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WAVE THEORY OF TURBULENCE IN COMPRESSIBLE MEDIA

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16. Abstract The generation of sound in turbulent flows cannot be determined without a solution of the problem of turbulence itself since both sound and turbulence are manifestations of the same phenomenon of random fluid fluctuations and because sound and turbulence are strongly coupled. In practical applications, one is usually interested either in the far-field noise away from turbulent flow regions or in the turbulent noise transmitted through solid boundaries in contact with turbulent flows. In the first case, it suffices to determine the acoustic mode of energy propagation at the edge of the turbulent region, and in the second case, both pressure and momentum fluctuations, whether radiating acoustically or being convected by a turbulent flow past an elastic solid durface, are of interest. Thus, a theory of noise generation in turbulent flows should be capable of predicting the radiating and the convected fluid fluctuations alike. Motivated by these requirements the "acoustical theory of turbulence" was developed by the author. The research effort was intensified under NASA Grant NGR 15-005-174, culminating in the present report which contains both the initial work and new results.

The statistical framework adopted is a quantum-like wave dynamical formulation in terms of complex distribution functions. This formulation results in nonlinear diffusion-type transport equations for the probability densities of the five modes of wave propagation: two vorticity modes, one entropy mode, and two acoustic modes. This system of nonlinear equations is closed and complete. The technique of analysis in this report is chosen such that direct applications to practical problems can be obtained wth relative ease.

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LIST OF SYMBOLS

$a = (\gamma RT)^{\frac{1}{2}}$ - adiabatic speed of sound

A - scalar potential of the probability velocity $\bar{V}$

$\bar{B}$ - vector potential of the probability velocity $\bar{V}$

$\mathcal{B}_i$ - vector function bilinear in the fluctuations defined by Eq.(4.15)

$c_\alpha = 0$, for $\alpha=1,2,3$; $= 1$ for $\alpha=4$; $= -1$ for $\alpha=5$

c_p, c_v - specific heats at constant pressure and volume, respectively

$C^{m_1 \ldots m_N}_{n_1 \ldots n_N}$ - cumulants of the wavenumber probability distributions

E - energy

$\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$ - force-like sources, defined by Eq.(4.13)

$f(\bar{x},\bar{k},t) = \phi^*\phi$ - probability distribution in the phase space

h - Planck's constant

$\hbar = h/2\pi$

H - Hamiltonian function

$\mathcal{I}_i$ - vector function linear in fluctuations, defined by Eq.(4.15)

$\bar{k}$ - wavenumber vector

$K = \gamma\nu k/(2aPr)$ - ratio of molecular mean free path to the wavelength

L - Lagrangian function

$\mathcal{L}_{ij}(u_j)$ - linear part of the Navier-Stokes differential operator

$\dot{m}_1$, $\dot{m}_2$ - mass-like sources, defined by Eq.(4.12)

$M^{m_1 \ldots m_N}_{n_1 \ldots n_N}$ - ordinary moments of the wavenumber distribution

$\bar{n}$ - unit normal vector

$N_{n_1 \ldots n_N}^{m_1 \ldots m_N}$ - central moments of the wavenumber distribution

p - pressure

$\bar{p}$ - quasi-particle momentum vector

$Pr = c_p \mu/\kappa$ - Prandtl number

$P_\alpha = R^2$ - spatial probability density of the α-th mode

$P_{\alpha j}$ - participation coefficients of the j-th unknown in the α-th mode of propagation

$\dot{q}_1, \dot{q}_2, \dot{q}_3$ - heat-like sources, defined by Eq(4.14)

R - gas constant, also amplitude of the characteristic function ψ

$s_\alpha = (\bar{x} \cdot \bar{k} - \omega_\alpha t)$ - phase function of the Fourier components

$S = \int (\nabla A + \nabla \times \bar{B}) \cdot d\bar{x}$ - phase function of the characteristic function ψ

t - time

T - absolute temperature

$\bar{T}$ - mean or average absolute temperature

$\mathcal{T}_i$ - vector function trilinear in the fluctuations, defined by Eq.(4.15)

$\bar{u}$ - velocity vector

$\bar{u}'$ - fluctuating component of the velocity vector

u - x-component of the velocity fluctuations

u_j - vector of unknowns, $j = 1,..,5$

$\bar{U}$ - mean or average velocity vector

$U_{\alpha j}$ - group velocity of the α-th mode

$\mathcal{U}$ - phase space energy density

v - y-component of the velocity fluctuations

$\tilde{V} = 2 + \mu_2/\mu_1$ - viscosity number

$\bar{V} = \nabla S$ - probability velocity

$\mathcal{V}$ - volume

V - potential energy of quasi-particles

$\alpha = 1,..,5$ - denotes the α-th mode of wave propagation

$\gamma = c_p/c_v$ - ratio of specific heats

Γ_α - attenuation factor, imaginary part of complex frequency

∇ - vector differential operator

ζ_α - real part of the random function ω^*_α

$\zeta(z)$ - Riemann zeta function

κ - heat conductivity coefficient

λ - wavelength

μ - coefficient of dynamic viscosity

μ_1, μ_2 - first and second coefficients of viscosity

ξ_α - imaginary part of the random function ω^*_α

$\bar{\rho}$ - mass density of the mean flow

ρ' - fluctuations in mass density

ϕ - wave amplitude; also the dissipation function in Chapter IV

$\psi_\alpha = Re^{iS}$ - characteristic function of the α-th mode, Eq.(6.2)

ω_α - circular frequency of the α-th mode, Eq.(4.19)

$\omega^*_\alpha = (\zeta_\alpha - i\xi_\alpha)k^2$ - random function modifying the frequency ω_α

Special Symbols

$(\ )^*$ - indicates complex conjugate

$<\ >_\alpha$ - indicates average with respect to the probability distribution of the α-th mode of wave propagation

$(\overline{\ \ })$ - indicates mean value, sum of averages with respect to all modes of wave propagation

"It seems to me that the test of 'Do we or do we not understand a particular subject in physics?' is 'Can we make a mechanical model of it?'...."

Lord Kelvin

Quoted in P. Duhem, "The Aim and Structure of Scientific Theory," Princeton University Press, p. 71, 1954.

I. INTRODUCTION

The generation of sound in turbulent flows cannot be determined without a solution of the problem of turbulence itself since both sound and turbulence are manifestations of the same phenomenon of random fluid fluctuations and because sound and turbulence are strongly coupled. As a matter of fact, carefully worded definitions are needed to distinguish the sound proper (a coherent or incoherent acoustic radiation) from a "pseudo-sound" or the non-radiating pressure and other fluctuations convected by the fluid and diffusing through it. Solving the problem of turbulent noise in terms of the properties of turbulent flows rather than as a particular aspect of such flows entails another difficulty. Turbulence is amendable only to a statistical description. Thus only statistical properties of sound generated by turbulence may be inferred from its statistics. Consequently, it may be concluded that real practical advantages in the analysis of turbulent noise lie in, 1º, a common theory that would treat sound and turbulence simultaneously as two different manifestations of the same random phenomenon, 2º, that the theory be based on the statistical methods in the form commonly used in the fields of acoustics and turbulence alike permitting a convenient representation of the "acoustical" and "turbulent" functions, and, 3º, that the theory be able to separate those aspects of the problem that are referred to as "acoustical" from those that are traditionally associated with the purely "turbulent" motion, even though this distinction is not clear and depends on the particular definition employed. In practical applications one is usually interested either in the far field noise away from turbulent flow regions or in the turbulent noise transmitted through solid boundaries in contact with turbulent flows. In the first case, it suffices to determine *the acoustic mode of propagation of energy at the edge of the turbulent* region, because only the acoustic mode is capable of radiating far away from its source. In the second case, both pressure and momentum fluctuations, whether radiating acoustically or being convected by a turbulent flow past an elastic solid surface, are of interest because sound waves in a solid material of the boundary may be excited by both the radiating and the convected fields. Thus a theory of noise generation in turbulent flows should be capable of predicting the radiating and the convected fluid fluctuations, the "acoustical" and the "turbulent" properties alike. Motivated by these requirements the "acoustical theory of turbulence" was developed by the author (Kentzer, 1974a, b), and this report addresses itself to the task of summarizing and extending the previously achieved results to a point where the theory is closed and complete and ready to be tested on some simple cases.

From its early conception the theory was based on the statistics of wave motions in viscous compressible fluids in order to take advantage of the Fourier mode of analysis which is in common use both in the statistical turbulence and in the field of acoustics. An added incentive to pursue the wave formulation of the theory of turbulence was the meeting and stimulating conversations with Academician A. A. Dorodnitsyn and Prof. M. Z. E. Krzywobłocki on the occasion of a round-table discussion of the numerical computations of turbulent flows, held during the 8th Symposium on Advanced Methods and Problems in Fluid Dynamics, Tarda, Poland, 1967. It became apparent at that time that what remained to be done was to choose a proper statistical framework in order to obtain kinetic equations for the time evolution of the wave distributions, and that such distributions would determine all statistical properties of turbulence including its acoustic modes. Since the Symposium at Tarda the author continued exchanging numerous communications, with Prof. Krzywobłocki and others, on the subject of analogies existing between the wave dynamics of turbulence and wave mechanics of quantum systems. The analogies suggested the quantum-like framework for the theory mainly on the basis of the availability of proven mathematical methods developed over the years for the purpose of treating quantum problems. Thus the sufficiency of the use of quantum methods in turbulence was recognized early. A survey of literature revealed many applications of quantum methods to the study of turbulence and many arguments for the necessity of quantum-like formulation of the theory. A discussion of these subjects is included in this report, Chapters II and III, in order to illuminate the background of the genesis of the present theory.

The wavedynamical formulation of turbulence, with its orthogonal decomposition of the fluid fluctuations into the vorticity, entropy, and acoustic modes, was found to be a natural tool for the study of the noise generation in turbulent flows. With the objectives of deriving expressions for the sound sources in turbulence, arising from interactions with the mean flow and with the vorticity and entropy waves, and the determination of the propagation properties of the acoustic mode, the research effort was intensified under NASA Grant NGR 15-005-174, culminating in the present report which contains both the previous work and new results.

The philosophy underlying the concepts that guided the development of the present theory will be discussed briefly. We observe first that laminar flows, considered as solutions of the Navier-Stokes equations, depend continuously on and are determined by the parameters contained in the differential equations and in the appropriate boundary and initial conditions. These parameters, which usually are grouped into nondimensional ratios, are macroscopic in nature. Once laminar flows become unstable, the initial-boundary value problems for the Navier-Stokes equations are not properly posed because the solutions cease to depend continuously on the initial and/or boundary data. These data are not sufficient for the determination of a unique solution. As soon as a disturbance in a laminar flow becomes sufficiently irregular so that a large number of wave components (eigenmodes) becomes excited, there arises the need for treating a continuous medium as a system with infinitely many degrees of freedom. To describe the behavior of such a system one must use an infinite number of generalized coordinates. It was then decided to choose the wave solutions of the Navier-Stokes equations, linearized around the local mean flow, to serve as a complete orthogonal set of basis vectors. Consequently, the coefficients of the expansion in terms of such orthogonal modes become the coordinates in the space spanned by the basis vectors. Further, the basis vectors are functions of the instantaneous local mean flow which plays the rôle of the space-time-dependent macroscopic parameters. The analogy to a system of harmonic oscillators becomes apparent and suggests the use of traditional statistical methods for treating such systems. In turn, the arguments of Ehrenfest (1911) applied to a system of oscillators in equilibrium with an energy reservoir convinced the author of the necessity of considering the statistical methods of quantum theory.

We will not review here the state of the theoretical knowledge of turbulent phenomena. A brief history of theories of turbulence and a description of modern theories are given in the Introduction, pp. 5-19 of the book by Monin and Yaglom (1971) to which the reader should refer for an extensive bibliography of the subject. Modern approaches to the theory of turbulence apply statistical methods to the ensemble of turbulent flows satisfying macroscopically identical external conditions. Theories that are rigorous and free from any ad hoc statistical approximations have their origin in the work of Hopf (1952) who derived a linear functional differential equation for a characteristic functional of incompressible turbulent fields. Hopf's formulation is closed and complete, but leads to considerable practical difficulties of solving equations in functional derivatives. With numerical solutions of turbulent flow problems in mind, the present theory follows a more tractable space-time formulation. Admittedly, the mathematical rigor is sacrificed in the process and traded for the chance to use more familiar mathematical techniques and for the relative ease of direct applications to practical problems.

The organization of the material presented here is as follows. In

Chapter II the apparent analogies between turbulence theories and quantum theories are discussed. The use of quantum concepts and known quantum-like formulations of theories of turbulence are reviewed briefly. Several arguments for the necessity of allowing for the discontinuous ("quantized") nature of turbulent energy exchange processes are given in Chapter III. The present wavedynamical theory is developed in Chapter IV, its quantum-like interpretation formalized in Chapter V. The derivation of the partial differential equations for the characteristic functions and for the field probabilities is carried out in Chapter VI. With the view toward applications to practical problems, Chapter VII gives the generalization of the statistical concepts required in the present formulation, and Chapter VIII gives some simple results in the form of expressions for averages of the squares of turbulent fluctuations which show separate contributions of the vorticity, entropy, and acoustic modes. Chapter VIII also contains suggestions for the method of obtaining distribution functions approximately.

Equations in this report are numbered consecutively within each chapter, with the number of the chapter followed by a period and the number of the equation in that chapter. References are listed in an alphabetical order by the surname of the first author and are cited in the text by the name(s) of the author(s) with the year of publication given in parentheses. Letters of the alphabet are further used to distinguish works of a given author which appeared in the same year.

...[a physical analogy may be defined as] *"that partial similarity between the laws of one science and those of another which makes each of them illustrate the other."*

James Clerk Maxwell

"On Faraday's Lines of Force," Trans. Cambr. Phil. Soc. 10 (1855), Sci. Papers, I, p. 155.

II. RELATIONS BETWEEN TURBULENCE THEORY AND QUANTUM THEORY

In previous publications by the author (1974a,b,c), denoted hereafter as I, II and III, a mathematical formalism of a theory of turbulence (TT) of compressible, viscous, heat conducting fluids was developed from the Navier-Stokes theory. In I, II and III the author alluded to the association of the developed theory to the quantum mechanics of single particles (QM) and to Planck's theory of thermal radiation. These allusions raise the question whether the association (or more properly, the isomorphism of mathematical structures) of the theory proposed in I and II with QM and, possibly, with other physical theories is strictly coincidental, or physically meaningful, mathematically significant, and, in general, useful.

Independently of each other, many researchers have observed in the past that there are analogies between fluid mechanics and physical processes studied by quantum theory. For instance, Madelung (1926) derived the equations for isentropic flows of an inviscid gas by separating real and imaginary parts of the Schroedinger equation of one-particle quantum mechanics and thus formulated quantum mechanics in the hydrodynamic form. A comprehensive discussion of the hydrodynamic picture of quantum mechanics is given by Wilhelm (1970a,b) who derives quantum-hydrodynamic uncertainty relations and relates the minimum uncertainty products to the interior quantum stresses. In quantum-hydrodynamics the quantum stresses are quadratic in the gradient of the logarithm of the position probability. Wilhelm mentions that *"a hidden turbulence (excited by the presence of a particle) kicking the particle to and fro in a random manner could lead to an explanation of the nonlinear quantum force,...; the mechanism giving rise to the uncertainty phenomenon in quantum systems would be similar to that in classical stochastic systems."* Terms analogous to quantum stresses appear naturally in the partial differential equations for the probability density of turbulence in the present formulation, Eq. (6.17).

The inverse problem, namely, the association of fluid mechanics with quantum mechanics, allows for the transformation of the fluid-mechanical set of nonlinear conservation equations into a complex scalar wave equation with nonlinearity appearing only in the expression for the potential of the pressure forces. Krzywobłocki (1958) used this approach to study diabatic flows with heat addition and followed later with a wavemechanical formulation of the theory of turbulence, (1971a, 1971b). Whilhelm (1971) in his formulation of the wavemechanics of compressible fluids points out that the transformed complex scalar equation (the Schroedinger equation) leads to a considerable simplification in the mathematical description of compressible fluids. As an illustration of a solution of a fluid problem formulated according to the wavemechanical theory he gives the example of the propagation of sound waves.

Green (1965) remarks that fluid mechanics has only statistical significance and that predictions based on equations of fluid mechanics are only confirmed exactly in an experimental ensemble. He further draws the attention to the fact (p. 174) that *"in the macroscopic context there are uncertainties which no amount of careful observation and calculation can remove. This situation is not fundamentally different from what is known to exist in quantum mechanics, where Heisenberg's uncertainty principle frustrates every attempt to predict the result of a single experiment."*

Spalding (1972, 1974), in discussing a turbulent transport of a scalar, observes that the gradient approximation for the turbulent flux relates the turbulent diffusion to local properties of the flow. Such an approximation is inadequate when the length scale of turbulence is not small in comparison to the distance over which the gradients of fluid properties vary. In some situations the coefficient relating the flux to the gradient may even change signs. Spalding (1974) remarks that *"the situation is similar in this regard to that encountered in radiative transfer; for often the 'mean free path of radiation' is of the same order of magnitude as the dimensions of the apparatus."* He then proceeds to model the turbulent transport after the radiative processes.

Millsaps (1974) turns to the fundamentals of the quantum theory to propose the extension of Poincaré's (1912) proof of the necessity of quantization to the case of hydrodynamical turbulence. Whitham (1965) in his work on waves in inhomogeneous media discovered the prominent rôle played by the adiabatic invariants. The same concepts of adiabatic invariants were used by Ehrenfest (1911) to provide the proof of the necessity of quantization of a system of oscillators.

Huggins (1971) gave an interesting interpretation of the classical vorticity field, such as the one given by the curl of the incompressible turbulent velocity field. First, he observes that one can represent the dynamics of the three-dimensional solenoidal field by a conserved two-dimensional vorticity current. Classically, one can have a continuous vorticity field flowing in space, while for a quantum fluid all vorticity is localized in quantized cores. Huggins, then, proposes that the cross section of the quantized core be treated as a two-dimensional quantum excitation of the classical vorticity field in a way similar to the treatment of phonons considered as quantum excitations of the classical sound field. To establish this picture, Huggins proposes that the classical vorticity field be treated as the intensity of the probability field for a quantized vortex, that the vorticity current be treated as a probability current, and that the classical hydrodynamic equation for vorticity be treated as a semiclassical equation for the vortex probability field. With $\vec{\omega}$ = curl of velocity, and Γ = circulation around a circuit C, the flux of $\vec{\omega}/\Gamma$ through the circuit C is the probability that a quantized vortex threads the circuit. This interpretation allows one to describe the dynamical behavior of the vorticity field and leads to an explicit hydrodynamic model for how the fluid fluctuations can create a distribution of vortex rings.

The most striking example of a mathematical analogy between fluid mechanics and quantum mechanics is provided by the normal modes of the ocean. Eckart (1961) showed that the depth of the ocean, at which the Väisälä-Brunt frequency rises to a maximum, defines a thin layer in which the ocean can sustain trapped waves with frequencies not exceeding the local cut-off frequency. The waves are governed by an equation formally identical to the Schroedinger equation. He further points out that this situation defines a set of normal trapped wave modes for the ocean mathematically analogous to the vibrational quantum states of the diatomic molecule. Eckart's normal modes of the ocean are a special case of waveguide effects in stratified fluids. As Tolstoy (1973), p. 124, observes, the general form of a characteristic equation for an internal waveguide in a stratified medium takes the form of the Bohr-Sommerfeld quantization condition.

Edwards and McComb (1969) studied the statistical mechanics of a system far from equilibrium in which the dominant process is a flow of energy through the normal modes of the system. They argued that in the case of a randomly excited fluid turbulence there is a strong mathematical analogy between the classical (i.e. turbulent) cascade of energy and the quantum field or the many-body problem.

On page 4 of their book on the mechanics of turbulence Monin and Yaglom (1971) point out mathematical similarities of statistical theories of turbulence (TT) and quantum field theory (QFT) stating that "*a far more fruitful, perhaps, is the analogy between the theory of turbulence and quantum field theory, which is connected with the fact that a system of interacting fields is also a nonlinear system with a theoretically infinite number of degrees of freedom. From this follows the similarity of the mathematical techniques used in both theories. This allows us to hope that the considerable advances in the one will also have a decisive effect on the development of the other.*" In particular, they observe, p. 19, that Hopf's (1952) equation for the characteristic functional of an incompressible turbulent field is formally similar to the Schwinger equations of quantum field theory, which are equations for the Green's function of interacting quantum fields.

Methods similar to those of the quantum field theory and the quantum mechanical many-body problem were used by Wyld (1961) to formulate the theory of turbulence in incompressible fluids. A systematic perturbation series is shown by Wyld to be in one-to-one correspondence with certain diagrams analogous to Feynman diagrams. The series is arranged and partially summed in such a way as to reduce the problem to the solution of three simultaneous integral equations in three functions, one of which is the second order velocity correlation. Truncation of the integral equations at the lowest nontrivial order yields Chandrasekhar's equation, and truncation at higher order yields the equations discussed by Kraichnan.

Kawasaki (1974), in studying the statistical mechanics of turbulence far from equilibrium, points out that the merit of his approach to the solution of the stochastic equations of turbulence is that it is formulated in the language of quantum field theory and many-body problems and, therefore, various techniques developed there should be also applicable to turbulence. He explores this aspect of his approach by developing a non-perturbative self-consistent scheme to obtain average values of the gross variables, the time-correlation functions of the fluctuations, the non-equilibrium steady state distribution function, and the response function to a small external disturbance. He finds particularly helpful the analogy to the condensed Bose systems. Similarly Ross (1969) develops a quantum-mechanical prescription, together with Feynman diagrams, for calculating wave spectra, statistical averages, and particle diffusion in a turbulent plasma. He concludes that *"the quantum method provides a relatively simple way of deriving and interpreting equations for the time development of the wave spectrum and particle diffusion."*

Piest (1974) attempts to develop a theory of turbulent fluid motion by means of a classical n-particle molecular statistical mechanics (which is a classical limit of a quantum mechanics of a system of particles), and derives closed system of equations which are nonlinear and nonlocal in space-time. Quantities are defined which resemble the mean fields of density, temperature, and velocity of turbulent flow. The nonlocal terms contain equilibrium correlation functions which are physical properties of matter, i.e., space-time-dependent counterparts of viscosity and heat conduction coefficients.

In a recent publication Gyarmati (1974) showed that the generalization of dissipative fields to complex scalar fields leads to a generalized variational principle for dissipative processes in media with linear constitutive equations, and that if and only if complex state vectors are used the variational formulation is isomorphic with (has the same mathematical structure as) the one-particle quantum mechanics. Kentzer (1974c) showed that the use of Fourier modes as complex state vectors in the representation of statistical turbulence, combined with the allowance for nondifferentiability of the phase function, establishes the operator algebra, the uncertainty principle, and the complementarity principle for the statistical theory of turbulence, in analogy to the similar corner-stones of the quantum theory. The classical limit in the quantum mechanical correspondence principle has as its counterpart in turbulence the case of low intensity turbulence in a steady homogeneous mean flow in which case turbulence may be described by the statistics of non-interacting wave packets that follow classical Hamiltonian trajectories.

Treating a general stochastic process in the mathematical framework of the quantum theory Santos (1974) shows that, if non-commutative complex algebra is used, an operator equation can be associated with every stochastic equation. The equations of motion derived by Santos for the Brownian motion and for a single particle in stochastic electrodynamics coincide with the basic ones of quantum mechanics. These two examples give credence to the belief that quantum-like formulation of stochastic processes, being a more general than the classical formulation, may be necessary for the description of some random processes. In words of Santos, "*the difficulty might be that the mathematical techniques developed to deal with stochastic systems are not suitable for the specific system...*" Turbulence might be just such a system for which the combination of classical fluid mechanics and statistics of *real* probability distribution functions, as opposed to *complex* wave functions, may be inadequate.

