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STRUCTURAL DYNAMICS, STABILITY, AND CONTROL OF HELICOPTERS

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DYNAMIC SYNTHESIS OF HELICOPTERS

by

L. Meirovitch, L. G. Kraige, and A. L. Hale

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Abstract

This investigation is concerned with the dynamic synthesis of a helicopter. The method of approach is a variation of the component-mode synthesis in the sense that it regards the aircraft as an assemblage of interconnected substructures. The equations of motion are derived in general form by means of the Lagrangian formulation in conjunction with an orderly kinematical procedure that takes into account the superposition of motion of various substructures, thus circumventing constraint problems.

Contents

1. Introduction

This investigation is concerned with the development of a mathematical model capable of simulating the vibrational characteristics of a helicopter in various flight regimes, such as hover and forward flight. The helicopter represents a very complicated structure consisting of a given number of interconnected substructures. The object is to produce a mathematical formulation which includes all the system dynamic characteristics and yet is not so cumbersome as to defy analysis.

The method of approach represents a variation of the component-mode synthesis (Refs. 1 and 2) in the sense that it regards the aircraft as an assemblage of interconnected substructures acting as a system. The various substructures identified are the airframe, the transmission shaft, the main rotor (consisting of the hub and the rotor blades), and the tail rotor (regarded as a rigid fan). To ensure that the various substructures are acting as parts of the whole structure, an orderly kinematical procedure is developed which takes into account automatically the superposition of motions. This procedure does away with the question of constraints.

The approach based on the substructure concept has the advantage that it permits a large measure of versatility in the mathematical modeling of the substructures. For example, due to the complex configuration, the airframe is best represented by a discrete model. On the other hand, the transmission shaft or a rotor blade can best be represented by a continuous model. In the final synthesis, each of the substructures is simulated by only a limited number of degrees of freedom. To this end, one expresses the displacements as a superposition of space-dependent modes multiplied by time-dependent generalized coordinates. For the space-dependent functions or vectors one must use rigid-body modes and deformation modes capable of describing the motion of the substructure with sufficient accuracy. This Rayleigh-Ritz type approach permits the simulation of the aircraft by a discrete system.

To derive the system equations of motion, it is desirable to use an approach devoid of unnecessary complications. An approach fitting this description is the Lagrangian approach which is capable of producing the equations of motion of the system without the need of calculating constraint forces acting at points connecting various substructures. To this end, one must use a consistent kinematical representation to calculate the inertial velocity of every mass point of the aircraft. Such a representation necessitates the introduction of a number of reference frames associated with the various substructures. The motion of these reference frames can be regarded as representing the rigid body motion, and the motion of a point relative to the frames can be regarded as representing the elastic motion of the substructure.

The Lagrangian approach requires the calculation of the kinetic energy, the potential energy, and the nonconservative virtual work for the elastic system. The potential energy is due to the structural elasticity and gravitational forces. On the other hand, the nonconservative virtual work is due mainly to aerodynamic forces. A relatively complicated procedure is necessary to transform the physical aerodynamic forces and torque into generalized forces.

The equations of motion are nonlinear and their general solution is beyond the state of the art. Fortunately, general solutions are not necessary, and in fact may not even be very informative. Indeed, quite often the interest lies in solutions in the neighborhood of certain particular solutions, where the latter are generally known as trim solutions. Trim solutions of special interest are those associated with hover and forward flight. To examine the motion in the neighborhood of trim solutions, it is necessary to derive the so-called variational equations, which are really perturbation equations about the trim.

In deriving variational equations, one must make certain simplifying assumptions. These assumptions require a judgement as to the expected order of magnitude of the various generalized coordinates and system parameters. Such a set of assumptions constitute a so-called ordering scheme and it applies primarily to the main rotor.

The derivation of the variational equations implies an extremely large number of matrix multiplications and differentiations with respect to the state variables and time. In addition, it implies the selection from the multitude of terms only those which the ordering scheme deems essential. The magnitude of the task demands an efficient approach. To this end, a procedure for the derivation of the variational equations by means of computer manipulation is developed.

