegible]496-04
6[illegible]	.93693583+00	-.94360099-05
61	.77486102+00	-.99146048-05
62	.77524973+00	-.24117916-04
63	.77617018+00	-.46903656-04
64	.77957501+00	-.27590024-03

ORIGINAL PAGE IS
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55	.77710512+09	-.77934738-03
56	.85519758+07	-.23463198-02
67	.19136132+01	.97507469-02
6[illegible]	.21447611+01	.[illegible]0[illegible]00
6[illegible]	.19129[illegible]2+01	.43452[illegible]18-[illegible]2
7[illegible]	.[illegible]5513927+0[illegible]	-.23758755-02
71	.79772007+07	-.72692608-03
72	.77457[illegible]1+09	-.2[illegible]467159-[illegible]3
73	.77617232+07	-.47273430-04
74	.77[illegible]28907+0[illegible]	-.2016[illegible]296-04
75	.77496159+07	-.[illegible]56[illegible]07-05
76	.7[illegible]7[illegible]0959+09	.251[illegible]159-05
77	.7[illegible]795812+02	.52121962-[illegible]5
7[illegible]	.7[illegible]99605+0[illegible]	.1[illegible]27[illegible]263-[illegible]4
7[illegible]	.71769512+07	-.[illegible]7517168-05
[illegible]	.73500[illegible]70+0[illegible]	-.[illegible]576[illegible]11-03
[illegible]1	.79406535+07	-.2732[illegible]53-02
[illegible]	.14213353+01	.4[illegible]26[illegible]635-02
[illegible]	.71713203+01	.[illegible]0[illegible]0[illegible]05
[illegible]	.10[illegible]17536+01	.9[illegible]11[illegible]45-02
[illegible]5	.794132[illegible]5+07	-.26517917-02
[illegible]	.7320307[illegible]+0[illegible]	-.6[illegible]677104-03
[illegible]7	.71282159+09	.37027350-04
[illegible]8	.7[illegible]389758+0[illegible]	.25696750-05
[illegible]	.71755967+07	.57053[illegible]1[illegible]-05
[illegible]	.7[illegible]701[illegible]5+0[illegible]	.2[illegible]72[illegible]567-05
91	.6320[illegible]131+07	.[illegible]9652[illegible]81-[illegible]5
9[illegible]	.65349109+07	.1[illegible]727979-[illegible]4
[illegible]	.62935699+07	.10622[illegible]47-[illegible]4
[illegible]	.6[illegible]17[illegible]65+07	.6[illegible]731[illegible]60-04
95	.66569736+07	-.10775112-04
[illegible]	.72[illegible]27[illegible]72+0[illegible]	-.[illegible]72[illegible]1-02
97	.13780622+01	.[illegible]6[illegible]01-02
[illegible]	.[illegible]78177+01	.[illegible]0[illegible]
[illegible]5	.1[illegible]9[illegible]4+01	.4[illegible]65[illegible]11-02
1[illegible]	.[illegible]25347[illegible]+0[illegible]	-.2[illegible]145571-0[illegible]
101	.64573686+0[illegible]	-.1[illegible]51[illegible]96-[illegible]4
1[illegible]2	.6[illegible] 17[illegible]93+0[illegible]	.[illegible]462[illegible]57-04
10[illegible]	.65435547+0[illegible]	.14372[illegible]06-[illegible]4
[illegible]	.6[illegible]3[illegible]55+0[illegible]	.1[illegible]56[illegible]381-04
10[illegible]	.6528[illegible]91+00	.57[illegible]36-[illegible]5
[illegible]6	.[illegible] 10715+0[illegible]	-.7[illegible]47080-05
107	.5[illegible]80755+0[illegible]	-.[illegible]7253097-05
1[illegible]	.5[illegible]50700+07	-.5[illegible]37[illegible]252-[illegible]4
10[illegible]	.55179618+09	-.37797[illegible]06-03
11[illegible]	.5[illegible]65900+07	.4467[illegible]30-03
111	.6465[illegible]462+03	-.32854025-02
112	.13794432+01	.3[illegible]91[illegible]527-02
11[illegible]	.2073[illegible]1+01	.[illegible]0[illegible]00
11[illegible]	.12770325+03	.3[illegible]501679-[illegible]2
115	.60696372+07	-.[illegible]293[illegible]71-[illegible]2
11[illegible]	.56[illegible]659[illegible]0+09	.00607797-03
117	.55179590+09	-.32325019-03
11[illegible]	.5[illegible]50676+09	-.569[illegible]031-04
11[illegible]	.[illegible] 80706+0[illegible]	-.52770060-05
12[illegible]	.5[illegible]7712+0[illegible]	-.77639077-05
[illegible]	[illegible]	[illegible]

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FINITE PLATE DATE 091075 PAGE 31

122	.44877660+00	.14529983-03
123	.44982868+00	.15187012-03
124	.45120241+00	-.14651273-03
125	.49412981+00	-.64916898-03
126	.55131104+00	-.36690759-02
127	.13649151+01	.30195750-02
128	.20493901+01	.00000000
129	.13667589+01	.33961498-02
130	.56141461+00	-.36782455-02
131	.49412796+00	-.64476176-03
132	.45120216+00	-.14651188-03
133	.44982848+00	.15183060-03
134	.44877636+00	.14530419-03
135	.44788764+00	.13637024-03
136	.31402631+00	.30661240-03
137	.31501483+00	.33295283-03
138	.31657765+00	.36753011-03
139	.31996447+00	.28723609-03
140	.38819173+00	-.14899262-03
141	.47789087+00	-.45548266-02
142	.13551628+01	.28577865-02
143	.20248457+01	.00000000
144	.13589257+01	.33204710-02
145	.47906439+00	-.45598721-02
146	.38819004+00	-.14963655-03
147	.31996456+00	.28725274-03
148	.31657765+00	.36753523-03
149	.31501483+00	.33295924-03
150	.31402631+00	.30661377-03
151	.24725767+00	-.15008318-02
152	.44501818+00	-.30222750-02
153	.13517915+01	.13566850-02
154	.13556429+01	.16404209-02
155	.44632914+00	-.30113188-02
156	.24726036+00	-.15008970-02

THE SUM OF SQUARES IS .63937107-03

*FIN

```
@FOR,IS MAIN
FOR 011A-04/17/75-18 47 07 (,0)

  MAIN PROGRAM

  STORAGE USED  CODE(1) 002244; DATA(0) 000627; BLANK COMMON(2) 000000

  EXTERNAL REFERENCES (BLOCK, NAME)

   0003   NINTR$
   0004   NRDU$
   0005   NIO1$
   0006   NIO2$
   0007   SQRT
   0010   NERR
   0011   ACOS
   0012   COS
   0013   NERR3$
   0014   NSTOP$

  STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

   0001   002373 1L      0001   001361 10L     0000   000536 100F    0001   000031 1166    0001   000043 1296
   0001   000345 2L      0001   000451 2376    0000   000540 26F     0001   000357 3L      0001   000207 30L
   0001   000362 31L     0001   000601 32L     0001   001017 33L     0001   001223 34L     0001   001377 35L
   0001   001725 4L      0001   001453 4566    0001   001527 4660    0001   001576 4776    0001   000737 5L
   0001   000052 50L     0001   001636 5056    0001   001676 5130    0001   001513 52L     0001   001733 5216
   0001   001776 5276    0001   002036 5356    0001   002061 5436    0001   002104 5516    0001   002126 5576
   0001   002150 5656    0001   002173 5736    0001   000751 6L      0001   002233 6116    0001   001347 7L
   0001   001076 70L     0001   000144 77L     0001   000145 78L     0001   000537 79L     0000   000551 8F
   0001   000540 80L     0001   001160 81L     0001   001161 82L     0001   000172 83L     0001   000173 84L
   0001   000564 85L     0001   000565 86L     0001   001206 87L     0001   001207 88L     0001   001373 9L
   0001   002215 99L     0000 R 000505 A       0000 R 000476 AL      0000 R 000507 ARG     0000 R 000506 B
   0000 R 000513 CAPA    0000 R 000517 CAPR    0000 R 000502 CO      0000 I 000501 COUNT   0000 R 000503 C1
   0000 R 000504 C2      0000 R 000472 D       0000 R 000521 E       0000 I 000535 I       0000 I 000531 IRP
   0000 I 000533 IS      0000 I 000534 IT      0000 I 000500 J       0000 I 000532 K       0000 R 000470 NO
   0000 I 000471 NN      0000 I 000477 N       0000 R 000510 FAT     0000 R 000522 PHI     0000 R 000511 POP
   0000 R 000475 PSI6    0000 R 000530 RT      0000 R 000514 SAT     0000 R 000527 SMALL   0000 R 000515 SOP
   0000 R 000520 T       0000 R 000512 TIP     0000 R 000516 TOP     0000 R 000523 T1      0000 R 000524 T2
   0000 R 000525 T3      0000 R 000473 VAR     0000 R 000474 WE      0000 R 000526 WT      0000 R 000000 Y
   0000 R 000234 Z

00100      1*      C      NO/L=1/2
00101      2*             DIMENSION Y(156)
00103      3*             DIMENSION Z(156)
00104      4*             REAL NO
00105      5*             NN=1000
00106      6*             D=.0204
00107      7*             VAR=.000000000001
00110      8*             WE=.4.
00111                     PSI6=.5
00112                     AL=PSI6/10.
```

```
00113          NT=156
00114          READ(5,100)(Y(J),J=1,N)
00122          READ(5,100)(Z(J),J=1,N)
00130      100 FORMAT(8(F10.2))
00131          KO=3.5
00132          INTEGER COUNT
00133          COUNT=0
00134       50 COUNT=COUNT+1
00135          IF(COUNT.GE.NN) GO TO 99
00137          CO=(4.*AL**2)/(D*(Z(2)+KO)**2)
00140          C1=1.-2./WE
00141          C2=2./(WE*Y(1))
00142          A=-CO/C1
00143          B=C2/C1
00144          ARG=B**2/4.+A**3/27.
00145          IF(ARG.LT.0.0) GO TO 30
00147          PAT=-B/2.+SQRT(ARG)
00150          POP=-PAT
00151          IF(POP.LE.VAR) GO TO 77
00153          TIP=POP**.333
00154          GO TO 78
00155       77 TIP=0.0
00156       78 CAPA=-TIP
00157          SAT=-B/2.-SQRT(ARG)
00160          SOP=-SAT
00161          IF(SOP.LE.VAR) GO TO 83
00163          TOP=SOP**.333
00164          GO TO 84
00165       83 TOP=0.0
00166       84 CAPB=-TOP
00167          T=CAPA+CAPB
00170          IF(T.GT.-1.) T=-1.
00172          GO TO 31
00173       30 E=3.*B/(2.*A)
00174          IF(E.LT.-1.) E=-1.
00176          PHI=ACOS(E)
00177          T1=2.*SQRT(-A/3.)*COS(PHI/3.)
00200          T2=2.*SQRT(-A/3.)*COS(PHI/3.+4.*3.14/3.)
00201          T3=2.*SQRT(-A/3.)*COS(PHI/3.+2.*3.14/3.)
00202          WY=ABS(T3)
00203          IF(T1.LT.1.) T1=100.
00205          IF(T2.LT.1.) T2=100.
00207          IF(WY.LT.1.) WY=100.
00211          IF(T1.GT.T2) GO TO 1
00213          IF(T1.GT.WY) GO TO 2
00215          SMALL=T1
00216          GO TO 3
00217        1 SMALL=T2
00220          IF(T2.GT.WY) SMALL=WY
00222          GO TO 3
00223        2 SMALL=WY
00224          IF(WY.GT.T2) SMALL=T2
00226          GO TO 3
00227        3 CONTINUE
00230          T=SMALL
00231          T=1.
00232       31 RT=(Z(2)+KO)*SQRT(D)*SQRT(T**2-1.)/Y(1))
```

```
      RT=-RT
      Z(2)=Y(1)**2*Z(16)/(Y(1)**2+D)+(Z(2)-HO)/(2.*(Y(1)**2/D+1.))*
     1RT*(2.*Y(1))**2*SQRT(D)*(Z(2)-HO))/(4.*(Y(1)**2+D))
      IRA=1
      DO 70 K=2,14
      IRA=IRA+1
      IF(IRA.EQ.8) GO TO 70
      C=2*.*AL**2/(D+(Z(K+1)-Z(K-1))**2)
      C1=1.-2./WE
      C2=2./(WE*Y(K))
      A=-C2/C1
      B=C2/C1
      ARG=B**2/4.+A**3/27.
      IF(ARG.LT.0.0) GO TO 72
      RAT=-B/2.+SQRT(ARG)
      TOP=-RAT
      IF(TOP.LE.VAR) GO TO 79
      TIP=TOP**.333
      GO TO 80
   79 TIP=0.0
   80 CAPA=-TIP
      SAT=-B/2.-SQRT(ARG)
      SOP=-SAT
      IF(SOP.LE.VAR) GO TO 85
      TOP=SOP**.333
      GO TO 86
   85 TOP=0.0
   86 CAPB=-TOP
      T=CAPA+CAPB
      IF(T.GT.-1.) T=-1.
      GO TO 31
   72 F=B/2./(...)
      IF(F.LT.-1.) F=-1.
      PHI=ACOS(F)
      T1=2.*SQRT(-A/3.)*COS(PHI/3.)
      T2=2.*SQRT(-A/3.)*COS(PHI/3.+4.*3.14/3.)
      T3=2.*SQRT(-A/3.)*COS(PHI/3.+2.*3.14/3.)
      AY=ABS(T3)
      IF(T1.LT.1.) T1=100.
      IF(T2.LT.1.) T2=100.
      IF(AY.LT.1.) AY=100.
      IF(T1.GT.T2) GO TO 4
      IF(T1.GT.AY) GO TO 5
      SMALL=T1
      GO TO 6
    4 SMALL=T2
      IF(T2.GT.AY) SMALL=AY
      GO TO 6
    5 SMALL=AY
      IF(AY.GT.T2) SMALL=T2
      GO TO 6
    6 CONTINUE
      T=SMALL
      IF(... ) T=1.
      IF(... ) T=1.
      IF(... ) T=1.
      IF(... ) T=1.
```

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```
      IF(IPA.EQ.13) T=1.
      IF(IPA.EQ.14) T=1.
   73 RT=SQRT(D)*(Z(K+1)-Z(K-1))*SQRT(T**2-1.)/Y(K)
      IF(Z(K+15).LT.0.0) RT=-RT
      Z(K)=Y(K)**2*Z(K+15)/(Y(K)**2+D)+
     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
     2RT*(2.*Y(K)**2+SQRT(D)*(Z(K+1)-Z(K-1)))/(4.*(Y(K)**2+D))
   70 CONTINUE
      C0=N.*AL**2/(D*(HO-Z(1+1))**2)
      C1=1.-2./HE
      C2=2./(WL*Y(15))
      A=-C?/C1
      B=C2/C1
      ARG=B**2/4.+A**3/27.
      IF(ARG.LT.0.0) GO TO 34
      RAT=-B/2.+SQRT(ARG)
      POP=-RAT
      IF(POP.LE.VAR) GO TO 91
      TIP=POP**.333
      GO TO 92
   91 TIP=0.0
   92 CAPS=-TIP
      SAT=-B/2.-SQRT(ARG)
      SOP=-SAT
      IF(SOP.LE.VAR) GO TO 67
      TOP=SOP**.333
      GO TO 68
   67 TOP=0.0
   68 CAPP=-TOP
      T=CAPP+CAPS
      IF(T.GT.-1.) T=-1.
      GO TO 35
   34 E=?.*?/(2.*A*)
      IF(E.LT.-1.) E=-1.
      PHI=ACOS(E)
      T1=2.*SQRT(-A/3.)*COS(PHI/3.)
      T2=2.*SQRT(-A/3.)*COS(PHI/3.+4.*3.14/3.)
      T3=2.*SQRT(-A/3.)*COS(PHI/3.+2.*3.14/3.)
      WY=ABS(T3)
      IF(T1.LT.1.) T1=100.
      IF(T2.LT.1.) T2=100.
      IF(WY.LT.1.) WY=100.
      IF(T1.GT.T2) GO TO 7
      IF(T1.GT.WY) GO TO 10
      SMALL=T1
      GO TO 9
    7 SMALL=T2
      IF(T2.GT.WY) SMALL=WY
      GO TO 9
   10 SMALL=WY
      IF(WY.GT.T2) SMALL=T2
      GO TO 9
    9 CONTINUE
      T=SMALL
      T=1.
   35 RT=(HO-Z(14))*SQRT(D)*SQRT(T**2-1.)/Y(15)
      Z(15)=Y(15)**2*Z(30)/(Y(15)**2+D)+(HO+Z(14))/(2.*(Y(15)**2/D+1.))+
```

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```
00059      1RT*(2.*Y(15)**2 *SQRT(D)*(K0-Z(19)))/(4.*(Y(15)**2+D))
00059      DO 51 K=16,136,15
00060      Z(K)=Y(K)**2*(Z(K-15)+Z(K+15))/(2.*(Y(K)**2+D))+
00060     1(Z(K+1)-K0)/(2.*(Y(K)**2/D+1.))+
00060     2(Z(K-15)-Z(K+15))*(Z(K+1)+K0)/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00061   51 CONTINUE
00061      IS=17
00064   52 IY=IS+12
00065      DO 53 K=IS,IY
00070      Z(K)=Y(K)**2*(Z(K-15)+Z(K+15))/(2.*(Y(K)**2+D))+
00070     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00070     2(Z(K-15)-Z(K+15))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00071   53 CONTINUE
00073      IS= IS + 15
00074      IF(IS .LE. 122) GO TO 52
00076      DO 54  K=30,135,15
00501      Z(K)= Y(K)**2*(Z(K-15)+Z(K+15))/(2.*(Y(K)**2+D))+
00501     1(K0+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00501     2(Z(K-15)-Z(K+15))*(K0-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00502   54 CONTINUE
00504      DO 55  K=137,138
00507      Z(K)= Y(K)**2*(Z(K-15)+Z(K+15))/(2.*(Y(K)**2+D))+
00507     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00507     2(Z(K-15)-Z(K+15))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00510   55 CONTINUE
00512      DO 56  K=139,147
00515      Z(K)= Y(K)**2*Z(K-15)/(2.*(Y(K)**2+D))+
00515     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00515     2Z(K-15)*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00516   56 CONTINUE
00523      DO 57  K=148,149
00523      Z(K)= Y(K)**2*(Z(K-15)+Z(K+6))/(2.*(Y(K)**2+D))+
00523     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00523     2(Z(K-15)-Z(K+6))*(Z(K+1)-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00524   57 CONTINUE
00526      DO 58  K=150,150
00531      Z(K)=Y(K)**2*(Z(K-15)+Z(K+6))/(2.*(Y(K)**2+D))+
00531     1(K0+Z(K-1))/(2.*(Y(K)**2/D+1.))+
00531     2(Z(K-15)-Z(K+6))*(K0-Z(K-1))/(4.*(Y(K)**2/SQRT(D)+SQRT(D)))
00532   58 CONTINUE
00534      DO 59  K=151,151
00537      Z(K)= Y(K)**2*Z(K-15)/(Y(K)**2+D)+
00537     1(Z(K+1)-K0)/(2.*(Y(K)**2/D+1.))
00540   59 CONTINUE
00542      DO 60  K=152,152
00545      Z(K)=Y(K)**2*Z(K-15)/(Y(K)**2+D)+
00545     1(Z(K+1)+Z(K-1))/(2.*(Y(K)**2/D+1.))
00546   60 CONTINUE
00552      DO 61 K=153,153
00553      Z(K)= Y(K)**2*Z(K-15)/(Y(K)**2+D)+
00553     1Z(K-1)/(2.*(Y(K)**2/D+1.))
00554   61 CONTINUE
00556      DO 62  K=154,154
00561      Z(K)= Y(K)**2*Z(K-6)/(Y(K)**2+D)+
00561     1Z(K+1)/(2.*(Y(K)**2/D+1.))
00562   62 CONTINUE
00564      DO 63  K=155,155
```

```
00567   279*        Z(N)=Y(N)**2*Z(N-6)/(Y(N)**2+D)+
00567   240*       1(Z(N+1)+Z(N-1))/(2.*(Y(N)**2/D+1.))
00570   241*     63 CONTINUE
00572   242*        DO 64  N=156,156
00575   243*        Z(N)=Y(N)**2*Z(N-6)/(Y(N)**2+D)+
00575   244*       1(N0+Z(N-1))/(2.*(Y(N)**2/D+1.))
00576   245*     64 CONTINUE
00600   246*        GO TO 57
00601   247*     99 CONTINUE
00602   *DIAGNOSTIC*  THIS STATEMENT HAS TOO MANY LEFT PARENTHESES.
00602   248*     26 FORMAT(1H ,AX7HAT THE I5,1X25HTH ITERATION WE HAVE)
00603   249*        WRITE(6,26) COUNT
00606   250*      8 FORMAT(1H ,4X1HI,9X4HZ(I)//(I5,F17.9))
00607   251*        WRITE(6,8) (I,Z(I),I=1,N)
00616   252*        END

      END OF COMPILATION          1 DIAGNOSTICS.


