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NONLINEAR EQUATIONS OF MOTION FOR CANTILEVER ROTOR BLADES IN HOVER WITH PITCH LINK FLEXIBILITY, TWIST, PRECONE, DROOP, SWEEP, TORQUE OFFSET, AND BLADE ROOT OFFSET

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16. Abstract

Nonlinear equations of motion for a cantilever rotor blade are derived for the hovering flight condition. The blade is assumed to have twist, precone, droop, sweep, torque offset and blade root offset, and the elastic axis and the axes of center of mass, tension, and aerodynamic center coincident at the quarter chord. The blade is cantilevered in bending, but has a torsional root spring to simulate pitch-link flexibility. Aerodynamic forces acting on the blade are derived from strip theory based on quasi-steady two-dimensional airfoil theory. The equations are hybrid, consisting of one integro-differential equation for root torsion and three integro-partial differential equations for flatwise and chordwise bending and elastic torsion. The equations are specialized for a uniform blade and reduced to nonlinear ordinary differential equations by Galerkin's method. They are linearized for small perturbation motions about the equilibrium operating condition. Modal analysis leads to formulation of a standard eigenvalue problem where the elements of the stability matrix depend on the solution of the equilibrium equations. Two different forms of the root torsion equation are derived that yield virtually identical numerical results. This provides a reasonable check for the accuracy of the equations.

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SYMBOLS

A	blade structure cross-section area, m^2
$A_i - Z_{ij}$	modal integrals, equations (77)
$A_x, A_{\dot{x}}, A_{\ddot{x}}, A_{\delta x}, A_{\delta \dot{x}}, A_y, A_{\dot{y}}, A_{\ddot{y}}, A_{\delta y}, A_{\delta \dot{y}}, A_z, A_{\dot{z}}, A_{\ddot{z}}, A_{\delta z}, A_{\delta \dot{z}}$	notation for writing the kinetic energy in a concise form, equation (11)
a	airfoil left curve slope, 2π/rad
b	number of blades
[C]	modal damping matrix, equation (76)
C(k)	Theodorsen's function
$C_{(\ )}$	cosine ( )
c	blade chord, m
c_{d_o}	airfoil profile drag coefficient
D	airfoil profile drag per unit length, equation (33), N/m
E	Young's modulus, N/m^2
e_o	torque offset, figure 2, m
e_1	blade root offset, figure 2, m
$\vec{F}$	vector of external forces per unit length, N/m
G	shear modulus, N/m^2
[G]	modal gyroscopic matrix, equation (84)
h	vertical displacement of two-dimensional airfoil section, normal to free-stream velocity component V, figure 6, m
[I]	identity matrix
$I_{y'}, I_{z'}$	blade structure cross-section area moments at inertia about the y' and z' axes, respectively, m^4

$\vec{i}_r, \vec{j}_r, \vec{k}_r$, $\vec{i}_c, \vec{j}_c, \vec{k}_c$, $\vec{i}_p, \vec{j}_p, \vec{k}_p$, $\vec{i}_s, \vec{j}_s, \vec{k}_s$, $\vec{i}, \vec{j}, \vec{k}$, $\vec{i}', \vec{j}', \vec{k}'$	orthogonal unit vector systems, equations (4)
J	torsional stiffness constant, m^4
$[K]$	modal stiffness matrix, equation (76)
$[K_v]$	modal stiffness matrix for free vibration about the equilibrium operating condition, equation (84)
k_A	blade cross-section polar radius of gyration, m
k_m	blade cross-section mass radius of gyration, m
k_{m_1}, k_{m_2}	principal mass radii of gyration, m
k_ϕ	torsional root spring constant, N-m/rad
L	aerodynamic lift per unit length, equation (29), N/m
L_u, L_v, L_w	generalized aerodynamic forces per unit length, equations (47)-(49), N/m
ℓ	blade length, $R - e_1$, m
M	number of rotating coupled modes; also, aerodynamic pitching moment per unit length, equation (29), N-m/m
$\vec{M}$	vector of external moments per unit length, N-m/m
M_v, M_w	bending moments at blade root due to v and w deflections, respectively, equation (13), N-m
M_ϕ, M_Φ	generalized aerodynamic moments per unit length, equations (50) and (51), N-m/m; also twisting moments at blade root due to ϕ and Φ deflections, respectively, equation (13), N-m
$[M]$	modal mass matrix, equation (76)
$[M_s]$	symmetric part of modal mass matrix, equation (84)
m	blade mass per unit length, kg/m

N	number of nonrotating modes for each of the elastic torsion, flap bending, and lead-lag bending deflections
[P],[P*]	matrices whose eigenvalues determine stability, equations (83) and (88)
Q	dimensionless pitch-link stiffness, equation (63)
R	blade radius, m
$\mathcal{R}$	flap-lag structural coupling parameter
S	aerodynamic force per unit length tangent to blade airfoil chordline, figure 7, N/m
$S_{(\ )}$	sine ( )
T	blade tension, equation (1f), N; also aerodynamic force per unit length normal to blade airfoil chordline, figure 7, N/m
[T]	deformed blade coordinate transformation matrix, equation (8)
$\mathcal{T}$	kinetic energy, equation (9), kg-m^2/sec^2
t	time, sec
U	blade airfoil velocity with respect to the fluid, component normal to spanwise x' axis, equations (31), figure 6, m/sec
[U]	matrix of eigenvectors for free vibration of blade about its equilibrium position, equation (86)
U_P,U_T	velocity components of blade airfoil section with respect to the fluid, parallel to the z' and y' axes, respectively, figure 6, m/sec
u,v,w	displacements of the blade elastic axis parallel to the $\vec{i}$, $\vec{j}$, $\vec{k}$, unit vectors, figure 5, m
V	free-stream velocity component of two-dimensional airfoil, figure 6, m/sec
$\vec{V}$	velocity of a point in the blade, equations (7) and (44), m/sec
V_i,W_i	lead-lag and flap bending generalized coordinates, equations (65)
v_i	induced downwash velocity, equation (52), m/sec
{X}	column vector of modal generalized coordinates, equation (74)
x,y,z	coordinate system for the undeformed blade, parallel to the unit vectors $\vec{i},\vec{j},\vec{k}$, m

x_1, y_1, z_1 coordinates of a generic point on the blade in the x,y,z system, equation (8), m

x', y', z' deformed blade coordinate system along the unit vectors $\vec{i}', \vec{j}', \vec{k}'$, m

Y_i column vector of equilibrium equations, equation (78)

α_i, β_i modal constants, equation (66)

β_d blade droop angle, positive down, figure 2, rad

β_{pc} blade precone angle, positive up, figure 2, rad

β_1 $\beta_{pc} + \zeta_s S_{\theta_o} - \beta_d C_{\theta_o}$, rad

β_2 $\zeta_s S_{2\theta_o} - \beta_d C_{2\theta_o}$, rad

γ $\frac{3\rho_\infty a c \ell}{m}$, Lock number for a blade with uniform mass distribution and no blade root offset ($e_1 = 0$, $\ell = R$). For $e_1 \neq 0$ Lock number $\cong \frac{\gamma(R^4 - e_1{}^4)}{\ell^4}$

γ_i $\pi\left(i - \frac{1}{2}\right)$

δ_{ij} Kronecker delta

$\delta\vec{q}$ vector of virtual displacements, m

δW virtual work of nonconservative forces, N-m

$\delta\vec{\omega}$ vector of virtual rotations, rad

ε small parameter of the order of bending slopes; also, airfoil section pitch angle with respect to free-stream velocity, figure 6, rad

η, ζ blade cross section principal axes coordinates, m

ζ_s blade sweep, figure 2, rad

ζ_1 $\zeta_s C_{\theta_o} + \beta_d S_{\theta_o}$, rad

ζ_2 $\zeta_s C_{2\theta_o} + \beta_d S_{2\theta_o}$, rad

$\Theta_i(\bar{x})$ nonrotating torsional mode shape, equations (66)

θ_{pt} pretwist (built-in twist) angle, rad

θ_t twist parameter, $\theta_{pt} = -\theta_t \bar{x}$, rad

θ_o blade pitch angle at the blade root, figure 4, rad

κ	dimensionless torsional rigidity, equation (63)
λ	warp function, m^2
Λ_1, Λ_2	dimensionless bending stiffnesses, equation (63)
ρ	blade structural density, kg/m^3
ρ_∞	air density, kg/m^3
σ	solidity $\frac{bc}{\pi R}$
τ	dimensionless tension, equation (54)
Φ	root torsion angle, figure 4, rad
Φ_i	torsion generalized coordinate, equations (65)
ϕ	elastic torsion deflection, figure 5, rad
$\Psi_i(\bar{x})$	nonrotating flap and lead-lag bending mode shapes, equation (66)
ψ	dimensionless time, Ωt
Ω	rotor blade angular velocity, rad/sec
$\vec{\omega}$	vector rotation of blade structure at any point on the elastic axis, equation (44), rad/sec
$(\ )'$	$\frac{\partial}{\partial x}$ (or $\frac{\partial}{\partial \bar{x}}$ in dimensionless equations)
$(\dot{\ })$	$\frac{\partial}{\partial t}$ (or $\frac{\partial}{\partial \psi}$ in dimensionless equations)
$(\ )_o, \Delta(\ )$	equilibrium and perturbation components of generalized coordinates
$(\ )_C$	circulatory aerodynamic term
$(\ )_{NC}$	noncirculatory aerodynamic term
$(\bar{\ })$	length quantity made dimensionless by ℓ, or velocity made dimensionless by $\Omega\ell$
$[\ ]^T$	transpose of a square matrix
$\lfloor\ \rfloor^T$	transpose of a row matrix

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SUMMARY

Nonlinear equations of motion for a cantilever rotor blade are derived for the hovering flight condition. The blade is assumed to have twist, precone, droop, sweep, torque offset and blade root offset, and the elastic axis and the axes of center of mass, tension, and aerodynamic center coincident at the quarter chord. The blade is cantilevered in bending, but has a torsional root spring to simulate pitch-link flexibility. Aerodynamic forces acting on the blade are derived from strip theory based on quasi-steady two-dimensional airfoil theory. The equations are hybrid, consisting of one integro-differential equation for root torsion and three integro-partial differential equations for flatwise and chordwise bending and elastic torsion. The equations are specialized for a uniform blade and reduced to nonlinear ordinary differential equations by Galerkin's method. They are linearized for small perturbation motions about the equilibrium operating condition. Modal analysis leads to formulation of a standard eigenvalue problem where the elements of the stability matrix depend on the solution of the equilibrium equations. Two different forms of the root torsion equation are derived that yield virtually identical results. This provides a reasonable check for the accuracy of the equations.

INTRODUCTION

The general problem of helicopter aeroelastic stability involves coupling between the motion of the individual blades and coupling between the rotor and the body of the helicopter. The complexity of the general problem poses a considerable challenge to the analyst, both in developing an analytical model of the system and in understanding its physical behavior. An important part of the general rotor-body dynamic system is the single blade rotating about an axis fixed in space. For many problems of practical interest, blade-to-blade and rotor-body couplings are not significant and the analysis of a single rotor blade constitutes an important problem by itself. Even when coupling with other blades and the body is significant, single blade behavior usually remains recognizable and can be helpful in understanding the behavior of the more complete system. For this reason, the dynamics of a single blade forms an important fundamental building block in the study of helicopter dynamics.

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Helicopter rotors with cantilever blades are commonly termed "hingeless rotors." In contrast with the more conventional articulated rotor, the cantilever blades of the hingeless rotor are attached directly to the hub without flap or lead-lag hinges. This configuration reduces mechanical complexity and improves helicopter flying qualities by increasing rotor control power and angular rate damping. The lack of hinge articulation also alters the structural characteristics of the rotor blade and can significantly influence aeroelastic stability.

Aeroelastic instability is possible because of the structural coupling between bending and torsion deflections of cantilever blades. This type of instability is usually characterized by coupled flap bending, lead-lag bending and torsion deflections with a frequency near the lead-lag bending natural frequency. The structural coupling of cantilever blades is significantly dependent on the specific configuration parameters of the rotor blade, and the magnitude and variability of this coupling make the analysis of cantilever rotor blades an important and complex subject.

A comprehensive study of hingeless rotor stability is a formidable task because of the many important configuration parameters. The cantilever blade structure treated here is shown in figure 1. The elastic blade can be rotated about the pitch change bearing by vertical movement of the pitch link from the swashplate controls. Pitch-link flexibility, represented by a spring element, will permit rigid body pitching motion of the blade (i.e., root torsion). Certain small offsets of the blade axis are often provided to reduce steady blade-bending stresses, to improve rotorcraft flying qualities, or to enhance rotor blade aeroelastic stability. Five of these offsets are considered in this report: precone, droop, sweep, torque offset, and blade root offset. Precone is the inclination of the pitch change bearing with respect to the plane of rotation (positive upward). Droop is an inclination (positive downward at zero pitch angle) of the blade segment outboard of the pitch change bearing. Sweep is a rotation of the blade in the plane of rotation (at zero pitch angle) about the blade root, positive in the direction of blade rotation. Torque offset is a lateral shift of the blade in the plane of rotation (positive in the direction of blade rotation). The blade root offset is the distance between the center of rotation and the root of the blade. These parameters are illustrated in figure 2.

The equations of motion developed in this report are an extension of the previous nonlinear equations of Hodges and Dowell (ref. 1) which treated a general nonuniform, twisted, torsionally elastic cantilever blade. The configuration of reference 1 included precone as well as chordwise offsets of the blade section mass center, tension center and elastic axes. The present configuration does not include offsets between the chordwise axes but does include the additional configuration parameters: droop, sweep, pretwist, torque offset, hub offset, and pitch-link flexibility. For additional background discussion of the aeroelastic stability of hingeless rotor blades, and a discussion of recent pertinent research, the reader is referred to reference 2.

The equations of motion are derived from Hamilton's principle. First, the structural terms are adapted from reference 1; the inertial terms are then formulated from the kinetic energy. Blade aerodynamic loads are developed in a way similar to that in reference 2 using strip theory and quasi-steady two-dimensional airfoil theory. The resulting hybrid equations are then specialized for a uniform blade. The three nonlinear integro-partial differential equations and one integro-differential equation are transformed into $3N + 1$ nonlinear ordinary differential equations by Galerkin's method, where N is the total number of mode shapes for each of the elastic torsion, lead-lag, and flap deflections. When these equations are linearized about the equilibrium operating condition, two sets of equations result. The equilibrium deflection is specified by $3N + 1$ nonlinear algebraic equations and the stability of small perturbation motions about equilibrium is determined by a set of $3N + 1$ homogeneous linear ordinary differential equations with constant coefficients that depend on the solution of the equilibrium equations. Preliminary calculations for stability are presented in reference 3 which treats the effects of twist, pitch-link flexibility, precone, and droop.

Technical discussions with Dr. Robert A. Ormiston and checking of the entire derivation by Dr. Donald L. Kunz are gratefully acknowledged.

DERIVATION OF THE EQUATIONS OF MOTION

In this section we consider a blade for which the structural, inertial, and aerodynamic properties may be treated independently. Here we assume that the cross section structural and inertial properties are doubly symmetric with respect to the blade cross section principal axes. The blade, shown schematically in figure 3, rotates at constant angular velocity Ω about the $\vec{k}_r$ axis fixed in space. The derivation of the equations of motion is fundamentally the same regardless of the assumption of double symmetry of the blade structural cross section. The aerodynamic center of the blade is assumed to be at the quarter chord and coincident with the blade elastic axis. The deformed blade position may be described in terms of several coordinate system rotations, the sequence of which is precone β_{pc}, pitch (θ_o prescribed and $\Phi(t)$ restrained by the pitch-link stiffness k_Φ, as shown in fig. 4), sweep ζ_s, droop β_d, and the blade elastic deformations u, v, w, and ϕ. The blade bending deflection coordinates v and w are defined parallel to and fixed to the blade principal axes at the blade root as pictured in figure 5 (along $\vec{j}$ and $\vec{k}$ in fig. 3). Also shown in figure 5 are the axial deflection u and the blade elastic torsion deformation ϕ. This notation is a slight modification from that of reference 1 where v and w were parallel to the horizontal and vertical planes, respectively. The present scheme is more convenient when assumed modes are used since the vertical and horizontal deflections are functions of precone, droop, sweep, torque offset, and the total root pitch angle ($\theta_o + \Phi$) as well as the blade elastic deflections.

Structural Terms

In this section the structural terms are written for a blade with arbitrary radial distribution of stiffness properties. These terms are taken directly from the final equations of reference 1. The parameters e_A, B_1^*, and C_1^*, defined in reference 1, are set equal to zero because of the double symmetry of the cross section assumed here. These terms were derived from integrating the variation of the strain energy by parts and collecting the coefficients of δu, δv, δw, and $\delta \phi$. For the coordinate system of this report we must let θ in the equations of reference 1 represent the blade pretwist angle θ_{pt}, defined to be zero at the blade root. We also assume that θ_{pt} is a small angle, $O(\varepsilon)$, where ε is of the order of magnitude of the bending slopes. The structural terms for the present configuration, neglecting warping rigidity C_1, are thus

$$\left.\begin{aligned}
&\delta u \text{ term:}\\
&\quad -T'\\
&\delta v \text{ terms:}\\
&\quad -(Tv')' + \{[EI_{z'} - \underline{(EI_{z'} - EI_{y'})(\theta_{pt} + \phi)^2}]v'' + (EI_{z'} - EI_{y'})(\theta_{pt} + \phi)w''\}''\\
&\delta w \text{ terms:}\\
&\quad -(Tw')' + \{(EI_{z'} - EI_{y'})(\theta_{pt} + \phi)v'' + [EI_{y'} + \underline{(EI_{z'} - EI_{y'})(\theta_{pt} + \phi)^2}]w''\}''\\
&\delta \phi \text{ terms:}\\
&\quad -k_A^2[T(\theta_{pt} + \phi)']' - (GJ\phi')' + (EI_{z'} - EI_{y'})[\underline{(w''^2 - v''^2)(\theta_{pt} + \phi)} + v''w'']
\end{aligned}\right\} \tag{1}$$

where $T \equiv EA[u' + (v'^2/2) + (w'^2/2)]$. The underlined terms in equations (1) and below are terms of $O(\varepsilon^3)$ associated with ϕ and $\delta\phi$. These terms are small and are retained only for the sake of completeness. The only structural term in the root torsion equation may be derived from the potential energy of the spring $(1/2)k_\phi \Phi^2$. Thus,

$$\delta\Phi \text{ term:} \qquad k_\phi \Phi \tag{2}$$

The boundary conditions, strictly from structural considerations are

$$\left.\begin{aligned}
x = 0:&\quad v = w = v' = w' = \phi = 0\\
x = \ell:&\quad v'' = w'' = v''' = w''' = \phi' = 0
\end{aligned}\right\} \tag{3}$$

Another boundary condition at the blade root is that of torsion moment equilibrium. This requires knowledge of the various blade axes systems. The rotations of axes are described fully by the following transformations:

$$\left\{\begin{matrix}\vec{i}_r\\ \vec{j}_r\\ \vec{k}_r\end{matrix}\right\} = \begin{bmatrix} C_{\beta_{pc}} & 0 & -S_{\beta_{pc}}\\ 0 & 1 & 0\\ S_{\beta_{pc}} & 0 & C_{\beta_{pc}}\end{bmatrix}\left\{\begin{matrix}\vec{i}_c\\ \vec{j}_c\\ \vec{k}_c\end{matrix}\right\}; \quad \left\{\begin{matrix}\vec{i}_c\\ \vec{j}_c\\ \vec{k}_c\end{matrix}\right\} = \begin{bmatrix} 1 & 0 & 0\\ 0 & C_{\theta_o+\phi} & -S_{\theta_o+\phi}\\ 0 & S_{\theta_o+\phi} & C_{\theta_o+\phi}\end{bmatrix}\left\{\begin{matrix}\vec{i}_p\\ \vec{j}_p\\ \vec{k}_p\end{matrix}\right\}$$

$$\left\{\begin{matrix}\vec{i}_p\\ \vec{j}_p\\ \vec{k}_p\end{matrix}\right\} = \begin{bmatrix} C_{\zeta_s} & -S_{\zeta_s} & 0\\ S_{\zeta_s} & C_{\zeta_s} & 0\\ 0 & 0 & 1\end{bmatrix}\left\{\begin{matrix}\vec{i}_s\\ \vec{j}_s\\ \vec{k}_s\end{matrix}\right\}; \quad \left\{\begin{matrix}\vec{i}_s\\ \vec{j}_s\\ \vec{k}_s\end{matrix}\right\} = \begin{bmatrix} C_{\beta_d} & 0 & S_{\beta_d}\\ 0 & 1 & 0\\ -S_{\beta_d} & 0 & C_{\beta_d}\end{bmatrix}\left\{\begin{matrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{matrix}\right\} \tag{4}$$

$$\left\{\begin{matrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{matrix}\right\} = [T]^T\left\{\begin{matrix}\vec{i}'\\ \vec{j}'\\ \vec{k}'\end{matrix}\right\}$$

where C and S are the cosine and sine, respectively, of the subscripted angles; [T] is the deformed blade coordinate transformation of reference 1 with $\theta = \theta_{pt}$; and for small angles and deflections the $\vec{i}$ vectors are approximately along the blade, the $\vec{j}$ vectors are approximately horizontal, the $\vec{k}$ vectors are approximately vertical; $\vec{i} \times \vec{j} = \vec{k}$. For determining the boundary conditions at the blade root, we note that the pitch-bearing axis is along $\vec{i}_p$. Thus, $\vec{M} = M_\phi\vec{i}_p$. At the root of the blade, however, $\vec{M} = M_\phi\vec{i} + M_w\vec{j} + M_v\vec{k}$ and this, of course, must be balanced by the spring moment $M_\phi\vec{i}_p$. Here M_ϕ, M_w, and M_v are twisting and bending moments due to ϕ, w, and v deflections, respectively. Thus,

$$\begin{aligned} M_\phi &= M_\phi\vec{i}\cdot\vec{i}_p + M_w\vec{j}\cdot\vec{i}_p + M_v\vec{k}\cdot\vec{i}_p \\ &= M_\phi C_{\zeta_s}C_{\beta_d} - M_w S_{\zeta_s} + M_v C_{\zeta_s}S_{\beta_d} \\ &\cong M_\phi - \zeta_s M_w + \beta_d M_v \end{aligned} \tag{5}$$

Therefore, in terms of deformation quantities

$$k_\phi\phi \cong GJ\phi'(0) + T(0)k_A^2[\theta_{pt}'(0) + \phi'(0)] + \zeta_s EI_{y'}w''(0) + \beta_d EI_{z'}v''(0) \tag{6}$$

This is the torsion moment equilibrium boundary condition assuming that $\theta_{pt}(0) = 0$ and θ_{pt} is a small angle. Equation (6) may be used as a replacement for the root torsion equation to be developed below. Numerical results are virtually identical regardless of which of the equations are used. All structural quantities in the above equations are defined the same as those of reference 1.

