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SYMMETRIES IN LAMINATED COMPOSITE PLATES

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Abstract

The different types of symmetry exhibited by laminated anisotropic fibrous composite plates are identified and contrasted with the symmetries of isotropic and homogeneous orthotropic plates. The effects of variations in the fiber orientation and the stacking sequence of the layers on the symmetries exhibited by composite plates are discussed. Both the linear and geometrically nonlinear responses of the plates are considered. A simple procedure is presented for exploiting the symmetries in the finite element analysis. Examples are given of square, skew and polygonal plates where use of symmetry concepts can significantly reduce the scope and cost of analysis.

Introduction

In the analysis of isotropic and orthotropic plates, computations can be substantially reduced in scope and cost if certain symmetries exist in geometry, material properties, loading and boundary conditions. For laminated composite plates made of anisotropic materials symmetry properties of a different nature exist and except for a recent study by the author [Ref. 1], these symmetries have not been utilized in finite element analyses. The present study focuses on this problem and is an extension of the work reported in Ref. 1. More specifically, the objectives of this paper are: a) to identify the different types of symmetry exhibited by the commonly-used fibrous composite plates in both the linear and nonlinear regimes and b) to present a simple procedure for exploiting these symmetries in the finite element analysis.

The plate is assumed to consist of a number of layers bonded together. Each layer has arbitrary thickness, elastic properties and fiber orientation with respect to the plate axes. The analytical formulation is based on a geometrically nonlinear von-Karman type plate theory with the effects of transverse shear deformation, anisotropic material behavior and bending-extensional coupling included. A displacement finite element formulation is assumed with the fundamental unknowns consisting of the displacement and rotation components of the middle plane of the plate (u_α, w and ϕ_α). For convenience, the nodal parameters are taken to be the five generalized displacements at each node. The sign convention for the different plate quantities is shown in Fig. 1. It should be emphasized, however, that the procedure outlined herein for exploiting the symmetries in the finite element analysis can be adapted for use with other plate theories and other finite element models (e.g. mixed models).

Examples are given of square, skew and polygonal composite plates where use of symmetry concepts can significantly reduce the computational cost of the analysis.

Types of Symmetry in Composite Plates

The fundamental definitions of symmetry, symmetry elements and symmetry operations are reviewed in Ref. 2. The "axiom of symmetry" introduced in Ref. 2 when applied to composite plates reads: Given a composite plate exhibiting certain types of symmetry and a system of loads which exhibits the same types of symmetry as those of the plate, the response obtained will exhibit the same types of symmetry as those inherent in the plate and the loading system. Here the symmetry of the plate refers to the symmetry of a) plate geometry, b) lamination parameters (e.g. fiber orientation and stacking of layers) and c) boundary conditions.

Symmetry of the plate is described by giving the set of all operations which preserves the distance between pairs of points of the plate and which takes the plate into an equivalent configuration (i.e. into a configuration which is indistinguishable from the original configuration but not necessarily identical with that configuration). Any such operation is called a *symmetry transformation*. The set of all symmetry transformations forms the *symmetry group* of the plate [Ref. 3].

The three types of symmetry frequently exhibited by composite plates are: 1) Reflection (or bilateral) symmetry; 2) rotational symmetry; and 3) inversion symmetry. The characteristics of these symmetries are examined subsequently.

Reflection (or bilateral) symmetry

A composite plate exhibits reflection symmetry (also called bilateral or mirror symmetry) with respect to a plane, a x_1 + b x_2 (a, b constants), if it can be brought into an equivalent configuration by mirror reflection in that plane. Obviously, the loading on the plate and the boundary conditions must possess mirror symmetry with respect to the same plane. Composite plates which are skew-symmetrically laminated with respect to their middle plane [Refs. 4 and 5] often exhibit a different type of bilateral symmetry, namely, reflection symmetry with respect to a line, $a x_1 + b x_2$, in their middle plane. The plate can thus be brought into an equivalent configuration through a 180° rotation about the line, $a x_1 + b x_2$.

The two types of symmetry, reflection in a plane and reflection in a line, will henceforth be called reflection symmetries of the first and second types, respectively. The symmetry relations for the loading components, generalized displacements and stress resultants for reflection types I and II are given in tables I and II. The first type of reflection symmetry, reflection type I, is the better known one and is the one exhibited by isotropic and homogeneous orthotropic plates with symmetric geometries. Note that the symmetry relations for the extensional group (i.e., in-plane loading components p_α ; in-plane displacements u_α and extensional stress resultants $N_{\alpha\beta}$) are the same in reflection types I and II. On the other hand, the symmetry relations for the bending group (i.e., transverse loading p ; transverse displacement w ; rotation components ϕ_α ; bending stress resultants $M_{\alpha\beta}$; and transverse shear stress resultants Q_α) in reflection type II differ in sign from the corresponding ones in reflection type I. Stated differently, the symmetry relations for the bending group in reflection type II are identical with the antisymmetry relations in reflection type I.

Composite plates can possess one or more planes (or lines) of reflection symmetry. Circular plates may be thought of as plates with infinite number of planes of reflection symmetry. Examples of composite plates exhibiting reflection symmetries of the types I and II are shown in figures 3a and 3b.

Figures 3a and 3b show contour plots for the in-plane and transverse displacement components in two-layered composite square plates subjected to uniform transverse loading. The plates of figures 3a and 3b have their fibers oriented at +45/-45 and 0/90, respectively. Note that the contour plots for u_1 in figure 3a can be brought into coincidence with u_2 by mirror reflection in the plane of reflection symmetry (plane $x_1 = x_2$). On the other hand, the contour plots for u_1 in figure 3b can be brought into coincidence with u_2 through a 180° rotation about the line of reflection symmetry (line $x_1 = x_2$), which is equivalent to a mirror reflection in the plane $x_1 = x_2$, followed by a change in sign. A mirror reflection in plane $x_1 = x_2$ or a rotation through 180° about line $x_1 = x_2$ leaves the contour plots for w, in figures 3a and 3b, unchanged.

Rotational symmetry

A composite plate is said to exhibit rotational symmetry with respect to an axis normal to its plane if it can be brought into an equivalent configuration by rotation around that axis. The axis of rotation is called an n-fold axis of symmetry if the smallest possible rotation which takes the plate into an equivalent configuration is $2\pi/n$ radians. An n-fold axis of symmetry has n symmetry operations associated with it, namely rotations of 2π, π, $2\pi/3$, ... $2\pi/n$ (see Ref. 2). The symmetry relations for the loading components, generalized displacements and stress resultants for rotational symmetry with n=2 and n=4 are given in tables I and II. The axis of rotation is assumed to coincide with the x_3-axis. Note that for n=4 the symmetry relations for the bending stress resultants in quasi-isotropic composite plates (see Ref. 4) differ in sign from the corresponding ones for isotropic plates.