*XQT
MAP U025-09/17-15 07
ABCM STATUS OF OUTPUT ELEMENT=UNKNOWN

ADDRESS LIMITS    001000 013544      5477 IBANK WORDS DECIMAL
                  040000 044566      2423 DBANK WORDS DECIMAL
STARTING ADDRESS  011701


            SEGMENT MAIN$         001000 013544     040000 044566

  NSWTCH$/FOR5-7      $(1)  001000 001024
  NSBLK$/FOR6-        $(1)  001025 001047
  NSBRCS/FOR6-        $(1)  001050 001131     $(2)  040000 040011
  NSERR$/FOR6-6       $(1)  001132 001325     $(2)  040012 040031
  NSDCV$/FOR64        $(1)  001736 001463     $(2)  040032 040074
  NSTCH$/FOR6E-5      $(1)  001464 001706     $(2)  040075 040110
  ERTUS/FOR           $(1)  001707 001771
  NCNVT$/FOR65        $(1)  001772 002213     $(2)  040111 040205
  NCLOS$/FOR67        $(1)  002214 002404     $(2)  040206 040236
  NSBLK$/FOR64        $(1)  002405 002516
  NS5DL$/FOR6?        $(1)  002517 002557
  NLPDA$/FOR5?        $(1)  002560 002613
  NSFD$/FOR                                   $(2)  040237 042990
  NOTIN$/FOR6F        $(1)  002614 003110     $(2)  042442 042444
  NCWT$/FOR62-10      $(1)  003111 004265     $(2)  042495 042503
  NIOC$/FOR6P-5       $(1)  004266 004455     $(2)  042504 042642
  NINL$/FOR68         $(1)  004456 004646     $(2)  042693 042696
  NINPT$/FOR6P-P      $(1)  004647 005655     $(2)  042647 042677
  NERI$/FOR68-8       $(1)  005556 006532     $(2)  042700 042754
  APCHK$/FOR68-8      $(1)  006533 007520     $(2)  042755 043130
                                              $(4)  043131 043202
```

```
SIAS$/FOR                                    $(2)  043203 043291
   N$NRS/FOR66-5              $(1)  007521 010121   $(2)  043242 043432
   SSIOPS/FOR69-E.            $(1)  010122 010146   $(2)  043433 043442
   NOBUFS/FOR68               $(1)  010147 010207
   SINCOS$/FOR59              $(1)  010210 010342   $(2)  043443 043464
   ASINCOSS/FOR59             $(1)  010343 010557   $(0)  043465 043512
   SLIPSS/FOR63               $(1)  010560 010754   $(2)  043513 043564
   SQRTS/FOR59                $(1)  010755 011015   $(2)  043565 043576
   NIERS/FOR62-B.             $(1)  011016 011177   $(2)  043577 043717
   NIBUFS/FOR65               $(1)  011200 011237   $(2)  043720 043720
   SINTMS/FOR62-3.            $(1)  011240 011300   $(2)  043721 043737
   BLANK$COMMON(COMMONBLOCK)
   MAIN                       $(1)  011301 013594   $(0)  043740 044566
                                                    $(2)  BLANK$COMMON

   SYS$RLIB$. LEVEL 168-11
   END OF COLLECTION - TIME 1.238 SECONDS

        AT THE  100C TH ITERATION WE HAVE)
      I          Z(I)

      1     -.31427282+01
      2     -.27789856+01
      3     -.24544140+01
      4     -.22741159+01
      5     -.17446865+01
      6     -.13891798+01
      7     -.58995286+00
      8      .00000000
      9      .59333497+00
     10      .13898634+01
     11      .17579312+01
     12      .21047935+01
     13      .24466541+01
     14      .27932792+01
     15      .31601284+01
     16     -.31467161+01
     17     -.27751887+01
     18     -.24397472+01
     19     -.20585250+01
     20     -.17115390+01
     21     -.13310619+01
     22     -.56309455+00
     23      .74868552-05
     24      .56043530+00
     25      .13318291+01
     26      .17125971+01
     27      .20656713+01
     28      .24326439+01
     29      .27862498+01
     30      .31557309+01
     31     -.31367126+01
     32     -.27672647+01
     33     -.24106251+01
     34     -.20345858+01
     35     -.16727932+01
     36     -.12718105+01
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40	.12717909+01
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42	.20334345+01
43	.28124058+01
44	.27749896+02
45	.31984780+01
46	-.31306890+01
47	-.27553717+01
48	-.23933555+01
49	-.20049667+01
50	-.16276062+01
51	-.12076806+01
52	-.30946263+00
53	.17270303-04
54	.40483208+00
55	.12082104+01
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61	-.31225157+01
62	-.27393304+01
63	-.23639691+01
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65	-.15795559+01
66	-.11385792+01
67	-.93776067+00
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69	.93710772+00
70	.11789132+01
71	.15155998+01
72	.19647164+01
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74	.27408289+01
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77	-.27199963+01
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79	-.19243069+01
80	-.15113077+01
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97	-.92330620+00
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110	-.13267599+01
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115	.85361176+00
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117	.17929958+01
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123	-.21338525+01
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125	-.11709884+01
126	-.68999996+00
127	-.17687156+00
128	.10266228-04
129	.17698256+00
130	.68413676+00
131	.11701255+01
132	.16630302+01
133	.21360199+01
134	.25955221+01
135	-.30805504+01
136	-.29939234+01
137	-.24051874+01
138	-.19664576+01
139	-.14788120+01
140	-.98290670+00
141	-.43381182+00
142	-.91348278-01
143	.52444900-05
144	.91393908-01
145	.43355163+00
146	.98190158+00
147	.14689578+01
148	.19511263+01
149	.24048959+01
150	.29939975+01


851 -.26250000+01
852 -.17500000+01
853 -.87499999+00
854 .87499999+00
855 .17500000+01
856 .26250000+01

X. SURFACE TENSION DOMINATED r FORMULATION

O M A

	1.0003	1.007	1.0015	1.0058	1.018	1.032	1.432	2.236	1.432	1.032	1.018	1.0058	1.0015	1.0007	1.0003	
1.0	y_{1} .9489	y_{2} .9493	y_{3} .9500	y_{4} .9540	y_{5} .9679	y_{6} .9970	y_{7} 1.431	y_{8} 2.213	y_{9} 1.431	y_{10} .9970	y_{11} .9679	y_{12} .9540	y_{13} .9500	y_{14} .9493	y_{15} .9489	1.0
.949	y_{16} .8947	y_{17} .8950	y_{18} .8959	y_{19} .8990	y_{20} .9141	y_{21} .9561	y_{22} 1.428	y_{23} 2.190	y_{24} 1.428	y_{25} .9561	y_{26} .9141	y_{27} .8990	y_{28} .8959	y_{29} .8950	y_{30} .8947	.949
.895	y_{31} .8368	y_{32} .8372	y_{33} .8381	y_{34} .8410	y_{35} .8576	y_{36} .9090	y_{37} 1.421	y_{38} 2.167	y_{39} 1.421	y_{40} .9090	y_{41} .8576	y_{42} .8410	y_{43} .8381	y_{44} .8372	y_{45} .8368	.895
.837	y_{46} .7748	y_{47} .7752	y_{48} .7761	y_{49} .7794	y_{50} .7972	y_{51} .8552	y_{52} 1.411	y_{53} 2.144	y_{54} 1.411	y_{55} .8552	y_{56} .7972	y_{57} .7794	y_{58} .7761	y_{59} .7752	y_{60} .7748	.837
.775	y_{61} .7073	y_{62} .7078	y_{63} .7087	y_{64} .7121	y_{65} .7331	y_{66} .7941	y_{67} 1.398	y_{68} 2.121	y_{69} 1.398	y_{70} .7941	y_{71} .7331	y_{72} .7121	y_{73} .7087	y_{74} .7078	y_{75} .7073	.775
.707	y_{76} .6327	y_{77} .6333	y_{78} .6342	y_{79} .6380	y_{80} .6656	y_{81} .7250	y_{82} 1.383	y_{83} 2.097	y_{84} 1.383	y_{85} .7250	y_{86} .6656	y_{87} .6380	y_{88} .6342	y_{89} .6333	y_{90} .6327	.707
.632	y_{91} .5481	y_{92} .5488	y_{93} .5495	y_{94} .5517	y_{95} .5886	y_{96} .6469	y_{97} 1.369	y_{98} 2.073	y_{99} 1.369	y_{100} .6469	y_{101} .5886	y_{102} .5517	y_{103} .5495	y_{104} .5488	y_{105} .5481	.632
.548	y_{106} .4477	y_{107} .4486	y_{108} .4497	y_{109} .4511	y_{110} .4941	y_{111} .5610	y_{112} 1.358	y_{113} 2.049	y_{114} 1.358	y_{115} .5610	y_{116} .4941	y_{117} .4511	y_{118} .4497	y_{119} .4486	y_{120} .4477	.548
.447	y_{121} .3139	y_{122} .3149	y_{123} .3164	y_{124} .3198	y_{125} .3880	y_{126} .4789	y_{127} 1.351	y_{128} 2.024	y_{129} 1.351	y_{130} .4789	y_{131} .3880	y_{132} .3198	y_{133} .3164	y_{134} .3149	y_{135} .3139	.447
.316	y_{136} 0.0	y_{137} 0.0	y_{138} 0.0	y_{139} 0.0	y_{140} .2471	y_{141} .4460	y_{142} 1.348	y_{143} 2.000	y_{144} 1.348	y_{145} .4460	y_{146} .2471	y_{147} 0.0	y_{148} 0.0	y_{149} 0.0	y_{150} 0.0	.316
0.0					y_{151}	y_{152}	y_{153}		y_{154}	y_{155}	y_{156}					0.0

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XI. SURFACE TENSION DOMINATED z FORMULATION

G A

K:=3.5	-3.14	-2.77	-2.45	-2.07	-1.74	-1.38	-.589	0	.590	1.38	1.75	2.10	2.44	2.79	3.16	z=K:
z=-K:	z_{1}	z_{2}	z_{3}	z_{4}	z_{5}	z_{6}	z_{7}	z_{8}	z_{9}	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}	z_{15}	
	-3.14	-2.77	-2.43	-2.05	-1.71	-1.33	-.560	0	.560	1.33	1.71	2.06	2.43	2.78	3.15	z=K:
z=-K:	z_{16}	z_{17}	z_{18}	z_{19}	z_{20}	z_{21}	z_{22}	z_{23}	z_{24}	z_{25}	z_{26}	z_{27}	z_{28}	z_{29}	z_{30}	
	-3.13	-2.76	-2.41	-2.03	-1.67	-1.27	-.525	0	.525	1.27	1.67	2.03	2.41	2.77	3.14	z=K:
z=-K:	z_{31}	z_{32}	z_{33}	z_{34}	z_{35}	z_{36}	z_{37}	z_{38}	z_{39}	z_{40}	z_{41}	z_{42}	z_{43}	z_{44}	z_{45}	
	-3.13	-2.75	-2.39	-2.00	-1.62	-1.20	-.484	0	.484	1.20	1.63	2.00	2.38	2.75	3.13	z=K:
z=-K:	z_{46}	z_{47}	z_{48}	z_{49}	z_{50}	z_{51}	z_{52}	z_{53}	z_{54}	z_{55}	z_{56}	z_{57}	z_{58}	z_{59}	z_{60}	
	-3.12	-2.73	-2.36	-1.96	-1.57	-1.13	-.437	0	.438	1.13	1.57	1.96	2.37	2.74	3.12	z=K:
z=-K:	z_{61}	z_{62}	z_{63}	z_{64}	z_{65}	z_{66}	z_{67}	z_{68}	z_{69}	z_{70}	z_{71}	z_{72}	z_{73}	z_{74}	z_{75}	
	-3.11	-2.71	-2.32	-1.92	-1.51	-1.06	-.384	0	.384	1.04	1.51	1.92	2.32	2.71	3.11	z=K:
z=-K:	z_{76}	z_{77}	z_{78}	z_{79}	z_{80}	z_{81}	z_{82}	z_{83}	z_{84}	z_{85}	z_{86}	z_{87}	z_{88}	z_{89}	z_{90}	
	-3.09	-2.69	-2.28	-1.86	-1.43	-.967	-.323	0	.323	.967	1.43	1.86	2.28	2.69	3.10	z=K:
z=-K:	z_{91}	z_{92}	z_{93}	z_{94}	z_{95}	z_{96}	z_{97}	z_{98}	z_{99}	z_{100}	z_{101}	z_{102}	z_{103}	z_{104}	z_{105}	
	-3.07	-2.65	-2.22	-1.79	-1.32	-.850	-2.54	0	.254	.850	1.32	1.79	2.22	2.65	3.07	z=K:
z=-K:	z_{106}	z_{107}	z_{108}	z_{109}	z_{110}	z_{111}	z_{112}	z_{113}	z_{114}	z_{115}	z_{116}	z_{117}	z_{118}	z_{119}	z_{120}	
	-3.04	-2.59	-2.13	-1.68	-1.17	-.688	-1.76	0	.176	.688	1.17	1.68	2.13	2.59	3.05	z=K:
z=-K:	z_{121}	z_{122}	123	z_{124}	z_{125}	z_{126}	z_{127}	z_{128}	z_{129}	z_{130}	z_{131}	z_{132}	z_{133}	z_{134}	z_{135}	
	-2.99	-2.48	-1.96	-1.47	-.882	-.433	-.091	0	.091	.433	.881	1.46	1.96	2.48	2.99	z=K:
z=-K:	z_{136}	z_{137}	z_{138}	z_{139}	z_{140}	z_{141}	z_{142}	z_{143}	z_{144}	z_{145}	z_{146}	z_{147}	z_{148}	z_{149}	z_{150}	
	-2.62	-1.75	-.875										.875	1.75	2.62	z=K:
z=-K:	z_{151}	z_{152}	z_{153}	z=0	z=0	z=0	z=0	z=0	z=0	z=0	z=0	z=0	z_{154}	z_{155}	z_{156}	

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TABLE 1. - LIQUID PROPERTIES AT 20° C

Liquid	Surface tension, N/cm	Density, g/cm^3	Absolute viscosity, g/cm-sec
Anhydrous ethanol	22.3×10^{-5}	0.789	1.200×10^{-2}
Trichloro-trifluoro-ethane	18.6	1.579	0.700
Distilled water	72.5	1.00	1.0037

TABLE 2. - SUMMARY OF PARAMETERS - ZERO GRAVITY

Test liquid	Nozzle radius, R_o, cm	Disk radius, L, cm	Ratio, R_o/L	Velocity of jet, cm/sec	Reynolds number, $\rho V R_o/\mu$	Weber number, $\rho V^2 R_o/\sigma$	Flow category
Freon TF	0.25	1.0	0.23	34.3	1934	24.8	S
Freon TF				38.0	2133	30.5	S
Freon TF				43.7	2464	40.4	T
Ethanol				77.0	1278	52.2	T
Freon TF				50.5	2848	54.0	T
Freon TF				58.8	3316	73.4	T
Ethanol				101.0	1662	90.1	T
Freon TF				79.0	4460	113.2	T
Ethanol				131.3	2160	151.9	I
Freon TF				88.6	5000	161.5	I
Ethanol				149.8	2465	197.8	I
Ethanol				237.7	3911	498	I
Ethanol				282.2	4642	703	I
Ethanol				365.6	6014	1180	I
Ethanol	.50	1.5	.33	26.4	868	12.2	S
Freon TF				19.5	1815	16.1	S
Ethanol				37.7	1240	25.1	S
Freon TF				27.4	3090	31.8	T
Ethanol				46.0	1513	37.2	T
Ethanol				53.9	1773	51.2	T
Freon TF				37.6	4240	60.0	T
Ethanol				59.6	1900	62.0	T
Freon TF				42.75	4830	77.6	I

TABLE 2. - Concluded.

Test liquid	Nozzle radius, R_o, cm	Disk radius, L, cm	Ratio, R_o/L	Velocity of jet, cm/sec	Reynolds number, $\rho VR_o/\mu$	Weber number, $\rho V^2R_o/\sigma$	Flow category
Ethanol	0.75	1.5	0.50	15.4	760	6.2	S
Ethanol	.75	1.5	↓	19.1	942	9.6	S
Ethanol	.75	1.5	↓	19.9	986	10.5	S
Ethanol	.50	1.0	↓	26.4	868	12.2	S
Freon TF	.50	1.0	↓	17.7	1996	13.2	S
Ethanol	.75	1.5	↓	28.2	1891	21.0	S
Ethanol	.50	1.0	↓	34.9	1148	21.3	T
Freon TF	.50	1.0	↓	23.3	2628	23.0	T
Ethanol	.75	1.5	↓	34.4	1697	31.2	T
Ethanol	.50	1.0	↓	46.1	1516	37.4	T
Freon TF	.50	1.0	↓	31.9	3490	43.2	T
Ethanol	.75	1.5	↓	41.2	2036	44.7	I
Ethanol	.50	1.0	↓	52.3	1720	48.2	T
Freon TF	.50	1.0	↓	37.5	4230	59.5	I
Ethanol	.75	1.5	↓	47.5	2347	59.6	I
Ethanol	.50	1.0	↓	85.7	2820	129.0	I
Ethanol	.75	1.0	.75	12.7	630	4.2	S
Ethanol	↓	↓	↓	22.5	1110	13.3	S
Ethanol	↓	↓	↓	24.8	1227	16.2	S
Ethanol	↓	↓	↓	27.3	1350	19.7	T
Ethanol	↓	↓	↓	29.4	1454	22.9	T
Ethanol	↓	↓	↓	32.0	1580	27.0	T
Ethanol	↓	↓	↓	35.0	1730	32.3	T
Ethanol	↓	↓	↓	39.1	1930	40.2	I

TABLE 3. - SUMMARY OF PARAMETERS - NORMAL GRAVITY

Test liquid	Nozzle radius, R_o, cm	Disk radius, L, cm	Ratio, R_o/L	Ratio, H/R_o	Weber number, $\rho V^2 R_o/\sigma$	Reynolds number, $\rho V R_o/\mu$	Bond number, $\rho g R_o^2/\sigma$	Ratio, R_F/R_o
Distilled water	0.25	0.50	0.50	4	121	4775	0.852	19.96
		.50	.50	20		5261		22.56
		.75	.33	4		4775		22.88
		.75	.33	20		5261		24.68
		1.0	.25	4		4775		23.38
		1.0	.25	20		5261		25.0
		1.5	.167	4		4775		20.96
		1.5	.167	20		5261		22.78
		2.0	.125	4		4775		18.42
		2.0	.125	20		5261		19.14
	.50	.75	.667	2	26	3775	3.42	5.76
		.75	.667	10		5815		9.26
		1.0	.50	2		3775		5.86
		1.0	.50	10		5815		9.32
		1.5	.33	2		3775		6.33
		1.5	.33	10		5815		9.47
		2.0	.25	2		3775		5.03
		2.0	.25	10		5815		8.80
		.75	.667	2	44	4550	3.39	7.07
		.75	.667	5		5300		9.04
		.75	.667	10		6350		10.90
		1.0	.50	2		4550		8.34
		1.0	.50	10		6330		10.40
		1.5	.333	2		4550		8.33
		1.5	.333	10		6330		11.62
		2.0	.25	2		4550		7.88
		2.0	.25	10		6330		10.62
		.75	.667	2	102	6470	3.40	15.89
		.75	.667	10		7850		19.38
		1.0	.50	2		6470		[illegible]
		1.0	.50	10		7850		20.71
		1.5	.333	2		6470		16.72
		1.5	.333	10		7850		[illegible]
		2.0	.250	2		6470		[illegible]
		2.0	.250	10		7850		19.18

TABLE 3. - Concluded.

Test liquid	Nozzle radius, R_o, cm	Disk radius, L, cm	Ratio, R_o/L	Ratio, H/R_o	Weber number, $\rho V^2 R_o/\sigma$	Reynolds number, $\rho V R_o/\mu$	Bond number, $\rho g R_o^2/\sigma$	Ratio, R_p/R_o
Distilled water	0.25	0.50	0.50	4	235	6573	0.852	35.14
Distilled water	0.25	.50	.50	20	235	6922	0.852	38.80
Distilled water	0.25	.75	.33	4	235	6573	0.852	38.50
Distilled water	0.25	.75	.33	20	235	6922	0.852	45.86
Distilled water	0.25	1.0	.25	4	235	6573	0.852	37.96
Distilled water	0.25	1.0	.25	20	235	6922	0.852	41.56
Distilled water	0.25	1.5	.167	4	235	6573	0.852	[illegible]
Distilled water	0.25	1.5	.167	20	235	6922	0.852	[illegible]
Distilled water	0.25	2.0	.125	4	235	6573	0.852	33.55
Ethanol	0.25	.50	.50	4	62.5	1558	2.16	12.08
Ethanol	0.25	.50	.50	20	62.5	2128	2.16	15.36
Ethanol	0.25	.75	.33	20	62.5	2128	2.16	16.98
Ethanol	0.25	.75	.33	4	62.5	1558	2.16	11.66
Ethanol	0.25	.75	.33	4	157	2307	2.16	24.20
Ethanol	0.25	.75	.33	20	157	2725	2.16	31.80
Ethanol	0.25	1.0	.25	20	157	2725	2.16	30.76
Ethanol	0.25	1.0	.25	4	157	2307	2.16	23.78
Ethanol	0.25	.50	.50	2	200	2520	2.16	24.02
Ethanol	0.25	.50	.50	4	200	2571	2.16	24.12
Ethanol	0.25	.50	.50	10	200	2720	2.16	27.12
Ethanol	0.25	.50	.50	20	200	2953	2.16	29.82
Ethanol	.125	.50	.25	2	224	1857	.54	32.80
Ethanol	.125	.50	.25	4	224	1866	.54	[illegible]
Ethanol	.125	.50	.25	10	224	1892	.54	[illegible]
Ethanol	.125	.50	.25	20	224	1936	.54	32.60
Ethanol	.125	.50	.25	2	177	1657	.54	[illegible]
Ethanol	.125	.30	.25	4	177	1667	.54	[illegible]
Ethanol	.125	.50	.25	10	177	1696	.54	[illegible]
Ethanol	.125	.50	.25	20	177	[illegible]	.54	[illegible]
Ethanol	.25	.50	.50	4	233	2761	2.16	[illegible]
Ethanol	.25	.50	.50	20	233	3119	2.16	[illegible]
Ethanol	.25	.75	.33	20	233	3119	2.16	[illegible]
Ethanol	.25	.75	.33	4	233	2761	2.16	[illegible]
Ethanol	.25	1.0	.25	4	233	2761	2.16	[illegible]
Ethanol	.25	1.0	.25	20	233	3119	2.16	[illegible]

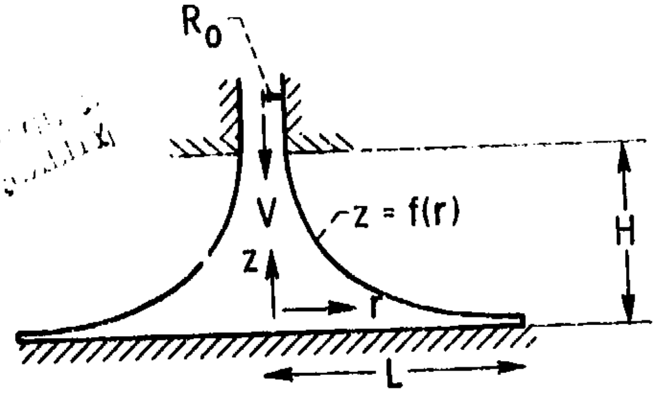


Figure 1. - Schematic of liquid jet impinging on a flat plate.

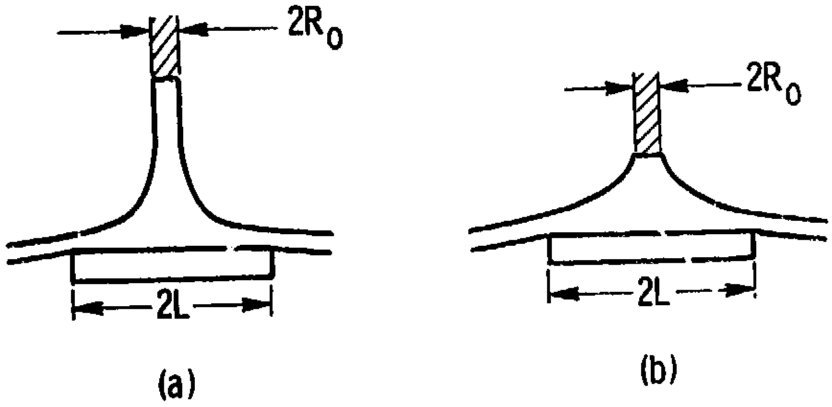


Figure 2. - Flow pattern as a function of nozzle height.

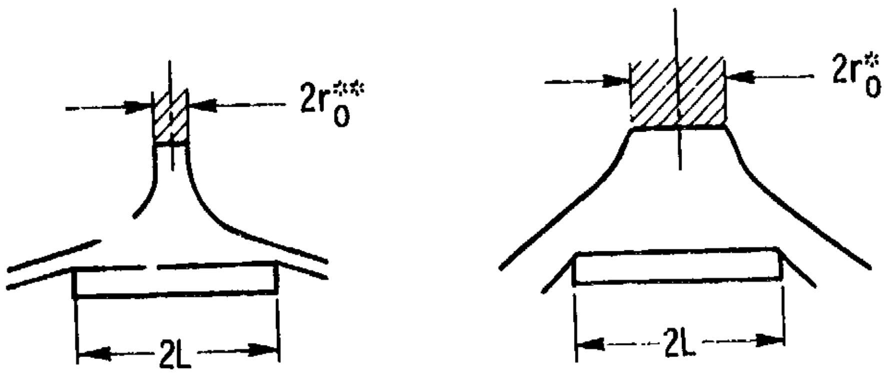


Figure 3. - Flow pattern as a function of jet radius.