Inertial Terms

In this section the inertial terms are developed for a blade with arbitrary radial distribution of mass properties. For the derivation of the structural terms we were able to utilize those derived previously in reference 1. This is not feasible with the inertial terms because of the many new parameters and the new coordinate system. Thus, we will derive the inertial terms independently from the kinetic energy following the method of reference 1.

The first step is to write down expressions for the velocity of a generic point in the blade. Symbolically, this is quite simple

$$\vec{V} = \Omega\vec{k}_r \times (e_1\vec{i}_r + e_o\vec{j}_r + x_1\vec{i} + y_1\vec{j} + z_1\vec{k})$$

$$+\dot{\phi}\vec{i}_p \times (x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) + \dot{x}_1\vec{i} + \dot{y}_1\vec{j} + \dot{z}_1\vec{k} \tag{7}$$

where x_1, y_1, and z_1 are the coordinates of a generic point in the blade along the $\vec{i}$, $\vec{j}$, and $\vec{k}$ vectors measured from the root of the beam (at the pitch change bearing). Here e_o is the torque offset, e_1 is the blade root offset, and the remaining unit vectors are given in equations (4). The first line of equation (7) is the contribution from blade rotation. The first term on the second line is the contribution from root torsion. The final three terms are the contributions from blade deformation rates. Expressions for x_1, y_1, and z_1 may be obtained from the deformed blade coordinate transformation [T] (ref. 1).

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} x+u \\ v \\ w \end{Bmatrix} + [T]^T \begin{Bmatrix} -\lambda(\theta_{pt}+\phi)' \\ \eta \\ \zeta \end{Bmatrix} \tag{8}$$

where x is the axial coordinate measured from the root of the undeformed beam to the tip; u, v, w, and ϕ are elastic displacements; λ is the warp function ($\lambda = 0$ at the elastic axis); and η and ζ are the beam cross-section principal axes centered at the elastic axis. The variation of the system kinetic energy is then

$$\delta\mathscr{T} = \int_0^{\ell} \iint_A \rho\vec{V} \cdot \delta\vec{V}\, d\eta\, d\zeta\, dx \tag{9}$$

We now write the velocity in the $\vec{i}_r$, $\vec{j}_r$, $\vec{k}_r$ system

$$\vec{V} = [-\Omega e_o - (\Omega + S_{\beta_{pc}}\dot{\phi})A_y + C_{\beta_{pc}}A_{\dot{x}} - S_{\beta_{pc}}A_{\dot{z}}]\vec{i}_r$$

$$+[\Omega e_1 + \Omega C_{\beta_{pc}}A_x - (\Omega S_{\beta_{pc}} + \dot{\phi})A_z + A_{\dot{y}}]\vec{j}_r$$

$$+[C_{\beta_{pc}}(A_{\dot{z}} + \dot{\phi}A_y) + S_{\beta_{pc}}A_{\dot{x}}]\vec{k}_r \tag{10}$$

where

$$
\left.\begin{aligned}
A_x &= x_1 C_{\zeta_s} C_{\beta_d} - y_1 S_{\zeta_s} + z_1 C_{\zeta_s} S_{\beta_d} \\
A_y &= x_1\left(C_{\theta_o+\Phi} S_{\zeta_s} C_{\beta_d} + S_{\theta_o+\Phi} S_{\beta_d}\right) + y_1 C_{\theta_o+\Phi} C_{\zeta_s} + z_1\left(C_{\theta_o+\Phi} S_{\zeta_s} S_{\beta_d} - S_{\theta_o+\Phi} C_{\beta_d}\right) \\
A_z &= x_1\left(S_{\theta_o+\Phi} S_{\zeta_s} C_{\beta_d} - C_{\theta_o+\Phi} S_{\beta_d}\right) + y_1 S_{\theta_o+\Phi} C_{\zeta_s} + z_1\left(S_{\theta_o+\Phi} S_{\zeta_s} S_{\beta_d} + C_{\theta_o+\Phi} C_{\beta_d}\right) \\
A_{\dot{x}} &= \dot{x}_1 C_{\zeta_s} C_{\beta_d} - \dot{y}_1 S_{\zeta_s} + \dot{z}_1 C_{\zeta_s} S_{\beta_d} \\
A_{\dot{y}} &= \dot{x}_1\left(C_{\theta_o+\Phi} S_{\zeta_s} C_{\beta_d} + S_{\theta_o+\Phi} S_{\beta_d}\right) + \dot{y}_1 C_{\theta_o+\Phi} C_{\zeta_s} + \dot{z}_1\left(C_{\theta_o+\Phi} S_{\zeta_s} S_{\beta_d} - S_{\theta_o+\Phi} C_{\beta_d}\right) \\
A_{\dot{z}} &= \dot{x}_1\left(S_{\theta_o+\Phi} S_{\zeta_s} C_{\beta_d} - C_{\theta_o+\Phi} S_{\beta_d}\right) + \dot{y}_1 S_{\theta_o+\Phi} C_{\zeta_s} + \dot{z}_1\left(S_{\theta_o+\Phi} S_{\zeta_s} S_{\beta_d} + C_{\theta_o+\Phi} C_{\beta_d}\right) \\
&\vdots
\end{aligned}\right\} \quad (11)
$$

Note that with this short hand notation $\dot{A}_y = A_{\dot{y}} - \dot{\Phi} A_z$; $\delta A_z = A_{\delta z} + A_y \delta\Phi$, etc. Thus, the variation of the velocity is simply

$$
\begin{aligned}
\delta\vec{V} = &\Big[-(\Omega + S_{\beta_{pc}}\dot{\Phi})(A_{\delta y} - A_z\delta\Phi) + C_{\beta_{pc}} A_{\delta\dot{x}} \\
&- S_{\beta_{pc}}\left(A_{\delta\dot{z}} + A_{\dot{y}}\delta\Phi + A_y\delta\dot{\Phi}\right)\Big]\vec{i}_r \\
&+ \Big[\Omega C_{\beta_{pc}} A_{\delta x} - (\Omega S_{\beta_{pc}} + \dot{\Phi})(A_{\delta z} + A_y\delta\Phi) - A_z\delta\dot{\Phi} + A_{\delta\dot{y}} - A_{\dot{z}}\delta\Phi\Big]\vec{j}_r \\
&+ \Big\{C_{\beta_{pc}}\Big[A_y\delta\dot{\Phi} + (A_{\delta y} - A_z\delta\Phi)\dot{\Phi} + A_{\delta\dot{z}} + A_{\dot{y}}\delta\Phi\Big] + S_{\beta_{pc}} A_{\delta\dot{x}}\Big\}\vec{k}_r
\end{aligned} \quad (12)
$$

Substitution of equations (10) and (12) into equation (9) yields (with a change of sign and integration by parts to accommodate Hamilton's Principle)

$$
\begin{aligned}
\delta\mathcal{T} = \int_0^{\ell}\iint_A \rho \Big\{&\Big[\Omega^2 S_{\beta_{pc}} C_{\beta_{pc}} A_z - \Omega^2 e_1 C_{\beta_{pc}} - \Omega^2 C^2_{\beta_{pc}} A_x - 2\Omega C_{\beta_{pc}}(A_{\dot{y}} - \dot{\Phi}A_z) + A_{\ddot{x}}\Big] A_{\delta x} \\
&+ \Big[-\Omega^2(e_o + A_y) + 2\Omega C_{\beta_{pc}} A_{\dot{x}} - 2\Omega S_{\beta_{pc}}(A_{\dot{z}} + \dot{\Phi}A_y) - (\ddot{\Phi}A_z + \dot{\Phi}A_{\dot{z}} + \dot{\Phi}^2 A_y) + A_{\ddot{y}}\Big] A_{\delta y} \\
&+ \Big[\Omega^2 S_{\beta_{pc}} C_{\beta_{pc}} A_x + \Omega^2 e_1 S_{\beta_{pc}} - \Omega^2 S^2_{\beta_{pc}} A_z + 2\Omega S_{\beta_{pc}}(A_{\dot{y}} - \dot{\Phi}A_z) \\
&+ (\ddot{\Phi}A_y + \dot{\Phi}A_{\dot{y}} - \dot{\Phi}^2 A_z) + A_{\ddot{z}}\Big] A_{\delta z} + \Big[\Omega^2 e_o A_z + \Omega^2 S_{\beta_{pc}} C_{\beta_{pc}} A_x A_y + \Omega^2 e_1 S_{\beta_{pc}} A_y \\
&+ \Omega^2 C^2_{\beta_{pc}} A_y A_z + 2\Omega S_{\beta_{pc}}(A_y A_{\dot{y}} + A_z A_{\dot{z}}) - 2\Omega C_{\beta_{pc}} A_z A_{\dot{x}} + 2\dot{\Phi}(A_y A_{\dot{y}} + A_z A_{\dot{z}}) \\
&+ A_y A_{\ddot{z}} - A_z A_{\ddot{y}} + \ddot{\Phi}(A_y^2 + A_z^2)\Big]\delta\Phi\Big\} d\eta\, d\zeta\, dx
\end{aligned} \quad (13)
$$

Equation (13) contains an exact representation of the inertial terms. It is at this point that one must make some assumptions regarding an ordering scheme in order to have a tractable set of equations. The same ordering scheme as that of reference 1 is assumed here, with the addition of β_d, ζ_s, e_o/ℓ, e_1/ℓ, and Φ all being $O(\varepsilon)$. Here ε is a small parameter of the order of magnitude of the bending slopes. We assume that $\delta\mathcal{T}$ is truncated so that terms $O(\varepsilon^2)$ are neglected with respect to unity. This scheme is not perfect and, hence, is not followed rigidly. It does, however, greatly simplify equation (13) and yields a set of nonlinear equations with only the most important nonlinear terms included.

Even though they are very small terms it has been suggested (ref. 4) that higher-order terms should be included in the root torsion equation. We will follow that suggestion for the sake of completeness. Unlike reference 4, however, corresponding terms in the other equations will also be retained to insure that modal mass and stiffness matrices are symmetric and that the modal gyroscopic matrix is antisymmetric (*in vacuo*). The analogous small terms associated with the blade elastic torsion equation, underlined in equation (1), are also retained on this basis - strictly for the benefit of the doubt in both cases.

We first consider terms in equation (13) which multiply δx_1, δy_1, and δz_1, consecutively. Since δx_1, δy_1, and δz_1 contain δu, δv, δw, and $\delta\phi$, the inertial terms from these respective partial differential equations may be recovered. We need all terms up through $O(\varepsilon)$ for δx_1 and up through $O(\varepsilon^2)$ for terms involving Φ. Thus,

δx_1 terms:

$$\int_0^\ell \iint_A -\rho\left[\Omega^2(A_x + e_1) + 2\Omega(A_{\dot{y}} - \dot{\Phi}A_z)\right]d\eta\, d\zeta\, dx \tag{14}$$

We need all terms up through $O(\varepsilon^2)$ for δy_1 and δz_1, and up through $O(\varepsilon^3)$ for terms involving Φ. Thus,

δy_1 terms:

$$\begin{aligned}\int_0^\ell \iint_A \rho\Big\{&\Omega^2\zeta_s(A_x + e_1) + 2\Omega\zeta_s(A_{\dot{y}} - \dot{\Phi}A_z)\\ &+\left[-\Omega^2(e_o + A_y) - 2\Omega\beta_{pc}(A_{\dot{z}} + \dot{\Phi}A_y) + 2\Omega A_{\dot{x}}\right.\\ &\left.-\ddot{\Phi}A_z + A_{\ddot{y}}\right]C_{\theta_o+\Phi} + \left[\Omega^2\beta_{pc}(A_x + e_1) + 2\Omega\beta_{pc}(A_{\dot{y}} - \dot{\Phi}A_z)\right.\\ &\left.+\ddot{\Phi}A_y + A_{\ddot{z}}\right]S_{\theta_o+\Phi}\Big\}d\eta\, d\zeta\, dx\end{aligned} \tag{15}$$

δz_1 terms:

$$\int_0^\ell \iint_A \rho\{-\Omega^2\beta_d(A_x+e_1) - 2\Omega\beta_d(A_{\dot y} - \dot\phi A_z)$$

$$+[\Omega^2(e_o+A_y) + 2\Omega\beta_{pc}(A_{\dot z}+\dot\phi A_y) - 2\Omega A_{\dot x} + \ddot\phi A_z - A_{\ddot y}]S_{\theta_o+\phi}$$

$$+[\Omega^2\beta_{pc}(A_x+e_1) + 2\Omega\beta_{pc}(A_{\dot y} - \dot\phi A_z)$$

$$+\ddot\phi A_y + A_{\ddot z}]C_{\theta_o+\phi}\} d\eta\ d\zeta\ dx \tag{16}$$

The $\delta\Phi$ contribution as it appears in equation (13) is unchanged for this step in the analysis. Rate product terms such as $\dot\phi^2 A_y$ or $\dot\phi A_{\dot z}$ are neglected since they do not contribute in a linearized stability analysis.

Second, we may substitute equations (11) into equations (14)-(16) and in the $\delta\Phi$ part of equation (13) to produce

δx_1 terms:

$$\int_0^\ell \iint_A -\rho[\Omega^2(x+e_1) + 2\Omega(\dot y_1 C_{\theta_o+\phi} - \dot z_1 S_{\theta_o+\phi})$$

$$-2\Omega\dot\phi(y_1 S_{\theta_o} + z_1 C_{\theta_o}) - 2\Omega x\dot\phi(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})] d\eta\ d\zeta\ dx \tag{17}$$

δy_1 terms:

$$\int_0^\ell \iint_A \rho[\Omega^2 e_1(\beta_{pc}S_{\theta_o+\phi} + \zeta_s) - \Omega^2 e_o C_{\theta_o+\phi} + \Omega^2 x(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi})S_{\theta_o+\phi}$$

$$+2\Omega\dot x_1 C_{\theta_o+\phi} - 2\Omega\dot z_1(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi}) - 2\Omega x\zeta_s\beta_1\dot\phi$$

$$-2\Omega\beta_{pc}y_1\dot\phi - 2\Omega\zeta_s\dot\phi(y_1 S_{\theta_o} + z_1 C_{\theta_o}) - \Omega^2 y_1 C^2_{\theta_o+\phi}$$

$$+\Omega^2 z_1 S_{\theta_o+\phi}C_{\theta_o+\phi} - (z_1 - \beta_d x)\ddot\phi + \ddot y_1] d\eta\ d\zeta\ dx \tag{18}$$

δz_1 terms:

$$\int_0^\ell \iint_A \rho[\Omega^2 e_1(\beta_{pc}C_{\theta_o+\Phi} - \beta_d) + \Omega^2 e_o S_{\theta_o+\Phi} + \Omega^2 x(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi})C_{\theta_o+\Phi}$$

$$-2\Omega\dot{x}_1 S_{\theta_o+\Phi} + 2\Omega\dot{y}_1(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi}) + 2\Omega x\beta_d\beta_1\dot{\Phi}$$

$$-2\Omega\beta_{pc}z_1\dot{\Phi} + 2\Omega\beta_d\dot{\Phi}(y_1 S_{\theta_o} + z_1 C_{\theta_o}) + \Omega^2 y_1 S_{\theta_o+\Phi}C_{\theta_o+\Phi}$$

$$-\Omega^2 z_1 S^2_{\theta_o+\Phi} + (y_1 + \zeta_s x)\ddot{\Phi} + \ddot{z}_1]d\eta\ d\zeta\ dx \tag{19}$$

$\delta\Phi$ terms:

$$\int_0^\ell \iint_A \rho[\Omega^2 x e_o(\zeta_s S_{\theta_o+\Phi} + \beta_d C_{\theta_o+\Phi}) + \Omega^2 e_o(y_1 S_{\theta_o+\Phi} + z_1 C_{\theta_o+\Phi})$$

$$+\Omega^2 x^2(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi})(\zeta_s C_{\theta_o+\Phi} + \beta_d S_{\theta_o+\Phi}) + \Omega^2 x e_1\beta_{pc}\zeta_1$$

$$+\Omega^2 x(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi})(y_1 C_{\theta_o+\Phi} - z_1 S_{\theta_o+\Phi}) + \Omega^2 e_1\beta_{pc}(y_1 C_{\theta_o} - z_1 S_{\theta_o})$$

$$+\Omega^2 x(\zeta_s C_{\theta_o+\Phi} + \beta_d S_{\theta_o+\Phi})(y_1 S_{\theta_o+\Phi} + z_1 C_{\theta_o+\Phi}) + \Omega^2(y_1{}^2 - z_1{}^2)S_{\theta_o+\Phi}C_{\theta_o+\Phi}$$

$$+\Omega^2 y_1 z_1 C_{2(\theta_o+\Phi)} + 2\Omega\beta_{pc}(y_1\dot{y}_1 + z_1\dot{z}_1) + 2\Omega x\beta_1(\zeta_s\dot{y}_1 - \beta_d\dot{z}_1)$$

$$-2\Omega x(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})\dot{x}_1 - 2\Omega(y_1 S_{\theta_o} + z_1 C_{\theta_o})\dot{x}_1$$

$$+2\Omega(y_1 S_{\theta_o} + z_1 C_{\theta_o})(\zeta_s\dot{y}_1 - \beta_d\dot{z}_1) + (y_1 + \zeta_s x)\ddot{z}_1 - (z_1 - \beta_d x)\ddot{y}_1$$

$$+x^2\ddot{\Phi}(\zeta_s{}^2 + \beta_d{}^2) + 2x\ddot{\Phi}(\zeta_s y_1 - \beta_d z_1) + (y_1{}^2 + z_1{}^2)\ddot{\Phi}]d\eta\ d\zeta\ dx \tag{20}$$

where $\beta_1 = \beta_{pc} + \zeta_s S_{\theta_o} - \beta_d C_{\theta_o}$; $\zeta_1 = \zeta_s C_{\theta_o} + \beta_d S_{\theta_o}$. In equations (17)-(20), the expressions have been truncated to the appropriate order of magnitude as prescribed by the ordering scheme discussed above. Finally we must write explicit expressions for x_1, y_1, and z_1 and use these expressions to formulate the inertial terms explicitly in terms of u, v, w, ϕ, and Φ. From reference 1 and equation (8)

$$x_1 \cong x + u - \lambda(\theta_{pt} + \phi)' - \eta[v' \cos(\theta_{pt} + \phi) + w' \sin(\theta_{pt} + \phi)]$$

$$-\zeta[-v' \sin(\theta_{pt} + \phi) + w' \cos(\theta_{pt} + \phi)]$$

$$y_1 \cong v + \eta \cos(\theta_{pt} + \phi) - \zeta \sin(\theta_{pt} + \phi)$$

$$z_1 \cong w + \eta \sin(\theta_{pt} + \phi) + \zeta \cos(\theta_{pt} + \phi) \tag{21}$$

