Simplified Theory of Large-Amplitude Wave Propagation

by

Hongjin Kim

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ABSTRACT

An orbit perturbation procedure is applied to the description of monochromatic, large-amplitude, electrostatic plasma wave propagation. In the lowest order approximation, untrapped electrons are assumed to follow constant-velocity orbits and trapped electrons are assumed to execute simple harmonic motion. The deviations of these orbits from the actual orbits are regarded as perturbations. The nonlinear damping rate and frequency shift are then obtained in terms of simple functions. The results are in good agreement with previous less approximate analyses.

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1. INTRODUCTION

There is currently strong interest in the nonlinear behavior of plasmas involving large-amplitude waves, or high levels of turbulence. The relevant theory has advanced from analyses of wave-wave and wave-particle interactions based on second- or third-order perturbation series in the wave amplitude (Kadomtsev 1965; Sagdeev and Galeev 1969; Tsytovich 1970), to the inclusion of higher-order nonlinear effects, which become important as the level of perturbation increases. The theory is quickly limited by mathematical difficulties, one of which stems from time secularities in the perturbation series caused by trapped particles (Davidson 1972). Since the larger the wave amplitude, the more particles are trapped, the series rapidly becomes non-convergent. Furthermore, trapped particles oscillate with periods proportional to a fractional power of the wave amplitude (proportional to the square root of the amplitude for particles at the bottom of a monochromatic wave potential). Thus, when trapped particle dynamics are significant, the series does not give a proper approximation.

A number of theories have been developed to include trapping effects for a coherent, electrostatic wave, when the wave amplitude is not too large, i.e. when $\alpha[= (v_\varphi/v_t)(e\phi/T)^{1/2}] \ll 1$ for a one-dimensional Maxwellian electron velocity distribution; here v_φ and $v_t[= (T/m)^{1/2}]$ are the wave phase velocity and electron thermal velocity, respectively (O'Neil 1965; Al'tshul and Karpman 1966; Taniuti 1969; Imamura, Sugihara and Taniuti 1969; Lee and Schmidt 1970; Bailey and Denavit 1970; Oei and Swanson 1972; Sugihara and Kamimura 1972). The various theories have predicted different time evolutions of the wave depending on the value of γ_L/ω_B, where γ_L is the linear Landau damping rate and $\omega_B[= (ekE_0/m)^{1/2}]$ is the electron bounce frequency. When $\gamma_L/\omega_B \gtrsim 3$, the wave damps out according to the Landau (1946) treatment, before trapped electrons can execute any significant bounce motions. For larger wave amplitudes, the trapped electrons tend to flatten the velocity distribution locally, so that the damping rate decreases with time from its initial value, γ_L. For a wave of even larger amplitude (typically $\gamma_L/\omega_B \ll 1$), the initial Landau damping rate decreases rapidly at about

$\omega_B t \approx \pi$, and oscillates thereafter as the trapped electrons bounce. This oscillation eventually disappears, and the wave evolves towards a steady-state Bernstein-Greene-Kruskal mode (1957). The wave frequency in this state is different from the linear value, the difference being proportional to ω_B (Manheimer and Flynn 1971; Morales and O'Neil 1972; Dewar 1972; Lee and Pocobelli 1972 and 1973). The foregoing characteristics are in general agreement with experiments (Malmberg and Wharton 1967; Franklin, Hamberger and Smith 1972; Vidmar, Malmberg and Starke 1975), account being taken of the fact that the theories treat temporal rather than spatial variations, and with computer simulations (Armstrong 1967; Tsai 1974; Canosa and Gazdag 1974; Matsuda and Crawford 1975).

There is considerable interest in cases for which the condition $\alpha \ll 1$ is unnecessarily restrictive. For example, sideband instabilities are prominent for $\alpha > 1$ (Wharton, Malmberg and O'Neil 1968; Franklin, Hamberger, Ikezi, Lampis and Smith 1972; Jahns and Van Hoven 1973). In fact, $\alpha \ll 1$ is more restrictive than the small-amplitude condition of the usual perturbation theory for the non-resonant region, $(\epsilon_0 E_0^2/nT)^{1/2} \ll 1$ or $\alpha \ll (v_\varphi/v_t)^{3/2}$ (Bud'ko, Karpman and Shklyar 1972).

For $\alpha \gtrsim 1$, the wave may have quite different characteristics from those described for $\alpha \ll 1$. For example, consider an arbitrarily large wave amplitude, so that $\gamma_L/\omega_B \ll 1$ even for low phase velocity. We might anticipate that this wave will propagate at an arbitrarily low phase velocity, and reach a steady state by trapping. However, this does not happen: since the distribution function increases rapidly with decreasing velocity, the number of trapped electron increases rapidly with the wave amplitude. These electrons take more energy from the wave, and thus enhance the wave damping (Armstrong 1967; Dawson and Shanny 1968; Sato, Ikezi, Takahashi and Yamashita 1969; Nakamura and Ito 1971; Tsai 1974; Vidmar, Malmberg and Starke 1975; Sugihara and Yamanaka 1975; Canosa 1975). Consequently, a wave of amplitude larger than a certain value damps out, the precise value depending on the phase velocity. Dawson and Shanny (1968) have shown that this eventual total damping occurs for all values of amplitude when $v_\varphi/v_t \gtrsim 3.5$.

It is extremely difficult to extend rigorous theories for $\alpha \ll 1$ to the case of $\alpha \gtrsim 1$. Here, we shall sacrifice exactness, and apply a perturbation procedure to the case of $\alpha \ll 1$. Specifically, we consider a coherent, monochromatic, electrostatic wave, and separate the dynamics of untrapped and trapped electrons, following O'Neil (1965). In the lowest approximation, we assume the untrapped and trapped electron orbits to be of constant velocity and simple harmonic motion, respectively. We then consider the difference between these orbits and the actual orbits as perturbations.

In §2, the damping rate and frequency shift are obtained from the lowest-order orbit approximation. In §3, an appropriate perturbation theory is developed. This is applied in §4 and §5 to untrapped and trapped electrons, respectively, to include the effect of the orbit perturbations. In §6, our results are compared with those of Morales and O'Neil (1972), and some general conclusions are drawn.

2. LOWEST ORDER APPROXIMATION FOR NONLINEAR DISPERSION RELATION

Consider an infinite, homogeneous, collisionless plasma, with a neutralizing background of immobile positive ions, driven by a single, longitudinal wave $E(x,t) = E_0(t) \sin kx$. We shall restrict our interest to an initial value problem, where k is a real constant, and will assume that temporal variations of the wave amplitude, E_0 , and frequency (hence the wave phase velocity) are small. In the wave frame, the electron velocity distribution function in phase space, $f(x,v,t)$, is described by the Vlasov equation

$$\frac{\partial f(x,v,t)}{\partial t} + v \frac{\partial f(x,v,t)}{\partial x} - \frac{e}{m} E_0 \sin kx \frac{\partial f(x,v,t)}{\partial v} = 0 . \qquad (1)$$

