Accepted by Journal of Plasma Physics

NGL-16-001-043

U. of Iowa 75-44

HIGH-BETA TURBULENCE IN TWO-DIMENSIONAL MAGNETOHYDRODYNAMICS

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December 1975

ABSTRACT

Incompressible turbulent flows are investigated in the framework of ideal magnetohydrodynamics. All the field quantities vary with only two spatial dimensions. Equilibrium canonical distributions are determined in a phase space whose coordinates are the real and imaginary parts of the Fourier coefficients for the field variables. In the geometry considered, the magnetic field and fluid velocity have variable x and y components, and all field quantities are independent of z. Three constants of the motion are found (one of them new) which survive the truncation in Fourier space and permit the construction of canonical distributions with three independent temperatures. Spectral densities are calculated. One of the more novel physical effects is the appearance of macroscopic structures involving long-wavelength, self-generated, magnetic fields ("magnetic islands") for a wide range of initial parameters. Current filaments show a tendency toward consolidation in much the same way that vorticity filaments do in the guiding-center plasma case. In the presence of finite dissipation, energy cascades to higher wave numbers can be accompanied by vector potential cascades to lower wave numbers, in much the same way that in the fluid dynamic (Navier-Stokes) case, energy cascades to lower wave numbers accompany enstrophy cascades to higher wave numbers. It is suggested that the techniques may be relevant to theories of the magnetic dynamo problem and to the generation of megagauss magnetic fields when pellets are irradiated by lasers.

1. INTRODUCTION

We have recently been involved in a number of investigations in the statistical theory of turbulence for two-dimensional flows in electrostatic guiding-center plasmas and/or inviscid Navier-Stokes fluids. Any tractable mathematical description of a turbulent continuum requires a discretization, and this has been carried out in one of two ways: either in terms of long rod-like "particles" (Onsager 1949; Taylor and McNamara 1971; Montgomery 1972; Joyce and Montgomery 1973; Montgomery and Joyce 1974; Seyler 1974; Montgomery 1975a, b; Lundgren and Pointin 1975; Pointin and Lundgren 1975), or in terms of truncated Fourier series representations of continuum fluid equations (Kraichnan 1967, 1975; Seyler, Salu, Montgomery, and Knorr 1975; Montgomery 1975a, b; Montgomery and Salu 1975).

The two discretizations lead to dynamical systems which differ in their predictions at the smallest spatial scales, but the most interesting physical phenomena have occurred at large spatial scales and have been common to both representations: for a wide range of initial conditions, the flows have shown a remarkable capacity to organize themselves into a few large, persistent vortices, comparable in cross-sectional area to the dimensions of the system. In the plasma interpretation, phase space considerations dictate that for the higher-energy states of the system, the more probable configurations consist of large regions of non-zero net charge density, around which the plasma $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts azimuthally in a direction normal to the dc magnetic field. These configurations have repeatedly appeared in numerical simulations (see the above references) and have also been studied from the point of view of time-independent self-consistent Vlasov equilibria (Book, McDonald, and Fisher 1975; McDonald 1974).

The techniques by means of which these phenomena have been investigated are of considerable generality, and the limits of their applicability have not been reached. Here we report some theoretical investigations into a situation in which the plasma flow is still two-dimensional, but with a high enough ratio of mechanical energy to magnetic energy that the magnetic field has to be self-consistently determined: colloquially, the "low beta" limit is no longer appropriate. (Heretofore, the plasma has been assumed to $\underset{\sim}{E} \times \underset{\sim}{B}$ drift across a given, static magnetic field, normal to the plane of variation of the field quantities.) Some important beginnings on the problem have been made by Frisch, Pouquet, Léorat, and Mazure (1975) who considered the <u>three-dimensional</u> magnetohydrodynamic case, with conclusions significantly different from the ones we report here for two dimensions. A two-dimensional calculation with spatial variation <u>across an external dc magnetic field</u> is due to Schumann (1975).