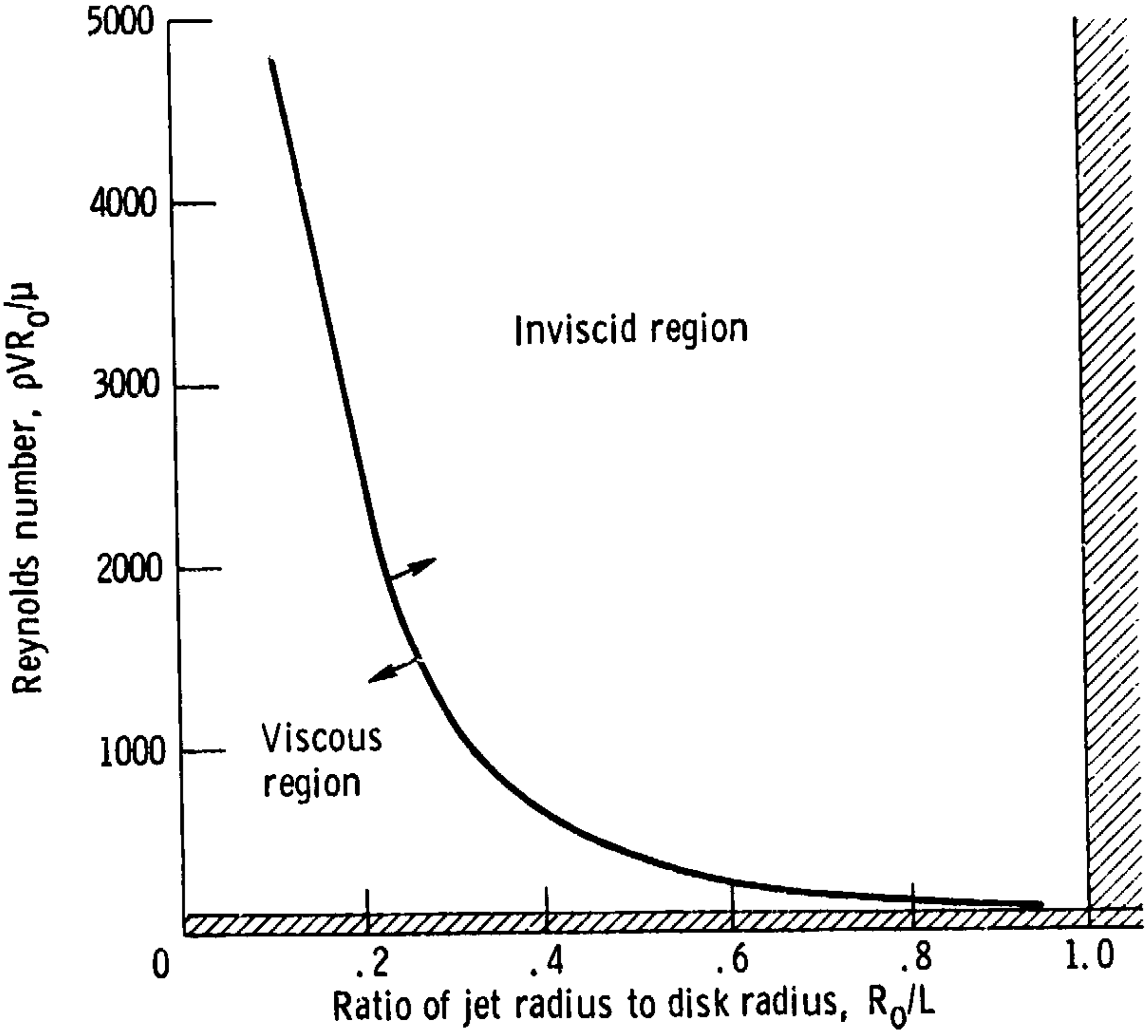


Figure 4. - Results of order of magnitude analysis.

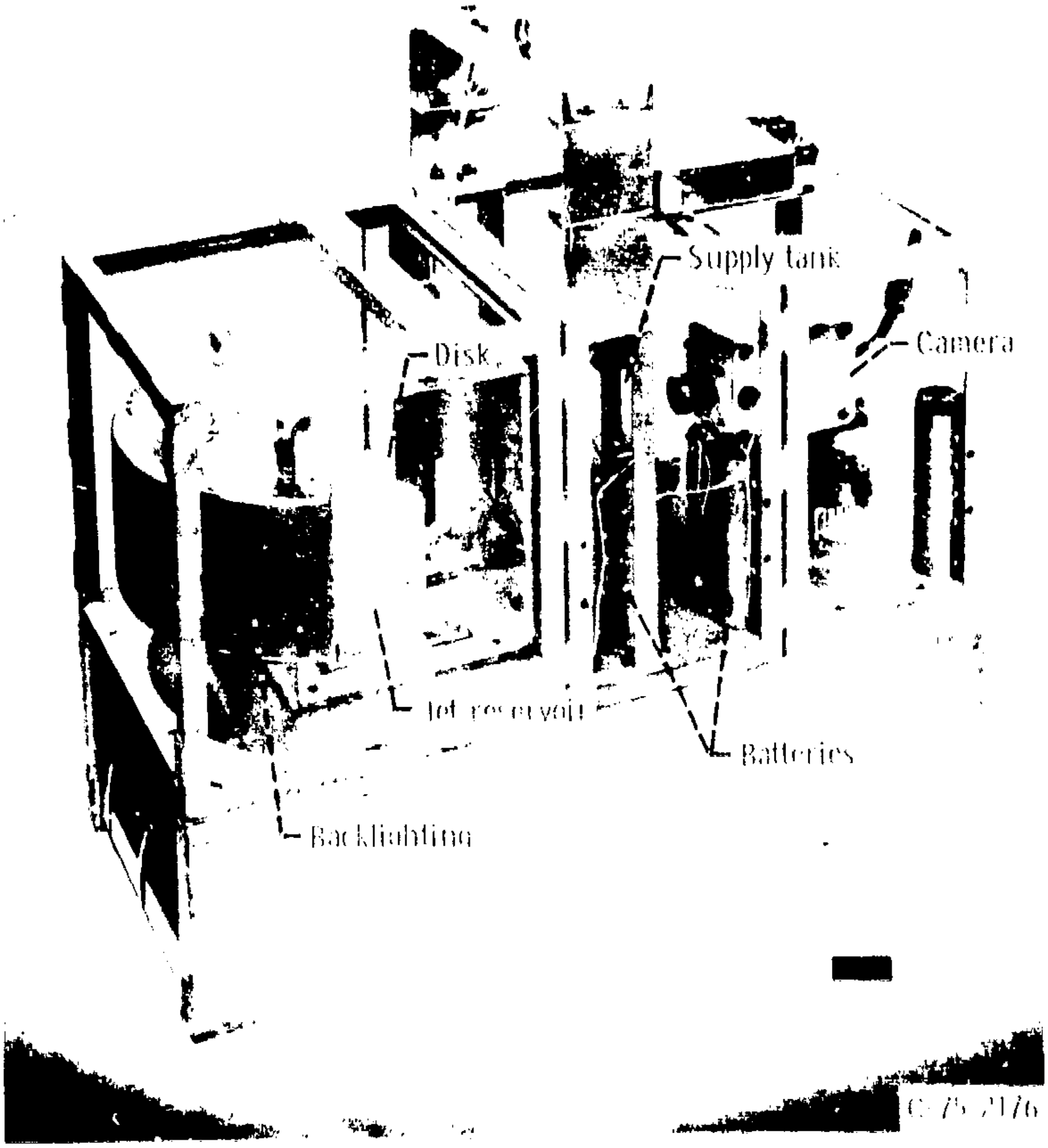


Figure 5. - Experiment package.

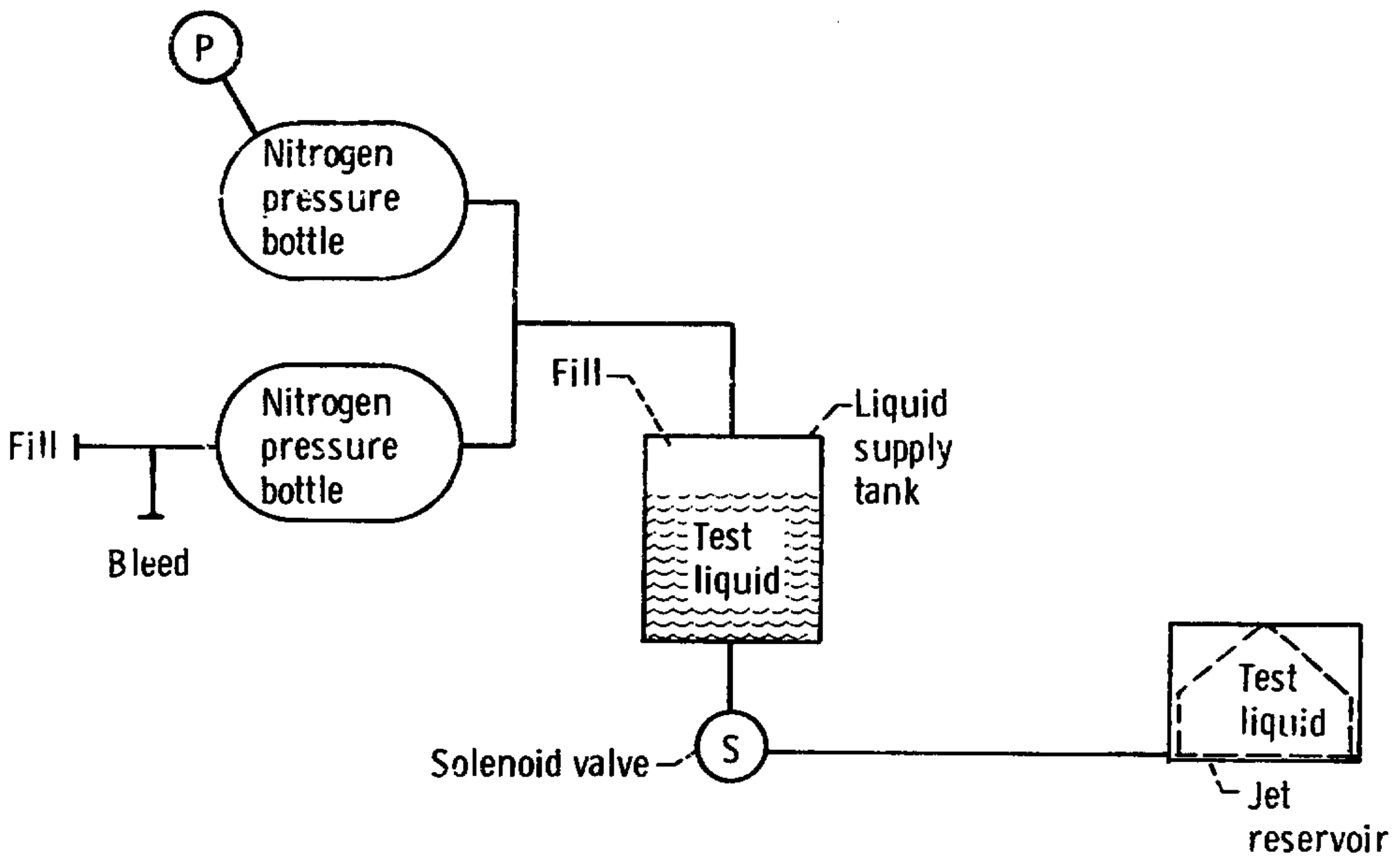


Figure 6. - Flow system schematic.

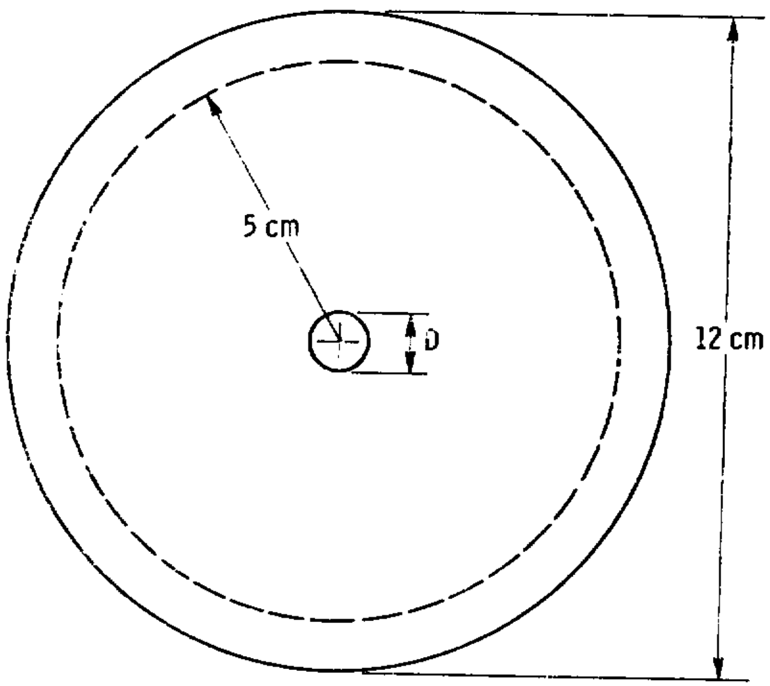


Figure 7(a). - Top view of jet reservoir.

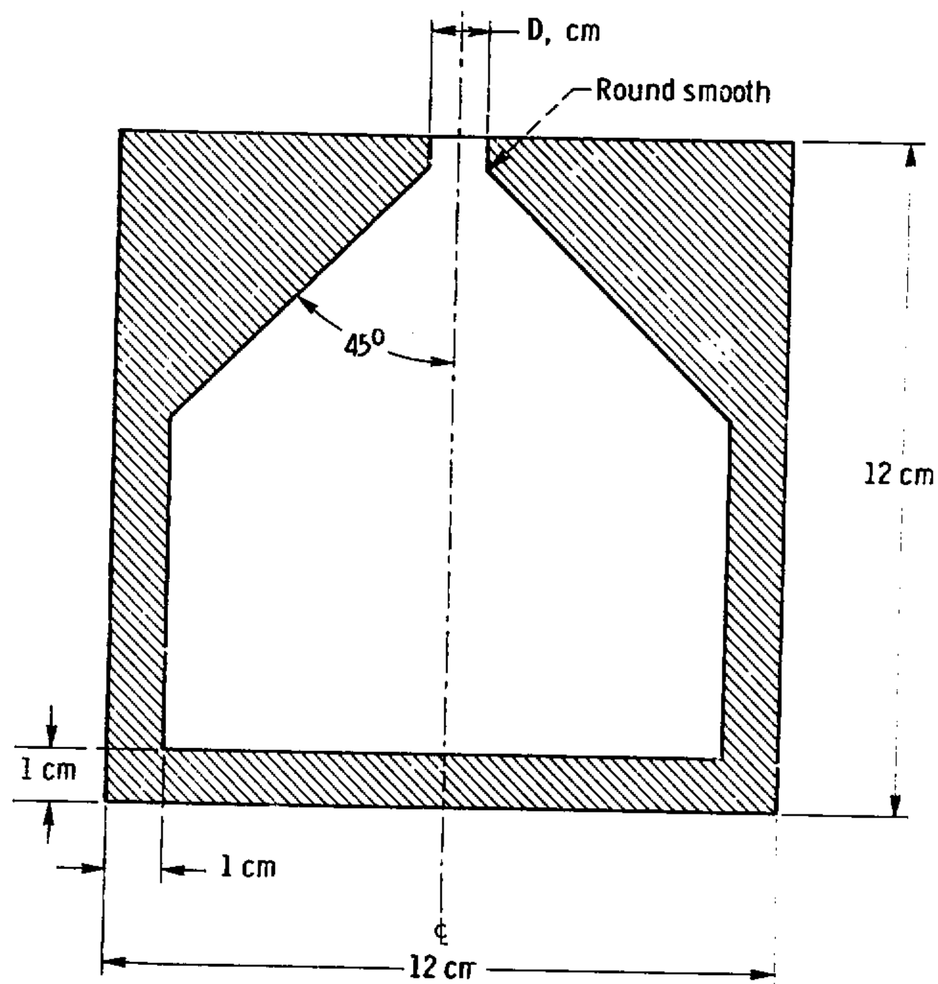


Figure 7(b). - Side view of jet reservoir.

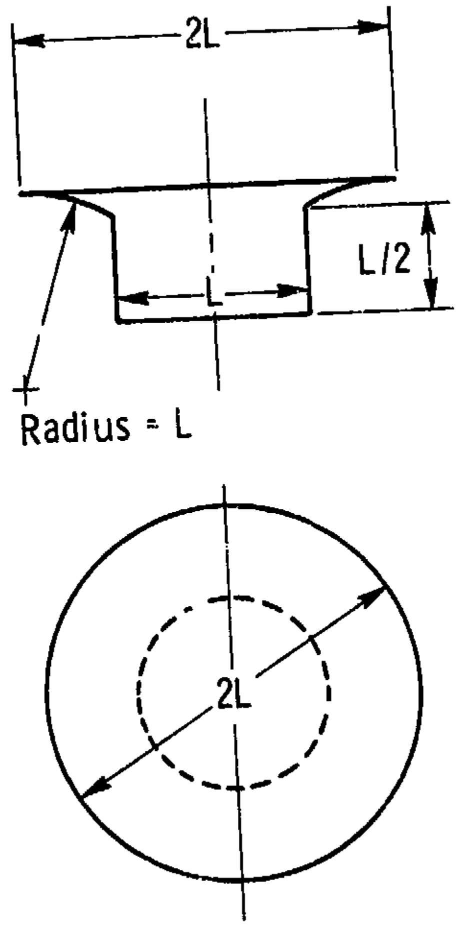


Figure 8. - Schematic diagram of sharp-edged disk.

(a) Surface tension flow. (b) Transition flow. (c) Inertia flow.

Figure 9. - Liquid jet impinging on flat solid plate.

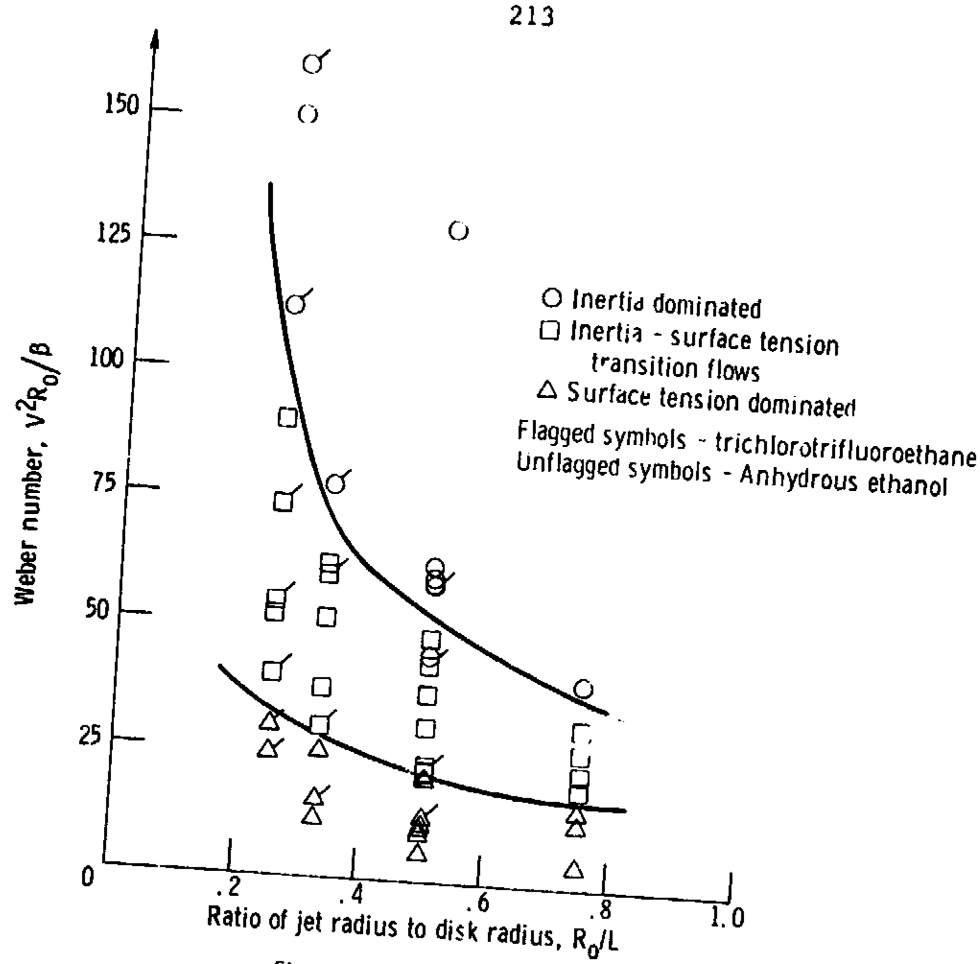


Figure 10. - Zero gravity experimental results.

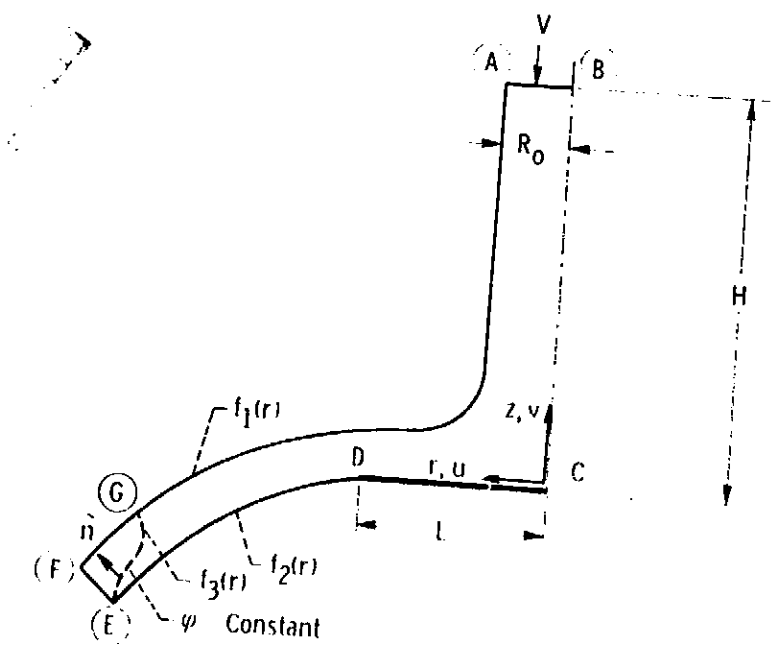


Figure 11. - Physical plane model of liquid jet impingement.