2. Generalized Coordinates and Velocities

We shall be concerned with the flight of a helicopter in the neighborhood of a given point on the earth's surface, so that the gravitational field can be regarded as uniform. It will prove convenient to introduce a set of inertial axes XYZ with the origin at a point O on the surface of the earth, so that axes X and Y are in the local horizontal plane and Z is aligned with the local vertical, and measure the motion relative to this inertial frame.

The mathematical formulation will be produced by regarding the helicopter as an assemblage of a given number of interconnected substructures. This requires a set of kinematically consistent coordinates, in the sense that the motion of one substructure must take into account the motion of another. A kinematically consistent formulation can be obtained by describing the motion of the substructures independently and then imposing constraints guaranteeing that points shared by two substructures undergo the same motion. It can also be obtained by means of an orderly kinematical procedure taking into account automatically the superposition of motions. For example, if the airframe, the transmission shaft, and the rotor are identified as different substructures, then the absolute motion of the rotor can be regarded as a superposition of the motion of the airframe relative to the inertial space, the motion of the transmission shaft relative to the airframe, and the motion of the rotor relative to the transmission shaft. This procedure eliminates the need of constraint equations and is the one used in this investigation.

To describe the motion of the airframe, it will prove advantageous to introduce a set of airframe body axes $x_A y_A z_A$ with the origin A at an arbitrary point in the undeformed airframe, so that x_A is along the forward direction, z_A is along the vertical, and y_A is normal to both so as to form a right-hand system. Then the motion of the airframe can be described in terms of the translation of the origin A, the rotation of the airframe body axes $x_A y_A z_A$ relative to the inertial axes XYZ, and the elastic motion of the airframe relative to $x_A y_A z_A$. The translation of A relative to XYZ is given simply by the position vector $\underset{\sim}{w}_{0A}$ from 0 to A and the absolute velocity vector $\dot{\underset{\sim}{w}}_{0A}$. They can be expressed in terms of the column matrices $\{w_{0A}\} = [w_{0AX} w_{0AY} w_{0AZ}]^T$ and $\{\dot{w}_{0A}\} = [\dot{w}_{0AX} \dot{w}_{0AY} \dot{w}_{0AZ}]^T$, respectively, where the meaning of the vector components is obvious. On the other hand, the rotation of axes $x_A y_A z_A$ relative to XYZ is fully determined by the matrix of direction cosines between these two sets of axes and by the angular velocity vector of axes $x_A y_A z_A$ relative to the inertial space. In defining the orientation of axes $x_A y_A z_A$ relative to axes XYZ, it is convenient to regard $x_A y_A z_A$ as a triad originally coincident with axes XYZ and moving relative to these axes. Then, the orientation of axes $x_A y_A z_A$ relative to axes XYZ can be obtained by means of three rotations: λ_z about z_A, λ_x about x_A, and λ_y about y_A in that order (see Fig. 2). This permits us to write the coordinate transformation from one set of axes to the other in the compact matrix form

$$\{r_A\} = [T_{A0}]\{R\} \tag{1}$$

where

$$\{r_A\} = [x_A \; y_A \; z_A]^T, \quad \{R\} = [X \; Y \; Z]^T \tag{2}$$

are position vectors and

$$[T_{A0}] = [\lambda_y][\lambda_x][\lambda_z] \tag{3a}$$

is the transformation matrix, in which

$$[\lambda_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\lambda_x & s\lambda_x \\ 0 & -s\lambda_x & c\lambda_x \end{bmatrix}, \quad [\lambda_y] = \begin{bmatrix} c\lambda_y & 0 & -s\lambda_y \\ 0 & 1 & 0 \\ s\lambda_y & 0 & c\lambda_y \end{bmatrix}, \quad [\lambda_z] = \begin{bmatrix} c\lambda_z & s\lambda_z & 0 \\ -s\lambda_z & c\lambda_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3b}$$

Note that Eqs. (3b) represent transformation matrices defining the rotations described above, in which $s\lambda_x = \sin \lambda_x$, $c\lambda_x = \cos \lambda_x$, etc. Also from Fig. 2 it can be easily verified that the absolute angular velocity vector $\underset{\sim}{\Omega}_A$, which defines the angular velocity of axes $x_A y_A z_A$ relative to axes XYZ and which is also equal to the relative angular velocity vector $\underset{\sim}{\omega}_A$, can be written in the matrix form

$$\{\Omega_A\} = \{\omega_A\} = \dot{\lambda}_z[\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[\lambda_y]\{e_1\} + \dot{\lambda}_y\{e_2\} \tag{4}$$

where

$$\{e_1\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \{e_2\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \{e_3\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{5}$$

are unit vectors written in matrix form.