We proceed by means of the equations of ideal magnetohydrodynamics, and consider incompressible flows in the following geometry. There is no externally imposed component of the magnetic field, but there may be a large internally-generated one. The magnetic field $\underset{\sim}{B}$ and the fluid velocity $\underset{\sim}{v}$ have only x and y components. All quantities are independent of the z-coordinate. The electric current vector $\underset{\sim}{j}$ and the vorticity vector $\underset{\sim}{\omega}$, where $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{v}$ and $\underset{\sim}{j} = c\nabla \times \underset{\sim}{B}/4\pi$, have only z-components. Some numerical simulations for this geometry have been carried out by Tappert (1971). At any stage in the development, we may add a uniform dc magnetic field in the z-direction without changing any of the manipulations. We omit it from the development to keep the equations as simple as possible.

The omission of dissipative effects (finite conductivity or viscosity, for example) will be expected to alter the quantitative conclusions significantly, just as it does in the Navier-Stokes theory. It is common knowledge that omission of viscosity, however small, alters the inertial range spectral behavior with k for the Navier-Stokes equation. [Compare, for example, the results of Seyler et al. (1975) or Basdevant and Sadourny (1975) with those of Herring, Orszag, Kraichnan, and Fox (1974).] But qualitative conclusions, independent of the presence of small dissipation at high wave numbers, are easier to draw in the non-dissipative limit, based as they are on the existence of constants of the motion which decay slowly in the presence of a finite dissipation. The purpose of this article is not quantitative comparison with data, which would in any case be premature, but the isolation of some qualitatively new gross physical effects which have not to our knowledge been previously calculated.

In section 2, the Lundquist equations for the present geometry are written down in an appropriate set of dimensionless variables.

Each field quantity is represented by a Fourier series involving a large but finite number of terms. The Fourier coefficients are then the basic time-dependent dynamical variables of the theory. Questions of the convergence of the Fourier series results to solutions of the original Lundquist equations are non-trivial, but they are as intractable here as they have been in fluid turbulence theory and are left open, as they must be at present.

We work without reference to normal modes, eigenfrequencies, waves, perturbation theory, or any of the paraphernalia which have so unfortunately limited many plasma turbulence calculations to minor corrections to linear theory. There is no attempt to follow the evolution of the dynamics from an initial to a final state. Rather, the procedures followed are basically those of Kraichnan (1967) [see also: Seyler et al. (1975)], in which classical equilibrium ensembles are built around constants of the motion identified from the Fourier transformed equations of motion. Constants of the motion which exist for the original equations but not for their truncated Fourier representations are regarded as insufficiently rugged to enter into the statistical formulation of the problem. Identification of constants of the motion is at first glance more an art than a science, and whether one has found them all is always slightly uncertain. Nevertheless, symmetry considerations, combined with an accumulation of practical experience, lead to confidence that the possibilities have been exhausted after awhile. The striking confirmation, by Seyler et al. (1975), of the Kraichnan theory's predictions lend considerable credibility to the conjecture that only two such constants (energy and enstrophy) exist for the truncated Navier-Stokes system. Until mathematical investigations of a considerably higher order of rigor than those that have been carried out so far are performed, this is probably the best we can do. Similar reservations apply to the work of Frisch et al. (1975).

Once the constants of the motion represented in terms of the Fourier coefficients are identified, the statistical formulation of the problem is presented in a phase space whose coordinates are the real and imaginary parts of the Fourier coefficients. Canonical ensembles are constructed in this phase space by classical arguments (e.g., ter Haar 1967) in which the probability of a state is maximized subject to the constraints that the constants of the motion retain their initial values. The method of Lagrange multipliers is utilized, with the Lagrange multipliers playing the role of inverse temperatures, one for each independent constant of the motion. The derivation of the ensemble by this conventional Boltzmann-Gibbs combinatorial method differs in no important particular from the primitive derivation of the Gibbs distribution offered in any first course in statistical mechanics. From the canonical ensemble so derived, expectation values of any functions of the amplitudes of the turbulent field can be computed by straightforward integration (e.g., the spectral densities). In section 3 some of these are computed, and their variation with initial values of the constants of the motion is discussed.