Together with (1), Poisson's equation

$$E_0 = -2 \frac{m_i^2 p}{ek} \int_{-\infty}^{\infty} dv \int_{-\lambda/2}^{\lambda/2} \frac{dx}{\lambda} \Delta f(x,v,t) \exp(ikx) , \qquad (2)$$

constitutes the complete set of equations. Here $\Delta f(x,v,t) = f(x,v,t) - f_0(v)$, $f_0(v)$ is the initial distribution function, and $\lambda (= 2\pi/k)$ is the wavelength. Equation (1) implies conservation of the distribution function along the electron orbit,

$$\dot{x} = v , \qquad \dot{v} = -\frac{e}{m} E_0 \sin kx . \qquad (3)$$

Although f is expressed conveniently in (Lagrangian) coordinates moving with the electrons according to (3), E_0 is better described by fixed (Eulerian) coordinates. This difficulty may be minimized by finding an invariant of the electron motion, and using this as an independent coordinate. If E_0 is infinitely small, we may neglect the right-hand side of the $\dot{v}$-equation in (3), regard v as a constant of the motion, and expand (1) in terms of E_0 to the desired order. This is the procedure of the usual small-signal theory (Sagdeev and Galeev 1969). For large amplitude, we can find several invariants, e.g. energy, the trapping parameter (O'Neil 1965), or the action variable (Goldstein 1950), and

transform the (x,v)-coordinates to this new coordinate for (1) and (2). This had been done by many authors cited in §1. The transformation is rather involved and, in most cases, the integrations in (2) can not be performed analytically. In order to simplify the procedure, we first obtain the orbit equation from (3) by multiplying the $\dot{v}$ equation by v, and integrating with respect to time,

$$\left(\frac{v}{v_r}\right)^2 + 4 \sin^2\left(\frac{kx}{2}\right) = 2\left(\frac{H}{mv_r^2} + 1\right) , \qquad (4)$$

where H is the total energy of an electron, and $v_r (= \omega_B/k)$ is the trapping velocity. The electron orbit described by (4) is drawn in figure 1(a).

To lowest-order, we may approximate (4) by a constant velocity orbit for the untrapped electrons $(H > mv_r^2)$. We can then follow the procedure of the usual small-signal theory to solve the Vlasov equation. Neglecting higher-order mode-mode coupling, we obtain the well known, linearized solution of (2),

$$\Delta f(x,v,t) = \left(\frac{eE_0}{mk}\right)\left(\frac{f_0'(v)}{v}\right)[\cos k(x-vt) - \cos kx] , \qquad (5)$$

where the prime denotes the velocity derivative. For trapped electrons $(H < mv_r^2)$, we replace the wave potential by a parabola as shown in figure 1(b). Near the bottom of the potential well, it is quite accurate. The largest difference arises near the wave crest. However, electrons travel very slowly near the crest and do not exert a significant effect on the wave. Therefore, we consider this potential difference a perturbation. In the zeroth order approximation, (4) reduces to

$$\left(\frac{v}{v_r}\right)^2 + (kx)^2 = 2\left(\frac{H}{mv_r^2} + 1\right) , \qquad (6)$$

which describes a circular orbit in the $(kx,\ v/v_r)$-plane.

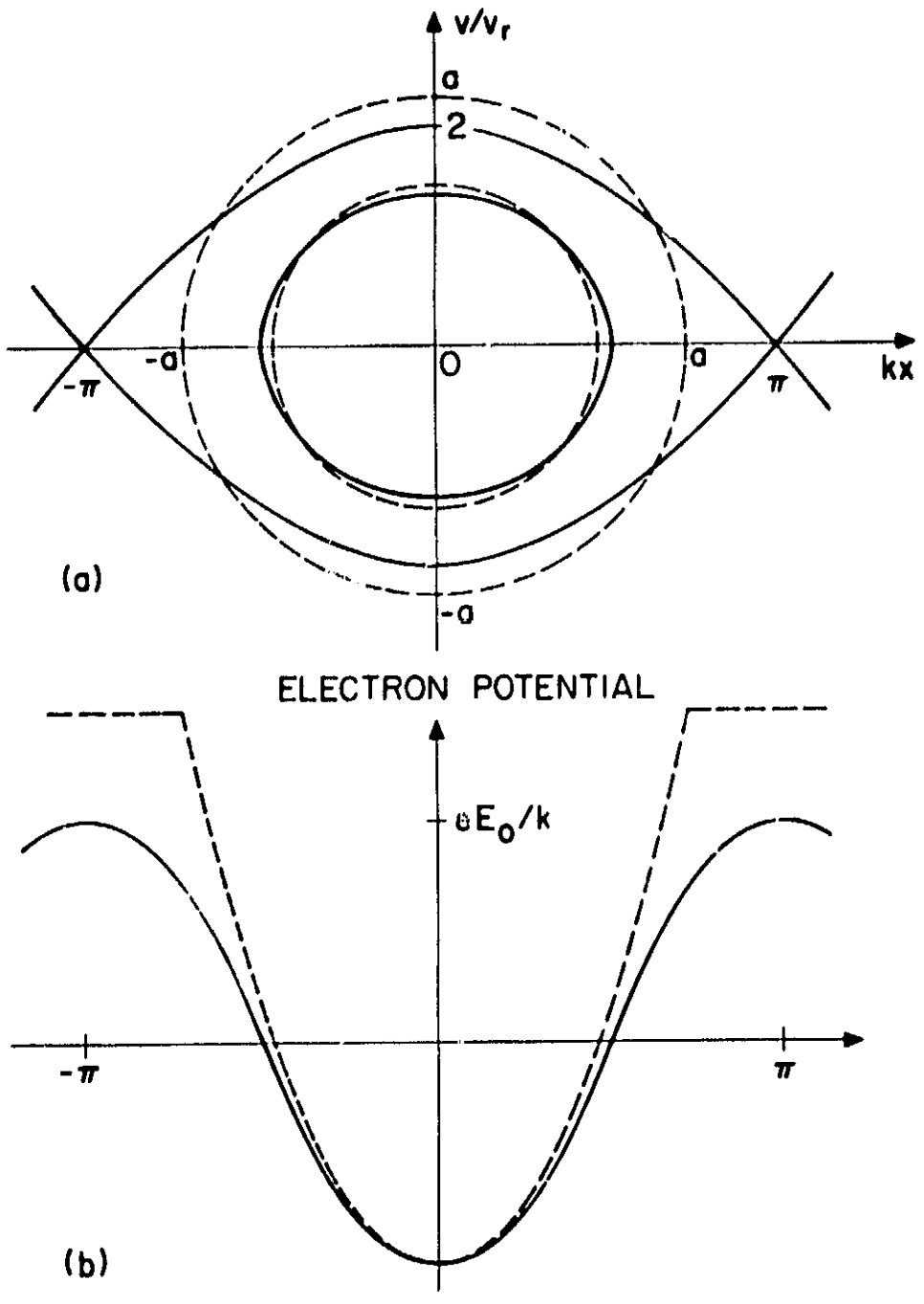

FIG 1. Actual (——) and approximate (---)
(a) trapped electron orbits and
(b) wave potentials.

With these approximations, the Vlasov equation may be written as

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \omega_B^2 x \frac{\partial f}{\partial v} = \omega_B^2 x \left(\frac{\sin kx}{kx} - 1 \right) \frac{\partial f}{\partial v} . \qquad (7)$$

The characteristic curve of the left-hand side is the circular orbit of (6). The right-hand side represents the perturbation due to the difference between the sinusoidal and parabolic potentials. Changing the variables (x,v) to circular coordinates (r,ψ) defined by

$$kx = r \sin \psi , \quad v/v_r = r \cos \psi , \qquad (8)$$

(7) can be put in the form

$$\frac{\partial F}{\partial t} + \omega_B \frac{\partial F}{\partial \psi} = \omega_B \left(\frac{\sin(r \sin\psi)}{r} - \sin\psi \right) \left(- \sin\psi \frac{\partial F}{\partial \psi} + r \cos\psi \frac{\partial F}{\partial r} \right) , \qquad (9)$$

where $F(r,\psi,t) = f\big(x(r,\psi,t), v(r,\psi,t), t\big)$. In the zeroth order approximation for trapped electrons, we neglect the right-hand side of (9). The solution is then

$$F(r,\psi,t) = f_0[r v_r \cos(\psi - \omega_B t)] . \qquad (10)$$

Following Morales and O'Neil (1972), we assume that the wave amplitude is small $(\alpha \ll 1)$ so that, in the trapped region, $f_0(v)$ can be expanded about $v = 0$ as

$$f_0(v) = f_0(0) + v f_0'(0) + \frac{v^2}{2} f_0''(0) + \ldots . \qquad (11)$$

Substituting this into (10) gives

$$\Delta F(r,\psi,t) \equiv F(r,\psi,t) - F(r,\psi,0)$$

$$= 2 r v_r f_0'(0) \sin\left(\frac{\omega_B t}{2}\right) \sin\left(\psi - \frac{\omega_B t}{2}\right) + \frac{(r v_r)^2}{2} f_0''(0) \sin \omega_B t \sin(2\psi - \omega_B t) . \qquad (12)$$

