Polytechnic Institute of New York

Department of
Aerospace Engineering
and Applied Mechanics

CALCULATION OF THE THREE-DIMENSIONAL, SUPERSONIC, INVISCID, STEADY FLOW PAST AN ARROW-WINGED AIRFRAME

PART I

By Gino Moretti

MAY 1976

(NASA-CR-147230) CALCULATION OF THE THREE-DIMENSION, SUPERSONIC, INVISCID, STEADY FLOW PAST AN ARROW-WINGED AIRFRAME, PART 1 Interim Report (Polytechnic Inst. of New York.) 38 p HC $4.00 CSCL 01A

N76-23160

Unclas

G3/02 28094

Grant No. NGL 1248

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POLY-AE/AM REPORT NO. 76-8

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER POLY-AE/AM Report No. 76-8	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CALCULATION OF THE THREE-DIMENSIONAL, SUPERSONIC, INVISCID, STEADY FLOW PAST AN ARROW-WINGED AIRFRAME. PART I		5. TYPE OF REPORT & PERIOD COVERED Scientific Interim 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) GINO MORETTI		8. CONTRACT OR GRANT NUMBER(s) NSG 1248
9. PERFORMING ORGANIZATION NAME AND ADDRESS Polytechnic Institute of New York Aerodynamics Laboratories Route 110, Farmingdale, New York 11735		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS NASA Langley Research Center Hampton, Virginia 23365		12. REPORT DATE May 1976 13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Supersonic flow
Numerical gas dynamics
Conformal mapping
Finite difference techniques

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (Contd.)

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This research was supported by the NASA Langley Research Center, Hampton, Virginia, under Grant No. NSG 1248.

POLYTECHNIC INSTITUTE OF NEW YORK

Department

of

Aerospace Engineering and Applied Mechanics

May 1976

POLY-AE/AM Report No. 76-8

CALCULATION OF THE THREE-DIMENSIONAL, SUPERSONIC INVISCID, STEADY FLOW PAST AN ARROW-WINGED AIRFRAME[+]

PART I

by

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ABSTRACT

A detailed description of the procedure used to compute three-dimensional, supersonic, inviscid, steady flows past airframes is given. No limitations are imposed on the geometry of the airplane. Suitable computational grids are generated by automatic conformal mappings. The equations of motion, with pressure, entropy, and velocity direction as basic unknowns, are written and discretized in the computational space. Special rules to approximate derivatives are given. Boundary points are treated by a modified method of characteristics. No provision is made for imbedded shocks. Description of sample geometries and results of calculation will be given in Part II.

[+]This research was supported by the NASA Langley Research Center, Hampton, Virginia, under Grant No. NSG 1248.

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I. INTRODUCTION

In the present Report a detailed description is given of a basic computational program for the evaluation of three-dimensional, supersonic, inviscid, steady flows past airplanes. For the sake of simplicity, no imbedded shocks are considered here. The emphasis is put instead on how a powerful, automatic mapping technique is coupled to the fluid mechanical analysis in order to assure a high degree of accuracy without increasing the number of computational nodes beyond reasonable limits.

Care has been taken to describe and to code each of the three constituents of the analysis (body geometry, mapping technique, and gas dynamical effects) separately, to facilitate applications to different geometries or substitution of the present set of unknowns and equations of motion by other sets. Sections V through X contain the outline of the code dealing with gas dynamical effects; all their statements and formulas are unaffected by changes in the mapping technique or mapping parameters or in the geometry of the airplane. All expressions related to the mapping are given in Sections XI and XII. Results of computations based on sample geometries, and discussions will be presented in a separate Report.

II. FRAMES OF REFERENCE

The free stream flow is assumed to be uniform, with a given Mach number, M_∞. A Cartesian, orthogonal frame of reference, (x, y, t) is defined as having the y and t-axes in the symmetry plane of the vehicle, the t-axis lying along the fuselage. The unit vectors of the x, y, and t-axis are called $\hat{I}, \hat{J}$ and $\hat{K}$, respectively. The free stream velocity vector, $\vec{V}_\infty$, is parallel to the (y, t)-plane; the angle of attack, α, is the angle between $\vec{V}_\infty$ and $\hat{K}$; therefore,

$$\vec{V}_\infty = V_\infty(\hat{J}\sin\alpha + \hat{K}\cos\alpha) \qquad (1)$$

In each cross-sectional plane, a complex variable, z, is defined as

$$z = x + iy \qquad (2)$$

A conformal mapping (details of which will be found in Section XII) defines a one-to-one correspondence between the portion of interest of the right-hand side of the z-plane and a portion of the right-hand side of a ζ-plane where, by and large, the image of the cross-section of the airplane is nearly circular; it is convenient, thus, to express the complex variable ζ in the form:

$$\zeta = \rho\, e^{i\theta} \qquad (3)$$

The analytic function $\zeta(z)$ implies that ρ and θ are functions of x and y, and vice versa. Such functions, in general, change from one cross-section to another; therefore, we may write:

$$\begin{cases} \rho = \rho(x, y, t) \\ \theta = \theta(x, y, t) \\ \tau = t \end{cases} \qquad \begin{cases} x = x(\rho, \theta, \tau) \\ y = y(\rho, \theta, \tau) \\ t = \tau \end{cases} \qquad (4)$$

We must take good care of denoting t by another symbol, τ, when considered in connection with ρ and θ since when t changes and x, y remain unchanged, ρ and θ generally change; consequently, derivatives with respect to t (at constant x and y) generally differ from derivatives with respect to τ (at constant ρ and θ). Let $\rho = b(\theta, \tau)$ and $\rho = c(\theta, \tau)$ be the equations of the image of the airplane body contour and of the image of the bow shock in the ζ-plane. A non-conformal mapping, defined by a suitable function of ρ, θ and τ:

$$\begin{cases} X = X(\rho, \theta, \tau) \\ Y = \theta \\ T = \tau \end{cases} \qquad \begin{cases} \rho = \rho(X, Y, T) \\ \theta = Y \\ \tau = T \end{cases} \qquad (5)$$

will transform the region of interest in the right-hand side of the ζ-plane bounded by $\rho = b$ and $\rho = c$ onto a rectangle, bounded by the lines:

$$\begin{cases} X = 0, & \text{corresponding to } \rho = b \quad \text{(body)} \\ X = 1, & \text{corresponding to } \rho = c \quad \text{(bow shock)} \\ Y = -\frac{\pi}{2}, & \text{corresponding to } \theta = -\frac{\pi}{2} \quad \text{(windward symmetry line)} \\ Y = \frac{\pi}{2}, & \text{corresponding to } \theta = \frac{\pi}{2} \quad \text{(leeward symmetry line)} \end{cases} \tag{6}$$

An example of such a function, $X(\rho, \theta, \tau)$ will be discussed in Section XI.

III. DERIVATIVES RELATED TO THE MAPPINGS

Let

$$g = \frac{d\zeta}{dz} = G\, e^{i\omega} \tag{7}$$

be the complex derivative of ζ with respect to z (at t, τ, T all constant); similarly, let

$$\varpi = \frac{\zeta}{g} \frac{d \log g}{dz} = \varpi_1 + i\varpi_2 \tag{8}$$

From (3) and (7) it follows that

$$\frac{G}{\rho}\frac{\zeta}{g} = \frac{\zeta/g}{|\zeta/g|} = e^{i(\theta - \omega)} = \mathcal{C} + i\mathcal{S} \tag{9}$$

where

$$\mathcal{C} = \cos(\theta - \omega), \quad \mathcal{S} = \sin(\theta - \omega) \tag{10}$$

We introduce now the notations, ψ and f, for two analytic functions obtained by differentiating g and ζ with respect to t (that is, at constant x and y):

$$\psi = \frac{\partial \log g}{\partial t} = \psi_1 + i\psi_2 \tag{11}$$

$$f = \frac{\partial \log \zeta}{\partial t} = f_1 + i f_2 \tag{12}$$

Recalling that

$$\frac{d\zeta}{dz} = \frac{\partial(\rho\cos\theta)}{\partial x} + i\frac{\partial(\rho \sin\theta)}{\partial x} = \frac{\partial(\rho\cos\theta)}{i\,\partial y} + \frac{\partial(\rho\sin\theta)}{\partial y} \tag{13}$$

and

$$\frac{\partial \log \zeta}{\partial t} = \frac{1}{\rho}\frac{\partial \rho}{\partial t} + i\,\frac{\partial \theta}{\partial t} \tag{14}$$

we obtain:

$$\begin{cases} \rho_x = G\mathcal{C}, & \rho_y = G\mathcal{S}, & \rho_t = \rho f_1 \\ \theta_x = -\frac{G}{\rho}\mathcal{S} & \theta_y = \frac{G}{\rho}\mathcal{C}, & \theta_t = f_2 \\ \tau_x = 0, & \tau_y = 0, & \tau_t = 1 \end{cases} \tag{15}$$

Conversely, noting that

$$x_\tau = -(x_\rho\, \rho_t + x_\theta\, \theta_t), \qquad y_\tau = -(y_\rho\, \rho_t + y_\theta\, \theta_t) \tag{16}$$

we obtain:

$$\begin{cases} x_\rho = \frac{1}{G}\mathcal{C}, & x_\theta = -\frac{\rho}{G}\mathcal{S}, & x_\tau = -(\mathcal{C}f_1 - \mathcal{S}f_2)\frac{\rho}{G} \\ y_\rho = \frac{1}{G}\mathcal{S}, & y_\theta = \frac{\rho}{G}\mathcal{C}, & y_\tau = -(\mathcal{S}f_1 + \mathcal{C}f_2)\frac{\rho}{G} \\ t_\rho = 0, & t_\theta = 0, & t_\tau = 1 \end{cases}$$