Of particular interest to us has been the question of the degree to which a highly conducting turbulent fluid in which there is an initially small amount of magnetic field energy can convert a significant fraction of its turbulent mechanical energy into magnetic field energy—i.e., the degree to which the plasma can "magnetize" itself. In a number of situations (e.g., in the appearance of megagauss magnetic fields in the irradiation of pellets by lasers, or in dynamo theories of the earth's magnetic field), spontaneous large macroscopic magnetic fields are known to appear in magnetohydrodynamic fluids. While our theory has not advanced to the point where a serious quantitative confrontation with data can be undertaken for either of these phenomena, we do show in principle how such large internally-generated $\underset{\sim}{B}$ fields can develop, a result believed to be among the first of its kind insofar as it is rigorously derived, albeit within a highly simplified framework.

A second phenomenon which can be extracted is the consolidation of electric current filaments, much in parallel to the consolidation of vortices in the guiding-center plasma in two dimensions.

These and other results are commented upon further in section 4.

2. BASIC EQUATIONS AND CONSTANTS OF THE MOTION

The Lundquist equations of ideal incompressible magnetohydrodynamics are:

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \quad ,$$

$$\rho\left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v}\right) = \frac{\underset{\sim}{j}}{c} \times \underset{\sim}{B} - \nabla p \quad ,$$

$$\nabla \cdot \underset{\sim}{v} = 0 \quad ,$$

$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c} \underset{\sim}{j} \quad , \qquad (1)$$

where $\underset{\sim}{j} = (0, 0, j_z)$ is the electric current density, $\underset{\sim}{B} = (B_x, B_y, 0)$ is the magnetic field, $\underset{\sim}{v} = (v_x, v_y, 0)$ is the fluid velocity, p is the pressure, ρ is the (constant) mass density, and c is the speed of light. The vorticity $\underset{\sim}{\omega} = (0, 0, \omega_z)$ is given by

$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{v} \quad .$$

We assume throughout that $\partial/\partial z \equiv 0$, and that periodic boundary conditions apply in x and y. In the given geometry, a uniform, constant, magnetic field in the z-direction can be added without altering the equations of motion. It simplifies the expressions to omit it, so we do. It can be re-inserted at any step of the development.

We measure velocities in units of a constant velocity U_o, which may be taken to characterize the mean initial speed of the turbulent field. We measure lengths in units of L_o, which may be taken as a characteristic macroscopic length of the turbulent field. We measure times in units of L_o/U_o. We measure magnetic fields in units of $B_o \equiv \sqrt{4\pi\rho_o U_o^2}$, where ρ_o is the mass density. We measure current densities in units of $j_o \equiv cB_o/4\pi L_o$. In these units, Eqs. (1) reduce to

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \quad , \tag{2a}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} - c_s^2 \nabla p \quad , \tag{2b}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \quad , \tag{2c}$$

$$\underset{\sim}{j} = \nabla \times \underset{\sim}{B} \quad , \qquad \underset{\sim}{\omega} = \nabla \times \underset{\sim}{v} \quad . \tag{2d}$$

Here, $c_s^2 \equiv p_o/\rho_o U_o^2$ is the ratio of the characteristic mechanical pressure p_o to $\rho_o U_o^2$. As usual, taking the divergence of Eq. (2b) and using Eq. (2c) enables us to solve a Poisson equation for p, so that p is in effect a known functional of $\underset{\sim}{B}$ and $\underset{\sim}{v}$.

All quantities are expanded as Fourier series in a large square box, assuming periodic boundary conditions:

$$\underset{\sim}{B} = \sum_{\underset{\sim}{k}} \underset{\sim}{B}(\underset{\sim}{k}, t) \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x}) \quad ,$$

$$\underset{\sim}{v} = \sum_{\underset{\sim}{k}} \underset{\sim}{v}(\underset{\sim}{k}, t) \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x}) \quad ,$$

$$\omega_z = \sum_{\underset{\sim}{k}} \omega(\underset{\sim}{k}, t) \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x})$$

$$j_z = \sum_{\underset{\sim}{k}} j(\underset{\sim}{k}, t) \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x}) \quad , \tag{3}$$

where $\underset{\sim}{k}$ has only x and y components;

$$\underset{\sim}{B}(\underset{\sim}{k}) = \frac{i\underset{\sim}{k} \times \hat{e}_z}{k^2} \, j(\underset{\sim}{k}) \quad ,$$

$$\underset{\sim}{v}(\underset{\sim}{k}) = \frac{i\underset{\sim}{k} \times \hat{e}_z}{k^2} \, \omega(\underset{\sim}{k}) \quad , \tag{4}$$

and $\hat{e}_z$ is a unit vector in the z-direction.