In order to use (7) and (12), we have to separate contributions to (2) of untrapped and trapped electrons, $E_0 = E_U + E_T$. For the untrapped electron contribution, we determine the lower bound $|v| = v_U$ of the untrapped region by considering the area between the x-axis and the orbit of the electron during one wave period. This area is the action variable (Goldstein 10.6). From (4), we see that the minimum of this area occurs when $H = mv_r^2$ and takes the value 8 in the $(kx, v/v_r)$-plane. Since the untrapped electron orbit is of constant velocity in the lowest order approximation, the lower bound is given by $k\lambda v_U/v_r = 8$, or $v_U = (4/\pi)v_r$. The untrapped electron contribution to (2) is then

$$E_U = -2\frac{m\omega_p^2}{ek}\left[\int_{-\infty}^{-v_U} + \int_{v_U}^{\infty}\right] dv\, f_k(v,t) ,$$

$$f_k(v,t) = \int_{-\pi}^{\pi} \frac{d(kx)}{2\pi}\, \Delta f(x,v,t)\, \exp(ikx) . \tag{13}$$

Substituting (5) gives f_k as

$$f_k(v,t) = \left(\frac{eE_0}{2mk}\right)\frac{f_0'}{v}\,(\exp ikvt - 1) . \tag{14}$$

The untrapped electron contribution, E_U, therefore becomes

$$E_U = -E_0\left(\frac{\omega_p}{k}\right)^2\left[\int_{-\infty}^{\infty} - \int_{-v_U}^{v_U}\right] dv\, \frac{f_0'(v)}{v}\,(\cos kvt - 1)$$

$$-iE_0\left(\frac{\omega_p}{k}\right)^2\left[\int_{-\infty}^{-v_U} + \int_{v_U}^{\infty}\right] dv\, \frac{f_0'(v)}{v}\,\sin kvt , \tag{15}$$

where the barred integral sign denotes the Cauchy principal value. In the real part, the first integration of the cosine term phase-mixes to zero. For the second integration, we may use (11). In the imaginary part, most of the contribution comes from near $|v| = v_U$, and we may again use (11) for $f_0'(v)$. Carrying out the integration reduces (15) to

$$E_U = E_0 \left(\frac{\omega_p}{k}\right)^2 \fint_{-\infty}^{\infty} \frac{f_0'(v)}{v}\, dv - 2 E_0 \left(\frac{\omega_p}{k}\right)^2 v_U f''_0(0) \left(1 - \frac{\sin kv_U t}{kv_U t}\right)$$

$$- i\pi E_0 \left(\frac{\omega_p}{k}\right)^2 f_0'(0) \left(1 - \frac{2}{\pi} \mathrm{Si}(kv_U t)\right) , \qquad (16)$$

where $\mathrm{Si}(\alpha)$ is defined by

$$\mathrm{Si}(\alpha) = \int_0^{\alpha} \frac{\sin x}{x}\, dx . \qquad (17)$$

Changing variables (x,v) to (r,ψ) gives the trapped electron contribution to (2) as

$$E_T = -2 \left(\frac{m\omega_p^2}{ek}\right) v_r \int_0^a r dr\, F_k(r,t) ,$$

$$F_k(r,t) = \int_{-\pi}^{\pi} \frac{d\psi}{2\pi} \Delta F(r,\psi,t) \exp(ir \sin\psi) , \qquad (18)$$

where the upper bound of the r-integration is determined by $\pi a^2/2 = 8$ or $a = 4/\pi^{1/2}$. Substituting (12), and performing the ψ-integration (Gradshteyn and Ryzhik, 1965) yields

$$F_k = -\frac{v_r^2}{4} f''_0(0)\, r^2 J_2(r)(1-\cos 2\omega_B t) + i v_r f_0'(0) r J_1(r) \sin \omega_B t . \qquad (19)$$

Using the derivative formula for Bessel functions (Abramowitz and Stegun 1965),

$$\frac{d}{dr}\left(r^n J_n(r)\right) = r^n J_{n-1}(r) , \tag{20}$$

the r-integrations in (18) can be performed to obtain

$$E_T = \frac{a^3}{2} J_3(a)\, E_0 \left(\frac{\omega_p}{k}\right)^2 v_r f_0''(0)(1-\cos 2\omega_B t) - i2a^2 J_2(a) E_0 \left(\frac{\omega_p}{k}\right)^2 f_0'(0)\sin \omega_B t . \tag{21}$$

The Poisson equation involves the sum of (16) and (21). Substitution yields the dispersion relation

$$\epsilon = \epsilon_L + \delta\epsilon = 0 , \tag{22}$$

where ϵ_L is the linear dispersion relation; ϵ_L and $\delta\epsilon$ are given by

$$\epsilon_L = 1 - \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{f_0'(v)}{v}\, dv ,$$

$$\delta\epsilon = 2\left(\frac{\omega_p}{k}\right)^2 v_r f_0''(0) \left[\frac{4}{\pi}\left(1 - \frac{\sin kv_U t}{kv_U t}\right) - \frac{a^3}{4} J_3(a)(1-\cos 2\omega_B t)\right]$$

$$+ i\pi \left(\frac{\omega_p}{k}\right)^2 f_0'(0) \left[1 - \frac{2}{\pi}\,\mathrm{Si}(kv_U t) + \frac{2a^2}{\pi} J_2(a) \sin \omega_B t\right] . \tag{23}$$

We now let $\omega = \omega_0 + \delta\omega + i\gamma$, where ω_0 satisfies the linear dispersion relation ($\epsilon_L = 0$), and expand (22) about ω_0. To first order in $\delta\epsilon$, $\delta\omega$, and γ, we obtain

$$\frac{\delta\omega}{\Omega_0} = -2.55\left(1 - \frac{\sin(1.27\omega_B t)}{1.27\omega_B t}\right) + 0.99\,(1 - \cos 2\omega_B t) ,$$

$$\frac{\gamma}{\gamma_L} = 1 - 0.64\,\mathrm{Si}(1.27\,\omega_B t) + 1.32 \sin \omega_B t , \tag{24}$$

where $\Omega_0 = (\omega_p/k)^2 v_r f_0''(0)\,(\partial\epsilon_L/\partial\omega_0)^{-1}$, the Landau damping coefficient $\gamma_L = -\pi(\omega_p/k)^2 f_0'(0)(\partial\epsilon_L/\partial\omega_0)^{-1}$, and $J_2(4/\pi^{1/2}) = 0.41$ and $J_3(4/\pi^{1/2}) = 0.17$ have been substituted. The first bracketed terms for $\delta\omega$, and the first two terms for γ, are the untrapped electron contributions; the remaining terms are the trapped electron contributions.