Between the two sets, (ρ, θ, τ) and (X, Y, T), the following relations hold:

$$\begin{cases} \rho_X = \frac{1}{X_\rho}, & \rho_Y = -\frac{X_\theta}{X_\rho}, & \rho_T = -\frac{X_\tau}{X_\rho} \\ \theta_X = 0, & \theta_Y = 1, & \theta_T = 0 \\ \tau_X = 0, & \tau_Y = 0, & \tau_T = 1 \end{cases} \tag{18}$$

$$\begin{cases} X_\rho = \frac{1}{\rho_X}, & X_\theta = -\frac{\rho_Y}{\rho_X}, & X_\tau = -\frac{\rho_T}{\rho_X} \\ Y_\rho = 0, & Y_\theta = 1, & Y_\tau = 0 \\ T_\rho = 0, & T_\theta = 0, & T_\tau = 1 \end{cases} \tag{19}$$

By combining (17) and (18), we obtain:

$$
\begin{cases}
x_X = \dfrac{\mathcal{C}}{GX_\rho}\,, & x_Y = -\dfrac{\rho}{G}\left(\mathcal{C}\,\dfrac{X_\theta}{\rho X_\rho} + \mathcal{S}\right)\,, & x_T = -\dfrac{\mathcal{C}}{G}\,\dfrac{X_\tau}{X_\rho} - \left(\mathcal{C}f_1 - \mathcal{S}f_2\right)\dfrac{\rho}{G} \\[2ex]
y_X = \dfrac{\mathcal{S}}{GX_\rho}\,, & y_Y = \dfrac{\rho}{G}\left(-\mathcal{S}\,\dfrac{X_\theta}{\rho X_\rho} + \mathcal{C}\right)\,, & y_T = -\dfrac{\mathcal{S}}{G}\,\dfrac{X_\tau}{X_\rho} - \left(\mathcal{S}f_1 + \mathcal{C}f_2\right)\dfrac{\rho}{G} \\[2ex]
t_X = 0\,, & t_Y = 0\,, & t_T = 1
\end{cases}
\tag{20}
$$

The following formulae are also obtained easily:

$$
\begin{cases}
G_\rho = \dfrac{G}{\rho}\omega_1\,, & G_\theta = -G\omega_2\,, & G_\tau = G[\psi_1 - \omega_1 f_1 + \omega_2 f_2] \\[2ex]
\omega_\rho = \dfrac{G}{\rho}\varphi_2\,, & \omega_\theta = G\omega_1\,, & \omega_\tau = \psi_2 - \varphi_1 f_2 + \omega_2 f_1
\end{cases}
\tag{21}
$$

$$
\begin{cases}
G_x = \dfrac{G^2}{\rho}(\mathcal{C}\omega_1 + \mathcal{S}\omega_2)\,, & G_y = \dfrac{G^2}{\rho}(\mathcal{S}\omega_1 - \mathcal{C}\varphi_2)\,, & G_t = G\psi_1 \\[2ex]
\omega_x = -\dfrac{G}{\rho}(\mathcal{S}\omega_1 - \mathcal{C}\varphi_2)\,, & \omega_y = \dfrac{G}{\rho}(\mathcal{C}\varphi_1 + \mathcal{S}\varphi_2)\,, & \omega_t = \psi_2
\end{cases}
\tag{22}
$$

IV. IMPORTANT UNIT VECTORS

We begin this section by defining a ρ-line on a τ=constant (physical) cross-sectional plane as a line along which θ=constant; similarly, a θ-line will be a line on the t=constant plane along which ρ=constant. The unit vectors, $\hat{i}$ and $\hat{j}$ will be used to identify the tangents to a θ-line and to a ρ-line respectively. Note that

$$
\begin{cases}
\hat{i} = \mathcal{C}\,\hat{I} + \mathcal{S}\,\hat{J} \\
\hat{j} = -\mathcal{S}\,\hat{I} + \mathcal{C}\,\hat{J}
\end{cases}
\qquad
\begin{cases}
\hat{I} = \mathcal{C}\,\hat{i} - \mathcal{S}\,\hat{j} \\
\hat{J} = \mathcal{S}\,\hat{i} + \mathcal{C}\,\hat{j}
\end{cases}
\tag{23}
$$

By using (20 and (23) for any point,

$$
Q = x\,\hat{I} + y\,\hat{J} + t\,\hat{K} \tag{24}
$$

we obtain:

$$\begin{cases} Q_X = \dfrac{1}{G X_\rho}\,\hat{i} \\ Q_Y = \dfrac{\rho}{G}\left(-\dfrac{X_\theta}{\rho X_\rho}\,\hat{i} + \hat{j}\right) \\ Q_T = -\dfrac{\rho}{G}\left(\dfrac{X_\tau}{\rho X_\rho} + f_1\right)\hat{i} - \dfrac{\rho}{G} f_2\,\hat{j} + \hat{K} \end{cases} \tag{25}$$

The unit vector, $\hat{N}$, normal to an X=constant surface, is important for the calculation of body and bow shock points. The body, indeed, is defined by X=0 and the bow shock by X=1. In general,

$$\hat{N} = N_1\,\hat{i} + N_2\,\hat{j} + N_3\,\hat{K} = \frac{1}{|Q_Y \times Q_T|}\, Q_Y \times Q_T \tag{26}$$

where Q is a point on the surface. From (25) it follows that

$$N_1 = \frac{1}{\nu}, \qquad N_2 = \frac{X_\theta}{\rho X_\rho} N_1, \qquad N_3 = N_1 d \tag{27}$$

with

$$d = \frac{1}{G}\left(\frac{X_\tau}{X_\rho} + \frac{X_\theta}{X_\rho} f_2 + \rho f_1\right), \qquad \nu = \sqrt{1 + \left(\frac{X_\theta}{\rho X_\rho}\right)^2 + d^2} \tag{28}$$

In particular, at the body, from (18),

$$\frac{X_\theta}{\rho X_\rho} = -\frac{b_Y}{b}, \qquad \frac{X_\tau}{X_\rho} = -b_T \qquad \text{(Body)} \tag{29}$$

Note also that we can write b_Y or b_θ, and b_T or b_τ, indifferently. Therefore, at the body (27) and (28) take on the form:

$$N_1 = \frac{1}{\nu}, \qquad N_2 = -\frac{b_Y}{b} N_1, \qquad N_3 = N_1 d \qquad \text{(Body)} \tag{30}$$

$$d = -\frac{1}{G}\left(b_T + b_Y f_2 - b f_1\right), \qquad \nu = \sqrt{1 + \left(\frac{b_Y}{b}\right)^2 + d^2} \qquad \text{(Body)} \tag{31}$$

Similarly, at the shock,

$$\text{(Shock)}\begin{cases} \dfrac{X_\theta}{cX_\rho} = -\dfrac{c_Y}{c}, \quad \dfrac{X_\tau}{X_\rho} = -c_T & (32) \\[2ex] N_1 = \dfrac{1}{\nu}, \quad N_2 = -\dfrac{c_Y}{c} N_1, \quad N_3 = N_1 d & (33) \\[2ex] d = -\dfrac{1}{G}\left(c_T + c_Y f_2 - cf_1\right), \quad \nu = \sqrt{1 + \left(\dfrac{c_Y}{c}\right)^2 + d^2} & (34) \end{cases}$$

Let

$$\mathcal{F}(x, y, t) = 0 \tag{35}$$

define the geometry of the body in the physical space. The image of the body in the (ρ, θ, τ) space is

$$\rho = b(\theta, \tau) \tag{36}$$

To evaluate (30) and (31), that is the normal to the body, we need b_θ/b and b_τ. At τ=constant,

$$\mathcal{F}_x(x_\rho b_\theta + x_\theta) + \mathcal{F}_y(y_\rho b_\theta + y_\theta) = 0 \tag{37}$$

Consequently, and using (17):

$$\frac{b_\theta}{b} = \frac{S\mathcal{F}_x - C\mathcal{F}_y}{C\mathcal{F}_x + S\mathcal{F}_y} \tag{38}$$

Similarly, at θ=constant,

$$\mathcal{F}_x(x_\rho b_\tau + x_\tau) + \mathcal{F}_y(y_\rho b_\tau + y_\tau) + \mathcal{F}_t = 0 \tag{39}$$

and

$$b_\tau = -f_2 b_\theta + b f_1 - G \frac{\mathcal{F}_t}{C\mathcal{F}_x + S\mathcal{F}_y} \tag{40}$$

Therefore, at the body,

$$d = \frac{\mathcal{F}_t}{C\mathcal{F}_x + S\mathcal{F}_y} \qquad \text{(Body)} \tag{41}$$

and (30, (31) can be replaced by the simpler expressions:

$$N_1 = \frac{C\mathcal{F}_x + S\mathcal{F}_y}{\nu}, \qquad N_2 = \frac{C\mathcal{F}_y - S\mathcal{F}_x}{\nu}, \qquad N_3 = \frac{\mathcal{F}_t}{\nu} \qquad \text{(Body)}$$

$$\nu = \sqrt{\mathcal{F}_x^2 + \mathcal{F}_y^2 + \mathcal{F}_t^2} \qquad \text{(Body)} \tag{42}$$

The above formulae are general. For any particular geometry, $\mathcal{F}_x$, $\mathcal{F}_y$, and $\mathcal{F}_\theta$ must be evaluated.

