A TRANSFORMATION METHOD FOR DERIVING, FROM A PHOTOGRAPH, POSITION AND HEADING OF A VEHICLE IN A PLANE

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Vehicle position is determined by transforming the perspectively viewed coordinate position of a representative vehicle target into runway coordinates. Vehicle heading is determined from the runway coordinates of two vehicle target points. When the targets are elevated above the plane of the reference grid, the computation of the heading angle is unaffected; however, the computation of the target position may require adjustment of two parameters. Methods are given for adjusting the parameters for elevation and an example is included for both nonelevated and elevated target conditions.

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SUMMARY

Equations have been derived that transform perspectively viewed planar surface coordinates, as seen in a photograph, into coordinates of the original plane surface. These transformation equations are developed in terms of nine geometric variables that define the photographic setup and are redefined in terms of eight parameters. The parameters are then treated as independent quantities that fully characterize the transformation and are expressed directly in terms of the four corner coordinates of a reference rectangle in the object plane and their coordinates as seen in a photograph.

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INTRODUCTION

Many situations exist which require monitoring the take-off, landing, and taxi behavior of aircraft or the ground handling performance of surface vehicles without special vehicle instrumentation and often under adverse operating conditions. A fundamental source of this type of information is the planar trajectory of the vehicle being examined. In the past, the trajectories have been straight lines and such devices as a trailing wheel or a phototheodolite have been adequate for providing the necessary trajectory data. Recently, however, interest has been generated in measuring vehicle behavior where the vehicle undergoes lateral displacements and/or changes in its heading as attributed, for example, to operations under cross-wind conditions. Since these trajectories do not lend themselves to the straight-line measurement techniques, a more general measurement technique is needed. Motion pictures in stereo were considered, but to the authors' knowledge derivation of position data from still stereo motion-picture frame pairs has

been unsuccessful because of synchronization problems of the twin views. Single frame motion pictures of a vehicle trajectory contain most of the positional data that the stereo process provides; however, no method is known to exist for reducing such data. The development of such a method was the objective of this report.

This paper develops and applies a transformation technique to the trajectory determination problem. The technique transforms coordinates in a single perspective view of a plane into coordinates of the original plane. The transformation uses parameters that are functions of several geometric variables which could, with difficulty, be measured. However, if a known rectangular reference grid is present in the test plane, the method of this report allows easy calculation of the parameters directly from the photograph. The requirement of a grid generally is not severe since such a grid is already present on many test surfaces as, for example, the expansion joints that exist on most concrete runways. For surfaces where no grid is present, one can be provided by the simple installation of markers, raised to the elevation of the test plane if necessary, at each corner of a measured rectangle.

Transformation equations are derived, a technique for determining values of the parameters is presented and, by using two reference target points on the vehicle, the position and heading are computed. For those cases where the targets are not in the plane of the grid, it may be necessary to adjust the parameters. Two example applications of the method have been included.

SYMBOLS

Measurements were made in U.S. Customary Units and values have been converted to SI Units.

C_j jth tilted perspective projection coefficient (see eqs. (12) to (17))

f distance from lens node to film plane

H height of lens node above object plane

ΔH elevation of target above runway

i reference rectangle corner index, 1 to 4

j tilted perspective projection coefficient index, 1 to 6

k variable index, 1 to 4

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N   lens node (figs. 1 and 2)

O   foot of perpendicular to object plane and passing through lens node (figs. 1 and 2)

o   projection of point O on image plane (figs. 1 and 2)

P   projection of principal point of image plane on object plane (figs. 1 and 2)

p   principal point of image plane (figs. 1 and 2)

U_k kth composite variable (eqs. (B9) to (B12))

(X,Y) coordinates in object plane

(X_i,Y_i) coordinates of ith corner of reference rectangle in plane of reference rectangle

(X_L,Y_L) coordinates of left target in object plane

(X_n,Y_n) coordinates of reference point in object plane

(X_o,Y_o) coordinates of lens node in object plane

(X_R,Y_R) coordinates of right target in object plane

(X_s',Y_s') coordinates of origin of image plane of an original projection system in object plane of a secondary projection system (see eqs. (B3) and (B4))

ΔX X-component of elevation error

(x,y) coordinates in image plane

(x_i,y_i) coordinates of ith corner of distorted reference rectangle in image plane

(x_L,y_L) coordinates of left target in image plane

(x_n,y_n) coordinates of reference point in image plane

(x_p, y_p) coordinates of principal point in image plane

(x_R, y_R) coordinates of right target in image plane

ΔY Y-component of elevation error

α rotation angle between image and object planes of perspective projections

θ angle of tilt (figs. 1 and 2)

Φ coordinate rotation angle between Y-axis and principal plane (fig. 1 and eq. (23))

ϕ coordinate rotation angle between y-axis and principal plane (fig. 1 and eq. (18))

ψ heading angle

A bar above a variable denotes principal coordinates in the image or object planes and a bar over a coefficient denotes an adjusted coefficient. Primes are used to denote quantities of different coordinate systems.

TRANSFORMATION DERIVATIONS

This report describes a method for determining the position and heading of an object on a flat surface from a single overhead photographic view of the surface. The method, when applied to motion pictures, offers a valuable tool for measuring data to describe the trajectory of an airplane on a runway. For such an application the camera may be tilted with respect to the runway and the vehicle is free to undergo lateral displacements and/or changes in heading. To meet the needs of the method, the only requirements are a rectangular grid marked on the runway and designated target points on the vehicle. The method applies a transformation for determining the positions of the target points relative to the runway grid from the motion-picture frames without iteration. The following sections present the mathematical development of the transformation where the target points are assumed to be in the plane of the reference grid, the formulas developed for calculating the parameters of the transformation from coordinates in the film and runway coordinate systems, and finally, an extension to the basic method to include the condition where the target points move in a parallel plane above the runway.

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Development of Transformation Equations

A fundamental premise of the technique is that coordinates of points of the object plane and the image plane of a photograph are related by a tilted perspective projection and coordinate rotations in the image and object planes as illustrated in figure 1. Both references 1 and 2, for example, provide a discussion of the perspective projection. In figure 1, N denotes a simple lens node and ON and pN are defined to be normal to the object and image planes, respectively. The point p on the image plane is called the principal point. Point P is the projection of the principal point through the lens node onto the object plane and point O is the foot of the perpendicular from the lens node to the object plane. The plane formed by the lines ON and pN is called the principal plane. Line pNP is called the optical axis; θ is the true angle of tilt of the lens optical axis with respect to the vertical; H denotes the lens node height above the runway; and f denotes a lens-to-film distance. Also shown are angles of coordinate rotations Φ and ϕ in the object and image planes, respectively. The rotations aline arbitrary coordinate (without bars) systems with the principal plane (with bars) coordinate systems. The coordinates with bars are called principal coordinates.

The origins of the coordinate systems shown in the figure have been positioned at points O and p for clarity. However, the same axes designations are used in the subsequent derivation for arbitrarily translated origins. In the translated systems, the coordinates of point p are denoted x_p, y_p and the coordinates of point O are denoted X_o, Y_o.

With the use of figures 1 and 2, the following relationships can be written between coordinates in the image and object planes:

$$\bar{x} = x \cos \phi + y \sin \phi \tag{1}$$

$$\bar{y} = -x \sin \phi + y \cos \phi \tag{2}$$

$$\bar{X} = X \cos \Phi - Y \sin \Phi \tag{3}$$

$$\bar{Y} = X \sin \Phi + Y \cos \Phi \tag{4}$$

$$\frac{\bar{x} - \bar{x}_p}{f \cos \theta - \left(\bar{y} - \bar{y}_p\right) \sin \theta} = \frac{\bar{X} - \bar{X}_o}{H} \tag{5}$$

$$\frac{\bar{y} - \bar{y}_p}{f} = \frac{\left(\bar{Y} - \bar{Y}_o\right) \cos \theta - H \sin \theta}{\left(\bar{Y} - \bar{Y}_o\right) \sin \theta + H \cos \theta} \tag{6}$$

where

$$\bar{x}_p = x_p \cos \phi + y_p \sin \phi$$

$$\bar{y}_p = -x_p \sin \phi + y_p \cos \phi$$

$$\bar{X}_O = X_O \cos \Phi - Y_O \sin \Phi$$

$$\bar{Y}_O = X_O \sin \Phi + Y_O \cos \Phi$$

For the special case when the tilt angle θ is zero, the image and object planes are parallel and the transformation reduces to a simple linear enlargement relationship. A method for deriving object plane data from image plane information for such a condition is straightforward; however, only the more general perspective transformations are treated herein.

Equations (5) and (6) may be written as

$$\bar{X} = \bar{X}_O + \frac{\left(\dfrac{H}{\sin \theta}\right)\left(\bar{x} - \bar{x}_p\right)}{\left(f \cot \theta + \bar{y}_p\right) - \bar{y}} \tag{7}$$

$$\bar{Y} = \left(\bar{Y}_O + H \tan \theta\right) + \frac{\left(\dfrac{H}{\sin \theta \cos \theta}\right)\left(\bar{y} - \bar{y}_p\right)}{\left(f \cot \theta + \bar{y}_p\right) - \bar{y}} \tag{8}$$

or

$$\bar{Y} = \left(\bar{Y}_O - H \cot \theta\right) + \frac{\left(\dfrac{f}{\sin \theta}\right)\left(\dfrac{H}{\sin \theta}\right)}{\left(f \cot \theta + \bar{y}_p\right) - \bar{y}} \tag{9}$$

These transformation equations are seen to be characterized by nine independent quantities ϕ, Φ, θ, f, H, x_p, y_p, X_O, and Y_O which henceforth will be called perspective variables.

Let the transformation equations (7) and (9) be redefined in terms of constant coefficients as

$$\bar{X} = C_5 + \frac{C_2\left(\bar{x} - C_4\right)}{C_1 - \bar{y}} \tag{10}$$

$$\bar{Y} = C_6 + \frac{C_2 C_3}{C_1 - \bar{y}} \tag{11}$$

where

$$C_1 = f \cot \theta + \bar{y}_p \tag{12}$$

$$C_2 = \frac{H}{\sin \theta} \tag{13}$$

$$C_3 = \frac{f}{\sin \theta} \tag{14}$$

$$C_4 = \bar{x}_p \tag{15}$$

$$C_5 = \bar{X}_o \tag{16}$$

$$C_6 = \bar{Y}_o - H \cot \theta \tag{17}$$

The six coefficients C_1, C_2, C_3, C_4, C_5, and C_6 are expressed here as simple functions of the perspective variables and determine the coordinate relationships of the perspective projection. In the next section, formulas are presented for computing these coefficients directly from coordinates measured in the image and object planes.

Note that equations (12) to (17) define six coefficients in terms of seven perspective variables; thus, it is necessary to specify a perspective variable before all the variables that completely characterize the photographic setup can be ascertained from a picture.

Solution for Transformation Parameters

In this section the parameters characterizing the transformation equations (1) to (4), (10), and (11) are expressed explicitly in terms of a set of four point coordinates measured in both the image and object planes. The parameters consist of two angles associated with the rotational coordinate transformations and six coefficients which are related to the perspective projection.

Solving for the transformation parameters requires that a reference rectangle in the object plane with sides parallel to an arbitrarily assigned X,Y axis system be visible in or be otherwise derivable from the image plane as illustrated in figure 3. Let the corners of the reference rectangle be assigned numbers as shown in the figure in a counterclockwise direction with the sides 1-4 and 2-3 parallel to the X-axis. The corner coordinates, so identified in both the image and object planes, are the only inputs required for the parameter solution. The details of how the angles and coefficients are determined from the set of input coordinate pairs are presented in appendix A; however, their derived expressions are given here with comments relative to the development.