Conversely to Santos' objectives, Hańčkowiak (1975) adapts methods of classical random fields, in particular that of Hopf (1952) and of Monin and Yaglom (1971), to the quantum field theory. Specifically, he constructs n-point functions (moments) describing quantum fields with the aid of solutions of the classical field equations. In the present work, and in previous publications, the author independently arrived at the same conclusion, namely, that if a strongly interacting turbulence is to be treated by a quantum field theory of strongly interacting, infinitely-many bodies, then such a theory should be formulated in terms of solutions of the classical field equations. Wave solutions of a compressible viscous fluid in a locally steady and homogeneous flow provide a complete orthogonal set of solutions which serve as vector basis for a formal expansion of the fields.

In the work of Santos and Hančkowiak we have examples of both the classical stochastic processes put in the form of quantum theory, and quantum field theory formulated in terms of solutions of classical random fields. Thus the two theories, the classical and quantum, have been used interchangeably to study physical processes from different points of view.

The above discussion of analogies and similarities that exist between the theories of turbulence (TT) and various physical theories, such as particle quantum mechanics (QM), quantum field theory (QFT), or radiative transfer and kinetic theory, raises an important question, namely, whether TT, in the form of competing theories (e.g. those of Hopf (1952), Wyld (1961), Kawasaki (1974), Piest (1974), Krzywobłocki (1971b) and the present theory), are isomorphic to QM, QFT, to an extension or generalization of these, or to other physical theories. This question may be discussed in the light of mathematical logic and foundations of the quantum theory as employed, e.g., by Strauss (1972).

As defined by Strauss, a physical theory is "*a union of mathematical structure and its physical interpretation.*" The equivalence of different theories thus has dual aspects, viz., mathematical equivalence (isomorphism) and physical equivalence. In the words of Strauss, "*mathematical isomorphism is not the same as physical equivalence...*" (p. 94) ... "*isomorphic formalisms can represent different physical theories, e.g., the well known isomorphism between geometrical optics and classical* *mechanics which may both be deduced from the same variational principle ... The view that mathematical formalisms have to be isomorphic if they are to represent the same theory is untenable."* (p. 94). (Yet)... *"different interpretations of the same formalism lead to inequivalent physical theories."* (p. 95). Consequently, mathematical isomorphism of TT and QM or QFT does not imply that they are physically equivalent and no such claims will be made. On the other hand, various competing theories may be physically equivalent in some, but not necessarily in all the aspects as they are not mathematically equivalent. Depending on their mathematical structures, the various competing theories may be generalizations or special cases (subsets) of each other.

In this work we are primarily interested in choosing a mathematical structure for the formulation of TT that will be more general than a formulation in terms of a classical theory of probability based on stochastic equations in real variables. Yet, we would like to avoid complexities of functional calculus as used in QFT. Thus we search for a convenient passage from the field equations of the Navier-Stokes theory to a statistical theory of field probabilities. Of great help and inspiration in this task are the Intertheory Relationships which serve as examples, guidelines, and storehouse of existing knowledge. *"Not all will agree that Intertheory Relations may become a heuristic instrument for finding new physical theories. However, we can extend our studies to relations of second order, viz., to relations between relations. We may have ground for believing that the new theory (T_4) looked for will stand in the same (or similar) relations to T_3 as T_2 stands to T_1:*

$$T_4 : T_3 \approx T_2 : T_1.$$

In fact, this was precisely the heuristic scheme by which Schroedinger obtained his 'wave equation':

wave mechanics : classical particle mechanics

≈ wave optics : geometrical optics."

(Strauss, 1972, p. 268). Essentially, this type of reasoning also inspired de Broglie to propose his famous matter-wave hypothesis by observing the formal analogy between Fermat's principle of optics and Hamilton's principle of dynamics on one hand, and the wave-particle nature of light on the other.

In the case of fluid turbulence, we have the field equations (the Navier-Stokes system) that determine all geometrical (causal) attributes of a single infinitesimal wave, the types of waves, their interactions, and we may average the equations and thus obtain partial differential equations for the average (the mean or expectation) values. Thus, we know, essentially, the quasi-particles (wave packets) and the forces of interaction, and we are faced with the many-body problem for strongly interacting random systems of such quasi-particles. We are aware of similarities and analogies to many physical theories, such as, e.g., the many-body classical and quantum mechanics, the quantum field theory, the theory of random systems, etc. We are thus faced with the choice of the Intertheory Relations, namely, which theories (and at which level of application) are in the $T_4 : T_3 = T_2 : T_1$ relation with the theory of turbulence. Since many such levels of application may be of interest, no single Intertheory Relation would suffice as a guide for the development of TT. Expecting many such relations to exist, one should, it appears, not to attempt to enumerate all or as many as possible of the Intertheory Relations relevant to turbulence, but, instead, one should proceed formally developing a TT guided by Intertheory Relations at whatever step or level of application such relations become apparent and helpful.

It is the intention of the author to mention now and then the particular theory from which he borrows the mathematical formalism in the hope that the readers will be able to follow his line of reasoning, extend it, or generalize it so as to help in further developments and improvements.

The steps to be taken in generalizing classical mechanics to quantum mechanics, namely, a replacement of commuting operators on an algebra of real functions by a non-commutative operator algebra and complex functions, will be outlined below. These steps appear to be sufficient, with the help of the correspondence principle, to postulate a system of equations for the characteristic functions that define the probability distributions in the wavenumber space for the several modes of wave propagation including the acoustic mode coupled to the vorticity and entropy modes.

..."we cannot at present compare the contents of a nonlinear classical field theory with experience... At the present time the opinion prevails that a [classical] *field theory must first, by "quantization", be transformed into a statistical theory of field probabilities according to more or less established rules. I see in this method only an attempt to describe relationships of an essentially nonlinear character by linear methods."*

Albert Einstein

"The Meaning of Relativity," p. 165, Princeton University Press, New Jersey, 1966.

III. NECESSITY OF QUANTIZATION OF TURBULENCE

As is well known, see, e.g., p. 292, Morse & Feshbach (1953), in the limit of large values of action and energy, the surfaces of constant phase for the wave function become the surfaces of constant action for the corresponding classical system. Thus wave mechanics goes over to geometrical mechanics just as wave optics goes over to geometrical optics for vanishingly small wavelengths. Bohm (1951, p. 264) shows that the classical (deterministic) treatment is a valid approximation if the spatial gradient of the wavelength λ is negligible as compared to unity, that is, if $|\nabla\lambda| \ll 1$. For one-dimensional steady motion and for the entropy or vorticity waves it may be shown that the condition on the gradient of the wavelength becomes

$$\lambda \frac{|\nabla \bar{u}|}{|\bar{u}|} << 1,$$

where $\bar{u}$ = velocity vector. This condition will be satisfied for $\lambda \to 0$ (short wavelength limit) or for a uniform flow, $\nabla\bar{u} = 0$. We should observe that the condition on the wavelength may be rewritten as $\Delta\lambda/\Delta x << 1$, or $\Delta x \cdot \Delta k >> 2\pi$ where $k = 2\pi/\lambda$ = wavenumber. In the latter form of the condition Δx is the scale of the inhomogeneities of the fluid-mechanical (classical) nature, and we have here a statement of the uncertainty in a simultaneous determination of mechanical and wave attributes of motion. Thus classical (deterministic) treatment is valid only in the limit as one of the scales becomes infinite, that is, as $\Delta x \to \infty$ or $\Delta k \to \infty$ $(\Delta\lambda \to 0)$. Turbulence, however, is a phenomenon in which the energy-containing wavelengths are of the same order of magnitude as the scale of the inhomogeneities of the material properties of turbulence, e.g., the turbulent velocity field. As a consequence, classical continuum mechanics must be extended to cope with such a situation. The extension should account for the fact that the wavenumber can be determined only within the uncertainty range $\Delta k \simeq 2\pi/\Delta x$, where Δx is a measure of the scale of the fluid inhomogeneities (length scale).

One may argue that the Navier-Stokes (NS) theory of viscous fluid is a valid description of all fluid phenomena on scales much larger than the molecular mean free path. Thus, in the continuum range there is no need to introduce wave representation and thus be restricted by the limitations on the wavelength. It may be argued that the limitations on the wavelength arise only in the wave representation and as a consequence of introducing the wave concepts. The penalty for not using the wave representation is the impossibility, at present and in the foreseeable future, of obtaining global solutions of the NS system of equations, which solutions would give the finest and minute details of turbulent microstructure together or simultaneously with the large scale features of fluid motion. Separation of the mean or averaged motion from the small scale turbulent fluctuations renders the solution of the average motion tractable provided that certain functions (correlations or statistical moments) of the turbulent fluctuations are known.

Attempts at replacing the NS description of the details of the turbulent microstructure by an averaged flow with slowly varying statistical properties as parameters avoid the necessity of non-classical treatment of the averaged flow only. The statistics of the turbulent microstructure, on the other hand, is characterized by two length, usually separated in magnitude, L = length scale over which statistical properties vary, and λ = characteristic length (e.g. correlation length) of the turbulent structure. If $L >> \lambda$, then the statistics may be determined under local conditions. If the statistical behavior of the turbulent structure is to be determined by the NS theory in regions of volume λ^3, then, again, we have to consider the motion of eddies of all sizes up to the dimension λ moving through a fluid with irregular (fluctuating) properties of all scales up to the dimension λ. Analytical solutions of the NS equations describing such motions, even in small volumes of order λ^3, are not feasible. Numerical studies of comparable situations usually proceed along two directions - computations in the physical space or in the transformed (Fourier) space. We may argue that both formulations are equivalent because of a one-to-one transformation of the two spaces. In the Fourier space, however, the limitations on the classical treatment apply and cannot be dispensed with. Thus, it is believed that similar limitations are present in fluid-mechanical calculations in the physical space. It is premature and counter-productive to philosophize on the impossibility or uncertainty in simultaneous determination of physical variables which are conjugate to each other. It is much easier to give mathematical arguments in Fourier representation in terms of wave variables and then translate the arguments to physical variables. This procedure necessitates the association of, e.g., the wavenumber and frequency with mechanical quantities, viz., the momentum and energy, respectively. This is essentially the approach used in the quantum mechanics.

In order not to create an impression that quantum effects on the scale of the quantum of action h (Planck's constant) have necessarily any significance in turbulence, we observe that the quantum of action h characterizes the magnitudes (scales) of phenomena on the atomic particle level. In particular, the Einstein-de Broglie relations,

$$E = \hbar\omega, \qquad \bar{p} = \hbar\bar{k}, \qquad \hbar = h/2\pi,$$

establish the exact and universal relations between the scale of mechanical properties E (energy) and $\bar{p}$ (momentum) and the scale of wave properties ω (frequency) and $\bar{k}$ (wavenumber) which hold in applications to atomic systems. Thus h plays the rôle of a conversion factor which changes mechanical units to those appropriate in wave representation. Obviously, the scale ratio, which would play the rôle of the quantum of action in turbulence, cannot be a universal constant since the scale effects in turbulence are governed by the non-dimensional Reynolds number. As a consequence, relations establishing proportionality between wave and particle attributes, such as those of Einstein-de Broglie, are either nonexistent or take a complicated form of a functional dependence on the boundary conditions of the hydrodynamical fields.

As a point of interest we may add that many familiar fluid dynamical concepts could play the rôle of the unit of action (or rather the unit of action density) for the purpose of converting the dimensions from those of frequency and wavenumber to those of energy and momentum, and vice versa. We observe that the quantities ρUL = the numerator of the Reynolds number, μ = dynamic viscosity or the denominator of the Reynolds number, $\rho\int\bar{U}\cdot d\bar{\ell}$ = density x circulation, $\rho\int\int\nabla x\bar{U}\cdot d\bar{s}$ = density x vorticity x area, $\rho\int\int\int\bar{U}\cdot(\nabla x\bar{U})dV/\int\bar{U}\cdot d\bar{\ell}$ = density x helicity x volume/circulation, etc., all have dimensions of action per unit volume. The variety of such quantities, their varied significance, the arbitrariness of the choice of line, surface or volume integrals, all seem to suggest the lack of any physical basis for establishing a physical relation between frequency and wavenumber of waves and their energy and momentum. The work of Ehrenfest (1916) suggests that the quantum of action is related to the adiabatic invariants which, in the case of nonlinear dispersive waves in inhomogeneous fluids, are given a prominent rôle by Whitham (1965).

Any physical theory of turbulence should lead to an agreement with experimental facts. At early stages of the development of a theory it is helpful to check for agreement in some simple or idealized cases. A state of equilibrium in, say, homogeneous, isotropic turbulence is such a case. One would expect that the energy spectrum should drop off to zero at both zero and infinite frequency, be positive everywhere, and have a finite integral (finite spatial density of energy). Thus the spectrum must have a maximum. At high frequencies (vanishingly small wavelengths) classical statistical behavior is expected, equipartition should prevail, and the energy spectrum should drop off exponentially with frequency. The presence of viscous dissipation at high frequencies would cause the spectrum to drop off even faster.

At this point we note that, by considering the entropy of a system of normal modes and by imposing the condition that the energy spectrum fall off faster than any power of frequency, Ehrenfest (1911) has shown that discreteness of energy spectrum is a necessary consequence, whence follows the necessity of quantization of systems of normal modes. It is believed that the same argument would hold in the case of weak turbulence represented by an infinite system of Fourier modes. However, Ehrenfest's proof of the necessity of quantization of the energy of a system of oscillators does not give the magnitude of the spacing of energy levels. Consequently, the spacing of energy levels need not be the same in theories which are not physically equivalent.

We may turn now to the case of turbulence expressed in terms of Fourier modes. Examination of the wave interaction terms, see, e.g., Davidson (1967) or Appendix C of Vedenov (1968), reveals the fact that the Rayleigh-Jeans distribution is sufficient for the existence of an equilibrium. Thus the classical Rayleigh-Jeans distribution appears as an equilibrium solution of the turbulent field analyzed classically. It was Ehrenfest (1911) who first coined the term "ultraviolet catastrophe" to describe the behavior of the classical energy spectrum at high frequencies. Classically, the energy increases as frequency squared or as a negative fourth power of wavelength. Since it is observed experimentally that the λ^{-4} behavior occurs in turbulent spectra at large wavelengths, and that the spectrum drops off approximately (or almost) exponentially as wavelength approaches zero, the question arises whether the departure from the classical Rayleigh-Jeans distribution and the approach to the Planck's shape of the energy spectrum is brought about either by, (1), appreciable "quantum effects" on scales far removed from those governed by the universal constant h, (2), by the presence of dissipation (an energy sink) at small wavelengths, or, (3), by the absence of equilibrium. The explanation (3) was proposed in the case of black body radiation by Jeans and was vigorously defended by him until the publication of Poincaré's proof (1912) of the necessity of discreteness of the energy transfer process. For our purposes we observe that the nonlinear wave interaction processes so dominate the flow of energy through the spectrum that the approach to equilibrium cannot slow down to a standstill and must be sufficiently rapid, or at least non trivial, so as to negate Jean's explanation.

Should (2) be the correct explanation, then the energy would have to increase monotonely as λ^{-4} until the viscous dissipation would become important. Thus one would expect the energy containing scales to be close to if not equal to the dissipation scales. This is not the situation observed in general where the energy containing scales and the dissipation scales are usually far apart. The separation of scales is especially dramatic at high Reynolds numbers, e.g. in atmospheric turbulence. This leaves us with the necessity of a discrete energy exchange process (the "quantum" effect) as the only plausible explanation.

The similarity of general features of the turbulent energy spectrum and Planck's distribution permits one to assess the behavior of turbulence spectrum in slowly decaying homogeneous and isotropic turbulence. It is known that during the decay process the energy distribution is slowly and constantly modified, with the maximum value of the spectral energy decreasing and shifting toward longer wavelengths. The same is true for Planck's distribution,

$$E(\lambda) = \frac{c_1}{\lambda^5} \frac{1}{e^{c_2/\lambda}-1} \quad , \quad c_2 \sim T^{-1} \; .$$

The radiation temperature T should be replaced in case of turbulence by the mean square of the turbulent velocity fluctuations. The maximum of E, E_{max}, occurs at $c_2/\lambda = 4.96$ and is given by

$$E_{max} = \frac{c_1}{\lambda^5_{max}} \frac{1}{e^{4.96}-1} \sim T^5 \sim (\overline{u'^2})^5 \; .$$

Thus, if a turbulent spectrum is approximated by a Planck's distribution, and if the decaying process is represented by decreasing temperature, then, since $\lambda_{max} \sim 1/T$, λ_{max} increases (maximum shifts toward longer wavelengths) and E_{max} decreases during decay. The total energy density, $E = \int E(\lambda) d\lambda$, integrates to give the Stefan-Boltzmann Law, $E \sim T^4$ or the Lighthill's "eights power of velocity law" of acoustic radiation in turbulence.

We conclude that the type of energy spectrum one may expect to obtain for equilibrium turbulence (modeled in terms of normal modes which exchange energy by a discrete process) possesses correct general characteristics, a proper behavior at zero and at infinite wavelengths, and a proper decay behavior. The classical Rayleigh-Jeans spectrum leads to the "ultraviolet catastrophe" and cannot represent the energy distribution except in the limit of long waves which is not of interest in turbulence.

The most general argument in favor of a discrete (as compared to a continuous) energy exchange process was given by Poincaré (1912). Millsaps (1974) was the first to observe that Poincaré's proof applies to turbulent spectra as well as to electromagnetic radiation. The conditions, as stated by Poincaré, are satisfied by turbulent spectra:

1. $E(\omega) \to 0$ as $\omega \to 0$,
2. $E(\omega) \to 0$ as $\omega \to \infty$,
3. $0 \leq \int_0^\infty E(\omega)d\omega < \infty$ for all time, $0 \leq t \leq \infty$,
4. The First and Second Laws of thermodynamics hold.

Poincaré has proved that, if conditions 1 - 4 hold, then it is necessary that the energy exchange in the spectrum occur in a discrete or discontinuous manner. Further, he gave the expression for the form of the distribution at large times, which form is a generalization of the Planck's distribution valid at equilibrium, $t = \infty$. Here again, the mathematical argument does not yield the numerical value of the quantum of energy, it even does not require that such a measure of discontinuity of energy exchanges be a constant. As a consequence, one would expect that in the case of turbulence the energy spectrum would approach the Poincaré's generalized form sufficiently far away from boundaries and far downstream from the transition point, and that such a distribution would have to be modified to account for lack of homogeneity and isotropy.

Another argument in favor of quantization of turbulent energy exchange processes follows from the statistical mechanics. We observe that the difference between the classical Boltzmann statistics (BS) and either the Bose-Einstein (BES) or Fermi-Dirac statistics (FDS) is dependent on the treatment of the elements of the statistical system (particles, oscillators, or normal modes) as distinguishable or indistinguishable entities. Which statistics applies must be determined by experiment. In quantum mechanics the choice between BES and FDS is made on the basis of symmetry or antisymmetry of the wavefunction. If one assumes that similar particles are distinguishable (or indistinguishable), then it follows as a consequence of such an assumption that BS (or either BES or FDS) must be used. In classical physics the identifying or "labeling" of particles in case of similar or equivalent particles depends critically on the continuity of their trajectories. In the Fourier representation there arises a possibility of indeterminate changes (in multiples of 2π) of the phase of the wave. The phase determines the trajectories of the wave packets. Thus the differentiability of the phase implies continuity of wave packet trajectories. The assumption of differentiability is not a necessary one. Thus, the distinction between a classical and nonclassical treatment lies in the arbitrary assumption of differentiability of the phase. The classical treatment may thus be readily generalized by relaxing the conditions of differentiability of the phase, integrability of the trajectories, and distinguishiability of particles. This amounts to a generalization of the geometrical mechanics of wave packet motion characterized by continuous trajectories to wave mechanics which admits discontinuities in trajectories, in the number of particles, in their energy, etc. We have mentioned earlier that the classical (geometric) mechanics fails to describe the system accurately when the scale of inhomogeneities of the medium is comparable to the wavelength considered, as is the case with turbulence. Consequently, it follows that it is necessary in wave (or Fourier) description of turbulence to generalize the geometric (deterministic) description to the nonclassical (probabilistic) mode of treatment. This is accomplished by not requiring that the phase be differentiable. Consequently, the trajectories of wave packets (and their numbers) become discontinuous and similar wave packets cannot be labeled and thus become indistinguishable. The statistics appropriate to indistinguishable wave packets is the Bose-Einstein or Fermi-Dirac statistics. The nonlinear interaction terms, derivable from the Navier-Stokes theory, show that a two-particle distribution function ("wave-function") is given by a product of one-particle distributions, at least at the level of the Navier-Stokes theory. Thus the "wavefunctions" are symmetric and we make the choice, subject to eventual experimental verification, of the Bose-Einstein statistics.

Later on, in discussing statistical moments with respect to the probability distribution in the phase space, it is mentioned (and may be readily verified) that without allowance for the uncertainty in the phase (discontinuity or indeterminacy of phase) the statistics reduces to a trivial case where all the correlations are symmetric and moments of various orders may be put in the form of products of moments of first order. Then all moments would reduce to zero in the case of isotropic turbulence in contradiction with experimental facts.

In conclusion, it is not only *sufficient*, according to Gyarmati's (1974) proof, to treat turbulence as a complex field with complex state vectors as a basis, but it is also *necessary* to allow for discontinuous ("quantized") energy exchange processes in turbulence. A formulation of a statistical theory of turbulence based on quantum mechanical principles generalizes classical statistics, extends and uplates it to a higher level. Thus, such an uplation offers hope of the eventual solution of the general problem of turbulent motion. In this work we outline an approach to the quantum-like wave dynamics of turbulence, which approach avoids the difficulties of solving functional partial differential equations.

"Seeing that I can find no subject specially useful or pleasing - since the men who have come before me have taken for their own every useful or necessary theme - I must do like one who, being poor, comes last to the fair, and can find no other way of providing for himself than by taking all the things already seen by other buyers..."

Leonardo da Vinci, Codice Atlantico, folio 117v

IV. WAVE REPRESENTATION OF TURBULENT MICRO-STRUCTURE

We shall consider a system of equations governing the flow of a viscous, heat-conducting compressible fluid,

$$\frac{\partial \rho}{\partial t} + (\bar{u}\cdot\nabla)\rho + \rho\nabla\cdot\bar{u} = 0,$$

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} + \frac{RT}{\rho}\nabla\rho + R\nabla T - \frac{\mu_1}{\rho}[\nabla\cdot\nabla\bar{u} + (\tilde{V}-1)\nabla\nabla\cdot\bar{u}] = 0, \qquad (4.1)$$

$$\frac{\partial T}{\partial t} + (\bar{u}\cdot\nabla)T + (\gamma-1)T\nabla\cdot\bar{u} - \frac{\gamma\mu_1}{\rho Pr}\nabla\cdot\nabla T - \frac{\mu_1}{\rho}\phi = 0,$$

where ρ = mass density, $\bar{u}$ = velocity vector, T = temperature, ϕ = dissipation function (rate of dissipation of mechanical energy), γ = ratio of specific heats = c_p/c_v, c_p = specific heat at constant pressure, c_v = specific heat at constant volume, Pr = Prandtl number = $c_p\mu_1/\kappa$, κ = heat conductivity, $\tilde{V}$ = viscosity number = $2 + \mu_2/\mu_1$, μ_1 and μ_2 = first and second coefficients of viscosity (then the bulk viscosity is $\frac{2}{3}\mu_1 + \mu_2 = \mu_1(\tilde{V} - \frac{4}{3}) \geq 0)$, R = gas constant = $c_p - c_v$.