V. EQUATIONS OF MOTION

Having chosen a suitable reference length, x_{ref}, the pressure, density and temperature of the free stream are chosen as reference pressure, density and temperature, respectively (p_{ref}, ρ_{ref}, $\mathcal{T}_{ref}$). With $p, \rho, \mathcal{T}$ measuring non-dimensional quantities, the equation of state is then

$$p = \rho \mathcal{T} \tag{43}$$

The reference velocity, u_{ref}, is defined by

$$u_{ref}^2 = p_{ref}/\rho_{ref} = \mathcal{R}\,\mathcal{T}_{ref} \tag{44}$$

where $\mathcal{R}$ is the gas constant divided by the molecular weight of air. The speed of sound in the free stream, in a non-dimensional form, is then

$$a_\infty = \sqrt{\gamma} \tag{45}$$

The logarithm of pressure is denoted by P:

$$P = \ln p \tag{46}$$

A non-dimensional entropy, S (which is the difference between the local entropy and the free stream entropy divided by c_v) is related to non-dimensional temperature and pressure by

$$S = \gamma \ln \mathcal{T} - (\gamma - 1)P, \qquad \mathcal{T} = \exp\left(\frac{\gamma-1}{\gamma}P + \frac{1}{\gamma}S\right) \tag{47}$$

Euler's equations of motion in non-dimensional form are:

$$\left\{\begin{array}{l} \vec{V}\cdot\nabla P + \gamma\nabla\cdot\vec{V} = 0 \\ \frac{1}{2}\nabla(\vec{V}^2) - \vec{V}\times\nabla\times\vec{V} + \mathcal{J}\nabla P = 0 \\ \vec{V}\cdot\nabla S = 0 \end{array}\right. \tag{48}$$

With $\hat{k} \equiv \hat{K}$, let

$$\vec{V} = w(\vec{\chi} + \hat{k}) \tag{49}$$

where

$$\vec{\chi} = \sigma\,\hat{i} + \eta\hat{j} \tag{50}$$

and let

$$\nabla_1 = \frac{\partial}{\partial x}\hat{I} + \frac{\partial}{\partial y}\hat{J} \tag{51}$$

Note that

$$\left\{\begin{array}{l} \vec{V}\cdot\nabla P = w(\vec{\chi}+\hat{k})\cdot(\nabla_1 P + P_t\hat{k}) = w\,\vec{\chi}\cdot\nabla_1 P + w P_t \\ \nabla\cdot\vec{V} = \nabla_1\cdot[w(\vec{\chi}+\hat{k})] + w_t = \vec{\chi}\cdot\nabla_1 w + w\nabla_1\cdot\vec{\chi} + w_t \\ \frac{1}{2}\nabla(\vec{V}^2) = w\,\nabla w\,(\vec{\chi}^2+1) + \frac{1}{2}w^2\,\nabla_1\,\vec{\chi}^2 + \frac{1}{2}w^2\,(\vec{\chi}^2)_t\,\hat{k} \\ \vec{V}\times\nabla\times\vec{V} = -\,w(\nabla_1 w\cdot\vec{\chi}+w_t)(\vec{\chi}+\hat{k}) + w\nabla w(\vec{\chi}^2+1) + w^2[(\vec{\chi}_t\cdot\vec{\chi})\hat{k} - \vec{\chi}_t + \vec{\chi}\times\nabla_1\times\vec{\chi}] \end{array}\right. \tag{52}$$

therefore, (48) take the form:

$$\left\{\begin{array}{l} w(\vec{\chi}\cdot\nabla_1 P + P_t) + \gamma(\vec{\chi}\cdot\nabla_1 w + w\nabla_1\cdot\vec{\chi} + w_t) = 0 \\ \frac{1}{2}w^2\,\nabla_1\,\vec{\chi}^2 + w(\nabla_1 w\cdot\vec{\chi} + w_t)\vec{\chi} + w^2\,\vec{\chi}_t - w^2\vec{\chi}\times\nabla_1\times\vec{\chi} + \mathcal{J}\nabla_1 P = 0 \\ w(\nabla_1 w\cdot\vec{\chi} + w_t) + \mathcal{J}P_t = 0 \\ \vec{\chi}\cdot\nabla_1 S + S_t = 0 \end{array}\right. \tag{53}$$

The third of these equations can be used to simplify the first and second equation; finally, the following system is obtained:

$$
\begin{cases}
(1 - \frac{a^2}{w^2})P_t + \vec{\chi} \cdot \nabla_1 P + \gamma \nabla_1 \cdot \vec{\chi} = 0 \\
\frac{1}{2} \nabla_1 \vec{\chi}^2 - \vec{\chi} \times \nabla_1 \times \vec{\chi} + \frac{\mathcal{T}}{w^2} (\nabla_1 P - P_t \vec{\chi}) + \vec{\chi}_t = 0 \\
\vec{\chi} \cdot \nabla_1 S + S_t = 0
\end{cases}
\tag{54}
$$

The third of (53) is not needed; the above system is composed of four scalar equations for the two unknown scalar, P and S and the two-component unknown vector, $\vec{\chi}$. Once P and S are determined, $\mathcal{T}$ is obtained from (47); the modulus of the velocity, q, is obtained from

$$
q^2 = \frac{2\gamma}{\gamma - 1} (\mathcal{T}_o - \mathcal{T}) \tag{55}
$$

where $\mathcal{T}_o$ is the (non-dimensional) stagnation temperature, and w follows from

$$
w^2 (1 + \sigma^2 + \tau^2) = q^2 \tag{56}
$$

There are definite advantages in using (54) as a basic system of equations (instead of (48) or of equations in divergence form). Not only it contains only four differential equations to be integrated, but it provides a clear separation of unknowns, S on one side and P and $\vec{\chi}$ on the other side, which is particularly welcome in problems where strong entropy gradients occur (Refs. 1, 2). Another advantage of (54) stems from the fact that ∇_1 operates on the (x, y) plane only; therefore, it can be expressed in terms of ρ and θ as independent variables, and using $\hat{i}$ and $\hat{j}$ as unit vectors. In particular, note that

$$\nabla_1 P = G(P_\rho \hat{i} + \frac{1}{\rho} P_\theta \hat{j})$$

$$\left\{\begin{aligned} \vec{\chi}\cdot\nabla_1 P &= G(\sigma P_\rho + \frac{\eta}{\rho} P_\theta) \\ \vec{\chi} \times \nabla_1 \times \vec{\chi} &= \frac{G^2}{\rho}[(\frac{\rho\eta}{G})_\rho - (\frac{\sigma}{G})_\theta](\eta\hat{i} - \sigma\hat{j}) \\ \nabla_1 \cdot \vec{\chi} &= \frac{G^2}{\rho}[(\frac{\rho\sigma}{G})_\rho + (\frac{\eta}{G})_\theta] \end{aligned}\right. \tag{57}$$

$$\frac{1}{2}\nabla_1 (\vec{\chi}^2) = G[(\sigma\sigma_\rho + \eta\eta_\rho)\hat{i} + \frac{1}{\rho}(\sigma\sigma_\theta + \eta\eta_\theta)\hat{j}]$$

$$\vec{\chi}_t = [\sigma_t + (\omega_t - \theta_t)\eta]\hat{i} + [\eta_t + (\theta_t - \omega_t)\sigma]\hat{j}$$

These expressions can be substituted into (54); in doing it, however, note that t also must be substituted by τ, and that, for any function, Φ:

$$\Phi_t = \Phi_\tau + \Phi_\rho \rho_t + \Phi_\theta \theta_t = \Phi_\tau + \rho f_1 \Phi_\rho + f_2 \Phi_\theta \tag{58}$$

Using the notations:

$$\left\{\begin{aligned} &\varkappa = 1 - \frac{a^2}{w^2} , \\ &A_1 = G\frac{\sigma}{\varkappa} + \rho f_1 , \qquad A_2 = \frac{G}{\rho}\frac{\eta}{\varkappa} + f_2 \\ &B_1 = G\sigma + \rho f_1 , \qquad B_2 = \frac{G}{\rho}\eta + f_2 \\ &D = \frac{G}{\rho}[\eta(1 - \varphi_1) - \sigma\varphi_2] + f_2 - \psi_2 \end{aligned}\right. \tag{59}$$

and taking (21) and (22) into account, (54) become:

$$\left\{\begin{aligned} &P_\tau + A_1 P_\rho + A_2 P_\theta + \frac{\gamma G}{\varkappa}(\sigma_\rho + \frac{1}{\rho}\eta_\theta) + \frac{\gamma G}{\varkappa\rho}[\sigma(1 - \varphi_1) + \eta\varphi_2] = 0 \\ &\sigma_\tau + B_1\sigma_\rho + B_2\sigma_\theta + \frac{\mathcal{J}}{w^2}[-\sigma P_\tau + (G - \sigma\rho f_1)P_\rho - \sigma f_2 P_\theta] - \eta D = 0 \\ &\eta_\tau + B_1\eta_\rho + B_2\eta_\theta + \frac{\mathcal{J}}{w^2}[-\eta P_\tau - \rho\eta f_1 P_\rho + (\frac{G}{\rho} - \eta f_2)P_\theta] + \sigma D = 0 \\ &S_\tau + B_1 S_\rho + B_2 S_\theta = 0 \end{aligned}\right. \tag{60}$$

The final form of the equations of motion is obtained by expressing the derivatives in terms of X, Y, and T, considering that, for any function, Φ:

$$\begin{cases} \Phi_\rho = \Phi_X X_\rho \\ \Phi_\theta = \Phi_Y + \Phi_X X_\theta \\ \Phi_\tau = \Phi_T + \Phi_X X_\tau \end{cases} \tag{61}$$

and, consequently:

$$\begin{cases} \Phi_\tau + B_1 \Phi_\rho + B_2 \Phi_\theta = \Phi_T + E \Phi_X + B_2 \Phi_Y \\ \Phi_\tau + A_1 \Phi_\rho + A_2 \Phi_\theta = \Phi_T + C \Phi_X + A_2 \Phi_Y \end{cases} \tag{62}$$

where

$$C = X_\tau + A_2 X_\rho + A_2 X_\theta , \qquad E = X_\tau + B_1 X_\rho + B_2 X_\theta \tag{63}$$

With the additional notations:

$$\begin{cases} L = \sigma(1 - \varphi_1) + \eta \varphi_2 \\ F = - \sigma X_\tau + (G - \sigma \rho f_1) X_\rho - \sigma f_2 X_\theta \\ H = - \eta X_\tau - \rho \eta f_1 X_\rho + (\frac{G}{\rho} - \eta f_2) X_\theta \end{cases} \tag{64}$$

the equations to be integrated at every grid point, except on the body and on the bow shock, are:

$$\begin{cases} P_T + C P_X + A_2 P_Y + \frac{\gamma G}{\varkappa} [X_\rho \sigma_X + \frac{1}{\rho}(\eta_Y + X_\theta \eta_X + L)] = 0 \\ \sigma_T + E \sigma_X + B_2 \sigma_Y + \frac{\Im}{w^2}[- \sigma P_T + F P_X - \sigma f_2 P_Y] - \eta D = 0 \\ \eta_T + E \eta_X + B_2 \eta_Y + \frac{\Im}{w^2}[- \eta P_T + H P_X + (\frac{G}{\rho} - \eta f_2) P_Y] + \sigma D = 0 \\ S_T + E S_X + B_2 S_Y = 0 \end{cases} \tag{65}$$

VI. CHARACTERISTIC EQUATION FOR BODY AND SHOCK POINTS

Equations (65) are not suited for the numerical analysis of points on the body and on the bow shock. A characteristic equation in an (X, T) plane is needed; note that in the physical space the image of such a plane is a surface almost parallel to the local velocity vector and almost normal to either the body or the bow shock.