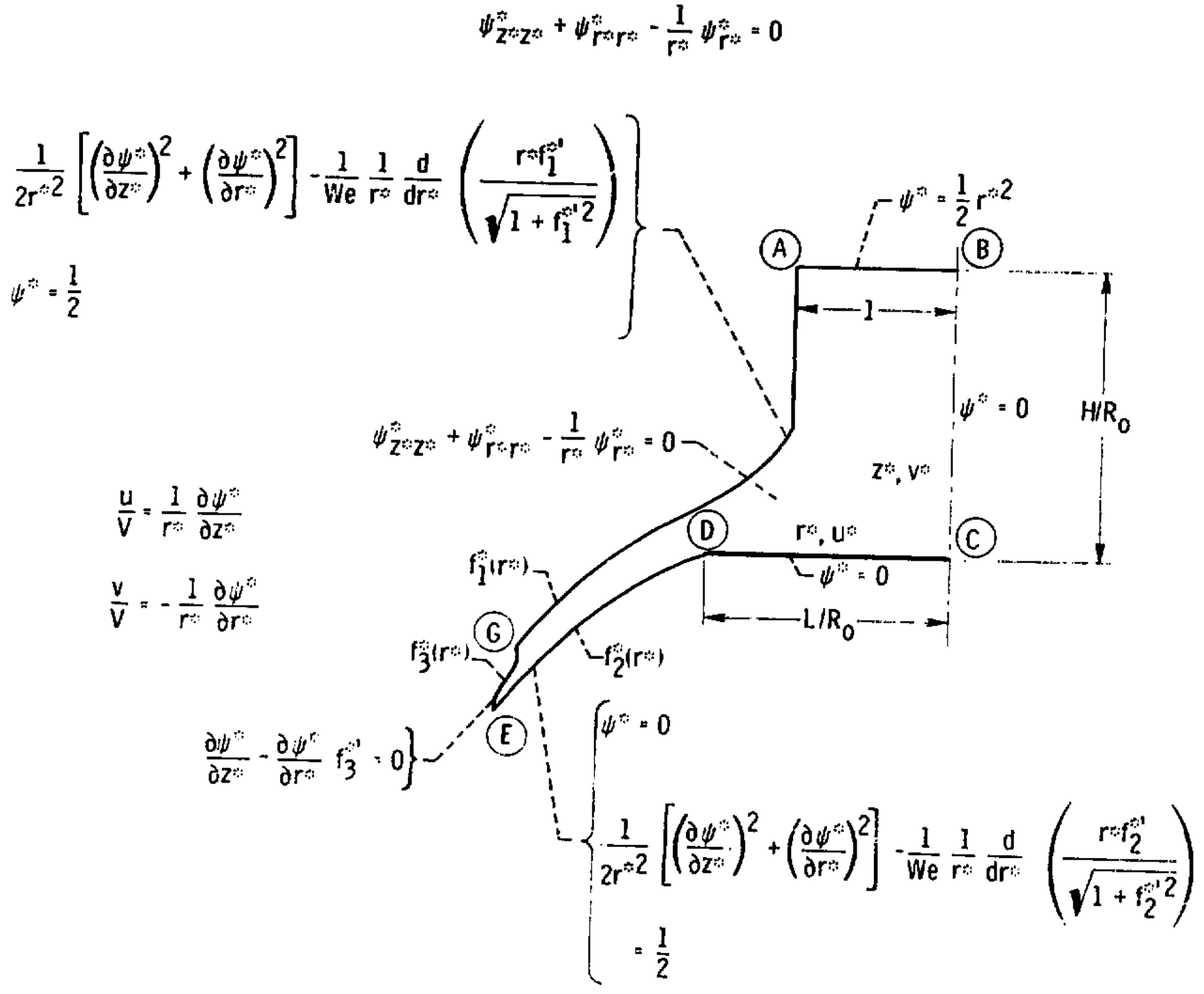


Figure 12. - Dimensionless governing equation and boundary conditions in physical plane employing Stokes' stream function.

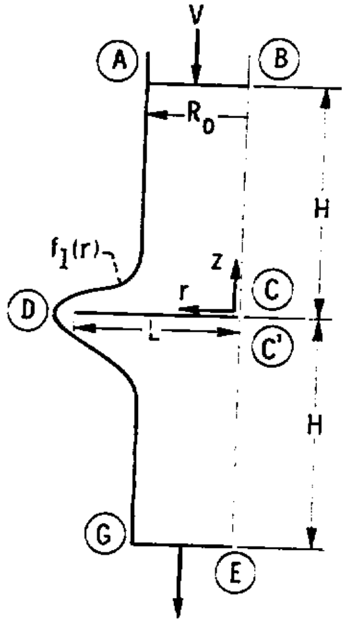


Figure 13. - Surface tension model in physical plane.

$$\psi^*_{z^*z^*} + \psi^*_{r^*r^*} - \frac{1}{r^*}\psi^*_{r^*} = 0$$

where $We = \dfrac{\rho V^2 R_0}{\sigma}$

$$u^* = \frac{1}{r^*}\frac{\partial \psi^*}{\partial z^*}$$

$$v^* = -\frac{1}{r^*}\frac{\partial \psi^*}{\partial r^*}$$

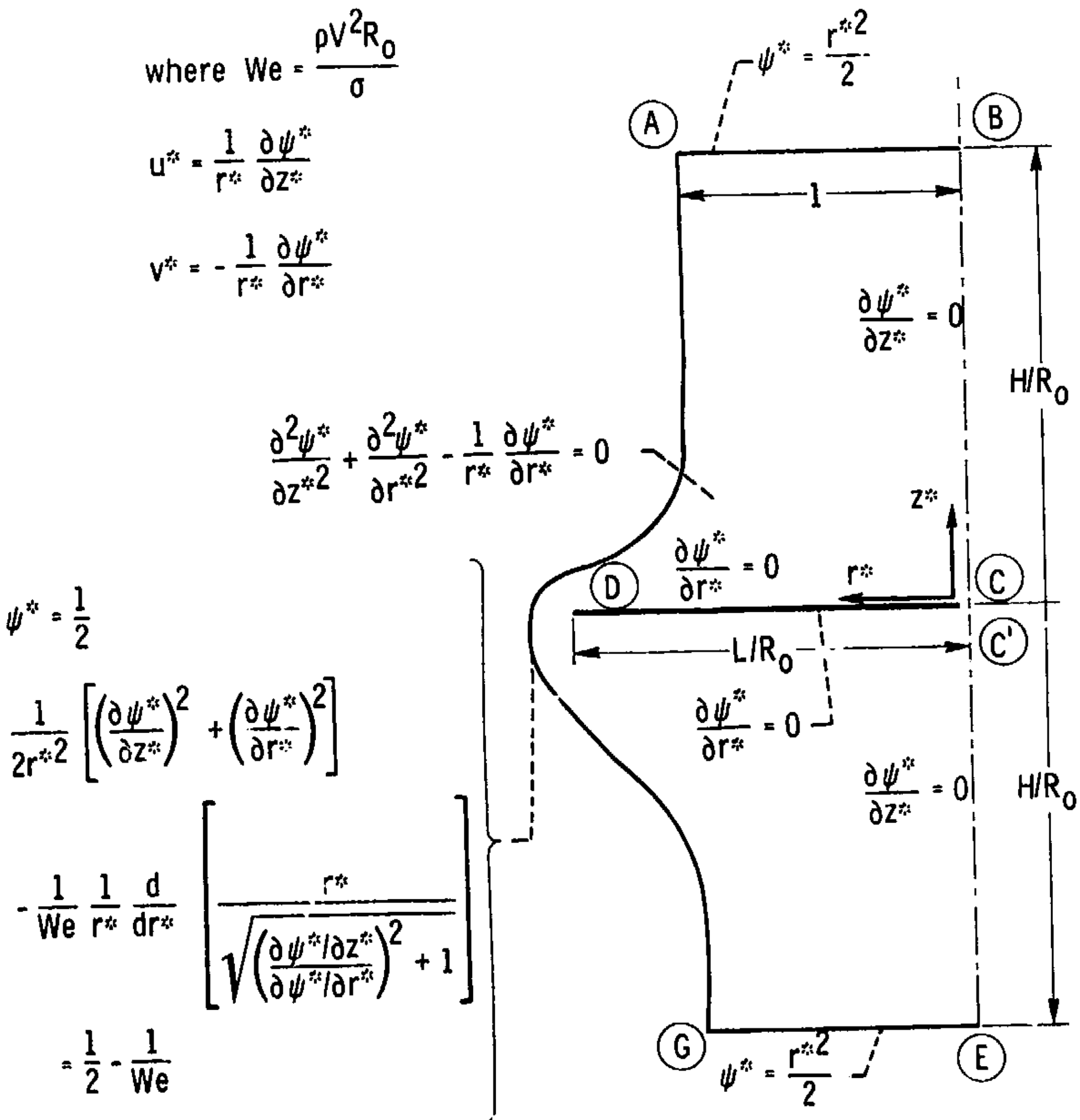


Figure 14. - Dimensionless governing equations and boundary conditions.

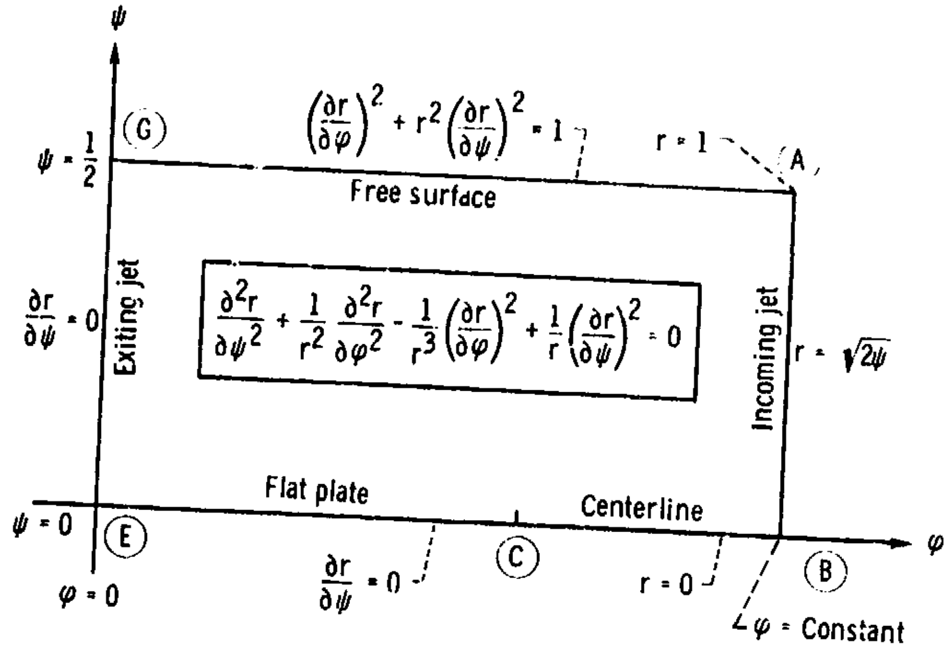


Figure 15(a). - Inverse formulation excluding surface tension (r solution) for infinite plate.

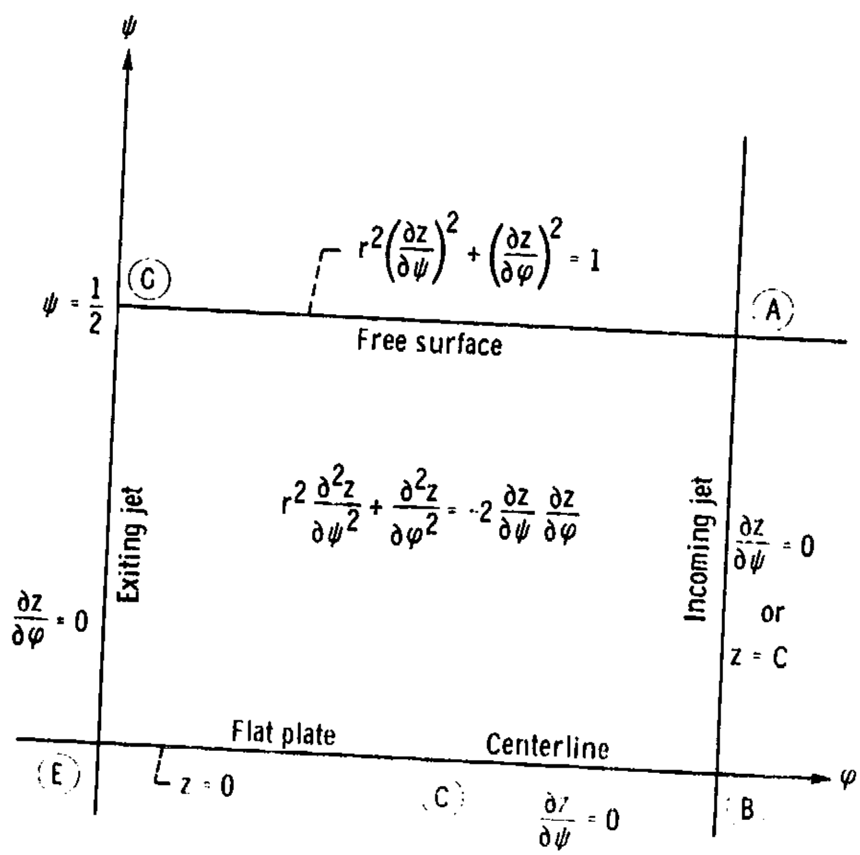


Figure 15(b). - Inverse formulation excluding surface tension (z solution) for infinite plate.

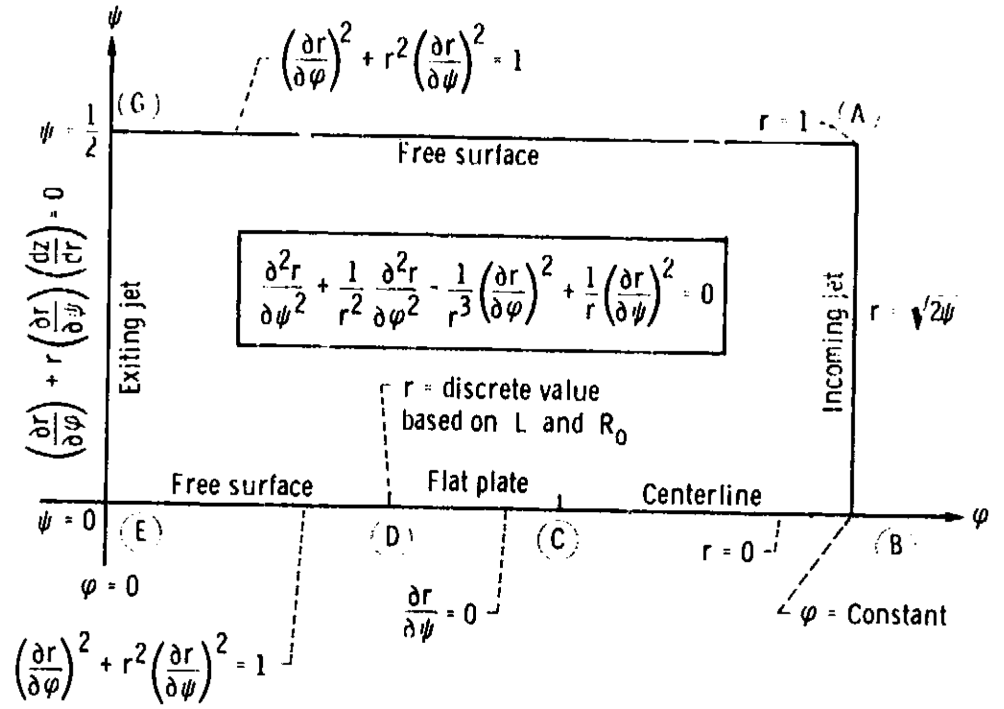


Figure 16(a). - Inverse formulation excluding surface tension (r solution) for finite plate.

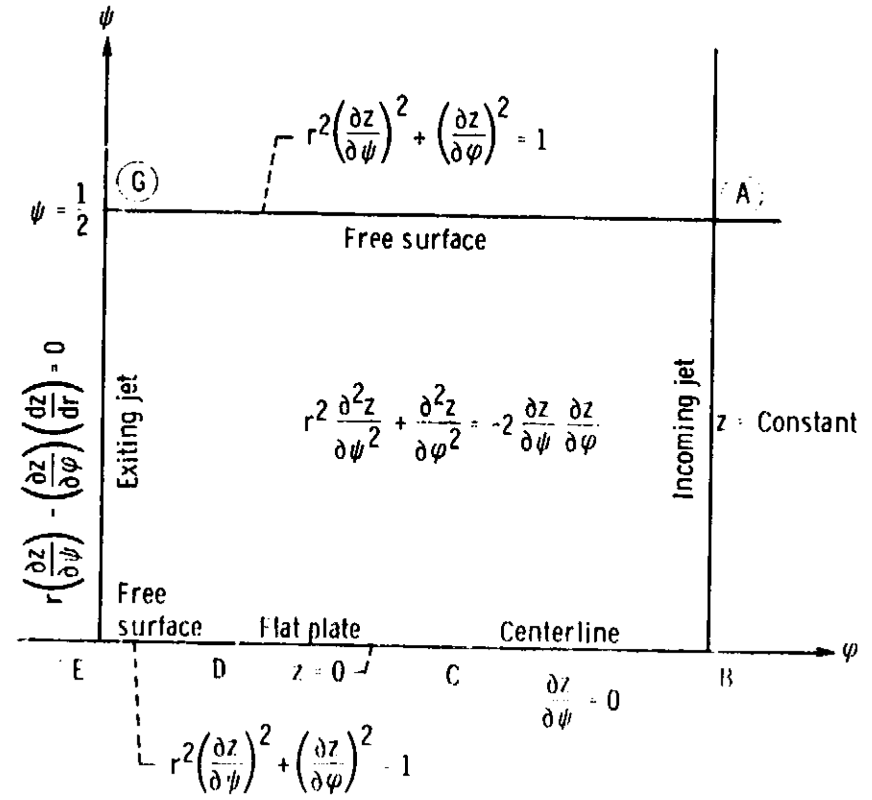


Figure 16(b). - Inverse formulation excluding surface tension (z solution) for finite plate.

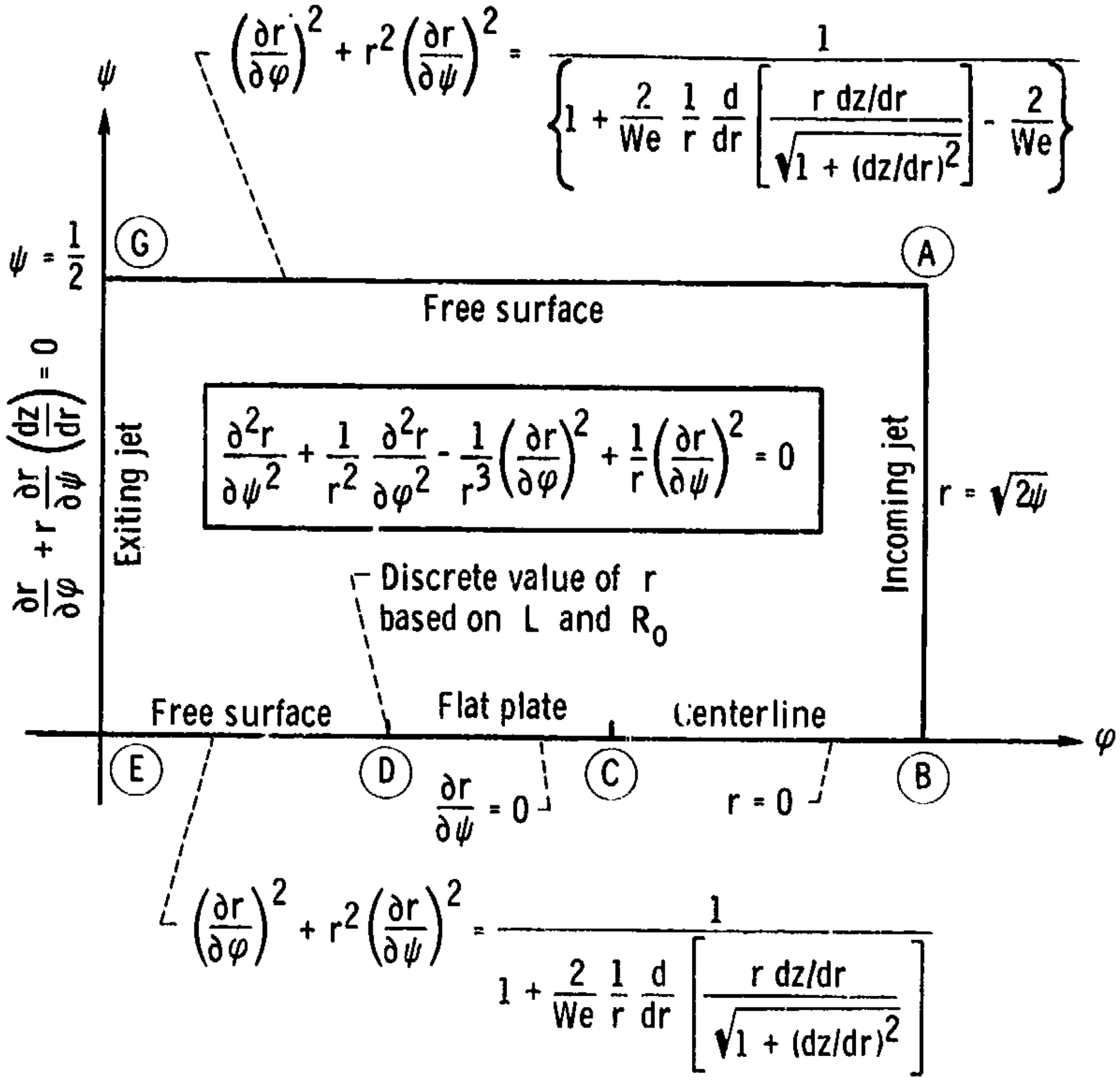


Figure 17. - Inverse formulation including surface tension (r solution) for finite plate.

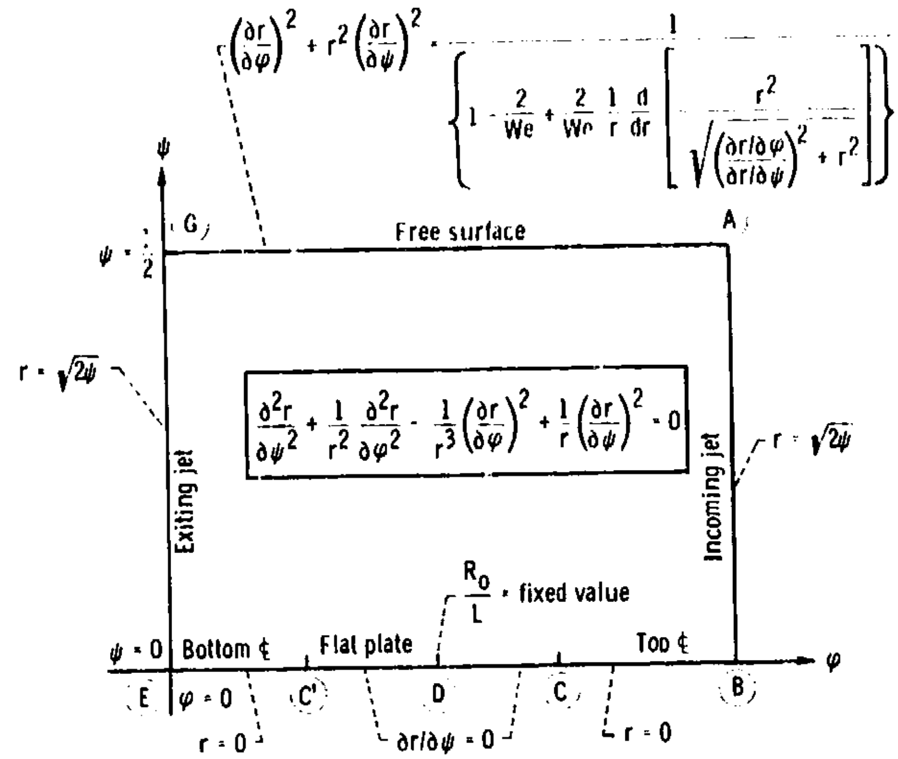


Figure 18(a). - Surface tension model - inverse formulation (r solution).