The δu terms may be formed directly from the δx_1 terms

$$\delta u \text{ terms:} \quad -m\Omega^2(x + e_1) + 2m\Omega x\dot{\phi}(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})$$

$$-2m\Omega(\dot{v}C_{\theta_o+\phi} - \dot{w}S_{\theta_o+\phi}) + 2m\Omega(vS_{\theta_o} + wC_{\theta_o})\dot{\phi} \tag{22}$$

where $m = \iint_A \rho \, d\eta \, d\zeta$ and $\iint_A \rho\eta \, d\eta \, d\zeta = \iint_A \rho\zeta \, d\eta \, d\zeta = 0$ for coincident elastic and mass center axes. The δv and δw terms may arise from both δx_1, δy_1, and δz_1. The terms from δx_1 are small, however, and are underlined below

δv terms:

$$m\Omega^2 e_1(\beta_{pc}S_{\theta_o+\phi} + \zeta_s) - m\Omega^2 e_o C_{\theta_o+\phi} + m\Omega^2 x(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi})S_{\theta_o+\phi}$$

$$-m\Omega^2 vC^2_{\theta_o+\phi} + m\Omega^2 wS_{\theta_o+\phi}C_{\theta_o+\phi} + 2m\Omega\dot{u}C_{\theta_o+\phi} - 2m\Omega\dot{w}(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi})$$

$$-2m\Omega x\zeta_s\beta_1\dot{\phi} - 2m\Omega\beta_{pc}v\dot{\phi} - 2m\Omega\zeta_s(vS_{\theta_o} + wC_{\theta_o})\dot{\phi}$$

$$-m(w - \beta_d x)\ddot{\phi} + m\ddot{v} + \underline{2\Omega S_{\theta_o}(mk^2_{m_2}\dot{\phi})'} + \underline{2\Omega\dot{\phi}S_{\theta_o}(mk^2_{m_2})'} \tag{23}$$

δw terms:

$$m\Omega^2 e_1(\beta_{pc}C_{\theta_o+\phi} - \beta_d) + m\Omega^2 e_o S_{\theta_o+\phi} + m\Omega^2 x(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi})C_{\theta_o+\phi}$$

$$+m\Omega^2 vS_{\theta_o+\phi}C_{\theta_o+\phi} - m\Omega^2 wS^2_{\theta_o+\phi} - 2m\Omega\dot{u}S_{\theta_o+\phi} + 2m\Omega\dot{v}(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi})$$

$$+2m\Omega x\beta_d\beta_1\dot{\phi} - 2m\Omega\beta_{pc}w\dot{\phi} + 2m\Omega\beta_d(vS_{\theta_o} + wC_{\theta_o})\dot{\phi}$$

$$+m(v + \zeta_s x)\ddot{\phi} + m\ddot{w} + \underline{2\Omega C_{\theta_o}(mk^2_{m_1}\dot{\phi})'} + \underline{2\Omega\dot{\phi}C_{\theta_o}(mk^2_{m_1})'} \tag{24}$$

Since the δv and δw terms from δx_1 arise from integration of $\delta\mathcal{T}$ by parts, there are additional terms evaluated at the boundary that do not necessarily satisfy equations (3). These terms are

$$-2m\Omega(\dot{\phi} + \dot{\phi})(k^2_{m_2}S_{\theta_o}\delta v + k^2_{m_1}C_{\theta_o}\delta w)\Big|_o^{\ell} \tag{25}$$

These terms must be included in the analysis when Galerkin's method is applied in the solution of the equations of motion.

The $\delta\phi$ terms arise from both δy_1 and δz_1 since contributions from δx_1 are negligibly small. The contributing terms from δy_1 and δz_1 are

$$\ddot{y}_1\delta y_1 + \ddot{z}_1\delta z_1 + (2\Omega\dot{x}_1 - \Omega^2 y_1 C_{\theta_o+\Phi} + \Omega^2 z_1 S_{\theta_o+\Phi})(\delta y_1 C_{\theta_o+\Phi} - \delta z_1 S_{\theta_o+\Phi}) + (\ddot{y}_1\delta z_1 - \ddot{z}_1\delta y_1)\ddot{\Phi} \tag{26}$$

The $\delta\phi$ terms follow immediately:

$\delta\phi$ terms:

$$\begin{aligned} & m\Omega^2(k_{m_2}^2 - k_{m_1}^2)[S_{\theta_o}C_{\theta_o} + \underline{(\theta_{pt} + \phi + \Phi)C_{2\theta_o}}] \\ & \underline{+2m\Omega(\dot{v}' S_{\theta_o} k_{m_2}^2 + \dot{w}' C_{\theta_o} k_{m_1}^2)} \\ & \underline{+mk_m^2(\ddot{\phi} + \ddot{\Phi})} \end{aligned} \tag{27}$$

Here as above the underlined terms are $0(\varepsilon^3)$. From equation (20) the $\delta\Phi$ terms may be written

$\delta\Phi$ terms:

$$\begin{aligned} \int_0^{\ell} \Big\{ & m\Omega^2 x e_o(\zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi}) + m\Omega^2 e_o(vS_{\theta_o+\Phi} + wC_{\theta_o+\Phi}) \\ & +m\Omega^2 x^2(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi})(\zeta_s C_{\theta_o+\Phi} + \beta_d S_{\theta_o+\Phi}) + m\Omega^2 e_1\beta_{pc}\zeta_1 \\ & +m\Omega^2 x(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi})(vC_{\theta_o+\Phi} - wS_{\theta_o+\Phi}) + m\Omega^2 e_1\beta_{pc}(vC_{\theta_o} - wS_{\theta_o}) \\ & +m\Omega^2 x(\zeta_s C_{\theta_o+\Phi} + \beta_d S_{\theta_o+\Phi})(vS_{\theta_o+\Phi} + wC_{\theta_o+\Phi}) + m\Omega^2(v^2 - w^2)S_{\theta_o+\Phi}C_{\theta_o+\Phi} \\ & +m\Omega^2 vwC_{2(\theta_o+\Phi)} + m\Omega^2(k_{m_2}^2 - k_{m_1}^2)[S_{\theta_o}C_{\theta_o} + (\theta_{pt} + \phi + \Phi)C_{2\theta_o}] \\ & +2m\Omega\beta_{pc}(v\dot{v} + w\dot{w}) + 2m\Omega x\beta_1(\zeta_s\dot{v} - \beta_d\dot{w}) \\ & -2m\Omega x(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})\dot{u} - 2m\Omega(vS_{\theta_o} + wC_{\theta_o})\dot{u} \\ & +2m\Omega(\dot{v}' S_{\theta_o} k_{m_2}^2 + \dot{w}' C_{\theta_o} k_{m_1}^2) + 2m\Omega(vS_{\theta_o} + wC_{\theta_o})(\zeta_s\dot{v} - \beta_d\dot{w}) \\ & +m(v + \zeta_s x)\ddot{w} - m(w - \beta_d x)\ddot{v} + mk_m^2(\ddot{\phi} + \ddot{\Phi}) \\ & +mx^2(\zeta_s{}^2 + \beta_d{}^2)\ddot{\Phi} + 2mx(\zeta_s v - \beta_d w)\ddot{\Phi} + m(v^2 + w^2)\ddot{\Phi}\Big\} dx \end{aligned} \tag{28}$$

The $\delta\Phi$ terms are under an integral because $\delta\Phi$ is not a function of x. (Hamilton's principle results in a partial differential equation only when the δ-quantity is a function of x.)

Aerodynamic Terms

The aerodynamic lift and pitching moment acting on the blade in hover are based on Greenberg's extension of Theodorsen's theory (ref. 5) for a two-dimensional airfoil undergoing sinusoidal motion in pulsating incompressible flow. The rotor blade aerodynamic forces are formulated from strip theory and only the velocity component perpendicular to the blade spanwise axis (the x'-axis in the deformed blade coordinate system x', y', z' in fig. 6) influences the aerodynamic forces. A quasi-steady approximation of the unsteady theory for low reduced frequency k is employed in which the Theodorsen function $C(k)$ is taken to be unity. The steady induced inflow for the rotor is calculated from classical blade-element momentum theory. These simplifying assumptions are judged to be adequate for low frequency (mainly determined by the blade bending frequencies) stability analyses of a hovering rotor.

In Greenberg's theory (ref. 5), a two-dimensional airfoil is assumed to be pivoted about an axis which may be distinct, in general, from the aerodynamic center axis. The airfoil is pitched at an angle $\varepsilon(t)$ to the free stream flowing at pulsating free-stream velocity $V(t)$. The airfoil is vertically displaced with velocity $\dot{h}(t)$ positive downward as shown in figure 6. The relations for lift and pitching moment per unit length may be expressed in terms of the circulatory and noncirculatory components

$$\left.\begin{aligned} L &= L_C + L_{NC} \\ M &= M_C + M_{NC} \end{aligned}\right\} \tag{29}$$

With the airfoil pivot axis (analogous to the rotor blade elastic axis) at the airfoil quarter chord (the airfoil aerodynamic center) these components are

$$\left.\begin{aligned} L_{NC} &= \frac{\rho_\infty ac}{2}\frac{c}{4}\left(\ddot{h} + V\dot{\varepsilon} + \dot{V}\varepsilon + \frac{c\ddot{\varepsilon}}{4}\right) \\ L_C &= \frac{\rho_\infty ac}{2} V\left(\dot{h} + V\varepsilon + \frac{c\dot{\varepsilon}}{2}\right) \\ M_{NC} &= -\frac{\rho_\infty ac}{2}\left(\frac{c}{4}\right)^3 \frac{\ddot{\varepsilon}}{2} - \left(\frac{c}{4}\right) L_{NC} \\ M_C &= -\frac{\rho_\infty ac}{2}\left(\frac{c}{4}\right)^2 V\dot{\varepsilon} \end{aligned}\right\} \tag{30}$$

The Theodorsen function $C(k)$ has been set equal to unity in the circulatory lift. It should be noted that ε is the angular position of the airfoil with respect to space; $\dot{\varepsilon}$ and $\ddot{\varepsilon}$ are the angular velocity and angular acceleration of the airfoil. The instantaneous angle of attack of the airfoil $\alpha = \tan^{-1}(U_P/U_T)$ is the angle between the airfoil chord line and the resultant fluid velocity U of the airfoil. The airfoil velocity components in the y', z' principal axis system are U_T and U_P shown in figure 6. It is desirable to express the aerodynamic forces and moments in terms of U_P and U_T. Assuming that the angles ε and α are small yields

$$\left.\begin{aligned} U_P &\cong -\dot{h} - V\varepsilon \\ U &= \sqrt{U_T^2 + U_P^2} \cong V \end{aligned}\right\} \tag{31}$$

Substitution of $\dot{h}$ and V from equations (31) into equations (30) yields

$$\left.\begin{aligned} L_{NC} &= \frac{\rho_\infty ac}{2} \frac{c}{4} \left(-\dot{U}_P + \frac{c}{4}\ddot{\varepsilon}\right) \\ L_C &= \frac{\rho_\infty ac}{2} U\left(-U_P + \frac{c}{2}\dot{\varepsilon}\right) \end{aligned}\right\} \tag{32}$$

Next we consider the total aerodynamic forces in directions parallel and perpendicular to the airfoil chord line. The noncirculatory lift is taken to act normal to the chord line, and the circulatory lift is taken to act normal to the resultant blade velocity U. An aerodynamic profile drag force per unit length, based on a constant profile drag coefficient c_{d_o} and acting parallel to the resultant blade velocity, is included.

$$D = \frac{\rho_\infty ac}{2} \frac{c_{d_o}}{a} (U_T^2 + U_P^2) \tag{33}$$

The force components and directions are shown in figure 7. The force components T, normal to the airfoil chord line, and S, parallel to the airfoil chord line, are therefore

$$\left.\begin{aligned} T &= L_C \cos\alpha + L_{NC} + D\sin\alpha \\ S &= -L_C \sin\alpha - D\cos\alpha \end{aligned}\right\} \tag{34}$$

From figure 5,

$$\left.\begin{aligned} \cos\alpha &= \frac{U_T}{U} \\ \sin\alpha &= \frac{U_P}{U} \end{aligned}\right\} \tag{35}$$

Substitution of equations (32), (33), and (35) into equations (34), with c_{d_o}/a neglected with respect to unity, yields

$$T = \frac{\rho_\infty ac}{2}\left[-U_P U_T + \frac{c}{2}\ U_T\dot{\varepsilon} - \frac{c}{4}\ \dot{U}_P + \left(\frac{c}{4}\right)^2\ \ddot{\varepsilon}\right]$$

$$S = \frac{\rho_\infty ac}{2}\left(U_P{}^2 - \frac{c}{2}\ U_P\dot{\varepsilon} - \frac{c_{d_o}}{a}\ U_T{}^2\right) \tag{36}$$

The expressions for aerodynamic pitching moment components may be written from equations (30) and (31) as

$$M_{NC} = -\frac{\rho_\infty ac}{2}\left(\frac{c}{4}\right)^2\left(-\dot{U}_P + \frac{3c}{8}\ \ddot{\varepsilon}\right)$$

$$M_C = -\frac{\rho_\infty ac}{2}\left(\frac{c}{4}\right)^2\ U_T\dot{\varepsilon} \tag{37}$$

where U has been approximated by U_T in M_C. The total pitching moment is then given by

$$M = -\frac{\rho_\infty ac}{2}\left(\frac{c}{4}\right)^2\left(U_T\dot{\varepsilon} - \dot{U}_P + \frac{3c}{8}\ \ddot{\varepsilon}\right) \tag{38}$$

The aerodynamic force and moment acting on the blade at a point on the deformed beam elastic axis (coincident with the blade airfoil section aerodynamic center) are

$$\vec{F} = S\vec{j}\,' + T\vec{k}\,' \tag{39}$$

$$\vec{M} = M\vec{i}\,' \tag{40}$$

The virtual displacement and virtual rotation are given, respectively, by

$$\delta\vec{q} = \delta u\vec{i} + \delta v\vec{j} + \delta w\vec{k} + \delta\Phi\vec{i}_p \times [(x+u)\vec{i} + v\vec{j} + w\vec{k}] \tag{41}$$

$$\delta\vec{\omega} = \delta\phi\vec{i}\,' + \delta\Phi\vec{i}_p \tag{42}$$

Thus, the total virtual work of the aerodynamic loads is

$$\delta W = \int_o^{\ell} (\vec{F}\cdot\delta\vec{q} + \vec{M}\cdot\delta\vec{\omega})dx \tag{43}$$

In order to write the aerodynamic terms in each equation we must express equation (43) in terms of u, v, w, ϕ, and Φ. This entails writing U_P, U_T, and $\dot{\varepsilon}$ in terms of u, v, w, ϕ, and Φ. The blade airfoil velocity and rotation are simply

$$\vec{V} = -\Omega e_o\vec{i}_r + \Omega e_1\vec{j}_r + v_i\vec{k}_r + \dot{u}\vec{i} + \dot{v}\vec{j} + \dot{w}\vec{k} + (\Omega\vec{k}_r + \dot{\Phi}\vec{i}_p) \times [(x+u)\vec{i} + v\vec{j} + w\vec{k}]$$

$$\vec{\omega} = \Omega\vec{k}_r + \dot{\Phi}\vec{i}_p + \dot{\phi}\vec{i}\,' \tag{44}$$

where v_i is the induced inflow velocity equivalent to that of reference 2 except that root torsion is included and the pitch angle is not necessarily a small angle. From figure 6

$$\left.\begin{aligned} U_P &= \vec{V} \cdot \vec{k}' \\ U_T &= \vec{V} \cdot \vec{j}' \\ \dot{\varepsilon} &= \vec{\omega} \cdot \vec{i}' \end{aligned}\right\} \tag{45}$$

Without writing all the details, substitution of equations (36), (38)-(42), (44) and (45) into equation (43) yields an expression for δW of the form

$$\delta W = \int_0^{\ell} (L_u\delta u + L_v\delta v + L_w\delta w + M_\phi\delta\phi)dx + \delta\Phi\int_0^{\ell} M_\Phi \, dx \tag{46}$$

where L_u, L_v, and L_w are aerodynamic forces per unit length and M_ϕ and M_Φ are aerodynamic moments per unit length. All these quantities appear as forcing functions on the right hand side of the equations of motion. We assume $\theta_{pt} = O(\varepsilon)$ and $\theta_o = O(\varepsilon^{1/2})$ so that the expressions are not overly complicated by small terms.