The dynamical equations can be written entirely in terms of the scalars $\omega(\underset{\sim}{k}, t)$ and $j(\underset{\sim}{k}, t)$, the Fourier transformed vorticity current density and electrical current density. A straightforward substitution of Eqs. (3) and (4) into (2) yields

$$\frac{\partial\omega(\underset{\sim}{k}, t)}{\partial t} = \sum M_1(\underset{\sim}{r}, \underset{\sim}{p})\, \delta(\underset{\sim}{p} + \underset{\sim}{r} - \underset{\sim}{k})[\omega(\underset{\sim}{r})\omega(\underset{\sim}{p}) - j(\underset{\sim}{r})j(\underset{\sim}{p})] \qquad (5)$$

and

$$\frac{\partial j(\underset{\sim}{k}, t)}{\partial t} = \sum M_2(\underset{\sim}{r}, \underset{\sim}{p})\, \delta(\underset{\sim}{p} + \underset{\sim}{r} - \underset{\sim}{k})[j(\underset{\sim}{r})\omega(\underset{\sim}{p}) - \omega(\underset{\sim}{r})j(\underset{\sim}{p})] \quad . \qquad (6)$$

$\delta(\underset{\sim}{p} + \underset{\sim}{r} - \underset{\sim}{k}) = 1$ when its argument is zero and is zero otherwise, and

$$M_1(\underset{\sim}{r}, \underset{\sim}{p}) = M_1(\underset{\sim}{p}, \underset{\sim}{r}) = \frac{1}{2}\, \hat{e}_z \cdot (\underset{\sim}{r} \times \underset{\sim}{p})\left(\frac{1}{p^2} - \frac{1}{r^2}\right) \quad , \qquad (7)$$

$$M_2(\underset{\sim}{r}, \underset{\sim}{p}) = -M_2(\underset{\sim}{p}, \underset{\sim}{r}) = \frac{1}{2}\, \hat{e}_z \cdot (\underset{\sim}{r} \times \underset{\sim}{p})\left(\frac{k^2}{r^2 p^2}\right) \quad . \qquad (8)$$

The sums in Eqs. (5) and (6) are over all non-zero wave vectors $\underset{\sim}{p}$ and $\underset{\sim}{r}$. Dropping Eq. (6) and the j-terms in Eq. (5) reduces the system to the inviscid Navier-Stokes equation in the same geometry (Seyler *et al.* 1975). Only wave numbers permitted by the periodic boundary conditions are, of course, allowed.

Equations (5) and (6) describe an incompressible flow in the phase space whose coordinates are defined by the real and imaginary parts of the $\omega(\underset{\sim}{k}, t)$ and the $j(\underset{\sim}{k}, t)$. Hence a Liouville equation obtains in this phase space; we may seek equilibrium ensembles in this phase space in the conventional fashion, providing that we are able to find the isolating integrals (which define hypersurfaces in the phase space) or "constants of the motion". In particular, we are interested in the constants which survive the truncation which represents Eqs. (3), (5), and (6) in terms of a large but <u>finite</u> number of terms.

Many constants of the motion exist for Eqs. (2) which do <u>not</u> survive the restriction, in Eqs. (5) and (6), of values of $|\underset{\sim}{k}|$, $|\underset{\sim}{p}|$, and $|\underset{\sim}{r}|$ to a large but finite maximum. In particular, the line integral $\int_1^2 \underset{\sim}{B} \times d\underset{\sim}{\ell}$, along any curve whose end points are 1 and 2 and which moves with the fluid velocity, is time-independent according to Eqs. (2) but not according to Eqs. (5) and (6), if any finite k_{max} is allowed to restrict the values of $|\underset{\sim}{k}|$, $|\underset{\sim}{p}|$, and $|\underset{\sim}{r}|$.