Equation (4) gives $\{\Omega_A\}$ in terms of components along axes $x_A y_A z_A$, whereas $\{\dot{w}_{0A}\} = [\dot{w}_{0AX} \dot{w}_{0AY} \dot{w}_{0AZ}]^T$ is expressed in terms of inertial components. However, there are equations requiring that $\{\dot{w}_{0A}\}$ and $\{\Omega_A\}$ be expressed in terms of components along the same system of axes. It is often convenient to write $\{\dot{w}_{0A}\}$ in terms of components along axes $x_A y_A z_A$, which can be done by simply premultiplying $\{\dot{w}_{0A}\}$ by the transformation matrix $[T_{A0}]$.

The elastic motion of the airframe requires first a definition of the airframe itself. The major question is whether the power train, consisting of the transmission assembly, the transmission shaft, and the control linkages, should be considered as part of the airframe or as a separate subsystem. Moreover, there is the question of the tail rotor. In view of the difficulties encountered in accounting properly for the flexibility of the transmission shaft in airframe free-free modes, and because we desire a formulation independent of the type of airframe modes used, the transmission assembly and shaft will be treated as a separate elastic subsystem connecting the airframe to the main rotor hub. As such, we will ignore the mass of the transmission itself* and account for the transmission mount flexibility by introducing two torsional springs at the base of the shaft acting along orthogonal axes which are perpendicular to the shaft. The flexibility of the transmission shaft itself will be described in terms of two bending and one torsional displacement. The tail rotor will be regarded as a rigid fan, not part of the airframe, so that the airframe is assumed to exclude the transmission, transmission shaft, control linkages, and tail rotor. Note also that the control linkages are not being considered explicitly but only through their kinematical effects.

Next, let us consider any arbitrary mass point in the airframe. The position of this point relative to the inertial space is given by

$$\underset{\sim}{w}_A = \underset{\sim}{w}_{0A} + \underset{\sim}{r}_A + \underset{\sim}{u}_A \tag{6}$$

where $\underset{\sim}{r}_A$ is the radius vector from A to the point in question when the airframe is undeformed and $\underset{\sim}{u}_A$ is the elastic displacement vector of that point. Recognizing that $\underset{\sim}{r}_A$ and $\underset{\sim}{u}_A$ are measured relative to the moving axes $x_A y_A z_A$, the inertial velocity of the point is simply

$$\dot{\underset{\sim}{w}}_A = \dot{\underset{\sim}{w}}_{0A} + \underset{\sim}{\Omega}_A \times (\underset{\sim}{r}_A + \underset{\sim}{u}_A) + \dot{\underset{\sim}{u}}_A = \dot{\underset{\sim}{w}}_{0A} - (\underset{\sim}{r}_A + \underset{\sim}{u}_A) \times \underset{\sim}{\Omega}_A + \dot{\underset{\sim}{u}}_A \tag{7}$$

where $\dot{\underset{\sim}{w}}_{0A}$ and $\underset{\sim}{\Omega}_A$ were defined earlier and $\dot{\underset{\sim}{u}}_A$ is the elastic velocity vector

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*This assumption may be discarded later if it is found that the inertia of the transmission is significant.

of the mass point measured relative to $x_A y_A z_A$. Equation (7) can be written in the matrix form

$$\{\dot{w}_A\} = [T_{A0}]\{\dot{w}_{0A}\} - [r_A + u_A]\{\Omega_A\} + \{\dot{u}_A\} \tag{8}$$

where $[r_A + u_A]$ is a skew symmetric matrix defined by

$$[r_A + u_A] = \begin{bmatrix} 0 & -(r_{Az} + u_{Az}) & r_{Ay} + u_{Ay} \\ r_{Az} + u_{Az} & 0 & -(r_{Ax} + u_{Ax}) \\ -(r_{Ay} + u_{Ay}) & r_{Ax} + u_{Ax} & 0 \end{bmatrix} \tag{9}$$

in which r_{Ax}, r_{Ay}, r_{Az} are the components of $\underset{\sim}{r}_A$ and u_{Ax}, u_{Ay}, u_{Az} are the components of $\underset{\sim}{u}_A$.