Image plane coordinate rotation angle ϕ:

$$\phi = \tan^{-1}\left(\frac{\dfrac{y_b}{x_b} - \dfrac{y_a}{x_b}}{1 - \dfrac{x_a}{x_b}}\right) \tag{18}$$

where

$$x_a = \frac{(x_2 - x_1)(x_3 - x_4)(y_4 - y_1) + x_1(x_3 - x_4)(y_2 - y_1) - x_4(x_2 - x_1)(y_3 - y_4)}{(x_3 - x_4)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_4)} \tag{19}$$

$$y_a = \frac{-(y_2 - y_1)(y_3 - y_4)(x_4 - x_1) - y_1(y_3 - y_4)(x_2 - x_1) + y_4(y_2 - y_1)(x_3 - x_4)}{(x_3 - x_4)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_4)} \tag{20}$$

$$x_b = \frac{(x_4 - x_1)(x_3 - x_2)(y_2 - y_1) + x_1(x_3 - x_2)(y_4 - y_1) - x_2(x_4 - x_1)(y_3 - y_2)}{(x_3 - x_2)(y_4 - y_1) - (x_4 - x_1)(y_3 - y_2)} \tag{21}$$

$$y_b = \frac{-(y_4 - y_1)(y_3 - y_2)(x_2 - x_1) - y_1(y_3 - y_2)(x_4 - x_1) + y_2(y_4 - y_1)(x_3 - x_2)}{(x_3 - x_2)(y_4 - y_1) - (x_4 - x_1)(y_3 - y_2)} \tag{22}$$

and the numerical subscripts refer to the corners of the distorted reference rectangle of figure 3(b). The denominator of equations (21) and (22) can vanish without inhibiting a solution for the angle ϕ. This vanishing denominator, however, implies that the two sides of the quadrilateral in the image plane, 1-4 and 2-3, are parallel. Note, however, that a similar vanishing of the denominator of equations (19) and (20) cannot exist since it would imply that sides 1-2 and 3-4 are parallel. Once ϕ has been determined, equations (1) and (2) may be applied to any image plane x,y coordinates to yield principal $\bar{x},\bar{y}$ coordinates which will be used in the remainder of the derivation of the transformation parameters.

Object plane coordinate rotation angle Φ:

$$\Phi = \tan^{-1}\left|\left(\frac{Y_1 - Y_2}{X_3 - X_2}\right)\left[\frac{(\bar{y}_4 - \bar{y}_1)(\bar{y}_3 - \bar{y}_2)}{(\bar{y}_4 - \bar{y}_3)(\bar{y}_1 - \bar{y}_2)}\right]^{1/2}\right| \, \text{sgn}\left(\frac{\bar{y}_3 - \bar{y}_2}{\bar{x}_3 - \bar{x}_2}\right) \tag{23}$$

Once Φ has been found, equations (12) and (13) may be applied to any object plane X,Y coordinates to yield principal $\bar{X},\bar{Y}$ coordinates. Equations for determining the

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perspective projection coefficients are presented. A general set of equations is given first; however, because the equations are indeterminant when Φ equals zero, a separate set is included for this case. Subsequent references will be made only to the general equations.

Perspective projection coefficients ($\Phi \neq 0$): When the object plane rotation angle Φ is not zero, the perspective projection coefficients are solved sequentially in the order of the numerical subscripts as follows:

$$C_1 = \frac{\bar{y}_3(\bar{y}_1 - \bar{y}_2)(X_3 - X_2)\tan\Phi - \bar{y}_1(\bar{y}_3 - \bar{y}_2)(Y_1 - Y_2)}{(\bar{y}_1 - \bar{y}_2)(X_3 - X_2)\tan\Phi - (\bar{y}_3 - \bar{y}_2)(Y_1 - Y_2)} \tag{24}$$

$$C_2 = \frac{(\bar{X}_1 - \bar{X}_2)(\bar{y}_3 - \bar{y}_2)(C_1 - \bar{y}_1) - (\bar{X}_2 - \bar{X}_3)(\bar{y}_2 - \bar{y}_1)(C_1 - \bar{y}_3)}{(\bar{x}_1 - \bar{x}_2)\bar{y}_3 - (\bar{x}_1 - \bar{x}_3)\bar{y}_2 + (\bar{x}_2 - \bar{x}_3)\bar{y}_1} \tag{25}$$

$$C_3 = \frac{(\bar{Y}_2 - \bar{Y}_1)(C_1 - \bar{y}_1)(C_1 - \bar{y}_2)}{C_2(\bar{y}_2 - \bar{y}_1)} \tag{26}$$

$$C_4 = \frac{C_2\left[C_1(\bar{x}_2 - \bar{x}_1) + (\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1)\right] + (\bar{X}_1 - \bar{X}_2)(C_1 - \bar{y}_1)(C_1 - \bar{y}_2)}{C_2(\bar{y}_2 - \bar{y}_1)} \tag{27}$$

$$C_5 = \frac{C_2(\bar{x}_2 - \bar{x}_1) - C_1(\bar{X}_2 - \bar{X}_1) + \bar{X}_2\bar{y}_2 - \bar{X}_1\bar{y}_1}{\bar{y}_2 - \bar{y}_1} \tag{28}$$

$$C_6 = \frac{C_1(\bar{Y}_1 - \bar{Y}_2) - (\bar{Y}_1\bar{y}_1 - \bar{Y}_2\bar{y}_2)}{\bar{y}_2 - \bar{y}_1} \tag{29}$$

Perspective projection coefficients ($\Phi = 0$): When the object plane rotation angle Φ is zero, the perspective projection coefficients are solved sequentially, not in the order of the numerical subscripts given below, but in the order C_4, C_5, C_2, C_3, C_6, and C_1 where

$$C_1 = \frac{\bar{y}_1(Y_1 - C_6) + C_2 C_3}{Y_1 - C_6} \tag{30}$$

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$$C_2 = \frac{(\bar{y}_1 - \bar{y}_2)(\bar{x}_1 - C_5)}{\bar{x}_2 - \bar{x}_1} \tag{31}$$

$$C_3 = \frac{(Y_1 - Y_2)(\bar{x}_1 - C_4)(\bar{x}_2 - C_4)}{(\bar{x}_1 - C_5)(\bar{x}_2 - \bar{x}_1)} \tag{32}$$

$$C_4 = \frac{(\bar{x}_2 - \bar{x}_3)(\bar{x}_4 x_1 - \bar{x}_1 x_4) - (\bar{x}_1 - \bar{x}_4)(\bar{x}_3 x_2 - \bar{x}_2 x_3)}{(\bar{x}_2 - \bar{x}_3)(x_1 - x_4) - (\bar{x}_1 - \bar{x}_4)(x_2 - x_3)} \tag{33}$$

$$C_5 = \frac{(x_1 - x_4)(\bar{x}_3 x_2 - \bar{x}_2 x_3) - (x_2 - x_3)(\bar{x}_4 x_1 - \bar{x}_1 x_4)}{(\bar{x}_2 - \bar{x}_3)(x_1 - x_4) - (\bar{x}_1 - \bar{x}_4)(x_2 - x_3)} \tag{34}$$

$$C_6 = \frac{Y_1(\bar{x}_1 - C_4) - C_3(x_1 - C_5)}{\bar{x}_1 - C_4} \tag{35}$$

Adjustment of Parameters for Target Elevation

In the development of the transformation, it has been assumed that the target is in the same plane as that containing the reference rectangle. However, for many applications the target will move in a plane parallel to and above this "reference plane" and the previously derived transformation is no longer strictly valid. For such conditions, it may be convenient to think of the object plane raised above the reference plane. A good example of an elevated target is the case where the target is a point on the wing of an airplane on a runway. One recourse is to eliminate the elevation by raising the reference corner markers to the height of the target. However, for those cases where it is impossible or impractical to obtain coplanar conditions, an adjustment of the transformation is required.

Examine first the nature of a target position error when the target is elevated above the runway plane that contains the grid. Let $(X + \Delta X)$ and $(Y + \Delta Y)$ be the apparent coordinates of an elevated target as determined by the transformation and X and Y be the coordinates of the same target orthogonally projected onto the runway. The difference between the projected and apparent positions is designated the error due to an elevated target. Thus, for a lens node position (X_0, Y_0) and lens node and target heights above the runway denoted by H and ΔH, respectively, the error components are given by

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$$\Delta X = \frac{\Delta H}{H}\left(X - X_0\right) \tag{36}$$

$$\Delta Y = \frac{\Delta H}{H}\left(Y - Y_0\right) \tag{37}$$

and the

$$\text{Total elevation error} = \left[(\Delta X)^2 + (\Delta Y)^2\right]^{1/2} = \frac{\Delta H}{H}\left[\left(X - X_0\right)^2 + \left(Y - Y_0\right)^2\right]^{1/2} \tag{38}$$

which is independent of the orientation of the object plane coordinate system. As one would expect, these relations reveal that the elevation error varies with the ratio of the target elevation to the camera height and the distance of the target from the lens foot-print coordinates $\left(X_0, Y_0\right)$. In the subsequent adjustment procedures, the effective elevation is reduced by analytically raising the grid plane to that of the target.

An interesting and important fact may also be noted with regard to vehicle heading. Let the heading angle ψ be calculated by the expression

$$\psi = \tan^{-1} \frac{Y_R - Y_L}{X_R - X_L} \tag{39}$$

where the subscripts refer to left and right points. By using equations (3), (4), (10), and (11), equation (39) can be written as

$$\psi = \tan^{-1} \frac{-\left[C_1\left(\bar{x}_R - \bar{x}_L\right) + C_4\left(\bar{y}_L - \bar{y}_R\right) + \bar{x}_L\bar{y}_R - \bar{x}_R\bar{y}_L\right]\sin\Phi + C_3\left(\bar{y}_R - \bar{y}_L\right)\cos\Phi}{\left[C_1\left(\bar{x}_R - \bar{x}_L\right) + C_4\left(\bar{y}_L - \bar{y}_R\right) + \bar{x}_L\bar{y}_R - \bar{x}_R\bar{y}_L\right]\cos\Phi + C_3\left(\bar{y}_R - \bar{y}_L\right)\sin\Phi} \tag{40}$$

Suppose that the reference rectangle, assumed to be on the runway, was raised to the elevation of the target points and the transformation was recomputed. It is apparent from equations (12) to (17) that the only coefficients that would be affected by the change in height H would be C_2 and C_6. Since neither of these coefficients appears in equation (40), it follows that target elevation introduces no error in the heading angle calculation.

The remainder of the section gives three rigorous methods of compensating for elevation error by adjusting the transformation parameters. Each of these methods requires that the transformation parameters for the runway be solved by using the transformation as given and, with additional information which includes the elevation, two of the coefficients adjusted.

Lens node height H known.- In principle, the adjustment can always be made if the elevation ΔH and one of the perspective variables are provided. The height H of the lens node above the reference plane is, perhaps, the least critical variable and the easiest to measure. Since the height required by the transformation is now $(H - \Delta H)$, C_2 and C_6 must be adjusted so that equation (13) becomes

$$\overline{C}_2 = \frac{H - \Delta H}{\sin \theta}$$

$$= \frac{H}{\sin \theta}\left(1 - \frac{\Delta H}{H}\right)$$

$$= C_2\left(1 - \frac{\Delta H}{H}\right) \tag{41}$$

and equation (17) becomes

$$\overline{C}_6 = \overline{Y}_0 - (H - \Delta H)\cot \theta$$

$$= \left(\overline{Y}_0 - H\cot \theta\right) + \Delta H \frac{\left[1 - \left(\frac{H}{C_2}\right)^2\right]^{1/2}}{H/C_2}$$

$$= C_6 + \frac{\Delta H}{H}\left(C_2^2 - H^2\right)^{1/2} \tag{42}$$

where the bar above the coefficient denotes an adjusted coefficient.