$$L_u = -\frac{\rho_\infty ac}{2} \Omega^2 x^2 S_{\theta_o} C_{\theta_o} w' \tag{47}$$

$$\begin{aligned} L_v = \frac{\rho_\infty ac}{2} \Big\langle & \Omega^2(x^2 + 2e_1x)S^2_{\theta_o} + \Omega^2x^2(\theta_{pt} + \phi)S_{\theta_o}C_{\theta_o} \\ & + v_i{}^2C^2_{\theta_o} - \Omega x v_i[S_{2\theta_o} + (\theta_{pt} + \phi)C_{2\theta_o}] - \frac{c_{d_o}}{a} \Omega^2x^2C^2_{\theta_o} \\ & + \left\{\Omega^2x^2S_{2\theta_o} + [\Omega^2x^2(\theta_{pt} + \phi + \Phi) - 2\Omega x v_i]C_{2\theta_o}\right\}\phi \\ & + \left\{(\theta_{pt} + \phi)[\Omega x(S_{\theta_o} + \Phi C_{\theta_o}) - v_iC_{\theta_o}] - 2\frac{c_{d_o}}{a}\Omega x C_{\theta_o}\right\}\dot{v} \\ & + \left\{2v_iC_{\theta_o} - \Omega x[2S_{\theta_o} + (2\Phi + \theta_{pt} + \phi)C_{\theta_o}]\right\}\dot{w}\Big\rangle \end{aligned} \tag{48}$$

$$\begin{aligned} L_w = \frac{\rho_\infty ac}{2} \Big\langle & \Omega^2(x^2 + 2e_1x)C_{\theta_o}[S_{\theta_o} + (\theta_{pt} + \phi)C_{\theta_o}] + \Omega^2x^2C^2_{\theta_o}\int_0^x v'w''\,dx \\ & - \Omega(x + e_1)v_iC_{2\theta_o} - \Omega^2e_ox(\beta_{pc}C_{2\theta_o} - \beta_dC_{\theta_o}) + \Omega^2\frac{xc}{2}(\beta_1 + w'C_{\theta_o})C_{\theta_o} \\ & - \Omega^2e_oxw'C_{\theta_o} - \Omega^2xvw'C^2_{\theta_o} - \Omega^2x\beta_1vC_{\theta_o} - \Omega^2x^2\beta_{pc}\zeta_sC_{\theta_o} + \Omega^2(x^2 + 2e_1x)\Phi C_{2\theta_o} \\ & + \left\{\Omega x[S_{\theta_o} + (\Phi + 2\theta_{pt} + 2\phi)C_{\theta_o}] - v_iC_{\theta_o}\right\}\dot{v} - \Omega(x + e_1)\dot{w}C_{\theta_o} - \frac{c}{4}\ddot{w} \\ & + \frac{3c}{4}\Omega xC_{\theta_o}\dot{\phi} + \left[\frac{3c}{4} - (v + \zeta_sx)\right]\Omega xC_{\theta_o}\dot{\Phi}\Big\rangle \end{aligned} \tag{49}$$

$$\underline{M_\phi = -\frac{\rho_\infty ac}{2}\left(\frac{c}{4}\right)^2 \left[2\Omega x C_{\theta_o}(\dot{\phi}+\dot{\Phi}) + \Omega^2 x C_{\theta_o}(\beta_1 + w' C_{\theta_o}) - \ddot{w}\right]} \tag{50}$$

$$\begin{aligned}
\int_0^\ell M_\phi\, dx = \int_0^\ell \frac{\rho_\infty ac}{2} \Bigg(& -\left(\frac{c}{4}\right)^2 \left[2\Omega x C_{\theta_o}(\dot{\phi}+\dot{\Phi}) + \Omega^2 x(\beta_1 + w' C_{\theta_o})\right] \\
& + \left(\frac{c}{4}\right)\left[\frac{c}{4} - (v + \zeta_s x)\right]\ddot{w} + \Omega^2 (x^2 + 2e_1 x) S_{\theta_o}\left[-S_{\theta_o}(w - \beta_d x) + C_{\theta_o}(v + \zeta_s x)\right] \\
& + \Omega^2 (x^2 + 2e_1 x) C^2_{\theta_o}(\theta_{pt} + \phi)(v + \zeta_s x) - \Omega^2 x^2 (\theta_{pt} + \phi) S_{\theta_o} C_{\theta_o}(w - \beta_d x) \\
& - \left(v_i{}^2 - \Omega^2 x^2 \frac{c_{d_o}}{a}\right) C_{\theta_o}(w - \beta_d x) + \Omega^2 x^2 C^2_{\theta_o}(v + \zeta_s x) \int_0^x v' w''\, dx \\
& - \Omega(x + e_1) v_i C_{2\theta_o}(v + \zeta_s x) + \Omega x v_i \left[S_{2\theta_o} + (\theta_{pt} + \phi) C_{2\theta_o}\right](w - \beta_d x) \\
& - \Omega^2 e_o x(\beta_{pc} C_{2\theta_o} - \beta_d C_{\theta_o})(v + \zeta_s x) + \Omega^2 \frac{xc}{2}(\beta_1 + w' C_{\theta_o}) C_{\theta_o}(v + \zeta_s x) \\
& - \Omega^2 x C_{\theta_o}(e_o w' + v w' C_{\theta_o} + \beta_1 v + x \beta_{pc} \zeta_s)(v + \zeta_s x) + 2\Omega^2 e_1 x \Phi C_{2\theta_o}(v + \zeta_s x) \\
& + \Omega^2 x^2 \Phi\left[(v + \zeta_s x) C_{2\theta_o} - (w - \beta_d x) S_{2\theta_o}\right] - \left[\Omega^2 x^2 (\theta_{pt} + \phi + \Phi)\right. \\
& \left. - 2\Omega x v_i\right] \Phi C_{2\theta_o}(w - \beta_d x) + \dot{v}\Big\langle \Big\{ 2 \frac{c_{d_o}}{a} \Omega x C_{\theta_o} - (\theta_{pt} + \phi)\left[\Omega x (S_{\theta_o} + \Phi C_{\theta_o})\right. \\
& \left. - v_i C_{\theta_o}\right] \Big\} (w - \beta_d x) + \Big\{\Omega x\left[S_{\theta_o} + (\Phi + 2\theta_{pt} + 2\phi) C_{\theta_o}\right] - v_i C_{\theta_o}\Big\}(v + \zeta_s x)\Big\rangle \\
& + \dot{w}\Big\langle \Big\{\Omega x\left[2S_{\theta_o} + (2\Phi + \theta_{pt} + \phi) C_{\theta_o}\right] - 2 v_i C_{\theta_o}\Big\}(w - \beta_d x) \\
& - \Omega(x + e_1) C_{\theta_o}(v + \zeta_s x)\Big\rangle + \frac{3c}{4}\Omega x (v + \zeta_s x) C_{\theta_o}\dot{\Phi} \\
& + \left[\frac{3c}{4} - (v + \zeta_s x)\right](v + \zeta_s x)\Omega x C_{\theta_o}\dot{\phi}\Bigg) dx
\end{aligned} \tag{51}$$

where

$$\left.\begin{aligned}
v_i &= \pm \Omega R \frac{\pi\sigma}{8}\left\{\sqrt{1 + \frac{12}{\pi\sigma}\left|S_{\theta_o} + \left[\theta_{pt}\left(\tfrac{3}{4}\right) + \phi_o\left(\tfrac{3}{4}\right) + \Phi\right] C_{\theta_o}\right|} - 1\right\} \\
\operatorname{sgn} v_i &= \operatorname{sgn}\left\{S_{\theta_o} + \left[\theta_{pt}\left(\tfrac{3}{4}\right) + \phi_o\left(\tfrac{3}{4}\right) + \Phi\right] C_{\theta_o}\right\} \\
\sigma &= \frac{bc}{\pi R}
\end{aligned}\right\} \tag{52}$$

Equations (1), (2), (22)-(24), (27), (28), and (47)-(51) when combined as

structural terms + inertial terms - aerodynamic terms = 0 yield the hybrid equations of motion. The boundary conditions are found in equations (3) and (6).

SOLUTION OF THE EQUATIONS OF MOTION

Simplification and Nondimensionalization

The equations are solved by Galerkin's method using mode shapes of a nonrotating uniform cantilever beam. For convenience we treat only the case with uniform radial distributions of mass and stiffness properties. In general, a flexible beam segment inboard of the pitch-bearing is present and nonuniformities in bending stiffnesses influence the flap-lag structural coupling. In this paper, the inboard beam segment is eliminated entirely by placing the pitch change bearing in the hub itself, thus leaving only the single outboard blade segment. However, the effect of the inboard beam segment on the structural coupling between flap and lead-lag bending is represented in an approximate fashion. Flap-lag structural coupling depends on the relative stiffness of the blade segments inboard and outboard of the pitch-bearing because the principal elastic axes of the outboard blade segment rotate through the angle θ_o as the blade pitch angle varies, while the inboard segment principal axes do not. The resultant effective orientation of principal axes depends on the blade geometry and distribution of bending stiffnesses inboard and outboard of the pitch-bearing. Although the variations in the structural coupling significantly influence stability, they are not present in a simple single-segment uniform beam treated here. They are difficult to include exactly without resorting to a more general blade configuration and a more sophisticated analysis. However, an approximate representation of these effects may be introduced with no increase in complexity. This is accomplished by arbitrarily assuming that the average inclination of the principal elastic axes of a nonuniform blade is equal to some fraction of the inclination of the principal axes of a uniform single-segment blade. This entails having the structural principal axes inclined at $\mathcal{R}\theta_o$ rather than θ_o while the mass and inertial terms are unchanged. The factor $\mathcal{R}$ is called the structural coupling parameter. When $\mathcal{R} = 1$, the original equations are retained, but as $\mathcal{R}$ is reduced to zero, the flap-lag structural coupling terms diminish and eventually vanish. Although this is only an approximation of the true effect of flap-lag structural coupling, it greatly simplifies the numerical model and does represent the type of behavior that would be exhibited by a general nonuniform blade. In order for the structural axes to be inclined at $\mathcal{R}\theta_o$ to the plane of rotation, however, we must substitute $\theta_{pt} - (1 - \mathcal{R})\theta_o$ for θ in the structural terms of reference 1 because of the change in the coordinate system.

The equations are further simplified by eliminating the δu equation and all terms containing u. This is accomplished by solving the δu equation (consisting of equations (1), (22), and (47) combined as discussed above) for T. The resulting expression for T may then be substituted into the other equations. An expression for $\dot{u}$ may be easily obtained using the definition of T. Proceeding in this manner we first write the δu equation

$$-T' - m\Omega^2(x+e_1) + 2m\Omega x\dot{\phi}(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})$$

$$-2m\Omega(\dot{v}C_{\theta_o+\phi} - \dot{w}S_{\theta_o+\phi}) + 2m\Omega\dot{\phi}(vS_{\theta_o} + wC_{\theta_o}) + \frac{\rho_\infty ac}{2}\ \Omega^2 x^2 S_{\theta_o} C_{\theta_o} w' = 0 \quad (53)$$

For a uniform blade we may nondimensionalize as follows

$$-\tau' - (\bar{x}+\bar{e}_1) + 2\bar{x}(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})\dot{\phi} + 2(\bar{v}S_{\theta_o} + \bar{w}C_{\theta_o})\dot{\phi}$$

$$-2(\dot{\bar{v}}C_{\theta_o+\phi} - \dot{\bar{w}}S_{\theta_o+\phi}) + \frac{\gamma}{6}\ S_{\theta_o} C_{\theta_o} \bar{x}^2 \bar{w}' = 0 \quad (54)$$

where

$$\tau = \frac{T}{m\Omega^2 \ell^2}$$

$$\bar{x} = \frac{x}{\ell}; \quad \bar{u} = \frac{u}{\ell}; \quad \bar{v} = \frac{v}{\ell}; \quad \bar{w} = \frac{w}{\ell}$$

$$\bar{e}_1 = \frac{e_1}{\ell}; \quad \gamma = \frac{3\rho_\infty ac\ell}{m} \equiv \frac{\text{Lock number}}{(R^4 - e_1{}^4)/\ell^4}$$

Now, integration yields

$$\tau = \frac{1-\bar{x}^2}{2} + (1-\bar{x})\bar{e}_1 - (1-\bar{x}^2)(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})\dot{\phi}$$

$$-2\dot{\phi}\int_{\bar{x}}^{1}(\bar{v}S_{\theta_o} + \bar{w}C_{\theta_o})d\bar{x} + 2\int_{\bar{x}}^{1}(\dot{\bar{v}}C_{\theta_o+\phi} - \dot{\bar{w}}S_{\theta_o+\phi})d\bar{x}$$

$$-\frac{\gamma}{6}\ S_{\theta_o} C_{\theta_o}\int_{\bar{x}}^{1}\bar{x}^2 \bar{w}'\ d\bar{x} \quad (55)$$

Equation (55) for τ may now be substituted into the bending and torsion equations retaining terms to the appropriate order of magnitude. In the bending equations the contribution of the last term, an aerodynamic term proportional to γ, is negligibly small. In the torsion equation only the first term is retained.

In order to eliminate $\dot{u}$ from the equations, we now consider the definition of T, equation (1f), nondimensionalized with respect to $m\Omega^2\ell^2$

$$\tau = \frac{EA}{m\Omega^2\ell^2}\left(\bar{u}' + \frac{\bar{v}'^2}{2} + \frac{\bar{w}'^2}{2}\right) \quad (56)$$

Differentiating equation (56) with respect to ψ yields

$$\dot{\bar{u}} = \frac{m\Omega^2\ell^2}{EA}\dot{\tau} - \bar{v}'\dot{\bar{v}}' - \bar{w}'\dot{\bar{w}}' \quad (57)$$

Typical hingeless rotor configurations have sufficiently large radial stiffness EA so that $m\Omega^2\ell^2/EA = O(\varepsilon^2)$. Thus the first term of equation (56) is quite small when substituted into the bending and root torsion equations containing $2m\Omega\dot{u}$ and for our purposes it is negligibly small. Therefore, $\dot{u}$ is obtained by integrating equation (57)

$$\dot{\bar{u}} = -\int_0^{\bar{x}} (\bar{v}'\dot{\bar{v}}' + \bar{w}'\dot{\bar{w}}')d\bar{x} \tag{58}$$

Equation (58) may now be substituted for $\dot{u}$ in the bending and root torsion equations. The nondimensional equations for v, w, ϕ, and Φ now may be written for a uniform beam

$\delta\bar{v}$ equation:

$$\Lambda_2\bar{v}'''' - (\Lambda_2 - \Lambda_1)\left\{\bar{v}''''S^2_{\bar{\theta}_o} - S_{2\bar{\theta}_o}[(\theta_{pt} + \phi)\bar{v}'']'' + \underline{C_{2\bar{\theta}_o}[(\theta_{pt} + \phi)^2\bar{v}'']''}\right\}$$

$$+(\Lambda_2 - \Lambda_1)\left\{-\bar{w}''''\frac{S_{2\bar{\theta}_o}}{2} + C_{2\bar{\theta}_o}[(\theta_{pt} + \phi)\bar{w}'']'' + \underline{S_{2\theta_o}[(\theta_{pt} + \phi)^2 w'']''}\right\}$$

$$-(\tau\bar{v}')' + \bar{e}_1(\beta_{pc}S_{\theta_o+\Phi} + \zeta_s) - \overline{e_o}C_{\theta_o+\Phi} + \bar{x}(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi})S_{\theta_o+\Phi}$$

$$-\bar{v}C^2_{\theta_o+\Phi} + \bar{w}S_{\theta_o+\Phi}C_{\theta_o+\Phi} - 2C_{\theta_o+\Phi}\int_0^{\bar{x}} (\bar{v}'\dot{\bar{v}}' + \bar{w}'\dot{\bar{w}}')d\bar{x} - 2\dot{\bar{w}}(\beta_{pc} + \zeta_s S_{\theta_o+\Phi} - \beta_d C_{\theta_o+\Phi})$$

$$-2\bar{x}\zeta_s\beta_1\dot{\Phi} - 2\beta_{pc}\bar{v}\dot{\Phi} - 2\zeta_s(\bar{v}S_{\theta_o} + \bar{w}C_{\theta_o})\dot{\Phi} - (w - \beta_d x)\ddot{\Phi} + \ddot{\bar{v}} + 2\bar{k}^2_{m_2}S_{\theta_o}\dot{\phi}'$$

$$+ \frac{\gamma}{6}\Big\langle(\bar{x}\bar{v}_i C_{2\theta_o} - \bar{x}^2 S_{\theta_o}C_{\theta_o})\phi - \left\{\bar{x}^2 S_{2\theta_o} + [\bar{x}^2(\theta_{pt} + \phi + \dot{\Phi}) - 2\bar{x}\bar{v}_i]C_{2\theta_o}\right\}\Phi$$

$$+\left\{2\frac{c_{d_o}}{a}\bar{x}C_{\theta_o} - (\theta_{pt} + \phi)[\bar{x}(S_{\theta_o} + \Phi C_{\theta_o}) - \bar{v}_i C_{\theta_o}]\right\}\dot{\bar{v}}$$

$$+\left\{\bar{x}[2S_{\theta_o} + (2\Phi + \theta_{pt} + \phi)C_{\theta_o}] - 2\bar{v}_i C_{\theta_o}\right\}\dot{\bar{w}}\Big\rangle$$

$$= \frac{\gamma}{6}\left[(\bar{x}^2 + 2\bar{e}_1\bar{x})S^2_{\theta_o} + \bar{x}^2\theta_{pt}S_{\theta_o}C_{\theta_o} + \bar{v}_i{}^2C^2_{\theta_o} - \bar{x}\bar{v}_i(S_{2\theta_o} + \theta_{pt}C_{2\theta_o}) - \frac{c_{d_o}}{a}\bar{x}^2C^2_{\theta_o}\right] \tag{59}$$

$\delta\bar{w}$ equation:

$$
\begin{aligned}
&(\Lambda_2 - \Lambda_1)\left\{-\bar{v}'''' \frac{S_{2\bar{\theta}_o}}{2} + C_{2\bar{\theta}_o}[(\theta_{pt} + \phi)\bar{v}'']'' + \underline{S_{2\bar{\theta}_o}[(\theta_{pt} + \phi)^2\bar{v}'']''}\right\} \\
&+\Lambda_1\bar{w}'''' + (\Lambda_2 - \Lambda_1)\left\{\bar{w}''''S^2_{\bar{\theta}_o} - S_{2\bar{\theta}_o}[(\theta_{pt} + \phi)\bar{w}'']'' + \underline{C_{2\bar{\theta}_o}[(\theta_{pt} + \phi)^2\bar{w}'']''}\right\} \\
&-(\tau\bar{w}')' + \bar{e}_1(\beta_{pc}C_{\theta_o+\phi} - \beta_d) + \bar{x}(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi})C_{\theta_o+\phi} + \bar{e}_o S_{\theta_o+\phi} \\
&+\bar{v}S_{\theta_o+\phi}C_{\theta_o+\phi} - \bar{w}S^2_{\theta_o+\phi} + 2S_{\theta_o+\phi}\int_o^{\bar{x}} (\bar{v}'\dot{\bar{v}}' + \bar{w}'\dot{\bar{w}}')d\bar{x} + 2\dot{\bar{v}}(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi}) \\
&+2\bar{x}\beta_d\beta_1\dot{\phi} - 2\beta_{pc}\bar{w}\dot{\phi} + 2\beta_d(\bar{v}S_{\theta_o} + \bar{w}C_{\theta_o})\dot{\phi} + (\bar{v} + \zeta_s\bar{x})\ddot{\phi} + \ddot{\bar{w}} + 2\bar{k}^2_{m_1}C_{\theta_o}\dot{\phi}' \\
&+ \frac{\gamma}{6}\Big\langle -(\bar{x}^2 + 2\bar{e}_1\bar{x})C^2_{\theta_o}\phi - \bar{x}^2C^2_{\theta_o}\int_o^{\bar{x}} \bar{v}'\bar{w}''\, d\bar{x} - \frac{\bar{x}\bar{c}}{2}\bar{w}C^2_{\theta_o} + \bar{e}_o\bar{x}\bar{w}'C_{\theta_o} + \bar{x}\bar{v}\bar{w}'C^2_{\theta_o} \\
&+\bar{x}\beta_1\bar{v}C_{\theta_o} - (\bar{x}^2 + 2\bar{e}_1\bar{x})\Phi C_{2\theta_o} - \left(\frac{\bar{c}}{4}\right)^2 (\ddot{\phi} + \ddot{\Phi}) + \left\{\bar{v}_i C_{\theta_o} - \bar{x}[S_{\theta_o} + (\Phi + 2\theta_{pt} + 2\phi)C_{\theta_o}]\right\}\dot{\bar{v}} \\
&+(\bar{x} + \bar{e}_1)C_{\theta_o}\dot{\bar{w}} + \frac{\bar{c}}{4}\ddot{\bar{w}} - \frac{3\bar{c}}{4}\bar{x}C_{\theta_o}\dot{\phi} - \left[\frac{3\bar{c}}{4} - (\bar{v} + \zeta_s\bar{x})\right]\bar{x}C_{\theta_o}\dot{\Phi}\Big\rangle \\
&\qquad = \frac{\gamma}{6}\Big[(\bar{x}^2 + 2\bar{e}_1\bar{x})C_{\theta_o}(S_{\theta_o} + \theta_{pt}C_{\theta_o}) \\
&\qquad\quad -(\bar{x} + \bar{e}_1)\bar{v}_i C_{2\theta_o} - \bar{e}_o\bar{x}(\beta_{pc}C_{2\theta_o} - \beta_d C_{\theta_o}) + \frac{\bar{x}\bar{c}}{2}\beta_1 C_{\theta_o} - \bar{x}^2\beta_{pc}\zeta_s C_{\theta_o}\Big] \qquad (60)
\end{aligned}
$$