We note that the vectors $\{\dot{w}_{0A}\}$ and $\{\Omega_A\}$ represent the translation of the origin A of axes $x_A y_A z_A$ and the rotation of these axes. They can be interpreted as the "rigid-body modes" of the airframe. On the other hand, $\{u_A\}$ represents the elastic displacement vector relative to the airframe axes $x_A y_A z_A$. As customary in the analysis of complex structures, we assume that the elastic displacements of the airframe can be represented as a linear combination of space dependent functions multiplying time dependent coordinates, where the first are referred to as "airframe modes". There are two main possibilities. One is to use "free-free modes", which can be obtained by regarding the airframe as being clamped at point A, where A is taken to coincide with the center of mass of the undeformed airframe. Another possibility is to use "cantilever modes", which can be obtained by regarding the airframe as being fixed at the base of the transmission. The implications of these possibilities will be examined in the next section.

Equation (8) defines the motion of any mass point of the airframe. One point of particular interest is that corresponding to the lower end of the transmission shaft, namely, the end attached to the transmission and trans-

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mission mount. We shall denote this point by S and attach a set of axes $x_{ES}y_{ES}z_{ES}$ to the airframe at this point such that z_{ES} coincides with the shaft axis before any deformation takes place and x_{ES} and y_{ES} are attached to the airframe and are normal to z_{ES} (Fig. 3). Another set of axes, the shaft axes $x_S y_S z_S$, have the origin at the same point S and are obtained from axes $x_{ES}y_{ES}z_{ES}$ by means of three rotations. Considering axes $x_S y_S z_S$ to be initially coincident with axes $x_{ES}y_{ES}z_{ES}$, these rotations are ψ_x about x_S, ψ_y about y_S, and ψ_z about z_S in that order, where ψ_x, ψ_y are small angles permitted by the transmission mount flexibility and $\dot{\psi}_z = \Omega$ is the constant angular velocity imparted to the shaft by the engine. The position of point S relative to point A is given by the vector $\underset{\sim}{r}_{AS} + \underset{\sim}{u}_{AS}$, where $\underset{\sim}{r}_{AS}$ is the radius vector from A to S when the airframe is undeformed and $\underset{\sim}{u}_{AS}$ is the elastic displacement vector of point S relative to axes $x_A y_A z_A$. The translational velocity vector $\dot{\underset{\sim}{w}}_{AS}$ of point S is obtained in matrix form by simply introducing the coordinates of point S in Eq. (8). On the other hand, the angular velocity vector of the frame $x_S y_S z_S$ is simply $\underset{\sim}{\Omega}_A + \underset{\sim}{\omega}_{ES} + \underset{\sim}{\omega}_S$, where $\underset{\sim}{\omega}_{ES}$ is the elastic angular velocity vector of axes $x_{ES}y_{ES}z_{ES}$ relative to axes $x_A y_A z_A$ and $\underset{\sim}{\omega}_S$ is the angular velocity vector of axes $x_S y_S z_S$ relative to axes $x_{ES}y_{ES}z_{ES}$.

The motion of point S is in terms of components along axes $x_A y_A z_A$. In particular, the translational velocity of S is

$$\{\dot{w}_{AS}\} = [T_{AO}]\{\dot{w}_{OA}\} - [r_{AS} + u_{AS}]\{\Omega_A\} + \{\dot{u}_{AS}\} \tag{10}$$