Point in the elevated plane known.- When the coordinate pairs of a single reference point on the elevated target plane are known, for example, $\left(x_n, y_n\right)$ and $\left(X_n, Y_n\right)$, another method for adjusting the transformation can be employed. The angles ϕ and Φ and the coefficients C_1 to C_6 are first determined from the transformation coefficient equations (18) and (23) to (29), based on the runway grid. The coordinates of the elevated reference point may then be inserted into equations (1) to (4) and the following manipulated forms of equations (10) and (11) may be used to compute adjusted values for C_2 and C_6:

$$\overline{C}_2 = \frac{\left(\overline{X}_n - C_5\right)\left(C_1 - \overline{y}_n\right)}{\overline{x}_n - C_4} \tag{43}$$

and

$$\overline{C}_6 = \overline{Y}_n - \frac{C_3\left(\overline{X}_n - C_5\right)}{\overline{x}_n - C_4} \tag{44}$$

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Principal point position known.- Cameras typically used in photogrammetry have the location of the principal point $\left(x_p, y_p\right)$ indicated by fiducial marks on the edge of the film. Knowledge of the location of this point permits a solution for all the perspective variables without additional information; when the elevation ΔH is supplied, the transformation can be adjusted for the elevation change in effective camera height. For this condition the transformation parameters are first computed from the transformation coefficient equations (18) and (23) to (29) based on the runway grid. Next, the coordinates of the principal point are converted into principal coordinates by using equations (1) and (2) to yield $\left(\bar{x}_p, \bar{y}_p\right)$. If sufficient photographic fidelity is present, the new value of $\bar{x}_p$ should agree with the value for C_4. To adjust the coefficients C_2 and C_6 for elevation, equations (12) and (14) may be combined to yield

$$\cos\theta = \frac{C_1 - \bar{y}_p}{C_3} \tag{45}$$

Equation (45) may be introduced into equations for the adjusted coefficients $\bar{C}_2$ and $\bar{C}_6$ as shown:

$$\bar{C}_2 = \frac{H - \Delta H}{\sin\theta}$$

$$= C_2 - \frac{\Delta H}{\sin\theta}$$

$$= C_2 - \frac{C_3\,\Delta H}{\left[C_3^2 - \left(C_1 - \bar{y}_p\right)^2\right]^{1/2}} \tag{46}$$

and

$$\bar{C}_6 = \bar{Y}_o - (H - \Delta H)\cot\theta$$

$$= C_6 + \Delta H \cot\theta$$

$$= C_6 + \frac{\left(C_1 - \bar{y}_p\right)\Delta H}{\left[C_3^2 - \left(C_1 - \bar{y}_p\right)^2\right]^{1/2}} \tag{47}$$

Equations (46) and (47), with known values of $\bar{y}_p$ and ΔH inserted, replace values of C_2 and C_6 derived from the runway grid.

Effect of Image Plane Projection on Usage

A single projection relationship has been assumed in the derivation of the preceding equations; however, most image planes are derived from more than one projection.

For instance, pictures which have been enlarged in photographic processing or film that has been projected in some other manner, such as for still or motion-picture projection, have been derived from two perspective projections. To show whether the perspective projection relationship between the image and object planes still exists for a projected image plane, transformation equations have been formulated in appendix B for two consecutively applied tilted projections. The parameters of the transformation were found to change with the additional projection but the form of the equations was unchanged. It was concluded that the perspective projection relationship is retained for additional projections and, hence, the transformation method is valid even when the photographic data have been altered by projection techniques.

APPLICATIONS AND DISCUSSION

Two applications of the transformation have been made to exercise and evaluate the method. One application had targets in the same plane as the reference grid to test the direct application of the transformation. The other application involved an elevated target and required adjustment of the coefficients.

Target and Grid Plane Coincident

The main objective of this test was to examine the accuracy of the direct application of the transformation method to photographs obtained from different size cameras. The reference rectangle selected for this example was 2.743 m wide and 4.572 m long and targets were designated as shown in figure 4, one of the data photographs. The camera lens node position for each photograph was approximately 4.3 m above the surface with approximate coordinates of (-2.7 m,-3.2 m) measured from the lower corner of the reference grid. The camera tilt was approximately 54°. Four general inventory still cameras of types standard for their format size were employed to represent four different camera format sizes and lens-to-film distances. Photographs were made with each camera and, as required, enlarged to a common size of 20 cm by 25 cm. In the cases where enlargement was required, the image plane was then derived from a double projection as discussed in a previous section.

The transformation equations were programed on a desktop calculator equipped with a digitizer to convert points on the photograph electronically into x,y coordinates to the nearest 0.25 mm.

Results are presented in table I which lists the actual coordinates of the two targets in the plane of the reference grid and the corresponding coordinates determined from the photographs by using the transformation directly. The maximum error is found to be 0.8 cm where the width of the grid markings was 1.3 cm and the errors are seen to vary little for different cameras.

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Targets Elevated Above the Grid Plane

The objective of this test was to measure from 16-mm motion-picture film the position and heading of an aircraft landing-gear model on a runway when the vehicle targets are elevated above the runway. By use of a general inventory camera, motion pictures were taken of the model placed at nine different positions on the runway and oriented with heading attitudes ψ of -30°, -15°, 0°, 15°, and 30°. Frames from this film are presented in figure 5 which shows the model with a heading attitude of -30° at each of the tested positions. Two circular targets 6.4 cm in diameter and elevated 0.271 m above the runway are identified on the model. (The targets were spaced 1.219 m apart and attached to outriggers to facilitate heading measurements.) Grids in the central section of the runway had a 1.219-m lateral and longitudinal spacing and corner coordinates as specified by the grid in the figure. In this coordinate system, the motion-picture camera which was 3.246 m above the runway had an approximate location of (0.7 m, 3.3 m).

The steps in determining the model position and heading were straightforward. The motion pictures of the tests were first projected onto the digitizer platen; this process enlarged the frame size to approximately 0.37 m by 0.52 m and then the film movement was stopped. (Like the first example, this application necessarily incorporated a double projection.) The coordinates of the four grid corners (indicated with circles for each position in fig. 5) and two targets shown in the projected picture were read by the digitizer and input directly into the programmable desktop calculator. Transformation parameters were computed from both of the image and object plane grid corner coordinates by using equations (24) to (29); then, by introducing the known camera height and target elevation into equations (41) and (42), the parameters were adjusted for elevation. Target point coordinates were computed by inserting the digitizer target readings into the transformation equations. Finally, the heading angle was determined from the target coordinates by using equation (39), and the model position was found by averaging the two coordinates.

Test results were computed from digitizer readings made by an operator unfamiliar with the test and no attempt was made to enhance original readings. These results are presented as determined vehicle positions and headings in table II. When compared with their actual values, most position differences are less than 1 or 2 cm and most heading discrepancies are less than a degree. (The width of the grid lines was about 3 cm.) Inaccuracies found in the table could generally be attributed to the poor photographic clarity of the films; nevertheless, the results obtained were judged to be sufficiently accurate for evaluations of landing-gear performance for which the method was intended.

General Comments

This photographic measurement technique has a number of advantages that should be emphasized. The technique utilizes basic equipment that is readily available, it does not interact with the vehicle being measured, and, because the technique does not require the vehicle to carry onboard equipment, it can be used equally well for full-scale vehicles and for models where weight and volume are limited. Furthermore, little advanced preparation of the model or the surface is required.

Special emphasis should be placed on the use of the technique when no elevation adjustment is necessary. Not only is less computation required but, for such conditions, only the reference grid coordinates need to be designated, and all the measurements for computing the transformation parameters can readily be made from the film. Since all measured transformation data are now inherent in the film, there is more photographic freedom. For instance, photographs need not be taken from a camera whose position is fixed, but moving reference bases, such as those provided by aircraft, where speed and attitude may be varying or unknown may be used. Furthermore, photographs may be taken with cameras employing such devices as zoom lenses which vary the lens-to-film distance. When there is no need for elevation adjustment, the only requirement of the technique is that a known reference rectangular grid be in the object plane and the field of view of, or otherwise derivable from, the photograph. This requirement is not considered to be severe since such a grid is already present on many surfaces as, for example, the expansion joints that exist on most concrete runways. For surfaces where no grid exists, one can be simply introduced with the addition of markers, elevated to the target height if necessary, at each corner of a measured rectangle. With special markers this method may even be applied to measure the position of a vehicle on an unusual surface such as water.

It should be noted, however, that errors may exist in the image plane due to photographic distortion, digitizer resolution, and reading inaccuracy; and such errors are proportionally amplified by the height of the camera above the grid plane. Another type of error multiplication also exists which is not bounded. This error occurs for points near the horizon trace where a position difference in the object plane can be imperceptible in the image plane and is caused by the singularity of the transformation in that region. For this reason data measurement near the horizon of a photograph should be avoided.

CONCLUDING REMARKS

Equations have been derived that transform coordinates from a perspectively viewed planar surface into coordinates of the original plane surface. These transformation equations are developed in terms of nine geometric variables that define the photographic setup

and are redefined in terms of eight parameters. The parameters are then treated as independent quantities that fully characterize the transformation and are expressed directly in terms of the four corner coordinates of a reference rectangle in the object plane and their coordinates as seen in a photograph.

Vehicle position is determined by transforming the perspectively viewed coordinate position of a representative vehicle target into runway coordinates. When the target is in the plane of the reference grid, only the coordinates of the grid and the four corner coordinates from the perspective view are needed to derive the transformation. However, if the planes of the target and grid are not coincident, it may be necessary to adjust two of the parameters of the transformation for the elevation difference. Three methods for adjusting the parameters for an elevated target are given. Vehicle heading, which is found by determining the runway coordinates of two selected points on the vehicle that have the same elevation, is shown to be independent of elevation difference.

Two tilted arbitrary projections applied in sequence were found to be equivalent to a single projection, from which it was deduced that the transformation method is valid even when there are supplemental projections as are common in the photographic process and occur anytime that still or motion-picture film is projected.

Several advantages of the technique are evident. Some of these advantages are

(1) It measures all planar trajectory data without interacting with the vehicle being measured and carries no onboard equipment.

(2) It requires little preparation of vehicle or surface for its application.

(3) It is equally useful on model or full-scale tests.

(4) It utilizes simple, inexpensive photographic equipment that is readily available.

(5) It permits the use of photographic projection techniques for improved data analysis.

(6) It requires only minimal computation capability.

The example applications of the technique showed little variation in the results for cameras with different size formats, and the technique was judged to be potentially useful for ground vehicle performance analyses.

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APPENDIX A

SOLVING FOR THE TRANSFORMATION PARAMETERS

In this appendix transformation parameters ϕ, Φ, C_1, C_2, C_3, C_4, C_5, and C_6 are derived in terms of a set of four corresponding image- and object-plane coordinate pairs for coordinates in the object plane positioned in a rectangular pattern. Figure 3(a) shows a reference rectangle arbitrarily oriented in the object plane. An X,Y axis system is established with axes parallel to the sides of the rectangle as shown and corresponding corners of the reference rectangle are numerically designated in a counterclockwise order with sides 1-4 and 2-3 parallel to the X-axis of the object plane. Also shown is a rotated $\overline{X},\overline{Y}$ axis system with its ordinate parallel to the principal plane intersection which will be determined. The amount of clockwise rotation of the $\overline{X},\overline{Y}$ axis with respect to the object plane X,Y coordinate system is denoted by Φ.

An image plane view of the reference rectangle of figure 3(a) is shown in figure 3(b). The coordinates of the four corner points of this quadrilateral may be referred to any convenient x,y axis system. In the image plane it can be seen that, in general, sides of the quadrilateral exhibit vanishing points such as those at a and b. These points are on the horizons of a photograph and define a line called the horizon trace.

Also shown in figure 3(b) is an $\overline{x},\overline{y}$ axis system rotated counterclockwise by an angle ϕ with respect to the x,y axis system. This axis system has its $\overline{x}$-axis parallel to the horizon trace as shown; its $\overline{y}$-axis must be parallel to the intersection of the image and principal planes; therefore, the angle ϕ specifies the orientation of this intersection in the image plane.

The rotational transformations associated with the angles ϕ and Φ convert the image and object plane coordinates to their respective principal coordinates. After the rotational transformations in the image and object planes have been determined, the coefficients of the perspective projection may be evaluated in terms of the known image and object plane coordinates converted to the principal coordinates.

The sets of four coordinate pairs then are the only requirements for a solution to the transformation parameters. These sets consist of the corner coordinates (X_1,Y_1), (X_2,Y_2), (X_3,Y_3), and (X_4,Y_4) of the reference rectangle in the object plane, and the corner coordinates (x_1,y_1), (x_2,y_2), (x_3,y_3), and (x_4,y_4) of the distorted rectangle as seen perspectively in the image plane. The first parameters to be determined are the angles of the rotational transformations.

APPENDIX A

Determination of Image Plane Rotation Angle

The image plane rotation angle ϕ is determined from the image plane coordinates by finding the vanishing points where sides of the quadrilateral intersect, by constructing the horizon trace line that passes through these points and by determining its slope. This slope corresponds directly to the angle ϕ.