An important special case of rotational symmetry is the case of complete or axial symmetry ($n=\infty$); which is exhibited by composite circular plates whose elastic characteristics and boundary conditions are independent of the circumferential coordinate (i.e. axisymmetric). The response of the plate will also be axisymmetric. However, in contradistinction to isotropic and orthotropic plates, the two sets of displacements and stress resultants (u_2, ϕ_2, N_{12}, M_{12}, Q_2) and (u_1, w, ϕ_1, N_{11}, N_{22}, M_{11}, M_{22}, Q_1), though axisymmetric, are coupled, i.e. they do not all vanish under any of the axisymmetric loadings shown in Fig. 2 (see Ref. 6).

It should be mentioned that reflection symmetry type II can be considered as a rotational symmetry, n=2, with respect to a line in the middle plane of the plate.

Inversion symmetry

A composite plate is said to exhibit inversion symmetry with respect to an axis x_3 normal to its plane if it can be brought into an equivalent configuration through 180° rotation about the axis. This amounts to changing the coordinates x_α of each material point of the plate into $-x_\alpha$. The x_3-axis is called the axis of symmetry and its intersection with the middle plane of the plate is called the center of symmetry. Inversion symmetry can be thought of as a reflection through a point, viz. the center of symmetry; or a rotational symmetry with n=2 (i.e. rotation angle=π).

The center of symmetry is located at the intersection of lines of geometric symmetry on the middle plane of the plate. For plates with skew planform the center of symmetry is located at the intersection of the two diagonals. An example of a composite skew plate subjected to uniform loading and exhibiting inversion type symmetry is given in Fig. 4. The inversion symmetry relations listed in table II are clearly obvious in the contour plots of Fig. 4.

It may be mentioned that for certain geometries and/or loadings the response of isotropic (and orthotropic) plates exhibits inversion symmetry similar to that exhibited by anisotropic (composite) plates. Examples of these situations include plates with skew planform and plates subjected to edge shear.

Nonlinear Problems

The arguments of reflection symmetry in plane (reflection type I), inversion and axial symmetry apply to both the linear and geometrically nonlinear responses of the plate. On the other hand, rotational symmetry with $n=4$ for quasi-isotropic plates and reflection symmetry in a line (reflection type II) hold only for the linear response. This can be explained for the latter symmetry by the fact that in the geometrically nonlinear case, a composite plate behaves like a shallow shell and a rotation of 180° about the axis of reflection symmetry (in its middle surface) does not bring the shell into an equivalent configuration.

Antisymmetric (or Skew-Symmetric) Loadings

For antisymmetric (or skew-symmetric) loadings, the response of the plate will be antisymmetric (or skew-symmetric). For plates with inversion skew-symmetry the transverse displacement w, the inplane stress-resultants $N_{\alpha\beta}$ and the bending stress resultants $M_{\alpha\beta}$ vanish at the center of symmetry.

Any loading system can be decomposed into symmetric and antisymmetric (or skew-symmetric) components. For linear problems, significant computational advantages result from such a decomposition.

Free Vibration Problems

In free vibration problems of composite plates, the mode shapes can be obtained by using a portion of the plate and applying the four different combinations of symmetry and antisymmetry (or skew-symmetry) conditions along the internal boundaries. Such an approach was found to significantly reduce the computational effort in the case of isotropic and orthotropic plates.

Effect of Lamination Parameters on Symmetries of Composite Plates

Composite plates having the same symmetric geometry, loading and boundary conditions, but different laminations can exhibit different types of symmetry. The lamination parameters which have the strongest effect on symmetries are fiber orientation of individual layers and stacking sequence of different layers. To illustrate the effect of these two parameters on the symmetries, table III gives the symmetry transformations for composite square plates having the following laminations (for a description of the characteristics of these laminates see Ref. 4):

i) Midplane symmetric laminates (cross-ply, angle-ply with $\theta=\pm 45$, and angle-ply with $\theta \neq 45$).

ii) Midplane antisymmetric laminates (cross-ply, angle-ply with $\theta=\pm 45$ and angle-ply with $\theta \neq 45$).

iii) Quasi-isotropic, four-layered laminate with fiber orientation $\theta=+45/0/90/-45$.

The loading on the plate and the boundary conditions are assumed to be symmetric. For the purpose of comparison, the size of the mathematical model required for the analysis as well as the symmetry transformations for isotropic and homogeneous orthotropic square plates are also given in table III. The symmetry relations for the generalized displacements and stress resultants, implied by the symmetry transformations of table III, are given in table II.

Figures 5 through 7 show contour plots for stress resultants and generalized displacements in composite square plates with symmetric, antisymmetric and quasi-isotropic laminations. The plates have clamped edges and are subjected to uniform transverse loading. The symmetry relations listed in table II are clearly obvious in these figures. Note that for quasi-isotropic plates (Fig. 7) there are no planes or lines of reflection symmetry and the symmetry group of the plate includes rotation through 90° (and 180°) about the x_3-axis. If these symmetries are utilized, only one quadrant of the plate needs to be analyzed.

Some of the symmetries listed in table III are not restricted to square plates but can be exhibited by plates having other geometries. Examples of such situations are shown in Fig. 8.

Exploiting Symmetries in Finite Element Analysis

Identification of symmetries

The first step in exploiting the symmetries inherent in composite plates in their finite element analysis is to identify these symmetries for a given plate and loading. This is accomplished by examining the plate for possible rotational, reflection or inversion symmetries (see table III and Figure 8). Again, it should be emphasized that mirror reflection in a line (reflection type II) is exhibited by the linear response of the plate. It disappears in the geometrically nonlinear response. The same remark applies to quasi-isotropic plates with rotational symmetry of the type $n=4$.

Finite element grids

In order to exploit the symmetries of composite plates in their analysis, the finite element grid chosen must exhibit the same types of symmetry as those of the plate. This means that the grid can be brought into self-coincidence by each of the symmetry transformations used for the plate (rotation about axis of revolution; mirror reflection in each of the planes or lines of reflection symmetry and inversion through each of the centers of symmetry).

Independent nodes and elements

After the finite element grid is selected, the next step is to identify the nodes associated with the independent degrees of freedom of the plate. These nodes will henceforth be referred to as *independent nodes*. The other nodes in the plate will be called *dependent nodes*. The dependent nodes will be designated by the symbols, R, I, P or L according to whether they are obtained from independent nodes by rotation about axis of revolution, inversion through center of symmetry, reflection in a plane or reflection in a line, respectively.