where $[r_{AS} + u_{AS}]$ can be obtained from Eq. (9) by replacing $r_{Ax} + u_{Ax}$ (x_A, y_A, z_A) by $r_{ASx} + u_{ASx}$ (x_{AS}, y_{AS}, z_{AS}), etc., and where $\{\dot{u}_{AS}\}$ is a vector having the components $\dot{u}_{ASx}$, $\dot{u}_{ASy}$, $\dot{u}_{ASz}$. If the interest lies in working with components along x_S, y_S, z_S, then these can be obtained by premultiplying the components along $x_A y_A z_A$ by the transformation matrix $[T_{SA}] = [\ell_S][\ell_{ES}][\ell_{GS}]$ representing the matrix of direction cosines between these two sets of axes, where $[\ell_{ES}][\ell_{GS}]$ represents the matrix of direction cosines between axes $x_A y_A z_A$ and $x_{ES} y_{ES} z_{ES}$ and $[\ell_S]$ is the matrix of direction cosines between axes $x_{ES} y_{ES} z_{ES}$ and $x_S y_S z_S$. Note that $[\ell_{GS}]$ represents the matrix of direction cosines between axes $x_A y_A z_A$ and $x_{ES} y_{ES} z_{ES}$ due to the geometrical configuration when the airframe is undeformed, whereas $[\ell_{ES}]$ is due to the airframe elastic deformations. To obtain the latter matrix, we assume that the orientation of axes $x_{ES} y_{ES} z_{ES}$ is defined by three infinitesimal rotations given by the curl of the elastic displacement vector $[\ell_{GS}]\{u_{AS}\}$. Therefore, we can write $[\ell_{ES}]$ in the form

$$[\ell_{ES}] = \begin{bmatrix} 1 & \left(\frac{\partial u^*_{ASy}}{\partial x_{ES}} - \frac{\partial u^*_{ASx}}{\partial y_{ES}}\right) & -\left(\frac{\partial u^*_{ASx}}{\partial z_{ES}} - \frac{\partial u^*_{ASz}}{\partial x_{ES}}\right) \\ -\left(\frac{\partial u^*_{ASy}}{\partial x_{ES}} - \frac{\partial u^*_{ASx}}{\partial y_{ES}}\right) & 1 & \left(\frac{\partial u^*_{ASz}}{\partial y_{ES}} - \frac{\partial u^*_{ASy}}{\partial z_{ES}}\right) \\ \left(\frac{\partial u^*_{ASx}}{\partial z_{ES}} - \frac{\partial u^*_{ASz}}{\partial x_{ES}}\right) & -\left(\frac{\partial u^*_{ASz}}{\partial y_{ES}} - \frac{\partial u^*_{ASy}}{\partial z_{ES}}\right) & 1 \end{bmatrix} \tag{11}$$

where u^*_{ASx}, u^*_{ASy}, u^*_{ASz} are the components of the vector $[\ell_{GS}]\{u_{AS}\}$ and where it is understood that the partial derivatives are to be evaluated at the point S. Furthermore, the matrix $[\ell_S]$ can be shown to have the explicit form

$$[\ell_S] = [\psi_z][\psi_y][\psi_x] \tag{12a}$$

in which

$$[\psi_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi_x & s\psi_x \\ 0 & -s\psi_x & c\psi_x \end{bmatrix}, \quad [\psi_y] = \begin{bmatrix} c\psi_y & 0 & -s\psi_y \\ 0 & 1 & 0 \\ s\psi_y & 0 & c\psi_y \end{bmatrix}, \quad [\psi_z] = \begin{bmatrix} c\psi_z & s\psi_z & 0 \\ -s\psi_z & c\psi_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12b}$$

The angular velocity vector of the frame $x_{ES}y_{ES}z_{ES}$ relative to $x_Ay_Az_A$ is recognized simply as the curl of the elastic velocity vector $\dot{\underset{\sim}{u}}_{AS}$ which can be shown to have the vector form

$$\underset{\sim}{\omega}_{ES} = \underset{\sim}{\nabla} \times \dot{\underset{\sim}{u}}_{AS} \tag{13}$$

where $\underset{\sim}{\nabla}$ is the well known del operator. Equation (13) can be written in the matrix form

$$\{\omega_{ES}\} = [\nabla]([\ell_{GS}]\{\dot{u}_{AS}\}) \tag{14}$$

in which

$$[\nabla] = \begin{bmatrix} 0 & -\frac{\partial}{\partial z_{ES}} & \frac{\partial}{\partial y_{ES}} \\ \frac{\partial}{\partial z_{ES}} & 0 & -\frac{\partial}{\partial x_{ES}} \\ -\frac{\partial}{\partial y_{ES}} & \frac{\partial}{\partial x_{ES}} & 0 \end{bmatrix} \tag{15}$$

is the skew symmetric matrix operator, which is the matrix counterpart of the $\underset{\sim}{\nabla}\times$ vector operator. On the other hand, the angular velocity of axes $x_Sy_Sz_S$ relative to axes $x_{ES}y_{ES}z_{ES}$ can be shown to be