Coordinates of the vanishing points are found by solving for the points of intersection of the lines that pass through adjacent corner points and are given by

$$x_a = \frac{(x_2 - x_1)(x_3 - x_4)(y_4 - y_1) + x_1(x_3 - x_4)(y_2 - y_1) - x_4(x_2 - x_1)(y_3 - y_4)}{(x_3 - x_4)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_4)} \tag{A1}$$

$$y_a = \frac{-(y_2 - y_1)(y_3 - y_4)(x_4 - x_1) - y_1(y_3 - y_4)(x_2 - x_1) + y_4(y_2 - y_1)(x_3 - x_4)}{(x_3 - x_4)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_4)} \tag{A2}$$

$$x_b = \frac{(x_4 - x_1)(x_3 - x_2)(y_2 - y_1) + x_1(x_3 - x_2)(y_4 - y_1) - x_2(x_4 - x_1)(y_3 - y_2)}{(x_3 - x_2)(y_4 - y_1) - (x_4 - x_1)(y_3 - y_2)} \tag{A3}$$

$$y_b = \frac{-(y_4 - y_1)(y_3 - y_2)(x_2 - x_1) - y_1(y_3 - y_2)(x_4 - x_1) + y_2(y_4 - y_1)(x_3 - x_2)}{(x_3 - x_2)(y_4 - y_1) - (x_4 - x_1)(y_3 - y_2)} \tag{A4}$$

where (x_a,y_a) denotes the vanishing point associated with the intersection of the extended lines 1-2 and 3-4 and (x_b,y_b) is associated with lines 1-4 and 2-3 of figure 3(b).

The equations for each vanishing point are shown to have a common denominator which is related to the difference between the slopes of opposite sides of the quadrilateral in the image plane. Thus, the denominator is zero and is said to have a vanishing point at infinity when opposite sides are parallel. However, for the specified orientation of the quadilateral only the sides 1-4 and 2-3 can be parallel (zero tilt excluded). Thus, only the denominators of equations (A3) and (A4) can vanish.

The angle ϕ may be written as

$$\phi = \tan^{-1} \frac{y_b - y_a}{x_b - x_a}$$

However, an indeterminant form exists when the denominator of equations (A3) and (A4) vanishes and the alternate form

$$\phi = \tan^{-1}\left(\frac{\frac{y_b}{x_b} - \frac{y_a}{x_b}}{1 - \frac{x_a}{x_b}}\right) \tag{A5}$$

which does not exhibit this characteristic will be more useful.

Once ϕ is determined, equations (1) and (2) may be applied to any image plane x,y coordinates to yield the principal $\bar{x},\bar{y}$ coordinates which are used in the derivation of the transformation parameters.

Determination of Object Plane Rotation Angle

For the solution of the object plane rotation angle Φ, it can be noted that the object plane coordinate system has been alined with the reference rectangle so that

$$X_4 - X_1 = X_3 - X_2 \tag{A6}$$

$$Y_4 - Y_1 = Y_3 - Y_2 = 0 \tag{A7}$$

and

$$X_1 - X_2 = X_4 - X_3 = 0 \tag{A8}$$

$$Y_1 - Y_2 = Y_4 - Y_3 \tag{A9}$$

Introducing a portion of relations (A7) and (A8) in equation (4) yields

$$\sin \Phi = \frac{\bar{Y}_3 - \bar{Y}_2}{X_3 - X_2} \tag{A10}$$

and

$$\cos \Phi = \frac{\bar{Y}_1 - \bar{Y}_2}{Y_1 - Y_2} \tag{A11}$$

or

$$\tan \Phi = \frac{\left(Y_1 - Y_2\right)\left(\bar{Y}_3 - \bar{Y}_2\right)}{\left(X_3 - X_2\right)\left(\bar{Y}_1 - \bar{Y}_2\right)} \tag{A12}$$

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where the quantity $\left(\overline{Y}_3 - \overline{Y}_2\right)\!\big/\!\left(\overline{Y}_1 - \overline{Y}_2\right)$ is an unknown ratio whose magnitude and sign will be determined.

When equations (A6) to (A9) are introduced into equation (4), it is also found that

$$\overline{Y}_4 - \overline{Y}_1 = \overline{Y}_3 - \overline{Y}_2 \tag{A13}$$

and

$$\overline{Y}_1 - \overline{Y}_2 = \overline{Y}_4 - \overline{Y}_3 \tag{A14}$$

From the identity (A13) and equation (11), the following relation may also be found:

$$\frac{C_2C_3}{C_1 - \overline{y}_4} - \frac{C_2C_3}{C_1 - \overline{y}_1} = \frac{C_2C_3}{C_1 - \overline{y}_3} - \frac{C_2C_3}{C_1 - \overline{y}_2} \tag{A15}$$

or since the product C_2C_3 can never be zero,

$$\frac{\overline{y}_4 - \overline{y}_1}{\left(C_1 - \overline{y}_4\right)\left(C_1 - \overline{y}_1\right)} = \frac{\overline{y}_3 - \overline{y}_2}{\left(C_1 - \overline{y}_3\right)\left(C_1 - \overline{y}_2\right)} \tag{A16}$$

By using the identity (A14) and equation (11) in a similar manner,

$$\frac{\overline{y}_1 - \overline{y}_2}{\left(C_1 - \overline{y}_1\right)\left(C_1 - \overline{y}_2\right)} = \frac{\overline{y}_4 - \overline{y}_3}{\left(C_1 - \overline{y}_4\right)\left(C_1 - \overline{y}_3\right)} \tag{A17}$$

Equations (A16) and (A17) may be combined to yield

$$\left(\frac{C_1 - \overline{y}_1}{C_1 - \overline{y}_3}\right)^2 = \left[\frac{\left(\overline{y}_4 - \overline{y}_1\right)\left(\overline{y}_1 - \overline{y}_2\right)}{\left(\overline{y}_3 - \overline{y}_2\right)\left(\overline{y}_4 - \overline{y}_3\right)}\right] \tag{A18}$$

or

$$\frac{C_1 - \overline{y}_1}{C_1 - \overline{y}_3} = \pm\left[\frac{\left(\overline{y}_4 - \overline{y}_1\right)\left(\overline{y}_1 - \overline{y}_2\right)}{\left(\overline{y}_3 - \overline{y}_2\right)\left(\overline{y}_4 - \overline{y}_3\right)}\right]^{1/2} \tag{A19}$$

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With the aid of equation (11) the unknown ratio of equation (A12) may now be expressed in terms of the ratio of equation (A19). Thus,

$$\overline{Y}_1 - \overline{Y}_2 = C_2 C_3 \frac{\overline{y}_1 - \overline{y}_2}{\left(C_1 - \overline{y}_1\right)\left(C_1 - \overline{y}_2\right)} \tag{A20}$$

$$\overline{Y}_3 - \overline{Y}_2 = C_2 C_3 \frac{\overline{y}_3 - \overline{y}_2}{\left(C_1 - \overline{y}_3\right)\left(C_1 - \overline{y}_2\right)} \tag{A21}$$

or

$$\frac{\overline{Y}_3 - \overline{Y}_2}{\overline{Y}_1 - \overline{Y}_2} = \frac{\left(C_1 - \overline{y}_1\right)\left(\overline{y}_3 - \overline{y}_2\right)}{\left(C_1 - \overline{y}_3\right)\left(\overline{y}_1 - \overline{y}_2\right)} \tag{A22}$$

or when equation (A19) is inserted,

$$\frac{\overline{Y}_3 - \overline{Y}_2}{\overline{Y}_1 - \overline{Y}_2} = \pm \left[\frac{\left(\overline{y}_4 - \overline{y}_1\right)\left(\overline{y}_3 - \overline{y}_2\right)}{\left(\overline{y}_4 - \overline{y}_3\right)\left(\overline{y}_1 - \overline{y}_2\right)}\right]^{1/2} \tag{A23}$$

Thus, upon substituting equation (A23) into equation (A12), the magnitude relationship of Φ may simply be expressed as

$$\Phi = \tan^{-1} \pm \frac{Y_1 - Y_2}{X_3 - X_2} \left[\frac{\left(\overline{y}_4 - \overline{y}_1\right)\left(\overline{y}_3 - \overline{y}_2\right)}{\left(\overline{y}_4 - \overline{y}_3\right)\left(\overline{y}_1 - \overline{y}_2\right)}\right]^{1/2} \tag{A24}$$

The sign of Φ may be determined by examining the following sketches of the reference grid in the object plane and the grid in the corresponding principal image plane for different values of Φ. (See sketches (a) and (b).) These sketches show that parallel sides of the quadrilateral in the image plane correspond to zero object plane rotation ($\Phi = 0$) and the edges of the reference grid parallel to the X-axis of the object plane have a slope direction in the principal image plane commensurate with the object
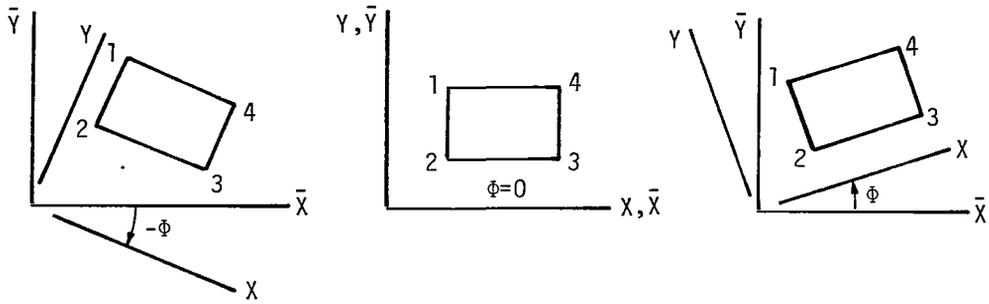
Object plane

Sketch (a)
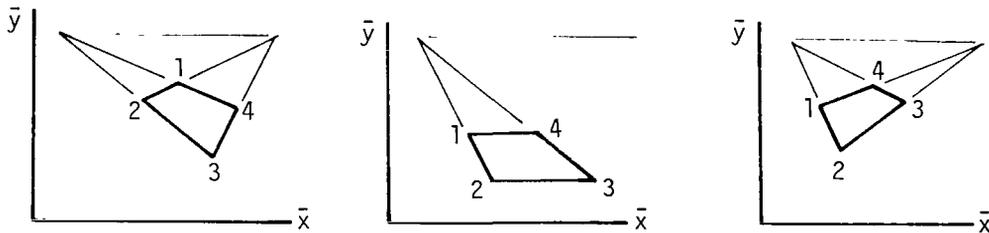
Image plane

Sketch (b)

plane rotation angle Φ. The sketches also indicate that for the specified corner designation, the magnitude of the angle is less than 90°. Thus,

$$\Phi = \tan^{-1} \left| \left(\frac{Y_1 - Y_2}{X_3 - X_2} \right) \left[\frac{(\bar{y}_4 - \bar{y}_1)(\bar{y}_3 - \bar{y}_2)}{(\bar{y}_4 - \bar{y}_3)(\bar{y}_1 - \bar{y}_2)} \right]^{1/2} \right| \, \text{sgn} \left(\frac{\bar{y}_3 - \bar{y}_2}{\bar{x}_3 - \bar{x}_2} \right) \tag{A25}$$

Once Φ has been specified, equations (3) and (4) may be used to convert object plane coordinates (X,Y) to the principal coordinates $(\bar{X}, \bar{Y})$ for the solution of the coefficients of the perspective projection which follows. A general solution for the perspective projection coefficients has been derived first; however, it exhibits an indeterminant form when Φ is zero. A separate set of equations is derived for this condition.