We shall first separate the flow variables into the mean and fluctuating components,

$$\rho = \langle\rho\rangle + \rho', \qquad \bar{u} = \langle\bar{u}\rangle + \bar{u}', \qquad T = \langle T\rangle + T'. \tag{4.2}$$

The definition of the averaging process will be made specific later. At this point it suffices to say that the averages of the fluctuations vanish by definition, $\langle\rho'\rangle = \langle\bar{u}'\rangle = \langle T'\rangle = 0$. Substitution of the sums (4.2) into Eqs. (4.1) gives

$$\{\frac{\partial\langle\rho\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle\rho\rangle + \langle\rho\rangle\nabla\cdot\langle\bar{u}\rangle\} + \{\frac{\partial\rho}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\rho' + \langle\rho\rangle\nabla\cdot\bar{u}'\}$$

$$+ \{\bar{u}'\cdot\nabla\langle\rho\rangle + \rho'\nabla\cdot\langle\bar{u}\rangle\} + \{\bar{u}'\cdot\nabla\rho' + \rho'\nabla\cdot\bar{u}'\} = 0 \tag{4.3}$$

$$\{\frac{\partial\langle\bar{u}\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle\bar{u}\rangle + R\nabla\langle T\rangle + R\,\frac{\langle T\rangle}{\langle\rho\rangle}\,\nabla\langle\rho\rangle$$

$$- \frac{\mu_1}{\langle\rho\rangle}\,[\nabla\cdot\nabla\langle\bar{u}\rangle + (\tilde{V}-1)\nabla\nabla\cdot\langle\bar{u}\rangle]\}$$

$$+ \{\frac{\partial\bar{u}'}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\bar{u}' + R\nabla T' + R\frac{\langle T\rangle}{\langle\rho\rangle}\,\nabla\rho' - \frac{\mu_1}{\langle\rho\rangle}\,[\nabla\cdot\nabla\bar{u}' + (\tilde{V}-1)\nabla\nabla\cdot\bar{u}']\}$$

$$+ \{\bar{u}'\cdot\nabla\langle\bar{u}\rangle + \frac{RT'}{\langle\rho\rangle}\,\nabla\langle\rho\rangle + \frac{\rho'}{\langle\rho\rangle}\,[\frac{\partial\langle\bar{u}\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle\bar{u}\rangle + R\nabla\langle T\rangle]\}$$

$$+ \{\bar{u}'\cdot\nabla\bar{u}' + \frac{RT'}{\langle\rho\rangle}\,\nabla\rho' + \frac{\rho'}{\langle\rho\rangle}[\frac{\partial\bar{u}}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\bar{u}' + R\nabla T' + \bar{u}'\cdot\nabla\langle\bar{u}\rangle]\}$$

$$+ \{\frac{\rho'}{\langle\rho\rangle}\,\bar{u}'\cdot\nabla\bar{u}'\} = 0 \tag{4.4}$$

$$\{\frac{\partial\langle T\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle T\rangle + (\gamma-1)\langle T\rangle\nabla\cdot\langle\bar{u}\rangle - \frac{\gamma\mu_1}{\langle\rho\rangle Pr}\nabla\cdot\nabla\langle T\rangle - \frac{2\mu_1}{\langle\rho\rangle}\langle(e_{ij} - \frac{1}{3}\Delta\delta_{ij})^2\rangle\}$$

$$+ \{\frac{\partial T'}{\partial t} + \langle\bar{u}\rangle\cdot\nabla T' + (\gamma-1)\langle T\rangle\nabla\cdot\bar{u}' - \frac{\gamma\mu_1}{\langle\rho\rangle Pr}\nabla\cdot\nabla T'\} + \{\bar{u}'\cdot\nabla\langle T\rangle + (\gamma-1)T'\nabla\cdot\langle\bar{u}\rangle$$

$$+ \frac{\rho'}{\langle\rho\rangle}[\frac{\partial\langle T\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle T\rangle + (\gamma-1)\langle T\rangle\nabla\cdot\langle\bar{u}\rangle] - \frac{4\mu_1}{\langle\rho\rangle}\langle e_{ij} - \frac{1}{3}\Delta\delta_{ij}\rangle(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})\}$$

$$+ \{\bar{u}'\cdot\nabla T' + (\gamma-1)T'\nabla\cdot\bar{u}' + \frac{\rho'}{\langle\rho\rangle}[\frac{\partial T'}{\partial t} + \langle\bar{u}\rangle\cdot\nabla T' + (\gamma-1)\langle T\rangle\nabla\cdot\bar{u}' + \bar{u}'\cdot\nabla\langle T\rangle$$

$$+ (\gamma-1)T'\nabla\cdot\langle\bar{u}\rangle] + \frac{2\mu_1}{\langle\rho\rangle}(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})^2\}$$

$$+ \frac{\rho'}{\langle\rho\rangle}\{\bar{u}'\cdot\nabla T' + (\gamma-1)T'\nabla\cdot\bar{u}' + 2\mu_1(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})^2\} \tag{4.5}$$

where $\phi = 2(e_{ij} - \frac{1}{3}\Delta\delta_{ij})^2$, $e_{ij} = \frac{1}{2}\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\}$, $\Delta = \nabla\cdot\bar{u} = \frac{\partial u_i}{\partial x_i}$, $i,j=1,2,3.$

When this system is averaged, we obtain the Reynolds equations for the mean (averaged) fluid flow:

$$\frac{\partial\langle\rho\rangle}{\partial t} + \nabla\cdot(\langle\rho\rangle\langle\bar{u}\rangle) + \langle\nabla\cdot(\rho'\bar{u}')\rangle = 0 \tag{4.6}$$

$$\frac{\partial\langle\bar{u}\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle\bar{u}\rangle + R\nabla\langle T\rangle + R\frac{\langle T\rangle}{\langle\rho\rangle}\nabla\langle\rho\rangle - \nu[\nabla^2\langle\bar{u}\rangle + (\tilde{V}-1)\nabla\nabla\cdot\langle\bar{u}\rangle]$$

$$+ \frac{1}{\langle\rho\rangle}[\langle\rho'\frac{\partial\bar{u}'}{\partial t}\rangle + \langle\bar{u}\rangle\cdot\rangle\rho'\nabla\bar{u}'\rangle + R\langle\rho'\nabla T'\rangle + \langle\rho'\bar{u}'\rangle\cdot\nabla\langle\bar{u}\rangle]$$

$$+ \langle\bar{u}'\cdot\nabla\bar{u}'\rangle + R\frac{\langle T'\nabla\rho'\rangle}{\langle\rho\rangle} + \frac{1}{\langle\rho\rangle}\langle\rho'\bar{u}'\cdot\nabla\bar{u}'\rangle = 0 \tag{4.7}$$

$$\frac{\partial\langle T\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle T\rangle + (\gamma-1)\langle T\rangle\nabla\cdot\langle\bar{u}\rangle - \frac{\gamma\nu}{Pr}\nabla^2\langle T\rangle - \frac{2\nu}{\langle\rho\rangle}\langle(e_{ij} - \frac{1}{3}\Delta\delta_{ij})^2\rangle$$

$$+ \frac{1}{\langle\rho\rangle}\{\langle\rho'\frac{\partial T'}{\partial t}\rangle + \langle\bar{u}\rangle\cdot\langle\rho'\nabla T'\rangle + (\gamma-1)\langle T\rangle\langle\rho'\nabla\cdot\bar{u}'\rangle + \langle\rho'\bar{u}'\rangle\cdot\nabla\langle T\rangle$$

$$+ (\gamma-1)\langle\rho'T'\rangle\nabla\cdot\langle\bar{u}\rangle - 2\mu_1\langle(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})^2\rangle\} + \langle\bar{u}'\cdot\nabla T'\rangle$$

$$+ (\gamma-1)\langle T'\nabla\cdot\bar{u}'\rangle + \frac{1}{\langle\rho\rangle}[\langle\rho'\bar{u}'\cdot\nabla T'\rangle + (\gamma-1)\langle\rho'T'\nabla\cdot\bar{u}'\rangle] = 0 \qquad (4.8)$$

where $\nu = \mu_1/\langle\rho\rangle$ = kinematic viscosity of the mean flow.

When Eqs. (4.6) - (4.8) are subtracted from (4.3)-(4.5), we have the system for the determination of the fluctuations:

$$\{\frac{\partial\rho'}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\rho' + \langle\rho\rangle\nabla\cdot\bar{u}'\} + \{\bar{u}'\cdot\nabla\langle\rho\rangle + \rho'\nabla\cdot\langle\bar{u}\rangle\}$$

$$+ \{\bar{u}'\cdot\nabla\rho' - \langle\bar{u}'\cdot\nabla\rho'\rangle + \rho'\nabla\cdot\bar{u}' - \langle\rho'\nabla\cdot\bar{u}'\rangle\} = 0 \qquad (4.9)$$

$$\{\frac{\partial\bar{u}'}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\bar{u}' + R\nabla T' + R\frac{\langle T\rangle}{\langle\rho\rangle}\nabla\rho' - \nu[\nabla^2\bar{u}' + (\tilde{V}-1)\nabla\nabla\cdot\bar{u}']\}$$

$$+ \{\bar{u}'\cdot\nabla\langle\bar{u}\rangle + \frac{RT'}{\langle\rho\rangle}\nabla\langle\rho\rangle + \frac{\rho'}{\langle\rho\rangle}[\frac{\partial\langle\bar{u}\rangle}{\partial t} + \langle\bar{u}\rangle\cdot\nabla\langle\bar{u}\rangle + R\nabla\langle T\rangle]\}$$

$$+ \{\bar{u}'\cdot\nabla\bar{u}' - \langle\bar{u}'\cdot\nabla\bar{u}'\rangle + \frac{RT'}{\langle\rho\rangle}\nabla\rho' - R\frac{\langle T'\nabla\rho'\rangle}{\langle\rho\rangle} + \frac{1}{\langle\rho\rangle}(\rho'\frac{\partial\bar{u}'}{\partial t} - \langle\rho'\frac{\partial\bar{u}'}{\partial t}\rangle)$$

$$+ \frac{\langle\bar{u}\rangle}{\langle\rho\rangle}\cdot(\rho'\nabla\bar{u}' - \langle\rho'\nabla\bar{u}'\rangle) + \frac{R}{\langle\rho\rangle}(\rho'\nabla T' - \langle\rho'\nabla T'\rangle)$$

$$+ (\rho'\bar{u}' - \langle\rho'\bar{u}\rangle\cdot\frac{\nabla\langle\bar{u}\rangle}{\langle\rho\rangle}\} + \frac{1}{\langle\rho\rangle}\{\rho'\bar{u}'\cdot\nabla\bar{u}' - \langle\rho'\bar{u}'\cdot\nabla\bar{u}'\rangle\} = 0 \qquad (4.10)$$

$$\{\frac{\partial T'}{\partial t} + <\bar{u}>\cdot\nabla T' + (\gamma-1)<T>\nabla\cdot\bar{u}' - \frac{\gamma\nu}{Pr}\nabla\cdot\nabla T'\} + \{\bar{u}'\cdot\nabla<T> + (\gamma-1)T'\nabla\cdot<\bar{u}>$$

$$+ \frac{\rho'}{<\rho>}[\frac{\partial<T>}{\partial t} + <\bar{u}>\cdot\nabla<T> + (\gamma-1)<T>\nabla\cdot<\bar{u}>] - 4\nu<e_{ij} - \frac{1}{3}\Delta\delta_{ij}>(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})\}$$

$$+ \{\bar{u}'\cdot\nabla T' - <\bar{u}'\cdot\nabla T'> + (\gamma-1)(T'\nabla\cdot\bar{u}' - <T'\nabla\cdot\bar{u}'>)$$

$$+ \frac{1}{<\rho>}[\rho'\frac{\partial T'}{\partial t} - <\rho'\frac{\partial T'}{\partial t}> + <\bar{u}>\cdot(\rho'\nabla T' - <\rho'\nabla T'>) + (\gamma-1)<T>(\rho'\nabla\cdot\bar{u}'$$

$$- <\rho'\nabla\cdot\bar{u}'>) + (\rho'\bar{u}' - <\rho'\bar{u}'>)\cdot\nabla<T> + (\gamma-1)\nabla\cdot<\bar{u}>(\rho'T' - <\rho'T'>)$$

$$+ 2\mu_1(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})^2 - 2\mu_1<(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})^2>]\}$$

$$+ \frac{1}{<\rho>}\{(\rho'\bar{u}'\cdot\nabla T' - <\rho'\bar{u}'\cdot\nabla T'>) + (\gamma-1)(\rho'T'\nabla\cdot\bar{u}' - <\rho'T'\nabla\cdot\bar{u}'>)$$

$$+ 2\mu_1[\rho'(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})^2 - <\rho'(e'_{ij} - \frac{1}{3}\Delta'\delta_{ij})^2>]\} = 0 . \qquad (4.11)$$

The curly brackets are used to collect terms in the following order: first, terms containing only linear terms in the fluctuations that are independent of the derivatives of the mean flow, then products of the fluctuations and the derivatives of the mean flow, terms bilinear in the fluctuations, and lastly the terms trilinear in the fluctuations.

Equations (4.9)-(4.11) will now be written as a linear system in the fluctuations subject to three kinds of forcing functions (sources), those that are linear in the fluctuations and vanish when all derivatives of the mean flow vanish (representing first order interaction with the mean flow), and those that are bilinear and trilinear in the fluctuations (representing three-wave and four-wave interactions, respectively). This amounts to taking all but the first curly brackets to the right-hand-side:

$$\frac{\partial \rho'}{\partial t} + \langle \bar{u} \rangle \cdot \nabla \rho' + \langle \rho \rangle \nabla \cdot \bar{u}' = (\dot{m}_1 + \dot{m}_2)\langle \rho \rangle \tag{4.12}$$

$$\frac{\partial \bar{u}'}{\partial t} + \langle \bar{u} \rangle \cdot \nabla \bar{u}' + R\nabla T' + R\frac{\langle T \rangle}{\langle \rho \rangle} \nabla \rho' - \nu[\nabla^2 \bar{u}' + (\tilde{V}-1)\nabla\nabla \cdot \bar{u}']$$

$$= (\bar{f}_1 + \bar{f}_2 + \bar{f}_3)c \tag{4.13}$$

$$\frac{\partial T'}{\partial t} + \langle \bar{u} \rangle \cdot \nabla T' + (\gamma-1)\langle T \rangle \nabla \cdot \bar{u}' - \frac{\gamma\nu}{Pr} \nabla^2 T' = (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)\langle T \rangle \sqrt{\gamma - 1} \tag{4.14}$$

where $c = (RT)^{\frac{1}{2}}$ = isothermal speed of sound. The symbols $\dot{m}_i$, $\bar{f}_i$, and $\dot{q}_i$ denote mass-like, force-like, and heat-like sources of i-th order in the fluctuations. The various sources as defined above have units of frequency, sec^{-1}. The expressions for the sources may be obtained by referring to Eqs. (4.9)-(4.11), for instance, for the mass-like sources we have from Eq. (4.9):

$$\dot{m}_1 = - [\bar{u}' \cdot \nabla \langle \rho \rangle + \rho' \nabla \cdot \langle \bar{u} \rangle]/\langle \rho \rangle$$

$$\dot{m}_2 = [\langle \bar{u}' \cdot \nabla \rho' \rangle - \bar{u}' \cdot \nabla \rho' + \langle \rho' \nabla \cdot \bar{u}' \rangle - \rho' \nabla \cdot \bar{u}']/\langle \rho \rangle$$

$$\dot{m}_3 = 0 .$$

The form of the system (4.12)-(4.14) is rather arbitrary inasmuch as the primitive physical variables ρ, $\bar{u}$, T could be replaced by any other set of unknowns, e.g., ρ, $\rho\bar{u}$, p, where $p = \rho RT$, etc. With the wave representation in mind, we observe that, in general, any initial disturbance in any one or all of the primitive variables will be split into several waves, each wave carrying perturbations in the several of the primitive variables. Thus the amplitudes of the waves, and the ratios of the amplitudes of the perturbations in the primitive variables carried by a given wave, become natural variables in the wave representation, and not the primitive variables themselves.

In order to simplify the notation, we shall use from now on the following:

$$\langle\rho\rangle = \bar{\rho}, \qquad \langle T\rangle = \bar{T}, \qquad \langle\bar{u}\rangle = \bar{U}, \qquad \bar{u}' = \bar{u} = \{u, v, w\},$$

and, dropping the primes on the fluctuations, we shall introduce a non-dimensionalized vector of the unknown fluctuations,

$$u_j = \{\frac{\rho'}{\bar{\rho}}, \frac{u'}{c}, \frac{v'}{c}, \frac{w'}{c}, \frac{T'}{\bar{T}\sqrt{\gamma-1}}\} = \{\rho, u, v, w, T\}, \; j=1,2,\ldots 5$$

This choice of the nondimensionalization symmetrizes the system (4.12)-(4.14) if the local reference quantities, $\bar{\rho}$, c, $\bar{T}\sqrt{\gamma-1}$, are regarded as constant and taken inside the derivatives, e.g., $\frac{1}{\bar{\rho}}\frac{\partial\rho'}{\partial t} = \frac{\partial}{\partial t}(\frac{\rho'}{\bar{\rho}}) = \frac{\partial\rho}{\partial t}$. This will facilitate the treatment of the linear part of the system of equations under the assumption that the flow is steady and homogeneous and the reference quantities are constant.

The symmetrized system becomes:

$$\frac{D\rho}{Dt} + c\nabla\cdot\bar{u} = \dot{m}_1 + \dot{m}_2$$

$$\frac{D\bar{u}}{Dt} + c\sqrt{\gamma-1}\nabla T + c\nabla\rho - \nu\{\nabla^2\bar{u} + \frac{1}{3}\nabla\nabla\cdot\bar{u}\} = \bar{f}_1 + \bar{f}_2 + \bar{f}_3$$

$$\frac{DT}{Dt} + c\sqrt{\gamma-1}\nabla\cdot\bar{u} - \frac{\gamma\nu}{Pr}\nabla^2 T = \dot{q}_1 + \dot{q}_2 + \dot{q}_3$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{U}\cdot\nabla$ is the substantial derivative. Symbolically, we may write

$$L_{ij}(u_j) = I_i(u_j) + B_i\ (u_j u_k) + T_i\ (u_j u_k u_m)$$

$$= \{\dot{m}_1, \bar{f}_1, \dot{q}_1\} + \{\dot{m}_2, f_2, \dot{q}_2\} + \{0,\ \bar{f}_3, \dot{q}_3\} \qquad (4.15)$$

where $L_{ij}(u_j)$ is a linear differential operator represented by a symmetric matrix, and I_i, B_i, T_i are vectors linear, bilinear, and trilinear in the fluctuations u_j, respectively.

We now expand the vector of the fluctuations u_j in terms of complex exponentials:

$$u_j = \sum_{\alpha=1}^{\alpha=5} \int \phi_\alpha P_{\alpha j} \exp\{-\Gamma_\alpha t + i(\bar{x}\cdot\bar{k} - \omega_\alpha t)\} d\bar{k} \qquad (4.16)$$

where the integration with respect to the volume in the wavenumber space ($d\bar{k} = dk_1 dk_2 dk_3$) is extended either to the cut-off $\bar{k} = \bar{k}_{max}$ or to infinity if $\phi_\alpha = 0$ for $|\bar{k}| = k > k_{max}$. Eq. (4.16) defines u_j as a Fourier transform, at $t = 0$, of $\phi_\alpha P_{\alpha j}$, $|P_{\alpha j}| = 1$. This is a local representation for, if ϕ_α and $P_{\alpha j}$ depend on space and time, then (4.16) can no longer be inverted by taking a Fourier transform. The subscript α denotes a contribution of the α-mode of wave propagation and $u_j = \sum_\alpha u_{\alpha j}$ is the sum of contributions to the fluctuations of the j-th quantity from all wave modes.

If ϕ_α, $P_{\alpha j}$, Γ_α, and ω_α are taken as local values (independent of $\bar{x}$ and t) then we may impose the condition that the form (4.16) solves the linear part of Eq. (4.15), that is,

$$L_{ij}(u_j) = 0 \tag{4.17}$$

which is linear and homogeneous in u_j. As the condition of the existence of non-trivial solutions of the linear system (4.17), see Kentzer (1974a,b), we obtain the characteristic equation

$$(\lambda + \nu k^2)^2[\lambda(\lambda + \frac{\gamma\nu k^2}{Pr})(\lambda + \tilde{V}\nu k^2) + a^2k^2(\lambda + \frac{\nu k^2}{Pr})] = 0$$

where $\lambda = -\Gamma + i(\bar{U}\cdot\bar{k} - \omega)$. Under the assumption that the viscosity number $\tilde{V}$, which must be no smaller than 4/3 in order that the bulk viscosity be non-negative, is equal to Pr^{-1}, that is $Pr = \tilde{V}^{-1} \leqq 3/4$, the above expression factors out into the product

$$(\lambda + \nu k^2)^2(\lambda + \nu k^2/Pr)[\lambda(\lambda + \gamma\nu k^2/Pr) + a^2k^2] = 0. \tag{4.18}$$

By analogy to the characteristic determinant of the theory of characteristics in the inviscid case, the first double root will be identified with the vorticity mode, the second linear factor with the entropy mode, and the quadratic factor gives rise to the acoustic mode of wave propagation.

Setting each factor in Eq. (4.18) separately equal to zero we impose the necessary and sufficient conditions for the existence of infinitesimal wave-type solutions for the locally steady and homogeneous mean flow. For these solutions we have

$$\begin{array}{llll} \alpha = 1,2 & \Gamma_\alpha = \nu k^2, & \omega_\alpha = \bar{U}\cdot\bar{k}, & \text{(vorticity modes)} \\ \alpha = 3 & \Gamma_\alpha = \frac{\nu k^2}{Pr}, & \omega_\alpha = \bar{U}\cdot\bar{k}, & \text{(entropy mode)} \\ \alpha = 4,5 & \Gamma_\alpha = \frac{\gamma\nu k^2}{2Pr} & \omega_\alpha = \bar{U}\cdot\bar{k} \pm ak(1-K^2)^{\frac{1}{2}} & \text{(acoustic modes)} \end{array} \tag{4.19}$$

where $K = \gamma\nu k/(2aPr)$ = Knudsen number based on the mean wavelength, $2\pi/k$, and $a = (\gamma RT)^{\frac{1}{2}}$ = adiabatic speed of sound. We observe here that all modes have the same frequency $\bar{U}\cdot\bar{k}$ when $K = 1$. This condition corresponds to $k = 2aPr/(\gamma\nu) \simeq 10^5\ cm^{-1}$, or to wavelengths approaching the molecular mean free path. This value of k will be taken as the cut-off value, k_{max}, beyond which the continuum fluid mechanics does not apply and any results predicted by the Navier-Stokes theory will loose their physical meaning.

The vorticity and entropy modes may be interpreted as standing waves convected by the mean motion, their frequencies having the standard form for a Doppler shift, $\bar{U}\cdot\bar{k}$. The acoustic modes may be viewed as plane waves traveling through a moving medium at a reduced speed of sound, $a^* = a(1-K^2)^{\frac{1}{2}}$, in the directions $+\bar{k}$ and $-\bar{k}$. This also implies that an acoustic wave is a quadratic surface, a convected expanding sphere.

Using relations (4.19), one may return to the linear system (4.17) to determine the eigenvectors $\phi_\alpha P_{\alpha j}$. This is possible only if $K < 1$. Due to homogeneity of (4.17), the components of the eigenvectors may be determined only up to a common factor. We may find, therefore, the participation coefficients, $P_{\alpha j}$ under the condition that they be of unit magnitude,

$$\sum_j P_{\alpha j} P^*_{\alpha j} = 1$$

where the asterisk denotes a complex conjugate.

Further, since the $P_{\alpha j}$ correspond to distinct eigenvalues of the system (4.17), they are also orthogonal, that is,

$$\sum_j P_{\alpha j} P_{\beta j} = 0 \text{ for } \alpha \neq \beta. \tag{4.20}$$

The vectors $P_{\alpha j}$ are given by Kentzer (1974a,b) in the form

$$P_{1j} = \{0,\ k_2k_3,\ -k_1k_3,\ 0,\ 0\}/\{k_3^2(k_1^2+k_2^2)\}^{\frac{1}{2}}$$

$$P_{2j} = \{0,-k_1k_3,-k_2k_3,\ k_1^2+k_2^2,\ 0\}/\{k^2(k_1^2+k_2^2)\}^{\frac{1}{2}}$$

$$P_{3j} = \{1,-2ik_mK/(k\sqrt{\gamma}),-1/\sqrt{\gamma-1}\}/\{\frac{\gamma}{\gamma-1} + \frac{4K^2}{\gamma}\}^{\frac{1}{2}} \tag{4.21}$$

$$P_{4j} = \{-g,-i\gamma ck_m,\ g^*\sqrt{\gamma-1}\}/\{2\gamma a^2k^2\}^{\frac{1}{2}}$$

$$P_{5j} = \{-g^*,-i\gamma ck_m,\ g\sqrt{\gamma-1}\}/\{2\gamma a^2k^2\}^{\frac{1}{2}}$$

where $m = 1,2,3$, $g = -ak\{K+i(1-K^2)^{\frac{1}{2}}\}$, $k = |\bar{k}|$.