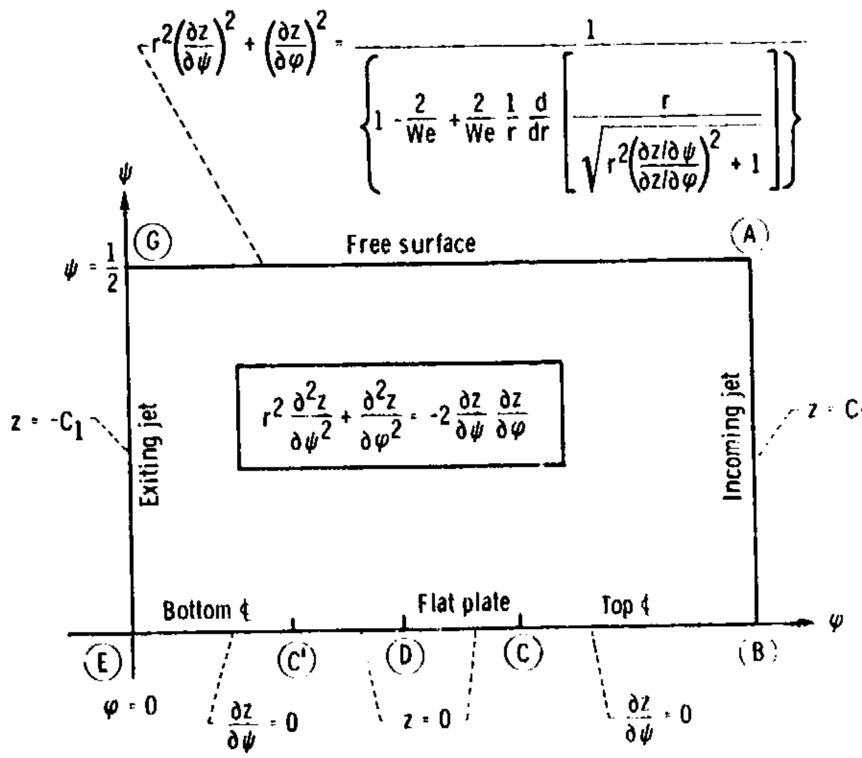


Figure 18(b). - Surface tension model - inverse formulation (z solution).

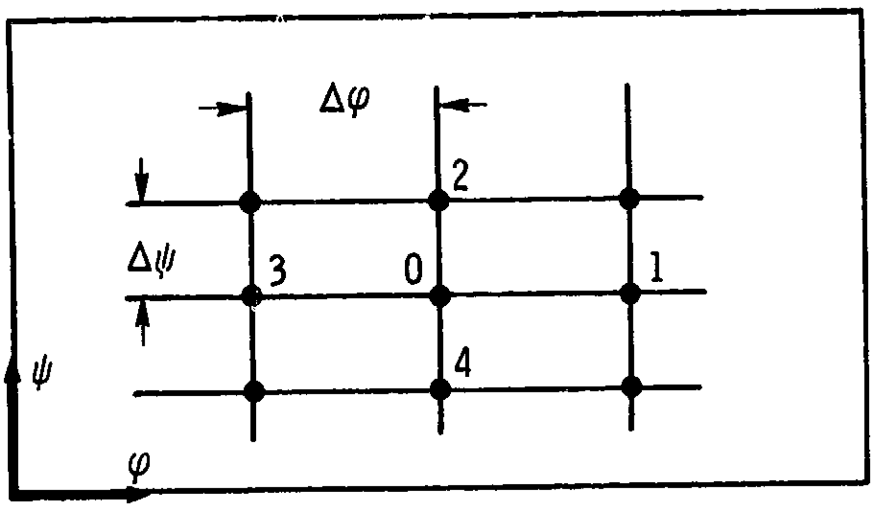


Figure 19. - Nodal point representation for rectangular mesh.

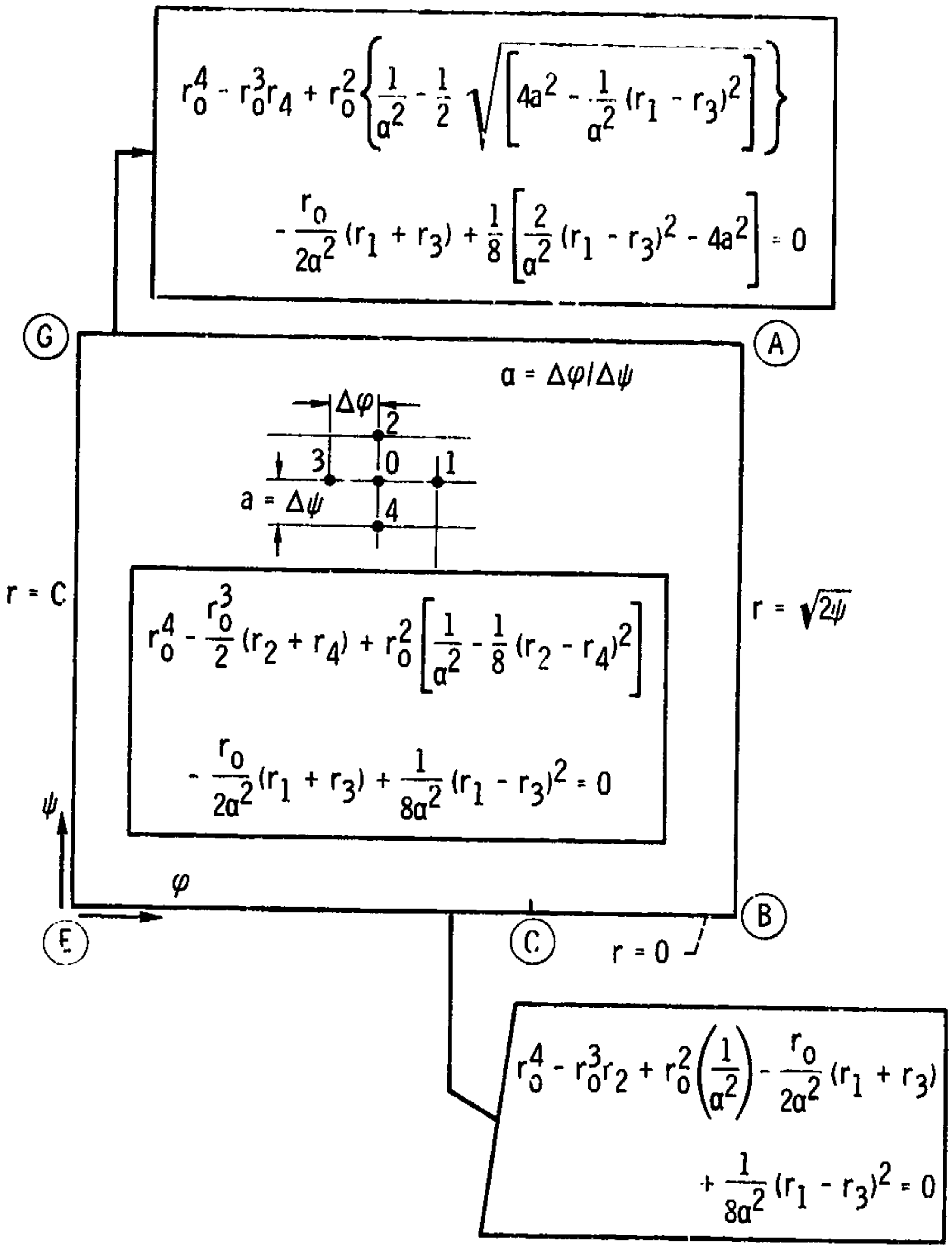


Figure 20(a). - Finite difference representation for infinite plate (r solution).

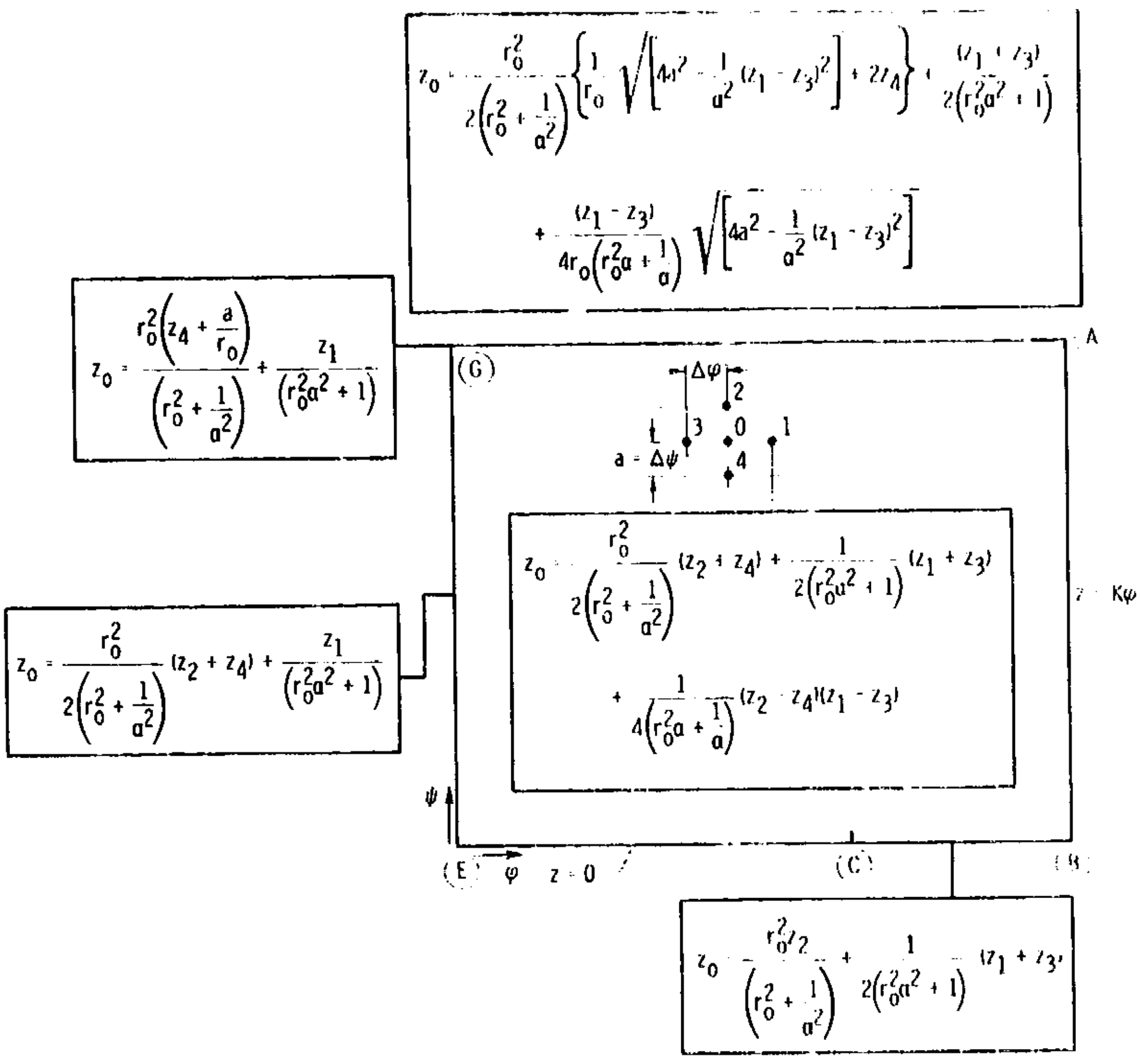


Figure 20(b). - Finite difference representation for infinite plate (z solution).

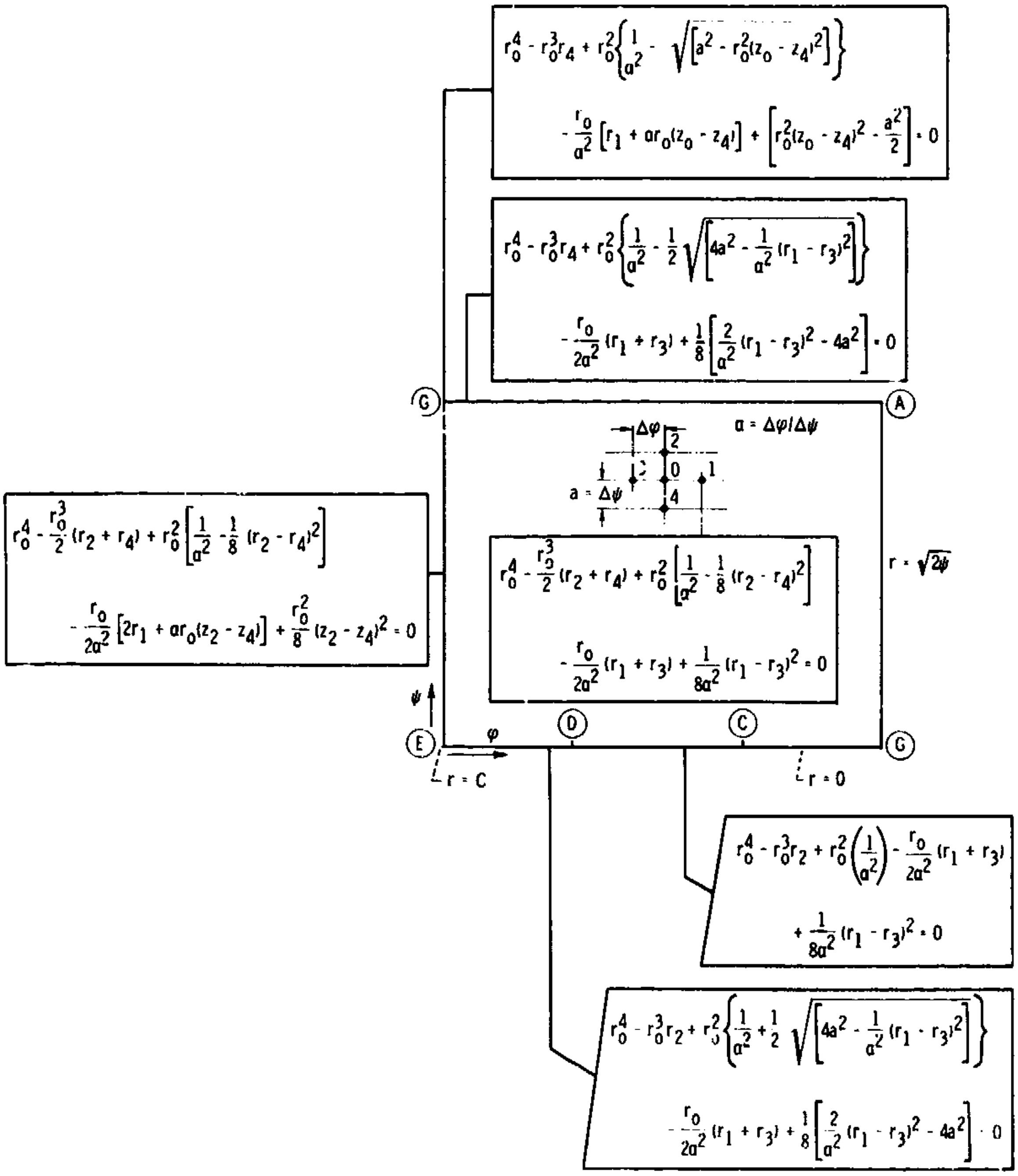
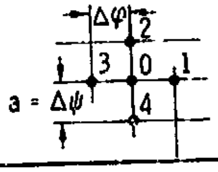


Figure 21(a). - Finite difference representation for finite plate (r solution).

$$z_0 = \frac{r_0^2}{2\left(r_0^2 + \frac{1}{\alpha^2}\right)}\left\{\frac{1}{r_0}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]} + 2z_4\right\} + \frac{(z_1 + z_3)}{2\left(r_0^2\alpha^2 + 1\right)}$$

$$+ \frac{(z_1 - z_3)}{4r_0\left(r_0^2\alpha + \frac{1}{\alpha}\right)}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]}$$

(G) (A)



$$z_0 = \frac{r_0^2}{2\left(r_0^2 + \frac{1}{\alpha^2}\right)}(z_2 + z_4) + \frac{1}{2\left(r_0^2\alpha^2 + 1\right)}(z_1 + z_3)$$

$$+ \frac{1}{4\left(r_0^2\alpha + \frac{1}{\alpha}\right)}(z_2 - z_4)(z_1 - z_3)$$

$z = K\varphi$

$$z_0 = \frac{r_0^2(z_2 + z_4)}{2\left(r_0^2 + \frac{1}{\alpha^2}\right)}$$

$$+ \frac{\left[2z_1 - r_0\alpha(r_2 - r_4)\right]}{2\left(r_0^2\alpha^2 + 1\right)}$$

$$+ \frac{(z_2 - z_4)\left[r_0\alpha(r_2 - r_4)\right]}{4\left(r_0^2\alpha + \frac{1}{\alpha}\right)}$$

ψ

(D) (C) (B)

(E) $\rightarrow \varphi$

$z = 0$

$$z_0 = \frac{r_0^2}{2\left(r_0^2 + \frac{1}{\alpha^2}\right)}\left\{2z_2 - \frac{1}{r_0}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]}\right\}$$

$$+ \frac{1}{2\left(r_0^2\alpha^2 + 1\right)}(z_1 + z_3)$$

$$+ \frac{(z_1 - z_3)}{4r_0\left(r_0^2\alpha + \frac{1}{\alpha}\right)}\sqrt{\left[4a^2 - \frac{1}{\alpha^2}(z_1 - z_3)^2\right]}$$

$$z_0 = \frac{r_0^2 z_2}{\left(r_0^2 + \frac{1}{\alpha^2}\right)} + \frac{1}{2\left(r_0^2\alpha^2 + 1\right)}(z_1 + z_3)$$

Figure 21(b). - Finite difference representation for finite plate (z solution).

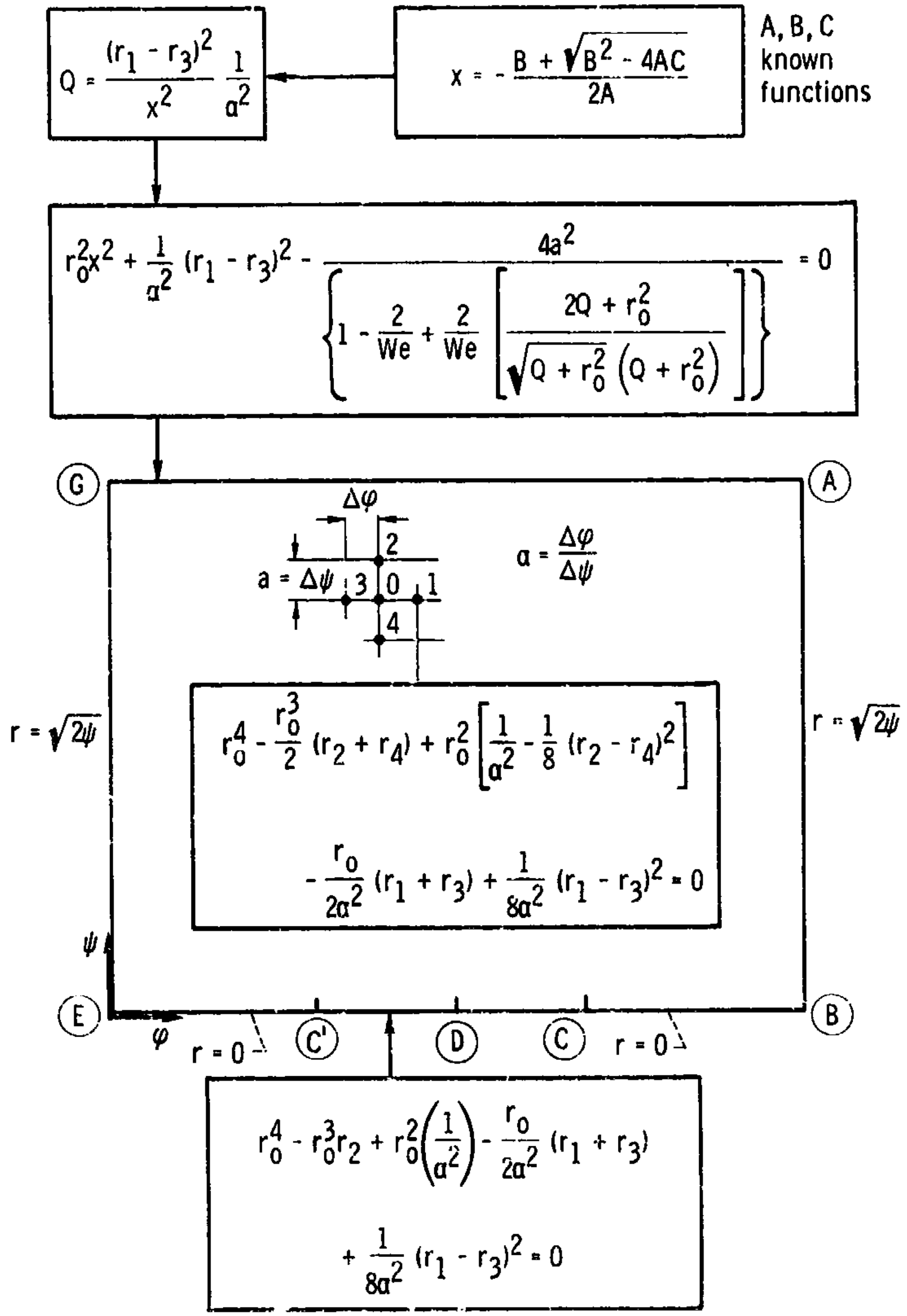


Figure 22(a). - Finite difference representation for surface tension dominated model (r solution).