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Solution for Perspective Projection Coefficients (Φ Nonzero)

Determination of C_1.- The coefficient C_1 is the composite quantity $(f \cot \theta + \bar{y}_p)$ and, in the principal coordinate system of the image plane, it defines the horizon trace or limiting positive ordinate position. This coefficient may be found by combining equations (A12) and (A22) to yield

$$\tan \Phi = \frac{(Y_1 - Y_2)(\bar{y}_3 - \bar{y}_2)(C_1 - \bar{y}_1)}{(X_3 - X_2)(\bar{y}_1 - \bar{y}_2)(C_1 - \bar{y}_3)} \tag{A26}$$

Thus,

$$C_1 = \frac{\bar{y}_3(\bar{y}_1 - \bar{y}_2)(X_3 - X_2)\tan \Phi - \bar{y}_1(\bar{y}_3 - \bar{y}_2)(Y_1 - Y_2)}{(\bar{y}_1 - \bar{y}_2)(X_3 - X_2)\tan \Phi - (\bar{y}_3 - \bar{y}_2)(Y_1 - Y_2)} \tag{A27}$$

or

$$C_1 = \frac{\bar{y}_3(\bar{y}_1 - \bar{y}_2)(X_3 - X_2) - \bar{y}_1(\bar{y}_3 - \bar{y}_2)(Y_1 - Y_2)\cot \Phi}{(\bar{y}_1 - \bar{y}_2)(X_3 - X_2) - (\bar{y}_3 - \bar{y}_2)(Y_1 - Y_2)\cot \Phi} \tag{A28}$$

Determination of C_2.- The coefficient C_2 is the composite height function $(H/\sin \theta)$ which is determined from three image-to-object plane coordinate pairs with the use of equation (10). Thus, for two corner coordinate pairs, for example, those identified by 1 and 2,

$$\bar{X}_1 - \bar{X}_2 = C_2\left(\frac{\bar{x}_1 - C_4}{C_1 - \bar{y}_1} - \frac{\bar{x}_2 - C_4}{C_1 - \bar{y}_2}\right) \tag{A29}$$

and for points 2 and 3,

$$\bar{X}_2 - \bar{X}_3 = C_2\left(\frac{\bar{x}_2 - C_4}{C_1 - \bar{y}_2} - \frac{\bar{x}_3 - C_4}{C_1 - \bar{y}_3}\right) \tag{A30}$$

The coefficient C_4 can be eliminated from equations (A29) and (A30) and, since C_1 has been previously determined,

$$C_2 = \frac{(C_1 - \bar{y}_2)\left[(\bar{y}_2 - \bar{y}_1)(\bar{X}_2 - \bar{X}_3)(C_1 - \bar{y}_3) - (\bar{y}_3 - \bar{y}_2)(\bar{X}_1 - \bar{X}_2)(C_1 - \bar{y}_1)\right]}{(\bar{y}_3 - \bar{y}_2)(\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1) - (\bar{y}_2 - \bar{y}_1)(\bar{x}_2\bar{y}_3 - \bar{x}_3\bar{y}_2) - C_1\left[(\bar{x}_1 - \bar{x}_2)(\bar{y}_3 - \bar{y}_2) - (\bar{x}_2 - \bar{x}_3)(\bar{y}_2 - \bar{y}_1)\right]} \tag{A31}$$

or after some further manipulation

$$C_2 = \frac{\left(\overline{x}_1 - \overline{x}_2\right)\left(\overline{y}_3 - \overline{y}_2\right)\left(C_1 - \overline{y}_1\right) - \left(\overline{x}_2 - \overline{x}_3\right)\left(\overline{y}_2 - \overline{y}_1\right)\left(C_1 - \overline{y}_3\right)}{\left(\overline{x}_1 - \overline{x}_2\right)\overline{y}_3 - \left(\overline{x}_1 - \overline{x}_3\right)\overline{y}_2 + \left(\overline{x}_2 - \overline{x}_3\right)\overline{y}_1} \tag{A32}$$

Determination of C_3.- The coefficient C_3 is the composite lens-to-film distance function $(f/\sin \theta)$ which can be determined by the application of equation (11) to two coordinate pairs, for example, those identified by 1 and 2. Thus,

$$\overline{Y}_1 - \overline{Y}_2 = C_2 C_3 \frac{\overline{y}_1 - \overline{y}_2}{\left(C_1 - \overline{y}_1\right)\left(C_1 - \overline{y}_2\right)} \tag{A33}$$

or since C_1 and C_2 have been previously determined,

$$C_3 = \frac{\left(\overline{Y}_2 - \overline{Y}_1\right)\left(C_1 - \overline{y}_1\right)\left(C_1 - \overline{y}_2\right)}{C_2\left(\overline{y}_2 - \overline{y}_1\right)} \tag{A34}$$

Determination of C_4.- The coefficient C_4 is the coordinate $\overline{x}_p$ which is the abscissa position of the principal point in the principal coordinate system of the image plane. Since C_1 and C_2 have been previously determined, equation (A29) may be readily rearranged to show that

$$C_4 = \frac{C_2\left[C_1\left(\overline{x}_2 - \overline{x}_1\right) + \left(\overline{x}_1 \overline{y}_2 - \overline{x}_2 \overline{y}_1\right)\right] + \left(\overline{x}_1 - \overline{x}_2\right)\left(C_1 - \overline{y}_1\right)\left(C_1 - \overline{y}_2\right)}{C_2\left(\overline{y}_2 - \overline{y}_1\right)} \tag{A35}$$

Determination of C_5.- The coefficient C_5 is the coordinate $\overline{X}_o$ which is the abscissa position of the camera lens node in the principal coordinate system of the object plane. To find its value, equation (10) has been expressed in terms of the corner coordinate pairs identified by 1 and 2, and C_4 has been eliminated from the relations to yield the expression

$$C_5 = \frac{C_2\left(\overline{x}_2 - \overline{x}_1\right) - C_1\left(\overline{X}_2 - \overline{X}_1\right) + \overline{X}_2 \overline{y}_2 - \overline{X}_1 \overline{y}_1}{\overline{y}_2 - \overline{y}_1} \tag{A36}$$

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<u>Determination of C_6.</u>- The coefficient C_6 is a composite quantity $\left(\overline{Y}_0 - H \cot \theta\right)$ which is the limiting negative ordinate position in the principal coordinate system of the object plane. It corresponds to a point at negative infinity for the principal coordinates in the image plane. In its solution, equation (11) may be expressed in terms of the coordinates identified by 1 and 2 and the coefficient C_3 eliminated from the two equations to yield

$$C_6 = \frac{C_1\left(\overline{Y}_1 - \overline{Y}_2\right) - \left(\overline{Y}_1\overline{y}_1 - \overline{Y}_2\overline{y}_2\right)}{\overline{y}_2 - \overline{y}_1} \tag{A37}$$

Solution of Perspective Projection Coefficients (Φ Zero)

When Φ is zero, object plane coordinates (X,Y) are also principal object plane coordinates. Furthermore, certain principal image ordinate values are equal; that is,

$$\overline{y}_2 = \overline{y}_3 \tag{A38}$$

and

$$\overline{y}_1 = \overline{y}_4 \tag{A39}$$

By using these relations in equations (10) and (11), the perspective projection coefficients may be solved in the order that follows.

<u>Determination of C_4 and C_5.</u>- Let equations (10) and (11) be expressed for the case when Φ equals zero as

$$C_1 - \overline{y} = \frac{C_2 C_3}{Y - C_6} \tag{A40}$$

$$C_1 - \overline{y} = \frac{C_2\left(\overline{x} - C_4\right)}{X - C_5} \tag{A41}$$

Then, since $\overline{y}_1 = \overline{y}_4$, equation (A41) yields

$$\frac{\overline{x}_1 - C_4}{X_1 - C_5} = \frac{\overline{x}_4 - C_4}{X_4 - C_5}$$

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and similarly for $\bar{y}_2 = \bar{y}_3$

$$\frac{\bar{x}_2 - C_4}{X_2 - C_5} = \frac{\bar{x}_3 - C_4}{X_3 - C_5}$$

or

$$C_4\left(X_1 - X_4\right) - C_5\left(\bar{x}_1 - \bar{x}_4\right) = \bar{x}_4 X_1 - \bar{x}_1 X_4$$

and

$$C_4\left(X_2 - X_3\right) - C_5\left(\bar{x}_2 - \bar{x}_3\right) = \bar{x}_3 X_2 - \bar{x}_2 X_3$$

These equations may be solved simultaneously to yield

$$C_4 = \frac{\left(\bar{x}_2 - \bar{x}_3\right)\left(\bar{x}_4 X_1 - \bar{x}_1 X_4\right) - \left(\bar{x}_1 - \bar{x}_4\right)\left(\bar{x}_3 X_2 - \bar{x}_2 X_3\right)}{\left(\bar{x}_2 - \bar{x}_3\right)\left(X_1 - X_4\right) - \left(\bar{x}_1 - \bar{x}_4\right)\left(X_2 - X_3\right)} \tag{A42}$$

and

$$C_5 = \frac{-\left(X_1 - X_4\right)\left(\bar{x}_3 X_2 - \bar{x}_2 X_3\right) + \left(X_2 - X_3\right)\left(\bar{x}_4 X_1 - \bar{x}_1 X_4\right)}{\left(\bar{x}_2 - \bar{x}_3\right)\left(X_1 - X_4\right) - \left(\bar{x}_1 - \bar{x}_4\right)\left(X_2 - X_3\right)} \tag{A43}$$

Determination of C_2.- Let equation (A41) be applied to two corner coordinate pairs having different abscissa values in the object plane. Then,

$$\bar{y}_1 - \bar{y}_2 = C_2\left(\frac{\bar{x}_2 - C_4}{X_2 - C_5} - \frac{\bar{x}_1 - C_4}{X_1 - C_5}\right)$$

and since $X_1 = X_2$ and C_5 is known

$$C_2 = \frac{\left(\bar{y}_1 - \bar{y}_2\right)\left(X_1 - C_5\right)}{\bar{x}_2 - \bar{x}_1} \tag{A44}$$

APPENDIX A

Determination of C_3.- Equations (A40) and (A41) may be used together on two corner coordinate pairs with different ordinates in the object plane to show that

$$Y_1 - Y_2 = C_3 \left(\frac{X_1 - C_5}{\overline{x}_1 - C_4} - \frac{X_2 - C_5}{\overline{x}_2 - C_4} \right)$$

or

$$C_3 = \frac{\left(Y_1 - Y_2\right)\left(\overline{x}_1 - C_4\right)\left(\overline{x}_2 - C_4\right)}{\left(X_1 - C_5\right)\left(\overline{x}_2 - \overline{x}_1\right)} \tag{A45}$$

Determination of C_6.- Since C_3 has been previously determined, equations (A40) and (A41) may be used together to solve for C_6. Thus,

$$C_6 = \frac{Y_1\left(\overline{x}_1 - C_4\right) - C_3\left(X_1 - C_5\right)}{\overline{x}_1 - C_4} \tag{A46}$$

Determination of C_1.- Since C_2, C_3, and C_6 are now known, equation (A40) can be used to yield

$$C_1 = \frac{\overline{y}_1\left(Y_1 - C_6\right) + C_2 C_3}{Y_1 - C_6} \tag{A47}$$

The eight transformation parameters that characterize the transformation have been solved in terms of four image and object plane coordinate pairs. A simple check of the transformation may now be performed by inserting the four image plane coordinates into the transformation equations and comparing the computed object plane coordinates with those used for the determination of the parameters.

APPENDIX B

EQUIVALENCE OF TWO SEQUENTIAL PERSPECTIVE
PROJECTIONS TO A SINGLE PROJECTION

In this appendix it is shown that two general perspective projections performed sequentially are equivalent to a single perspective projection. This relationship is proved by deriving the image-to-object plane relationship and showing that its equations have the same form as those for a single perspective projection.