$\delta\phi$ equation:

$$
\begin{aligned}
&-\bar{k}_A^2\left[\left(\frac{1 - \bar{x}^2}{2}\right)(\theta_{pt} + \phi)'\right]' - \kappa\phi'' + (\Lambda_2 - \Lambda_1)\left\{\left(\frac{\bar{w}''^2 - \bar{v}''^2}{2}\right)\left[-S_{2\bar{\theta}_o} + \underline{(\theta_{pt} + \phi)C_{2\theta_o}}\right]\right. \\
&\left.+\bar{v}''\bar{w}''\left[C_{2\bar{\theta}_o} + \underline{(\theta_{pt} + \phi)S_{2\bar{\theta}_o}}\right]\right\} + (\bar{k}^2_{m_2} - \bar{k}^2_{m_1})[S_{\theta_o}C_{\theta_o} + (\theta_{pt} + \phi + \Phi)C_{2\theta_o}] \\
&+2(\bar{k}^2_{m_2}S_{\theta_o}\dot{\bar{v}}' + \bar{k}^2_{m_1}C_{\theta_o}\dot{\bar{w}}') + \bar{k}_m^2\ddot{\phi} + \frac{\gamma\bar{c}^2}{96}[2\bar{x}C_{\theta_o}(\dot{\phi} + \dot{\Phi}) + \bar{x}C^2_{\theta_o}\bar{w}' - \ddot{\bar{w}}] \\
&\qquad = -\frac{\gamma\bar{c}^2}{96}\bar{x}C_{\theta_o}\beta_1 \qquad (61)
\end{aligned}
$$

$\delta\Phi$ **equation:**

$$
\begin{aligned}
& Q\phi + \frac{\bar{e}_o}{2}\,(\zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi}) + \bar{e}_o \int_o^1 (\bar{v} S_{\theta_o+\phi} + \bar{w} C_{\theta_o+\phi})\,d\bar{x} + \frac{1}{2}\,\bar{e}_1 \beta_{pc} \zeta_1 \\
& + \frac{1}{3}\,(\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi})(\zeta_s C_{\theta_o+\phi} + \beta_d S_{\theta_o+\phi}) + \bar{e}_1 \beta_{pc} \int_o^1 (\bar{v} C_{\theta_o} - \bar{w} S_{\theta_o})\,d\bar{x} \\
& + (\beta_{pc} + \zeta_s S_{\theta_o+\phi} - \beta_d C_{\theta_o+\phi}) \int_o^1 \bar{x}(\bar{v} C_{\theta_o+\phi} - \bar{w} S_{\theta_o+\phi})\,d\bar{x} \\
& + (\zeta_s C_{\theta_o+\phi} + \beta_d S_{\theta_o+\phi}) \int_o^1 \bar{x}(\bar{v} S_{\theta_o+\phi} + \bar{w} C_{\theta_o+\phi})\,d\bar{x} + S_{2(\theta_o+\phi)} \int_o^1 \left(\frac{\bar{v}^2 - \bar{w}^2}{2}\right) d\bar{x} \\
& + C_{2(\theta_o+\phi)} \int_o^1 \bar{v}\bar{w}\;d\bar{x} + (\bar{k}_{m_2}^2 - \bar{k}_{m_1}^2)\left[S_{\theta_o} C_{\theta_o} + \phi C_{2\theta_o} + C_{2\theta_o} \int_o^1 (\theta_{pt} + \phi)\,d\bar{x}\right] \\
& + 2\beta_{pc} \int_o^1 (\bar{v}\dot{\bar{v}} + \bar{w}\dot{\bar{w}})\,d\bar{x} + 2\beta_1 \int_o^1 \bar{x}(\zeta_s \dot{\bar{v}} - \beta_d \dot{\bar{w}})\,d\bar{x} \\
& + 2(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o}) \int_o^1 \left(\frac{1 - \bar{x}^2}{2}\right)(\bar{v}'\dot{\bar{v}}' + \bar{w}'\dot{\bar{w}}')\,d\bar{x} + 2\bar{k}_{m_2}^2 S_{\theta_o} \dot{\bar{v}}(1) + 2\bar{k}_{m_1}^2 C_{\theta_o} \dot{\bar{w}}(1) \\
& + 2\int_o^1 (\bar{v} S_{\theta_o} + \bar{w} C_{\theta_o}) \int_o^{\bar{x}} (\bar{v}'\dot{\bar{v}}' + \bar{w}'\dot{\bar{w}}')\,d\bar{x}_1\;d\bar{x} + 2\int_o^1 (\bar{v} S_{\theta_o} + \bar{w} C_{\theta_o})(\zeta_s \dot{\bar{v}} - \beta_d \dot{\bar{w}})\,d\bar{x} \\
& + \int_o^1 (\bar{v} + \zeta_s \bar{x})\ddot{\bar{w}}\;d\bar{x} - \int_o^1 (\bar{w} - \beta_d \bar{x})\ddot{\bar{v}}\;d\bar{x} + \bar{k}_m{}^2 \int_o^1 \ddot{\phi}\;d\bar{x} + \left(\frac{\zeta_s{}^2 + \beta_d{}^2}{3}\right)\ddot{\phi} \\
& + 2\ddot{\phi} \int_o^1 \bar{x}(\zeta_s \bar{v} - \beta_d \bar{w})\,d\bar{x} + \ddot{\phi} \int_o^1 (\bar{v}^2 + \bar{w}^2)\,d\bar{x} + \bar{k}_m{}^2 \ddot{\phi} \\
& + \frac{\gamma}{6}\Bigg\{\frac{\bar{c}^2}{16}\left[C_{\theta_o}\dot{\phi} + 2C_{\theta_o} \int_o^1 \bar{x}\dot{\phi}\;d\bar{x} + C_{\theta_o}^2 \int_o^1 \bar{x}\bar{w}'\;d\bar{x} - \int_o^1 \ddot{\bar{w}}\;d\bar{x}\right] + \frac{\bar{c}}{4} \int_o^1 (\bar{v} + \zeta_s \bar{x})\ddot{\bar{w}}\;d\bar{x} \\
& - S_{\theta_o} \int_o^1 (\bar{x}^2 + 2\bar{e}_1 \bar{x})(\bar{v} C_{\theta_o} - \bar{w} S_{\theta_o})\,d\bar{x} - \zeta_1 S_{\theta_o}\left(\frac{1}{4} + \frac{2\bar{e}_1}{3}\right) + \bar{v}_i C_{2\theta_o} \int_o^1 (\bar{x} + \bar{e}_1)\bar{v}\;d\bar{x} \\
& - C_{\theta_o}^2 \int_o^1 (\bar{x}^2 + 2\bar{e}_1 \bar{x})(\theta_{pt} + \phi)(\bar{v} + \zeta_s \bar{x})\,d\bar{x} + S_{\theta_o} C_{\theta_o} \int_o^1 \bar{x}^2(\theta_{pt} + \phi)(\bar{w} - \beta_d \bar{x})\,d\bar{x} \\
& + C_{\theta_o} \int_o^1 \left(\bar{v}_i{}^2 - \bar{x}^2\,\frac{c_{d_o}}{a}\right)(\bar{w} - \beta_d \bar{x})\,d\bar{x} - C_{\theta_o}^2 \int_o^1 \bar{x}^2(\bar{v} + \zeta_s x) \int_o^{\bar{x}} \bar{v}'\bar{w}''\;d\bar{x}_1\;d\bar{x} \\
& + \bar{v}_i C_{2\theta_o} \zeta_s \left(\frac{1}{3} + \frac{\bar{e}_1}{3}\right) - \bar{v}_i S_{2\theta_o}\left(\int_o^1 \bar{x}\bar{w}\;d\bar{x} - \frac{\beta_d}{3}\right) - \bar{v}_i C_{2\theta_o} \int_o^1 \bar{x}(\theta_{pt} + \phi)(\bar{w} - \beta_d \bar{x})\,d\bar{x} \\
& + \bar{e}_o(\beta_{pc} C_{2\theta_o} - \beta_d C_{\theta_o})\left(\frac{\zeta_s}{3} + \int_o^1 \bar{x}\bar{v}\;d\bar{x}\right) - \frac{\bar{c}}{2}\,\beta_1 C_{\theta_o}\left(\frac{\zeta_s}{3} + \int_o^1 \bar{x}\bar{v}\;d\bar{x}\right) \\
& - \frac{\bar{c}}{2}\,C_{\theta_o}^2 \int_o^1 \bar{x}\bar{w}'(\bar{v} + \zeta_s \bar{x})\,d\bar{x} + C_{\theta_o} \int_o^1 \bar{x}(\bar{e}_o \bar{w}' + \bar{v}\bar{w}' C_{\theta_o} + \beta_1 \bar{v})(\bar{v} + \zeta_s \bar{x})\,d\bar{x} \\
& + C_{\theta_o} \beta_{pc} \zeta_s \left(\frac{\zeta_s}{4} + \int_o^1 \bar{x}^2 \bar{v}\;d\bar{x}\right) - \phi \int_o^1 \bar{x}^2(\bar{v} C_{2\theta_o} - \bar{w} S_{2\theta_o})\,d\bar{x} - \frac{\zeta_2 \phi}{4} \\
& - 2\bar{e}_1 \phi\left(\frac{\zeta_s}{3} + \int_o^1 \bar{x}\bar{v}\;d\bar{x}\right) + \phi C_{2\theta_o} \int_o^1 \bar{x}^2(\theta_{pt} + \phi)(\bar{w} - \beta_d \bar{x})\,d\bar{x} + \phi^2 C_{2\theta_o}\left(\int_o^1 \bar{x}^2 \bar{w}\;d\bar{x} - \frac{\beta_d}{4}\right) \\
& - 2\bar{v}_i \phi C_{2\theta_o}\left(\int_o^1 \bar{x}\bar{w}\;d\bar{x} - \frac{\beta_d}{3}\right) - 2\,\frac{c_{d_o}}{a}\,C_{\theta_o} \int_o^1 \bar{x}\dot{\bar{v}}(\bar{w} - \beta_d x)\,d\bar{x} \\
& + (S_{\theta_o} + \phi C_{\theta_o}) \int_o^1 \bar{x}(\theta_{pt} + \phi)\dot{\bar{v}}(\bar{w} - \beta_d \bar{x})\,d\bar{x} - \bar{v}_i C_{\theta_o} \int_o^1 (\theta_{pt} + \phi)\dot{\bar{v}}(\bar{w} - \beta_d \bar{x})\,d\bar{x} \\
& + \bar{v}_i C_{\theta_o} \int_o^1 (\bar{v} + \zeta_s \bar{x})\dot{\bar{v}}\;d\bar{x} - (S_{\theta_o} + \phi C_{\theta_o}) \int_o^1 \bar{x}(\bar{v} + \zeta_s \bar{x})\dot{\bar{v}}\;d\bar{x} \\
& - 2C_{\theta_o} \int_o^1 \bar{x}(\theta_{pt} + \phi)\dot{\bar{v}}(\bar{v} + \zeta_s \bar{x})\,d\bar{x} - 2(S_{\theta_o} + \phi C_{\theta_o}) \int_o^1 \bar{x}\dot{\bar{w}}(\bar{w} - \beta_d \bar{x})\,d\bar{x} \\
& - C_{\theta_o} \int_o^1 \bar{x}(\theta_{pt} + \phi)\dot{\bar{w}}(\bar{w} - \beta_d x)\,d\bar{x} + 2\bar{v}_i C_{\theta_o} \int_o^1 \dot{\bar{w}}(\bar{w} - \beta_d x)\,d\bar{x} \\
& + C_{\theta_o} \int_o^1 (\bar{x} + \bar{e}_1)(\bar{v} + \zeta_1 \bar{x})\dot{\bar{w}}\;d\bar{x} - \frac{3\bar{c}}{4}\,C_{\theta_o} \int_o^1 \bar{x}(\bar{v} + \zeta_s \bar{x})\dot{\phi}\;d\bar{x} - \frac{3\bar{c}}{4}\,C_{\theta_o}\dot{\phi}\left(\int_o^1 \bar{x}\bar{v}\;d\bar{x} + \frac{\zeta_s}{3}\right) \\
& + C_{\theta_o}\dot{\phi}\left(\int_o^1 \bar{x}\bar{v}^2\;d\bar{x} + 2\zeta_s \int_o^1 \bar{x}^2 \bar{v}\;d\bar{x} + \frac{\zeta_s{}^2}{4}\right)\Bigg\} = -\frac{\gamma\bar{c}^2}{192}\,C_{\theta_o}\beta_1 \qquad (62)
\end{aligned}
$$

where

$$\left.\begin{aligned}
&\Lambda_2 = \frac{EI_{z'}}{m\Omega^2\ell^4}; \quad \Lambda_1 = \frac{EI_{y'}}{m\Omega^2\ell^4}; \quad \kappa = \frac{GJ}{m\Omega^2\ell^4}; \quad Q = \frac{k_\Phi}{m\Omega^2\ell^3}; \quad \bar{V}_i = \frac{V_i}{\Omega\ell} \\
&(\bar{\ }) = \frac{(\ )}{\ell}; \quad (\ )' = \frac{\partial}{\partial\bar{x}}; \quad (\dot{\ }) = \frac{\partial}{\partial\psi}; \quad \zeta_2 = \zeta_s C_{2\theta_o} + \beta_d S_{2\theta_o}; \quad \bar{\theta}_o = (1-R)\theta_o
\end{aligned}\right\} \tag{63}$$

Terms not satisfying equation (3) are

$$-2(\dot{\phi}+\dot{\Phi})(\bar{k}_{m_2}^2 S_{\theta_o}\delta\bar{v} + \bar{k}_{m_1}^2 C_{\theta_o}\delta\bar{w})\Big|_0^1 \tag{64}$$

The terms of equation (64) must be included in the analysis as discussed above in order to obtain the correct results. Equations (59)-(62) are the hybrid nonlinear equations of motion that will be solved by Galerkin's method.

Application of Galerkin's Method

In transforming equations (59)-(62) into modal equations we use the mode shapes for a nonrotating cantilever beam. We also assume that the motion is characterized by small perturbation motions about a steady equilibrium operating condition that depend on dimensionless time $\psi(=\Omega t)$. Thus,

$$\left.\begin{aligned}
\bar{v} &= \sum_{i=1}^{N} [V_{oi} + \Delta V_i(\psi)]\Psi_i(\bar{x}) \\
\bar{w} &= \sum_{i=1}^{N} [W_{oi} + \Delta W_i(\psi)]\Psi_i(\bar{x}) \\
\phi &= \sum_{i=1}^{N} [\Phi_{oi} + \Delta\Phi_i(\psi)]\,\Theta_i(\bar{x})
\end{aligned}\right\} \tag{65}$$

where

$$\left.\begin{aligned}
\Psi_i(\bar{x}) &= \cosh(\beta_i\bar{x}) - \cos(\beta_i\bar{x}) + \alpha_i[\sinh(\beta_i\bar{x}) - \sin(\beta_i\bar{x})] \\
\Theta_i(\bar{x}) &= \sqrt{2}\,\sin(\gamma_i\bar{x})
\end{aligned}\right\} \tag{66}$$

The constants α_i and β_i are defined in reference 6 and $\gamma_i = \pi[i-(1/2)]$. We also assume that

$$\Phi = \Phi_o + \Delta\Phi \tag{67}$$

and that

$$\theta_{pt} = -\theta_t\bar{x} \tag{68}$$

Nonlinear algebraic equations for V_{oj}, W_{oj}, Φ_{oj}, and Φ_o determine the equilibrium deflections. The perturbation deflections are governed by a set of homogeneous ordinary differential equations for ΔV_j, ΔW_j, $\Delta\Phi_j$, and $\Delta\Phi$, with constant coefficients depending on V_{oj}, W_{oj}, Φ_{oj}, and Φ_o. The modal equilibrium equations are as follows:

δV_{oi} equation:

$$
\begin{aligned}
&\sum_{j=1}^{N} \Big\langle \Lambda_2 \beta_j{}^4 \delta_{ij} V_{oj} - (\Lambda_2 - \Lambda_1)\Big[S^2_{\theta_o}\beta_j{}^4\delta_{ij} - S_{2\bar{\theta}_o}\Big(\sum_{k=1}^{N} \Phi_{ok}V_{kij} - \theta_t P_{ij}\Big) \\
&\underline{+C_{2\bar{\theta}_o}\sum_{k=1}^{N} \Phi_{ok}\Big(\sum_{\ell=1}^{N} \Phi_{o\ell}Y_{k\ell ij} - 2\theta_t V'_{kij}\Big)}\Big]V_{oj} + (\Lambda_2 - \Lambda_1)\Big[-\frac{S_{2\bar{\theta}_o}}{2}\beta_j{}^4\delta_{ij} \\
&+C_{2\bar{\theta}_o}\Big(\sum_{k=1}^{N} \Phi_{ok}V_{kij} - \theta_t P_{ij}\Big) + \underline{S_{2\bar{\theta}_o}\sum_{k=1}^{N} \Phi_{ok}\Big(\sum_{\ell=1}^{N} \Phi_{o\ell}Y_{k\ell ij} - 2\theta_t V'_{kij}\Big)}\Big]W_{oj} \\
&+(M_{ij} + \bar{e}_1 L_{ij})V_{oj} - V_{oj}\delta_{ij}(C^2_{\theta_o} - \Phi_o S_{2\theta_o} - \Phi_o{}^2 C_{2\theta_o}) + W_{oj}\delta_{ij}(S_{\theta_o}C_{\theta_o} + \Phi_o C_{2\theta_o} - \Phi_o{}^2 S_{2\theta_o}) \\
&+A_j\Big[\bar{e}_1\beta_{pc}\Phi_o C_{\theta_o} + \bar{e}_o\Phi_o\Big(S_{\theta_o} + \frac{\Phi_o C_{\theta_o}}{2}\Big)\Big]\delta_{ij} + B_j\Phi_o(\beta_2 + \Phi_o\zeta_2)\delta_{ij} \\
&+\frac{\gamma}{6}\Big\{(\bar{v}_i C_{2\theta_o}Q_{ij} - S_{\theta_o}C_{\theta_o}R_{ij})\Phi_{oj} - \Phi_o[S_{2\theta_o}C_j + C_{2\theta_o}(\Phi_o C_j - \theta_t D_j - 2\bar{v}_i B_j)]\delta_{ij} - \Phi_o R_{ij}\Phi_{oj}\Big\}\Big\rangle \\
&\qquad = [\bar{e}_o C_{\theta_o} - \bar{e}_1(\beta_{pc}S_{\theta_o} + \zeta_s)]A_i - \beta_1 S_{\theta_o}B_i + \frac{\gamma}{6}\Big[(C_i + 2\bar{e}_1 B_i)S^2_{\theta_o} - \theta_t C_i S_{\theta_o}C_{\theta_o} \\
&\qquad\quad +\bar{v}_i{}^2 C^2_{\theta_o}A_i - \bar{v}_i(S_{2\theta_o}B_i - \theta_t C_{2\theta_o}C_i) - \frac{c d_o}{a} C^2_{\theta_o}C_i\Big] \qquad i = 1,2,\ldots,N \qquad (69)
\end{aligned}
$$