The vectors P_{1j} and P_{2j} are unit polarization vectors perpendicular to the wave vector $\bar{k}$. This indicates that the vorticity mode is represented by a velocity field in the plane of the wave, or that the wave is a transverse one. A more general form, with an arbitrary constant vector $\bar{n}$, is

$$P_{1j} = \frac{\bar{n}\times\bar{k}}{|\bar{n}\times\bar{k}|}, \qquad P_{2j} = \frac{\bar{k}\times(\bar{n}\times\bar{k})}{|\bar{k}\times(\bar{n}\times\bar{k})|} .$$

If for practical purposes and ease of calculations P_{3j}, P_{4j}, and P_{5j} are approximated by their inviscid limits:

$$P_{3j} = \{1,\ 0,\ 0,\ 0,-1/\sqrt{\gamma-1}\}/\{\gamma/(\gamma-1)\}^{\frac{1}{2}}$$

$$P_{4j} = \{ak,\ -\gamma ck_m,\ ak\sqrt{\gamma-1}\}/\{2\gamma a^2k^2\}^{\frac{1}{2}} \tag{4.22}$$

$$P_{5j} = \{ak,\ +\gamma ck_m,\ ak\sqrt{\gamma-1}\}/\{2\gamma a^2k^2\}^{\frac{1}{2}}$$

then all $P_{\alpha j}$ are real and orthonormal, $\sum_j P_{\alpha j}P_{\beta j} = \delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{if } \alpha\neq\beta \end{cases}$

We may now observe that in the expansion (4.16) the amplitudes ϕ_α are the Fourier coefficients of u_j with respect to the state vectors

$$P_{\alpha j}\exp\{-\Gamma_\alpha t + i(\bar{x}\cdot\bar{k} - \omega_\alpha t)\} \tag{4.23}$$

as basis. The five-fold infinite system of state vectors (4.23) is complete and may be used to represent any piecewise continuous, square-integrable function of $\bar{x},t$.

Since, in a general non-stationary inhomogeneous case, ϕ_α, Γ_α, and ω_α are functions of $\bar{k}$ and of $\bar{x}$ and t, the expansion (4.16), while still valid, is not a Fourier transform and is not a solution of Eq. (4.15), we generalize the results and proceed as follows. We first write the integrand in the expression (4.16) as

$$\Phi_\alpha(\bar{x},\bar{k},t)\phi^o_\alpha(\bar{k})P_{\alpha j}(k)e^{-\Gamma^o_\alpha t+i(\bar{x}\cdot\bar{k}-\omega^o_\alpha t)} \tag{4.24}$$

where ϕ^o_α, Γ^o_α, and ω^o_α are local values, functions of $\bar{k}$ only, and

$$\Phi_\alpha = \frac{\phi_\alpha(\bar{x},\bar{k},t)}{\phi^o_\alpha(\bar{k})}\exp\{-(\Gamma_\alpha-\Gamma^o_\alpha)t-i(\omega_\alpha-\omega^o_\alpha)t\}.$$

The space-time dependence of ϕ_α, Γ_α, and ω_α is implicit through their dependence on the functions of the mean flow. We try to avoid introducing functional differentiation as done, e.g., by Hopf (1952) and avoid replacing the relation (4.16) by a functional analog of a Fourier transform. It should be observed that, treating the integrand (4.24) above as a product, the linear operator L_{ij} operating on u_j will give a zero contribution when operating on the state vectors (4.23) while keeping

Φ_α constant, and a non-zero contribution will come only from L_{ij} operating on Φ_α while keeping the state vectors constant. But this contribution must equal the source terms on the right-hand-side of Eq. (4.15). Let

$$L_{ij}(\Phi_\alpha) = - i\omega^*_\alpha$$

where the random function $\omega^*_\alpha = \omega^*_\alpha(\bar{x},\bar{k},t)$ is, in general, complex. Then

$$L_{ij}(u_j) = -i \sum_{\alpha=1}^{\alpha=5} \int \omega^*_\alpha \phi^o_\alpha P_{\alpha j} e^{-\Gamma^o_\alpha t + i(\bar{x}\cdot\bar{k} - \omega^o_\alpha t)} d\bar{k}$$

$$= I_i + B_i + T_i \ .$$

The new variable ω^*_α is intended to account indirectly for the inhomogeneity and unsteadiness of the mean flow (for the "slow" variation of ϕ_α, Γ_α, and ω_α with $\bar{x}$ and t) while, at the same time, ω^*_α is to be chosen so as to satisfy Eq. (4.15). This means that the "slow" changes in the amplitude ϕ_α and in the parameters Γ_α and ω_α are due to the source terms only. For convenience we shall modify the state vectors (4.23) by adding ω^*_α to the frequency ω^o_α,

$$P_{\alpha j}\exp\{-\Gamma^o_\alpha t + i[\bar{x}\cdot\bar{k} - (\omega^o_\alpha+\omega^*_\alpha)t]\} \tag{4.25}$$

and, instead of Eq. (4.15), we obtain

$$-i \sum_{\alpha=1}^{\alpha=5} \int \omega^*_\alpha \phi^o_\alpha P_{\alpha j} e^{-\Gamma^o_\alpha t + i[\bar{x}\cdot\bar{k} - (\omega^o_\alpha+\omega^*_\alpha)t]} d\bar{k} = I_j + B_j + T_j \tag{4.26}$$

because the time-derivatives of u_j appear only along the diagonal of the symmetric operator L_{ij}.

Omitting the superscript $(\ )^{o}$, we multiply both sides of Eq. (4.26) by

$$\phi^*_\beta(\bar{k}')P^*_{\beta j}(\bar{k}')e^{-\Gamma_\beta(\bar{k}')t-i[\bar{x}\cdot\bar{k}'-\omega_\beta(\bar{k}')t-\omega^*_\beta(\bar{k}')t]}$$

and we sum over j and integrate with respect to $\bar{k}'$. The result, under the condition that $\phi_\alpha(0)$, $\phi_\beta(0) = 0$, is

$$-i\int\omega^*_\beta|\phi_\beta|^2 d\bar{k}$$

$$= \sum_j\int\phi^*_\beta(\bar{k})P^*_{\beta j}(\bar{k})(\mathcal{I}_j+\mathcal{B}_j+\mathcal{T}_j)e^{-\Gamma_\beta t-i(\bar{x}\cdot\bar{k}-\omega_\beta t-\ddot{\omega}^*_\beta t)}d\bar{k} .$$

If we define the "local" mean value of any function of $\bar{k}$ by

$$\langle F(\bar{k})\rangle = \int F(\bar{k})|\phi_\beta(\bar{k})|^2 d\bar{k}/\int|\phi_\beta(\bar{k})|^2 d\bar{k},$$

then

$$\langle\omega^*_\beta\rangle = i\frac{\sum_j\int \phi^*_\beta P^*_{\beta j}(\mathcal{I}_j+\mathcal{B}_j+\mathcal{T}_j)\exp\{-\Gamma_\beta t-i[\bar{x}\cdot\bar{k}-\omega_\beta t-\omega^*_\beta t]\}d\bar{k}}{\int|\phi_\beta(\bar{k})|^2 d\bar{k}} . \qquad (4.27)$$

Equation (4.27) is an integral equation for ω^*_β because ω^*_β appears on both sides of the equation. In particular, ω^*_β appears in the arguments of the exponentials and in $\mathcal{B}_j$ (the bilinear interaction terms $\bar{f}_2$ and $\dot{q}_2$ contain the time derivatives of $\bar{u}$ and T). Of greater importance is the influence of ω^*_β on the resonance conditions. We note that a substitution of the expansion for the perturbations into $\mathcal{I}_j$, $\mathcal{B}_j$, and $\mathcal{T}_j$ will result in the appearance in the numerator of Eq. (4.27) of terms of the general form

$$\iint \phi_\alpha(\bar{k}')\phi^*_\beta(\bar{k})P_{\alpha i}(\bar{k}')P^*_{\beta j}(\bar{k})\delta(\bar{k}'-\bar{k})\times$$

$$\times\delta[\omega_\alpha(\bar{k}')+\omega^*_\alpha(\bar{k}')-\omega_\beta(\bar{k})-\omega^*_\beta(\bar{k})]d\bar{k}d\bar{k}',$$

$$\iiint \phi_\alpha(\bar{k}')\phi^*_\beta(\bar{k})\phi_\gamma(\bar{k}'')P_{\alpha i}(\bar{k}')P^*_{\beta j}(\bar{k})P_{\gamma k}(\bar{k}'')\delta(\bar{k}'+\bar{k}''-\bar{k})\times$$

$$\times\delta[\omega_\alpha(\bar{k}')+\omega^*_\alpha(\bar{k}')+\omega_\gamma(\bar{k}'')+\omega^*_\gamma(\bar{k}'')-\omega_\beta(\bar{k})-\omega^*_\beta(\bar{k})]d\bar{k}d\bar{k}'d\bar{k}'',$$

$$\iiiint \phi_\alpha(\bar{k}')\phi^*_\beta(\bar{k})\phi_\gamma(\bar{k}'')\phi_\delta(\bar{k}''')P_{\alpha i}(\bar{k}')P^*_{\beta j}(\bar{k})P_{\gamma k}(\bar{k}'')P_{\delta m}(\bar{k}''')\delta(\bar{k}'+\bar{k}''+\bar{k}'''-\bar{k})\times$$

$$\times\delta[\omega_\alpha(\bar{k}')+\omega^*_\alpha(\bar{k}')+\omega_\gamma(\bar{k}'')+\omega^*_\gamma(\bar{k}'')+\omega_\delta(\bar{k}''')+\omega^*_\delta(\bar{k}''')-\omega_\beta(\bar{k})-\omega^*_\beta(\bar{k})]d\bar{k}d\bar{k}'d\bar{k}''d\bar{k}''',$$

corresponding to, respectively, the linear, bilinear, and trilinear wave interaction terms.

Setting the arguments of the Dirac delta functions separately equal to zero gives the two-, three-, and four-wave resonance conditions which would have to be studied in great detail for a given choice of the wavenumber dependence of $\omega^*(\bar{k})$. For the purpose of a brief discussion we shall assume that a cross-mode coupling by resonance is negligible. In that case, we observe that if ω^* is not a function of $\bar{k}$ then we have a contribution only from I_j. If $\omega^* = \text{const.}\times\bar{k}$, then ω^* does not affect the resonance conditions. Of special interest to us will be the case $\omega^* = \text{const.}\times k^2$ for which there will be contributions from I_j for all $\bar{k}$, contributions from B_i only from those wavenumber vector triangles $\bar{k}=\bar{k}'+\bar{k}''$ for which $k^2=(k')^2+(k'')^2$ (right triangles only), and contributions from T_j for sets of three mutually perpendicular wave vectors $\bar{k}'+\bar{k}''+\bar{k}'''=\bar{k}$ for only then $(k')^2+(k'')^2+(k''')^2= k^2$.

For the vorticity and entropy modes $\omega_\alpha=\bar{U}\cdot\bar{k}$ and the resonance conditions are independent of the Doppler term and depend only on ω^*_α.

For the acoustic modes, besides the Doppler term $\bar{U}\cdot\bar{k}$, we have the intrinsic frequency, $\pm ak(1-K^2)^{\frac{1}{2}} \simeq \pm ak$. Acoustic waves with the like sign (waves of mode $\alpha=4$ or $\alpha=5$) would contribute only if all the vectors $\bar{k}$, $\bar{k}'$, $\bar{k}''$, $\bar{k}'''$ are parallel. Only acoustic waves of opposite sign (e.g. $\alpha=4$ interacting with $\alpha=5$) could resonate. The addition of the modifying term ω^*_α, and presence of vorticity and entropy waves, enlarge the resonance possibilities. This observation is important in calculating sound generation in turbulence. Only numerical calculations would assess the effects of neglecting ω^*_α in the interaction terms.

Similar considerations arise in the calculations of the various mean (average or expectation values) terms that enter into the Reynolds equations (4.6)-(4.8). The number of terms that need evaluation is very large due to double and triple summation over the five modes. In modeling of turbulence it will be necessary to reduce the number of such terms. It is believed, for instance, that the effect of the acoustic wave field on the vorticity mode is negligible. Similarly, at low Mach numbers, entropy mode will have negligible effect on vorticity. However, the acoustic field will be determined primarily by the vorticity mode and, to a lesser extent, by the entropy mode.

In conclusion we should state that the partial differential equations for the mean flow should be solved simultaneously with the equations for the fluctuations. The latter are disposed off by introduction of the random variable ω^*, and the problem of the fluctuations reduces to the solution for the amplitude function $\phi(\bar{x},\bar{k},\tilde{t})$. The square of its magnitude plays the rôle of the probability distribution function in the wavenumber space, which will be determined in terms of the characteristic function, a probability density in the physical space.

Finally, we note that Eq. (4.26) may be interpreted as

$$\sum_{\alpha} \langle L_{ij}[\ln\phi_{\alpha}]P_{\alpha j}\rangle = I_j + B_j + T_j$$

where $\langle L_{ij}[\ln\phi_{\alpha}]P_{\alpha j}\rangle = \int \phi_{\alpha}^{-1} L_{ij}(\phi_{\alpha})P_{\alpha j}\phi_{\alpha} e^{-i\Gamma_{\alpha}t+i(\bar{x}\,\bar{k}-\omega_{\alpha}t)}\,d\bar{k}$

is the average, with respect to the distribution ϕ_{α}, of the differential operator $\phi_{\alpha}^{-1}L_{ij}(\phi_{\alpha})P_{\alpha j}$. All attempts at separating or factoring out this operator failed. It was then decided to use a heuristic approach, to derive a system of nonlinear partial differential equations for the characteristic functions

$$\psi_{\alpha}(\bar{x},t) = \int \phi_{\alpha} e^{-\Gamma_{\alpha}t+i(\bar{x}\,\bar{k}-\omega_{\alpha}t)}\,d\bar{k}$$

under the assumption that ϕ_{α} and ω_{α} are independent of $\bar{x}$ and t, and then postulate that such a system, suitably modified, is valid also when the dependence on $\bar{x}$ and t is allowed for. The next chapters develop this approach.

> *"We have done considerable mountain climbing. Now we are in the rarefied atmosphere of theories of excessive beauty and we are nearing a high plateau on which geometry, optics, mechanics, and wave mechanics meet on common ground. Only concentrated thinking, and a considerable amount of re-creation, will reveal the full beauty of our subject in which the last word has not yet been spoken."*
>
> [Cornelius Lanczos]
>
> "The Variational Principles of Mechanics," p. 228, University of Toronto Press, Toronto, 1949.

V. WAVE-PARTICLE DUALITY

Wave Packets as Quasi-Particles

The complex exponentials, $e^{i(\bar{x}\cdot\bar{k}-\omega t)}$, (the phase factors) afford the following interpretation, see, e.g., Lighthill (1965), p. 15. Assume that the phase $s = \bar{x}\cdot\bar{k}-\omega t$ is differentiable at least twice. Then, regarding time as a fourth coordinate, x_4, and frequency as minus the fourth wavenumber component, $\omega = -k_4$, we write $s = x_m k_m$, m=1,2,3,4, and we have $k_m = \partial s/\partial x_m$. From the assumption of differentiability it follows that

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial^2 s}{\partial x_i \partial x_j} = \frac{\partial^2 s}{\partial x_j \partial x_i} = \frac{\partial k_i}{\partial x_j}\ , \qquad \oint k_i dx_i = 0\ ,$$

that is, the wavenumber vector is irrotational. If we define the frequency as a given function of $\bar{x}$, t, and $\bar{k}$ by

$$F(x_m,k_m) = \omega(x_i,k_i,t) + k_4 = 0, \qquad i = 1,2,3,$$

then

$$\sum_{m=1}^{4} \frac{\partial F}{\partial k_m} \cdot \frac{\partial k_m}{\partial x_n} + \frac{\partial F}{\partial x_n} = \sum_{m=1}^{4} \frac{\partial F}{\partial k_m} \cdot \frac{\partial k_n}{\partial x_m} + \frac{\partial F}{\partial x_n} = 0, \ n=1,\ldots,4.$$

If we now introduce a parameter τ such that

$$\frac{dx_m}{d\tau} = \frac{\partial F}{\partial k_m}, \tag{5.1}$$

then

$$\sum_{m=1}^{4} \frac{\partial k_n}{\partial x_m}\frac{dx_m}{d\tau} \quad \frac{dk_n}{d\tau} = -\frac{\partial F}{\partial x_n}. \tag{5.2}$$

With $m=4$, Eq. (5.1) gives $dt/d\tau = 1$, or $\tau = t+\text{const.}$, and we may write Eq. (5.1) and (5.2) as

$$\frac{dx_j}{dt} = \frac{\partial \omega}{\partial x_j}, \qquad \frac{dk_j}{dt} = -\frac{\partial \omega}{\partial x_j}, \qquad j = 1,2,3. \tag{5.3}$$

This is a Hamiltonian canonical form of the equations of motion of a wave packet having a wavenumber k_j and frequency ω and located at x_j at time t. This set of canonical equations forms the basis of geometrical mechanics. Lighthill (1965) observes that J. L. Synge pointed out a more general principle from which Eqs. (5.3) follow, namely, that the motion of a wave packet is such as to make the integral

$$\int(k_1dx_1+k_2dx_2+k_3dx_3+k_4dx_4) = \int(k_jdx_j-\omega dt)$$

stationary along the path in space-time between two fixed points.

The connection between geometrical mechanics and wave motion is a subject of an inspiring book by Synge (1954). To display this connection in the present case, we need only to introduce a conversion factor, h = energy x time = action, and write for the phase factor

$$e^{\frac{2\pi i}{h}[\bar{x}\cdot\frac{h\bar{k}}{2\pi} - \frac{h\omega}{2\pi}t]} = e^{\frac{2\pi i}{h}(\bar{x}\cdot\bar{p} - Ht)}$$

where, using the Einstein-de Broglie relations, we have set

$$\bar{p} = \frac{h\bar{k}}{2\pi} = \text{momentum}, \qquad H = \frac{h\omega}{2\pi} = \text{the Hamiltonian (energy)}.$$

With these substitutions, the variational principle of Synge becomes

$$\delta\int[k_1dx_1+k_2dx_2+k_3dx_3-\omega dt] = \delta\int\frac{2\pi}{h}[p_1\dot{q}_1+p_2\dot{q}_2+p_3\dot{q}_3-H]dt$$

$$= \frac{2\pi}{h}\,\delta\int L\,dt = 0$$

where $\dot{q}_j = \frac{dx_j}{dt}$ = velocity, L = the Lagrangian function.

As a consequence of the various interpretations given above, one may look at integrals of the form

$$\int \phi_\alpha e^{is_\alpha} d\bar{k} = \langle e^{is_\alpha}\rangle$$

as a sum of contributions to the average value of e^{is_α} due to infinitely many plane waves with number density in wavenumber space ϕ_α, or due to a system of wave packets that, in absence of interactions, follow Hamiltonian trajectories, or as a sum of contributions of a system of quasi-particles that obey Hamiltonian equations of motion.

This wave-particle duality permits one to look at the waves as particles, and vice versa.

Since k_j = spatial frequency (number of waves per unit distance along the x_j-coordinate), and ω = temporal frequency (number of waves per unit time), then the Synge principle is the principle of conservation of the number of waves or of the number of wave packets.

In presence of wave interactions, not only the wave amplitude changes (this may be interpreted as a change in the number density of wave packets), but also waves are created (excited) or annihilated (de-excited) so that the number of waves at a given location and at a given time having frequency ω and wavenumber $\bar{k}$ may change due to wave resonances in which the frequency and wavenumber (but not the number of wave packets) are preserved, i.e.,

$$k_j = k_j' + k_j'', \qquad \omega = \omega' + \omega'',$$

$$k_j = k_j' + k_j'' + k_j''', \qquad \omega = \omega' + \omega'' + \omega'''.$$

These relations, multiplied by $h/2\pi$, represent conservation of momentum and energy of the quasi-particles.

The group velocities $U_{\alpha j}$ and the generalized forces $F_{\alpha j}$ acting on the wave packets are given by the Hamiltonian equations (5.3). Using Eqs. (5.3) we obtain

$$\frac{dx_j}{dt} = U_{\alpha j} = U_j + c_\alpha a (k_j/k)(1-2K^2)(1-K^2)^{-\frac{1}{2}}$$

$$\frac{dk_j}{dt} = F_{\alpha j} = -[k_m \frac{\partial U_m}{\partial x_j} + c_\alpha k (1-K^2)^{-\frac{1}{2}} \frac{\partial a}{\partial x_j}] \ , \quad j = 1,2,3$$

where $c_\alpha = 0$ for $\alpha = 1,2,3$, and $c_4 = 1$, $c_5 = -1$, and U_j = velocity of the mean flow.

The group velocities of the acoustic waves, $\alpha = 4,5$, change discontinuously during an interaction, while those of the vorticity and entropy waves do not. A modification of the expressions for the frequencies, to be introduced later, will add a random component to all group velocities. It will be the random component that will change during resonant interactions.

The Uncertainty Principle

In order to allow for wave interactions we must relax the condition of conservation of the number of quasi-particles. This will require that the phase function change discontinuously along quasi-particle trajectory. Further, the observable physical quantities, which are functions of the phase factor (and of the state vectors), should remain continuous, that is, remain independent of the random discontinuities in the phase. Thus we generalize previous results to the discontinuous phase,

$$s = \bar{x}\cdot\bar{k} - \omega t + 2\pi n, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Conversely, if the phase s is allowed to change discontinuously by arbitrary integral multiples of 2π, then the integrals of the state vectors, e.g., $\int \phi e^{i(s+2\pi n)} d\bar{k}$, will remain unaffected and no amount of experimental measurement of the functions of such integrals will determine the actual value of the phase. Thus the phase will remain indeterminate and uncertain, with the uncertainty in the determination of phase being $\Delta s \geqq 2\pi$.

Correspondingly, due to the fact that now the phase s is not differentiable, the phase function is not single-valued function of space-time, and

$$\oint k_j dx_j \geqq 2\pi \quad , \quad \oint \omega dt \geqq 2\pi. \tag{5.4}$$

The wave number is no longer irrotational and the phase-space integrals satisfy the inequality

$$\iint dk_j dx_j \geqq 2\pi. \tag{5.5}$$

From Eqs. (5.4) and (5.5) it follows that

$$\Delta k\ \Delta x \geqq 2\pi, \qquad \Delta\omega\ \Delta t \geqq 2\pi. \tag{5.6}$$

Multiplying both sides by $h/2\pi$ we obtain the Bohr-Sommerfeld quantization of action conditions familiar from quantum mechanics:

$$\oint p_j dx_j = \iint dp_j dx_j \geqq h.$$

Equation (5.6) corresponds to the Heisenberg's form of the uncertainty principle, $\Delta p \Delta q \geqq h$, and $\Delta E \Delta t \geqq h$. We note that the mechanical picture requires introduction of the conversion factor h (Planck's constant) and that such a constant factor plays only the rôle of a scale factor which assigns a particular numerical value of momentum and energy to a wave packet of a given wavenumber and frequency. We introduced here the Planck's constant only to show the common mathematical structure (isomorphism) of wave and particle motions. We only assign a particular numerical value to the uncertainty of phase, viz., $\Delta s \geqq 2\pi$. Thus, we "quantize" the phase only, and do not introduce any scale factors, which factors must depend on the boundary conditions in the case of turbulent flow.

The uncertainty of phase, being a property of the Fourier transform, must be carried through the development of the theory. Thus, we adopt the uncertainty of phase as the *uncertainty principle* to be incorporated into the mathematical structure of the theory of turbulence.

The Complementarity Principle

We note that using the Fourier transform one introduces the products $x_m k_m$ and ωt into the phase function s. Thus x_m and k_m, or ω and t, become conjugate to each other forming "*complementary pairs of variables each of which can be better defined only at the expense of a corresponding loss in the degree of definition of the other.*" These words are used by Bohm (1951), p. 160, to define the quantum mechanical principle of complementarity which carries over into the turbulence theory in wave (that is, in Fourier) representation.