Let equations (1) to (4), (10), and (11) be combined to give two explicit relations for the object plane coordinates in terms of the image plane coordinates. Thus,

$$X = \frac{1}{C_1 + (x \sin \phi - y \cos \phi)} \left\{ \left(C_1 C_5 - C_2 C_4\right) \cos \Phi + \left(C_1 C_6 + C_2 C_3\right) \sin \Phi \right.$$

$$+ x \left[\left(C_2 \cos \phi + C_5 \sin \phi\right) \cos \Phi + C_6 \sin \phi \sin \Phi \right]$$

$$\left. + y \left[\left(C_2 \sin \phi - C_5 \cos \phi\right) \cos \Phi - C_6 \cos \phi \sin \Phi \right] \right\} \tag{B1}$$

and

$$Y = \frac{1}{C_1 + (x \sin \phi - y \cos \phi)} \left\{ \left(C_1 C_6 + C_2 C_3\right) \cos \Phi - \left(C_1 C_5 - C_2 C_4\right) \sin \Phi \right.$$

$$+ x \left[C_6 \sin \phi \cos \Phi - \left(C_2 \cos \phi + C_5 \sin \phi\right) \sin \Phi \right]$$

$$\left. - y \left[C_6 \cos \phi \cos \Phi + \left(C_2 \sin \phi - C_5 \cos \phi\right) \sin \Phi \right] \right\} \tag{B2}$$

Let the image coordinates (x,y) be expressed in terms of new coordinates (X',Y') which are rotated by an angle α and translated by distances X'_S and Y'_S in a manner such that

$$x = \left(X' - X'_S\right) \cos \alpha + \left(Y' - Y'_S\right) \sin \alpha \tag{B3}$$

$$y = -\left(X' - X'_S\right) \sin \alpha + \left(Y' - Y'_S\right) \cos \alpha \tag{B4}$$

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Finally, let the coordinates (X',Y') be treated as the object plane of a second perspective projection relationship distinguished by primes and given by

$$X' = \frac{1}{C_1' + (x' \sin \phi' - y' \cos \phi')} \left\{ \left(C_1' C_5' - C_2' C_4'\right) \cos \Phi' + \left(C_1' C_6' + C_2' C_3'\right) \sin \Phi' \right.$$

$$+ x' \left[\left(C_2' \cos \phi' + C_5' \sin \phi'\right) \cos \Phi' + C_6' \sin \phi' \sin \Phi' \right]$$

$$\left. + y' \left[\left(C_2' \sin \phi' - C_5' \cos \phi'\right) \cos \Phi' - C_6' \cos \phi' \sin \Phi' \right] \right\} \tag{B5}$$

and

$$Y' = \frac{1}{C_1' + (x' \sin \phi' - y' \cos \phi')} \left\{ \left(C_1' C_6' + C_2' C_3'\right) \cos \Phi' - \left(C_1' C_5' - C_2' C_4'\right) \sin \Phi' \right.$$

$$+ x' \left[C_6' \sin \phi' \cos \Phi' - \left(C_2' \cos \phi' + C_5' \sin \phi'\right) \sin \Phi' \right]$$

$$\left. - y' \left[C_6' \cos \phi' \cos \Phi' + \left(C_2' \sin \phi' - C_5' \cos \phi'\right) \sin \Phi' \right] \right\} \tag{B6}$$

Expressions relating the original object plane coordinates (X,Y) and the image plane coordinates of the second perspective projection (x',y') may be found by combining equations (B1) to (B6) and after considerable algebraic manipulation

$$X = \left(\left[C_5 \left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) + C_2 \left(C_3' U_2 - C_1' U_3 - C_4' U_4\right) \right] \cos \Phi \right.$$

$$+ \left[C_6 \left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) + C_1' C_2 C_3 \right] \sin \Phi + x' \left\{ \left[C_5 \left(U_1 \sin \phi' + U_2 \cos \phi'\right) \right. \right.$$

$$\left. - C_2 \left(U_3 \sin \phi' - U_4 \cos \phi'\right) \right] \cos \Phi + \left[\left(C_6 U_1 + C_2 C_3\right) \sin \phi' + C_6 U_2 \cos \phi' \right] \sin \Phi \right\}$$

$$+ y' \left\{ \left[C_5 \left(U_2 \sin \phi' - U_1 \cos \phi'\right) + C_2 \left(U_3 \cos \phi' + U_4 \sin \phi'\right) \right] \cos \Phi \right.$$

$$\left. \left. + \left[C_6 U_2 \sin \phi' - \left(C_6 U_1 + C_2 C_3\right) \cos \phi' \right] \sin \Phi \right\} \right) \left[\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) \right.$$

$$\left. + x' \left(U_1 \sin \phi' + U_2 \cos \phi'\right) - y' \left(U_1 \cos \phi' - U_2 \sin \phi'\right) \right]^{-1} \tag{B7}$$

and

$$Y = \frac{1}{C_1'' + (x' \sin \phi'' - y' \cos \phi'')} \left\{ \left(C_1'' C_6'' + C_2'' C_3'' \right) \cos \Phi'' - \left(C_1'' C_5'' - C_2'' C_4'' \right) \sin \Phi'' \right.$$

$$+ x' \left[C_6'' \sin \phi'' \cos \Phi'' - \left(C_2'' \cos \phi'' + C_5'' \sin \phi'' \right) \sin \Phi'' \right]$$

$$\left. - y' \left[C_6'' \cos \phi'' \cos \Phi'' + \left(C_2'' \sin \phi'' + C_5'' \cos \phi'' \right) \sin \Phi'' \right] \right\} \tag{B14}$$

where the double primed quantities denote equivalent transformation parameters of a single perspective projection.

It will now be shown that equations (B7) and (B8) have the same form as equations (B13) and (B14). To show this equivalence, it is convenient to first satisfy the denominator expressions by dividing the numerator and denominator expressions of equations (B7) and (B8) by $\left(U_1^2 + U_2^2 \right)^{1/2}$. Thus when like coefficients in the denominator are compared,

$$\sin \phi'' = \frac{U_1 \sin \phi' + U_2 \cos \phi'}{\left(U_1^2 + U_2^2 \right)^{1/2}} \tag{B15}$$

and

$$\cos \phi'' = \frac{U_1 \cos \phi' - U_2 \sin \phi'}{\left(U_1^2 + U_2^2 \right)^{1/2}} \tag{B16}$$

or

$$\phi'' = \tan^{-1} \frac{U_1 \sin \phi' + U_2 \cos \phi'}{U_1 \cos \phi' - U_2 \sin \phi'} \tag{B17}$$

and

$$C_1'' = \frac{C_1' U_1 - C_4' U_2 - C_3' U_4}{\left(U_1^2 + U_2^2 \right)^{1/2}} \tag{B18}$$

Similarly, by comparing numerator coefficients,

$$\left(C_1'' C_5'' - C_2'' C_4''\right) \cos \Phi'' + \left(C_1'' C_6'' + C_2'' C_3''\right) \sin \Phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_5\left(C_1' U_1 - C_4' U_2 - C_3' U_4 + C_2\left(C_3' U_2 - C_1' U_3 - C_4' U_4\right)\right] \cos \Phi \right. $$

$$\left. + \left[C_6\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) + C_1' C_2 C_3 \right] \sin \Phi \right\} \tag{B19}$$

$$\left(C_2'' \cos \phi'' + C_5'' \sin \phi''\right) \cos \Phi'' + C_6'' \sin \phi'' \sin \Phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_5\left(U_1 \sin \phi' + U_2 \cos \phi'\right) - C_2\left(U_3 \sin \phi' - U_4 \cos \phi'\right) \right] \cos \Phi \right. $$

$$\left. + \left[\left(C_6 U_1 + C_2 C_3\right) \sin \phi' + C_6 U_2 \cos \phi' \right] \sin \Phi \right\} \tag{B20}$$

$$\left(C_2'' \sin \phi'' - C_5'' \cos \phi''\right) \cos \Phi'' - C_6'' \cos \phi'' \sin \Phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_5\left(U_2 \sin \phi' - U_1 \cos \phi'\right) + C_2\left(U_3 \cos \phi' + U_4 \sin \phi'\right) \right] \cos \Phi \right. $$

$$\left. + \left[C_6 U_2 \sin \phi' - \left(C_6 U_1 + C_2 C_3\right) \cos \phi' \right] \sin \Phi \right\} \tag{B21}$$

$$-\left(C_1'' C_5'' - C_2'' C_4''\right) \sin \Phi'' + \left(C_1'' C_6'' + C_2'' C_3''\right) \cos \Phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_6\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) + C_1' C_2 C_3 \right] \cos \Phi \right. $$

$$\left. - \left[C_5\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) + C_2\left(C_3' U_2 - C_1' U_3 - C_4' U_4\right) \right] \sin \Phi \right\} \tag{B22}$$

$$C_6'' \sin \phi'' \cos \Phi'' - \left(C_2'' \cos \phi'' + C_5'' \sin \phi''\right) \sin \Phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[\left(C_6 U_1 + C_2 C_3\right) \sin \phi' + C_6 U_2 \cos \phi'\right] \cos \Phi \right.$$

$$\left. - \left[C_5\left(U_1 \sin \phi' + U_2 \cos \phi'\right) - C_2\left(U_3 \sin \phi' - U_4 \cos \phi'\right)\right] \sin \Phi \right\} \quad \text{(B23)}$$

$$-C_6'' \cos \phi'' \cos \Phi'' - \left(C_2'' \sin \phi'' - C_5'' \cos \phi''\right) \sin \Phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_6 U_2 \sin \phi' - \left(C_6 U_1 + C_2 C_3\right) \cos \phi'\right] \cos \Phi \right.$$

$$\left. - \left[C_5\left(U_2 \sin \phi' - U_1 \cos \phi'\right) + C_2\left(U_3 \cos \phi' + U_4 \sin \phi'\right)\right] \sin \Phi \right\} \quad \text{(B24)}$$

Equations (B19) and (B22) can be solved to yield

$$C_1'' C_5'' - C_2'' C_4''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_5\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) - C_2\left(C_1' U_3 - C_3' U_2 + C_4' U_4\right)\right] \cos \left(\Phi'' - \Phi\right) \right.$$

$$\left. - \left[C_6\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) + C_1' C_2 C_3\right] \sin \left(\Phi'' - \Phi\right) \right\} \quad \text{(B25)}$$

and

$$C_1'' C_6'' + C_2'' C_3''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_6\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) + C_1' C_2 C_3\right] \cos \left(\Phi'' - \Phi\right) \right.$$

$$\left. + \left[C_5\left(C_1' U_1 - C_4' U_2 - C_3' U_4\right) - C_2\left(C_1' U_3 - C_3' U_2 + C_4' U_4\right)\right] \sin \left(\Phi'' - \Phi\right) \right\}$$

$$\text{(B26)}$$

Similarly, equations (B20) and (B23) yield

$$C_2'' \cos \phi'' + C_5'' \sin \phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[\left(C_5 U_1 - C_2 U_3\right) \sin \phi' + \left(C_5 U_2 + C_2 U_4\right) \cos \phi' \right] \cos \left(\Phi'' - \Phi\right) \right.$$

$$\left. - \left[\left(C_6 U_1 + C_2 C_3\right) \sin \phi' + C_6 U_2 \cos \phi' \right] \sin \left(\Phi'' - \Phi\right) \right\} \qquad \text{(B27)}$$

and

$$C_6'' \sin \phi'' = \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[\left(C_6 U_1 + C_2 C_3\right) \sin \phi' + C_6 U_2 \cos \phi' \right] \cos \left(\Phi'' - \Phi\right) \right.$$

$$\left. + \left[\left(C_5 U_1 - C_2 U_3\right) \sin \phi' + \left(C_5 U_2 + C_2 U_4\right) \cos \phi' \right] \sin \left(\Phi'' - \Phi\right) \right\}$$

$$\text{(B28)}$$

and equations (B21) and (B24) yield

$$C_2'' \sin \phi'' - C_5'' \cos \phi''$$

$$= \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[\left(C_5 U_2 + C_2 U_4\right) \sin \phi' - \left(C_5 U_1 - C_2 U_3\right) \cos \phi' \right] \cos \left(\Phi'' - \Phi\right) \right.$$

$$\left. - \left[C_6 U_2 \sin \phi' - \left(C_6 U_1 + C_2 C_3\right) \cos \phi' \right] \sin \left(\Phi'' - \Phi\right) \right\} \qquad \text{(B29)}$$

and

$$-C_6'' \cos \phi'' = \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[C_6 U_2 \sin \phi' - \left(C_6 U_1 + C_2 C_3\right) \cos \phi' \right] \cos \left(\Phi'' - \Phi\right) \right.$$

$$\left. + \left[\left(C_5 U_2 + C_2 U_4\right) \sin \phi' - \left(C_5 U_1 - C_2 U_3\right) \cos \phi' \right] \sin \left(\Phi'' - \Phi\right) \right\}$$

$$\text{(B30)}$$

APPENDIX B

When equations (B15) and (B16) and equations (B28) and (B30) are expressed as an equivalent ratio, the angle $(\Phi'' - \Phi)$ may be found to be

$$(\Phi'' - \Phi) = \tan^{-1} \frac{C_3 U_2}{U_1 U_4 + U_2 U_3} \tag{B31}$$

Equations (B28) and (B30) also yield

$$C_6'' = \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[\left(C_5 U_1 - C_2 U_3\right) \sin (\Phi'' - \Phi) + \left(C_6 U_1 + C_2 C_3\right) \cos (\Phi'' - \Phi)\right] \cos (\phi'' - \phi') \right.$$

$$\left. + \left[\left(C_5 U_2 + C_2 U_4\right) \sin (\Phi'' - \Phi) + C_6 U_2 \cos (\Phi'' - \Phi)\right] \sin (\phi'' - \phi') \right\} \tag{B32}$$