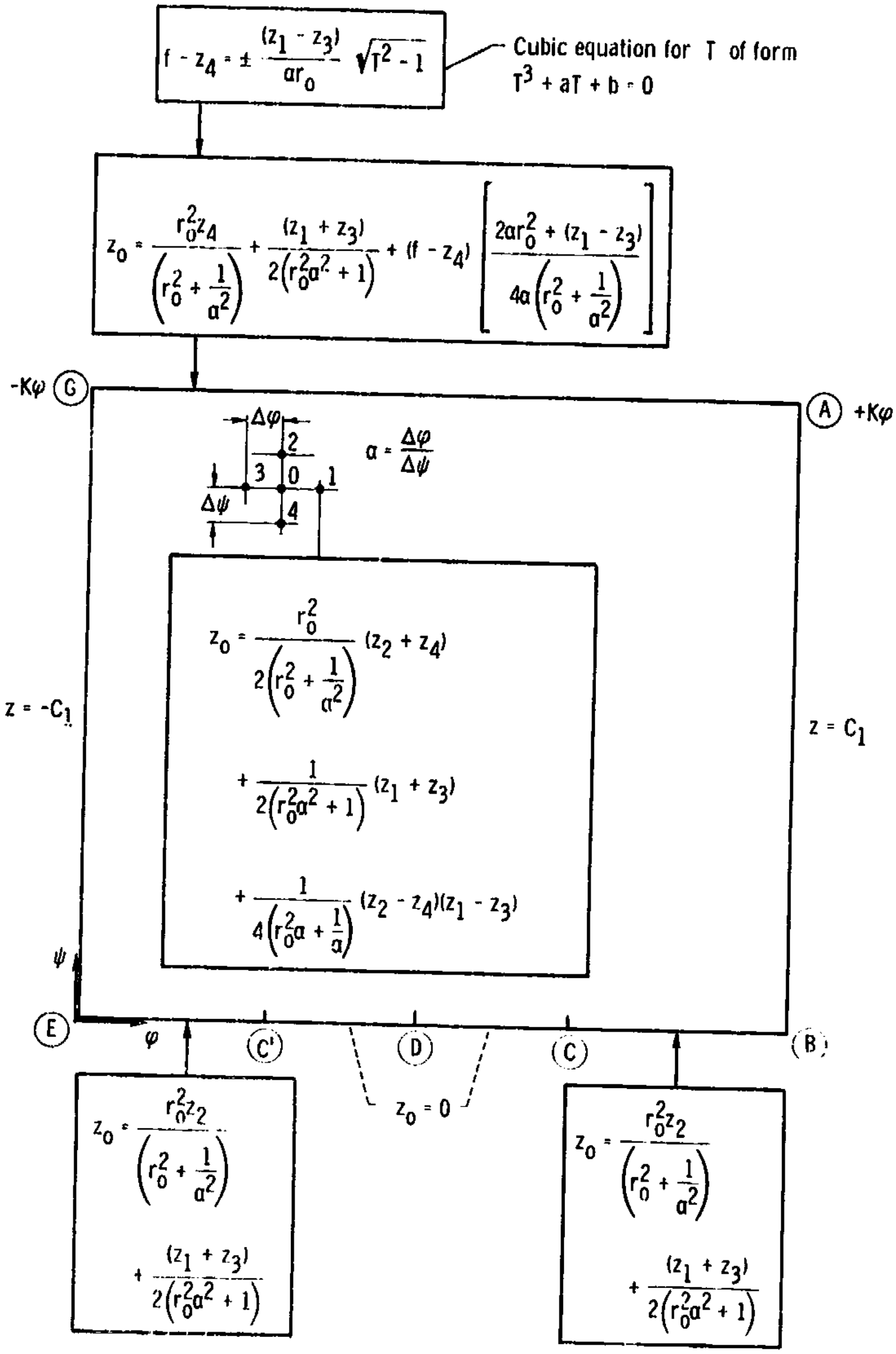


Figure 22(b). - Finite difference representation for surface tension dominated model (z solution).

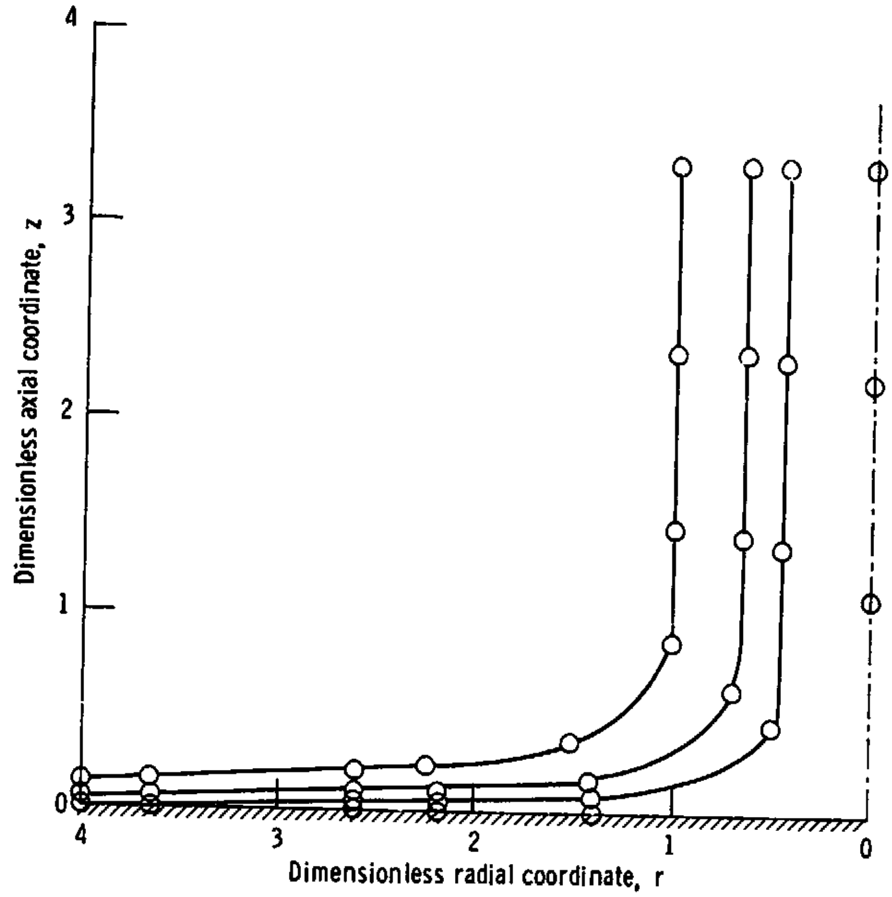


Figure 23. - Numerical solution of liquid jet impinging on infinite flat plate (coarse mesh).

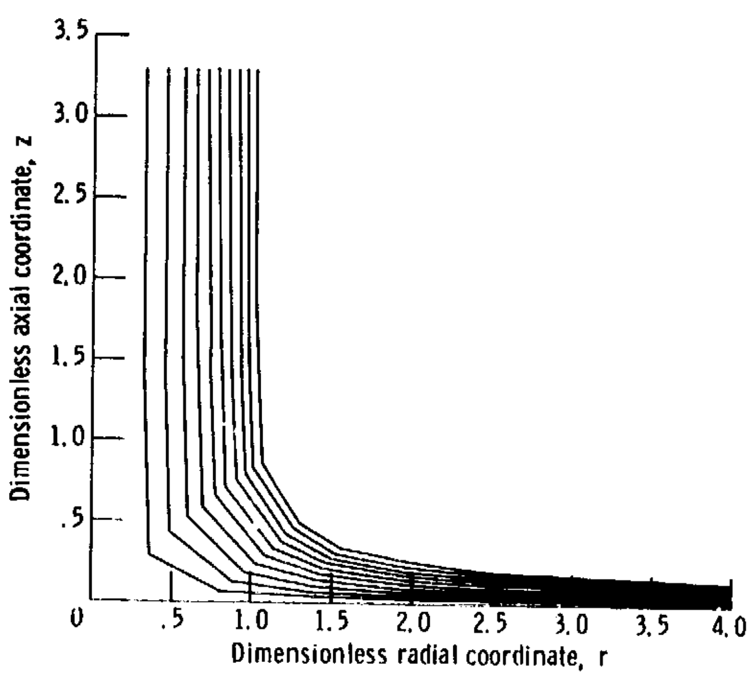


Figure 24. - Print-plot of liquid jet impinging on infinite flat plate (fine mesh).

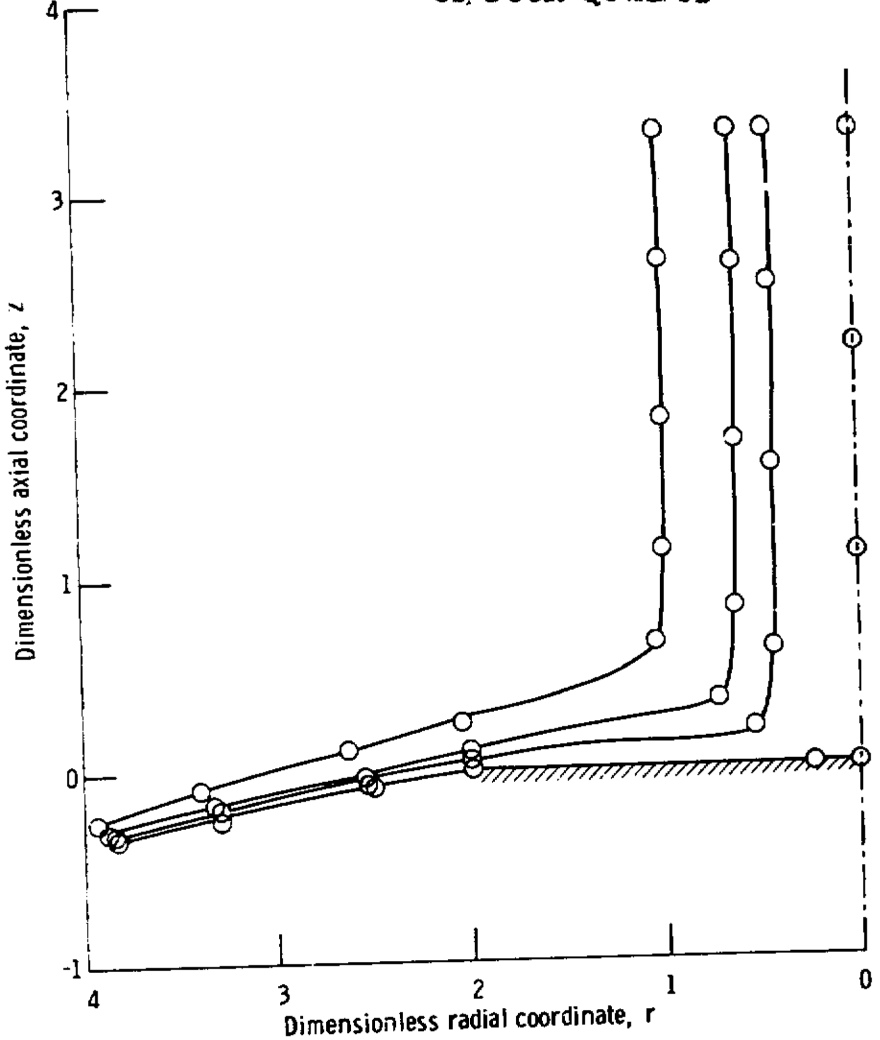


Figure 25. - Numerical solution of liquid jet impinging on finite plate (R_0/L = 1/2).

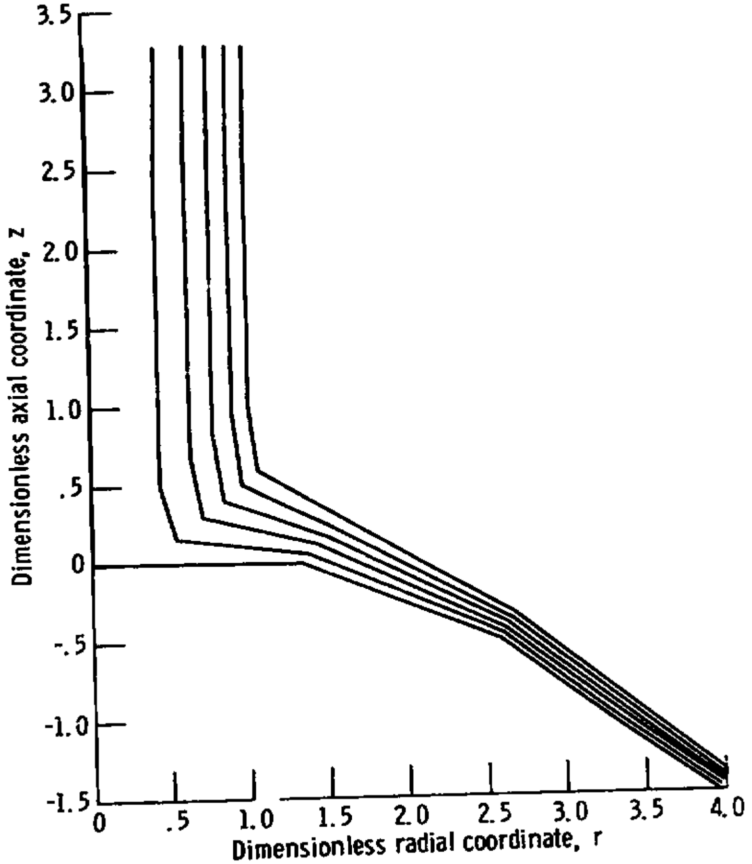


Figure 26. - Print-plot of numerical solution for impingement on a finite plate (R_0/L = 3/4).

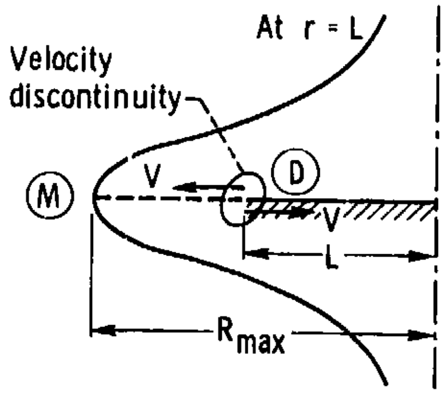


Figure 27. - Schematic diagram of velocity discontinuity occurring in surface tension model.

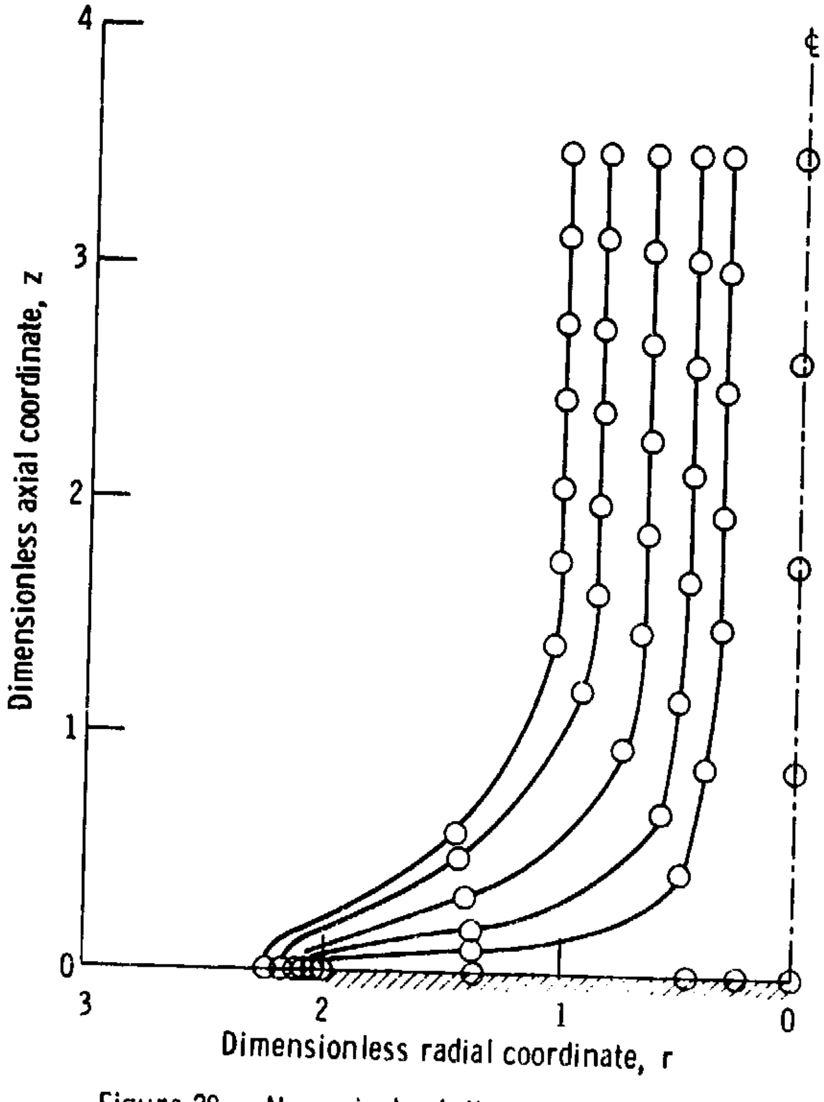


Figure 28. - Numerical solution of surface tension dominated model (We = 4, R_0/L = 1/2).

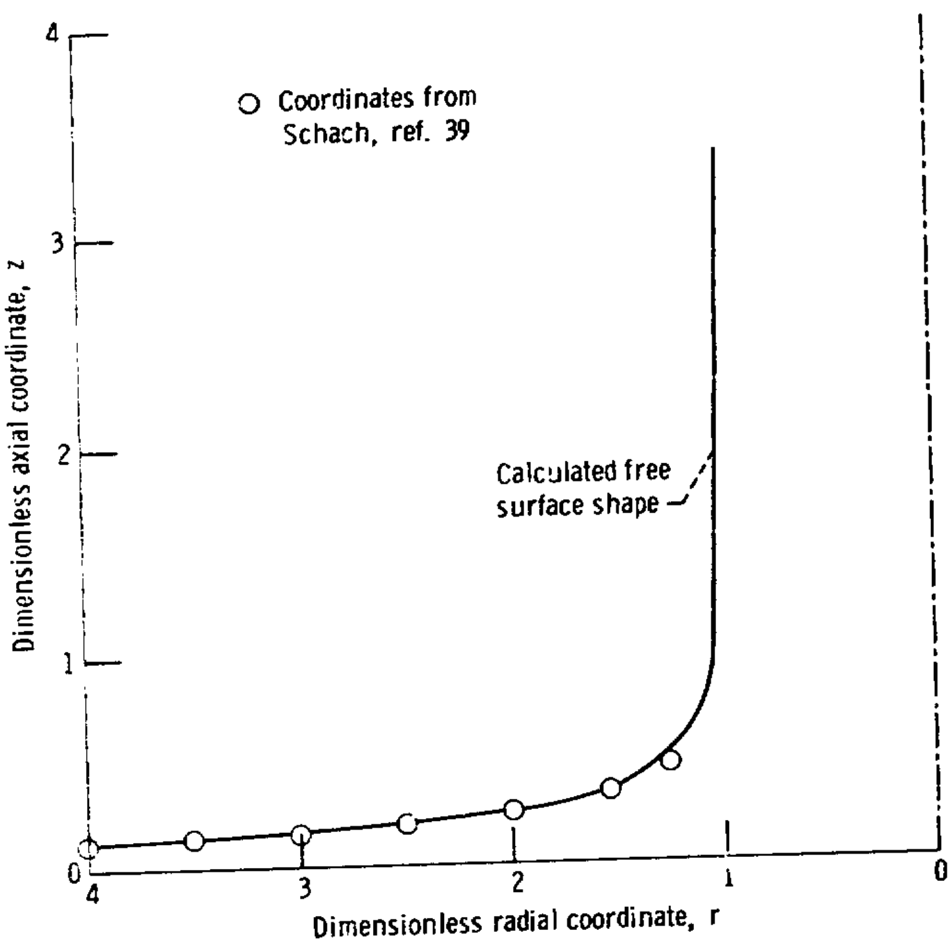


Figure 29. - Comparison of numerical results for infinite plate with reference 39.

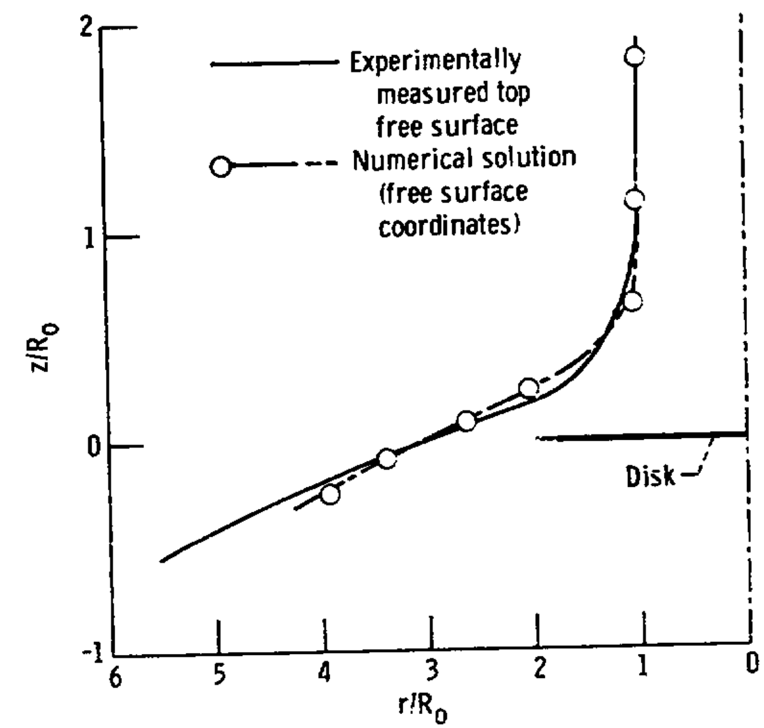


Figure 30. - Comparison of numerical results for finite plate - inertial flow with experiments ($R_0/L = 1/2$).

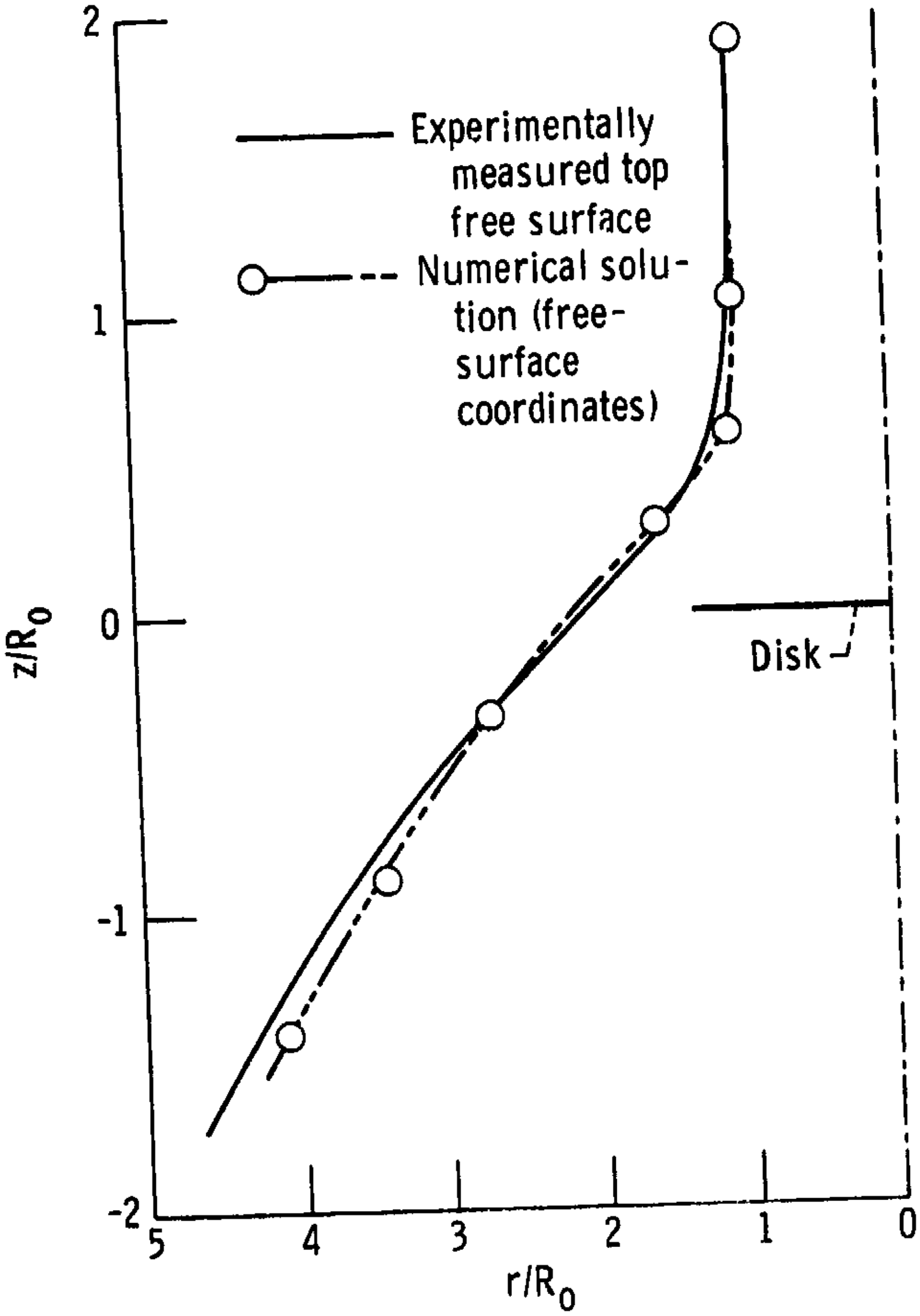


Figure 31. - Comparison of numerical results for finite plate - inertial flow with experiments (R_0/L = 3/4).