We have found three constants of the motion for Eqs. (2) which <u>do</u> survive the restriction that $|\underset{\sim}{k}|$, $|\underset{\sim}{p}|$, $|\underset{\sim}{r}|$ are required to lie in an annulus between k_{min} and k_{max} (k_{min} is just 2π divided by the box size). We believe that they are the only ones, but cannot rigorously prove it. They are the energy ,

$$\mathcal{E} = \frac{1}{2}\int(\underset{\sim}{v}^2 + \underset{\sim}{B}^2)\, dxdy \quad ; \qquad (9a)$$

the "cross helicity" P (Woltjer 1958, and Frisch *et al.* 1975),

$$P \equiv \frac{1}{2}\int \underset{\sim}{B} \cdot \underset{\sim}{v}\, dxdy \quad ; \tag{9b}$$

and a new integral, the mean square vector potential,

$$A \equiv \frac{1}{2}\int \mathcal{Q}^2\, dxdy \quad , \tag{9c}$$

where $\underset{\sim}{B} = \nabla \times \mathcal{Q}$, and $\mathcal{Q}$ is the vector potential in the gauge for which $\nabla \cdot \mathcal{Q} = 0$. The integrals in (9) run over the basic box.

The Fourier series representations of $\mathcal{E}$, P, and A are

$$\mathcal{E} = \frac{1}{2}\sum_{\underset{\sim}{k}} \frac{|\omega(\underset{\sim}{k})|^2 + |j(\underset{\sim}{k})|^2}{k^2} \quad ,$$

$$P = \frac{1}{2}\sum_{\underset{\sim}{k}} \frac{\omega(\underset{\sim}{k})\ j(-\underset{\sim}{k})}{k^2} \quad ,$$

$$A = \frac{1}{2}\sum_{\underset{\sim}{k}} \frac{|j(\underset{\sim}{k})|^2}{k^4} \quad . \tag{10}$$

(The time arguments are conveniently omitted.) The sums in Eqs. (10)

are over all the $\underset{\sim}{k}$ values allowed by the periodic boundary conditions between k_{min} and a large but finite k_{max}, which may be chosen arbitrarily. It is an easy manipulation to show that the expressions given in Eqs. (10) are conserved by Eqs. (5) and (6).

The canonical distribution, to be inferred by entirely conventional arguments, is then

$$D = \eta \exp\left(-\alpha\mathcal{E} - \beta P - \gamma A\right) \quad , \tag{11}$$

where D is the probability distribution in the many-dimensional phase space whose coordinates are the real and imaginary parts of

$$\omega(\underset{\sim}{k}) = \omega_r(\underset{\sim}{k}) + i\omega_i(\underset{\sim}{k})$$

and

$$j(\underset{\sim}{k}) = j_r(\underset{\sim}{k}) + ij_i(\underset{\sim}{k}) \quad .$$

η is a normalizing constant to be determined by the normalization

$$\int dX\, D = 1 \quad , \tag{12}$$

where $\int dX$ is an integral over all the independent ω_r, ω_i, j_r, j_i.

[Since $\omega(\underset{\sim}{k}) = \omega^*(-\underset{\sim}{k})$, $j(\underset{\sim}{k}) = j^*(-\underset{\sim}{k})$, independent ω_r, ω_i, j_r, j_i are associated with only half the $\underset{\sim}{k}$-vectors.] α, β, γ play the roles of inverse temperatures, and are constrained by the requirements that Eq. (11) be normalizeable—i.e., that for large $|j(\underset{\sim}{k})|^2$ and $|\omega(\underset{\sim}{k})|^2$, the argument of the exponential is a monotonically decreasing quantity. This implies right away that

$$\alpha > 0 \quad . \tag{13}$$

Both γ and β may be negative, but not α.

Another class of constants of the motion which do <u>not</u> survive the truncation of the Fourier series are integrals of the form $I_n \equiv \int a^n \, dxdy$, where n is some integer and the integral is over the basic box. The constancy of I_n can be proved by using the governing equation for the vector potential,

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla\right) a = 0 \quad .$$

However, only for n = 2 does I_n remain constant in the face of the truncation.