δW_{oi} equation:

$$\sum_{j=1}^{N}\Big\langle(\Lambda_2-\Lambda_1)\Big[-\frac{S_{2\bar\theta_o}}{2}\beta_j^4\delta_{ij}+C_{2\bar\theta_o}\Big(\sum_{k=1}^{N}\Phi_{ok}V_{kij}-\theta_tP_{ij}\Big)$$

$$\underline{+S_{2\bar\theta_o}\sum_{k=1}^{N}\Phi_{ok}\Big(\sum_{\ell=1}^{N}\Phi_{o\ell}Y_{k\ell ij}-2\theta_tV'_{kij}\Big)}\Big]V_{oj}+\Lambda_1\beta_j^4\delta_{ij}W_{oj}+(\Lambda_2-\Lambda_1)\Big[S^2_{\bar\theta_o}\beta_j^4\delta_{ij}$$

$$-S_{2\bar\theta_o}\Big(\sum_{k=1}^{N}\Phi_{ok}V_{kij}-\theta_tP_{ij}\Big)\underline{+C_{2\bar\theta_o}\sum_{k=1}^{N}\Phi_{ok}\Big(\sum_{\ell=1}^{N}\Phi_{o\ell}Y_{k\ell ij}-2\theta_tV'_{kij}\Big)}\Big]W_{oj}$$

$$+(M_{ij}+\bar e_1L_{ij})W_{oj}+V_{oj}\delta_{ij}(S_{\theta_o}C_{\theta_o}+\Phi_oC_{2\theta_o}-\Phi_o^{\,2}S_{2\theta_o})-W_{oj}\delta_{ij}(S^2_{\theta_o}+\Phi_oS_{2\theta_o}+\Phi_o^{\,2}C_{2\theta_o})$$

$$+A_j[-\bar e_1\beta_{pc}\Phi_oS_{\theta_o}+\bar e_o\Phi_o(C_{\theta_o}-\Phi_oS_{\theta_o})]\delta_{ij}+B_j\Phi_o(\zeta_2-\Phi_o\beta_2)\delta_{ij}$$

$$+\frac{\gamma}{6}\Big\{-(R_{ij}+2\bar e_1Q_{ij})C^2_{\theta_o}\Phi_{oj}+\sum_{k=1}^{N}U_{ijk}V_{oj}W_{ok}C^2_{\theta_o}+\Big(\bar e_o-\frac{\bar c}{2}C_{\theta_o}\Big)C_{\theta_o}O_{ij}W_{oj}$$

$$+\beta_1C_{\theta_o}I_{ij}V_{oj}-\Phi_oC_{2\theta_o}(C_j+2\bar e_1B_j)\delta_{ij}\Big\}\Big\rangle$$

$$=-[\bar e_1(\beta_{pc}C_{\theta_o}-\beta_d)+\bar e_oS_{\theta_o}]A_i-\beta_1C_{\theta_o}B_i+\frac{\gamma}{6}\Big[(C_i+2\bar e_1B_i)S_{\theta_o}C_{\theta_o}-\theta_tC^2_{\theta_o}(D_i+2\bar e_1C_i)$$

$$-\bar v_iC_{2\theta_o}(B_i+\bar e_1A_i)-\bar e_o(\beta_{pc}C_{2\theta_o}-\beta_dC_{\theta_o})B_i+\beta_1\frac{\bar c}{2}C_{\theta_o}B_i-\beta_{pc}\zeta_sC_{\theta_o}C_i\Big]\qquad(70)$$

$\delta\Phi_{oi}$ equation:

$$\sum_{j=1}^{N}\Big\langle(\bar k_A^{\,2}N_{ij}+\kappa\gamma_j^{\,2}\delta_{ij})\Phi_{oj}+(\Lambda_2-\Lambda_1)\Big\{\sum_{k=1}^{N}V_{ijk}\Big[-\frac{S_{2\bar\theta_o}}{2}(W_{oj}W_{ok}-V_{oj}V_{ok})$$

$$+C_{2\bar\theta_o}V_{oj}W_{ok}\Big]+\underline{\theta_t\sum_{k=1}^{N}V'_{ijk}[C_{2\bar\theta_o}(W_{oj}W_{ok}-V_{oj}V_{ok})+S_{2\bar\theta_o}V_{oj}W_{ok}]}$$

$$+\sum_{\ell=1}^{N}\sum_{k=1}^{N}\underline{Y_{ijk\ell}\Phi_{oj}[C_{2\bar\theta_o}(W_{ok}W_{o\ell}-V_{ok}V_{o\ell})+S_{2\bar\theta_o}V_{ok}W_{o\ell}]}\Big\}$$

$$+(\bar k^2_{m_2}-\bar k^2_{m_1})(\Phi_{oj}+E_j\Phi_o)\delta_{ij}C_{2\theta_o}+\frac{\gamma\bar c^2}{96}C^2_{\theta_o}Q'_{ij}W_{oj}\Big\rangle$$

$$=\frac{\gamma\bar c^2}{96}C_{\theta_o}\beta_1F_i-(\bar k^2_{m_2}-\bar k^2_{m_1})(S_{\theta_o}C_{\theta_o}E_i-\theta_tF_iC_{2\theta_o})+\bar k_A^{\,2}\theta_tF_i\qquad(71)$$

$\delta\Phi_o$ equation:

$$
\begin{aligned}
&Q\Phi_o + \frac{\bar{e}_o\zeta_1}{2}\Phi_o + \bar{e}_o\sum_{i=1}^{N} A_i[V_{oi}S_{\theta_o} + W_{oi}C_{\theta_o} + \Phi_o(V_{oi}C_{\theta_o} - W_{oi}S_{\theta_o})] \\
&+[\zeta_1^2 - \beta_1(\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})]\frac{\Phi_o}{3} + (\beta_1 + \zeta_1\Phi_o)\sum_{i=1}^{N} B_i[V_{oi}C_{\theta_o} - W_{oi}S_{\theta_o} - \Phi_o(V_{oi}S_{\theta_o} + W_{oi}C_{\theta_o})] \\
&+\bar{e}_1\beta_{pc}\sum_{i=1}^{N} A_i(V_{oi}C_{\theta_o} - W_{oi}S_{\theta_o}) + [\zeta_1 - (\zeta_s S_{\theta_o} - \beta_d C_{\theta_o})\Phi_o]\sum_{i=1}^{N} B_i[V_{oi}S_{\theta_o} + W_{oi}C_{\theta_o} \\
&+\Phi_o(V_{oi}C_{\theta_o} - W_{oi}S_{\theta_o})] + \sum_{i=1}^{N}\left[\frac{V_{oi}^2 - W_{oi}^2}{2}(S_{2\theta_o} + 2\Phi_o C_{2\theta_o}) + V_{oi}W_{oi}(C_{2\theta_o} - 2\Phi_o S_{2\theta_o})\right] \\
&+(\bar{k}_{m_2}^2 - \bar{k}_{m_1}^2)\left(\Phi_o + \sum_{i=1}^{N} E_i\Phi_{oi}\right)C_{2\theta_o} + \frac{\gamma}{6}\Big\langle\frac{\bar{c}^2}{16}C_{\theta_o}^2\sum_{i=1}^{N} W_{oi}[2(-1)^{i+1} - A_i] \\
&-S_{\theta_o}\sum_{i=1}^{N}(C_i + 2\bar{e}_1B_i)(V_{oi}C_{\theta_o} - W_{oi}S_{\theta_o}) + C_{\theta_o}^2\theta_t\sum_{i=1}^{N}(D_i + 2\bar{e}_1C_i)V_{oi} \\
&-C_{\theta_o}^2\sum_{i=1}^{N}\sum_{j=1}^{N}(Q_{ij} + 2\bar{e}_1R_{ij})V_{oi}\Phi_{oj} - C_{\theta_o}^2\zeta_s\sum_{i=1}^{N}(H_i + 2\bar{e}_1G_i)\Phi_{oi} - \theta_t S_{\theta_o}C_{\theta_o}\sum_{i=1}^{N} D_iW_{oi} \\
&-\beta_d S_{\theta_o}C_{\theta_o}\sum_{i=1}^{N} H_i\Phi_{oi} + S_{\theta_o}C_{\theta_o}\sum_{i=1}^{N}\sum_{j=1}^{N} R_{ij}W_{oi}\Phi_{oj} + C_{\theta_o}^2\bar{v}_i^2\sum_{i=1}^{N} A_iW_{oi} \\
&-C_{\theta_o}\frac{cd_o}{a}\sum_{i=1}^{N} C_iW_{oi} + C_{\theta_o}^2\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} U_{ijk}V_{oi}V_{oj}W_{ok} + C_{\theta_o}^2\zeta_s\sum_{i=1}^{N}\sum_{j=1}^{N} Z_{ij}V_{oi}W_{oj} \\
&+\bar{v}_iC_{2\theta_o}\sum_{i=1}^{N}(B_i + \bar{e}_1A_i)V_{oi} - \bar{v}_iS_{2\theta_o}\sum_{i=1}^{N} B_iW_{oi} + \bar{v}_i\theta_tC_{2\theta_o}\sum_{i=1}^{N} C_iW_{oi} \\
&+\bar{v}_i\beta_dC_{2\theta_o}\sum_{i=1}^{N} F_i\Phi_{oi} - \bar{v}_iC_{2\theta_o}\sum_{i=1}^{N}\sum_{j=1}^{N} Q_{ij}W_{oi}\Phi_{oj} + \bar{e}_o(\beta_{pc}C_{2\theta_o} - \beta_dC_{\theta_o})\sum_{i=1}^{N} B_iV_{oi} \\
&-\frac{\bar{c}}{2}\beta_1C_{\theta_o}\sum_{i=1}^{N} B_iV_{oi} + C_{\theta_o}\beta_1\sum_{i=1}^{N}\left(\sum_{j=1}^{N} I_{ij}V_{oi}V_{oj} + \zeta_sC_iV_{oi}\right) \\
&+\left(\bar{e}_o - \frac{\bar{c}}{2}C_{\theta_o}\right)C_{\theta_o}\sum_{i=1}^{N}\left\{\sum_{j=1}^{N} O_{ij}V_{oi}W_{oj} + 2\zeta_s[(-1)^{i+1} - B_i]W_{oi}\right\} + C_{\theta_o}\beta_{pc}\zeta_s\sum_{i=1}^{N} C_iV_{oi} \\
&-\Phi_o\sum_{i=1}^{N} C_i(V_{oi}C_{2\theta_o} - W_{oi}S_{2\theta_o}) - \frac{\zeta_2\Phi_o}{4} - 2\bar{e}_1\Phi_o\left(\frac{\zeta_s}{3} + \sum_{i=1}^{N} B_iV_{oi}\right) + \frac{\theta_t\beta_d\Phi_oC_{2\theta_o}}{5} \\
&+\Phi_oC_{2\theta_o}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} R_{ij}W_{oi}\Phi_{oj} - \theta_tC_iW_{oi} - \beta_dF_i\Phi_{oi}\right) + \Phi_o^2C_{2\theta_o}\left(\sum_{i=1}^{N} C_iW_{oi} - \frac{\beta_d}{4}\right) \\
&-2\bar{v}_i\Phi_oC_{2\theta_o}\left(\sum_{i=1}^{N} B_iW_{oi} - \frac{\beta_d}{3}\right)\Big\rangle \\
&\quad = -\frac{\bar{e}_o}{2}(\zeta_sS_{\theta_o} - \beta_dC_{\theta_o}) - \frac{\beta_1\zeta_1}{3} \\
&\quad - \frac{\bar{e}_1\beta_{pc}\zeta_1}{2} - (\bar{k}_{m_2}^2 - \bar{k}_{m_1}^2)\left(S_{\theta_o}C_{\theta_o} - \frac{\theta_tC_{2\theta_o}}{2}\right) + \frac{\gamma}{6}\Bigg[\zeta_1S_{\theta_o}\left(\frac{1}{4} + \frac{2\bar{e}_1}{3}\right) - \frac{\bar{c}^2}{32}C_{\theta_o}\beta_1 \\
&\quad -C_{\theta_o}^2\theta_t\zeta_s\left(\frac{1}{5} + \frac{\bar{e}_1}{4}\right) + \frac{S_{\theta_o}C_{\theta_o}\theta_t\beta_d}{5} + \frac{C_{\theta_o}\bar{v}_i^2\beta_d}{2} - \frac{C_{\theta_o}\frac{cd_o}{a}\beta_d}{4} - \bar{v}_i\zeta_sC_{2\theta_o}\left(\frac{1}{3} + \frac{\bar{e}_1}{2}\right) \\
&\quad - \frac{\bar{v}_i\beta_dS_{2\theta_o}}{3} + \frac{\bar{v}_i\theta_t\beta_dC_{2\theta_o}}{4} - \bar{e}_o(\beta_{pc}C_{2\theta_o} - \beta_dC_{\theta_o})\frac{\zeta_s}{3} + \frac{\bar{c}\beta_1\zeta_sC_{\theta_o}}{6} - \frac{\beta_{pc}\zeta_s^2C_{\theta_o}}{4}\Bigg]
\end{aligned}
\tag{72}
$$

The linearized perturbation equations are easily expressed in matrix form as

$$[M]\{\Delta\ddot{X}\} + [C]\{\Delta\dot{X}\} + [K]\{\Delta X\} = 0 \tag{73}$$

where [M], [C], and [K] are, respectively, the mass, gyroscopic and damping, and stiffness matrices given below and

$$\{\Delta X\} = \begin{Bmatrix} \Delta V_i \\ \Delta W_i \\ \Delta\Phi_i \\ \Delta\Phi \end{Bmatrix} \tag{74}$$

$$[M] = \left[\begin{array}{c|c|c|c} \delta_{ij} & 0 & 0 & -(W_{oi}-\beta_d B_i) \\ \hline 0 & \delta_{ij}\left(1+\frac{\gamma\bar{c}}{24}\right) & -\frac{\gamma\bar{c}^2}{96} S_{ij} & V_{oi}+\zeta_s B_i - \frac{\gamma\bar{c}^2}{96} A_i \\ \hline 0 & -\frac{\gamma\bar{c}^2}{96} S_{ji} & \bar{k}_m^2\delta_{ij} & \bar{k}_m^2 E_i \\ \hline -(W_{oi}-\beta_d B_i) & \begin{array}{c} (V_{oi}+\zeta_s B_i)\left(1+\frac{\gamma\bar{c}}{24}\right) \\ -\frac{\gamma\bar{c}^2}{96} A_i \end{array} & \bar{k}_m^2 E_i & \begin{array}{l} \bar{k}_m^2 + \frac{\beta_d^2+\zeta_s^2}{3} \\ +\sum_{i=1}^{N} [V_{oi}^2 + W_{oi}^2 \\ +2B_i(\zeta_s V_{oi}-\beta_d W_{oi})] \end{array} \end{array}\right] \tag{75}$$

$$
[C] = \left[\begin{array}{l|l|l|l}
\begin{array}{l}
2(C_{\theta_o}-\Phi_o S_{\theta_o})\sum\limits_{k=1}^{N} V_{ok}(T_{kij}-T_{kji}) \\
+\frac{\gamma}{6}\Big[2\frac{c_{d_o}}{a}C_{\theta_o}I_{ij}+\theta_t(S_{\theta_o}+\Phi_o C_{\theta_o})J_{ij} \\
-\theta_t\bar{v}_i C_{\theta_o}I_{ij}-\sum\limits_{k=1}^{N} X_{ijk}\Phi_{ok}(S_{\theta_o}+\Phi_o C_{\theta_o}) \\
+\bar{v}_i C_{\theta_o}\sum\limits_{k=1}^{N} W_{ijk}\Phi_{ok}\Big]
\end{array}
&
\begin{array}{l}
-2(S_{\theta_o}+\Phi_o C_{\theta_o})\sum\limits_{k=1}^{N} V_{ok}T_{kij} \\
-2(C_{\theta_o}-\Phi_o S_{\theta_o})\sum\limits_{k=1}^{N} W_{ok}T_{kji} \\
-2(\beta_1+\zeta_1\Phi_o)\delta_{ij}+\frac{\gamma}{6}\Big[2(S_{\theta_o}+\Phi C_{\theta_o})I_{ij} \\
-\theta_t C_{\theta_o}J_{ij}+\sum\limits_{k=1}^{N} X_{ijk}\Phi_{ok}C_{\theta_o}-2\bar{v}_i C_{\theta_o}\delta_{ij}\Big]
\end{array}
&
-2\bar{k}_{m_2}^2 S_{\theta_o}S'_{ji}
&
\begin{array}{l}
-2B_i\zeta_s\beta_1-4\bar{k}_{m_2}^2 S_{\theta_o}(-1)^{i+1} \\
-2\beta_{pc}V_{oi}-2\zeta_s(V_{oi}S_{\theta_o}+W_{oi}C_{\theta_o}) \\
-2(\zeta_s S_{\theta_o}-\beta_d C_{\theta_o})\sum\limits_{j=1}^{N} M_{ij}V_{oj} \\
-2\sum\limits_{j=1}^{N}\sum\limits_{k=1}^{N} T_{ijk}V_{oj}(V_{ok}S_{\theta_o}+W_{ok}C_{\theta_o})
\end{array}
\\ \hline
\begin{array}{l}
2(S_{\theta_o}+\Phi_o C_{\theta_o})\sum\limits_{k=1}^{N} V_{ok}T_{kji} \\
+2(C_{\theta_o}-\Phi_o S_{\theta_o})\sum\limits_{k=1}^{N} W_{ok}T_{kij} \\
+2(\beta_1+\zeta_1\Phi_o)\delta_{ij}+\frac{\gamma}{6}\Big[v_i C_{\theta_o}\delta_{ij} \\
-(S_{\theta_o}+\Phi_o C_{\theta_o})I_{ij}+2\theta_t C_{\theta_o}J_{ij} \\
-2\sum\limits_{k=1}^{N} X_{ijk}\Phi_{ok}\Big]
\end{array}
&
\frac{\gamma}{6}\left[C_{\theta_o}(I_{ij}+\bar{e}_1\delta_{ij})-\Phi_o S_{\theta_o}I_{ij}\right]
&
\begin{array}{l}
-2\bar{k}_{m_1}^2 C_{\theta_o}S'_{ji} \\
-\frac{\gamma\bar{c}}{8}C_{\theta_o}Q_{ij}
\end{array}
&
\begin{array}{l}
2B_i\beta_d\beta_1-4\bar{k}_{m_1}^2 C_{\theta_o}(-1)^{i+1} \\
-2\beta_{pc}W_{oi}+2\beta_d(V_{oi}S_{\theta_o}+W_{oi}C_{\theta_o}) \\
-2(\zeta_s S_{\theta_o}-\beta_d C_{\theta_o})\sum\limits_{j=1}^{N} M_{ij}W_{oj} \\
-2\sum\limits_{j=1}^{N}\sum\limits_{k=1}^{N} T_{ijk}W_{oj}(V_{ok}S_{\theta_o}+W_{ok}C_{\theta_o}) \\
+\frac{\gamma}{6}C_{\theta_o}\left(\sum\limits_{j=1}^{N} I_{ij}V_{oj}+\zeta_s C_i-\frac{3\bar{c}}{4}B_i\right)
\end{array}
\\ \hline
2\bar{k}_{m_2}^2 S_{\theta_o}S'_{ij}
&
2\bar{k}_{m_1}^2 C_{\theta_o}S'_{ij}
&
\frac{\gamma\bar{c}^2}{48}C_{\theta_o}K_{ij}
&
\frac{\gamma\bar{c}^2}{48}C_{\theta_o}F_i
\\ \hline
\begin{array}{l}
2B_i\zeta_s\beta_1+4\bar{k}_m^2 S_{\theta_o}(-1)^{i+1}+2\beta_{pc}V_{oi} \\
+2\zeta_s(V_{oi}S_{\theta_o}+W_{oi}C_{\theta_o})+2(\zeta_s S_{\theta_o}-\beta_d C_{\theta_o})\sum\limits_{j=1}^{N} M_{ij}V_{oj} \\
+2\sum\limits_{j=1}^{N}\sum\limits_{k=1}^{N} T_{jik}V_{oj}(V_{ok}S_{\theta_o}+W_{ok}C_{\theta_o}) \\
+\frac{\gamma}{6}\Big\{-2\frac{c_{d_o}}{a}C_{\theta_o}\left(\sum\limits_{j=1}^{N} I_{ij}W_{oj}-\beta_d C_i\right) \\
+(S_{\theta_o}+\Phi_o C_{\theta_o})\Big[\sum\limits_{j=1}^{N}\left(\sum\limits_{k=1}^{N} X_{ijk}\Phi_{ok}-\theta_t J_{ij}\right)W_{oj} \\
-\beta_d\left(\sum\limits_{j=1}^{N} R_{ij}\Phi_{oj}-\theta_t D_i\right)\Big]+\bar{v}_i C_{\theta_o}(V_{oi}+\zeta_s B_i) \\
-\bar{v}_i C_{\theta_o}\sum\limits_{j=1}^{N}\left(\sum\limits_{k=1}^{N} W_{ijk}\Phi_{ok}-\theta_t I_{ij}\right)W_{oj} \\
+\beta_d\bar{v}_i C_{\theta_o}\left(\sum\limits_{j=1}^{N} Q_{ij}\Phi_{oj}-\theta_t C_i\right) \\
-(S_{\theta_o}+\Phi_o C_{\theta_o})\left(\sum\limits_{j=1}^{N} I_{ij}V_{oj}+\zeta_s C_i\right) \\
-2C_{\theta_o}\Big[\sum\limits_{j=1}^{N}\left(\sum\limits_{k=1}^{N} X_{ijk}\Phi_{ok}-\theta_t J_{ij}\right)V_{oj} \\
+\zeta_s\left(\sum\limits_{j=1}^{N} R_{ij}\Phi_{oj}-\theta_t D_i\right)V_{oj}\Big]\Big\}
\end{array}
&
\begin{array}{l}
-2B_i\beta_d\beta_1+4\bar{k}_{m_1}^2 C_{\theta_o}(-1)^{i+1}+2\beta_{pc}W_{oi} \\
-2\beta_d(V_{oi}S_{\theta_o}+W_{oi}C_{\theta_o})+2(\zeta_s S_{\theta_o}-\beta_d C_{\theta_o})\sum\limits_{j=1}^{N} M_{ij}W_{oj} \\
+2\sum\limits_{j=1}^{N}\sum\limits_{k=1}^{N} T_{jik}W_{oj}(V_{ok}S_{\theta_o}+W_{ok}C_{\theta_o}) \\
+\frac{\gamma}{6}\Big\{-2(S_{\theta_o}+\Phi_o C_{\theta_o})\left(\sum\limits_{j=1}^{N} I_{ij}W_{oj}-\beta_d C_i\right) \\
+2\bar{v}_i C_{\theta_o}(W_{oi}-\beta_d B_i)+C_{\theta_o}\Big[\sum\limits_{j=1}^{N} I_{ij}V_{oj} \\
+\bar{e}_1 V_{oi}+\zeta_s(C_i+\bar{e}_1 B_i)-\sum\limits_{j=1}^{N}\Big(\sum\limits_{k=1}^{N} X_{ijk}\Phi_{ok} \\
-\theta_t J_{ij}\Big)W_{oj}-\beta_d\left(\sum\limits_{j=1}^{N} R_{ij}\Phi_{oj}-\theta_t D_i\right)\Big]\Big\}
\end{array}
&
\begin{array}{l}
\frac{\gamma}{6}C_{\theta_o}\Big[\frac{\bar{c}^2}{8}F_i \\
-\frac{3\bar{c}}{4}\Big(\zeta_s G_i \\
+\sum\limits_{j=1}^{N} Q_{ji}V_{oj}\Big)\Big]
\end{array}
&
\begin{array}{l}
\frac{\gamma}{6}C_{\theta_o}\Big[\sum\limits_{i=1}^{N}\Big(\sum\limits_{j=1}^{N} I_{ij}V_{oi}V_{oj} \\
+2\zeta_s C_i V_{oi}-\frac{3\bar{c}}{4}B_i V_{oi}\Big) \\
+\frac{\zeta_s^2}{4}+\frac{\bar{c}^2}{16}-\frac{\zeta_s\bar{c}}{4}\Big]
\end{array}
\end{array}\right] \qquad (76)
$$