The minimum number of finite elements which, by successive applications of the symmetry transformations can cover the whole plate will be referred to as the set of *independent elements*. Other elements in the plate will be called dependent elements. According to this definition an independent element cannot be obtained from other independent elements by symmetry transformations (e.g. rotation about axis of revolution, reflection or inversion).

The *multiplicity* of an element is defined as the number of times this element can appear in the plate by symmetry transformations. It is equal to one plus the number of dependent elements that can be obtained from that independent element by symmetry transformations. The size of the finite element model and the number of simultaneous algebraic equations required in the analysis are governed by the number of independent elements and nodes, respectively. Figures 9 and 10 show the independent nodes and elements for plates with reflection and inversion symmetries modelled by rectangular and parallelogram elements. Note that the independent nodes and elements are not uniquely given and can be selected in many different ways (see Fig. 10).

Procedure for exploiting the symmetry

A simple procedure for exploiting the symmetry in the finite element analysis of composite plates is outlined in Fig. 11. The key elements in this procedure are discussed subsequently.

Input data

The following modifications are made to the input data:

1. The total number of nodes and elements are set equal to the number of independent nodes and elements, respectively. As

a consequence of this, the coordinates of the independent nodes only are input. Also, the connectivity lists of the independent elements only are prescribed.

2. The multiplicity of each element must be prescribed. This can be input as an additional entry in the element connectivity table.

3. If any of the independent elements has nodes of the type R, I, P or L, they must be identified, in the element connectivity list (see Ref. 1).

Element modules

The elemental matrices for the independent elements only are generated. For elements with nodes lying on an axis of reflection symmetry $a\,x_1 + b\,x_2$ (a, b are nonzero constants) or with dependent nodes of the type R, I, P or L, the stiffness, mass and load matrices are modified by simple transformations of the form:

$$\begin{aligned} [\bar{K}] &= [\Gamma]^T[K]\,[\Gamma] \\ [\bar{M}] &= [\Gamma]^T[M]\,[\Gamma] \\ [\bar{P}] &= [\Gamma]^T[P] \end{aligned} \tag{1}$$

where $[K]$, $[M]$ and $[P]$ are the element stiffness, mass and load matrices, $[\Gamma]$ is a transformation matrix and a bar over a matrix denotes transformed (or modified) matrix.

If the nodal parameters are listed node by node, the matrix $[\Gamma]$ will have a block diagonal structure, i.e.

$$[\Gamma] = \begin{bmatrix} [\Gamma]_1 & & & \\ & [\Gamma]_2 & & \\ & & \ddots & \\ & & & [\Gamma]_m \end{bmatrix} \tag{2}$$

where m is the number of nodes in the element. For shear-flexible elements with nodal parameters consisting of the displacement and rotation components, the $[\Gamma]_i$ ($i=1\rightarrow m$) are 5x5 submatrices. The forms of these submatrices at I, P and L nodes are given in Appendix A.

Finally, the elemental matrices (stiffness, mass and load matrices) of each of the independent elements are multiplied by the multiplicity of that element. This step can be bypassed if all the independent elements have the same multiplicity.

Assembly modules

The elemental matrices for the independent elements only are assembled. In the assembly process no distinction is made between the dependent nodes (e.g., R , I , P and L) within the independent elements and their corresponding independent nodes. This can be accomplished by setting the numbers of the dependent nodes equal to the numbers of the corresponding independent nodes prior to the assembly and then the assembly is done in the usual manner.

Application of symmetry conditions

The boundary conditions along the edges of independent elements are applied in the usual manner. In addition, the symmetry and skew-symmetry conditions *must* be applied at each of the centers of inversion symmetry and along the axes and lines of reflection symmetry. The symmetry and skew-symmetry conditions for shear flexible elements are listed in table IV. The resulting reduced set of algebraic equations is solved for the independent degrees of freedom.

The foregoing procedure for exploiting the symmetry in the finite element analysis of composite plates has been implemented in a small research-oriented finite element system and was found to result in considerable savings in the computer time required for the analysis. The savings are primarily due to: a) the generation and assembly of the elemental matrices of independent elements only and b) the reduction in the size of the resulting system of simultaneous algebraic equations (their number, and possibly, their bandwidth). The savings can be particularly significant for eigenvalue and nonlinear problems.

Concluding Remarks

The different types of symmetry commonly exhibited by composite plates for various loadings and boundary conditions, are identified and contrasted with the symmetries of isotropic and orthotropic plates. Both the linear and geometrically nonlinear responses of the plate are considered. A simple procedure is presented for exploiting the symmetries in finite element analysis. The analytical formulation is based on a geometrically nonlinear plate theory with the effects of material anisotropy, bending-extensional coupling and transverse shear deformation included. A displacement (stiffness) finite element formulation is assumed with the nodal parameters consisting of the displacement and rotation components of the plate middle plane. However, the procedure outlined for exploiting the symmetry can be readily used with other plate theories and other finite element models.

Examples are given of composite square, skew and polygonal plates where use of symmetry can significantly reduce the number of independent degrees of freedom, and hence the computational time required for their finite element analysis.

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6. E. Reissner and F. Y. M. Wan, (1971), "On Rotationally Symmetric Stress and Strain in Anisotropic Shells of Revolution," Studies in Applied Mathematics, Vol. 50, No. 4. p. 391.

Appendix A - Transformation Matrices for I, P and L Nodes

The transformation matrices $[\Gamma]_i$ at an I node is given by:

$$[\Gamma]_{Ii} = \begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & +1 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix} \tag{A1}$$

If the plane (or line) of reflection symmetry is $x_1 = x_2$, the matrices $[\Gamma]_i$ at P and L nodes are given by:

$$[\Gamma]_{Pi \text{ or } Li} = \begin{bmatrix} . & 1 & & & \\ 1 & . & & & \\ & & K & & \\ & & & . & K \\ & & & K & . \end{bmatrix} \tag{A2}$$

where $K = +1$ at a P node and -1 at an L node.

The corresponding matrices $[\Gamma]_{Pi \text{ or } Li}$ for the case when the plane (or line) of reflection symmetry is $x_1 = -x_2$ are obtained by multiplying the right hand side matrix of Eq. (A1) with that of Eq. (A2).

For antisymmetric (or skew-symmetric) response the right hand sides of Eqs. (A1) and (A2) are multiplied by a minus sign.