3. CHARACTERISTICS OF THE DIFFERING TEMPERATURE REGIMES

$\mathcal{E}$, P, and A are sums of terms, each of which is indexed by a single wave number; therefore Eq. (11) predicts that in equilibrium there is no correlation between coefficients corresponding to different $\underset{\sim}{k}$'s. D factors into a product of distributions, one for each $\underset{\sim}{k}$, and the single-$\underset{\sim}{k}$ distribution function is

$$f_{\underset{\sim}{k}}[\omega_r(\underset{\sim}{k}),\ \omega_i(\underset{\sim}{k}),\ j_r(\underset{\sim}{k}),\ j_i(\underset{\sim}{k})]$$

$$= \eta_{\underset{\sim}{k}} \exp\left\{-\alpha\left[\omega_r^2(\underset{\sim}{k}) + \omega_i^2(\underset{\sim}{k}) + j_r^2(\underset{\sim}{k}) + j_i^2(\underset{\sim}{k})\right]/k^2\right.$$

$$-\beta\left[\omega_r(\underset{\sim}{k})\ j_r(\underset{\sim}{k}) + \omega_i(\underset{\sim}{k})\ j_i(\underset{\sim}{k})\right]/k^2$$

$$\left.-\gamma\left[j_r^2(\underset{\sim}{k}) + j_i^2(\underset{\sim}{k})\right]/k^4\right\} \quad , \tag{14}$$

where $\eta_{\underset{\sim}{k}}$ is a normalizing constant, given by:

$$\eta_{\underset{\sim}{k}} = \frac{\alpha}{\pi^2 k^4}\left(\alpha - \frac{\beta^2}{4\alpha} + \frac{\gamma}{k^2}\right) \quad .$$

The additional inequality required in order that the distributions be normalizable is that

$$\alpha > \frac{\beta^2}{4\alpha} - \frac{\gamma}{k^2} \tag{15}$$

for all $\underset{\sim}{k}$. If $\gamma < 0$, Eq. (15) requires that

$$\alpha > \frac{\beta^2}{4\alpha} - \frac{\gamma}{k_{min}^2} \quad , \tag{16a}$$

while if $\gamma > 0$, Eq. (15) requires that

$$\alpha > \frac{\beta^2}{4\alpha} - \frac{\gamma}{k_{max}^2} \quad . \tag{16b}$$

Expectation values computed with respect to Eq. (14) are:

$$\langle j_r^2(\underset{\sim}{k})\rangle = \langle j_i^2(\underset{\sim}{k})\rangle = \frac{k^2}{2}\left(\alpha - \frac{\beta^2}{4\alpha} + \frac{\gamma}{k^2}\right)^{-1} \tag{17}$$

and

$$\langle \omega_r^2(\underset{\sim}{k}) \rangle = \langle \omega_i^2(\underset{\sim}{k}) \rangle = \frac{1}{2}\left[\frac{k^2}{\alpha} + \frac{\beta^2}{4\alpha^2}\left(\frac{\alpha}{k^2} + \frac{\gamma}{k^4} - \frac{\beta^2}{4k^2\alpha}\right)^{-1}\right] \quad . \tag{18}$$

In terms of the velocity field and magnetic field we can determine the energy spectra; for the magnetic field, we find:

$$\langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle = \left(\alpha + \frac{\gamma}{k^2} - \frac{\beta^2}{4\alpha}\right)^{-1} \quad , \tag{19}$$

and for the velocity field:

$$\langle |\underset{\sim}{v}(\underset{\sim}{k})|^2 \rangle = \frac{1}{\alpha} + \frac{\beta^2}{4\alpha^2} \langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle = \frac{1}{\alpha} + \frac{\beta^2}{4\alpha^2}\left(\alpha + \frac{\gamma}{k^2} - \frac{\beta^2}{4\alpha}\right)^{-1} \quad . \tag{20}$$

Similarly, the expectation value $\langle \underset{\sim}{B}(\underset{\sim}{k}) \cdot \underset{\sim}{v}(-\underset{\sim}{k}) \rangle$ is

$$\langle \underset{\sim}{B}(\underset{\sim}{k}) \cdot \underset{\sim}{v}(-\underset{\sim}{k}) \rangle = \frac{\beta}{\alpha}\left(\alpha + \frac{\gamma}{k^2} - \frac{\beta^2}{4\alpha}\right)^{-1} = -\frac{\beta}{\alpha} \langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle \quad . \tag{21}$$

Summing over $\underset{\sim}{k}$ gives the immediate consequence that $\langle \underset{\sim}{B} \cdot \underset{\sim}{v} \rangle = -(\beta/\alpha) \langle \underset{\sim}{B}^2 \rangle$.