The frequency shift and damping coefficient of (24) are shown in figure 2. They exhibit the essential features of experimental observations (Malmberg and Wharton 1967; Vidmar, Malmberg and Starke 1975): $\delta\omega/\Omega_0$ and γ/γ_L decrease within or near $\omega_B t = \pi$; oscillatory behavior follows with periods of about $2\omega_B t$ for $\delta\omega/\Omega_0$ and $\omega_B t$ for γ/γ_L. The time-asymptotic value, $\delta\omega/\Omega_0 = -1.56$ is in reasonable agreement with the more accurate value of $\delta\omega/\Omega_0 = -1.63$ obtained by Morales and O'Neil (1972).

To gain insight into the relative importance of trapped and untrapped electron contributions, they are shown separately in figure 2, as well as combined. The sinusoidal behavior of the former is well understood from energy and momentum considerations for the trapped electrons and the wave (Morales and O'Neil 1972). The behavior of the latter has not been clarified previously: in the linear theory, the constant velocity orbit is assumed for the entire region of velocity. In the present case, of a large-amplitude wave, it can be assumed only for $|v| > v_U$. The Poisson equation therefore assumes the form of a finite Fourier transform [see (13)]. This contributes the $(1 - \sin x/x)$ and $\mathrm{Si}(x)$ [$x = 1.27\,\omega_B t$] terms in (24). The initial increase of $\delta\omega/\Omega_0$ and γ/γ_L of the trapped electron part is thus offset by the contribution from the untrapped electrons.

It should be noted that, whereas Morales and O'Neil show $\delta\omega/\Omega_0 = 0$ and $\gamma/\gamma_L = 1$ for some time before they decrease, figure 2 shows $\delta\omega/\Omega_0 > 0$ and $\gamma/\gamma_L > 1$ for $\omega_B t \lesssim 0.6\pi$. This difference is mainly due to our approximation that all trapped electrons have the same bounce frequency, and hence exchange energy and momentum with the wave coherently. The actual bounce frequency is, of course, lower for electrons closer to the separatrix. If we include this different bounce frequency, the first peaks of $\delta\omega/\Omega_0$ and γ/γ_L will be reduced; the amplitudes of their oscillations will be decreased; the maxima and minima will be delayed,

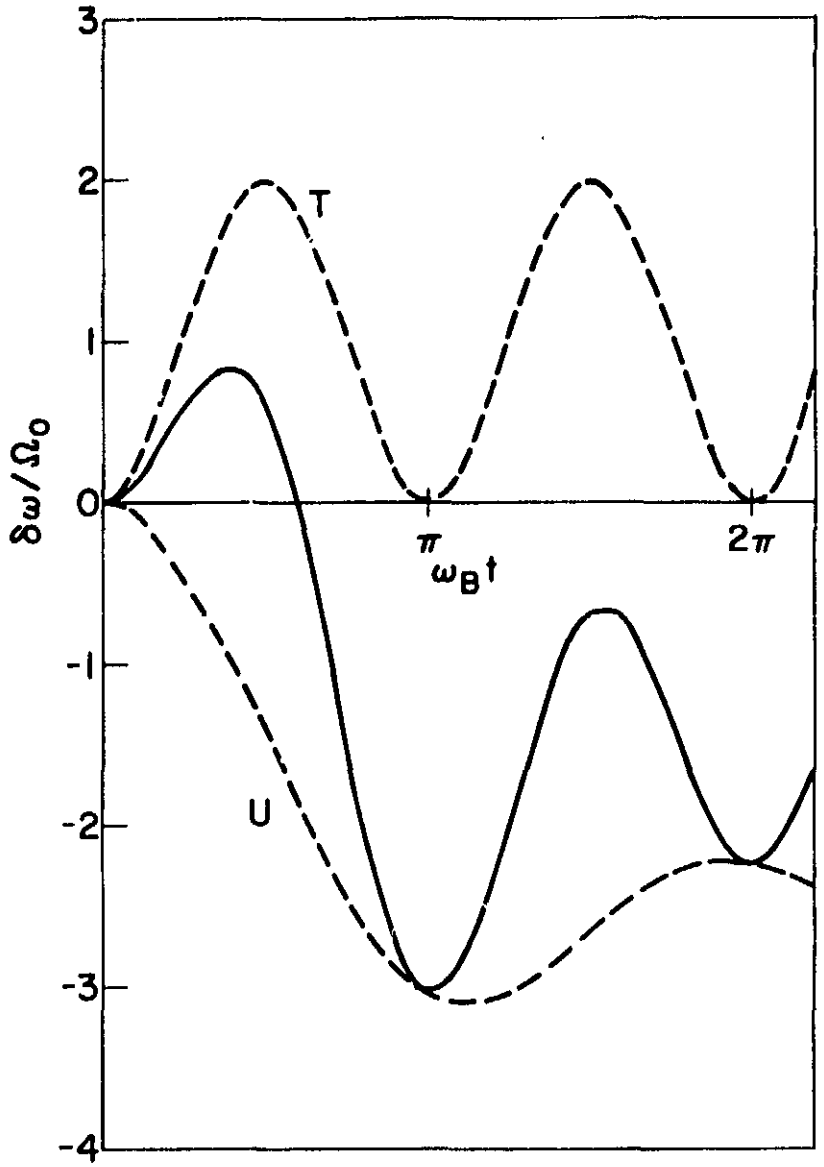

FIG. 2(a). Frequency shift in the lowest order approximation. Dashed lines are contributions of untrapped (U) and trapped (T) electrons.

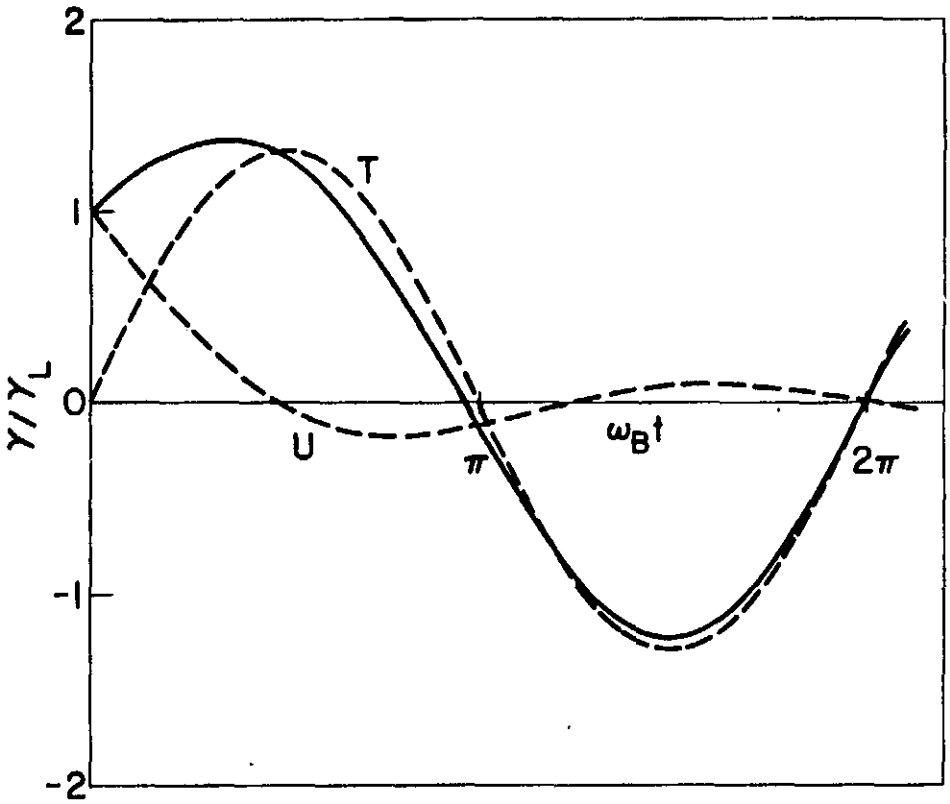

FIG. 2(b). Damping rate in the lowest order approximation. Dashed lines are contributions of untrapped (U) and trapped (T) electrons.

and the oscillations will eventually decay away, as demonstrated by Morales and O'Neil (1974). To include these effects, we shall take account of the difference between the actual and the lowest order orbit, as a perturbation effect, in succeeding sections.