Equations (B27) and (B29) yield

$$C_2'' = \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[\left(C_5 U_2 + C_2 U_4\right) \cos (\Phi'' - \Phi) - C_6 U_2 \sin (\Phi'' - \Phi)\right] \cos (\phi'' - \phi') \right.$$

$$\left. - \left[\left(C_5 U_1 - C_2 U_3\right) \cos (\Phi'' - \Phi) - \left(C_6 U_1 + C_2 C_3\right) \sin (\Phi'' - \Phi)\right] \sin (\phi'' - \phi') \right\} \tag{B33}$$

and

$$C_5'' = \frac{1}{\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left[\left(C_5 U_2 + C_2 U_4\right) \cos (\Phi'' - \Phi) - C_6 U_2 \sin (\Phi'' - \Phi)\right] \sin (\phi'' - \phi') \right.$$

$$\left. + \left[\left(C_5 U_1 - C_2 U_3\right) \cos (\Phi'' - \Phi) - \left(C_6 U_1 + C_2 C_3\right) \sin (\Phi'' - \Phi)\right] \cos (\phi'' - \phi') \right\} \tag{B34}$$

and finally equations (B25) and (B26) yield

$$C_4'' = \frac{1}{C_2''\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ \left(C_4' U_2 + C_3' U_4\right)\left[C_5 \cos (\Phi'' - \Phi) - C_6 \sin (\Phi'' - \Phi)\right] \right.$$

$$- C_1'\left[\left(C_5 U_1 - C_2 U_3\right) \cos (\Phi'' - \Phi) - \left(C_6 U_1 + C_2 C_3\right) \sin (\Phi'' - \Phi)\right]$$

$$\left. + C_1'' C_5''\left(U_1^2 + U_2^2\right)^{1/2} - C_2\left(C_3' U_2 - C_4' U_4\right) \cos (\Phi'' - \Phi) \right\} \tag{B35}$$

and

$$C_3'' = \frac{1}{C_2''\left(U_1^2 + U_2^2\right)^{1/2}} \left\{ C_1'\left[\left(C_5 U_1 - C_2 U_3\right) \sin\left(\Phi'' - \Phi\right) + \left(C_6 U_1 + C_2 C_3\right) \cos\left(\Phi'' - \Phi\right)\right]\right.$$

$$- \left(C_4' U_2 + C_3' U_4\right)\left[C_6 \cos\left(\Phi'' - \Phi\right) + C_5 \sin\left(\Phi'' - \Phi\right)\right]$$

$$\left. - C_2\left(C_3' U_2 - C_4' U_4\right) \sin\left(\Phi'' - \Phi\right) - C_1'' C_6''\left(U_1^2 + U_2^2\right)^{1/2} \right\} \tag{B36}$$

Since equations (B7) and (B8) represent the sequential application of two general perspective projection relationships and are found to exhibit the same form of equations as a single perspective projection, two (or more) sequentially performed perspective projections must be equivalent to a single perspective projection.

REFERENCES

1. Merritt, Everett L.: Analytical Photogrammetry. Pitman Pub. Corp., c.1958.

2. Moffitt, Francis H.: Photogrammetry. Second ed. Int. Textbook Co., c.1967.

TABLE I.- TARGET POSITIONS DETERMINED FROM DIRECT APPLICATION OF TRANSFORMATION

TO PHOTOGRAPHS FROM DIFFERENT CAMERAS

Camera format size	Target A				Target B			
	Lateral position, X, m		Longitudinal position, Y, m		Lateral position, X, m		Longitudinal position, Y, m	
	Actual	Determined	Actual	Determined	Actual	Determined	Actual	Determined
35 mm	0.920	0.923	3.658	3.652	1.530	1.531	0.914	0.911
6 cm × 8 cm	↓	.915	↓	3.650	↓	1.532	↓	.913
10 cm × 13 cm		.926		3.655		1.537		.918
20 cm × 25 cm	↓	.922	↓	3.662	↓	1.530	↓	.916

TABLE II.- MODEL POSITION[a] AND HEADING ANGLE DETERMINED FROM TRANSFORMATION ADJUSTED FOR ELEVATION OF MODEL TARGETS

(a) Far-left model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
1	-0.610	-0.600	13.411	13.373	-30.00	-30.12
2		-.614		13.375	-15.00	-15.60
3		-.577		13.360	.00	4.34
4		-.601		13.378	15.00	17.62
5		-.631		13.446	30.00	31.80

[a]Model positions are depicted in figure 5.

(b) Far-central model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
6	0.610	0.600	13.411	13.405	-30.00	-29.95
7		.607		13.387	-15.00	-14.22
8		.600		13.404	.00	.60
9		.580		13.406	15.00	15.26
10		.608		13.411	30.00	30.31

(c) Far-right model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
11	1.829	1.765	13.411	13.354	-30.00	-32.49
12		1.829		13.377	-15.00	-12.69
13		1.830		13.398	.00	-.42
14		1.856		13.453	15.00	15.34
15		1.783		13.344	30.00	32.17

TABLE II.- Continued

(d) Intermediate-left model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
16	-0.610	-0.599	10.973	10.957	-30.00	-30.33
17		-.598		10.968	-15.00	-14.26
18		-.609		10.967	.00	3.54
19		-.616		10.973	15.00	17.67
20		-.588		10.963	30.00	32.19

(e) Intermediate-central model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
21	0.610	0.588	10.973	10.964	-30.00	-29.00
22		.604		10.949	-15.00	-13.09
23		.599		10.972	.00	.62
24		.604		10.963	15.00	15.99
25		.612		10.974	30.00	31.28

(f) Intermediate-right model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
26	1.829	1.806	10.973	10.946	-30.00	-29.95
27		1.800		10.941	-15.00	-14.92
28		1.828		10.951	.00	.67
29		1.824		10.961	15.00	15.62
30		1.821		10.947	30.00	29.84

TABLE II.- Concluded

(g) Near-left model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
31	-0.610	-0.609	8.534	8.528	-30.00	-31.02
32		-.610		8.534	-15.00	-15.76
33		-.621		8.547	.00	-.47
34		-.591		8.527	15.00	15.68
35		-.584		8.539	30.00	29.81

(h) Near-central model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
36	0.610	0.603	8.534	8.530	-30.00	-30.26
37		.612		8.524	-15.00	-16.66
38		.606		8.533	.00	.38
39		.610		8.533	15.00	15.19
40		.615		8.518	30.00	29.46

(i) Near-right model position

Case	Lateral position, X, m		Longitudinal position, Y, m		Heading angle, ψ, deg	
	Actual	Determined	Actual	Determined	Actual	Determined
41	1.829	1.814	8.534	8.535	-30.00	-29.11
42		1.818		8.531	-15.00	-14.00
43		1.830		8.533	.00	.86
44		1.811		8.520	15.00	15.85
45		1.817		8.522	30.00	30.88

42
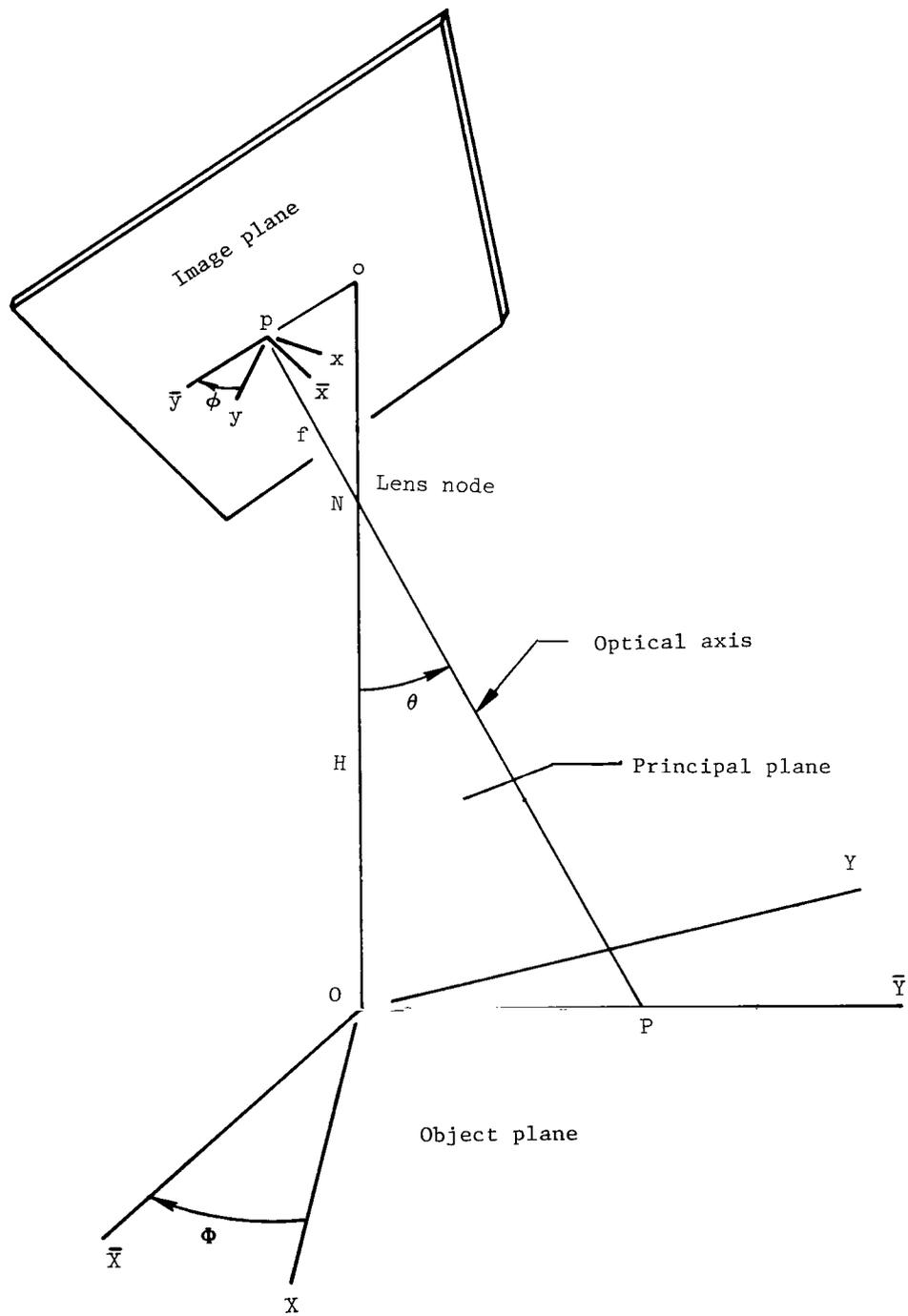
Figure 1.- Tilted perspective projection with coordinate rotations.
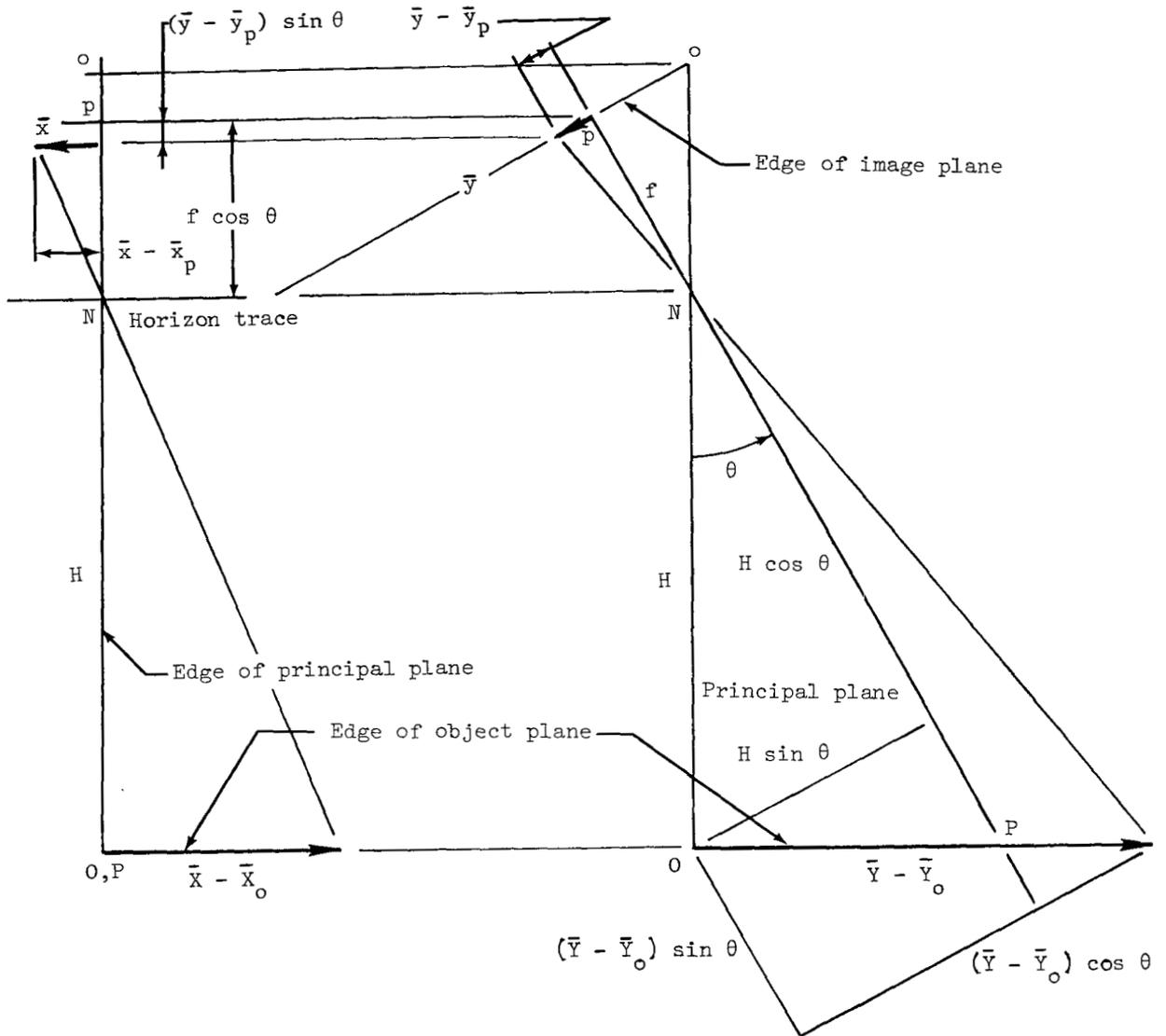
Figure 2.- Geometric relations between image and object planes.
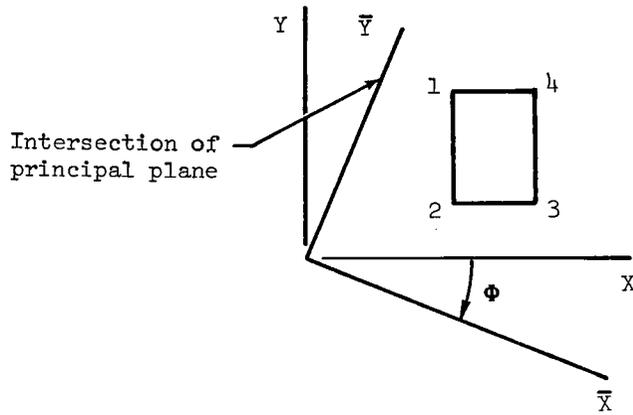
(a) Object plane.
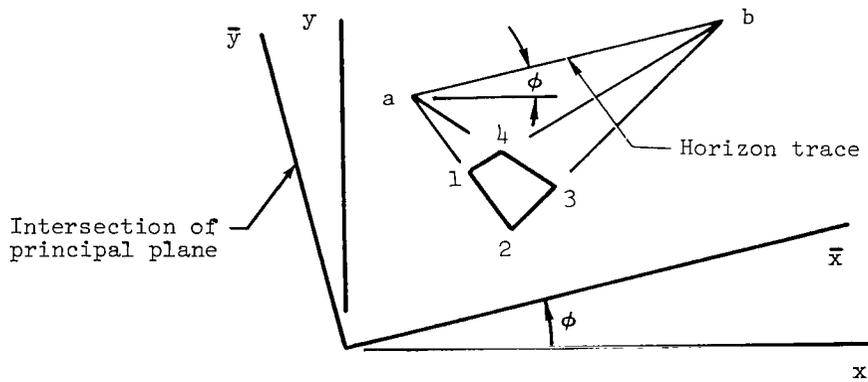
(b) Image plane.