The Correspondence Principle

As the amplitudes ϕ_α of the waves diminish to zero, one may neglect the higher order terms (the bilinear and trilinear interaction terms), and if the mean flow becomes homogeneous and steady, then the plain wave solutions satisfy the limiting (linear) form of the equations for the fluctuations. In that case the wave amplitudes ϕ_α remain rigorously constant in space-time and the wave resonances may be neglected. We will refer to the small amplitude steady homogeneous case as the "classical limit" to which a turbulence theory in wave representation must reduce. As a guide line in the development of the theory we shall adopt the *correspondence principle,* namely, the principle that the results of the theory should reduce to the classical form of the linear, small amplitude wave mechanics of steady homogeneous media.

Operator Formalism

In the Fourier analysis of functions of the phase factors $e^{i(\bar{x}\cdot\bar{k}-\omega t)}$ we have

$$\frac{\partial F}{\partial t} = -i\omega F, \qquad \nabla F = i\bar{k}F, \qquad F = F^{\circ}e^{i(\bar{x}\cdot\bar{k}-\omega t)} ,$$

from which follows the equivalence of operators

$$\omega \leftrightarrow -\frac{1}{i}\frac{\partial}{\partial t}, \qquad \bar{k} \leftrightarrow \frac{1}{i}\nabla .$$

We may introduce now a characteristic function ψ, which in the "classical" limit is defined as

$$\psi(\bar{x},t) = \int \phi e^{i(\bar{x}\cdot\bar{k}-\omega t)}d\bar{k} = \int \phi e^{is}d\bar{k} .$$

Here we omit the subscript α and allow ω to be complex in general. We observe the following properties of ψ:

$$|\psi|^2 = \psi^*\psi = \int\int \phi^*(\bar{k})e^{-is(\bar{k})}\ \phi(\bar{k})e^{is(\bar{k})}d\bar{k}d\bar{k}' = \int|\phi|^2 d\bar{k},$$

$$\psi^*c\psi = \int \phi^* c\phi d\bar{k} = c\int|\phi|^2 d\bar{k}, \quad c = \text{constant},$$

$$\nabla\psi = i\int \bar{k}\phi e^{is}d\bar{k}, \qquad \nabla^2\psi = -\int k^2\phi e^{is}d\bar{k}, \text{ etc.}$$

$$\psi^*\frac{\partial\psi}{\partial t} = -i\int \phi^*\omega\phi d\bar{k} = -i\langle\omega\rangle,$$

$$\psi^*\nabla\psi = i\int \phi^*\bar{k}\phi d\bar{k} = i\langle\bar{k}\rangle.$$

We see, therefore, that to any dispersion relation $\omega = H(\bar{x},\bar{k},t)$ there will correspond an operator equation or a differential equation of the form

$$i \frac{\partial\psi}{\partial t} = H(\bar{x},\frac{1}{i}\nabla,t)\psi,$$

which equation has a formal solution

$$\psi = e^{iHt} .$$

As long as the dispersion relation is a polynomial in $\bar{k}$, the operator H will be a proper spatial differential operator. The introduction of the characteristic function has two important advantages. First, a dispersion relation for a given mode of wave propagation determines a partial differential equation for the corresponding characteristic function. Second, the derivatives of the characteristic function determine the averages or moments of functions of wavenumber with respect to the distribution function $\phi^*(\bar{k})\phi(\bar{k}) \equiv f(\bar{k}) \geqq 0$. In turn, the moments of the distribution determine the distribution, thus we have here a method of computing the distribution function $f(\bar{k})$ and of calculating all the required functions of the distribution, in particular, the interaction integrals and the source functions. The characteristic function in the general, inhomogeneous, nonsteady, large amplitude case will be obtained by generalization of the above results for the limiting, "classical" case.

Modification of Wave Frequency

We shall now interpret the wave packet motion in terms of the particle mechanics in order to adopt a form of the frequency modification term that is consistent with the wave-particle duality.

Consider a particle motion in which the momentum $\bar{p} = h\bar{k}/2\pi$ is reckoned relative to the moving medium. Then $\bar{p} = h\bar{k}/2\pi + m\bar{U}$ is the momentum relative to the coordinate frame in which the fluid carrying the waves has velocity $\bar{U}$. The particle mass is m. The energy of the particle is then

$$H = \frac{p^2}{2m} + mV = \frac{1}{2}mU^2 + \frac{h}{2\pi}\,\bar{U}\cdot\bar{k} + \frac{1}{2m}\left(\frac{h}{2\pi}\right)^2 k^2 + mV$$

where V is the potential energy per unit mass. Since $H = h\omega/2\pi$, we have for the frequency

$$\omega = \bar{U}\cdot\bar{k} + \frac{h}{4\pi m}\,k^2 + \frac{2\pi m}{h}\left(\frac{1}{2}U^2 + V\right).$$

The first contribution to the frequency is the Doppler term. The second term is the kinetic energy of the particle motion relative to the moving medium. The last term is the sum of the kinetic energy of the particle as if it were moving with the velocity of the medium plus the potential energy.

In quantum mechanics the classical limit is obtained by allowing the quantum of action h to tend to zero. If we use the same argument here and require that ω remain finite, then we must set $V = -\frac{1}{2}U^2$. Thus, we are left with

$$\omega = \bar{U}\cdot\bar{k} + \frac{h}{4\pi m}\,k^2.$$

The term proportional to k^2 leads in quantum mechanics to a highly dispersive wave motion. In the case of turbulence this term offers means of introducing turbulent dissipation and dispersion due to a random motion of the wave packets relative to the mean flow.

The frequencies of vorticity and entropy modes, Eqs. (4.19), have only one contribution, namely the Doppler term, and thus represent a convected pattern of standing waves. A modification of frequency for the purpose of accounting for the wave interactions, see p. 43, allows one to model the wave motion because the form of the random function $\omega^*(\bar{k})$ introduced there is arbitrary. Consider a series expansion for $\omega^*(\bar{k})$,

$$\omega^* = \omega_o + \omega_i k_i + \omega_{ij} k_i k_j + \omega_{ijk} k_i k_j k_k + \ldots$$

The constant ω_o could be absorbed into the potential V and, by the previous argument, terms independent of $\bar{k}$ should vanish being inversely proportional to h. The $\omega_i k_i$ term would introduce a drift velocity relative to the mean velocity of the fluid. Thus the first meaningful term is $\omega_{ij} k_i k_j$. If ω is to be regarded as analogous to the energy of quasi-particles, a function quadratic in momenta, then such an analogy would require that we use only the diagonal terms of ω_{ij}. Further, the simplest expression of second order in k_i containing one parametric constant is $\omega^* = (\langle\omega^*\rangle/\langle k^2\rangle)k^2$ where $\langle\omega^*\rangle$ is given by Eq. (4.27). Admittedly, such a drastic approximation to the wave interaction terms, which puts all the information about the complicated interaction processes into one parametric constant, leaves a lot to be desired. However, even such a simple approximation has some redeeming features, namely:

1. the wave-particle analogy is preserved;

2. with $\langle\omega^*\rangle$ determined from Eq. (4.27), the interaction terms are accounted for approximately even in a non-steady, inhomogeneous flow with large fluctuations, and $\langle\omega^*\rangle$ vanishes for small fluctuations in steady homogeneous flow;

3. the differential equation corresponding to the choice of ω^* is the Schroedinger equation

$$\frac{\partial\psi}{\partial t} = i\nabla\cdot\left\{\frac{\langle\omega^*\rangle}{\langle k^2\rangle}\nabla\psi\right\} \simeq i\frac{\langle\omega^*\rangle}{\langle k^2\rangle}\nabla^2\psi,$$

   which is a wave equation with strong dispersion. Thus the various types of waves, in particular the vorticity and entropy waves, will not only be convected by the fluid but will also be dispersed and spread out relative to the moving fluid. The dispersion is governed by wave interactions. Both the interaction of waves with the mean flow and wave resonances (higher order effects or wave "collisions") contribute to the dispersion. With ω^* complex, there will also be attenuation or amplification due to interactions.

In conclusion, using the wave-particle duality we model the turbulent motion by allowing the wave packets to have frequency modified by a term that represents kinetic energy of the motion relative to the mean flow. The added term has the desired properties and it depends on the wave interactions.

"Shall I refuse my dinner because I do not fully understand the process of digestion?"

O. Heaviside

quoted by T. V. Kármán and M. A. Biot, Mathematical Methods in Engineering, McGraw-Hill Book Co., New York, 1940.

VI. TURBULENT TRANSPORT EQUATIONS

Differential Equations for the Characteristic Functions

For a given linear system $L\ (\psi_\alpha) = 0$ we have the plane wave solutions $\psi_\alpha = \phi_\alpha e^{i(\bar{x}\cdot\bar{k} - \omega_\alpha t)}$ with ϕ_α and ω_α independent of space-time. Because the system is linear, a weighted sum or an integral over the $\bar{k}$-space

$$\psi_\alpha(\bar{x},t) = \int \phi_\alpha(\bar{k})\ e^{i[\bar{x}\cdot\bar{k} - \omega_\alpha(\bar{k})t]}\,d\bar{k}$$

is also a solution of $L(\psi_\alpha) = 0$ for any distribution $\phi_\alpha(\bar{k})$ independent of $\bar{x}$ and t.

An equation for ψ may be obtained with local mean values of the flow properties, e.g., the mean velocity $\bar{U}(\bar{x},t)$, as parameters, by substituting the differential operators $i\frac{\partial}{\partial t}$ and $-i\nabla$ for ω and $\bar{k}$, respectively, into the dispersion relation $\omega = \omega[\bar{k};\bar{U}(\bar{x},t)]$, that is, the differential equation for ψ takes the symbolic form

$$i\frac{\partial\psi}{\partial t} = \omega[-i\nabla;\bar{U}(\bar{x},t)]. \tag{6.1}$$

In order to generalize these results to unsteady inhomogeneous flows we shall write the characteristic function ψ as

$$\psi(\bar{x},t) \equiv \int \phi(\bar{x},\bar{k},t) e^{i[\bar{x}\cdot\bar{k} - \omega(\bar{x},\bar{k},t)t]} d\bar{k} \tag{6.2}$$

where we omit the subscript α. By writing ϕ and ω as functions of $\bar{x}$, $\bar{k}$, and t, rather than as functions of $\bar{k}$ and of functions of $\bar{x}$ and t, we avoid treating ψ as a functional of the mean flow properties. A functional equation for such a functional could be derived by Hopf's (1952) formalism for any case of fluid flow. In avoiding the use of functional calculus we have no formal way of deriving an equation for the characteristic function (6.2) in the general case. Observing, however, that ψ and ϕ may be viewed as wave functions in space and momentum representations, and that in the quantum mechanics the Schroedinger equation may be derived formally for the noninteracting (free particle) case as done here in steps leading towards Eq. (6.1), and that the Schroedinger equation for the interacting particle case is postulated to be the same equation with a proper space-time dependent interaction term, we proceed likewise and *postulate* that ψ in its generalized form, Eq. (6.2) continues to satisfy Eq. (6.1). An additional heuristic argument behind this postulate is the observation that, following Madelung's (1926) steps, we may change the wave-function-description of our system of waves into a hydrodynamic description of a transport of the probability density $|\psi|^2$ in a medium of constant translational velocity and constant density and pressure. A generalization of such a transport equation to a transport of the probability density in nonuniform medium is a natural step if the effects of wave interactions are taken care of even approximately.

Realizing that the simplest expression for the term ω^* modifying the frequency to account for wave interactions, which expression would lead to dispersive wave motion, is $\omega^* = (\zeta - i\xi)k^2$, we may write the dispersion relation for the vorticity waves, $\alpha = 1,2$, as

$$\omega = [\bar{U}\cdot\bar{k} + \zeta_\alpha k^2 - i(\nu+\xi_\alpha)k^2], \qquad \alpha = 1,2$$

where ν = molecular kinematic viscosity, ξ_α = vorticity diffusion coefficient, ζ_α = vorticity dispersion coefficient.

Before writing the corresponding differential operator, we observe that $\bar{U}$, ζ_α, ν, and ξ_α are, in general, functions of $\bar{x}$ and, therefore, the order in which the functions of $\bar{x}$ and the operator $\bar{k} \leftrightarrow -i\nabla$ appear will affect the result; following the practice of quantum mechanics, we shall assume, subject to experimental verification, that the dispersion relation must be first symmetrized and only then $\bar{k}$ should be replaced by the operator $-i\nabla$. The operator corresponding to $\omega = [\frac{1}{2}(\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})+\bar{k}\cdot\zeta_\alpha\bar{k}-i\bar{k}\cdot(\nu+\xi_\alpha)\bar{k}]$ is

$$\omega(\bar{x},-i\nabla,t) = -\frac{i}{2}[\bar{U}\cdot\nabla(\;)+\nabla\cdot(\bar{U}\;)]-\nabla\cdot[\zeta_\alpha\nabla(\;)]+i\nabla\cdot[(\nu+\xi_\alpha)\nabla(\;)]. \tag{6.3}$$

Dividing ν by the Prandtl number we have the corresponding operator for the entropy mode, $\alpha = 3$:

$$\omega(\bar{x},-i\nabla,t) = -\frac{i}{2}[\bar{U}\cdot\nabla(\;)+\nabla\cdot(\bar{U}\;)]-\nabla\cdot[\zeta_3\nabla(\;)]+i\nabla\cdot[(\frac{\nu}{Pr}+\xi_3)\nabla(\;)]. \tag{6.4}$$

In the case of the acoustic waves the dispersion relation contains the term $ak[1-(\frac{\gamma\nu k}{2Pr})^2]^{\frac{1}{2}}$ which leads to an improper operator. Even with the term approximated by the binomial expansion or in the inviscid limit we would need the operator corresponding to $|\bar{k}| = [k_1^2+k_2^2+k_3^2]^{\frac{1}{2}}$. Since the acoustic waves give rise to a quadratic wave surface, we return for the moment to the quadratic term of the characteristic equation (4.18) which must vanish as a condition for the existence of acoustic waves. With $\lambda = -i[\omega-\bar{U}\cdot\bar{k}-(\zeta-i\xi)k^2]$, the quadratic dispersion relation becomes

$$\omega^2-2\bar{U}\cdot\bar{k}\omega-2(\zeta-i\xi)k^2\omega+(\bar{U}\cdot\bar{k})^2+2\bar{U}\cdot\bar{k}(\zeta-i\xi)k^2+[(\zeta-i\xi)k^2]^2+i\frac{\gamma\nu}{Pr}k^2\omega$$

$$-\ i\bar{U}\cdot\bar{k}\frac{\gamma\nu}{Pr}k^2-i\frac{\gamma\nu}{Pr}k^2(\zeta-i\xi)k^2-\left(\frac{\gamma\nu}{2Pr}\right)^2k^4-a^2k^2 = 0 . \tag{6.5}$$

We choose to symmetrize the dispersion relation (6.5) as follows:

$$\omega^2-\frac{1}{2}(\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})\omega-\frac{1}{2}\omega(\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})-\omega\bar{k}\cdot(\zeta-i\xi)\bar{k}-\bar{k}\cdot(\zeta-i\xi)\bar{k}\omega+\frac{1}{4}(\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})^2$$

$$+\frac{1}{2}(\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})[\bar{k}\cdot(\zeta-i\xi)\bar{k}]+\frac{1}{2}[\bar{k}\cdot(\zeta-i\xi)\bar{k}](\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})-[\bar{k}\cdot(\zeta-i\xi)\bar{k}]^2$$

$$+\frac{i}{2}\omega\bar{k}\cdot\left(\frac{\gamma\nu}{Pr}\bar{k}\right)+\frac{i}{2}\bar{k}\cdot\left(\frac{\gamma\nu}{Pr}\bar{k}\right)\omega-\frac{i}{4}(\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})\left[\bar{k}\cdot\left(\frac{\gamma\nu}{Pr}\bar{k}\right)\right]-\frac{i}{4}\left[\bar{k}\cdot\left(\frac{\gamma\nu}{Pr}\bar{k}\right)\right](\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U})$$

$$-\frac{i}{2}\left[\bar{k}\cdot\left(\frac{\gamma\nu}{Pr}\bar{k}\right)\right][\bar{k}\cdot(\zeta-i\xi)\bar{k}]-\frac{i}{2}[\bar{k}\cdot(\zeta-i\xi)\bar{k}]\left[\bar{k}\cdot\left(\frac{\gamma\nu}{Pr}\bar{k}\right)\right]-\left[\bar{k}\cdot\left(\frac{\gamma\nu}{2Pr}\right)\bar{k}\right]^2$$

$$-\ a^2k^2 = 0. \tag{6.6}$$

The differential equations for the characteristic functions ψ are obtained by substitution of the operators $i\frac{\partial}{\partial t}$ and $-i\nabla$ for ω and $\bar{k}$, respectively. From Eqs. (6.3), (6.4) and (6.6) we have

$\alpha = 1,2$ (vorticity mode):

$$\frac{\partial\psi_\alpha}{\partial t} = -\ \frac{1}{2}[\bar{U}\cdot\nabla\psi_\alpha+\nabla\cdot(\bar{U}\psi_\alpha)]+i\nabla\cdot[\zeta_\alpha\nabla\psi_\alpha]+\nabla\cdot[(\nu+\xi_\alpha)\nabla\psi_\alpha], \tag{6.7}$$

$\alpha = 3$ (entropy mode):

$$\frac{\partial\psi_3}{\partial t} = -\frac{1}{2}[\bar{U}\cdot\nabla\psi_3+\nabla\cdot(\bar{U}\psi_3)]+i\nabla\cdot[\zeta_3\nabla\psi_3]+\nabla\cdot[(\frac{\nu}{Pr}+\xi_3)\nabla\psi_3], \tag{6.8}$$

$\alpha = 4,5$ (acoustic mode):

$$\begin{aligned}\frac{\partial^2\psi}{\partial t^2} = &-\frac{1}{2}\{\bar{U}\cdot\nabla\frac{\partial\psi}{\partial t}+\nabla\cdot(\bar{U}\frac{\partial\psi}{\partial t})+\frac{\partial}{\partial t}(\bar{U}\cdot\nabla\psi)+\frac{\partial}{\partial t}[\nabla\cdot(\bar{U}\ )])+i\frac{\partial}{\partial t}\{\nabla\cdot[(\zeta-i\xi)\nabla\psi]\}\\
&+i\nabla\cdot[(\zeta-i\xi)\nabla\frac{\partial\psi}{\partial t}]+\frac{1}{4}\bar{U}\cdot\nabla[\bar{U}\cdot\nabla\psi+\nabla\cdot(\bar{U}\psi)]+\frac{1}{4}\nabla\cdot[\bar{U}(\bar{U}\cdot\nabla\psi)+\bar{U}\nabla\cdot(\bar{U}\psi)]\\
&+\frac{i}{2}\bar{U}\cdot\nabla\{\nabla\cdot[(\zeta-i\xi)\nabla\psi]\}+\frac{1}{2}\nabla\cdot\{\bar{U}\nabla\cdot[(\zeta-i\xi)\bar{U}\psi]+(\zeta-i\xi)\nabla[\bar{U}\cdot\nabla\psi+\nabla\cdot(\bar{U}\psi)]\}\\
&+\nabla\cdot\left\{(\zeta-i\xi)\nabla\{\nabla\cdot[(\zeta-i\xi)\nabla\psi]\}\right\}+\frac{1}{2}\frac{\partial}{\partial t}[\nabla\cdot(\frac{\gamma\nu}{Pr}\nabla\psi)]+\frac{1}{2}\nabla\cdot[\frac{\gamma\nu}{Pr}\frac{\partial\psi}{\partial t}]\\
&+\frac{1}{4}\bar{U}\cdot\nabla[\nabla\cdot(\frac{\gamma\nu}{Pr}\nabla\psi)]+\frac{1}{4}\nabla\cdot\{\bar{U}\nabla\cdot(\frac{\gamma\nu}{Pr}\nabla\psi)+\frac{\gamma\nu}{Pr}\nabla[\bar{U}\cdot\nabla\psi+\nabla\cdot(\bar{U}\psi)]\}\\
&-\frac{i}{2}\nabla\cdot\left\{(\zeta-i\xi)\nabla[\nabla\cdot(\frac{\gamma\nu}{Pr}\nabla\psi)]+\frac{\gamma\nu}{Pr}\nabla\{\nabla\cdot[(\zeta-i\xi)\nabla\psi]\}\right\}+\nabla\cdot(a^2\nabla\psi).\end{aligned} \tag{6.9}$$

In an inviscid medium at rest, and with $\xi_\alpha = 0$ and constant ζ_α, Eqs. (6.7)-(6.9) reduce to

$$\frac{\partial\psi_\alpha}{\partial t} = i\zeta_\alpha\nabla^2\psi_\alpha, \quad \alpha = 1,2,3 \tag{6.10}$$

$$\frac{\partial^2\psi}{\partial t^2} - a^2\nabla^2\psi = +\,i\zeta_\alpha\nabla^2[2\frac{\partial\psi}{\partial t} - i\zeta_\alpha\nabla^2\psi], \quad \alpha = 4,5\,. \tag{6.11}$$

Equations (6.10) and (6.11) formally resemble the free-particle Schroedinger equation,

$$\frac{\partial\psi}{\partial t} = i\frac{h}{2\pi}\nabla^2\psi,$$

and the relativistic Klein-Gordon equation (often referred to as the "relativistic Schroedinger equation"),

$$\frac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi = -\frac{4\pi iec}{h}\{\frac{\Phi}{c}\frac{\partial\psi}{\partial t} + \bar{A}\cdot\nabla\psi\}$$
$$+ \frac{4\pi^2}{h}\{e^2(\phi^2-h^2)-(m_oc^2)^2\}\psi,$$

where c = speed of light, e = electric charge, m_o = rest mass, ϕ and $\bar{A}$ = scalar and vector potentials. We may observe at this point that for vorticity and entropy waves ζ_α plays the rôle of the reduced Planck's constant $h/2\pi$. This is a consequence of choosing $\omega^*_\alpha=(\zeta_\alpha-i\xi_\alpha)k^2$.

In absence of electromagnetic fields the non-relativistic Hamiltonian is $H = p^2/2m$, and in the relativistic case we have $H = c(p^2+m_o^2c^2)^{\frac{1}{2}}$. Evidently, the acoustic mode leads to the analogy with the relativistic Klein-Gordon equation because the real part of the intrinsic frequency,

$$\omega-\bar{U}\cdot\bar{k} = a[k^2-(\frac{\gamma\nu k^2}{2aPr})^2]^{1/2}+\zeta k^2$$

contains the square root term in which $-(\frac{\gamma\nu k^2}{2aPr})^2$ replaces $m_o^2c^2$. Thus the analogy would call for an imaginary rest mass that varies inversely with the square of the wavelength. The modification of the frequency, the term ζk^2, behaves non-relativistically. In absence of interactions, $\zeta = 0$, and Eq. (6.11) reduces to the wave equation which is also a limit of the Klein-Gordon equation for vanishing electromagnetic field.

The Klein-Gordon equation is generally accepted as valid for spinless particles. The main objection to its use is the fact that the equation is of second order in time and that, as a consequence, it requires specification of two initial conditions. But this was to be expected because, instead of two equations for the separate acoustic modes, we obtained a single equation for the function which is in an unknown relation to ψ_4 and ψ_5 introduced earlier. If we consider ψ to be a two-component vector wave function and follow Pauli's procedure, see, e.g., pp. 388-398 of Bohm (1951), we could, in principle at least, reduce the problem to two first order equations for the two components of an acoustic vector wave function. There would always remain, of course, the difficulty of interpreting the meaning of the components of such a vector wave function and their relation to the characteristic functions ψ_4 and ψ_5. We propose therefore to derive an approximate set of first order equations for ψ_4 and ψ_5, which equations retain some of the arbitrariness expected for a two-component treatment by Pauli's procedure, but do not present any difficulty in interpreting their meaning.

If we introduce a unit normal $\bar{n}$ so that $|\bar{k}| = \bar{k}\cdot\bar{n}$ if $\bar{n} = \frac{\bar{k}}{|\bar{k}|}$, and neglect K^2 as compared to unity, then we have from Eq. (4.19) for the two acoustic modes

$$\omega_4 = \bar{U}\cdot\bar{k} + a\bar{k}\cdot\bar{n}_4, \qquad \omega_5 = \bar{U}\cdot\bar{k} - a\bar{k}\cdot\bar{n}_5,$$

so that $\bar{n}_5 = -\bar{n}_4$. We need only two orthogonal modes and two linearly independent equations for ψ_4 and ψ_5. Such equations may be obtained for any two distinct vectors $\bar{n}_4$ and $\bar{n}_5$, in particular, two vectors with opposite directions.