where the mode shape integrals are

$$A_i = \int_0^1 \Psi_i \, d\bar{x}$$

$$B_i = \int_0^1 \bar{x}\Psi_i \, d\bar{x}$$

$$C_i = \int_0^1 \bar{x}^2\Psi_i \, d\bar{x}$$

$$D_i = \int_0^1 \bar{x}^3\Psi_i \, d\bar{x}$$

$$E_i = \int_0^1 \Theta_i \, d\bar{x}$$

$$F_i = \int_0^1 \bar{x}\Theta_i \, d\bar{x}$$

$$G_i = \int_0^1 \bar{x}^2\Theta_i \, d\bar{x}$$

$$H_i = \int_0^1 \bar{x}^3\Theta_i \, d\bar{x}$$

$$I_{ij} = \int_0^1 \bar{x}\Psi_i\Psi_j \, d\bar{x}$$

$$J_{ij} = \int_0^1 \bar{x}^2\Psi_i\Psi_j \, d\bar{x}$$

$$K_{ij} = \int_0^1 \bar{x}\Theta_i\Theta_j \, d\bar{x}$$

$$L_{ij} = \int_0^1 (1-\bar{x})\Psi_i'\Psi_j' \, d\bar{x}$$

$$M_{ij} = \int_0^1 \left(\frac{1-\bar{x}^2}{2}\right) \Psi_i'\Psi_j' \, d\bar{x}$$

$$N_{ij} = \int_0^1 \left(\frac{1-\bar{x}^2}{2}\right) \Theta_i'\Theta_j' \, d\bar{x}$$

$$O_{ij} = \int_0^1 \bar{x}\Psi_i\Psi_j' \, d\bar{x}$$

$$P_{ij} = \int_0^1 \bar{x}\Psi_i''\Psi_j'' \, d\bar{x}$$

$$Q_{ij} = \int_0^1 \bar{x}\Psi_i\Theta_j \, d\bar{x}$$

$$Q_{ij}' = \int_0^1 \bar{x}\Theta_i\Psi_j' \, d\bar{x}$$

$$R_{ij} = \int_0^1 \bar{x}^2\Psi_i\Theta_j \, d\bar{x}$$

$$S_{ij} = \int_0^1 \Psi_i\Theta_j \, d\bar{x}$$

$$S_{ij}' = \int_0^1 \Theta_i\Psi_j' \, d\bar{x}$$

$$T_{ijk} = -\frac{1}{\beta_k^4} \int_0^1 \Psi_i'\Psi_j'\Psi_k''' \, d\bar{x}$$

$$U_{ijk} = \int_0^1 \bar{x}\Psi_i\Psi_j\Psi_k' \, d\bar{x} - \int_0^1 \bar{x}^2\Psi_i \int_0^{\bar{x}} \Psi_j'\Psi_k'' \, d\bar{x}_1 \, d\bar{x}$$

$$V_{ijk} = \int_0^1 \Theta_i\Psi_j''\Psi_k'' \, d\bar{x}$$

$$V_{ijk}' = \int_0^1 \bar{x}\Theta_i\Psi_j''\Psi_k'' \, d\bar{x}$$

$$W_{ijk} = \int_0^1 \Psi_i\Psi_j\Theta_k \, d\bar{x}$$

$$X_{ijk} = \int_0^1 \bar{x}\Psi_i\Psi_j\Theta_k \, d\bar{x}$$

$$Y_{ijk\ell} = \int_0^1 \Theta_i\Theta_j\Psi_k''\Psi_\ell'' \, d\bar{x}$$

$$Z_{ij} = \int_0^1 \bar{x}^2\Psi_i\Psi_j' \, d\bar{x} - \int_0^1 \bar{x}^3 \int_0^{\bar{x}} \Psi_i'\Psi_j'' \, d\bar{x}_1 \, d\bar{x} \qquad (77)$$

Many of these integrals have been evaluated in closed form by use of references 7 and 8. Additionally, these and all the remaining integrals were evaluated numerically. The matrix [K] is simply the Jacobian of equations (69)-(72) and will not be written out in detail. Representing equations (69)-(72) as

$$Y_i = 0 \qquad i = 1,2,\ldots,3N+1 \tag{78}$$

Then

$$[K] = \left[\frac{\partial Y_i}{\partial X_{oj}}\right] \tag{79}$$

where

$$\{X_{oj}\} = \lfloor V_{oj}, W_{oj}, \Phi_{oj}, \Phi_o \rfloor^T$$

We note that [M] and [K] are symmetric and [C] is antisymmetric for $\gamma = 0$ (*in vacuo*). Thus, equation (73), a standard eigenvalue problem, governs the stability of small motions about the equilibrium operating condition.

As described above, it is possible to use the torsion moment boundary condition, equation (6), instead of the root torsion equation (62). If this is done, the dimensionless form of equation (6) becomes

$$\begin{aligned}Q\Phi = {} & \kappa\phi'(0) + \frac{\bar{k}_A{}^2}{2}\,[\phi'(0) + \theta_{pt}'(0)] \\ & + \zeta_s\{\Lambda_1\bar{w}''(0) - (\Lambda_2 - \Lambda_1)S_{\bar{\theta}_o}[\bar{v}''(0)C_{\bar{\theta}_o} - \bar{w}''(0)S_{\bar{\theta}_o}]\} \\ & + \beta_d\{\Lambda_2\bar{v}''(0) - (\Lambda_2 - \Lambda_1)S_{\bar{\theta}_o}[\bar{v}''(0)S_{\bar{\theta}_o} + \bar{w}''(0)C_{\bar{\theta}_o}]\}\end{aligned} \tag{80}$$

The modal equilibrium equation is

$$\begin{aligned}Q\Phi_o = {} & \sqrt{2}\left(\kappa + \frac{\bar{k}_A{}^2}{2}\right)\sum_{i=1}^{N}\gamma_i\Phi_{oi} - \frac{\bar{k}_A{}^2}{2}\,\theta_t \\ & + 2\zeta_s\sum_{i=1}^{N}\beta_i{}^2[\Lambda_1 W_{oi} - (\Lambda_2 - \Lambda_1)S_{\bar{\theta}_o}(V_{oi}C_{\bar{\theta}_o} - W_{oi}S_{\bar{\theta}_o})] \\ & + 2\beta_d\sum_{i=1}^{N}\beta_i{}^2[\Lambda_2 V_{oi} - (\Lambda_2 - \Lambda_1)S_{\bar{\theta}_o}(V_{oi}S_{\bar{\theta}_o} + W_{oi}C_{\bar{\theta}_o})]\end{aligned} \tag{81}$$

The perturbation equations may be easily solved for $\Delta\Phi$ and substituted into the other equations since there are no time derivations. Thus,

$$\Delta\Phi = \frac{1}{Q}\left[\sqrt{2}\left(\kappa + \frac{\bar{k}_A{}^2}{2}\right)\sum_{i=1}^{N} \gamma_i\Delta\Phi_i + 2\zeta_s\Lambda_1 \sum_{i=1}^{N} \beta_i{}^2\Delta W_i\right.$$

$$+2\beta_d\Lambda_2 \sum_{i=1}^{N} \beta_i{}^2\Delta V_i - 2\zeta_s(\Lambda_2 - \Lambda_1)S_{\bar{\theta}_o} \sum_{i=1}^{N} \beta_i{}^2(\Delta V_i C_{\bar{\theta}_o} - \Delta W_i S_{\bar{\theta}_o})$$

$$\left. -2\beta_d(\Lambda_2 - \Lambda_1)S_{\bar{\theta}_o} \sum_{i=1}^{N} \beta_i{}^2(\Delta V_i S_{\bar{\theta}_o} + \Delta W_i C_{\bar{\theta}_o})\right] \tag{82}$$

Numerical results obtained with equations (81) and (82) are virtually identical with those using the $\delta\Phi_o$ and $\delta\Delta\Phi$ equations derived above based on integrated torsion moments. The use of equations (81) and (82) is considerably simpler and provides a reasonable check for numerical results.

Modal Analysis

We now describe a modal analysis that greatly simplifies numerical computation. From equation (73) the stability of small motions about the equilibrium operating condition is determined by the eigenvalues of the $6N + 2 \times 6N + 2$ matrix [P] where

$$\begin{Bmatrix} \Delta\dot{X} \\ \Delta\ddot{X} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} \Delta X \\ \Delta\dot{X} \end{Bmatrix} = [P] \begin{Bmatrix} \Delta X \\ \Delta\dot{X} \end{Bmatrix} \tag{83}$$

Since we are primarily concerned with lower frequency instabilities (first lead-lag, first flap, and first torsion frequencies), there is a value of N for which any increase in N will not appreciably change the eigenvalues associated with these lower frequencies. It is at this value of N that the eigenvalues are considered to be converged. For practical hingeless rotor configurations, N = 5 gives suitably converged eigenvalues; the matrix [P] is thus 32 × 32. By a change of modal coordinates, the size of the matrix [P] may be greatly reduced without significantly changing the eigenvalues of interest. Such a transformation may be found by first considering free vibrations *in vacuo* of the blade about the equilibrium deflected state. The equation of motion, analogous to equation (73), is

$$[M_s]\{\Delta\ddot{X}\} + [G]\{\Delta\dot{X}\} + [K_v]\{\Delta X\} = 0 \tag{84}$$

where the subscripts s and v imply the symmetric part and the vacuum case, respectively. Both $[M_s]$ and $[K_v]$ are thus symmetric; $[K_v]$ is equal to [K] with all the aerodynamic terms set equal to zero. The matrix [G] is antisymmetric and equal to [C] with all aerodynamic terms set equal to zero. The presence of [G] causes the eigenvectors of free vibration to be complex. This may be avoided for computational efficiency by approximating equation (84) as

$$[M_s]\{\Delta\ddot{X}\} + [K_v]\{\Delta X\} = 0 \tag{85}$$

The matrix of the eigenvectors [U] is orthogonal with respect to $[M_s]$. It is, therefore, approximately orthogonal with respect to [M] as well since the elements of the antisymmetric part of [M] are very small. Thus,

$$[U^T M_s U] = [I] \doteq [U^T M U] \tag{86}$$

According to Meirovitch (ref. 9) a so-called principal coordinate transformation for equation (73) may be determined by replacing $\{\Delta X\}$ by $[U]\{\Delta X\}$. We may then premultiply equation (73) by $[U]^T$ to take advantage of the form of equation (86) yielding

$$[I]\{\Delta\ddot{X}\} + [U^T CU]\{\Delta\dot{X}\} + [U^T KU]\{\Delta X\} = 0 \tag{87}$$

Hence,

$$\begin{Bmatrix} \Delta\dot{X} \\ \Delta\ddot{X} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -U^T KU & -U^T CU \end{bmatrix} \begin{Bmatrix} \Delta X \\ \Delta\dot{X} \end{Bmatrix} = [P^*] \begin{Bmatrix} \Delta X \\ \Delta\dot{X} \end{Bmatrix} \tag{88}$$

The matrices [P] and [P*] have virtually the same eigenvalues. However, because of the nature of this modal coordinate transformation from [P] to [P*], the rows and columns corresponding to high frequency modes of both $[U^T KU]$ and $[U^T CU]$ may be removed without affecting the eigenvalues of the low frequency modes of interest. These $3N + 2 \times 3N + 2$ matrices are thus reduced to $M \times M$ matrices whose rows and columns correspond to the M low frequency modes that are retained. The rows and columns that are retained in $[U^T KU]$ and $[U^T CU]$ may be chosen in two ways: (1) the M rows and columns that correspond to the M lowest frequency modes of the blade may be retained, or (2) the M rows and columns that correspond to M modes selected arbitrarily from the lowest lead-lag, the lowest flap, and the lowest torsion frequency modes are retained. For the second case, under certain conditions, M = 4 or 5 will result in converged eigenvalues. In either case, suitably converged results do not require M > 8.

The reduced matrices are analogous to stiffness and damping matrices generated from M coupled, rotating modes. Since the analysis is formulated in terms of standard cantilever mode shapes, however, repeated numerical integration of modal integrals is not necessary for different values of blade stiffnesses. Instead, the matrix operations described above lead to a net savings in CPU time.

CONCLUDING REMARKS

Hybrid equations of motion are developed for an elastic blade cantilevered in bending and having a torsional root spring to simulate pitch link flexibility. The blade is assumed to have coincident mass center, tension center, aerodynamic center, and elastic axes. Droop, precone, twist, sweep, torque offset, and blade root offset are included in the model. Quasi-steady aerodynamic loading is assumed to be adequate to investigate the low frequency type of unstable motion common in hingeless rotor systems. The solution is obtained by Galerkin's method and a modal analysis. The stability of small motions about the equilibrium operating condition is governed by a standard eigenvalue problem where the elements of the stability matrix depend on the solution of the equilibrium equations. In the analysis, two different forms of the root torsion equation are developed. One is based on the torsion moment boundary condition at the root of the blade and the other is based on integrated torsion moments derived from the kinetic energy. Numerical results for the two cases are virtually identical providing a reasonable check of the equations.

REFERENCES

1. Hodges, D. H.; and Dowell, E. H.: Nonlinear Equations of Motion for the Elastic Bending and Torsion of Twisted Nonuniform Rotor Blades. NASA TN D-7818, Dec. 1974.

2. Hodges, Dewey H.; and Ormiston, Robert A.: Stability of Elastic Bending and Torsion of Uniform Cantilever Rotor Blades in Hover with Variable Structural Coupling. NASA TN D-8192, April 1976.

3. Hodges, Dewey H.; and Ormiston, Robert A.: Stability of Hingeless Rotor Blades in Hover with Pitch-Link Flexibility. Proceedings of the 17th AIAA Structures, Structural Dynamics and Materials Conference, Valley Forge, Pennsylvania, May 1976, pp. 412-420.

4. Friedmann, P.; and Tong, P.: Dynamic Nonlinear Elastic Stability of Helicopter Rotor Blades in Hover and Forward Flight. (ASRL-TR-116-3, Massachusetts Institute of Technology; NAS2-6175.) NASA CR-114485, 1972.

5. Greenberg, J. Mayo: Airfoil in Sinusoidal Motion in a Pulsating Stream. NACA TN 1326, 1947.

6. Chang, Tish-Chun; and Craig, R. R., Jr.: On Normal Modes of Uniform Beams. EMRL 1068, Univ. of Texas (Austin), 1969.

7. Hodges, Dewey Harper: Nonlinear Bending and Torsion of Rotating Beams with Application to Linear Stability of Hingeless Helicopter Rotors. Ph.D. Thesis, Stanford University, Dec. 1972.

8. Felgar, R. P., Jr.: Formulas for Integrals Containing Characteristic Functions of a Vibrating Beam. Circular 14, Bur. Eng. Res., Univ. of Texas (Austin), 1950.

9. Meirovitch, Leonard: Elements of Vibration Analysis. McGraw-Hill Book Co., New York, 1975.

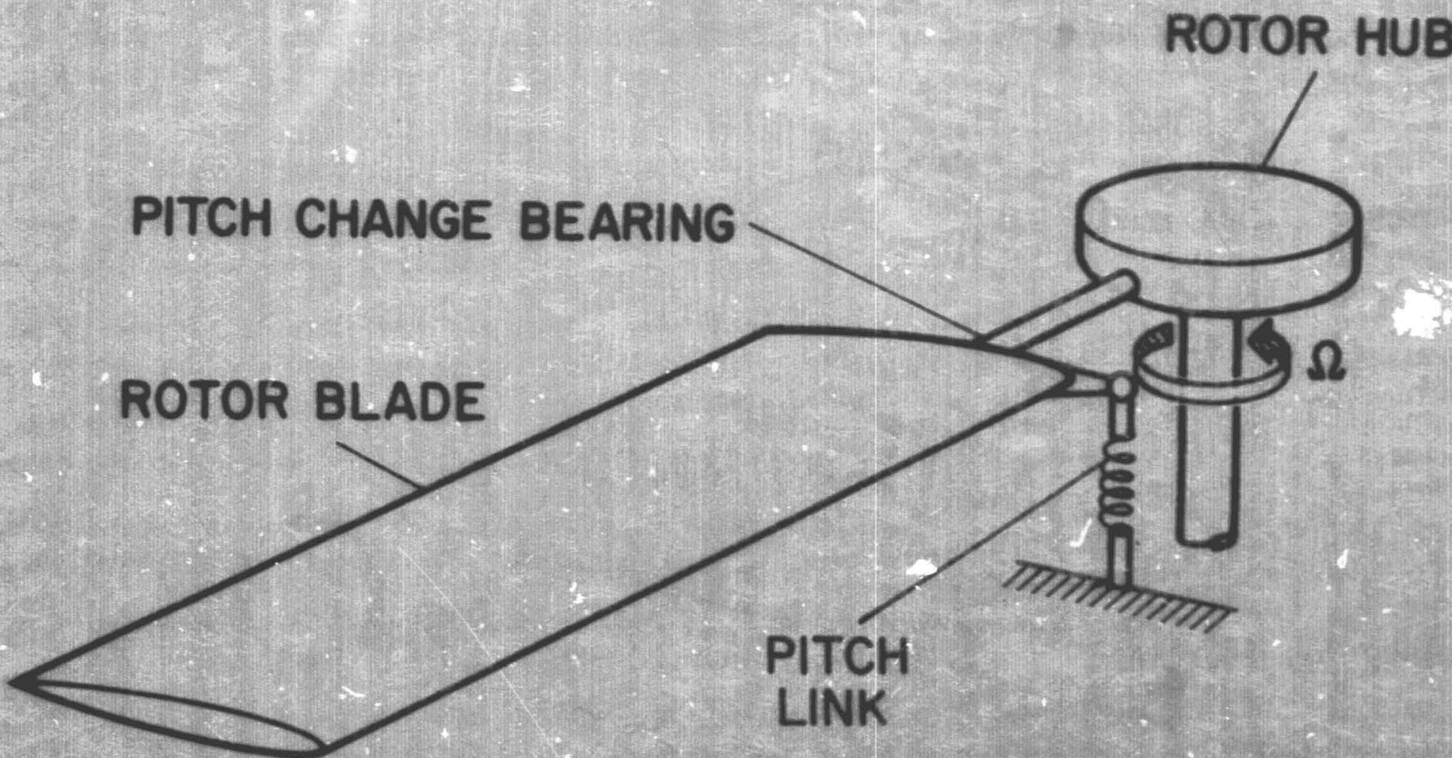

Figure 1.- Rotor blade configuration.