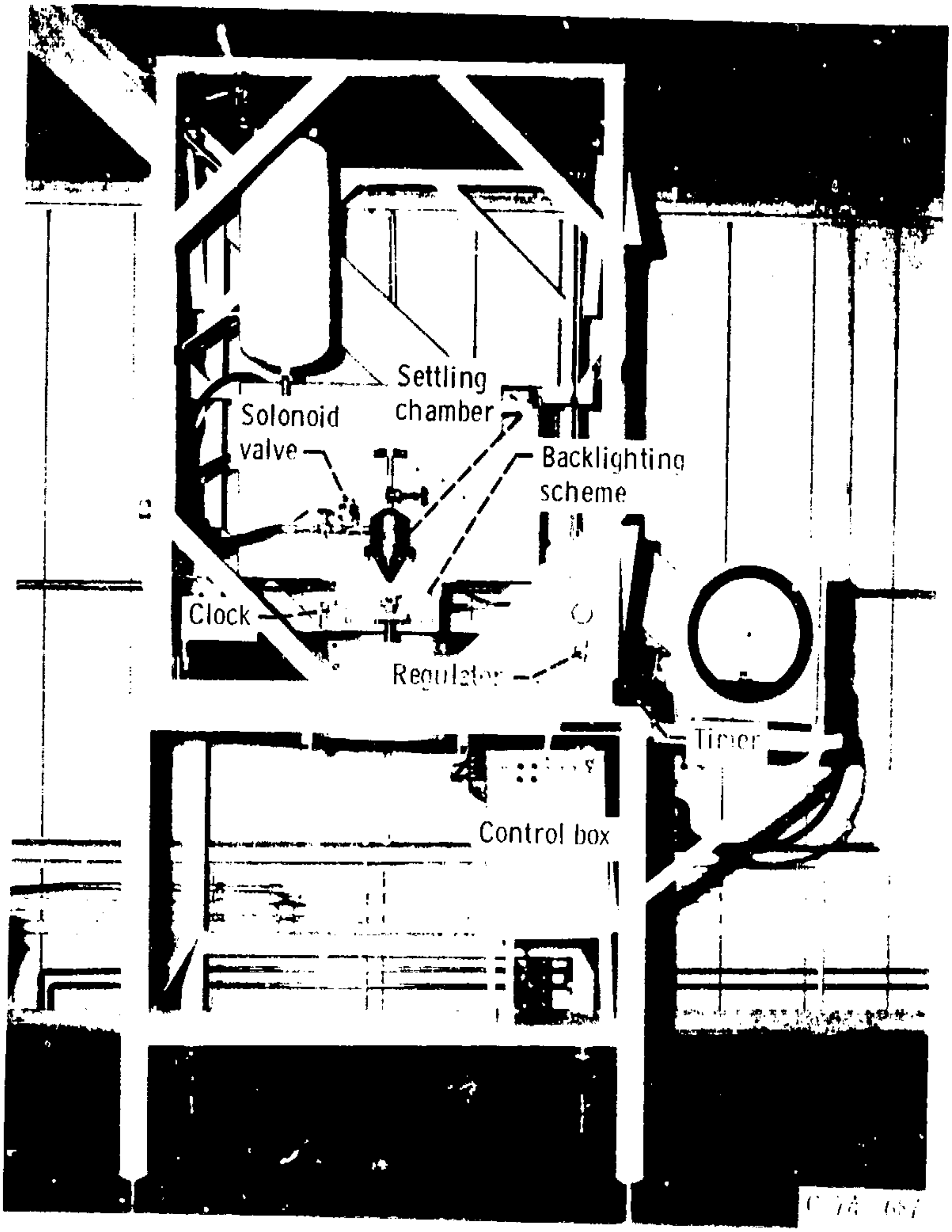


Figure 32(a). - Experimental test rig - front view.

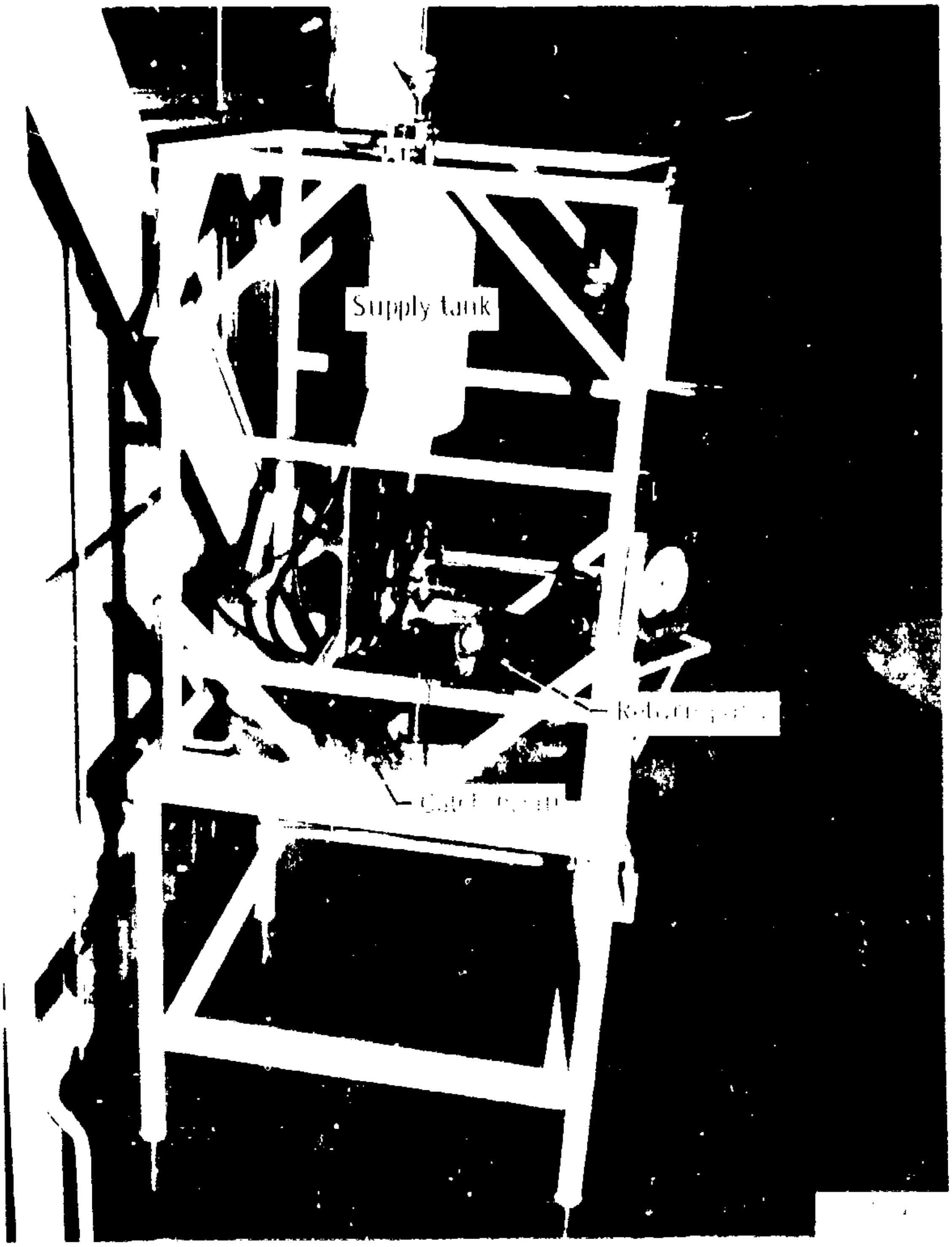


Figure 32(b). - Experimental test rig - side view.

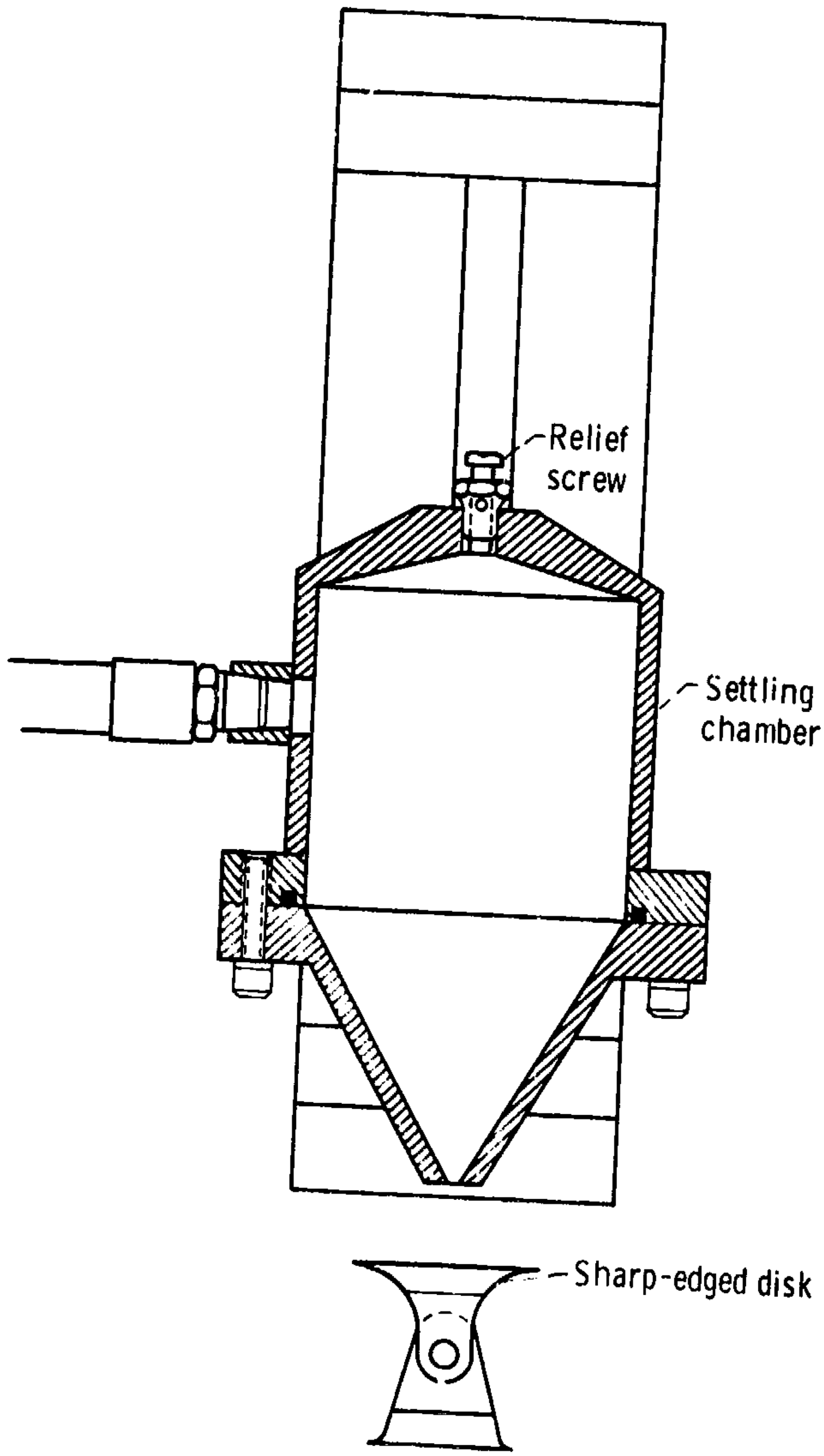


Figure 33. - Cross-section of settling chamber.

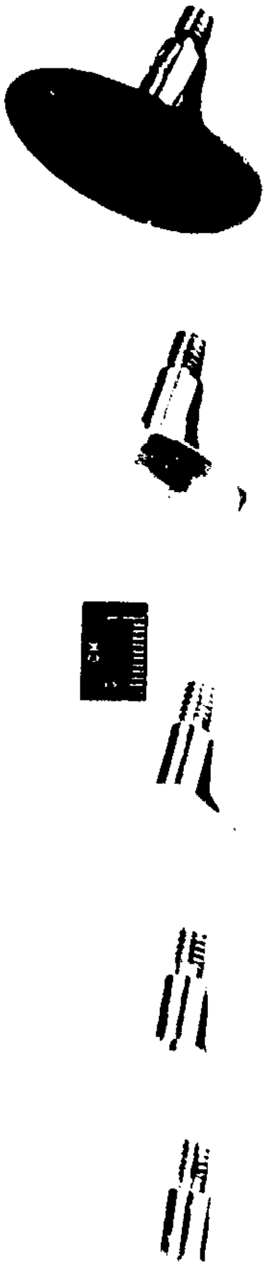


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Figure 34. - Photograph of sharp edge disks.

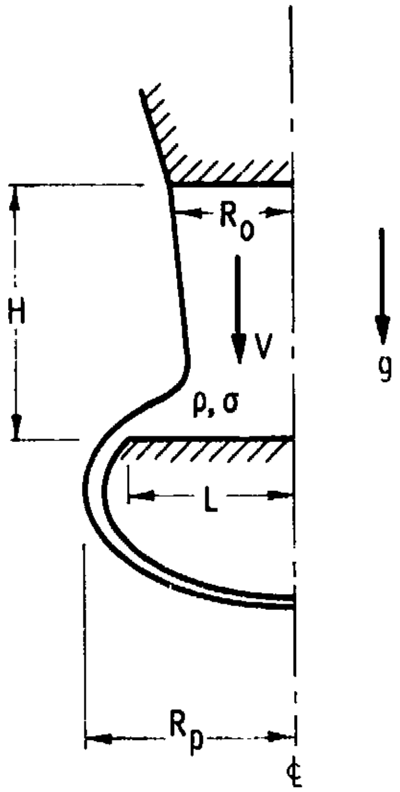
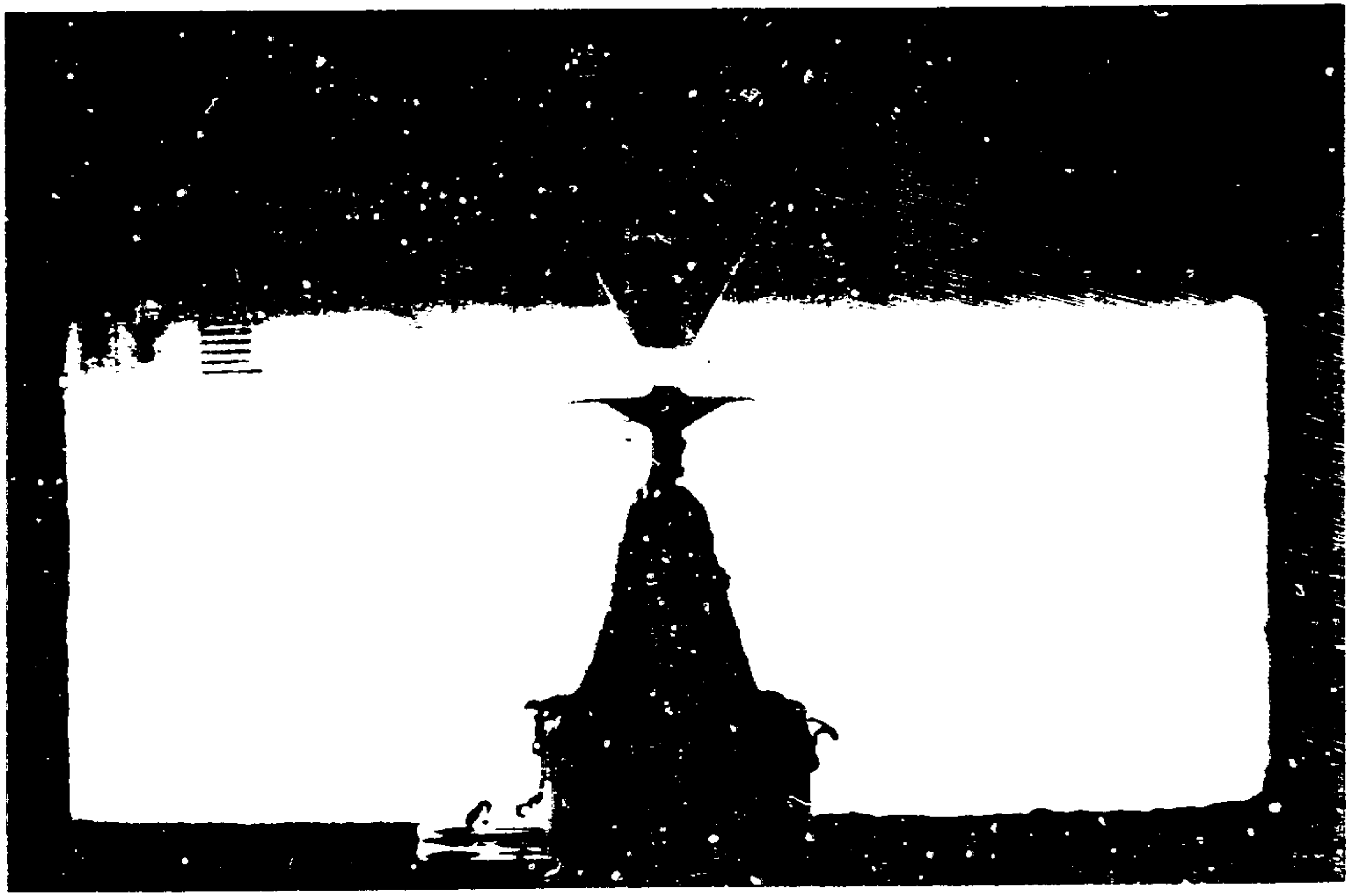
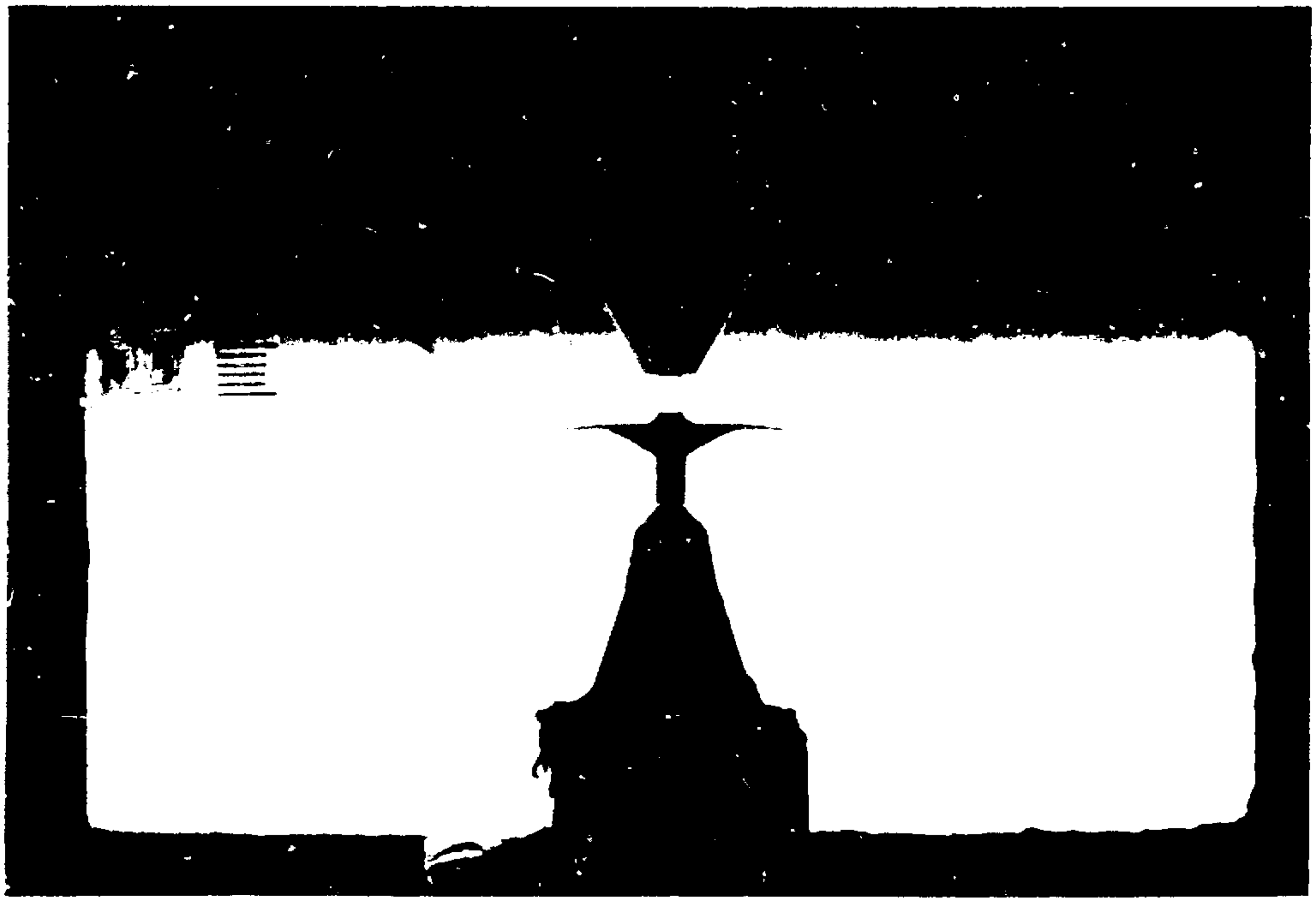


Figure 35. - Schematic of steady state normal gravity impingement.



(a) Disc radius, 3.0 cm.

Figure 36. - Test liquid, distilled water; nozzle diameter, 0.5 cm; jet velocity, 286 cm/sec; distance between nozzle and disk, 1 cm.