Figure 3.- Views of reference grid in object and image planes.
Rotated principal axes are denoted by bars.
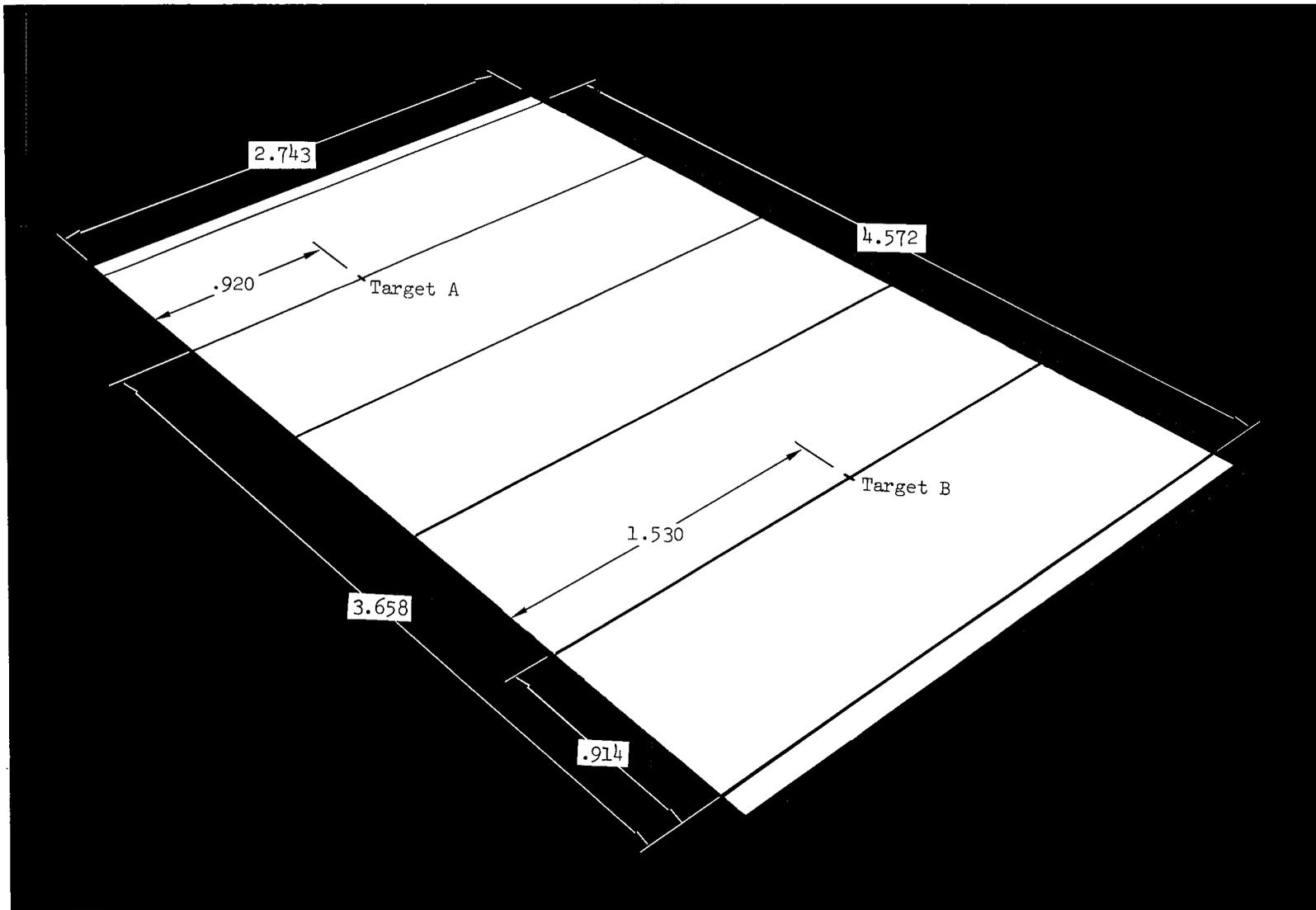
Figure 4.- View of grid and targets for direct application of transformation.   Dimensions are given in meters.
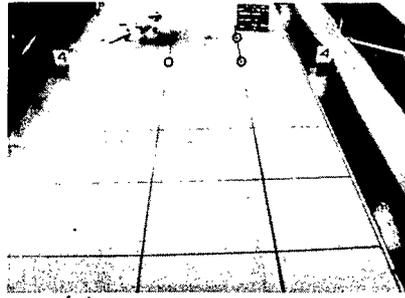
(a) Far-left position.
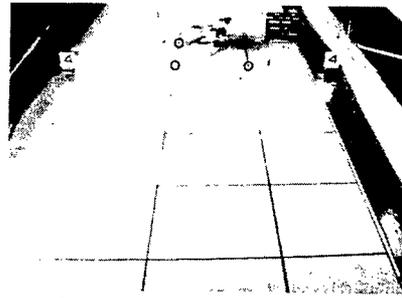
(b) Far-central position.
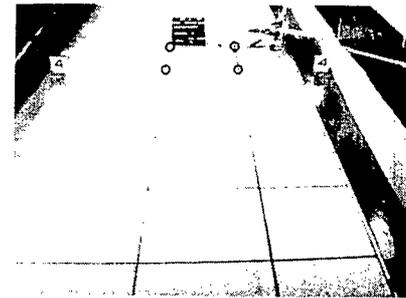
(c) Far-right position.
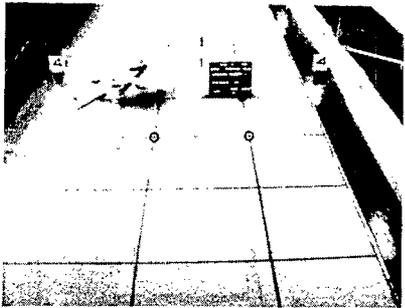
(d) Intermediate-left position.
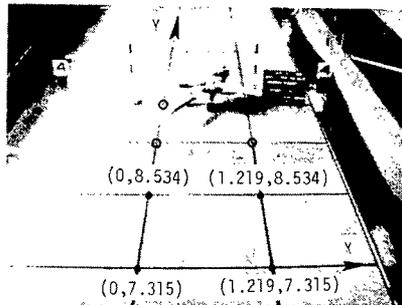
(e) Intermediate-central position.
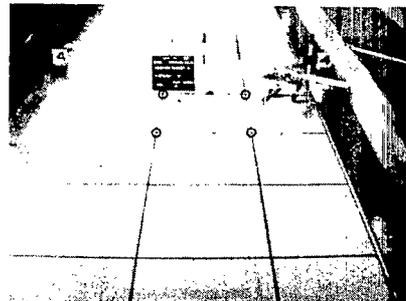
(f) Intermediate-right position.

(0,8.534)   (1.219,8.534)

(0,7.315)   (1.219,7.315)
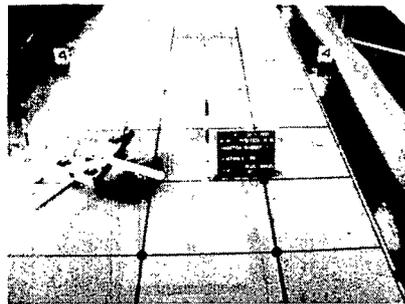
(g) Near-left position.

(h) Near-central position.
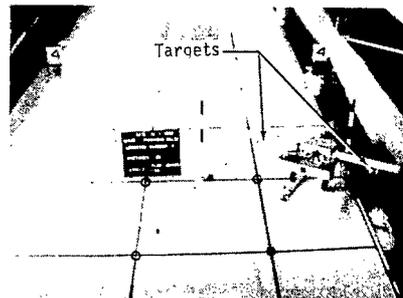
(i) Near-right position.

Targets

L-76-121

Figure 5.- Positions of landing gear model with raised targets. Model is shown with a 30° clockwise heading (ψ = -30°). Corners of reference grid are indicated by circles and the specified grid coordinates are given in meters.

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