3. ORBIT PERTURBATION

So far, we have considered the lowest order orbits: constant velocity for untrapped electrons, and circular orbits for trapped electrons. Under the influence of the wave, however, an untrapped electron changes its velocity as it travels. Similarly, a trapped electron changes its radius and the angular frequency because of the difference between the parabolic and actual sinusoidal wave potential. We shall assume the time-scales of these changes to be large compared to the time for an untrapped electron to traverse one wavelength, or for a trapped electron to complete one cycle of its trapped orbit.

We write the Vlasov equation in operator form,

$$\frac{\partial f(\underset{\sim}{x},t)}{\partial t} + Lf(\underset{\sim}{x},t) = 0 , \qquad L = g_i(\underset{\sim}{x},t)\,\frac{\partial}{\partial x_i} \tag{25}$$

where $\underset{\sim}{x}$ represents both spatial and velocity coordinates, $\underset{\sim}{g}$ is the generalized velocity, and the summation convention on the repeated index is understood. We divide f into a slowly-varying part, $\langle f\rangle$, and a rapidly-varying part, f_1, where $\langle\ \rangle$ denotes a time average over the shorter of the two time-scales. We assume $\langle g\rangle = 0$. [If this is not the case, we need only consider $\underset{\sim}{x}$ to be a coordinate moving with $\langle g\rangle$, and $\langle \underset{\sim}{g}\rangle$ then disappears from (25)]. Taking the time average of (25) gives the secular part. Subtracting this from (25) gives the rapidly-varying part,

$$\frac{\partial\langle f\rangle}{\partial t} + \langle Lf_1\rangle = 0 , \qquad \frac{\partial f_1}{\partial t} + (I-A)Lf_1 + L\langle f\rangle = 0 , \tag{26}$$

where I and A are identity and averaging operators. These equations are identical to those used in the renormalization procedure of plasma turbulence (Dupree 1966; Orszag and Kraichman 1967; Frisch 1968; Weinstock 1969; Rudakov and Tsytovich 1971; Tsytovich 1972; Benford and Thomson 1972; Misguich and Balescu 1975).

Following Tsytovich (1972), we introduce an average collision operator, ν, such that

$$\nu\langle f\rangle \equiv -\langle Lf_1\rangle \ , \tag{27}$$

and rewrite (26) as

$$\frac{\partial\langle f\rangle}{\partial t} = \nu\langle f\rangle \ . \tag{28}$$

This expression is then integrated to obtain

$$\langle f\rangle = \langle f(0)\rangle + \int_0^t dt_1 \ \nu(t_1)\langle f(t_1)\rangle$$

$$= \Bigg(1 + \int_0^t dt_1 \ \nu(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 \ \nu(t_1)\ \nu(t_2) + \ldots$$

$$+ \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} \nu(t_1)\nu(t_2\ldots\nu(t_n) + \ldots\Bigg) \quad \langle f(0)\rangle \ . \tag{29}$$

Unless indicated otherwise, throughout this section the argument of f is time. Introducing a time-ordered exponential function defined by (Kubo 1962)

$$\exp_0 \int_0^t dt_1 \nu(t_1) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n O[\nu(t_1)\ldots\nu(t_n)]$$

$$= \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \ \nu(t_1)\ldots\nu(t_n) \ , \tag{30}$$

we may cast (29) in the compact form

$$\langle f(t)\rangle = U(t,0)\ \langle f(0)\rangle \ , \tag{31}$$

where the average propagator, U, is given by

$$U(t,t_0) = \exp_0 \int_{t_0}^{t} dt_1\, \nu(t_1) , \qquad (32)$$

the operator 0 denotes the chronological ordering in ν's, and U satisfies the group property $U(t,t_0) = U(t,t_1)\, U(t_1,t_0)$ (Weinstock 1969).

For the rapidly-varying part, f_1, we may use the quasilinear form of (26) (Benford and Thomson 1972; Misguich and Balescu 1975)

$$\frac{\partial f_1}{\partial t} - \nu f_1 + L\langle f\rangle = 0 . \qquad (33)$$

The solution is then

$$f_1(t) = -\int_0^t dt_1\, U(t,t_1)\, L(t_1)\, U(t_1,0)\, \langle f(0)\rangle , \qquad (34)$$

where we have assumed the initial condition $f_1(0) = 0$. Substituting this into (27) gives

$$\nu(t) = \int_0^t dt_1\, \langle L(t)\, U(t,t_1)\, L(t_1)\rangle\, U(t_1,t) . \qquad (35)$$

The effect of the modified orbit on f_1 is contained in U, which can be found by iterating (32) and (35). If $U = I$, (34) reduces to the result for the lowest order orbit.

4. UNTRAPPED ELECTRON CONTRIBUTION TO ELECTRIC FIELD

In a coordinate system moving with the lowest order straight line orbit (y,w), defined by

$$y = x-vt , \qquad w = v , \tag{36}$$

the Vlasov equation (1) takes the form of (25) with

$$L(y,w,t) = \frac{e}{m} E_0 \sin k(y+wt) \left(t \frac{\partial}{\partial y} - \frac{\partial}{\partial w}\right) . \tag{37}$$

To carry out the iteration of (32) and (35), we first let $U = I$. Noting that $\sin k(y+wt)$ and $(t\partial/\partial y - \partial/\partial w)$ commute, we can write (35) as

$$\nu = \left(\frac{e}{m} E_0\right)^2 \left(t \frac{\partial}{\partial y} - \frac{\partial}{\partial w}\right) \int_0^t dt_1 \langle \sin k(y+wt) \sin k(y+wt_1)\rangle \left(t_1 \frac{\partial}{\partial y} - \frac{\partial}{\partial w}\right)$$

$$= \frac{1}{2} \left(\frac{e}{m} E_0\right)^2 \left(t \frac{\partial}{\partial y} - \frac{\partial}{\partial w}\right) \int_0^t d\tau \cos kw\tau \left((t-\tau) \frac{\partial}{\partial y} - \frac{\partial}{\partial w}\right) . \tag{38}$$

Since the time-scale of slow variations is assumed to be much longer than $1/kw$, the integrand contributes only for a very short time. The upper limit of the time integration may therefore be taken as $t \to \infty$. Noting that

$$\int_0^\infty dt \cos kwt = \pi\delta(kw) = 0 , \quad \int_0^\infty dt \sin kwt = \frac{1}{kw} , \tag{39}$$

which follow from the fact that $kw \neq 0$ for untrapped electrons, we can reduce (38) to

$$\nu = \frac{1}{2} \left(\frac{e}{m} E_0\right)^2 \left(\frac{1}{kw}\right)^2 \left(t \frac{\partial^2}{\partial y^2} - (kw)^2 \frac{\partial}{\partial w} \frac{1}{(kw)^2} \frac{\partial}{\partial y}\right) . \tag{40}$$