The conditions that the three expectation values of the invariants match their initial values,

$$\mathcal{E} = \frac{1}{2}\sum_{\underset{\sim}{k}}\left(\frac{1}{\alpha} + \frac{1 + \frac{\beta^2}{4\alpha^2}}{\alpha + \frac{\gamma}{k^2} - \frac{\beta^2}{4\alpha}}\right) = \frac{1}{2\alpha}\sum_{\underset{\sim}{k}} \frac{2 + \frac{\gamma}{k^2\alpha}}{1 + \frac{\gamma}{k^2\alpha} - \frac{\beta^2}{4\alpha^2}} ,$$

$$P = \frac{1}{2}\frac{\beta}{\alpha}\sum_{\underset{\sim}{k}} \frac{1}{\alpha + \frac{\gamma}{k^2} - \frac{\beta^2}{4\alpha}} ,$$

$$A = \sum_{\underset{\sim}{k}} \frac{1}{k^2\alpha + \gamma - \frac{\beta^2 k^2}{4\alpha}} \tag{22}$$

are to be regarded as three simultaneous algebraic equations which determine α, β, γ in terms of $\mathcal{E}$, P, A. Roughly speaking, β/α measures the correlation energy of the magnetic field and velocity field. γ/α is a more subtle parameter, but it will be seen presently that it measures indirectly the ratio of long-wavelength magnetic energy to total energy to be expected for the equilibrium coefficients.

A wide range of spectral behavior is possible for the different ranges of the initial parameters $\mathcal{E}$, P, and A. One regime of importance is the case $P \cong 0$. This is the case in which the cross correlation $\langle \underset{\sim}{B} \cdot \underset{\sim}{v} \rangle = 0$, either because $\underset{\sim}{B}$ is initially uncorrelated with $\underset{\sim}{v}$ or else is perpendicular to it. For this case, $\beta = 0$ and the spectral densities simply reduce to

$$\langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle = (\alpha + \gamma k^{-2})^{-1} \quad ,$$

$$\langle |\underset{\sim}{v}(\underset{\sim}{k})|^2 \rangle = \alpha^{-1} \quad . \tag{24}$$

Both $\langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle$ and $\langle |\underset{\sim}{v}(\underset{\sim}{k})|^2 \rangle$ approach constants (equipartition spectra) at large $|\underset{\sim}{k}|$, but $\langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle$ can be peaked at $|\underset{\sim}{k}| = k_{min}$, the longest wavelengths, for $\gamma < 0$ and $\alpha + \gamma/k^2_{min}$ small. This regime always occurs if, for fixed $\mathcal{E}$ and A, we allow k_{max} to become large enough. This corresponds closely to the "Regime I" sharply-peaked energy spectra for the Navier-Stokes case (Kraichnan 1975, Seyler et al. 1975), in which large macroscopic vortices occur; here, it is "magnetic islands" which appear. The vector potential spectral density, $\langle |\underset{\sim}{B}(\underset{\sim}{k})|^2/k^2 \rangle$, is here dominated by the lowest values of $|\underset{\sim}{k}|$, while the extra energy is divided approximately equally among the highest $\underset{\sim}{k}$-values. The magnetic field configuration for this $\gamma < 0$ regime is dominated by the largest wavelength contributions, $\underset{\sim}{k} = (k_{min}, 0)$ and $(0, k_{min})$, and can be graphically described as "magnetic islands" formed in the basic box, with essentially two regions of current densities of opposite sign dominating the flow pattern. From Eqs. (24) we can readily show that

$$\sum_{\underset{\sim}{k}} \langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle - \sum_{\underset{\sim}{k}} \langle |\underset{\sim}{v}(\underset{\sim}{k})|^2 \rangle = \frac{-2\gamma A}{\alpha} \quad . \tag{25}$$