$$\{\omega_S\} = \dot{\psi}_x[\psi_z][\psi_y]\{e_1\} + \dot{\psi}_y[\psi_z]\{e_2\} + \Omega\{e_3\} \tag{16}$$

where $\{e_i\}$ $(i = 1,2,3)$ are given by Eqs. (5). Hence, the total angular velocity of axes $x_Sy_Sz_S$ has the form

$$\{\Omega_S\} = [T_{SA}]\{\Omega_A\} + [\ell_S]\{\omega_{ES}\} + \{\omega_S\} \tag{17}$$

Next, we wish to define the motion of the shaft. To this end, let us consider an arbitrary point on the shaft. The position of this point relative to axes $x_Sy_Sz_S$ is given by the radius vector $\underset{\sim}{r}_S + \underset{\sim}{u}_S$, where $\underset{\sim}{r}_S$ is the position of the point in question when the shaft is undeformed and $\underset{\sim}{u}_S$

is the elastic displacement of that point relative to axes $x_S y_S z_S$. Denoting by $\dot{\underset{\sim}{u}}_S$ the elastic velocity of the point relative to axes $x_S y_S z_S$, the inertial velocity of this point can be written as

$$\dot{\underset{\sim}{w}}_S = \dot{\underset{\sim}{w}}_{AS} + \underset{\sim}{\Omega}_S \times (\underset{\sim}{r}_S + \underset{\sim}{u}_S) + \dot{\underset{\sim}{u}}_S = \dot{\underset{\sim}{w}}_{AS} - (\underset{\sim}{r}_S + \underset{\sim}{u}_S) \times \underset{\sim}{\Omega}_S + \dot{\underset{\sim}{u}}_S \tag{18}$$

But $\dot{\underset{\sim}{w}}_{AS}$ is given in terms of components along axes $x_A y_A z_A$, whereas the remaining two terms are in terms of components along axes $x_S y_S z_S$. To express all terms in Eq. (10) in terms of components along axes $x_S y_S z_S$, we premultiply $\{\dot{w}_{AS}\}$ by the transformation matrix $[T_{SA}]$, so that Eq. (18) can be written in the matrix form

$$\{\dot{w}_S\} = [T_{SA}]\{\dot{w}_{AS}\} - [r_S + u_S]\{\Omega_S\} + \{\dot{u}_S\} \tag{19}$$

To describe the elastic motion, it is convenient to regard the shaft as a one-dimensional member. Note that this is in direct contrast with the airframe which was regarded as a three-dimensional structure. We can write the position of any point on the deformed shaft relative to axes $x_S y_S z_S$ in the matrix form (Fig. 4)

$$\{r_S + u_S\} = \begin{Bmatrix} u_S \\ v_S \\ z_S \end{Bmatrix} \tag{20}$$

where u_S, v_S are elastic displacements parallel to axes x_S, y_S respectively, of a point originally on the axis z_S and at a distance z_S from S. Axial elastic displacements have been assumed to be smaller than u_S or v_S and have been ignored. The shaft also undergoes torsion, but this does not affect $\{\dot{w}_S\}$, which, in view of Eq. (20), is the velocity of a typical point on the shaft, originally on the axis z_S. The torsion does, however, affect the orientation of a set of axes attached to the shaft at any point z_S

which were originally parallel with axes $x_S y_S z_S$ and are moving with the shaft during deformation. Denoting by $[T_S]$ the transformation matrix from the undeformed shaft axes to the deformed axes, we can write

$$[T_S] = \begin{bmatrix} 1 & \phi_S & -u'_S \\ -\phi_S & 1 & -v'_S \\ u'_S & v'_S & 1 \end{bmatrix} \tag{21}$$

in which ϕ_S is the elastic torsional displacement about axis z_S and where primes denote differentiations with respect to the spatial variable z_S.