Appendix B - Notation

The following symbols are used in this paper:

a_1, a_2	side lengths of the plate
h	thickness of the plate
$[K], [\bar{K}]$	original and modified stiffness matrices of the plate element
m	number of nodes in the element
$[M], [\bar{M}]$	original and modified mass matrices of the plate element
$M_{\alpha\beta}$	bending stress resultants
$N_{\alpha\beta}$	extensional (in-plane) stress resultants
p_α, p	external load intensities in the x_α and x_3 directions
$[P], [\bar{P}]$	original and modified load matrices of the plate element
Q_α	transverse shear stress resultants
u_α, w	displacement components in the coordinate directions
u_s, u_n	displacement components along and normal to the axis of reflection symmetry
x_α, x_3	Cartesian coordinate system (x_3 normal to the middle plane of the plate)
ϕ_α	rotation components
$\kappa, \bar{\kappa}$	tracing constants which can have the values +1 and -1
θ	fiber orientation angle
$[\Gamma]$	transformation matrix
$[\Gamma]_i$	5x5 transformation submatrices
$[\ ]$	rectangular or square matrix
R, I, P and L	denote dependent nodes obtained from the independent nodes by rotation, inversion reflection in a plane or reflection in a line.

The range of the Greek indices α, β is 1,2.

Table I

Symmetry relations for in-plane and transverse loading components.

	Mirror reflection in plane (or line)				Rotational symmetry w.r.t. x_3 axis	
	$x_1=0$	$x_2=0$	$x_1=+x_2$	$x_1=-x_2$	n = 2 (Inversion symmetry)	n = 4
$\begin{bmatrix} p_1 \\ p_2 \\ p \end{bmatrix}_{x_1,x_2} =$	$\begin{bmatrix} -p_1 \\ p_2 \\ Kp \end{bmatrix}_{-x_1,x_2}$	$\begin{bmatrix} p_1 \\ -p_2 \\ Kp \end{bmatrix}_{x_1,-x_2}$	$\begin{bmatrix} p_2 \\ p_1 \\ Kp \end{bmatrix}_{x_2,x_1}$	$\begin{bmatrix} -p_2 \\ -p_1 \\ Kp \end{bmatrix}_{-x_2,-x_1}$	$\begin{bmatrix} -p_1 \\ -p_2 \\ p \end{bmatrix}_{-x_1,-x_2}$	$\begin{bmatrix} -p_2 \\ p_1 \\ \bar{K}p \end{bmatrix}_{x_2,-x_1}$

Notes: $K = +1$ for reflection in a plane and -1 for reflection in a line,

$\hat{K} = +1$ for isotropic and -1 for quasi-isotropic plates,

For antisymmetric (or skew symmetric) loadings the entries in columns 2 through 6 must be multiplied by a minus sign.

Table II

Symmetry relations for generalized displacements and stress resultants.

	Mirror reflection in plane (or line)				Rotational symmetry w.r.t. x_3 axis	
	$x_1=0$	$x_2=0$	$x_1=+x_2$	$x_1=-x_2$	n=2 (inversion symmetry)	n=4
Generalized Displacements $\begin{bmatrix} u_1 \\ u_2 \\ w \\ \phi_1 \\ \phi_2 \end{bmatrix}_{x_1,x_2} =$	$\begin{bmatrix} -u_1 \\ u_2 \\ Kw \\ -K\phi_1 \\ K\phi_2 \end{bmatrix}_{-x_1,x_2}$	$\begin{bmatrix} u_1 \\ -u_2 \\ Kw \\ K\phi_1 \\ -K\phi_2 \end{bmatrix}_{x_1,-x_2}$	$\begin{bmatrix} u_2 \\ u_1 \\ Kw \\ K\phi_2 \\ K\phi_1 \end{bmatrix}_{x_2,x_1}$	$\begin{bmatrix} -u_2 \\ -u_1 \\ Kw \\ -K\phi_2 \\ -K\phi_1 \end{bmatrix}_{-x_2,-x_1}$	$\begin{bmatrix} -u_1 \\ -u_2 \\ w \\ -\phi_1 \\ -\phi_2 \end{bmatrix}_{-x_1,-x_2}$	$\begin{bmatrix} -u_2 \\ u_1 \\ \bar{K}w \\ -\bar{K}\phi_2 \\ \bar{K}\phi_1 \end{bmatrix}_{x_2,-x_1}$
Stress Resultants $\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \\ Q_1 \\ Q_2 \end{bmatrix}_{x_1,x_2} =$	$\begin{bmatrix} N_{11} \\ N_{22} \\ -N_{12} \\ KM_{11} \\ KM_{22} \\ -KM_{12} \\ -KQ_1 \\ KQ_2 \end{bmatrix}_{-x_1,x_2}$	$\begin{bmatrix} N_{11} \\ N_{22} \\ -N_{12} \\ KM_{11} \\ KM_{22} \\ -KM_{12} \\ KQ_1 \\ -KQ_2 \end{bmatrix}_{x_1,-x_2}$	$\begin{bmatrix} N_{22} \\ N_{11} \\ N_{12} \\ KM_{22} \\ KM_{11} \\ KM_{12} \\ KQ_2 \\ KQ_1 \end{bmatrix}_{x_2,x_1}$	$\begin{bmatrix} N_{22} \\ N_{11} \\ N_{12} \\ KM_{22} \\ KM_{11} \\ KM_{12} \\ -KQ_2 \\ -KQ_1 \end{bmatrix}_{-x_2,-x_1}$	$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \\ -Q_1 \\ -Q_2 \end{bmatrix}_{-x_1,-x_2}$	$\begin{bmatrix} N_{22} \\ N_{11} \\ -N_{12} \\ \bar{K}M_{22} \\ \bar{K}M_{11} \\ -\bar{K}M_{12} \\ -\bar{K}Q_2 \\ \bar{K}Q_1 \end{bmatrix}_{x_2,-x_1}$

Notes: $K = +1$ for reflection in a plane and -1 for reflection in a line,
$\bar{K} = +1$ for isotropic and -1 for quasi-isotropic plates,
For antisymmetric (or skew-symmetric) response the entries in columns 2 through 6 must be multiplied by a minus sign.