Next, we may obtain $U(t,t_0)$ from (32) as

$$U(t,t_0) = \exp\left\{\frac{1}{2}\left(\frac{e}{m}E_0\right)^2\left(\frac{1}{kw}\right)^2\left[\frac{t^2-t_0^2}{2}\frac{\partial^2}{\partial y^2} - (t-t_0)(kw)^2\frac{\partial}{\partial w}\left(\frac{1}{kw}\right)^2\frac{\partial}{\partial y}\right]\right\}, \tag{41}$$

where the time-ordering subscript, 0 , has been dropped because the exponent no longer involves the time-integration. We now substitute (41) in (34) to obtain f_1 . For the $U(t,0)$ operating on $\langle f(0)\rangle$, we first note that, since $\langle f(0)\rangle$ is homogeneous, $\partial/\partial y = 0$. Therefore, $U(t_1,0) = I$. For the same reason, $L(t_1)$ becomes $-(e E_0/m) \sin k(y+wt_1)(\partial/\partial w)$. In the first U , we have $\partial/\partial y = \pm ikw$. Transforming back to the (x,v)-coordinate consequently reduces (34) to

$$f_1(x,v,t) = \frac{i}{2}\left(\frac{eE_0}{m}\right)\exp(-ikx)$$

$$\times\int_0^t d\tau \exp\left\{\left(\frac{eE_0\tau}{2mv}\right)^2\left[1 - \frac{i2}{kv\tau}\left(2 - v\frac{\partial}{\partial v}\right)\right]\right\}\exp(ikv\tau)f_0'(v) + c.c. \tag{42}$$

Since we assume that the time for an untrapped electron to traverse one wavelength, $T \approx 1/kv$, is much shorter than other variations, we may put $\tau >> T$ in (42) . Therefore, we neglect the second term in the square bracket in (42). The first exponential function in the integrand then takes the form $\exp(A+B)$, which can be expanded as (Misguich and Balescu 1974; Louisell 1964)

$$\exp(A+B) = \exp(A)\exp(B)\exp(-[A,B]/2)\ \ldots\ , \tag{43}$$

where $[A,B] \equiv AB - BA$. Since $[A,B] = O(E_0^4)$, and hence negligible, we may factorize the exponential function. Since $kv\tau >> 1$, the most significant contribution to $\partial/\partial v$ comes from operating on $\exp(ikv\tau)$. Therefore, we replace $\partial/\partial v$ by $ik\tau$. The Fourier transform of (42) then becomes

$$f_k(v,t) = \frac{i}{2}\left(\frac{eE_0}{m}\right) f_0'(v) \int_0^t d\tau \exp\left[-\frac{1}{4}\left(\frac{\omega_B \tau}{v/v_r}\right)^2 + i\left(\frac{v}{v_r}\right)\omega_B \tau\right]. \tag{44}$$

Since the first term in the exponent gives only a slow variation, the integration yields approximately

$$f_k(v,t) = \frac{i}{2}\left(\frac{eE_0}{m}\right) t\, f_0'(v) \left\{ \frac{\exp\left[\left(-\frac{\omega_B t}{2v/v_r}\right)^2 + i\left(\frac{v}{v_r}\right)\omega_B t\right]}{-\left(\frac{\omega_B t}{2v/v_r}\right)^2 + i\left(\frac{v}{v_r}\right)\omega_B t} - \frac{1}{i\left(\frac{v}{v_r}\right)\omega_B t} \right\} \tag{45}$$

Note that, in the limit $\omega_B t/(v/v_r) \sim O(E_0^2) \to 0$, we recover from (45) the linear result of (14). For simplicity, we shall neglect this small term in the denominator. Since we see that, when (45) is substituted in (13), the dominant contribution of the exponential term is from the small-v region, we shall write v_U for v in the first term of the exponent in (45). With these considerations, (16) is finally modified to

$$E_U = E_0\left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{f_0'(v)}{v}\, dv$$

$$- \frac{8}{\pi} E_0 \left(\frac{\omega_p}{k}\right)^2 v_r f_0''(v) \left(1 - \exp\left[-\left(\frac{\pi\omega_B t}{8}\right)^2\right] \frac{\sin(4\omega_B t/\pi)}{4\omega_B t/\pi}\right)$$

$$- i\pi E_0 \left(\frac{\omega_p}{k}\right)^2 f_0'(0) \exp\left[-\left(\frac{\pi\omega_B t}{8}\right)^2\right] \left(1 - \frac{2}{\pi}\,\mathrm{Si}\left(\frac{4\omega_B t}{\pi}\right)\right). \tag{46}$$

5. TRAPPED ELECTRON CONTRIBUTION TO ELECTRIC FIELD

In order to cast the Vlasov equation for trapped electrons, (9), in the form of (25), we average the right-hand side of (9) with respect to ψ, over the range $(-\pi, \pi)$, and subtract this averaged part from both sides. We then obtain

$$\frac{\partial F}{\partial t} + \omega_{B1} \frac{\partial F}{\partial \psi} + LF = 0 , \qquad L = \frac{\omega_B}{r} \left(R \frac{\partial}{\partial r} + S \frac{\partial}{\partial \psi} \right) , \tag{47}$$

where the functions R and S are defined by,

$$R(r,\psi) = \frac{r^2}{2} \left(\sin 2\psi - \frac{2}{r} \cos \psi \sin (r \sin \psi) \right) ,$$

$$S(r,\psi) = \frac{r}{2} \left(\cos 2\psi + \frac{2}{r} \sin \psi \sin (r \sin \psi) - \frac{2J_1(r)}{r} \right) , \tag{48}$$

and ω_{B1} is given by

$$\omega_{B1} = \omega_B (1+\delta_1) , \qquad \delta_1 = -\frac{1}{2} \left(1 - \frac{2J_1(r)}{r} \right) . \tag{49}$$

Trapped electrons traveling according to

$$\dot{\psi} = \omega_{B1} , \quad \dot{r} = 0 ; \quad \psi = \omega_{B1} t + \psi_0 , \quad r = r_0 , \tag{50}$$

feel the rapidly-varying force exerted by the L-term in (47). To find the orbit correction due to this force, we shall assume $\omega_{B1} = \omega_B$ in (50) for simplicity. In terms of the coordinates (r_0, ψ_0), (47) can then be reduced to the form of (25), with L given by

$$L = \frac{\omega_B}{r_0} \left(R \frac{\partial}{\partial r_0} + S \frac{\partial}{\partial \psi_0} \right) = \frac{\omega_B}{r_0} \left(\frac{\partial}{\partial r_0} R + \frac{\partial}{\partial \psi_0} S \right) , \tag{51}$$

where R and S are given by (48) with (r,ψ) replaced by (r_0, ψ_0) according to (50).

Substituting (51) and U 1 in (5), and taking the time average over one bounce period, $2\pi/\omega_B$, we obtain

$$\nu = \int_0^\infty d\tau \langle L(t)\ L(t-\tau)\rangle = -\ \omega_B \delta_2 \frac{\partial}{\partial \psi_0} ,$$

$$\delta_2 = \frac{1}{4r_0} \frac{\partial}{\partial r_0}\left[\frac{r_0^2}{4} - r_0 J_1(r_0) + \frac{4}{r_0} J_0(r_0) - \frac{2}{r_0} J_1(2r_0)\right] . \quad (52)$$

Transforming (r_0,ψ_0) to (r,ψ) , (47) now becomes

$$\frac{\partial F}{\partial t} + \Omega_B \frac{\partial F}{\partial \psi} = -(L + \nu)F , \quad (53)$$

where $\Omega_B = \omega_B (1 + \delta_1 + \delta_2)$ and δ_2 is given by (52) with r_0 replaced by r .