By multiplying the first three equations (65) by μ_1, μ_2, and μ_3, respectively, adding, and calling λ the slope of the characteristic in the (X, T) plane ($\lambda = dX/dT$), we obtain the compatibility equation:

$$\left[\mu_1 - \frac{\mathcal{J}}{w^2}(\sigma\mu_2 + \nu\mu_3)\right](P_T + \lambda P_X) + \mu_2(\sigma_T + \lambda\sigma_X) + \mu_3(\nu_T + \lambda\nu_X) =$$

$$= \mu_1 R_1 + \mu_2 R_2 + \mu_3 R_3 \tag{66}$$

where

$$\begin{cases} R_1 = - A_2 P_Y - \frac{\gamma G}{\kappa\rho}(\nu_Y + L) \\ R_2 = - B_2 \sigma_Y + \frac{\mathcal{J}}{w^2}\sigma f_2 P_Y + \nu D \\ R_3 = - B_2 \nu_Y - \frac{\mathcal{J}}{w^2}\left(\frac{G}{\rho} - \nu f_2\right)P_Y - \sigma D \end{cases} \tag{67}$$

In turn, λ is defined by

$$\begin{vmatrix} C - \lambda & \frac{\mathcal{J}}{w^2}(F + \sigma\lambda) & \frac{\mathcal{J}}{w^2}(H + \nu\lambda) \\ \frac{\gamma G}{\kappa}X_\rho & E - \lambda & 0 \\ \frac{\gamma G}{\kappa\rho}X_\theta & 0 & E - \lambda \end{vmatrix} = 0 \tag{68}$$

that is,

$$(E - \lambda)(C - \lambda) - \frac{a^2}{w^2}\frac{G}{\kappa}\left[\frac{X_\theta}{\rho}(H + \nu\lambda) + X_\rho(F + \sigma\lambda)\right] = 0$$

or

$$\lambda^2 - 2(X_T + A_1 X_\rho + A_2 X_\theta)\lambda + E\,C - \frac{a^2}{w^2}\frac{G}{\kappa}(F X_\rho + H\frac{X_\theta}{\rho}) = 0$$

which, after some manipulations, yields:

$$\lambda = C \pm \beta\frac{aG}{w\kappa} \tag{69}$$

$$\beta = \sqrt{\left(\sigma X_\rho + \tau \frac{X_\theta}{\rho}\right)^2 + \varkappa\left(X_\rho^2 + \frac{X_\theta^2}{\rho^2}\right)} \tag{70}$$

The lower sign and the upper sign must be used at body points and bow shock points, respectively.

From (68), we obtain

$$\begin{cases} \mu_1 = E - \lambda \\ \mu_2 = -\frac{\gamma G}{\varkappa} X_\rho \\ \mu_3 = -\frac{\gamma G}{\varkappa \rho} X_\theta \end{cases} \tag{71}$$

$$\mu_1 - \frac{\mathcal{J}}{w^2}(\sigma\mu_2 + \tau\mu_3) = \mp \beta \frac{aG}{w\varkappa} \tag{72}$$

The compatibility equation (66) is then

$$\mp (P_T + \lambda P_X) = \frac{\gamma w}{\beta a}\Big\{X_\rho(\sigma_T + \lambda\,\sigma_X) + \frac{X_\theta}{\rho}(\tau_T + \lambda\,\tau_X)$$
$$+ \left[\frac{a^2}{\gamma w^2}\left(\sigma X_\rho + \frac{\tau}{\rho}X_\theta\right) \mp \frac{a\beta}{\gamma w}\right] R_1 - X_\rho R_2 - \frac{X_\theta}{\rho} R_3\Big\} \tag{73}$$

VII. EQUATIONS FOR BODY POINTS

At body points, the boundary condition,

$$\vec{V} \cdot \hat{N} = 0 \tag{74}$$

yields

$$\sigma - \tau \frac{b_Y}{b} + d = 0 \tag{75}$$

if (49), (50), (30) and (31) are taken into account. On the other hand, (63) and (29) give

$$E = X_\rho(-b_T + B_1 - B_2\, b_Y) \tag{76}$$

and, using (59) and (31), it is easy to see that

$$E = 0 \tag{77}$$

It is also important to note that (75) is identically satisfied for any value of Y and T; in particular,

$$\sigma_T - r_T \frac{b_Y}{b} = r\left(\frac{b_Y}{b}\right)_T - d_T \tag{78}$$

The compatibility equation (73) becomes

$$P_T + \lambda P_X = \frac{\nu w}{\beta a} X_\rho \left\{ r\left(\frac{b_Y}{b}\right)_T - d_T + \lambda\left(\sigma_X - \frac{b_Y}{b} r_X\right) - R_2 + \frac{b_Y}{b} R_3 \right\} - \frac{\varkappa w \lambda}{\beta a G} R_1 \tag{79}$$

with

$$\lambda = C - \beta \frac{aG}{w\varkappa} \tag{80}$$

It remains to provide explicit expressions for $\left(\frac{b_Y}{b}\right)_T$ and d_T.

The former is easy to write:

$$\left(\frac{b_Y}{b}\right)_T = \frac{1}{b}\left(b_{YT} - \frac{b_Y}{b} b_T\right) \tag{81}$$

To get the latter, we may start noting that, by (20),

$$d_T = d_t + d_x x_T + d_y y_T = d_t + \frac{1}{G}\Big\{\left[\mathcal{C} b_T - b(\mathcal{C} f_1 - \mathcal{S} f_2)\right] d_x$$

$$+ \left[\mathcal{S} b_T - b(\mathcal{S} f_1 + \mathcal{C} f_2)\right] d_y \Big\} \tag{82}$$

Using d in the form (41), we obtain

$$d_T = \frac{1}{\mathcal{C}\mathcal{F}_x + \mathcal{S}\mathcal{F}_y}\Big\{\mathcal{F}_{tt} - d(\mathcal{C}_t\mathcal{F}_x + \mathcal{S}_t\mathcal{F}_y + \mathcal{C}\mathcal{F}_{xt} + \mathcal{S}\mathcal{F}_{yt}) +$$

$$+ \frac{1}{G}\Big\langle\left[\mathcal{C} b_T - b(\mathcal{C} f_1 - \mathcal{S} f_2)\right]\left[\mathcal{F}_{xt} - d(\mathcal{C}_x\mathcal{F}_x + \mathcal{S}_x\mathcal{F}_y + \mathcal{C}\mathcal{F}_{xx} + \mathcal{S}\mathcal{F}_{xy})\right] +$$

$$+ \left[\mathcal{S} b_T - b(\mathcal{S} f_1 + \mathcal{C} f_2)\right]\left[\mathcal{F}_{yt} - d(\mathcal{C}_y\mathcal{F}_x + \mathcal{S}_y\mathcal{F}_y + \mathcal{C}\mathcal{F}_{xy} + \mathcal{S}\mathcal{F}_{yy})\right]\Big\rangle\Big\}$$

$$= \frac{1}{\mathcal{C}\mathcal{F}_x + \mathcal{S}\mathcal{F}_y}\Big\{\mathcal{F}_{tt} + d(\mathcal{S}\mathcal{F}_x - \mathcal{C}\mathcal{F}_y)(f_2 - \psi_2) - d(\mathcal{C}\mathcal{F}_{xt} + \mathcal{S}\mathcal{F}_{yt}) +$$

$$+ \frac{1}{G}\Big\langle\left[\mathcal{C} b_T - b(\mathcal{C} f_1 - \mathcal{S} f_2)\right]\Big[\mathcal{F}_{xt} - d(\mathcal{S}\mathcal{F}_x - \mathcal{C}\mathcal{F}_y)\frac{G}{b}\left[\mathcal{S}(1-\omega_1) + \mathcal{C}\omega_2\right] - d(\mathcal{C}\mathcal{F}_{xx} + \mathcal{S}\mathcal{F}_{xy})\Big] +$$

$$+ \left[\mathcal{S} b_T - b(\mathcal{S} f_1 + \mathcal{C} f_2)\right]\Big[\mathcal{F}_{yt} + d(\mathcal{S}\mathcal{F}_x - \mathcal{C}\mathcal{F}_y)\frac{G}{b}\left[\mathcal{C}(1-\omega_1) - \mathcal{S}\omega_2\right] - d(\mathcal{C}\mathcal{F}_{xy} + \mathcal{S}\mathcal{F}_{yy})\Big]\Big\rangle\Big\}$$

That is,

$$d_T = \frac{1}{\alpha \mathcal{F}_x + \beta \mathcal{F}_y}\Big\{\mathcal{F}_{tt} - d(\alpha \mathcal{F}_{xt} + \beta \mathcal{F}_{yt}) + \frac{1}{G}\Big\langle\Big[\alpha b_T - b(\alpha f_1 - \beta f_2)\Big]\Big[\mathcal{F}_{xt} - d(\alpha \mathcal{F}_{xx} + \beta \mathcal{F}_{xy})\Big] +$$

$$+ \Big[\beta b_T - b(\beta f_1 + \alpha f_2)\Big]\Big[\mathcal{F}_{yt} - d(\alpha \mathcal{F}_{xy} + \beta \mathcal{F}_{yy})\Big]\Big\rangle\Big\} +$$

$$+ d\Big[\frac{b_Y}{b}\tau_2 - \frac{b_T}{b}\tau_2 + f_1 \tau_2 + f_2 \tau_1\Big] \tag{83}$$

Having used (79) to determine P_T, the last of (65) yields S and from updated values of P and S, updated values of $\mathcal{J}$ and q^2 are obtained from (47) and (55).