To define the motion of the rotor, let us assume that the upper end of the shaft coincides with the geometric center H of the hub. Then let us introduce the hub axes $x_H y_H z_H$ with the origin at H and with axis z_H along the rotor spin axis; axes x_H and y_H are attached to the hub and are normal to z_H. Note that axes x_H and y_H are parallel to axes x_S and y_S when the shaft is undeformed. The relation between the direction of axes $x_H y_H z_H$ and $x_S y_S z_S$ has the general form

$$\{r_H\} = [T_{HS}]\{r_S\} \tag{22}$$

where the matrix $[T_{HS}]$ of direction cosines is simply the matrix $[T_S]$ given by Eq. (21) evaluated at $z_S = L_S$, where L_S is the length of the shaft. Moreover, the angular velocity of the hub axes $x_H y_H z_H$ is

$$\{\Omega_H\} = [T_{HS}]\{\Omega_S\} + [T_{HS}]\{\omega_{EH}\} \tag{23}$$

where

$$\{\omega_{EH}\} = [-\dot{v}'_S(L_S,t) \quad \dot{u}'_S(L_S,t) \quad \dot{\phi}_S(L_S,t)]^T \tag{24}$$

is the angular velocity of axes $x_H y_H z_H$ relative to axes $x_S y_S z_S$ due to the elastic angular motions at the upper end of the shaft. The absolute velocity of point H is simply Eq. (19) evaluated at $z_S = L_S$ and can be written in the symbolic matrix form

$$\{\dot{w}_{SH}\} = [T_{SA}]\{\dot{w}_{AS}\} - [r_{SH} + u_{SH}]\{\Omega_S\} + \{\dot{u}_{SH}\} \tag{25}$$

The tail rotor is assumed to be a rigid fan spinning at the angular velocity $\underset{\sim}{\omega}_T$ relative to the airframe. Using the analogy with the transmission shaft, let $x_{ET}y_{ET}z_{ET}$ be a set of axes attached to the frame at the point T, coinciding with the center of the fan (Fig. 5) and let $x_Ty_Tz_T$ be a set of axes rotating relative to $x_{ET}y_{ET}z_{ET}$, so that the translational velocity of T is

$$\{\dot{w}_{AT}\} = [T_{AO}]\{\dot{w}_{OA}\} - [r_{AT} + u_{AT}]\{\Omega_A\} + \{\dot{u}_{AT}\} \tag{26}$$

where all the quantities are as in Eq. (10) except that the coordinates of point T replace those of point S. Similarly, the angular velocity of axes $x_{ET}y_{ET}z_{ET}$ relative to axes $x_Ay_Az_A$ is

$$\{\omega_{ET}\} = [\nabla]([\ell_{GT}]\{\dot{u}_{AT}\}) \tag{27}$$

and the angular velocity of the tail rotor is

$$\{\Omega_T\} = [T_{TA}]\{\Omega_A\} + [\ell_T]\{\omega_{ET}\} + \Omega_{TR}\{e_3\} \tag{28}$$

where there are no counterparts of ψ_x and ψ_y for the tail rotor and the spin axis is taken as coincident with axis z_T.

The main rotor is assumed to have M identical articulated blades ($M \geq 2$), where the blades are assumed to have the flap-lag-pitch configuration (Fig. 6).

The subsequent derivations are concerned with a typical blade i (i = 1, 2,...,M). Ordinarily, we would identify every quantity pertaining to the blade by the subscript i. However, with the tacit understanding that the subscript is implied throughout, considerable simplification of notation can be achieved by omitting the subscript during the derivation stage and rein-

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troducing it when it becomes necessary. Hence, let us consider a typical blade and assume that the flap hinge is at point F at a distance $\underset{\sim}{r}_{HF}$ from H, the lag hinge is at point L at a distance $\underset{\sim}{r}_{FL}$ from F, and the pitch hinge is at point B at a distance $\underset{\sim}{r}_{LB}$ from L. First, let us introduce the set of axes $x_F y_F z_F$ obtained through a rotation α_B about axis z_H and a rotation $-\beta$ about axis y_H, where α_B is known as the azimuth angle and β is known as the flapping angle (Fig. 6a). Note that α_B is constant for every blade. The relation between axes $x_F y_F z_F$ and $x_H y_H z_H$ is given by

$$\{r_F\} = [T_{FH}]\{r_H\} \tag{29}$$

where

$$[T_{FH}] = [\beta][\alpha_B] \tag{30}$$

is a transformation matrix, in which

$$[\alpha_B] = \begin{bmatrix} c\alpha_B & s\alpha_B & 0 \\ -s\alpha_B & c\alpha_B & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad [\beta] = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \tag{31}$$