Table III

Symmetry transformations for composite plates**

Lamination type		Mirror reflection in plane (or line)*			Rotation about x_3 through an angle		Size of model required for square plate	Symmetry type designation
		$x_1=0$	$x_2=0$	$x_1=\pm x_2$	$\pi/2$	π (inversion symmetry)		
Symmetric or unsymmetric laminates	cross ply	●	●			●	1/4	I
	angle ply, $\theta=\pm 45$			●		●	1/4	II
	angle ply, $\theta\neq 45$					●	1/2	III
antisymmetric laminates	cross ply	●	●	○		●	1/8	IVa
	angle ply, $\theta=\pm 45$	○	○	●		●	1/8	IVb
	angle ply, $\theta\neq 45$	○	○			●	1/4	Ia
Quasi-isotropic ($\theta=\pm 45/0/90/-45$)					○	●	1/4	Ib
Homogeneous orthotropic		●	●			●	1/4	I
Isotropic		●	●	●	●	●	1/8	IV

* Solid circles for reflection in a plane and open circles for reflection in a line

** Symmetry transformations denoted by open circles do not apply to nonlinear problems

Table IV

Symmetry and antisymmetry (or skew-symmetry) conditions for shear-flexible displacement finite element models.

Center of inversion symmetry	Line of reflection symmetry*	Plane of reflection* symmetry
	a) symmetry conditions	
$u_1 = u_2 = 0$	$u_n = 0$	$u_n = 0$
$\phi_1 = \phi_2 = 0$	$w = 0$	$\phi_n = 0$
	$\phi_s = 0$	
	b) skew-symmetry conditions	
$w = 0$	$u_s = 0$	$u_s = 0$
	$\phi_n = 0$	$w = 0$
		$\phi_s = 0$

* subscripts s and n denote the tangential and normal components to the axis of reflection symmetry respectively.

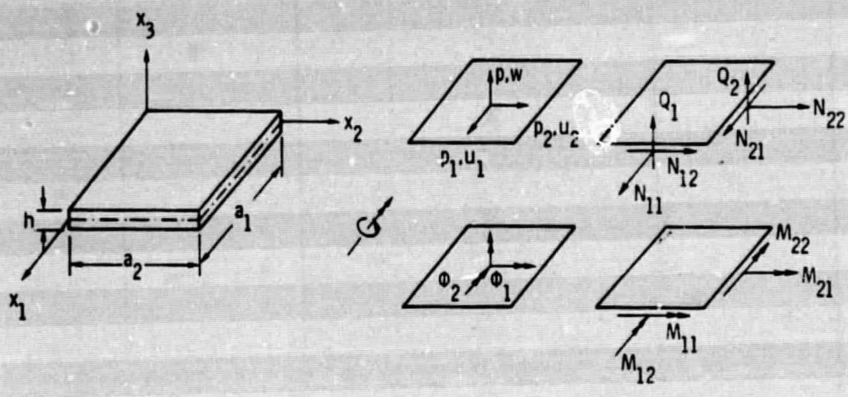


Figure 1 : Sign convention for stress resultants and generalized displacements.

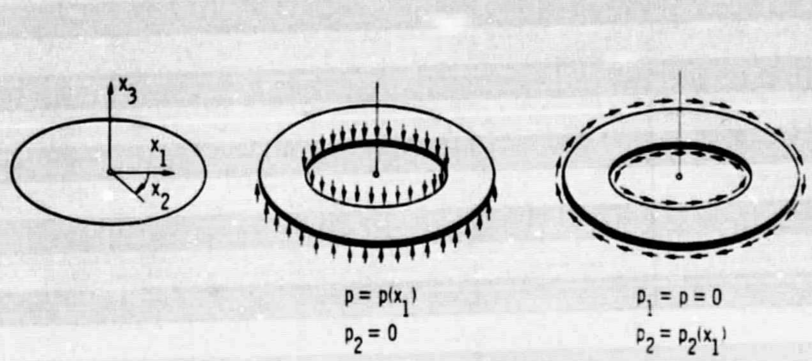


Figure 2 : Symmetric and antisymmetric loadings on axisymmetric plates.

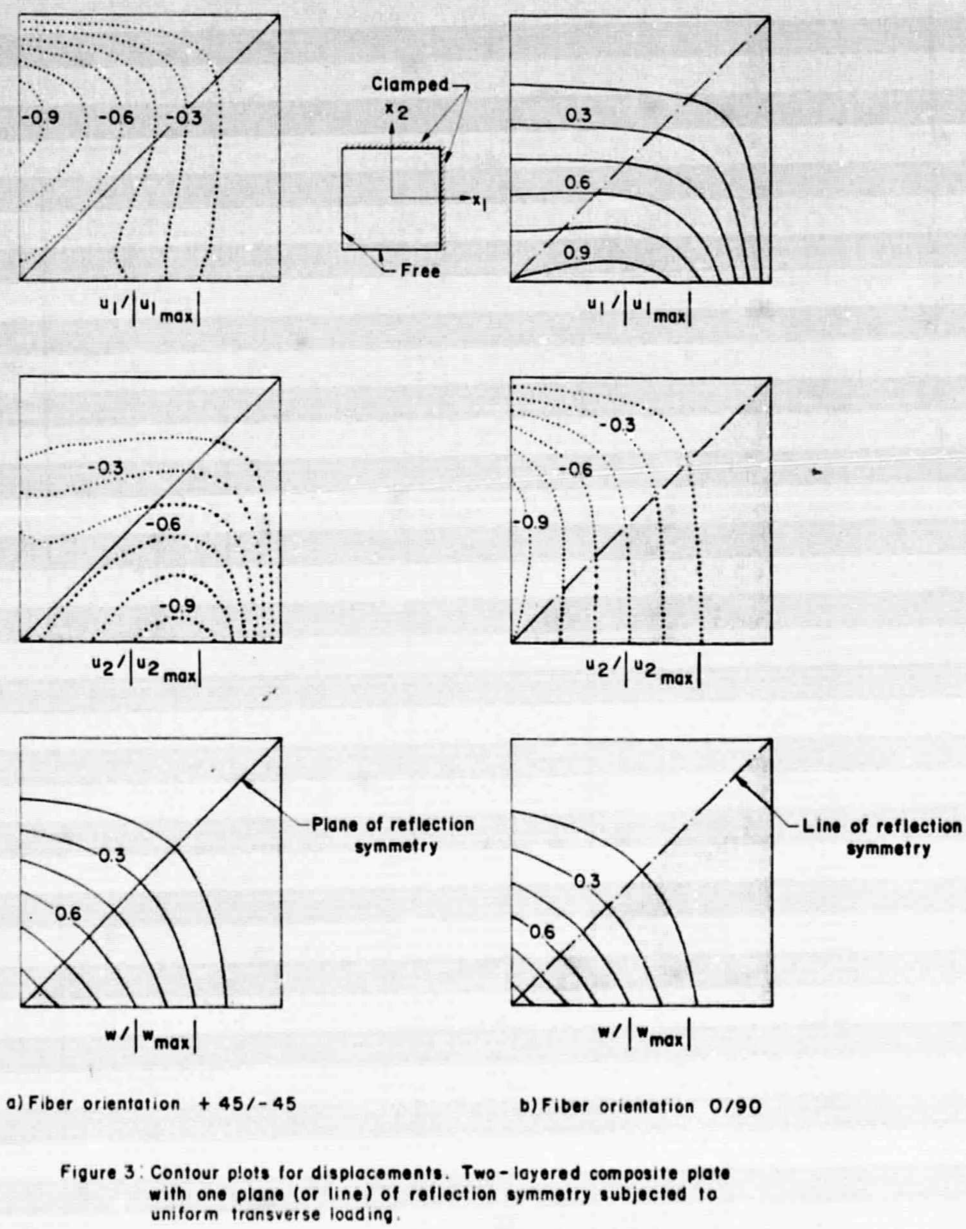


Figure 3: Contour plots for displacements. Two-layered composite plate with one plane (or line) of reflection symmetry subjected to uniform transverse loading.