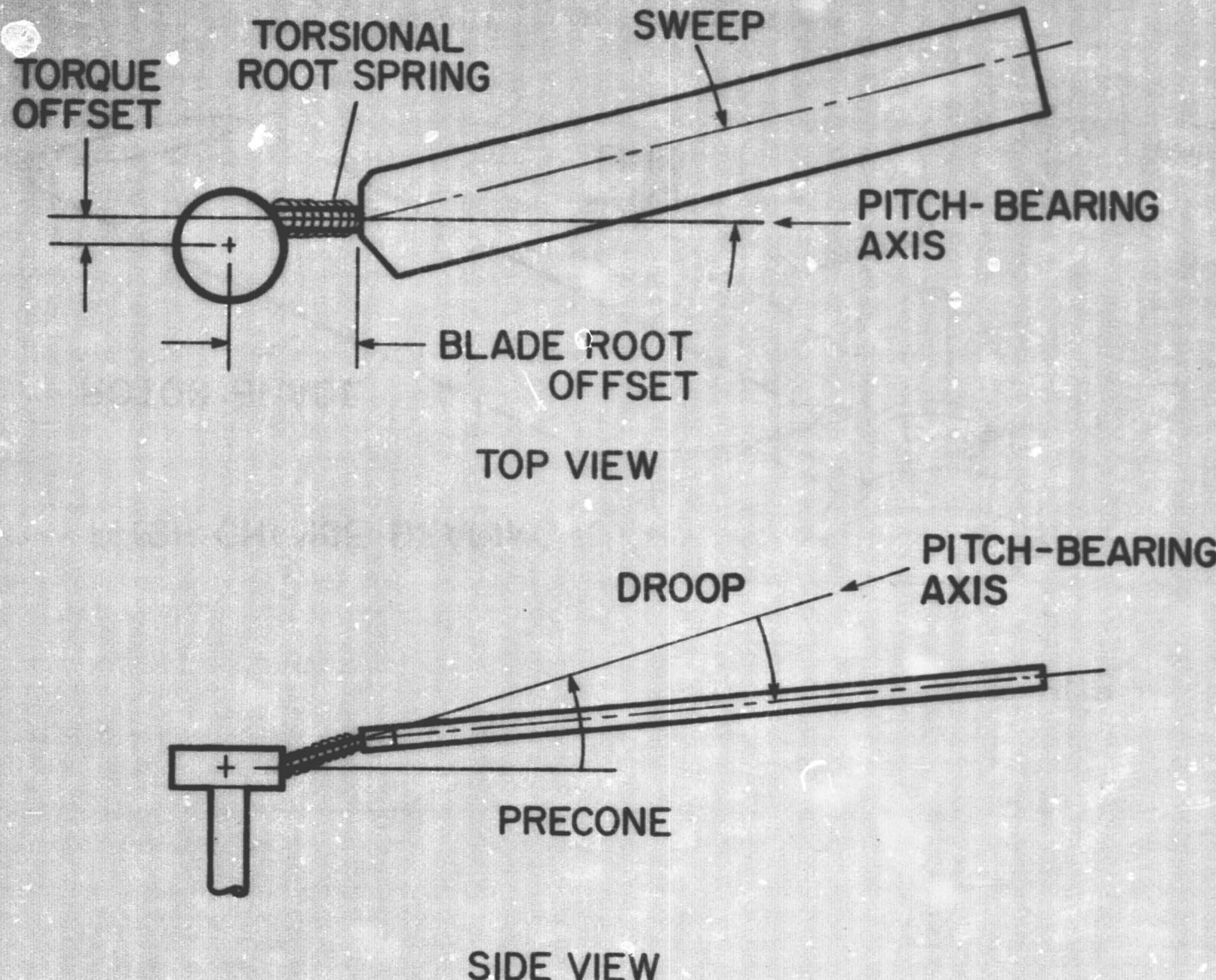

Figure 2.- Orientation of precone, droop, sweep, torque offset, and blade root offset.

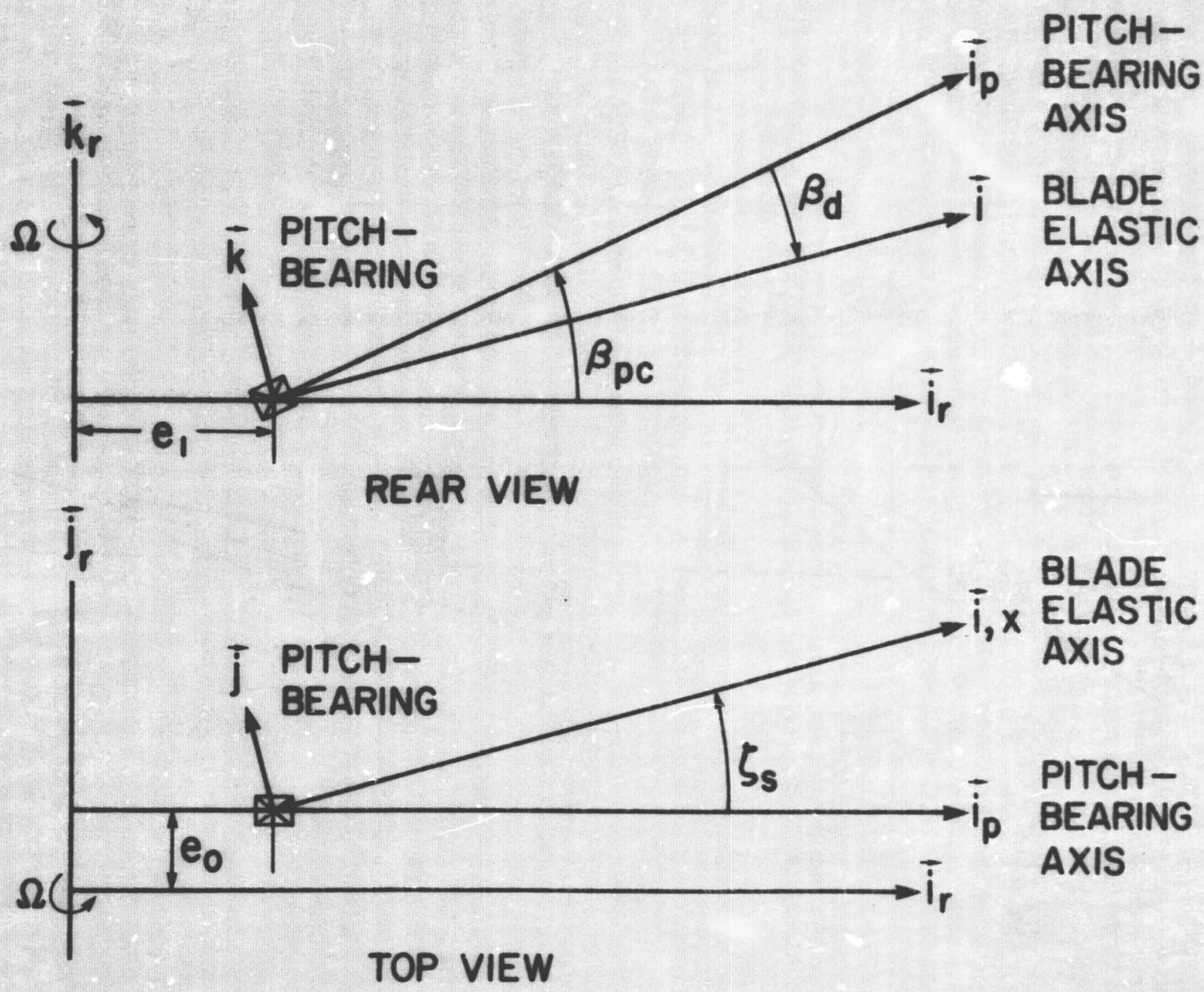

Figure 3.- Blade orientation with unit vectors, $\theta_o + \Phi = 0$.

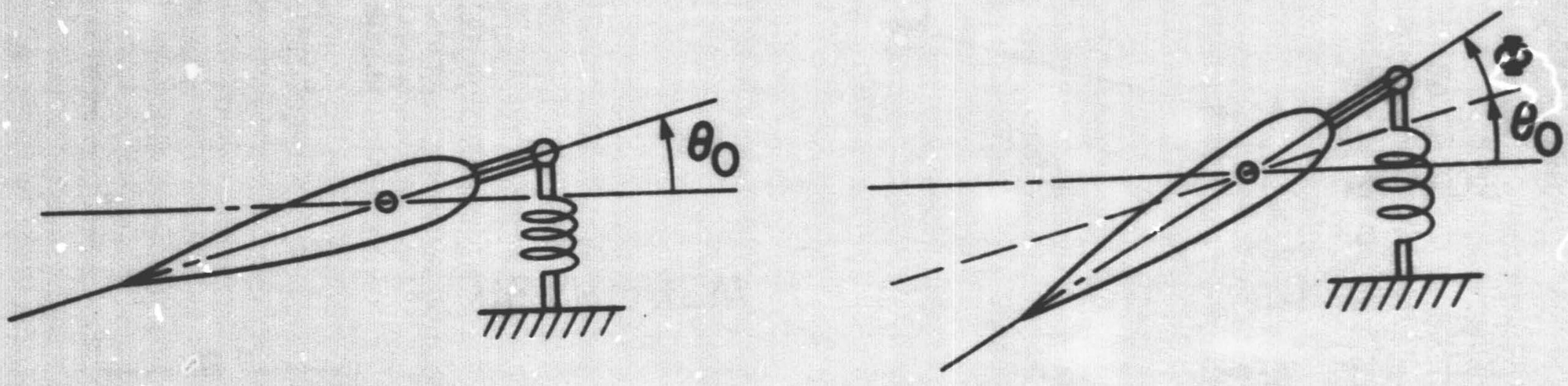

Figure 4.- Blade root pitch angle with and without pitch-link (spring) deformation.

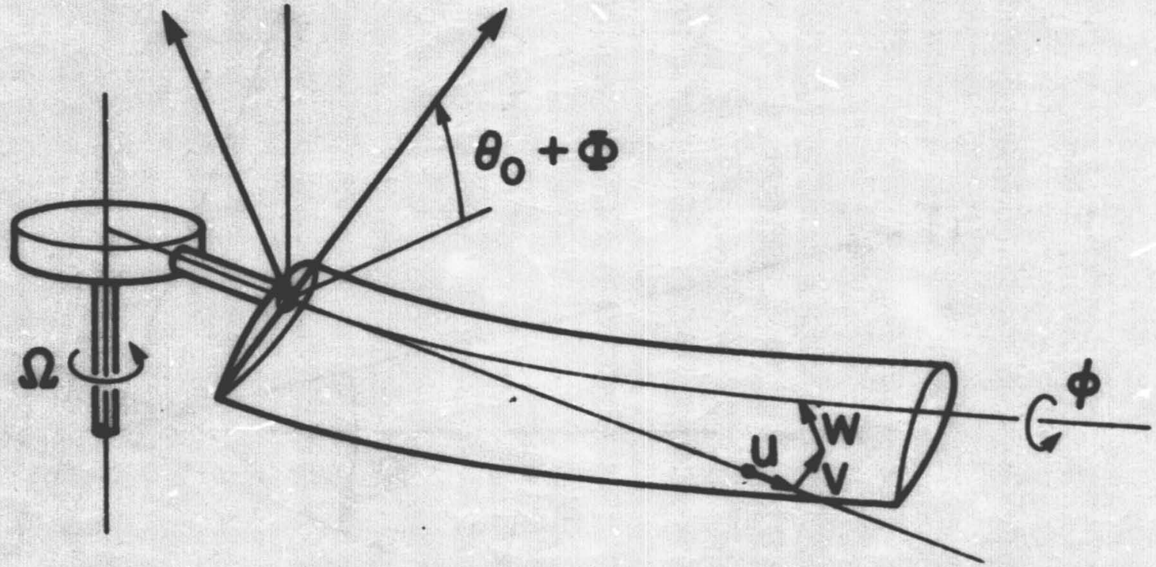

Figure 5.- Deformed blade showing orientation of elastic deformations u, v, w, and ϕ.

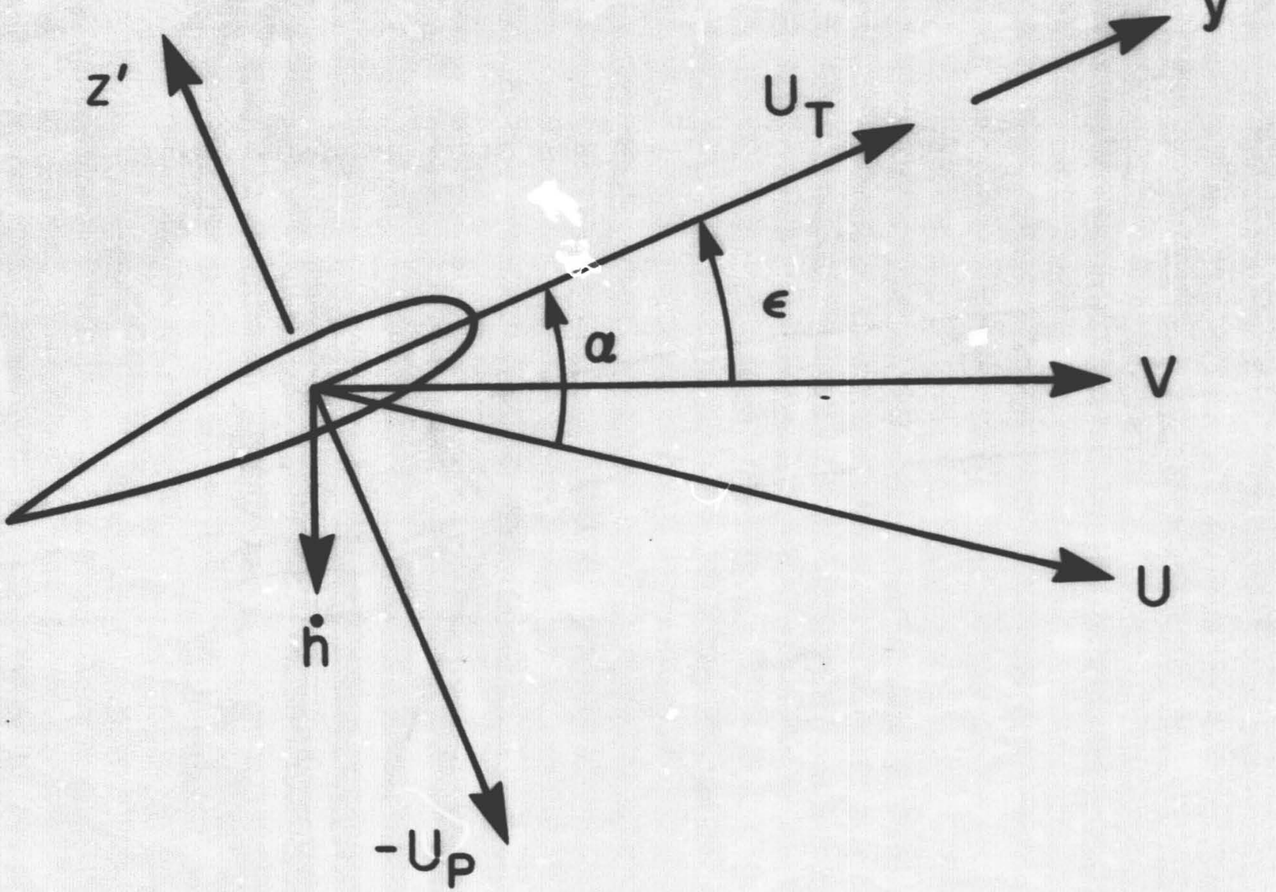

Figure 6.- Rotor blade airfoil section in general unsteady motion.

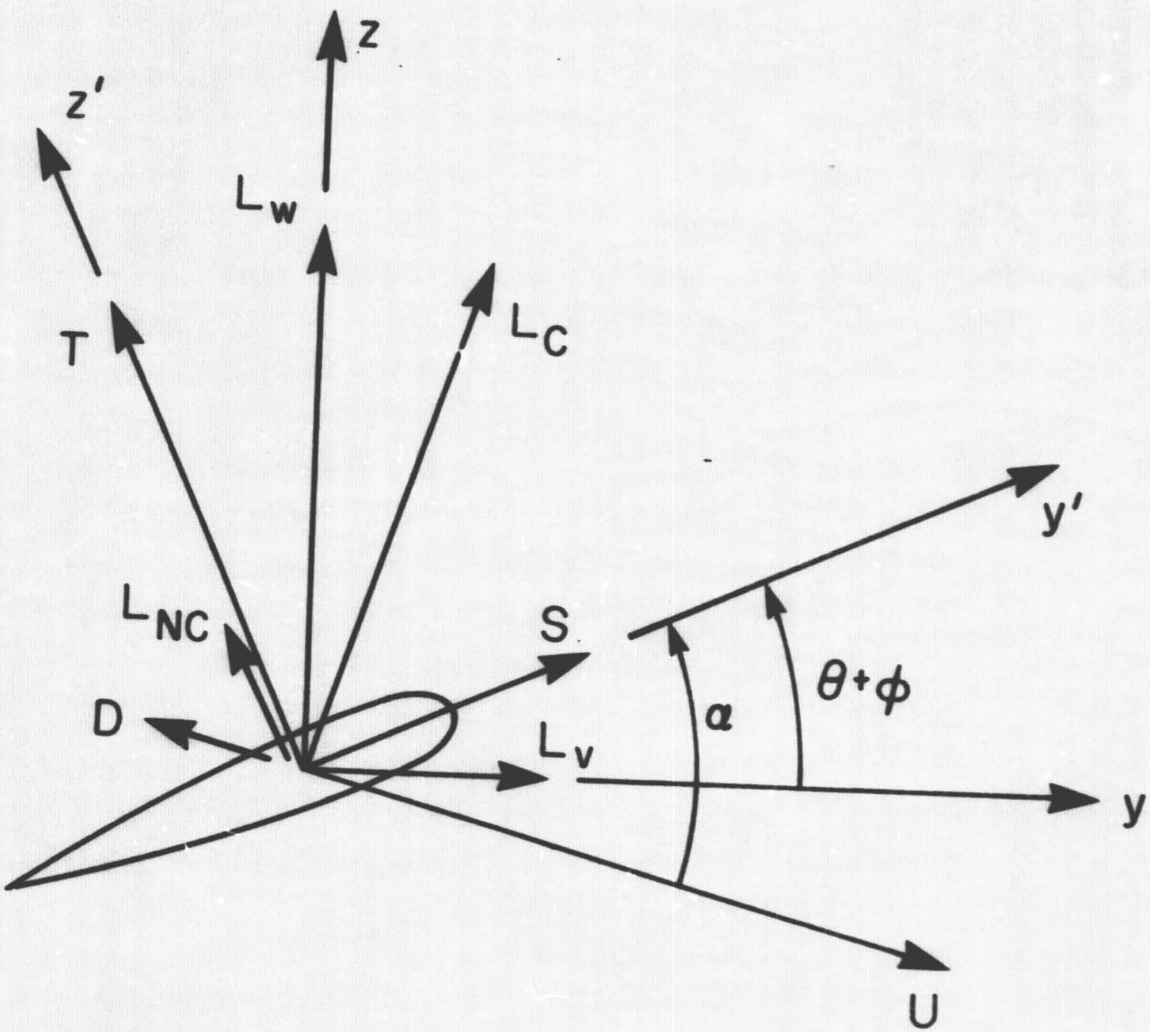

Figure 7.- Orientation of components of aerodynamic loading.