In the present notation the operator corresponding to $a\bar{k}\cdot\bar{n}$ is $-ia\bar{n}\cdot\nabla$. If the speed of sound, $a = a(\bar{x},t)$, is not constant, then we should symmetrize this operator to $-\frac{1}{2}\bar{n}\cdot[a\nabla(\ )+\nabla(a\ )]$. The equations for ψ_4 and ψ_5 will then correspond to $\bar{n}_4 = \bar{n}$, and $\bar{n}_5 = -\bar{n}$. The dispersion relation for the acoustic modes $\alpha = 4,5$ is

$$\omega = \bar{U}\cdot\bar{k} + \zeta_\alpha k^2 - i(\frac{\gamma\nu}{2Pr} + \xi_\alpha)k^2 + c_\alpha(\bar{k}\cdot\bar{n})a$$

where $c_\alpha = 1$ for $\alpha = 4$ and $c_\alpha = -1$ for $\alpha = 5$. We symmetrize this relation as follows

$$\omega = \frac{1}{2}(\bar{U}\cdot\bar{k}+\bar{k}\cdot\bar{U}) + \bar{k}\cdot(\xi_\alpha\bar{k}) - i\bar{k}\cdot[(\frac{\gamma\nu}{2Pr} + \xi_\alpha)\bar{k}] + \frac{c_\alpha}{2}\bar{n}\cdot(\bar{k}a+a\bar{k}).$$

The corresponding differential equation for ψ_α is

$$i\frac{\partial\psi_\alpha}{\partial t} = -\frac{i}{2}[\bar{U}\cdot\nabla\psi_\alpha+\nabla\cdot(\bar{U}\psi_\alpha)] - \nabla\cdot(\zeta_\alpha\nabla\psi_\alpha) + i\nabla\cdot[(\frac{\gamma\nu}{2Pr} + \xi_\alpha)\nabla\psi_\alpha]$$

$$-\frac{i}{2}c_\alpha\bar{n}\cdot[\nabla(a\psi_\alpha)+a\nabla\psi_\alpha], \quad \alpha = 4,5 .$$

In order to display the properties of this equation we rewrite it as follows,

$$\frac{\partial\psi_\alpha}{\partial t} + (\bar{U}+c_\alpha a\bar{n})\cdot\nabla\psi_\alpha = \nabla\cdot[(\frac{\gamma\nu}{2Pr} + \xi_\alpha)\nabla\psi_\alpha] + i\nabla\cdot(\zeta_\alpha\nabla\psi_\alpha)$$

$$- \frac{1}{2}[\nabla\cdot\bar{U} + c_\alpha\bar{n}\cdot\nabla a]\psi_\alpha. \qquad (6.12)$$

The left-hand-side of Eq. (6.12) is a total (directional) derivative along a characteristic cone of an inviscid mean flow taken in the direction of $\frac{d\bar{x}}{dt} = \bar{U}+c_\alpha a\bar{n}$. This direction, if $\bar{n}$ is arbitrary, corresponds to the direction of the group velocity for an acoustic wave with wavenumber $\bar{k}$ parallel to $\bar{n}$. The group velocity is

$$\frac{d\bar{x}}{dt} = U+c_\alpha a\frac{\bar{k}}{k}(1-2K^2)(1-K^2)^{-\frac{1}{2}} \rightarrow \bar{U}+c_\alpha a\frac{\bar{k}}{k} \text{ as } K^2 \rightarrow 0.$$

The first term on the right-hand-side of Eq. (6.12) represents dissipation due to molecular and turbulent random motions. The second term, as in the case of the Schroedinger equation, gives a turbulent dispersion, ζ_α=const. being the analog of the Planck's constant $h/2\pi$ for the acoustic case. The last term takes the form of the forcing function with the force potential

$$V(\bar{x},t) = -\frac{1}{2}\,(\nabla\cdot\bar{U}+c_\alpha\bar{n}\cdot\nabla a) = -\frac{1}{2}\,\nabla\cdot(\bar{U}+c_\alpha a\bar{n}).$$

Note that the potential V and, therefore, also the force, $-\nabla V$ depend on the direction of the unit normal $\bar{n}$. Such a force has, therefore, two components, $\frac{1}{2}\nabla\nabla\cdot\bar{U}$ and $\frac{1}{2}c_\alpha\nabla(\bar{n}\cdot\nabla a)$, an isotropic and anisotropic components, respectively. Thus the acoustic modes are subject to a dispersive force in presence of gradients of the "index of refraction" a/a_o where a_o is a reference value.

The dependence of the equation (6.12) on the choice of the direction of the normal $\bar{n}$ should have no effect on the statistical properties of sound in turbulence, partly, because in computing statistical functions (moments of the distribution) we must sum over the two acoustic modes, and partly because, with different directions of the normal, the equations for ψ_4 and ψ_5 are linearly independent. Similar situation arises in the theory of characteristics of inviscid gas dynamics where the particular choice of the characteristic normal is immaterial as long as the chosen directions are not parallel to assure linear independence of the resulting equations.

Some comments, regarding justification of the manner in which Eq. (6.12) was derived, are in order. We note that the derivation starts with the assumption that $\bar{n}$ is parallel to $\bar{k}$. Only after $\bar{k}$ is replaced by its associated operator, $-i\nabla$, the direction of $\bar{n}$ becomes arbitrary because any connection with a particular direction of the wavenumber vector disappears. We could derive (6.12) under the assumption that $\bar{n}$ is antiparallel to $\bar{k}$ (in the direction opposite to $\bar{k}$) and would obtain, for the same given acoustic mode, the equation (6.12) with the sign of $\bar{n}$ reversed. Thus each mode, $\alpha = 4$ and $\alpha = 5$, has to satisfy Eq. (6.12) with signs reversed, and only when the equations for ψ_4 and ψ_5 are to be solved simultaneously, the opposite signs have to be used. We conclude that each mode satisfies Eq. (6.12) with either (+) or (-) sign in front of $\bar{n}$. As a consequence, we may apply to ψ successively the operator corresponding to Eq. (6.12) twice, first with (+) and then with (-) sign in order to eliminate the normal $\bar{n}$. The result will be a second order equation of Klein-Gordon type for ψ_4 and an identical equation for ψ_5, which equation may not agree with Eq. (6.9) due to different symmetrization used. With constant $\bar{U}$, a, ζ, and ξ identical results would be obtained as the product of the operators corresponding to (+) and (-) signs is identical to the operator derived from the product of the dispersion relations (the quadratic dispersion relation), i.e.,

$$[\omega-\bar{U}\cdot\bar{k}-(\zeta_\alpha-i\xi_\alpha-\frac{i\gamma\nu}{2Pr})k^2+c_\alpha(\bar{k}\cdot\bar{n})a]\times[\omega-\bar{U}\cdot\bar{k}-(\zeta_\alpha-i\xi_\alpha-\frac{i\gamma\nu}{2Pr})k^2-c_\alpha(\bar{k}\cdot\bar{n})a]$$

$$= \omega^2-2\bar{U}\cdot\bar{k}\ -2(\zeta_\alpha-i\xi_\alpha)k^2\ +(\bar{U}\cdot\bar{k})^2+2\bar{U}\cdot\bar{k}(\zeta_\alpha-i\xi_\alpha)k^2+[(\zeta_\alpha-i\xi_\alpha)k^2]^2$$

$$+\ i\frac{\gamma\nu}{Pr}k^2\ -i\bar{U}\cdot\bar{k}\frac{\gamma\nu}{Pr}k^2-i\frac{\gamma\nu}{Pr}(\zeta_\alpha-i\xi_\alpha)k^4-(\frac{\gamma\nu}{2Pr})^2k^4\ -\ a^2k^2\ =\ 0.$$

This should be compared with Eq. (6.5). It is felt that Eq. (6.12) is not only correct, but that it could be obtained by following Pauli's procedure of introducing a two-component wave function and reducing the Klein-Gordon-type equation to two equations for the two "components" of the acoustic wave function, namely ψ_4 and ψ_5.

It remains to interpret the two acoustic wave modes. First, we observe that the directions of propagation of the two waves for a given wavenumber $\bar{k}$ are equal and opposite. Physically, the two waves represent, relative to a plane normal to $\bar{k}$, an incident and a reflected wave. The two types of acoustic waves (or wave packets) are distinguishable by the property of their "parity", and the "conservation of parity" would imply that a given wave cannot reverse its direction of propagation except upon collision (interaction) with other waves or with boundaries. Using two separate acoustic modes in calculations may be viewed as equivalent to the imposition of the radiation condition on the wave packet micromotion. Further, it may be concluded that a Klein-Gordon-type equation cannot describe the propagation of sound correctly in presence of turbulence with the turbulent scale comparable to the wavelength of sound.

Diffusion Equations for the Field Probabilities

Following Madelung (1926) we will now put the partial differential equations for the characteristic function ψ (which equations are *linear* in ψ) into a *nonlinear hydrodynamic form.*

We observe that Eqs. (6.7), (6.8) and (6.12) are in a common form corresponding to Eq. (6.12):

$$\frac{\partial\psi_\alpha}{\partial t} + (\bar{U}+c_\alpha a\bar{n})\cdot\nabla\psi_\alpha = \nabla\cdot[(\nu_\alpha+\xi_\alpha)\nabla\psi_\alpha]+i\nabla\cdot(\zeta_\alpha\nabla\psi_\alpha)$$

$$-\frac{1}{2}(\nabla\cdot\bar{U}+c_\alpha\bar{n}\cdot\nabla a), \qquad = 1,\ldots,5 \tag{6.13}$$

where $c_\alpha = 0$ for $\alpha = 1,2,3$, $c_4=1$, $c_5=-1$, $\nu_1=\nu_2=\nu$ = kinematic viscosity $\nu_3 = \nu/Pr$, $\nu_4=\nu_5=\gamma\nu/(2Pr)$.

If the coefficients of the above equation are constant, then the solution is of the form $\psi = \phi(\bar{k})\exp\{-\Gamma t+i[\bar{x}\cdot\bar{k}-\omega(k)t]\}$. In the general case, however, we shall allow ψ to have the most general form of a complex function, namely

$$\psi(\bar{x},t) = R(\bar{x},t)e^{iS(\bar{x},t)} \tag{6.14}$$

where R and S are arbitrary real functions of space-time to be determined from Eq. (6.13), and S may be multiple-valued. In particular, departing from Madelung's formalism we shall write the phase S as an integral

$$S = \int^{\bar{x}} (\nabla A+\nabla\times\bar{B})\cdot d\bar{x} = \int^{\bar{x}} \bar{V}\cdot d\bar{x}$$

so that

$$\nabla S = \bar{V} = \nabla A+\nabla\times\bar{B}, \qquad \oint \bar{V}\cdot d\bar{x} = \oint \nabla\times\bar{B}\cdot d\bar{x} \neq 0.$$

We shall keep in mind that ∇S has an irrotational part, ∇A, and a rotational component $\nabla\times\bar{B}$. This is a natural generalization of Madelung's formalism that is consistent with the fact that the phase of the state vectors, $s = \bar{x}\cdot\bar{k}-\omega t$ has as its gradient $\nabla s = \bar{k}$ which is assumed to be in general a rotational vector, $\oint \bar{k}\cdot d\bar{x} \neq 0$. Consequently, S is a multiple-valued function but its gradient is not.

Differentiating Eq. (6.14) we have

$$\frac{\partial\psi}{\partial t} = [\frac{\partial R}{\partial t} + iR\frac{\partial S}{\partial t}]\, e^{iS} = [\frac{\partial lnR}{\partial t} + i\frac{\partial S}{\partial t}]\psi \ ,$$

$$\nabla\psi = [\nabla lnR + i\bar{V}]\psi.$$

Substituting into Eq. (6.13), dividing by ψ, and separating real and imaginary parts, we obtain dropping the subscript α

$$\frac{\partial lnR}{\partial t} + [\bar{U}+ca\bar{n}+2\zeta\bar{V}]\cdot\nabla lnR$$

$$= \nabla\cdot[(\nu+\xi)\nabla lnR]+(\nu+\xi)[(\nabla lnR)^2-V^2]- \frac{1}{2}[\nabla\cdot\bar{U}+c\bar{n}\cdot\nabla a], \qquad (6.15)$$

$$\frac{\partial S}{\partial t} + [\bar{U}+ca\bar{n}+2(\nu+\xi)\nabla lnR]\cdot\bar{V}$$

$$= \nabla\cdot[(\nu+\xi)\bar{V}]+\nabla\cdot[\zeta\nabla lnR]+\zeta[\frac{1}{4}(\nabla lnR)^2-V^2]. \qquad (6.16)$$

Multiplying Eq. (6.15) by $2R^2$ and introducing the probability density, or "turbulent intensity", P_α,

$$P_\alpha \equiv R_\alpha^2 = \psi_\alpha^*\psi_\alpha = \int\phi_\alpha^*(\bar{x},\bar{k},t)\phi_\alpha(\bar{x},\bar{k},t)d\bar{k} = \int f_\alpha(\bar{x},\bar{k},t)d\bar{k},$$

we have omitting the subscript α:

$$\frac{\partial P}{\partial t} + [\bar{U}+ca\bar{n}+2\zeta\bar{V}]\nabla\cdot P = \nabla\cdot[(\nu+\xi)\nabla P]-(\nu+\xi)[\frac{1}{2}(\nabla lnP)^2+V^2]P$$

$$- [\nabla\cdot\bar{U}+c\bar{n}\cdot\nabla a]P.$$

The above equation may be put in the form of a conservation law for the probability density P:

$$\frac{\partial P}{\partial t} + \nabla\cdot\bar{j} = 2P\{\nabla\cdot(\zeta\bar{V}) - (\nu+\xi)[\tfrac{1}{4}(\nabla \ln P)^2 + V^2]\} \tag{6.17}$$

where the probability current in Eq. (6.17) is

$$\bar{j} = [\bar{U} + ca\bar{n} + 2\zeta\bar{V} - (\nu+\xi)\nabla \ln P]P. \tag{6.18}$$

Equation (6.17) is of the form similar to a conservation of chemical species equation,

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot[\rho_\alpha(\bar{U} + \bar{v}_\alpha)] = \dot{w}_\alpha,$$

where $\bar{v}_\alpha$ = species diffusion velocity, and $\dot{w}_\alpha$ = net rate of production of species α. Consequently, we may use the following terminology

$$\bar{v}_\alpha = c_\alpha a\bar{n} + 2\zeta\bar{V}_\alpha - (\nu_\alpha+\xi_\alpha)\nabla \ln P = \text{probability diffusion velocity},$$

$$\dot{w}_\alpha = 2P_\alpha\{\nabla\cdot[\zeta_\alpha\bar{V}_\alpha] - (\nu_\alpha+\xi_\alpha)[\tfrac{1}{4}(\nabla \ln P_\alpha)^2 + V_\alpha^2]\}$$

= net rate of probability production of species α.

An alternate form of Eq. (6.17) is

$$\frac{\partial P}{\partial t} + (\bar{U}\cdot\nabla)P = - c\bar{n}\cdot\nabla(aP) - 2\zeta\bar{V}\cdot\nabla P$$

$$- P\{\nabla\cdot\bar{U} + 2(\nu+\xi)[\tfrac{1}{4}(\nabla \ln P)^2 + V^2]\}. \tag{6.19}$$

Since we are not interested, per se, in the multiple-valued phase function S and since only its derivatives have physical significance, we take the gradient of Eq. (6.16) with the understanding that $\nabla S = \bar{V}$ may have a rotational component. The result is

$$\frac{\partial \bar{V}}{\partial t} + (\bar{U}\cdot\nabla)\bar{V} = -\nabla\{\zeta[V^2 - \frac{1}{4}(\nabla \ln P)^2] + [c a\bar{n} + (\nu+\xi)\nabla \ln P]\cdot\bar{V}\}$$

$$+ \nabla\cdot\{\nabla[(\nu+\xi)\bar{V}] + \frac{1}{2}\nabla[\zeta\nabla \ln P]\}$$

$$- (\bar{V}\cdot\nabla)\bar{U} - \bar{U}\times(\nabla\times\bar{V}) - \bar{V}\times(\nabla\times\bar{U}). \qquad (6.20)$$

For the purposes of interpretation we shall add to both sides of Eq. (6.20) the term $\frac{\partial \bar{U}}{\partial t} + \nabla[\frac{1}{2}(U^2+V^2)]$ to obtain

$$\frac{\partial(\bar{U}+\bar{V})}{\partial t} + [(\bar{U}+\bar{V})\cdot\nabla](\bar{U}+\bar{V}) = -\nabla\Pi + \nabla\cdot\bar{\bar{\sigma}}$$

$$- (\bar{U}+\bar{V})\times[\nabla\times(\bar{U}+\bar{V})] + \{\frac{\partial \bar{U}}{\partial t} + \nabla[\frac{1}{2}(U^2+V^2)]\}.$$

This may be compared to the momentum equation for a compressible gas in a moving frame of reference which, at a given instant, is rotating with angular velocity $\bar{\Omega} = -\frac{1}{2}\nabla\times(\bar{U}+\bar{V})$ and has an acceleration $\bar{f} = -\{\frac{\partial \bar{U}}{\partial t} + \nabla[\frac{1}{2}(U^2+V^2)]\}$ relative to a Newtonian (inertial)frame. Such a fictitious flowing medium, a "probability gas", is subject to the pressure force per unit mass, $-\nabla\Pi$,

$$-\nabla\Pi = -\nabla\{\zeta[V^2 - \frac{1}{4}(\nabla \ln P)^2] + [c a\bar{n} + (\nu+\xi)\nabla \ln P]\cdot\bar{V}\},$$

and a viscous force per unit mass $\nabla\cdot\bar{\bar{\sigma}}$, where the viscous stress tensor is

$$\bar{\bar{\sigma}} = \nabla[(\nu+\xi)\bar{V}] + \frac{1}{2}\nabla[\zeta\nabla \ln P].$$

We conclude that the "probability gas" flows relative to and is convected by an accelerating and rotating medium and, as a consequence, there appear additional fictitious forces acting on the "probability gas". We also note that the pressure and viscous forces are solely due to the micromotion (wave motion relative to the mean flow), and that the coupling between the several modes, $\alpha = 1,\ldots,5$, is implicit through the dependence of the diffusion and dispersion coefficients, ξ_α and ζ_α, on the interactions among the several modes. The probability density of each mode, P_α, is not conserved but is diffused and dissipated by the velocity $\bar{V}_\alpha$, by the random micromotion, and by the molecular viscosity. The probability density field P_α and the probability velocity $\bar{V}_\alpha$ are strongly coupled.

At this point we may add that nonlinear, diffusion-type equations were proposed for the study of turbulence by many authors. These phenomenological theories postulate an equation of the general form

$$\frac{\partial P}{\partial t} + (\bar{U} + \bar{v})\cdot\nabla P = \nabla\cdot(E\nabla P) + AP, \tag{6.21}$$

where P = "turbulent intensity", $\bar{v}$ = self-diffusion velocity, E = "eddy" viscosity, A = turbulent source function, a function of $\bar{U}$ and P. Equations of type (6.21) were proposed by, e.g., Kolmogorov (1942), Prandtl (1945), Nee & Kovásznáy (1969) and are discussed in great detail by Saffman (1970). The present theory derives equations of the form (6.21) with explicit expressions for the functions $\bar{v}$, E, and A. Further, the present theory couples the probability transport equation to the equation for the probability velocity $\bar{V}$ which velocity contributes to the probability transport in a substantial way. Further, the present theory is developed for the compressible case so as to isolate the acoustic mode of propagation explicitly. The theory may be easily extended to chemically reacting gases and to ionized gases (plasmas).

..."the logical operations of the calculus of probability cannot be immitated by the averaging operations. The relation between the calculus of probability and the calculus of mean values is not one-one; only the former determines the latter."

M. Strauss

Modern Physics and its Philisophy, D. Reidel Publ. Co., Dordrecht, Holland, 1972, p. 188.

VII. AVERAGES, MOMENTS, AND CUMULANTS

Generalization of Definitions

Given the characteristic function

$$\psi(\bar{x},t) = \int \phi(\bar{x},\bar{k},t)\exp\{i[\bar{x}\cdot\bar{k} - \omega(\bar{x},\bar{k},t)t]\}\, d\bar{k}$$

with ω complex in general, we seek various expressions in which $\phi^*\phi=|\phi|^2$ plays the rôle of a probability distribution function in the phase space $(\bar{x},\bar{k})$. It is assumed that the function ψ is known as a solution of a partial differential equation, and that ϕ may be specified in terms of its own moments.

We define the moments of powers of the components of the wavenumber vector $\bar{k} = \{k_1,k_2,k_3\}$ as

$$M_{n_1\ldots n_N}^{m_1\ldots m_N} = \int k_{n_1}^{m_1}\ldots k_{n_N}^{m_N}\, f(\bar{x},\bar{k},t)\, d\bar{k}$$

where $m_1+m_2+\ldots+m_N$ = K = order of the moment, $n_1\ldots n_N$ indicates the sequence in which the components k_{n_j}, $N \leqq K$, are arranged, and $f(\bar{x},\bar{k},t) = \phi^*(\bar{x},\bar{k},t)\phi(\bar{x},\bar{k},t)$.

The averages or expectations are defined as

$$<k_{n_1}^{m_1}\ldots k_{n_N}^{m_N}> = M_{n_1\ldots n_N}^{m_1\ldots m_N} / \int f(\bar{x},\bar{k},t)\, d\bar{k}$$

where $\int f\, d\bar{k} = \psi^*(\bar{x},t)\psi(\bar{x},t)$. If the distribution function is normalizėd, e.g. $f' = (\psi^*\psi)^{-1}f$, then the averages and moments are equal to each other.

If all the moments of arbitrary order are known, then the distribution function $f(\bar{x},\bar{k},t)$ may be considered also known and given in terms of its moments. The distribution function f may be called accurate to order K if the distribution gives correctly all moments up to and including those of order K. It will be necessary now to express the moments and averages in terms of the characteristic function ψ which is assumed known.

The numerical values of the moments in the *steady homogeneous case* are given in terms of the spatial derivatives of $\psi(\bar{x},t)$, the Fourier transform of $\phi(\bar{k},\omega)$. For, at t = 0 and with $s = \bar{x}\cdot\bar{k} - \omega(\bar{k})t$

$$\frac{\partial\psi}{\partial x_j} = \int ik_j\phi(\bar{k})e^{is(\bar{x},\bar{k})}d\bar{k}$$

and $$\psi^*\frac{\partial\psi}{\partial x_j} = \int\phi^*(\bar{k}')e^{-is(\bar{x},\bar{k}')} \int ik_j\phi(\bar{k})e^{is(\bar{x},\bar{k})}d\bar{k}\, d\bar{k}'$$

$$\psi^*\psi\frac{\partial\, ln\psi}{\partial x_j} = i\int k_j\phi^*\phi\, d\bar{k} = i\, M_j^1$$

$$\frac{\partial\, ln\psi}{\partial x_j} = i<k_j> = iM_j^1/(\psi^*\psi).$$

We observe that for $\nabla ln\psi$ to be equal to a pure imaginary number we must have $|\psi|$ = constant.