The dominant orbit correction is obtained by neglecting the right-hand side of (53), yielding the equation

$$\frac{\partial F_0}{\partial t} + \Omega_B \frac{\partial F_0}{\partial \psi} = 0 , \quad (54)$$

whose solution is

$$F_0(r,\psi,t) = f_0\left[rv_r \cos(\psi - \Omega_B t)\right] . \quad (55)$$

Substituting this in (18) gives

$$F_{k0}(r,t) = iv_r f_0'(v)\ rJ_1(r) \sin \Omega_B t - v_r^2 f_0''(0) \left(\frac{r^2}{4}\right) J_2(r)\ (1-\cos 2\Omega_B t) . \quad (56)$$

For the quasilinear correction, we write (53) as

$$\frac{\partial F_1}{\partial t} + \Omega_B \frac{\partial F_1}{\partial t} = -LF_0 . \quad (57)$$

Assuming $\partial\Omega_B/\partial r \approx 0$, for simplicity, we obtain

$$F_1(r,\psi,t) = - v_r \int_0^t d\tau \left\{\frac{1}{r} R(r,\psi-\Omega_B\tau)\cos(\psi-\Omega_B t) - S(r,\psi-\Omega_B\tau)\sin(\psi-\Omega_B t)\right\}$$

$$\times f_0'\left[rv_r \cos(\psi-\Omega_B\tau)\right] . \qquad (58)$$

Substituting this in (18), and using (11), (48),and integration formulas, (Gradshteyn and Ryzhik 1965), gives

$$F_{kl}(r,t) = - iv_r f_0'(0) \left(\frac{\Omega_B}{2\Omega_B}\right) \int_0^{\Omega_B t} d\theta \left\{rJ_1(r)\cos(\Omega_B t-2\theta) + 2J_1^2(r)t - P(r,\theta)\cos(\Omega_B t-\theta)\right\}$$

$$- v_r^2 f_0''(0) \left(\frac{\Omega_B}{4\Omega_B}\right) \int_0^{\Omega_B t} d\theta \left\{r^2 J_0(r) \sin 2\theta - r^2 J_2(r) \sin(2\Omega_B t-\theta)\right.$$

$$\left. - 2rJ_1(r)J_2(r) \sin 2\Omega_B t + 2rQ(r,\theta)\cos(\Omega_B t-\theta)\right\} . \qquad (59)$$

Here, $P(r,\theta)$ and $Q(r,\theta)$ are given by

$$P(r,\theta) = J_0\left(2r \sin\frac{\theta}{2}\right) - J_0\left(2r \cos\frac{\theta}{2}\right),$$

$$Q(r,\theta) = J_1\left(2r \sin\frac{\theta}{2}\right) \cos\left(\Omega_B t - \frac{\theta}{2}\right) - J_1\left(2r \cos\frac{\theta}{2}\right) \sin\left(\Omega_B t - \frac{\theta}{2}\right) . \qquad (60)$$

Using standard Bessel function identities (Abramowitz and Stegun 1965) we may express P and Q in series form as

$$P(r,\theta) = 4\sum_{n=0}^{\infty} J_{2n+1}^2(r) \cos (2n+1)\theta ,$$

$$Q(r,\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(r) \left\{ J_{2(n+1)}(r) \sin\left[\Omega_B t + (2n+1)\theta\right] - J_{2n}(r)\sin\left[\Omega_B t-(2n+1)\theta\right]\right\} . \tag{61}$$

The terms with $n \neq 0$ can be shown to make negligibly small contributions. Retaining $n=0$ terms in (A.5), substituting in (A.3), and performing θ-integrations gives

$$F_{k1}(r,t) = iv_r f_0'(0)\ rJ_1(r)\delta_1 \sin \Omega_B t$$

$$+ v_r^2 f_0''(0) \left(\frac{r^2}{4}\right)\left(J_0(r)-J_2(r)\right)\delta_1 (1-\cos 2\Omega_B t) , \tag{62}$$

where, since F_{k1} is small, we have used $\omega_B \approx \Omega_B$. Adding this to F_{k0} in (56) gives

$$\begin{aligned} F_k(r,t) &= F_{k0} + F_{k1} \\ &= iv_r f_0'(0)\ rJ_1(r)\left[1+\delta_1(r)\right] \sin \Omega_B t \\ &\quad - v_r^2 f''(0) \left(\frac{r^2}{4}\right)\left(J_2(r)\left[1+\delta_1(r)\right] - J_0(r)\right)(1-\cos \Omega_B t) . \end{aligned} \tag{63}$$

The contribution to the electric field due to trapped electrons, E_T , is then obtained from (18) as

$$E_T = - 2iE_0 \left(\frac{\omega_p}{k}\right)^2 f_0'(0)\int_0^a dr\ r^2 J_1(r)\left[1+\delta_1(r)\right] \sin \Omega_B t \tag{64}$$

$$+ \frac{E_0}{2}\left(\frac{\omega_p}{k}\right)^2 v_r f_0''(0)\int_0^a dr\ r^3 \left(J_2(r)\left[1+\delta_1(r)\right] - J_0(r)\delta_1(r)\right) (1-\cos 2\Omega_B t) .$$

6. CONCLUSIONS

The results of §4 and §5 are the equations (46) and (58). Combining them gives the dispersion relation (22) with $\delta\epsilon$ given by

$$\delta\epsilon = \left(\frac{\omega_p}{k}\right)^2 v_r f_0''(0) \left\{ 2.55 \left(1 - \exp\left[-(0.39\,\omega_B t)^2\right] \frac{\sin(1.27\,\omega_B t)}{1.27\,\omega_B t}\right) \right.$$

$$\left. - 0.5 \int_0^a dr\, r^3 \left(J_2(r)\left[1+\delta_1(r)\right] - J_0(r)\delta_1(r)\right)(1-\cos 2\Omega_B t) \right\}$$

$$+ i\pi \left(\frac{\omega_p}{k}\right)^2 f_0'(0) \left\{ \exp\left[-(0.39\,\omega_B t)^2\right] \left(1 - 0.64\ \mathrm{Si}\ (1.27\,\omega_B t)\right) \right.$$

$$\left. + 0.64 \int_0^a dr\, r^2\, J_1(r)\left[1+\delta_1(r)\right] \sin \Omega_B t \right\} . \qquad (64)$$

The frequency shift and damping rate are found by expanding ϵ about $\omega = \omega_0$. We obtain

$$\frac{\delta\omega}{\Omega_0} = -2.55 \left(1 - \exp\left[-(0.39\,\omega_B t)^2\right] \frac{\sin(1.27\,\omega_B t)}{1.27\,\omega_B t}\right)$$

$$+ 0.5 \int_0^a dr\, r^3 \left(J_2(r)\left[1+\delta_1(r)\right] - J_0(r)\delta_1(r)\right)(1-\cos 2\Omega_B t) ,$$

$$\frac{\gamma}{\gamma_L} = \exp\left[-(0.39\,\omega_B t)^2\right] \left(1 - 0.64\ \mathrm{Si}\ (1.27\,\omega_B t)\right)$$

$$+ 0.64 \int_0^a dr\, r^2 J_1(r)\left[1+\delta_1(r)\right] \sin \Omega_B t . \qquad (65)$$

Figure 3 shows $\delta\omega/\Omega_0$ and γ/γ_L calculated from (65), with contributions of trapped and untrapped electrons shown separately and combined. Comparison with the results of Morales and O'Neil (1972) generally shows good agreement. The asymptotic value of $\delta\omega/\Omega_0 = -1.62$ is very close to their result of $\delta\omega/\Omega_0 = -1.63$. The figure indicates that the untrapped electron contributions to $\delta\omega/\Omega_0$ and γ/γ_L change rapidly for $\omega_B t < \pi$, but remain substantially constant for $\omega_B t > \pi$. As was mentioned in §2, the origin of this characteristic can be easily seen by dividing the velocity integration over the region $|v| < v_U$, as shown in (15): the first integral, over the entire velocity region, gives the linear dispersion relation and the Landau damping rate; it is the second integral, over $|v| < v_U$, that gives the time variation of the untrapped parts of $\delta\omega/\Omega_0$ and γ/γ_L. In the work of Morales and O'Neil, the second integral has been combined with that of trapped electrons in (18), and is consequently observed in the subtraction of the linear charge density from that of the actual perturbed density.