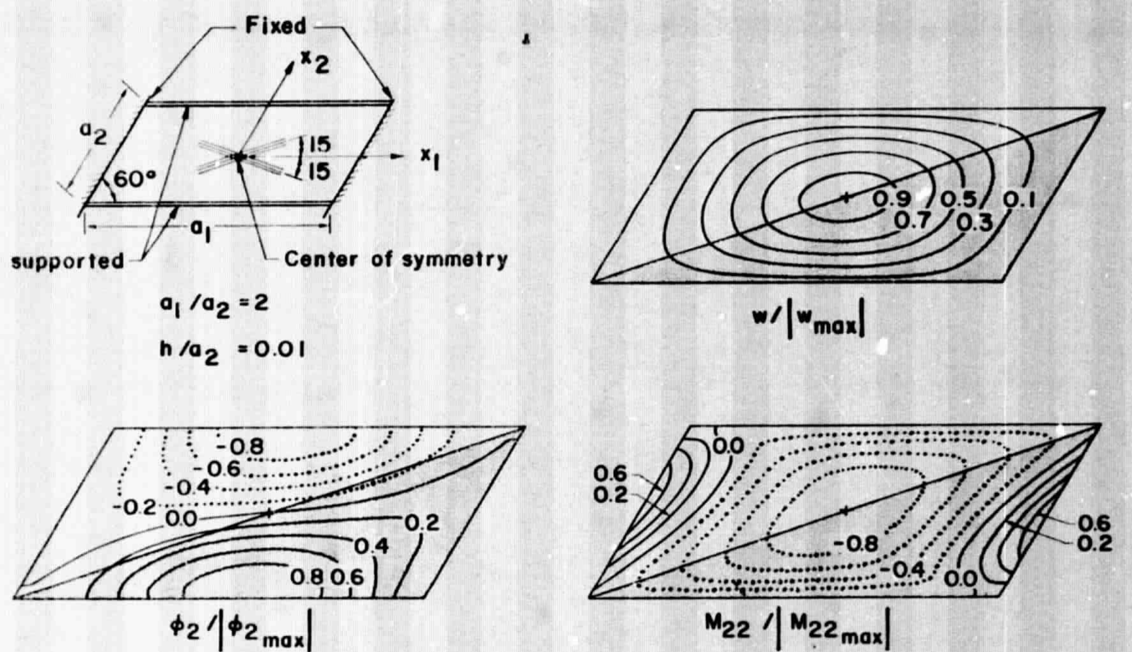
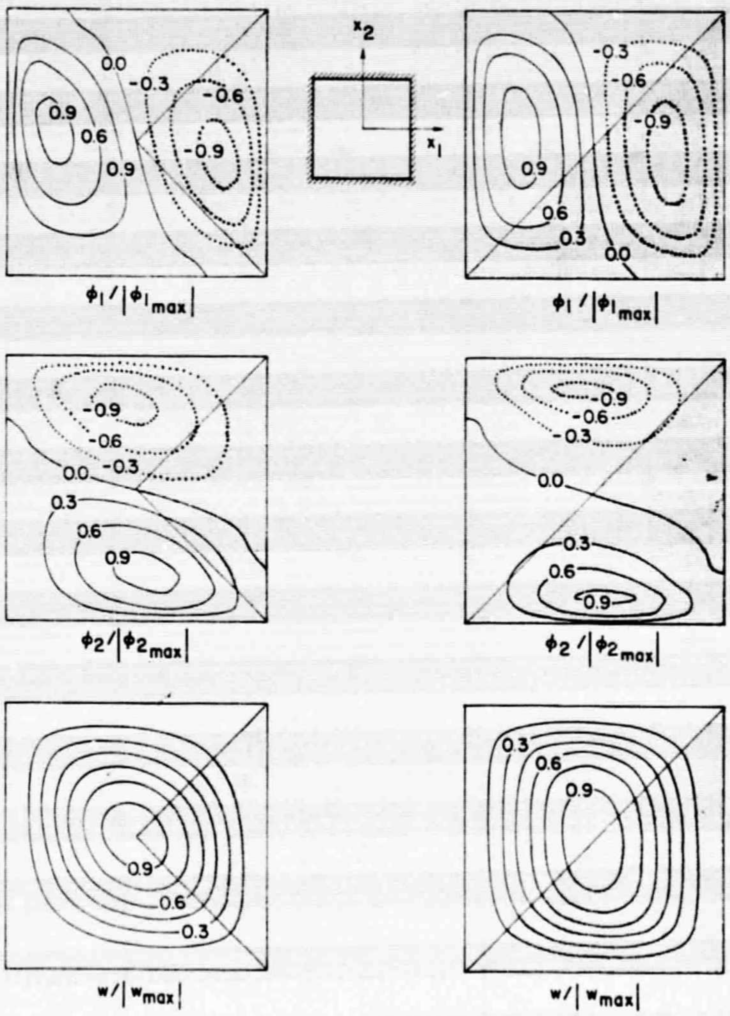