Since $i\bar{k}$ coresponds to the vector operator ∇, higher order products of components of $\bar{k}$ are given by corresponding derivatives of ψ. Thus

$$\frac{\partial^K \psi}{\partial x_{n_1}^{m_1} \cdots \partial x_{n_M}^{m_N}} = (i)^K \int k_{n_1}^{m_1} \cdots k_{n_N}^{m_N} \phi(\bar{k}) e^{is(\bar{x},\bar{k})} d\bar{k}.$$

We then multiply both sides by ψ^* to get

$$\psi^* \frac{\partial^K \psi}{\partial x_{n_1}^{m_1} \cdots \partial x_{n_N}^{m_N}} = (i)^K \int k_{n_1}^{m_1} \cdots k_{n_N}^{m_N} \phi^*\phi \, d\bar{k} = (i)^K M_{n_1 \cdots n_N}^{m_1 \cdots m_N}.$$

Dividing by $(i)^K \psi^*\psi$ we have

$$\langle k_{n_1}^{m_1} \cdots k_{n_N}^{m_N} \rangle = \frac{1}{\psi^*\psi} M_{n_1 \cdots n_N}^{m_1 \cdots m_N} = (-i)^K \frac{1}{\psi} \frac{\partial^K \psi}{\partial x_{n_1}^{m_1} \cdots \partial x_{n_N}^{m_N}} . \qquad (7.1)$$

In the probability theory one usually deals with a characteristic function F which is a Fourier transform of a *real* distribution function $f(\bar{k}) \geqq 0$, that is

$$F(\bar{x}) = \int f(\bar{k}) e^{i\bar{x}\cdot\bar{k}} d\bar{k}.$$

Then

$$\nabla F = i\int \bar{k} f d\bar{k} = i\langle \bar{k} \rangle = iM/\int f d\bar{k},$$

and the expectations and moments of various orders are given by

$$(-i)^K \frac{\partial^K F}{\partial x_1^p \partial x_2^q \partial x_3^r} = \langle k_1^p k_2^q k_3^r \rangle = M_{pqr}.$$

One also considers cumulants C_{pqr} related to the moments and defined as

$$C_{pqr} = (-i)^K \frac{\partial^K lnF}{\partial x^p \partial x^q \partial x^r} .$$

The cumulants are polynomials of order K in the various moments of all orders up to and including K.

In the present case these definitions must be generalized to reflect the fact that ϕ and its Fourier transform ψ are complex functions the squares of whose absolute values play the rôle of probability densities, and the fact that, in the most general case, ψ is not a single-valued function of space coordinates. Thus the order of differentiation cannot be changed without affecting the results.

A generalization to the *inhomogeneous time-dependent case* follows the practice of a formal generalization of the operator relationships. Observe that if $\phi = \phi(\bar{x},\bar{k},t)$ and $\omega = \omega(\bar{x},\bar{k},t)$, then, evaluating the derivatives at $t = 0$ (at the local time), we have

$$\nabla\psi(\bar{x},t) = \int\nabla[\phi e^{is}]d\bar{k} = \int[i\bar{k} + i\nabla arg\phi + \nabla ln|\phi|]\phi e^{is}d\bar{k}$$

$$= i\int[\bar{k} + \nabla arg\phi - i\nabla ln|\phi|]\phi(\bar{x},\bar{k},t)e^{is}d\bar{k}.$$

According to the correspondence principle, this expression must reduce to the "classical" one, that is, we must recover the case of the *steady homogeneous* wave motion in the limit as $\phi(\bar{x},\bar{k},t) \rightarrow \phi(\bar{k})$,

$$\lim_{\nabla\phi\to 0} \nabla\psi = i\int \bar{k}\phi(\bar{k})e^{is}d\bar{k}.$$

Thus we *postulate* that, in the *inhomogeneous case*, the role of the wavenumber vector $\bar{k}$ (which is proportional to the momentum of the quasi-particles) is taken over by the effective wavenumber vector $(\bar{k} + \nabla arg\phi - i\nabla ln|\phi|)$ where $\nabla arg\phi$ is a random function arising from the uncertainty in the phase $s = \bar{x}\cdot\bar{k} - \omega(\bar{x},\bar{k},t)t \pm 2\pi n$, $n = 1,2,3,\ldots$ The imaginary component, $-i\nabla ln|\phi|$, represents the attenuation or amplification of the momentum of the quasi-particles in wave interactions which result in the change in the number density of the quasi-particles. The sum $(\bar{k} + \nabla arg\phi - i\nabla ln|\phi|)$ gives three contributions to $\nabla\psi$ due to, respectively, the changes along the Hamiltonian trajectories of the quasi-particles, changes due to the uncertainty of phase (uncertainty in the number of particles), and changes due to interactions among the waves. With the understanding that in evaluating various integrals of functions of $\bar{k}$ in the inhomogeneous case we will express them in terms of the derivatives of ψ, then, because $\bar{k}$ and the effective wavenumber vector are in the same relationship to the derivatives of ψ, the distinction between the two wavenumbers is immaterial. As a consequence, we *postulate* that (7.1) holds also in the *inhomogeneous case* and, due to the fact that $|\psi|$ may vary in space, the righthand side of Eq. (7.1) is complex in general. Thus we should write as a generalization of Eq. (7.1):

either $$\langle k_{n_1}^{m_1}\ldots k_{n_N}^{m_N}\rangle = Re\left\{(-i)^K \frac{1}{\psi} \frac{\partial^K\psi}{\partial x_{n_1}^{m_1}\ldots\partial x_{n_N}^{m_N}}\right\}_{t=0} \tag{7.2}$$

$$\text{or} \quad <k_{n_1}^{m_1} \ldots k_{n_N}^{m_N}> = \frac{1}{2i}\left\{\frac{1}{\psi}\,\frac{\partial^K \psi}{\partial x_{n_1}^{m_1} \ldots \partial x_{n_N}^{m_N}} - \frac{1}{\psi^*}\,\frac{\partial^K \psi^*}{\partial x_{n_1}^{m_1} \ldots \partial x_{n_N}^{m_N}}\right\}_{t=0} \quad . \tag{7.3}$$

Both of the above equations give a real value for the average of a real quantity. However, Eq. (7.2) leads to symmetric tensors, e.g. $<k_q k_p> \equiv <k_p k_q>$, and is, therefore, not general enough. Thus we shall adopt Eq. (7.3) as the correct generalization of the definition of the average. Further, if the asymmetry of tensors such as $<k_q k_p>$ is to be allowed for, we should introduce additional generality in the permissible form of ψ, $\psi = Re^{iS}$.

The wave function ψ by itself has no physical meaning and only its amplitude squared, $R^2 = P$, has a significance when interpreted as a probability intensity. Consequently, the generalization should leave R unaltered so as to rendered P single-valued. This leaves us with a modification of the phase S as the only alternative. But the phase may be left arbitrary up to any integral multiple of 2π, or in general, could be represented by any multiple-valued real function of $\bar{x}$ and t. That is why we made the choice of writing S as a path-dependent (multiple-valued) function of $\bar{x}$, that is,

$$S(\bar{x},t) = \int^{\bar{x}} \bar{V}(\bar{x},t)\cdot d\bar{x}, \quad \bar{V} = \nabla A + \nabla x \bar{B}, \tag{7.4}$$

so that, in general,

$$\oint dS = \oint \bar{V}\cdot d\bar{x} = \oint (\nabla x \bar{B})\cdot d\bar{x} = \iint \nabla x(\nabla x \bar{B})\cdot d\bar{\sigma}$$

is different from zero and the spatial derivative of S (gradient of S) has both the rotational and irrotational components.

We may now give some examples of expressions for the moments and averages of powers of the wave vector $\bar{k}$. Differentiating Eq. (6.14), we have

$$\nabla\psi = (\nabla R + iR\bar{V})e^{iS} = (\nabla \ln R + i\bar{V})\psi .$$

From the definition of the average, Eq. (7.3),

$$\langle\bar{k}\rangle = \bar{V},$$

and we may interpret the phase S as the line integral of the average wave vector:

$$S = \int^{\bar{x}} \langle\bar{k}\rangle\cdot d\bar{x}.$$

Differentiating first with respect to x_p and then with respect to x_q, $p,q = 1,2,3$, we obtain

$$\frac{1}{\psi}\frac{\partial}{\partial x_q}\left[\frac{\partial\psi}{\partial x_p}\right] = \frac{\partial^2 \ln R}{\partial x_q \partial x_p} + \frac{\partial \ln R}{\partial x_q}\frac{\partial \ln R}{\partial x_p} - V_q V_p$$

$$+ i\left[\frac{\partial V_p}{\partial x_q} + iV_q\frac{\partial \ln R}{\partial x_p} + V_p\frac{\partial \ln R}{\partial x_q}\right] .$$

It should be observed that the differentiation of ψ is noncommutative, for

$$\frac{1}{\psi}\left[\frac{\partial^2\psi}{\partial x_q \partial x_p} - \frac{\partial^2\psi}{\partial x_p \partial x_q}\right] = i\left[\frac{\partial V_p}{\partial x_q} - \frac{\partial V_q}{\partial x_p}\right]$$

is different from zero if $\nabla\times\bar{V} \neq 0$.

Applying formula (7.3) and adopting the convention that the order of differentiation is indicated by the ordering of the subscripts when read from right to left, we obtain

$$<k_q k_p> = \frac{\partial V_p}{\partial x_q} + V_q \frac{\partial ln R}{\partial x_p} + V_p \frac{\partial ln R}{\partial x_q}$$

$$= \frac{\partial}{\partial x_q} <k_p> + V_q \frac{\partial ln R}{\partial x_p} + V_p \frac{\partial ln R}{\partial x_q} .$$

The skew-symmetric part of this tensor is

$$\frac{1}{2}[<k_q k_p> - <k_p k_q>] = \frac{1}{2}[\frac{\partial V_p}{\partial x_q} - \frac{\partial V_q}{\partial x_p}] = \frac{1}{2}[\frac{\partial}{\partial x_q} <k_p> - \frac{\partial}{\partial x_p} <k_q>].$$

The three non-zero components of the skew-symmetric part are given by the three components of the vector $\bar{\Omega} = \frac{1}{2} \nabla x \bar{V}$. The vector $\bar{\Omega}$ has units of the inverse square of length (cm^{-2}). If we multiply the phase S by an appropriate scaling factor h with dimensions of action (energy x time), then $h\bar{V}$ = momentum, and $h\bar{\Omega}$ = angular momentum.

The tensor invariant, $<k^2>$, is obtained by contraction and is equal to

$$<k^2>= \nabla \cdot \bar{V} + 2\bar{V} \cdot \nabla ln R . \tag{7.5}$$

A general expression for an average of a third order product is

$$<k_r k_q k_p> = \frac{\partial^2 V_p}{\partial x_r \partial x_q} + \frac{\partial V_q}{\partial x_r} \frac{\partial ln R}{\partial x_p} + \frac{\partial V_p}{\partial x_r} \frac{\partial ln R}{\partial x_q} + V_q \frac{\partial^2 ln R}{\partial x_r \partial x_p} + V_p \frac{\partial^2 ln R}{\partial x_r \partial x_q}$$

$$+ [\frac{\partial V_p}{\partial x_q} + V_q \frac{\partial ln R}{\partial x_p} + V_p \frac{\partial ln R}{\partial x_q}] \frac{\partial ln R}{\partial x_r}$$

$$+ V_r [\frac{\partial^2 ln\ R}{\partial x_q \partial x_p} + \frac{\partial ln R}{\partial x_q} \frac{\partial ln R}{\partial x_p} - V_q V_p]$$

$$= \frac{\partial}{\partial x_r} <k_q k_p> + <k_q k_p> \frac{\partial ln R}{\partial x_r} + V_r [\frac{\partial^2 ln\ R}{\partial x_q \partial x_p} + \frac{\partial ln R}{\partial x_q} \frac{\partial ln R}{\partial x_p} - V_q V_p].$$

Note that if S were single-valued ($\bar{B} = 0$), then the above third order tensor would be symmetric with respect to any pair of indices and could be expressed in terms of products of lower order averages. Thus it is due to the non-single-valuedness of $S = \arg\psi$ that the differentiation of ψ is noncommutative, and the differential operators follow the rules of the complex noncommutative algebra. Such an algebra was proposed as a mathematical structure which permits a representation of any stochastic process in a quantum-mechanical framework and permits interpretation of quantum mechanics as a stochastic process, see Santos (1974) who developed a quantum-like formalism to deal with general stochastic systems.

The central moments of various orders, defined as moments of various powers of deviations of components of wavenumber vector k_p from their mean values $\langle k_p \rangle$,

$$N^{m_1 \ldots m_N}_{n_1 \ldots n_N} = \int [k_{n_1} - \langle k_{n_1} \rangle]^{m_1} \ldots [k_{n_N} - \langle k_{n_N} \rangle]^{m_N} f(\bar{x},\bar{k},t)\,d\bar{k} ,$$

may be expressed in terms of ordinary (non-central) moments. For example, we have

$$N^{11}_{qp} = M^{11}_{qp} - M^1_q M^1_p, \qquad N^2_p = M^2_p - (M^1_p)^2$$

$$N^{111}_{rqp} = M^{111}_{rqp} - M^{11}_{rp} M^1_q - M^{11}_{rq} M^1_p - M^{11}_{qp} M^1_r + 2M^1_r M^1_q M^1_p$$

$$N^{21}_{qp} = M^{21}_{qp} - 2M^{11}_{qp} M^1_q - M^2_q M^1_p + 2(M^1_q)^2 M^1_p$$

$$N^3_p = M^3_p - 3M^2_p M^1_p + 2(M^1_p)^3, \text{ etc.}$$

The cumulants will be defined as successive space derivatives of the lowest order average,

$$C^{m_1 \ldots m_N}_{n_1 \ldots n_N} = \frac{\partial^{m_1}}{\partial x_{n_1}} \cdots \frac{\partial^{m_N}}{\partial x_{n_N}} [\frac{1}{2i}(ln\psi - ln\psi^*)]$$

$$= \frac{\partial^{m_1}}{\partial x_{n_1}} \quad \frac{\partial^{m_{N-1}}}{\partial x_{n_{N-1}}} <k_{n_N}>.$$

For example, $C^{1}_{p} = <k_p>$,

$$C^{11}_{qp} = \frac{\partial}{\partial x_q} C^{1}_{p} = \frac{\partial}{\partial x_q} <k_p> = <k_q k_p> - <k_q> \frac{\partial lnR}{\partial x_p} - <k_p> \frac{\partial lnR}{\partial x_q} ,$$

$$C^{111}_{rqp} = \frac{\partial}{\partial x_r} C^{11}_{qp}$$

$$= <k_r k_q k_p> - <k_q k_p> \frac{\partial lnR}{\partial x_r} - <k_r k_p> \frac{\partial lnR}{\partial x_q} - <k_r k_q> \frac{\partial lnR}{\partial x_p}$$

$$- <k_r>[\frac{\partial^2 \, lnR}{\partial x_q \partial x_p} - \frac{\partial lnR}{\partial x_q} \frac{\partial lnR}{\partial x_p}] - <k_q>[\frac{\partial^2 \, lnR}{\partial x_r \partial x_p} - \frac{\partial lnR}{\partial x_r} \frac{\partial lnR}{\partial x_p}]$$

$$- <k_p>[\frac{\partial^2 \, lnR}{\partial x_r \partial x_q} - \frac{\partial lnR}{\partial x_r} \frac{\partial lnR}{\partial x_q}] - <k_r><k_q><k_p>.$$

Ambiguity in the Definitions of Moments, Averages and Cumulants

Some general remarks, concerning the practical use of moments, averages and cumulants, are in order. First, we shall discuss the ambiguity that arises in the definitions of higher than the first order correlations and the reasons for making tentatively the choices of the definitions as given on preceding pages.

We will appeal to the mathematical principles as used in quantum mechanics. The basic principle is the fact that the average or moment of a real quantity should be real, and the mathematical method of calculation (the definition) must be such as to assure the reality. We shall start with the simplest case and return to the earlier observation that

$$\frac{\partial ln\psi}{\partial x_j} = i \langle k_j \rangle$$

in the steady homogeneous case, and that $|\psi|$ = constant in order that $\langle k_j \rangle$ be real. We may remove this restriction by a symmetrization of the integrand $\phi^* k_j \phi$ and writing

$$\langle k_j \rangle = \int \frac{1}{2}[\phi^* k_j \phi + \phi k_j \phi^*] \, d\bar{k} \; / \; \psi^* \psi$$

$$= \frac{1}{2}\left\{\int \phi^* e^{-is} \, \frac{1}{i} \, \frac{\partial}{\partial x}(\phi e^{is}) d\bar{k} - \int \phi e^{is} \, \frac{1}{i} \, \frac{\partial}{\partial x}(\phi^* e^{-is}) d\bar{k}\right\} / \; \psi^* \psi$$

$$= \frac{1}{2i}\left\{\frac{1}{\psi} \, \frac{\partial \psi}{\partial x_j} - \frac{1}{\psi^*} \frac{\partial \psi^*}{\partial x_j}\right\} = \frac{\partial}{\partial x_j} \, arg \; \psi .$$

The above implies that, while $\langle k_j \rangle$ is real, it depends only on the imaginary part of $\partial ln\psi/\partial x_j$, and, therefore, $|\psi|$ need not be constant. We see here the importance of symmetrization of expressions such as $\phi^* k_j \phi$ in view of the fact that $e^{i\bar{x}\cdot\bar{k}}$ is a function of $\bar{x}$ and is therefore canonically conjugate to $\bar{k}$. The difficulty arises, however, due to the ambiguity in the particular choice of symmetrization of higher order powers of k_j. The list below gives partial results for second order averages.

SYMMETRIC FORM	EXPRESSION FOR $\langle k_q k_p \rangle$
$\{\phi^* k_q k_p \phi + \phi k_q k_p \phi^*\}/2$	$-\left\{ \frac{1}{\psi} \frac{\partial^2 \psi}{\partial x_q \partial x_q} + \frac{1}{\psi^*} \frac{\partial^2 \psi^*}{\partial x_q \partial x_p} \right\}$
$\{k_q \phi^* k_p \phi + k_q \phi k_p \phi^*\}/2$	$-\left\{ \frac{1}{\psi} \frac{\partial^2 \psi}{\partial x_q \partial x_p} + \frac{1}{\psi^*} \frac{\partial^2 \psi^*}{\partial x_q \partial x_p} \right\} + \frac{\partial ln\psi}{\partial x_q} \frac{\partial ln\psi}{\partial x_p} + \frac{\partial ln\psi}{\partial x_p} \frac{\partial ln\psi^*}{\partial x_q}$

Obviously, any combination of the symmetrized forms may be used to define the average value.

Another possibility exists in an anti-symmetric ordering of terms followed by a multiplication by the imaginary unit i. This corresponds to a generation of a Hermitean operator by first obtaining a Hermitean conjugate operator and then multiplying it by i, which is also a Hermitean conjugate operator. The result is a Hermitean operator and only such will give a real function. Thus we may write

$$\langle k_j \rangle = \frac{i}{2\psi\psi^*} \int (\phi^* k_j \phi - \phi k_j \phi^*) d\bar{k} = \frac{1}{2} \left\{ \frac{1}{\psi} \frac{\partial \psi}{\partial x_j} + \frac{1}{\psi^*} \frac{\partial \psi^*}{\partial x_j} \right\} = \frac{\partial}{\partial x_j} ln\ |\psi|,$$

$$\langle k_q k_p \rangle = \frac{i}{2\psi\psi^*} \int (\phi^* k_q k_p \phi - \phi k_q k_p \phi^*) d\bar{k} = \frac{1}{2i} \left\{ \frac{1}{\psi} \frac{\partial^2 \psi}{\partial x_q \partial x_p} - \frac{1}{\psi^*} \frac{\partial^2 \psi^*}{\partial x_q \partial x_p} \right\}.$$

The last expression corresponds to Eq. (7.3). We also notice that $\langle k_j \rangle = \frac{\partial}{\partial x_j} ln\ |\psi|$ or $\frac{\partial}{\partial x_j} arg\ \psi$ depending on whether we form a symmetric integrand or whether we take an anti-symmetric integrand multiplied by i. Since k_j is the derivative of the phase $s = \bar{x} \cdot \bar{k}$ under the integral sign, and *arg* ψ is the phase of ψ, the choice here is obvious. Following this line of thought, we shall associate products of components of $\bar{k}$ with derivatives of the argument of ψ, and we shall use symmetric ordering for odd powers of $\bar{k}$ and anti-symmetric ordering for even powers. Then Eq. (7.3) may be used for all orders of the averages. This choice is somewhat heuristic and requires verification by a reference to experiment. Admittedly, anti-symmetric ordering is not in a common use and is introduced here to achieve a desired result.

All the expressions for moments, averages and cumulants are understood to be evaluated at a given instant of time (t = 0). The time-evolution of the various correlations should not be studied in terms of the time-dependence of the state vectors $\phi \exp\{i(\bar{x}\cdot\bar{k} - \omega t)\}$, because such a representation of quantum fields in terms of the active Schroedinger picture is known to be unstable in time, see, e.g. p. 5 of Dirac (1966). However, the time-history of the correlations is known if the space distribution of the wave function ψ is known at every instant of time. Thus the correlations become implicit functions of time through the time-dependence of ψ governed by the postulated non-linear partial differential equations. The state vectors are used here only for manipulative purposes to provide an instantaneous coordinate basis for the representation of functions of time at a given instant.

The averages or expectations of arbitrary functions of $\bar{k}$ are

$$F(\bar{k}) = \int \phi^* e^{-is} F(\bar{k}) \phi e^{is}\, d\bar{k} \,/ \int \phi^* \phi\, d\bar{k} \qquad (7.6)$$

If $F(\bar{k})$ is a polynomial in $\bar{k}$, then the expectation value of $F(\bar{k})$ may be expressed in terms of a corresponding polynomial of averages of $\bar{k}$.

If the coefficients of the polynomial depend on $\bar{x}$, then each term of the polynomial should be symmetrized first before the wavenumber vector is replaced by a corresponding differential operator.

If $F(\bar{k})$ is not a polynomial, one is faced with the use of improper operators (e.g., with derivatives of fractional order, etc.). Expressions of the type (7.6) arise in wave interaction terms. An evaluation of such terms, in cases where $F(\bar{k})$ is not a polynomial, may be carried out approximately by a suitable choice of an approximation to the probability distribution function $f = \phi^*\phi$. If the approximate distribution $f(\bar{x},\bar{k},t)$ contains a number of arbitrary functions of $\bar{x}$ and t, such functions may be determined by the requirement that f generates correct values of an equal number of moments. Thus, in principle, all interaction integrals of the form (7.6) may be determined with arbitrary, but finite accuracy.

"Neither seeking nor avoiding mathematical exercitations we enter into problems solely with a view to possible usefulness for physical science."

Lord Kelvin and Peter Guthrie Tait,

"Treatise on Natural Philosophy," Part II. Cambridge University Press, 1895.