Now, let

$$\tilde{v} = w\,(\tau + \frac{b_Y}{b}\sigma) \tag{84}$$

The third of (53) in the (X, Y, T) frame, with E=0, reads:

$$w_T + B_2\ w_Y + \frac{\mathcal{J}}{w}\Big[P_T + (X_\tau + \rho X_\rho f_1 + X_\theta f_2)P_X + f_2 P_Y\Big] = 0 \tag{85}$$

If (85) is multiplied by $\frac{\tilde{v}}{w^2}$ and added to the third of (65) and the second of (65) multiplied by $\frac{b_Y}{b}$, the following equation is obtained

$$\tilde{v}_T + B_2\ \tilde{v}_Y - \Big[(\frac{b_Y}{b})_T + B_2\ (\frac{b_Y}{b})_Y\Big]\sigma w + \frac{\mathcal{J}G}{\rho w}P_Y + Dw(\sigma - \frac{b_Y}{b}\tau) = 0 \tag{86}$$

which can be used to update $\tilde{v}$. Note that $\tilde{v}$ is the velocity component tangent to the body in the cross-sectional plane. Its Lagrangean derivative expressed in the (X, Y, T) frame by $\tilde{v}_T + B_2\ \tilde{v}_Y$, depends on the geometry of the body and on the Y-derivative of P only. After updating $\tilde{v}$, updated values of w, σ and τ at the body can be obtained by solving (84), (56) and (75). One obtains:

$$w = \frac{1}{\nu}\sqrt{q^2\Big(1 + \frac{b_Y^2}{b^2}\Big) - \tilde{v}^2} \tag{87}$$

with ν defined by (31),

$$\sigma = \frac{(\tilde{v}/w)(b_Y/b) - d}{1 + (b_Y/b)^2} \tag{88}$$

$$\tau = \frac{\tilde{v}}{w} - \sigma \frac{b_Y}{b} \tag{89}$$

VIII. BOW SHOCK CALCULATION

Let

$$\hat{\mathcal{n}} = \mathcal{n}_1 \hat{i} + \mathcal{n}_2 \hat{j} + \mathcal{n}_3 \hat{k} \tag{90}$$

be the unit vector normal to the bow shock surface. The values of $\mathcal{n}_1, \mathcal{n}_2, \mathcal{n}_3$ are the same as the values of N_1, N_2, N_3 defined by (33). The velocity component normal to the shock in front of it, $\tilde{u}_\infty$, is

$$\tilde{u}_\infty = \vec{V}_\infty \cdot \hat{\mathcal{n}} = u_\infty \mathcal{n}_1 + v_\infty \mathcal{n}_2 + w_\infty \mathcal{n}_3 \tag{91}$$

where

$$u_\infty = V_\infty \mathcal{S} \sin\alpha$$

$$v_\infty = V_\infty \mathcal{C} \sin\alpha$$

$$w_\infty = V_\infty \cos\alpha \tag{92}$$

If we denote by $\tilde{u}$ the corresponding velocity component behind the shock, the velocity vector, $\vec{V}$ behind the shock is:

$$\vec{V} = \vec{V}_\infty + (\tilde{u} - \tilde{u}_\infty)\hat{\mathcal{n}} \tag{93}$$

The Rankine-Hugoniot conditions provide the increment in P and the ratio $\tilde{u}/\tilde{u}_\infty$ across the shock (here P is the logarithm of pressure behind the shock; let us keep in mind that $P_\infty = 0$):

$$P = \ln \frac{2}{\gamma + 1} + \ln\left(\tilde{u}_\infty^2 - \frac{\gamma - 1}{2}\right) \tag{94}$$

$$u = \frac{\gamma - 1}{\gamma + 1} \tilde{u}_\infty + \frac{2\gamma}{\gamma + 1} \frac{1}{\tilde{u}_\infty} \tag{95}$$

Since (94) and (95) are identically satisfied at any T,

$$P_T = \frac{2\tilde{u}_\infty}{\tilde{u}_\infty^2 - \frac{\gamma - 1}{2}} \tilde{u}_{\infty T} \tag{96}$$

$$\tilde{u}_T = \left(\frac{\nu - 1}{\nu + 1} - \frac{2\nu}{\nu + 1} \frac{1}{\tilde{u}_\infty^2} \right) \tilde{u}_{\infty T} \tag{97}$$

In turn, from (92), (9), (18) and (21):

$$\begin{cases} u_{\infty T} = - v_\infty \omega_T \\ v_{\infty T} = u_\infty \omega_T \\ w_{\infty T} = 0 \end{cases} \tag{98}$$

and

$$\omega_T = \psi_2 - f_2 \varphi_1 + f_1 \varphi_2 + G \varphi_2 \frac{c_T}{c} \tag{99}$$

Therefore,

$$\tilde{u}_{\infty T} = - (v_\infty \mathfrak{n}_1 - u_\infty \mathfrak{n}_2)\omega_T + u_\infty \mathfrak{n}_{1T} + v_\infty \mathfrak{n}_{2T} + w_\infty \mathfrak{n}_{3T} \tag{100}$$

In all the above expressions, we will now separate c_{TT} from the rest, letting first d_T in the form:

$$d_T = C_1 + C_2 c_{TT} \tag{101}$$

The values of C_1 and C_2 will be computed later on. It follows that

$$\nu_T = C_3 + C_4 c_{TT} \tag{102}$$

$$C_3 = \frac{1}{\nu}\left[\frac{c_Y}{c}\left(\frac{c_Y}{c}\right)_T + d\, C_1\right], \quad C_4 = \frac{d}{\nu} C_2 \tag{103}$$

From (33),

$$\begin{cases} \mathfrak{n}_{1T} = C_5 + C_6 c_{TT}, & C_5 = -\frac{1}{\nu^2} C_3, & C_6 = -\frac{1}{\nu^2} C_4 \\ \mathfrak{n}_{2T} = C_7 + C_8 c_{TT}, & C_7 = -\left[\left(\frac{c_Y}{c}\right)_T \mathfrak{n}_1 + \frac{c_Y}{c} C_5\right], & C_8 = -\frac{c_Y}{c} C_6 \\ \mathfrak{n}_{3T} = C_9 + C_{10} c_{TT}, & C_9 = C_5 d + \mathfrak{n}_1 C_1, & C_{10} = C_6 d + \mathfrak{n}_1 C_2 \end{cases} \tag{104}$$

Then

$$\tilde{u}_{\infty T} = C_{11} + C_{12}\, c_{TT} \tag{105}$$

$$\begin{cases} C_{11} = (u_\infty \mathcal{N}_2 - v_\infty \mathcal{N}_1)\omega_T + v_\infty C_6 + v_\infty C_7 + w_\infty C_9 \\ C_{12} = u_\infty C_6 + v_\infty C_8 + w_\infty C_{10} \end{cases} \tag{106}$$

From (96) and (97)

$$\begin{cases} P_T = C_{13} + C_{14}\, c_{TT} \quad , \quad C_{13} = \dfrac{2\tilde{u}_\infty}{\tilde{u}_\infty^2 - \frac{\gamma-1}{2}} C_{11} \, , \qquad C_{14} = \dfrac{2\tilde{u}_\infty}{\tilde{u}_\infty^2 - \frac{\gamma-1}{2}} C_{12} \\ \tilde{u}_T = C_{15} + C_{16}\, c_{TT} \quad , \quad C_{15} = \left(\dfrac{\gamma-1}{\gamma+1} - \dfrac{2\gamma}{\gamma+1}\dfrac{1}{\tilde{u}_\infty^2}\right) C_{11} \, , \quad C_{16} = \left(\dfrac{\gamma-1}{\gamma+1} - \dfrac{2\gamma}{\gamma+1}\dfrac{1}{\tilde{u}_\infty^2}\right) C_{12} \end{cases} \tag{107}$$

Then,

$$\left(\tilde{u} - \tilde{u}_\infty\right)_T = C_{17} + C_{18}\, c_{TT} \quad , \quad C_{17} = C_{15} - C_{11} \quad , \quad C_{18} = C_{16} - C_{12} \tag{108}$$

With

$$\vec{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} \tag{109}$$

it follows from (93) that

$$\begin{cases} u = u_\infty + (\tilde{u}-\tilde{u}_\infty)\mathcal{N}_1 \\ v = v_\infty + (\tilde{u}-\tilde{u}_\infty)\mathcal{N}_2 \\ w = w_\infty + (\tilde{u}-\tilde{u}_\infty)\mathcal{N}_3 \end{cases} \tag{110}$$