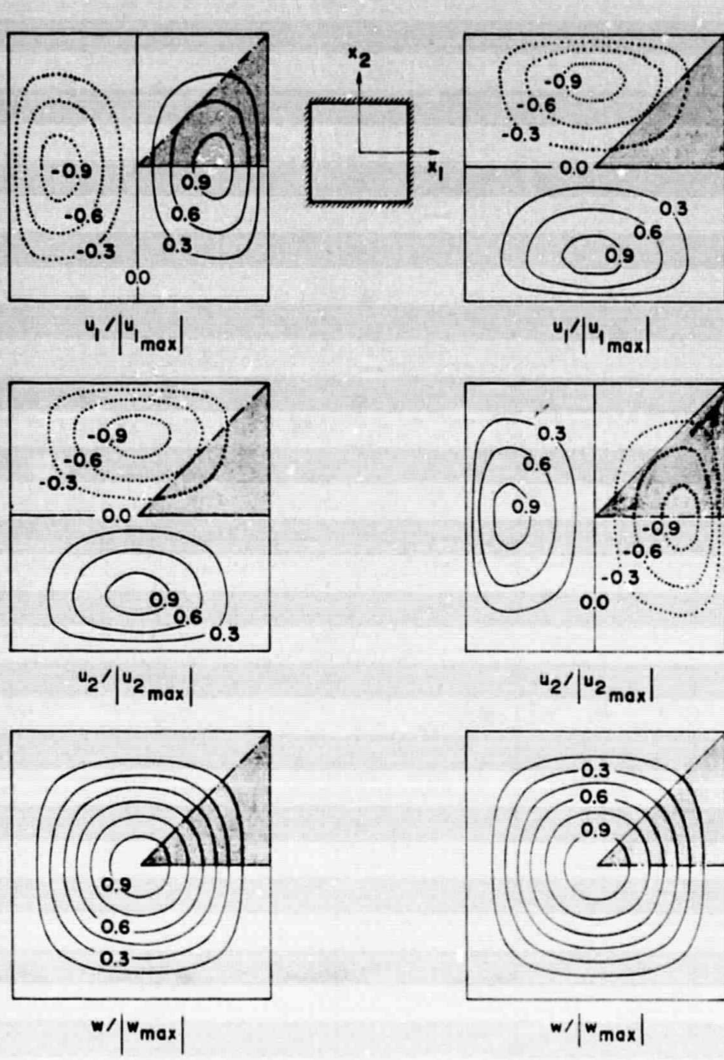
Figure 4: Contour plots for displacements and stress resultants. Two-layered composite skew plate subjected to uniform transverse loading.



a) Fiber orientation 45/-45/45

b) Fiber orientation 15/-15/15

Figure 5: Contour plots for displacements. Three-layered symmetrically-laminated clamped square plate subjected to uniform transverse loading.



a) Cross ply 0/90

b) Angle ply 45/-45

Figure 6 : Contour plots for displacements. Two-layered antisymmetrically-laminated clamped square plate subjected to uniform transverse loading.

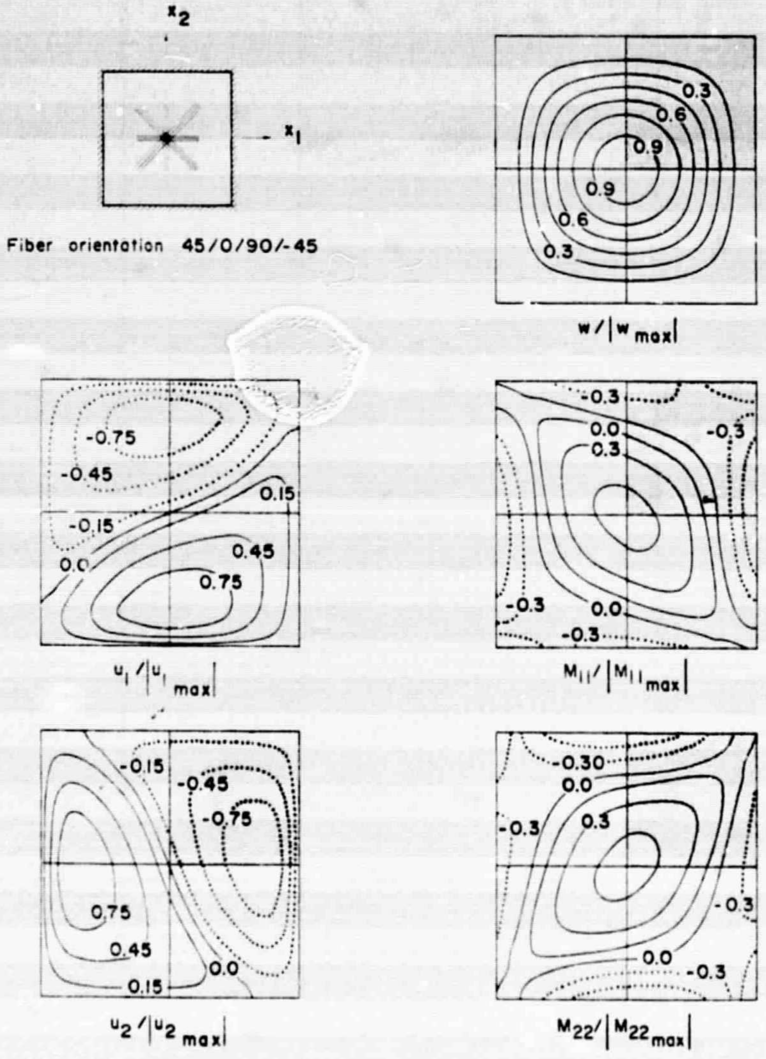
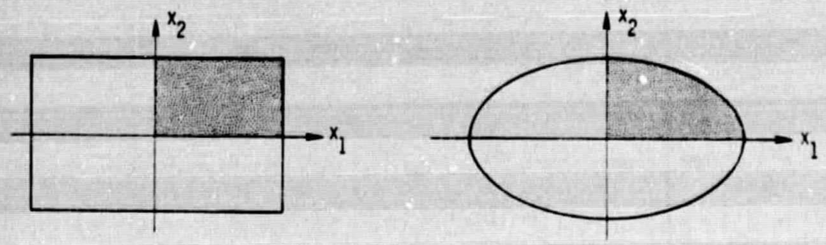
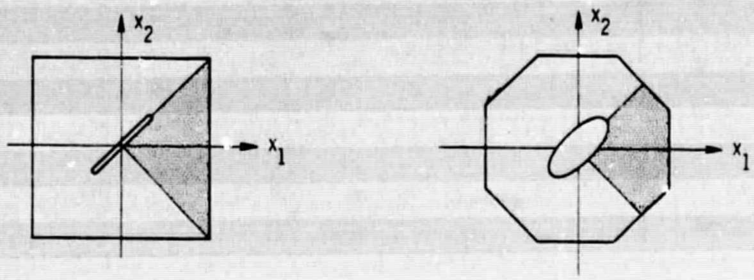