The damping rate, γ/γ_L, and frequency shift, $\delta\omega/\Omega_0$, are proportional to $f_0'(0)$ and $f_0''(0)$, respectively. Morales and O'Neil explain this from momentum and energy considerations. If four conjugate points keep their symmetry during motion in phase space about the potential well (which is the case of a wave with constant amplitude and phase velocity), the total momentum change experienced by them derives from the odd-order velocity derivative terms in the Taylor series of (11); the contributions from the even order derivative terms cancel out because the momentum is odd in velocity. On the other hand, since the energy is even in velocity, the total kinetic energy change of these conjugate points derives from the even-order derivative terms. Since we keep only terms up to the second velocity derivative of the distribution function, γ/γ_L and $\delta\omega/\Omega_0$ are consequently proportional to $f_0'(0)$ and $f_0''(0)$. For a larger amplitude wave than has been considered here, it is apparent from the above picture that, because of additional contributions from higher-order terms of the Taylor series, the frequency shift and damping rate can be larger than the present results.

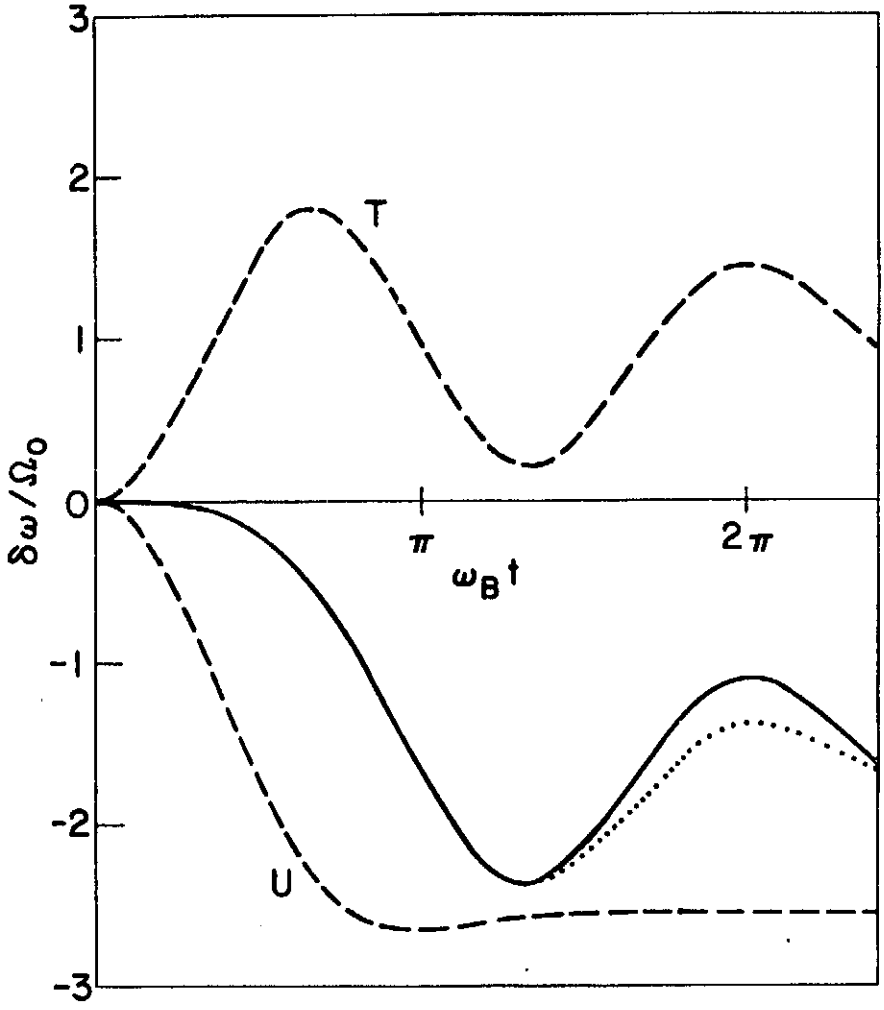

FIG. 3(a). Frequency shift obtained from the orbit perturbation. Dots are results of Morales and O'Neil (1972).

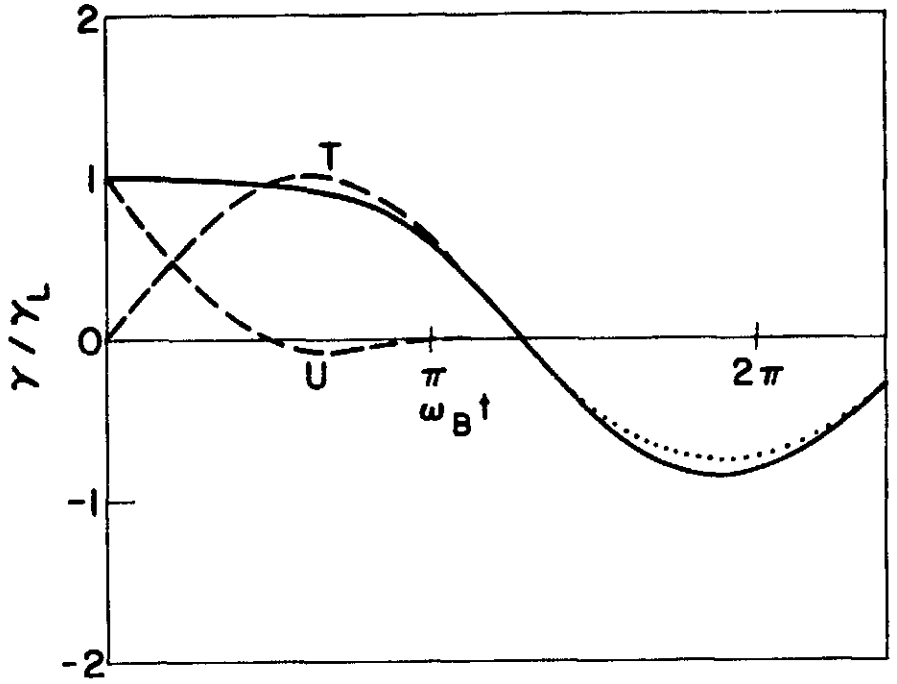

FIG. 3(b). Damping rate obtained from the orbit perturbation. Dots are results of Morales and O'Neil (1972).

Wave amplitude and phase velocity changes may also enhance the frequency shift and damping rate. It should be remembered that the present theory assumes constant wave amplitude and phase velocity, i.e. the momentum and energy gain by those four conjugate points during acceleration is returned to the wave during deceleration. However, when the wave damps initially, some of the trapped electrons which have gained momentum and energy during initial acceleration are detrapped, and a corresponding amount of momentum and energy is not recovered by the wave during the next decelerating period. Similarly, the phase-velocity change resulting from the frequency shift disrupts the symmertrical motion of the four conjugate points. The damping rate and frequency shift are then no longer necessarily functions of only odd and even orders, respectively, of the velocity derivatives of the distribution function.

Although it has been used for a monochromatic wave, the perturbation procedure developed in §3 is identical to that previously used for random waves. This is an important point since the literature developed in §3 contains many analyses specialized to one or other of these two extremes, but there is no easy way to combine them. The method used here effectively links them, and is potentially applicable to such important situations as slight randomness accompanying coherence, and vice versa.

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