Therefore,

$$\begin{cases} u_T = C_{19} + C_{20}\, c_{TT} \quad , \quad C_{19} = -v_\infty \omega_T + C_{17}\mathcal{N}_1 + (\tilde{u}-\tilde{u}_\infty)C_6 \quad , \quad C_{20} = C_{18}\mathcal{N}_1 + (\tilde{u}-\tilde{u}_\infty)C_6 \\ v_T = C_{21} + C_{22}\, c_{TT} \quad , \quad C_{21} = u_\infty \omega_T + C_{17}\mathcal{N}_2 + (\tilde{u}-\tilde{u}_\infty)C_7 \quad , \quad C_{22} = C_{18}\mathcal{N}_2 + (\tilde{u}-\tilde{u}_\infty)C_8 \\ w_T = C_{23} + C_{24}\, c_{TT} \quad , \quad C_{23} = C_{17}\mathcal{N}_3 + (\tilde{u}-\tilde{u}_\infty)C_9 \quad , \quad C_{24} = C_{18}\mathcal{N}_3 + (\tilde{u}-\tilde{u}_\infty)C_{10} \end{cases} \tag{111}$$

and

$$\begin{cases} \sigma_T = C_{25} + C_{26}\, c_{TT} \;, & C_{25} = \frac{1}{w}(C_{10} - \sigma C_{23}) \;, \quad C_{26} = \frac{1}{w}(C_{20} - \sigma C_{24}) \\ \tau_T = C_{27} + C_{28}\, c_{TT} \;, & C_{27} = \frac{1}{w}(C_{21} - \tau C_{23}) \;, \quad C_{28} = \frac{1}{w}(C_{22} - \tau C_{24}) \end{cases}$$

(112)

By substitution into the compatibility equation (73), we obtain

$$- (C_{13} + C_{14}\, c_{TT} + \lambda P_X) = \frac{wX_\rho}{\beta r}\Big\{C_{25} + C_{26}\, c_{TT} + \lambda\sigma_X - \frac{c_Y}{c}(C_{27} + C_{28}\, c_{TT} + \lambda\tau_X) - \left[\frac{a^2}{w}(\sigma - \tau\frac{c_Y}{c}) + a\beta\right]R_1 - R_2 + \frac{c_Y}{c}R_3\Big\}$$

which allows c_{TT} to be computed, as follows:

$$c_{TT} = D_7(D_8 + D_9 + \boldsymbol{\pi}) \tag{113}$$

$$D_7 = - \frac{1}{C_{14} + \frac{\gamma w X_\rho}{\beta a}(C_{26} - \frac{c_Y}{c}C_{28})} \tag{114}$$

$$D_8 = C_{13} + \frac{\gamma w X_\rho}{\beta a}(C_{25} - \frac{c_Y}{c}C_{27}) \tag{115}$$

$$D_9 = \frac{\gamma w X_\rho}{\beta a}\left\{-R_2 + \frac{c_Y}{c}R_3\right\} + \left[\frac{aX_\rho}{\beta w}(\sigma - \frac{c_Y}{c}\tau) - 1\right]R_1 \tag{116}$$

$$\boldsymbol{\pi} = \lambda\left[P_X + \frac{\gamma w X_\rho}{\beta a}(\sigma_X - \frac{c_Y}{c}\tau_X)\right] \tag{117}$$

Having computed c_{TT}, c_T and c are obtained by successive integrations. Once a new shock geometry, $\rho = c(\theta, \tau)$, is obtained, the new $\hat{n}$ and $\tilde{u}_\infty$ are evaluated; then, (94) and (95) can be applied to compute P and $\tilde{u}$. The three velocity components follow from (93); therefore $\sigma(=u/w)$ and $\tau(=v/w)$ are made known. Finally, S is given by

$$S = P - \gamma \ln \frac{\tilde{u}_\infty}{\tilde{u}} \tag{118}$$

To complete the information needed for coding the shock calculation, we need to evaluate C_1 and C_2. From (34),

$$d_T = -\frac{d}{G} G_T - \frac{1}{G}\left(c_{TT} + c_{YT} f_2 - c_T f_1 + c_Y f_{2T} - c f_{1T}\right)$$

$$= -d\left[\psi_1 + \frac{1}{c}(\mathcal{C}\varphi_1 + \mathcal{S}\varphi_2)\left\{\mathcal{C} c_T - c(\mathcal{C} f_1 - \mathcal{S} f_2)\right\} + \frac{1}{c}(\mathcal{S}\varphi_1 - \mathcal{C}\varphi_2)\left\{\mathcal{S} c_T - c(\mathcal{S} f_1 + \mathcal{C} f_2)\right\}\right]$$

$$-\frac{1}{G} c_{TT} - \frac{1}{G}(c_{YT} f_2 - c_T f_1) - \frac{1}{G}(c_Y f_{2T} - c f_{1T})$$

$$= -d\left[\psi_1 + \frac{c_T}{c}\varphi_1 - \varphi_1 f_1 + \varphi_2 f_2\right] - \frac{1}{G} c_{TT} - \frac{1}{G}(c_{YT} f_2 - c_T f_1) -$$

$$-\frac{1}{G}\left[c_Y \operatorname{Imag}(f_T) - c \operatorname{Real}(f_T)\right] \tag{119}$$

Now,

$$f_T = f_t + f_x x_T + f_y y_T$$

$$= \frac{\partial^2 \log\zeta}{\partial t^2} + \frac{1}{G}\frac{\partial^2 \log\zeta}{\partial x\,\partial t}\left[\mathcal{C} c_T - c(\mathcal{C} f_1 - \mathcal{S} f_2)\right] + \frac{1}{G}\frac{\partial^2 \log\zeta}{\partial y\,\partial t}\left[\mathcal{S} c_T - c(\mathcal{S} f_1 + \mathcal{C} f_2)\right]$$

$$= \frac{\partial^2 \log\zeta}{\partial t^2} + \frac{1}{G}\frac{\partial}{\partial t}\left(\frac{g}{\zeta}\right)\left[\mathcal{C} c_T - c(\mathcal{C} f_1 - \mathcal{S} f_2)\right] + \frac{i}{G}\frac{\partial}{\partial t}\left(\frac{g}{\zeta}\right)\left[\mathcal{S} c_T - c(\mathcal{S} f_1 - \mathcal{C} f_2)\right]$$

$$= \frac{\partial^2 \log\zeta}{\partial t^2} + \frac{1}{G}\frac{\partial}{\partial t}\left(\frac{g}{\zeta}\right)\left[(c_T - cf_1 - i\,cf_2)(\mathcal{C} + i\mathcal{S})\right]$$

$$= \frac{\partial^2 \log\zeta}{\partial t^2} + \frac{1}{G}\left[\frac{g}{\zeta}(\psi - f)\right]\left[\frac{G}{\rho}\frac{\zeta}{g}(c_T - c f)\right] \tag{120}$$

Therefore,

$$\left\{\begin{aligned} C_1 &= -d\left[\psi_1 + \frac{c_T}{c}\varphi_1 - \varphi_1 f_1 + \varphi_2 f_2\right] - \frac{1}{G}(c_{YT} f_2 - c_T f_1) - \\ &\quad -\frac{1}{G}\left[c_Y \operatorname{Imag}\frac{\partial^2 \log\zeta}{\partial t^2} + \frac{c_Y}{c}\operatorname{Imag}\left\{(\psi - f)(c_T - c f)\right\} - \right. \\ &\quad \left. - c \operatorname{Real}\frac{\partial^2 \log\zeta}{\partial t^2} - \operatorname{Real}\left\{(\psi - f)(c_T - c f)\right\}\right] \\ C_2 &= -\frac{1}{G} \end{aligned}\right. \tag{121}$$

IX. GENERAL OUTLINE OF ONE INTEGRATION STEP -

The equations obtained in the preceding sections are used to proceed from a station, t, to a station, $t + \Delta t$, using a predictor-corrector integration scheme, as follows.

Predictor stage

Given, original values of P, σ, τ, S, q^2, $\mathfrak{J}$, w, c, c_T, b, b_Y, b_T, ζ, z, g, f, τ and ψ

At all value of θ, compute c_Y, c_{YT}, b_{YY}, b_{YT}.

At all value of θ and ρ

Compute X- and Y- derivatives of P, σ, τ, S (see next section)

Compute $\rho = \bmod \zeta$, X_ρ, X_θ, X_t (see Section XI)

Compute G, $\varkappa$, A_1, A_2, B_1, B_2, D, E, C, L, F, H

For all points except body and shock points, determine P_T, σ_T, τ_T and S_T from (65)

For body points,compute R_1, R_2, R_3 from (67), λ from (80) and d_T from (83). Then, use (79) to obtain P_T, (86) to obtain $\tilde{v}_T$, and the last of (65) to obtain S_T.

For shock points, compute R_1, R_2, R_3 from (67), evaluate $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ and compute all coefficients (C_1 through C_{28}).

Compute D_7, D_8, D_9 and Π, and determine c_{TT} from (113).

Update P, σ, τ and S at all interior points, P, $\tilde{v}$ and S at all body points, as well as c_T and c, using the following rule (where Φ is an arbitrary function):

$$\Phi(t + \Delta t) = \Phi(t) + \Phi_T \Delta t \tag{122}$$

Move to the station defined by $t + \Delta t$; compute the geometry of the body; determine the basic parameters for a new mapping. Determine new values of b, b_Y, b_T. From the updated values of c, determine new values of c_Y; using the updated values of c_T, evaluate the new values of $\mathcal{H}_1, \mathcal{H}_2$, $\mathcal{H}_3$, the corresponding $\tilde{u}_\infty$ from (91) and updated values of P and $\tilde{u}$ behind the shock from (94) and (95); then use (110) to get updated values of u, v, and w and update σ and τ accordingly. Use (118) to get an updated value of S behind the shock. At the body, use (87), (88) and (89) with the updated values of q^a, $\tilde{v}$ and b. At all points, updated values of $\mathcal{J}, q^a$ and w must be computed.