Since $2A/\alpha$ is intrinsically positive, it is apparent that the magnetic field energy can have an expectation value which is greater or less than that of the kinetic energy, depending upon the sign of γ. $\gamma = 0$ is a natural boundary in the parameter space of the problem. For these $\langle \underset{\sim}{B} \cdot \underset{\sim}{v} \rangle = 0$ situations, the sign of γ is basically determined by the ratio $A/\mathcal{E}$, going from a minimum at $A/\mathcal{E} = k_{min}^{-2}$ to a maximum at $A/\mathcal{E} = 0$. $\gamma = 0$ when

$$\frac{A}{\mathcal{E}} = \frac{\left(\sum_{\underset{\sim}{k}} k^{-2} \right)}{\sum_{\underset{\sim}{k}}} ,$$

and this ratio is, to a good approximation, $k_{max}^{-2} \ln (k_{max}/k_{min})$; for this value, $\langle v^2 \rangle = \langle B^2 \rangle$. For $A/\mathcal{E}$ greater than this value (which $\to 0$ as $k_{max} \to \infty$), $\gamma < 0$ and thus for fixed A and $\mathcal{E}$, we always enter this regime if k_{max} is large enough. This regime is closely analogous to Kraichnan's Regime I in the Navier-Stokes case. Note that Eq. (24) predicts that when $\gamma < 0$, $\langle |\underset{\sim}{B}(\underset{\sim}{k})|^2 \rangle$ is greater than $\langle |\underset{\sim}{v}(\underset{\sim}{k})|^2 \rangle$ for all $\underset{\sim}{k}$, though only at the longer wavelengths is the difference significant.

As γ becomes more negative, the configuration increasingly resembles the limiting case of $A/\mathcal{E} = k_{min}^{-2}$, which keeps all the energy locked into the two longest wavelength magnetic contributions. Two macroscopic electric current filaments of opposite senses occupy the basic box in which the Fourier analysis is performed and have cross-sectional areas comparable to it. To the extent that the periodic boundary conditions could be thought of as simulating a finite physical system, the external magnetic field to be observed would be that due to two parallel current carrying wires with currents in opposite directions: a quadrupole field in two dimensions.

The limiting case of $A/\mathcal{E} = 0$ corresponds to $\gamma \to \infty$, α = [total number of allowed $\underset{\sim}{k}$'s]/[total energy]. There is no magnetic field energy at all, and a flat equipartition spectrum results for the velocity field.

Between these two limits lies a continuous variation of configurations characterized, as γ decreases, by a larger degree of long-range order, with a more and more sharply peaked deposition of magnetic field energy at the longest wavelengths. The parallel with the formation of large vortices in the Navier-Stokes case, as the ratio of energy to entropy increases, is striking.

4. DISCUSSION

Qualitatively, the phenomena predicted in the previous section are similar to a number of real situations in which a magnetic field of macroscopic dimensions has been observed to appear in a highly conducting fluid. Among these, one may list the spontaneous generation of megagauss magnetic fields in laser-irradiated pellets, the persistence of "magnetic dynamo" fields produced in the earth's core, and the consolidation of current filaments in numerical simulations of energetic electron-beam situations (Lee and Lampe 1973). For all of these situations, the geometry here assumed is an over-simplification, and detailed comparison with data would be premature, though the parallels with Lee and Lampe, in particular, are interesting.

Nevertheless, we find the emergence of the development of these spontaneously generated magnetic field structures from the mathematics to have occurred in an unstrained and natural fashion. The phenomena would seem much less convincing to us if they had emerged as the end product of a time-dependent theory, which would undoubtedly have required any number of hard-to-justify approximations, than they are, emerging as they have only from arguments of an essentially thermodynamic character. At this stage, we are in the process of trying to discover the outlines of new dynamical effects, and the process of making them precise by including more realistic geometries and finite dissipation terms is likely to be as slow as it has been for Navier-Stokes fluids.

However, we believe the phenomena are interesting enough to stand on their own at this point as new predictions which may be subject to verification by numerical solution of the magnetohydrodynamic equations. What the theory seems to be pointing toward is a class of macroscopic non-uniform states of non-trivial geometry, which the plasma regards as somehow natural and most probable. Because of the thermodynamic nature of their derivation, they almost certainly are stable. The class of configurations so described may well include some which various confinement schemes are and have been in search of by less systematic means.

Acknowledgment

This work has been supported in part by NASA Grant NGL 16-001-043.

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