VIII. ENERGIES AND DISTRIBUTIONS

Modal Energies

We are now able to write the expressions for the squares of the fluctuations. From Eq. (4.16) we have

$$|u_j^{\prime}|^2 = (u_j^{\prime})^*(u_j^{\prime}) = \sum_{\alpha}\sum_{\beta}\int\int \phi_{\beta}^*(\bar{k}^{\prime})\phi_{\alpha}(\bar{k})P_{\beta j}^*(\bar{k}^{\prime})P_{\alpha j}(\bar{k})$$

$$\times\delta(\bar{k}-\bar{k}^{\prime})\delta[\omega_{\alpha}(\bar{k})-\omega_{\beta}(\bar{k}^{\prime})]d\bar{k}d\bar{k}^{\prime}.$$

Due to the fact that ω_{α} contain the term $(\frac{h}{m})_{\alpha}k^2$, the two delta functions will vanish simultaneously only if $\alpha = \beta$ or at $\bar{k} = 0$. Assuming that $\phi(0) = 0$, we obtain

$$|u_j^{\prime}|^2 = \sum_{\alpha} \int \phi_{\alpha}^*\phi_{\alpha}P_{\alpha j}^*P_{\alpha j}d\bar{k} = \sum_{\alpha} \int |P_{\alpha j}|^2 f_{\alpha}(\bar{k})d\bar{k} \tag{8.1}$$

where the participation coefficients $P_{\alpha j}$ are given by Eq. (4.21) or, approximately, by Eq. (4.22). Since $u_j' = \{\frac{\rho'}{\bar{\rho}}, \frac{u'}{c}, \frac{v'}{c}, \frac{w'}{c}, \frac{t'}{\bar{T}\sqrt{\gamma-1}}\}$, we may obtain the squares of the fluctuations by applying Eq. (8.1) for a particular value of the index j. Thus, with $\int f_\alpha(k)dk = \psi_\alpha^* \psi_\alpha = P_\alpha$, where P_α is the probability density assumed known as a solution of the partial differential equation (6.19), we have using Eq. (4.22)

$$|\rho'|^2 = \bar{\rho}^2 |u_1'|^2 = \bar{\rho}^2 \left[\frac{\gamma-1}{\gamma} P_3 + \frac{1}{2\gamma}(P_4+P_5)\right],$$

$$|u'|^2 = c^2 \left\{P_1 \left\langle \frac{k_2^2}{k_1^2+k_2^2} \right\rangle_1 + P_2 \left\langle \frac{k_1^2 k_3^2}{k^2 (k_1^2+k_2^2)} \right\rangle_2 + \frac{1}{2} P_4 \left\langle \frac{k_1^2}{k^2} \right\rangle_4 + \frac{1}{2} P_5 \left\langle \frac{k_1^2}{k^2} \right\rangle_5 \right\},$$

$$|v'|^2 = c^2 \left\{P_1 \left\langle \frac{k_1^2}{k_1^2+k_2^2} \right\rangle_1 + P_2 \left\langle \frac{k_2^2 k_3^2}{k^2 (k_1^2+k_2^2)} \right\rangle_2 + \frac{1}{2} P_4 \left\langle \frac{k_2^2}{k^2} \right\rangle_4 + \frac{1}{2} P_5 \left\langle \frac{k_2^2}{k^2} \right\rangle_5 \right\},$$

$$|w'|^2 = c^2 \left\{P_2 \left\langle \frac{k_1^2+k_2^2}{k^2} \right\rangle_2 + \frac{1}{2} P_4 \left\langle \frac{k_3^2}{k^2} \right\rangle_4 + \frac{1}{2} P_5 \left\langle \frac{k_3^2}{k^2} \right\rangle_5 \right\},$$

$$|T'|^2 = \frac{\gamma-1}{\gamma} \bar{T}^2 \left[P_3 + \frac{\gamma-1}{2}(P_4+P_5)\right],$$

where the symbol $\langle\ (\ )\ \rangle_\alpha$ indicates the average with respect to the α-th distribution function, and where $c^2 = R\bar{T} = \bar{p}/\bar{\rho}$. The expressions $|u_j'|^2$ are, actually, sums of moments of $|P_{\alpha j}|^2$ with respect to the distributions $f_\alpha(\bar{k})$ that include all non-trivial contributions of all five modes, $\alpha = 1,\ldots,5$.

Summing up the squares of the velocity components we obtain for the turbulent kinetic energy

$$(|u'|^2+|v'|^2+|w'|^2) = \frac{\bar{p}}{\bar{\rho}} \left[P_1+P_2+ \frac{1}{2}(P_4+P_5)\right].$$

Here, it is expected that the contribution of the vorticity modes will be dominant, so that the turbulent energy, neglecting acoustic contributions, would become $(\bar{p}/\bar{\rho})(P_1+P_2)$.

Fluctuations of other physical variables may be expressed in terms of the u_j'. For example, using the perfect gas law, $p = \rho RT$, we have

$$\frac{p'}{\bar{p}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}} = u_1' + \sqrt{\gamma-1}\, u_5'$$

and

$$|\frac{p'}{\bar{p}}|^2 = |u_1'|^2 + (\gamma-1)\ |u_5'|^2 + 2\sqrt{\gamma-1} \sum_{\alpha} \int P^*_{\alpha 1} P_{\alpha 5} f_{\alpha}(\bar{k})\, d\bar{k}$$

$$= \frac{\gamma-1}{\gamma} P_3 + \frac{1}{2\gamma}(P_4+P_5) + \frac{2(\gamma-1)}{\gamma}\, [P_3 + \frac{1}{2}(P_4+P_5)],$$

$$|p'|^2 = \bar{p}^2[\frac{3(\gamma-1)}{\gamma}\, P_3 + \frac{2\gamma-1}{2\gamma}\, (P_4+P_5)].$$

The above expression for the expectation value of the square of pressure fluctuations illustrates the fact that contributions of the several modes of wave propagation are separated out and are given in terms of the probability densities P_α. For instance, the expectation of the square of the pressure carried by the acoustic modes is $\frac{2\gamma-1}{2\gamma}\,\bar{p}^{2}\,(P_4+P_5)$. Obviously, this quantity is radiating acoustically, while the contribution of the entropy mode, $\frac{3(\gamma-1)}{\gamma}\bar{p}^2 P_3$, represents the effects of the "pseudo-sound" that is convected by the turbulent medium. Thus the separation into *radiated* and *convected* contributions amounts to separation of the contributions of the acoustic and non-acoustic modes.

Finally, we observe that linear combinations of the diffusion equations for the field probabilities P_α, Eq. (6.19), would serve as transport equations for the turbulent kinetic energies. Such a turbulent energy transport equation was proposed by Nee and Kovásznáy (1969). In the present theory the transport equations for P_α are coupled with the equations for the probability density velocity field $\bar{V}_\alpha$ due to the use of a complex wave function (the characteristic function ψ). The use of a complex wave function and the allowance for multiple-valuedness of the argument of the wave functions, make present results more general.

Of special interest are the spectral distributions of the fluctuations. We turn now to the problem of an approximate determination of the wavenumber distribution functions. With the help of the distribution functions one may evaluate expressions such as $\langle\frac{k_1^2}{k_1^2+k_2^2}\rangle$ which cannot be given in terms of derivatives of the characteristic function ψ_α.

Approximate Distribution Functions

The knowledge of the distributions is required for evaluation of various statistical correlations which enter into the equations for the mean flow, Eqs. (4.6)-(4.8), and for the evaluation of the random functions ω_α^* which are given in terms of the interaction integrals, Eq.(4.27). We observe that the mean flow depends on the correlations of second and third order. As a consequence, a distribution function which is fitted to give correctly all first, second, third,... order moments of the wavenumber vector, would be expected also to predict accurately various correlations up to and including those of the first, second, third,...order. The procedure for fitting an approximate form of the distribution function $f(\bar{x},\bar{k},t)$ will be illustrated below.

At a given instant (t=const.) the wavenumber distribution function

$\phi^*_\alpha\phi_\alpha = f_\alpha(\bar{x},\bar{k},t)$ is a function of six variables. We know that, in the case of noninteracting linear oscillators in thermal contact with a reservoir of constant energy, the principle of stationary value of entropy leads to the Planck's distribution of energy (Bose-Einstein statistics) which gives the energy spectrum in terms of the wavenumber or frequency. By the correspondence principle, infinitesimal fluctuations in a steady homogeneous mean flow should approach Planck's distribution. Thus the limiting form of $f(\bar{k})$ is known and we may simply generalize it to the nonsteady inhomogeneous case retaining the general features of the $\bar{k}$-dependence of $f(\bar{x},\bar{k},t)$ and allowing for space-time dependence and for anisotropy.

By arguments leading to the derivation of Planck's distribution, e.g., see Bohm (1951) p. 19, one may show that the energy dE in the physical volume V contained in waves of intrinsic frequency $\omega' = \omega - \bar{U}\cdot\bar{k} = \omega'(\bar{k})$ in the wavenumber range $d\bar{k}$ centered at $\bar{k}$ is at a thermodynamic equilibrium

$$dE = \frac{Vh}{(2\pi)^4} \frac{\omega'(\bar{k})\ d\bar{k}}{e^{h\omega'/2\pi\bar{E}} - 1}$$

where $\bar{E}$ is the average energy per particle of the background with which the wave modes are in thermal equilibrium. This distribution gives the Planck's function for the acoustic waves, $\omega' \simeq ak$, when ω' is linear in k, and a generalized Planck's form for ω' quadratic in k, $\omega' = \zeta k^2$, for vorticity and entropy waves.

The phase space density of energy, $U = \frac{1}{V}\frac{dE}{d\bar{k}}$, is

$$U(\bar{k}) = \frac{h}{(2\pi)^4} \frac{\omega'}{e^{h\omega'/2\pi\bar{E}} - 1} ,$$

and the phase space density of the adiabatic invariant is

$$\frac{U(\bar{k})}{\omega'(\bar{k})} = \frac{h}{(2\pi)^4} \frac{1}{e^{h\omega'/2\pi\bar{E}} - 1} = \frac{h}{2\pi} \; n(\bar{x},t) ,$$

where n = phase space number density of the excited states. Thus the adiabatic invariant U/ω' is proportional to the quantum of action h.

If we introduce a wavenumber distribution function $f=\phi^*\phi$ such that $U/\omega' = h^* f(\bar{k})$ with a new constant h^*, then at equilibrium we may write

$$f(\bar{k}) = \{\exp(h^*\omega'/\bar{E}) - 1\}^{-1}. \tag{8.2}$$

For vorticity and entropy waves we may write in general

$$h^*\omega'/\bar{E} = \beta_1^2(k_1-c_1)^2 + \beta_2^2(k_2-c_2)^2 + \beta_3^2(k_3-c_3)^2 \tag{8.3}$$

with h^*, β_i^2, c_i being functions of space and time to be chosen so as to satisfy the following seven moment conditions:

$$P = \int f d\bar{k}, \quad P\langle k_i\rangle = \int k_i f d\bar{k}, \quad P\langle k_i^2\rangle = \int k_i^2 f d\bar{k}.$$

Carrying out the integrations with the approximate form (8.3) used in equation (8.2), we have

$$P = 8\pi\zeta(3)h^*/(\beta_1\beta_2\beta_3) \tag{8.4}$$

$$P\langle k_i\rangle = PV_i = P\{c_i + \frac{\zeta(2)}{4\beta_i\zeta(3)}\} \tag{8.5}$$

$$P\langle k_i^2\rangle = P\{V_i\frac{\partial \ln P}{\partial x_i} + \frac{\partial V_i}{\partial x_i}\}$$

$$= P\{[c_i^2 + \frac{c_i\zeta(2)}{4\beta_i\zeta(3)}] - c_i^2 + \frac{\zeta(5/2)}{8\sqrt{\pi}\zeta(3)\beta_i^2}\} \tag{8.6}$$

where $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ = Riemann zeta function for $Re(z) > 1$. The solution of the above nonlinear algebraic system gives

from (8.4) $\qquad h^* = \beta_1\beta_2\beta_3 P/\{8\pi\zeta(3)\}$

from (8.5), $\qquad c_i = V_i - \frac{\zeta(2)}{4\beta_i\zeta(3)}$.

Substitution in Eq. (8.6) gives a quadratic in β_i with two roots

$$\beta_i = \frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac},$$

where $a = \langle k_i^2\rangle$, $b = \frac{\zeta(2)}{4\zeta(3)} V_i$, $c = \frac{b^2}{V_i^2} - \frac{\zeta(5/2)}{8\sqrt{\pi}\zeta(3)}$, $V_i = \langle k_i\rangle$.

Real solutions are possible for $b^2 - 4ac \geqq 0$. This places an upper bound on the second moments,

$$\langle k_i^2\rangle \leqq V_i\left\{4 - \frac{8\zeta(5/2)\zeta(3)}{\sqrt{\pi}\zeta^2(2)}\right\}^{-1} = 0.7625\langle k_i\rangle^2.$$

This example illustrates the following facts.

1. An approximate distribution function may be expressed in terms of moments such as P, $\langle k_i\rangle$, $\langle k_i^2\rangle$, etc., which are known functions of

the probability density function P and the probability velocity V_i, and of their gradients. The functions P and V_i are solutions of the transport equations discussed in the preceding sections.

2. Fitting of the generalized Planck's type distribution functions requires a solution of a simultaneous set of nonlinear algebraic equations leading to nonunique and, for certain values of the moments, to nonexistent solutions.

3. The rôle played by the quantum of action h is taken over by $h^* = \beta_1\beta_2\beta_3 P/[8\pi\zeta(3)]$ in the present example.

Possible alternatives to the use of Planck's type distributions are suggested by the following observation,

$$f(\bar{k}) = (e^{ck^2} - 1)^{-1} = \sum_{n=1}^{\infty} e^{-nck^2} = \sum_{n=1}^{\infty} (e^{-ck^2})^n .$$

Thus a Planck's type distribution is a particular power series in the Gaussian distribution e^{-ck^2}. Frankiel and Klebanoff (1973) have shown that fourth- and six-order Gram-Charlier distributions give excellent approximations to carefully determined experimental data in turbulent boundary layers. The Gram-Charlier distributions of j-th order (one-dimensional) are defined as

$$f(k) = e^{-\frac{1}{2}k^2} \sum_{0}^{j} \frac{1}{j!} \bar{H}_j(k) H_j(k)$$

where

$$H_j(k) = (-1)^j e^{\frac{1}{2}k^2} \frac{d^j}{dk^j}(e^{-\frac{1}{2}k^2}),$$

and $\bar{H}_j(k)$ is the averaged value of $H_j(k)$. The Gram-Charlier distributions are of the form of a sum of products of Hermite polynomials $\bar{H}_j(k)H_j(k)$ times the Gaussian term. The advantage of Gram-Charlier distribution is that the polynomials $\bar{H}_j(k)$ are given explicitly in terms of the averages of various powers of k and the tedious algebra involved in fitting a given distribution is thus avoided.

It is interesting to note that Hermite expansions in terms of Gaussian variables were interpreted by Wiener as expansions around the state of perfect disorder ("white noise"), see, e.g., the discussions of Wiener-Hermite expansions in Canavan (1970), Crow and Canavan (1970), and Meecham (1970). It appears on the basis of the experimental data of Frankiel and Klebanoff that some almost-Gaussian distributions may serve as good approximations for the vorticity modes which contain most of the turbulent energy.

It is further suggested that for the acoustic modes, in absence of experimental data, the Planck's type distributions be used even though the state of equilibrium is not likely to be approached unless the turbulent flow is enclosed by the walls of a duct in internal flows. In external flows the acoustic energy will be radiated outwards into the infinite space leading to large fluxes of acoustic energy and to the absence of equilibrium.

"For with slight efforts, how should one obtain great results? It is foolish even to desire it."

Thomas Jefferson's favorite quotation from EURIPIDES.

IX. SUMMARY

The wave theory of turbulence formulated here leads to a closed system of nonlinear diffusion-type equations for the probability densities of each separate mode of wave propagation and for the associated probability velocity fields. The steps in the derivation of these equations will be summarized briefly.

Separating the primitive physical variables into the averages with respect to a probability distribution and into the turbulent fluctuations, the Navier-Stokes equations for a viscous compressible fluid are split into two coupled systems of equations for the averaged values, Eqs. (4.6)-(4.8), and for the fluctuations, Eqs.(4.9)-(4.11). The latter system is put in the form (4.15) in which the linear part of the equations for the fluctuations is equated to terms which are interpreted as sources for the fluctuations (or forcing functions) arising from the interactions with the mean flow and interactions among the fluctuations.

The linear part of the system for the fluctuations is then Fourier-analysed resulting in a five-fold infinite set of eigen-solutions corresponding to five orthogonal modes of fluid oscillations identified as: two vorticity modes, an entropy mode, and two acoustic modes. The eigensolutions, which represent infinitesimal waves in a steady homogeneous mean flow at instantaneous, local conditions, are then employed as a complete vector basis for the purpose of forming an integral representation of the solutions. The absolute values of the Fourier amplitudes squared are interpreted as the probability densities in the wavenumber space. The wavenumber vector and the frequency of the Fourier modes are associated with momentum and energy of quasi-particles (wave packets). This association is used to model the wave interaction terms as a stochastic function, $\omega^* = (\zeta - i\xi)k^2$, quadratic in the wavenumber (momentum) to be added to the expression for the frequency (energy) of non-interacting wave packets. Conditions imposed on the stochastic function ω^*_α, Eq. (4.27), are that the average of ω^*_α satisfies the averaged interaction terms so that each orthogonal mode α is coupled through the interaction terms with the remaining four modes, $\alpha = 1,\ldots,5$. Thus ω^*_α models the interactions of the fluctuations with the mean flow and the interactions of the wave-wave type through the wave resonance. It is then observed that the interaction terms may be determined if the probability distributions of the orthogonal modes in the wavenumber space are known. Thus the central problem in the development of the theory is the determination of the distributions.

An operator formalism is introduced in Chapter V so as to associate differential operators with dispersion relations for each orthogonal mode. A characteristic function, that reduces to a Fourier transform of the Fourier amplitudes in the steady homogeneous mean flow, is then sought as a solution of the Schroedinger-type differential equation determined by the operator formalism from the dispersion relations. The solution, analogous to a wave function of the quantum theory, is taken to have a general form of a complex function with a multiple-valued phase. Separation of real and imaginary parts, and differentiation of the imaginary part with respect to space, result in transport equations for the spatial probability density (for the squares of the absolute values of the amplitudes of the characteristic function), Eq. (6.17), and vector equations for the "probability velocities," Eq. (6.20), which velocities are shown to equal the averages of the wavenumber vector with respect to the probability distribution in the wavenumber space. The probability transport equations are nonlinear and of the diffusion type, similar to conservation of chemical species equations in reactive flows, and also similar to the equations of the quantum theory in the hydrodynamical form. The latter similarity is clearly apparent because of the presence of terms analogous to quantum stresses of the hydrodynamical form of the quantum theory.

The transport equations are nonlinear even in the limiting case of infinitesimal fluctuations in a steady homogeneous mean flow, the "classical" limit of the present theory. It is then postulated that these transport equations may be generalized to the nonlinear wave motion encountered in strong turbulence in presence of non-steady inhomogeneous mean flow. Further, the transport equations for the probability density are of the same form as the nonlinear equations for the transport of turbulent intensities proposed in the past by many researchers. The novel feature of the present theory absent in phenomenological theories is the strong coupling of the probability density transport equations to the equations for the probability velocity fields, which fields are, in general, rotational. The latter fact is a generalization of the hydrodynamical form of the quantum theory in which the velocity field is irrotational having been obtained by taking a gradient of a single-valued function.

Because the spatial probability distribution is obtained by squaring the absolute value of the characteristic function, the standard definitions of moments, averages, and cumulants of the theory of probability had to be generalized to the present case of complex characteristic functions. In Chapter VII these definitions are carefully developed with due regard for the uncertainty principle. The microscopic uncertainty principle arising from the Fourier representation carries over into the macroscopic uncertainty in the form of the multiple-valuedness of the phase of the characteristic function and rotationality of the probability velocity field. Thus the circulation of the probability velocity becomes a macroscopic analog (an expectation value) of the phase integrals of the quantum theory,

$$\oint_C \bar{V}\cdot d\bar{x} = \oint_C \langle\bar{k}\rangle\cdot d\bar{x} = \oint_C [\int \phi^* \bar{k} \phi \, d\bar{k}]\cdot d\bar{x} = \int \phi^* [\oint_C \bar{k}\cdot d\bar{x}] \phi \, d\bar{k}$$

$$= \langle \oint_C \bar{k}\cdot d\bar{x} \rangle .$$

Many important averages (expectation values of the squares of the fluctuations) are expressible directly in terms of the probability densities of the various orthogonal modes. Thus, conveniently, the contributions of the vorticity, entropy, and acoustic modes are given separately and explicitly in terms of the solutions of the probability density transport equations. Likewise, various moments of arbitrary powers of the wavenumber vector are expressible in terms of the probability densities and probability velocities. In turn, the moments of a distribution determine the distribution function, and the central problem of the proposed theory is solved. For practical purposes it suffices to determine the distributions only approximately. Thus the modeling of turbulence and sound generated by it would involve the use of assumed forms of the distribution functions which forms should contain enough arbitrary functions to meet a finite number of moment conditions. Two such forms are discussed in Chapter VIII, the Planck's type distributions that maximized the entropy of a system of waves at equilibrium, and Gram-Charlier distributions which were found successful and accurate in representing extensive experimental data.

Before the partial differential equations of the present theory could be applied to test cases, one has to consider in detail the boundary conditions to be imposed on the probability densities and probability velocities. Numerical integration of the partial differential equations would involve a simultaneous solution of the Reynolds system for the averaged mean flow, Eqs. (4.6)-(4.8), the probability density transport equations, Eq. (6.17), and the equations for the probability velocities, Eq. (6.20), altogether twenty-five nonlinear partial differential equations which replace the original Navier-Stokes system of five equations for density, velocity components, and temperature. This large number of equations may be reduced to only five in the case of a one-dimensional incompressible turbulent flow, e.g., in pipes and channels. Extensive numerical testing of various simplified turbulence models based on the equations presented here will be necessary before the present theory could be accepted as describing the physical processes in turbulent flows. Extensions of the theory to chemically reacting flows, to radiative gas-dynamics, and to magnetogasdynamics would be straightforward as it would suffice to introduce additional orthogonal modes. Likewise, the present theory could be easily reduced to a quantum-like statistical theory of sound in inhomogeneous media in which the ratio of the length scale of the inhomogeneities to the wavelength of sound is arbitrary.

In applications to the acoustics of turbulent noise the present theory offers means of determining, (1), the intensity of pressure fluctuations in the acoustic modes and, approximately, its spectral and directional distributions, and, (2), the intensities of fluctuations of arbitrary functions in any of the modes of wave propagation present. We observe here that the far field noise outside of a turbulent region may be calculated if the acoustic field at the edge of a turbulent region is known. The present theory is capable of providing such information. Secondly, the problem of determining the noise transmitted from the turbulent boundary layer through the walls constructed from a solid material requires the detailed knowledge of turbulent fluctuations not only in the acoustic mode, but also in the remaining modes convected along the wall. The interaction of turbulence with a solid boundary may be visualized as the excitation of sound waves in the solid material by the momentum and energy exchange with the turbulent flow. Thus also the fluctuations in the vorticity mode, and to a lesser degree, in the entropy mode would be capable of exciting the sound field in the solid wall. The present theory provides means of treating such problems.

X. CONCLUDING REMARKS

It should be observed that the theory of sound in turbulent flows proposed here is a necessary consequence of the adopted point of view. The point of view held was that sound is just one of the several aspects of wave propagation in turbulence and that the wave representation is therefore both necessary and convenient. The Fourier decomposition into interacting wave fields lead naturally to a quantum-like formulation in terms of complex distribution functions. The formulation imposes the conditions of reality of the *expectation values* of the fluctuating physical observables and does not require that the *turbulent fluctuations themselves* be real quantities. Thus the theory was formulated from the start as a statistical theory and the analogy to wave mechanics was exploited for the purpose of using a well established mathematical framework as an analytical tool.

On the other hand, it becomes clear in retrospect that the question of 'how to transform the deterministic Navier-Stokes fluid dynamics formally into a statistical fluid mechanics' was answered already in 1926 by Madelung. All that remained to be done was to obtain his hydrodynamical form of quantum mechanics from the equations of the Navier-Stokes theory. This was accomplished here rigorously only for the limiting case of infinitesimal fluctuations in steady homogeneous mean flow. The derivation of the nonlinear Madelung's equations for the hydrodynamic transport of field probabilities in the limiting case from the nonlinear field equations of fluid mechanics provided the information on how to decompose the physical variables into interacting waves and also provided the form to which the postulated equations must reduce in the limiting case.

Statistical theories of turbulence are plagued by the so called closure problem which arises from the fact that the equations for the lower order moments (statistical correlations) contain higher order moments as unknowns. It is necessary to comment on how the closure problem is avoided in the present formulation.

In order to avoid the closure problem one should not treat the infinitely many moments as unknowns. The moments determine the distributions, and vice versa. If the probability densities P_α and their velocities $\bar{V}_\alpha$ are treated as the dependent variables, then the moments become known functions of the dependent variables and of their derivatives, and only a finite number of dependent variables has to be determined from a closed system of equations.

Solutions of the turbulent transport equations must satisfy appropriate boundary conditions. A discussion of boundary conditions for turbulent transport equations and the conditions for the occurrence of sharp turbulent-nonturbulent interfaces may be found in the paper by Saffman (1970).

Because the governing equations of the theory are of the familiar diffusion type, existing numerical integration techniques could be used with the cost of computing increasing only five-fold as compared to the laminar case. Standard laminar boundary layer computer programs could be adapted to turbulent flow calculations without the need for an extensive research into computational methods.

The theory as proposed here remains unproven until its predictive capabilities are demonstrated on the basis of some computed examples. The turbulent transport equations may be readily simplified by application of boundary layer concepts or by reduction to special cases. For instance, disregarding all but the two vorticity modes gives the theory of incompressible turbulent flows, while the retention of only the two acoustic modes results in a statistical theory of sound of inhomogeneous irrotational (potential) flows of importance in cases where appreciable changes in wavelengths of sound occur over distances of one wavelength.

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