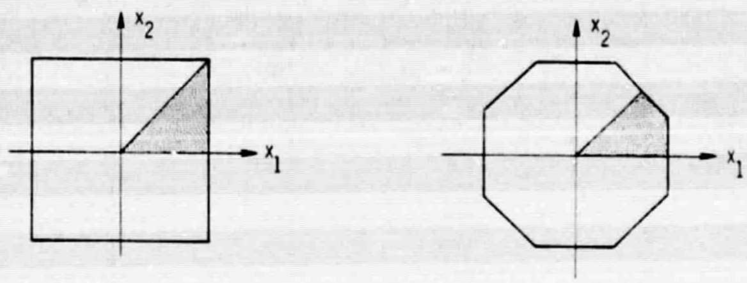
Figure 7 : Contour plots for displacements and stress resultants. Four-layered quasi-isotropic square plate with fiber orientation 45/0/90/-45 subjected to uniform transverse loading.



a) Plates with two planes (or lines) of reflection symmetry $x_1 = 0$, $x_2 = 0$. Symmetry types I and Ia.



b) Plates with two planes (or lines) of reflection symmetry $x_1 = \pm x_2$. Symmetry type II.



c) Plates with four planes (or lines) of reflection symmetry $x_1 = 0$, $x_2 = 0$ and $x_1 = \pm x_2$. Symmetry types IV and IVa.

Figure 8 : Examples of plates with two and four planes (or lines) of reflection symmetry and their possible symmetry types (see table III).

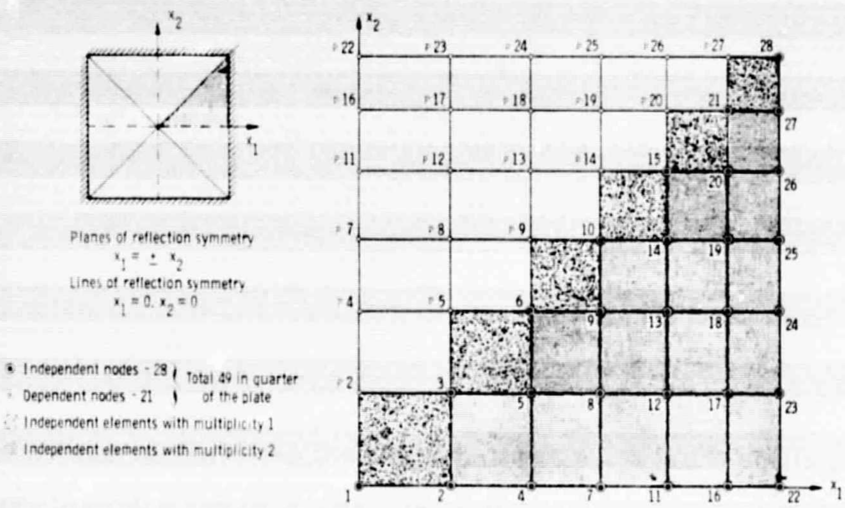


Figure 9: Independent nodes and elements for composite plates with two planes of reflection symmetry and two lines of reflection symmetry.

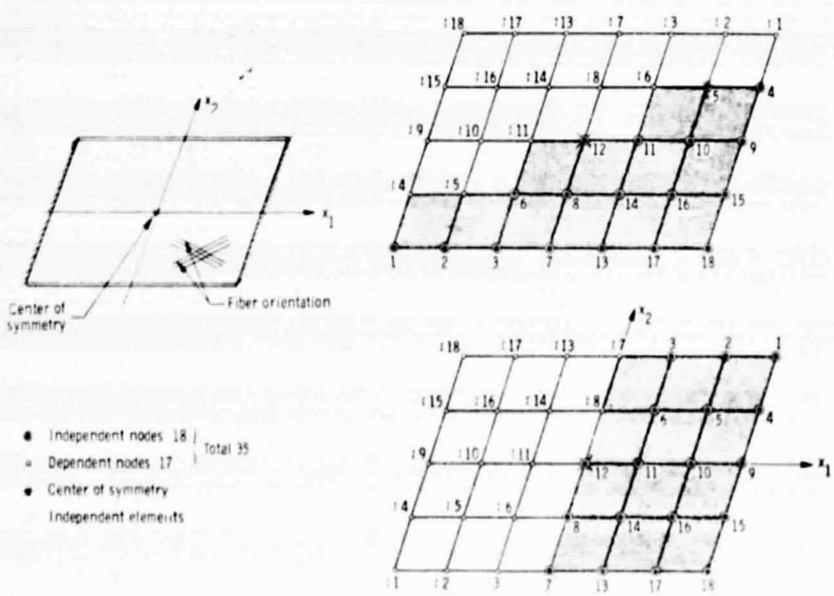


Figure 10: Independent nodes and elements for composite skew plates with one center of symmetry.

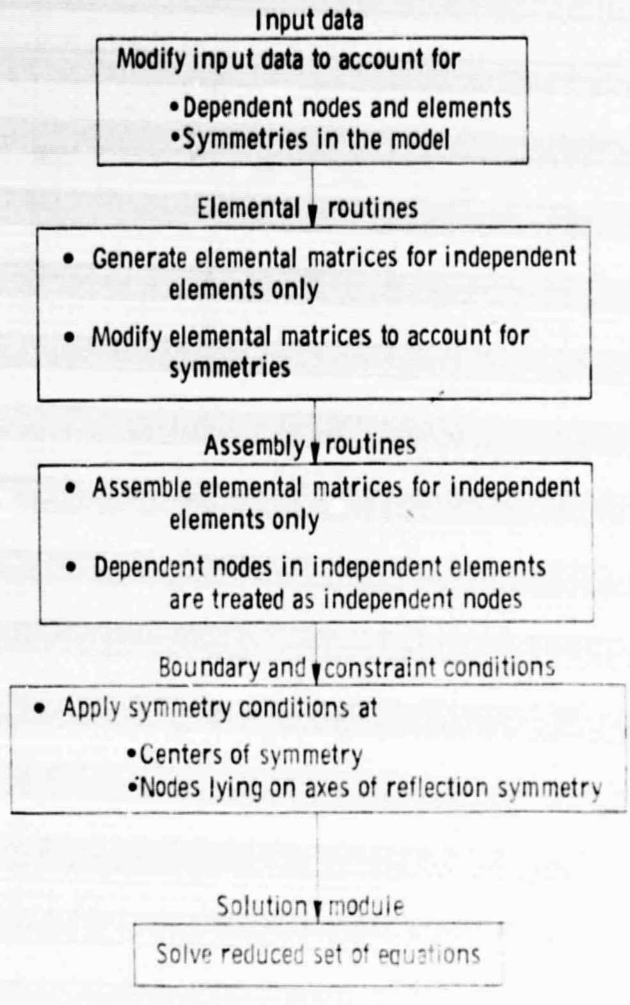


Figure 11 : Flow chart of the procedure for exploiting symmetries in finite element analysis.