Symmetry conditions are now imposed where necessary. Finally, the original values of P, σ, η S, $\tilde{v}$, c and c_T are temporarily saved. A new computational grid (including values of g, f, φ and ψ) is generated.

Corrector stage

The computation is restarted as at the beginning of the predictor stage. The geometry and the grid, however, are now those of station $t + \Delta t$; and all the variables have their predicted value at $t + \Delta t$ as well. The updating in the corrector stage is performed using the following rule in lieu of (122):

$$\Phi(t + \Delta t) = \frac{1}{2}\left[\Phi(t) + \overline{\Phi}(t + \Delta t) + \Phi_T \Delta t\right] \tag{123}$$

values $\overline{\Phi}(t + \Delta t)$ is the left-hand side of (122) and Φ_T is the T-derivative computed in the corrector stage. A new evaluation of the body geometry is not necessary; the computational grid, however, has to be re-evaluated since the bow shock location may have changed slightly. The values at the shock themselves have to be recomputed as at the end of the predictor stage. Symmetry conditions are imposed again. Finally, the updated values of $P, \sigma, \eta, S, \tilde{v}, c$ and c_T are stored as initial values for a new step.

X. DISCRETIZATION OF X- AND Y- DERIVATIVES

In principle, the MacCormack scheme[3] for integrating the equations of motion at all interior point is adopted. This is reflected by the use of (122) and (123) at the predictor and corrector level, respectively. In addition, whenever possible, the X- and Y- derivatives should be approximated by 2-point differences taken, for example, forwards at the predictor level and backwards at the corrector level.

There are several exceptions to the above rule.

1) At body boundary points, where $X=0$, backwards approximations cannot be taken and are replaced by the approximation:

$$\Phi_X \approx \frac{-2\Phi_0 + 3\Phi_1 - \Phi_2}{\Delta X} \tag{124}$$

which maintains a second order accuracy if the equations are linear throughout the integration step[4]. Here, $\Phi_i = \Phi(i\,\Delta X, Y)$

2) Similarly, at bow shock boundary points, where $X=1$, forward approximations cannot be taken and are replaced by the approximation:

$$\Phi_X \approx \frac{2\Phi_0 - 3\Phi_1 + \Phi_2}{\Delta X} \tag{125}$$

where $\Phi_i = \Phi(1 - i\,\Delta X, Y)$.

3) The last of (65) simply expresses the fact that entropy is transported unaltered by the moving particles. That means that no entropy signals travel backwards along streamlines. Consequently, no approximations to X- or Y- derivatives are allowed which imply forward differences of S along a streamline.

In most of the cases the rule does not have to be strictly enforced; in other words, the violation of the physical principle produced by alternating backward and forward differences (as in the MacCormack scheme) does not produce sizable errors, so long as the entropy distribution is smooth and essentially flat. Difficulties and inaccuracies appear, however, when entropy layers tend to build-up[1]. It is therefore advisable to adopt the following scheme systematically:

$$\begin{cases} \text{Predictor level:} & S_X \approx \operatorname{sgn}\sigma \dfrac{S_{00} - S_{10}}{\Delta X}, & S_Y \approx \operatorname{sgn}\eta \dfrac{S_{00} - S_{01}}{\Delta Y} \\ \text{Correction level:} & S_X \approx \operatorname{sgn}\sigma \dfrac{2S_{00} - 3S_{10} + S_{20}}{\Delta X}, & S_Y \approx \operatorname{sgn}\eta \dfrac{2S_{00} - 3S_{01} + S_{02}}{\Delta Y} \end{cases} \tag{126}$$

where

$$\begin{cases} S_{00} = S(X, Y), \quad S_{10} = S(X - \Delta X \operatorname{sgn} \sigma, Y) \\ S_{20} = S(X - 2\Delta X \operatorname{sgn} \sigma, Y) \\ S_{01} = S(X, Y - \Delta Y \operatorname{sgn} \tau), \quad S_{02} = S(X, Y - 2\Delta Y \operatorname{sgn} \tau) \end{cases} \tag{127}$$

At body and bow shock boundary points, the rules given at (1) and (2) above should be used for approximating S_X as well as the X-derivatives of the other physical parameters.

4) Similarly, the terms $\tilde{v}_T + B_2 \tilde{v}_Y$ in (86) express the Lagrangean derivative of $\tilde{v}$. Forward differencing of $\tilde{v}$ is thus forbidden and $\tilde{v}_Y$ should be approximated as follows:

$$\begin{cases} \text{Predictor level:} \quad \tilde{v}_Y \approx \operatorname{sgn} \tilde{v} \dfrac{\tilde{v}_0 - \tilde{v}_1}{\Delta Y} \\ \text{Corrector level:} \quad \tilde{v}_Y \approx \operatorname{sgn} \tilde{v} \dfrac{2\tilde{v}_0 - 3\tilde{v}_1 + \tilde{v}_2}{\Delta Y} \end{cases} \tag{128}$$

with

$$\begin{cases} \tilde{v}_0 = \tilde{v}(0, Y) \\ \tilde{v}_1 = \tilde{v}(0, Y - \Delta Y \operatorname{sgn} \tilde{v}) \\ \tilde{v}_2 = \tilde{v}(0, Y - 2\Delta Y \operatorname{sgn} \tilde{v}) \end{cases} \tag{129}$$

5) Y-derivatives at the bow shock boundary are conveniently approximated by centered differences .

XI. COORDINATE NORMALIZATION AND GRID STRETCHING ALONG ρ-LINES

The object of the transformation (5) is twofold. It defines a variable X which is constant (equal to zero) along the body and also constant (equal to 1) along the bow shock. In addition, it provides a stretching of coordinates according to which evenly spaced grid points on the X-axis correspond to unevenly spaced points on the ρ-lines. The latter property is used to accumulate θ-lines in the vicinity of the body where a stronger resolution is needed.

The values of the derivatives, X_ρ, X_θ, and X_τ depend on the choice of the stretching function $X(\rho, \theta, \tau)$. In this section we give an example of such a function. If the definition of $X(\rho, \theta, \tau)$ is changed, the definitions of X_ρ, X_θ and X_τ must be changed accordingly. The rest of the program does not need to be altered.

Let

$$\rho = c + \mathcal{D} \tanh[\alpha(X-1)] \tag{130}$$

with

$$\mathcal{D} = \frac{c-b}{\tanh \alpha} \tag{131}$$

Obviously, X as defined by (130) satisfied conditions (6). Different values of α provide different degrees of accumulation of θ-lines near the body. To give an idea of the effect of α, let b=0, c=1. Fig. 1 plots X vs. ρ for various values of α. Clearly, strong stretching effects begin to appear for values of α larger than 2.

It is easily proven that:

$$X_\rho = \mathcal{D} \frac{1}{\alpha[\mathcal{D}^2 - (\rho-c)^2]} \tag{132}$$

$$X_\theta = - \left\{ \frac{\rho-c}{\tanh \alpha} (c_Y - b_Y) + \mathcal{D} c_Y \right\} \frac{X_\rho}{\mathcal{D}} \tag{133}$$

$$X_\tau = - \left\{ \frac{\rho-c}{\tanh \alpha} (c_T - b_T) + \mathcal{D} c_T \right\} \frac{X_\rho}{\mathcal{D}} \tag{134}$$