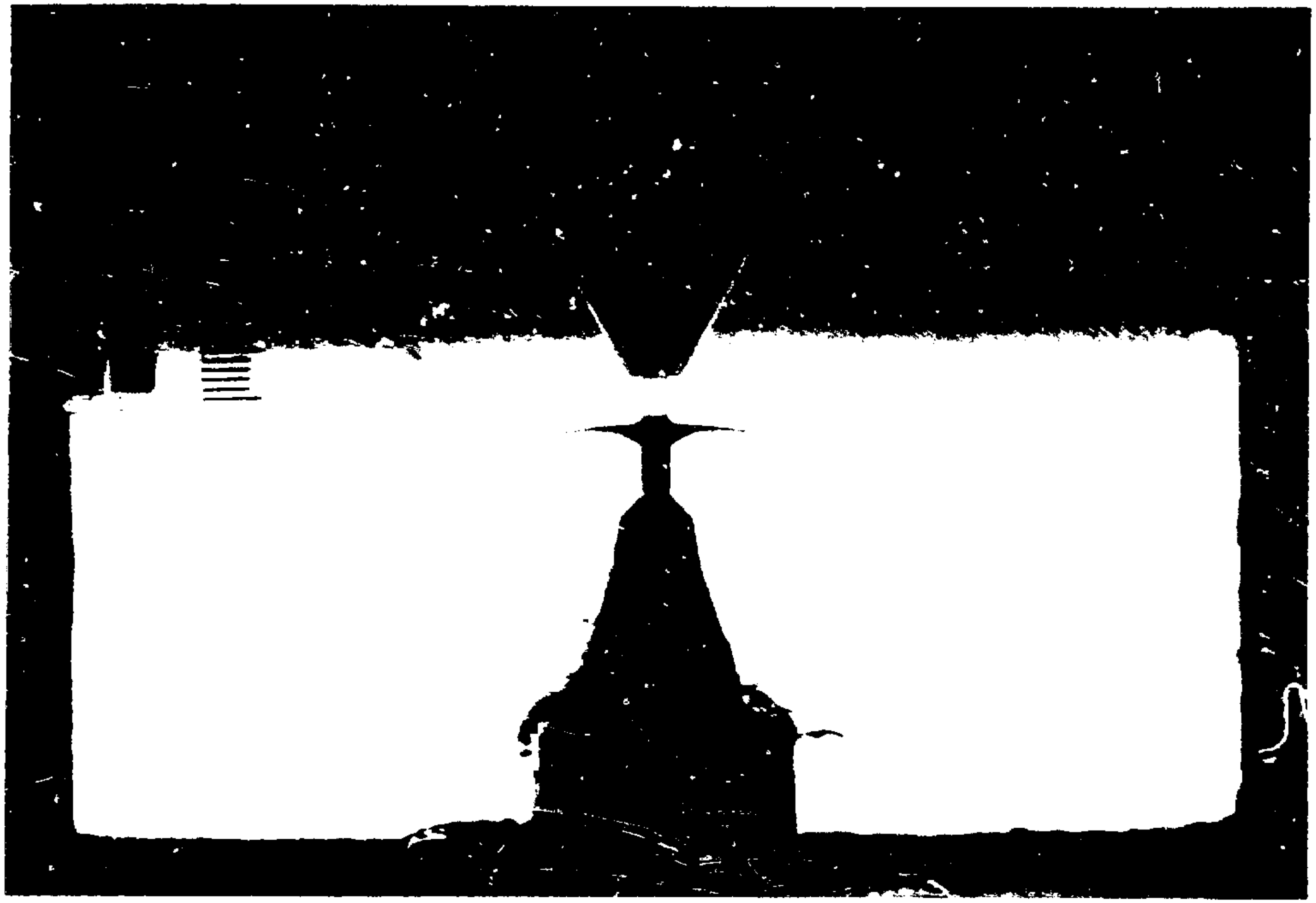


(b) Disk radius, 3.0 cm.

Figure 36. - Continued.

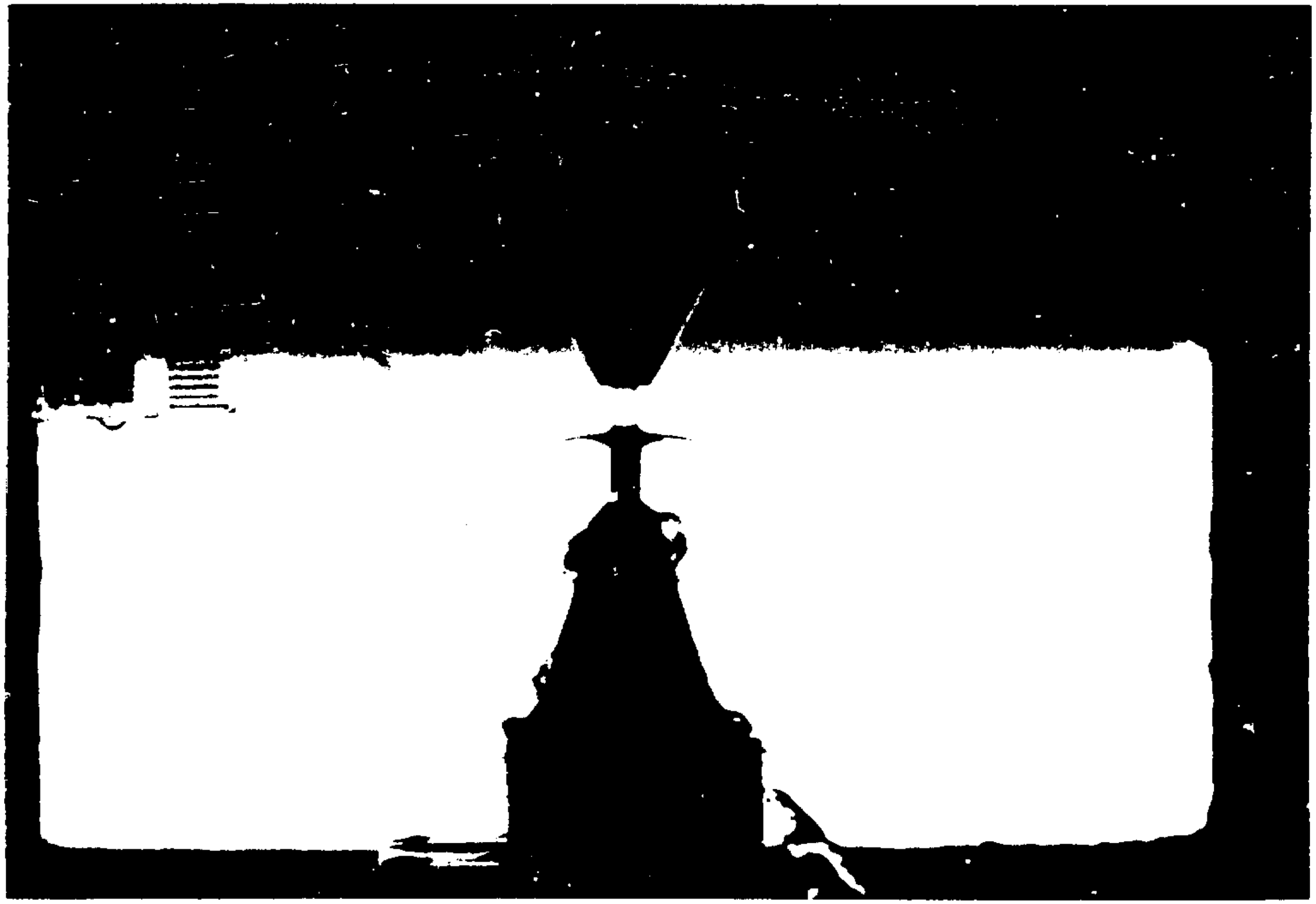
(c) Disk radius, 2.0 cm.

Figure 36. - Continued.



(d) Disk radius, 2.0 cm.

Figure 36. - Continued.



(e) Disk radius, 1.5 cm.

Figure 36. - Continued.

(f) Disk radius, 1.5 cm.

Figure 36. - Concluded.

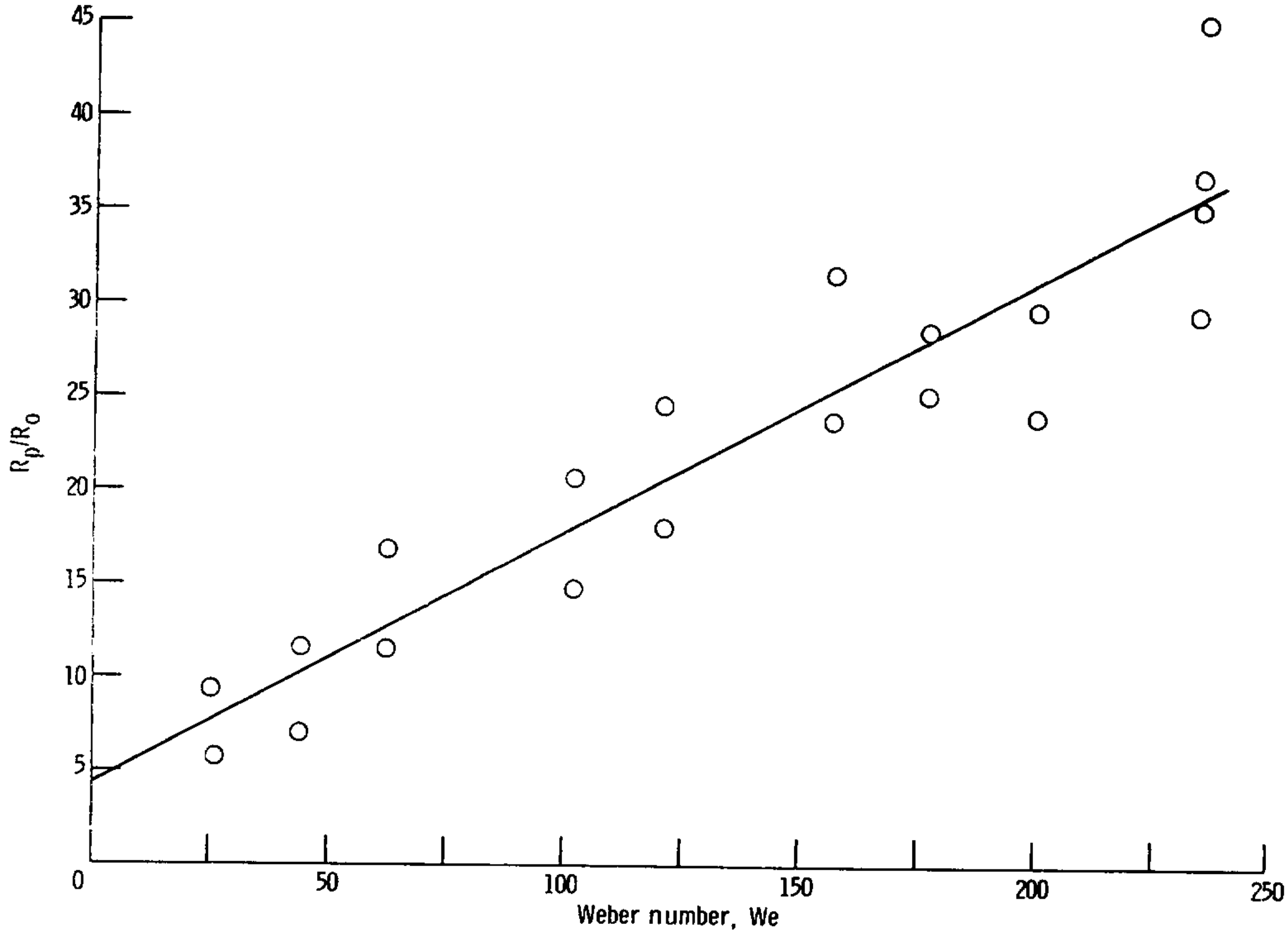


Figure 37. - Variation of plume width with Weber number $R_p/R_o = 4.3968 + 0.1334\ We$.

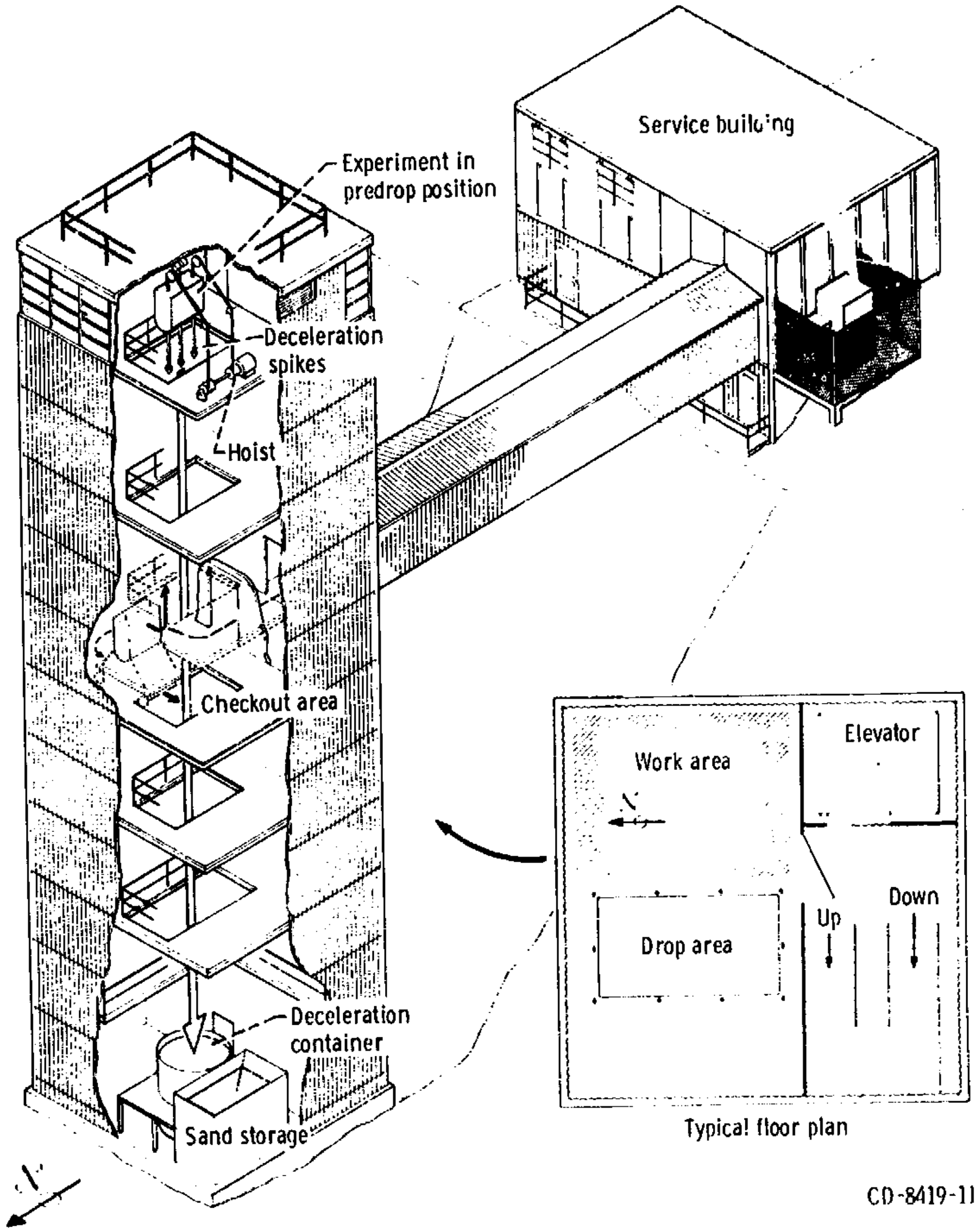
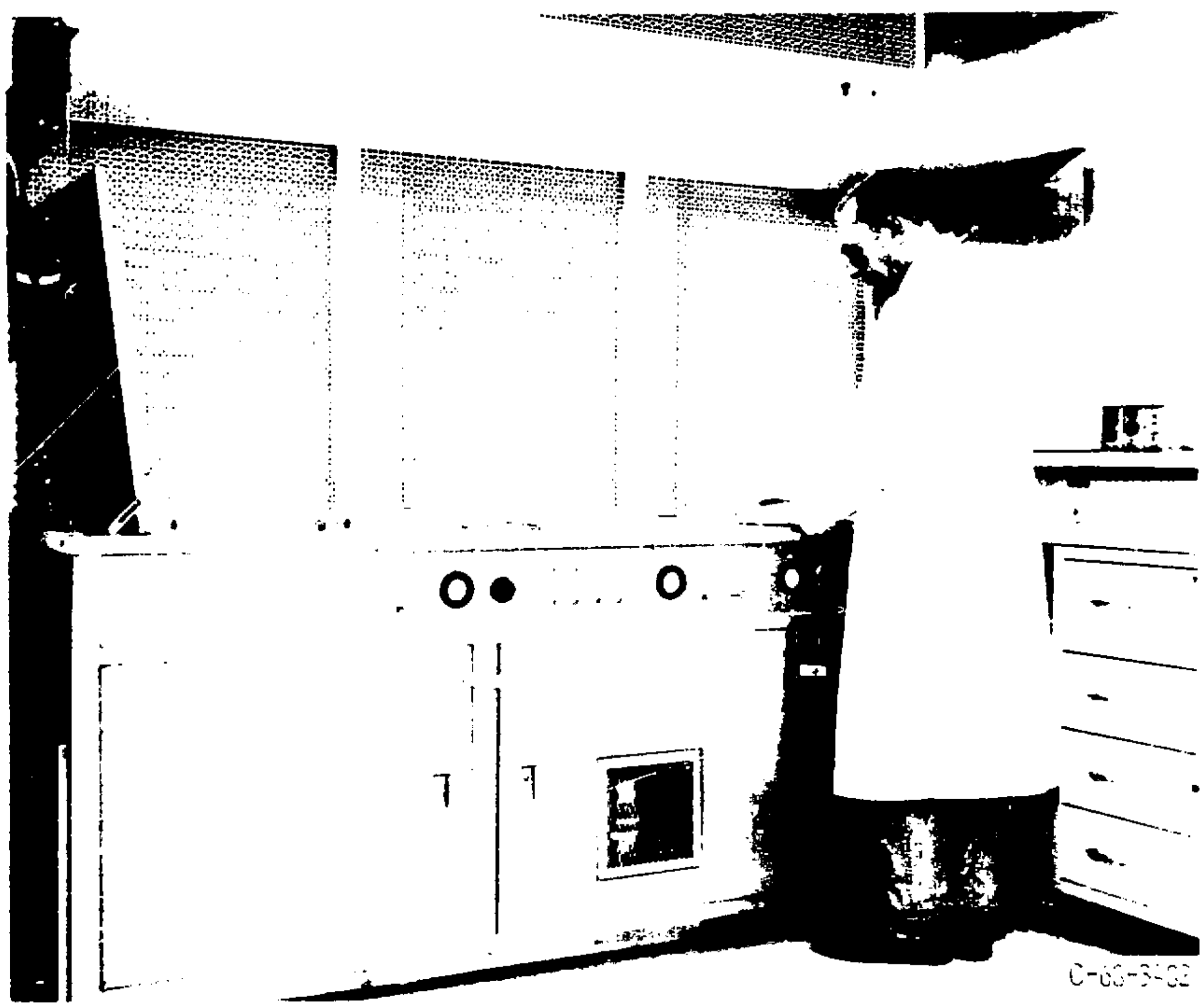


Figure B1. - 2.2-Second zero-gravity facility.



(a) Ultrasonic cleaning system.

Figure B2. - Controlled enviroment room.

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(b) Laboratory equipment.

Figure B2. - Concluded.

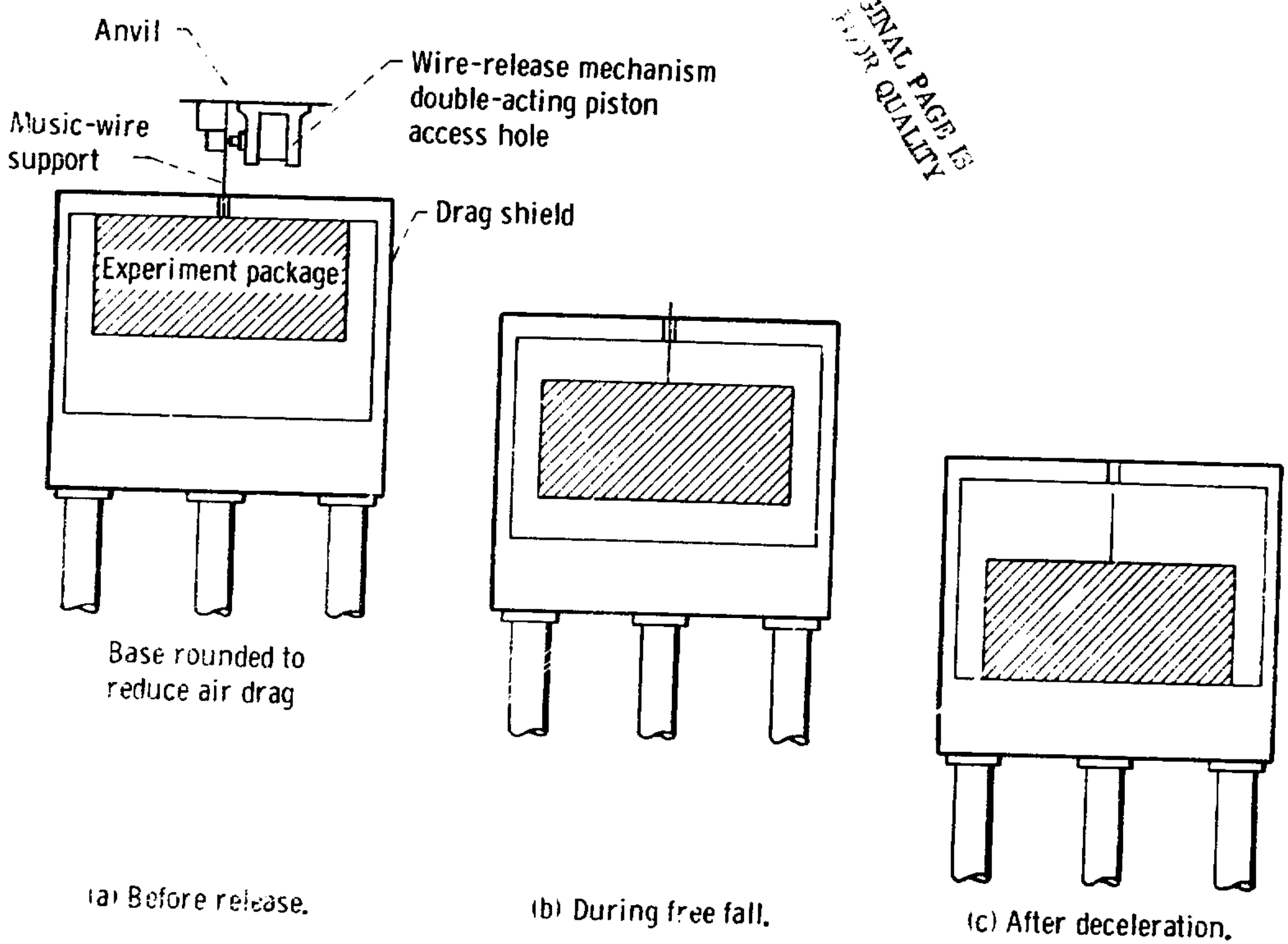

Figure B3. - Position of experiment package and drag shield before, during, and after test drop.