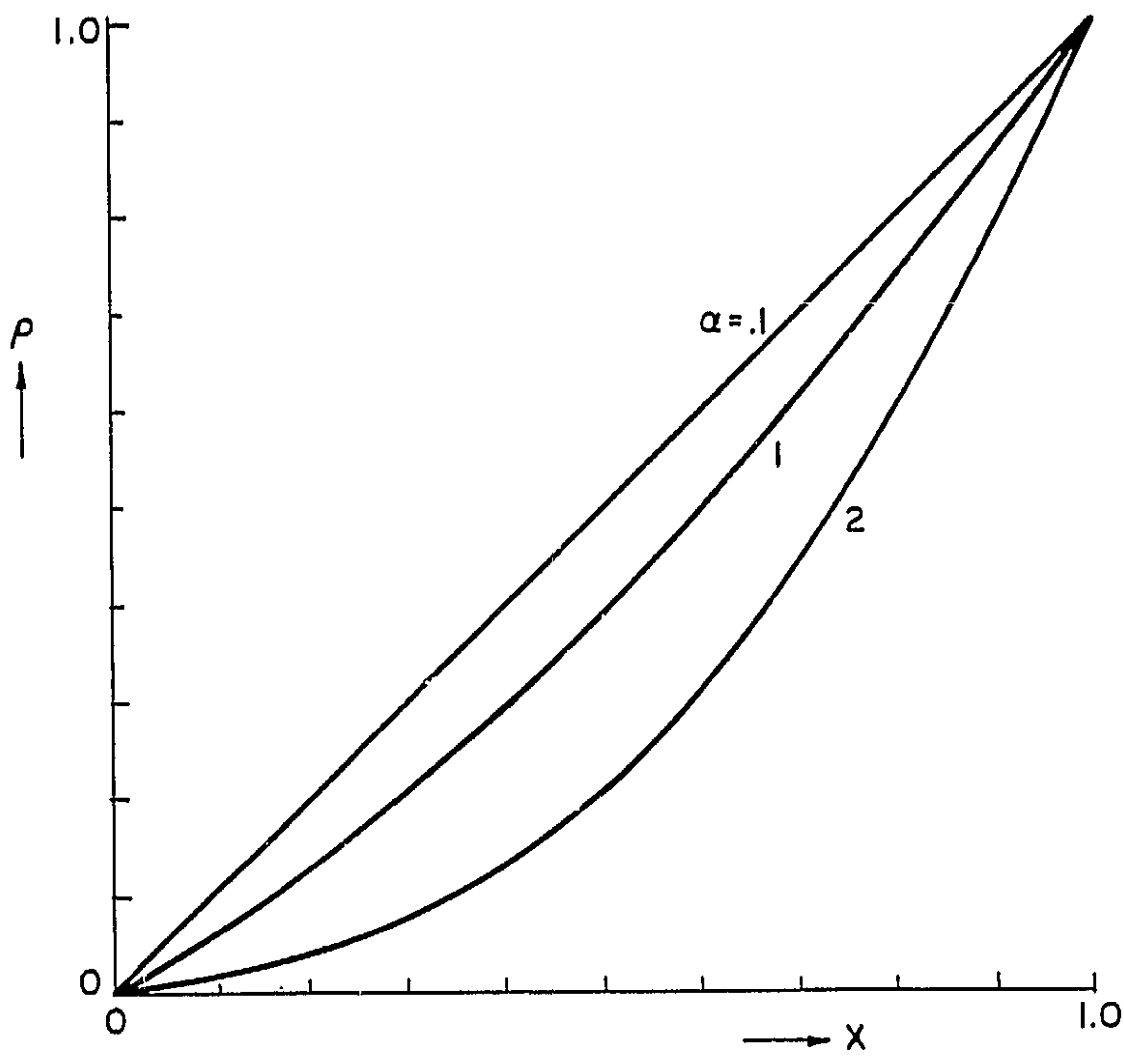

FIG. 1

XII. EXPLICIT COMPUTATION OF TERMS RELATED TO THE MAPPINGS

The mapping of the z-plane onto the ζ-plane, mentioned in Section II, is performed according to the general scheme exposed in Ref. 5. In the z-plane, the "hinge-points" are denoted by $h_{\ell 1}$ (ℓ=1 through J). In addition, $h_{J+1,1}$ and $h_{J+2,J}$ are the affixes of the lower and upper intersection of the cross-sectional contour with the y-axis.

Let

$$z_1 = z \tag{135}$$

and a sequence of J mappings be used, each defined by the equation:

$$\frac{z_{j+1} - 1}{z_{j+1} + 1} = \left(\frac{z_j - h_{jj}}{z_j + h_{jj}^*}\right)^{\delta_j} \qquad (j=1, 2, \ldots, J) \tag{136}$$

or its inverse:

$$\frac{z_j - h_{jj}}{z_j + h_{jj}^*} = \left(\frac{z_{j+1} - 1}{z_{j+1} + 1}\right)^{1/\delta_j} \tag{137}$$

The mappings are automatically performed in order of increasing δ_j.

With

$$\Sigma = \frac{1}{2}(h_{J+1,\,J+1} + h_{J+2,J+1}) \tag{138}$$

the ζ-plane is defined as

$$\zeta = z_{J+1} - \Sigma \tag{139}$$

To obtain the derivatives used in the computation of the flow field, the following definitions are used:

$$h_{jj} = \alpha_j + i\beta_j \tag{140}$$

$$
\begin{aligned}
k_1 &= \frac{h_{jj} - z_j}{\alpha_j} \\
k_2 &= z_{j+1}^2 - 1 \\
k_3 &= -\alpha_j k_1 = z_j - h_{jj} \\
k_4 &= z_j + h_{jj}^* \\
k_5 &= \log (k_3/k_4) \\
k_6 &= \dot{\alpha}_j \\
k_7 &= \dot{z}_j - \dot{h}_{jj} \\
k_8 &= z_j - i\beta_j = k_3 + \alpha_j \\
k_9 &= \dot{z}_j - i\dot{\beta}_j = k_7 + \dot{\alpha}_j \\
k_{10} &= \dot{\delta}_j/\delta_j + \dot{\alpha}_j/\alpha_j \\
k_{11} &= 1/(k_3 k_4)
\end{aligned}
\tag{141}
$$

where dots mean partial differentiations with respect to t at constant x and y, that is, at constant z).

We also define

$$
g_j = \frac{d z_{j+1}}{d z_j} = \delta_j \alpha_j k_2 k_{11} \tag{142}
$$

and we note that

$$
\frac{\partial z_{j+1}}{\partial h_{jj}} \dot{h}_{jj} + \frac{\partial z_{j+1}}{\partial h_{jj}^*} \dot{h}_{jj}^* = g_j (k_1 k_6 - \dot{h}_{jj}) \tag{143}
$$

Therefore,

$$
\dot{z}_{j+1} = g_j (k_7 + k_1 k_6) + \frac{1}{2} k_2 k_5 \dot{\delta}_j \tag{144}
$$

and

$$
\dot{\zeta} = \dot{z}_{J+1} - \frac{1}{2} (\dot{h}_{J+1, J+1} + \dot{h}_{J+2, J+1}) = \dot{z}_{J+1} - \dot{\Sigma} \tag{145}
$$

The t-derivatives of the $h_{\ell j}$ can be obtained as particular cases of (144), with $h_{\ell, j+1}$, $h_{\ell j}$ in lieu of z_{j+1}, z_j respectively. In particular, if $\ell = j$, $h_{j, j+1} = 0$, as evident from (136) and $\dot{h}_{j, j+1} = 0$.

Second derivatives are also needed to evaluate $\partial^2 \log \zeta / \partial t^2$ which is used in (121). From (142), it follows that

$$\dot{g}_j = g_j \left[k_{10} + \frac{2}{k_8} z_{j+1} \dot{z}_{j+1} - 2k_{11} (k_9 k_8 - k_8 \alpha_j) \right] \tag{146}$$

and, from (144),

$$\ddot{z}_{j+1} = \dot{g}_j (k_7 + k_1 k_6) + g_j \left[\ddot{z}_j - \ddot{h}_{jj} - \frac{k_6}{\tau_j} (k_7 + k_1 k_8) + k_1 \ddot{\alpha}_j \right] +$$

$$+ \left[z_{j+1} \dot{z}_{j+1} k_5 + (k_9 \alpha_j - k_8 \dot{\alpha}_j) k_2 k_{11} \right] \dot{\delta}_j + \frac{1}{2} k_2 k_5 \ddot{\delta}_j \tag{147}$$

As above, the second t-derivatives of the h_{ℓ_j} can be obtained as particular cases of (147), with $h_{\ell, j+1}$, $h_{\ell j}$ in lieu of z_{j+1}, z_j respectively.

At every new cross-section, the geometry defines the values of $h_{\ell 1}$, $\dot{h}_{\ell 1}$ and $\ddot{h}_{\ell 1}$ ($\ell = 1$ through J+2) as well as of δ_j, $\dot{\delta}_j$, and $\ddot{\delta}_j$ (j=1 through J). According to the procedure explained in Ref. 5, the mappings are performed in order of increasing δ_j. First, the values of $h_{\ell, j+1}$ (j=1 through J) and found by repeated applications of the mapping routine, which also provides the values of $\partial h_{\ell, j+1} / \partial h_{\ell j}$.

Then, Σ is obtained from (138). Repeated applications of (144) and (147) allow all the values of $\dot{h}_{\ell j}$ and $\ddot{h}_{\ell j}$ to be determined. It is possible, thus, to evaluate $\dot{\Sigma}$ and $\ddot{\Sigma}$ as well.

Grid points on the contour of the body are determined next. For each value of Y (that is, of θ), a guess of b is made and $\zeta = b e^{i\theta}$ is used to get z_{J+1} through (139); then, the sequence of mappings (137) with decreasing values of j is applied to obtain a value z which should match a body point in the physical cross-section. Matching within a prescribed tolerance is obtained by a trial-and-error procedure.

Knowing $c(\theta)$ and having just determined $b(\theta)$, values of ρ are obtained at every grid point outside the body by applying (130) or any equivalent expression if another stretching function is used. For such points the mappings are applied again as for the body points but no trial-and-error procedure is necessary. The entire grid is thus obtained both in the ζ-plane and in the z-plane (the physical cross-sectional plane). In addition, at this stage all values of g_j have been determined by the mapping routine. Consequently, it is possible to apply (144), in order of increasing j, to obtain all the values of $\dot{z}_{j+1}$ and, similarly, (146) and (147) to obtain all the values of $\ddot{z}_{j+1}$. The latter calculations are actually performed only at shock points, the sole points where second t-derivatives are needed. The functions g, φ, f and ψ necessary in the aerodynamical part of the code are now determined at each grid point as follows:

$$g = \frac{d\zeta}{dz} = \frac{dz_{J+1}}{dz_1} = \prod_{j=1}^{J} g_j \tag{148}$$

$$\varphi = \frac{\zeta}{g}\frac{d\log g}{dz} = \frac{\zeta}{g}\sum_{j=1}^{J}\frac{d\log g_j}{dz} = \frac{\zeta}{g}\sum_{j=1}^{J}\left(\frac{d\log g_j}{dz_j}\prod_{\ell=0}^{j-1} g_\ell\right)$$

$$= \frac{2\zeta}{g}\sum_{j=1}^{J}\frac{\prod_{\ell=0}^{j} g_\ell}{z_{j+1}^2 - 1}\left[z_{j+1} + \frac{1}{\delta_j}\left(\frac{h_{jj} - z_j}{\alpha_j} - 1\right)\right] \tag{149}$$

$$(g_o = 1)$$

$$f = \frac{1}{\zeta}\frac{\partial\zeta}{\partial t} = \frac{1}{\zeta}\left[\dot{z}_{J+1} - \dot{\Sigma}\right] \tag{150}$$

$$\psi = \frac{\partial\log g}{\partial t} = \sum_{j=1}^{J}\frac{\dot{g}_j}{g_j} \tag{151}$$

where the right hand side is obtained from (146).

Finally, the term $\partial^2 \log \zeta / \partial t^2$ which appears in (121), is computed as follows:

$$\frac{\partial^2 \log \zeta}{\partial t^2} = \frac{\partial f}{\partial t} = \frac{1}{\zeta}\left[\ddot{z}_{J+1} - \ddot{\zeta}\right] - f^2 \qquad (152)$$

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