Although it is more convenient to define the components of the radius vector $\underset{\sim}{r}_{HF}$ in terms of components along an intermediate set of axes, for consistency of notation, we express $\underset{\sim}{r}_{HF}$ in terms of components along axes $x_H y_H z_H$. Hence, we write

$$\{r_{HF}\} = [\alpha_B]^T[d_o \ \ell_o \ h_o]^T \tag{32}$$

The inertial velocity of point F has the matrix form

$$\{\dot{w}_{HF}\} = [T_{HS}]\{\dot{w}_{SH}\} - [r_{HF}]\{\Omega_H\} \tag{33}$$

where $[r_{HF}]$ is the skew symmetric matrix corresponding to $\{r_{HF}\}$. Also from Fig. 6a, it can be verified that the angular velocity of axes $x_F y_F z_F$ has the form

$$\{\Omega_F\} = [T_{FH}]\{\Omega_H\} - \dot{\beta}\{e_2\} \tag{34}$$

Following the same pattern, we conclude from Fig. 6b that the relation between the lag axes $x_L y_L z_L$ and the flap axes $x_F y_F z_F$ is

$$\{r_L\} = [T_{LF}]\{r_F\} \tag{35}$$

where

$$[T_{LF}] = \begin{bmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

in which α is known as the lag angle. Moreover, the inertial velocity of the lag hinge is

$$\{\dot{w}_{FL}\} = [T_{FH}]\{\dot{w}_{HF}\} - [r_{FL}]\{\Omega_F\} \tag{37}$$

in which $[r_{FL}]$ is the matrix obtained from $\{r_{FL}\} = L_F\{e_1\}$, where the latter is the vector from F to L whose magnitude is L_F. The angular velocity of axes $x_L y_L z_L$ is

$$\{\Omega_L\} = [T_{LF}]\{\Omega_F\} + \dot{\alpha}\{e_3\} \tag{38}$$

Finally, we wish to define the blade axes $x_B y_B z_B$ such that x_B is along the axial direction of the blade, y_B is the plane of the blade and in the direction of the leading edge, and z_B is normal to the blade (Fig. 6c). The relation between axes $x_B y_B z_B$ and $x_L y_L z_L$ is

$$\{r_B\} = [T_{BL}]\{r_L\} \tag{39}$$

where

$$[T_{BL}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \tag{40}$$

in which θ is known as the pitch angle. The inertial velocity of point B is

$$\{\dot{w}_{LB}\} = [T_{LF}]\{\dot{w}_{FL}\} - [r_{LB}]\{\Omega_L\} \tag{41}$$

where $[r_{LB}]$ is the matrix obtained from $\{r_{LB}\} = L_L\{e_1\}$, where $\{r_{LB}\}$ is the vector from L to B whose magnitude is L_L. The angular velocity of the blade axes $x_B y_B z_B$ is

$$\{\Omega_B\} = [T_{BL}]\{\Omega_L\} + \dot{\theta}\{e_1\} \tag{42}$$

At this point, we note the implicit assumption that the hub and its connecting links are rigid, with the connecting links being considered as one-dimensional bodies. In addition, we shall assume that the connecting links are small compared to the airframe and the rotor blades, so that they can be regarded as massless. Hence, there is no direct contribution from these links to the system energy. In contrast, the hub can be much more massive, particularly for large helicopters, and hence will contribute to the system energy as a rigid body.

The rotation angles α_B, β, α, and θ described above deserve further discussion. In particular, the angle α_B is a constant design parameter. On the other hand, the angles β and α representing flap and lag, respectively, are time-dependent generalized coordinates. The angle θ is the sum of three parts, namely, the collective pitch θ_{co}, the cyclic pitch θ_{cy}, and the pitch change θ_{cp} due to coupling, where θ_{co} and θ_{cy} are given functions of time and θ_{cp} is a function of β and/or α and provides the flap and/or lag kinematical coupling, respectively. Hence, θ does not introduce additional degrees of freedom.

We now wish to define the motion of a typical rotor blade. By analogy with the airframe and the transmission shaft, we can write the absolute velocity of an arbitrary point on the rotor blade in the general vector form

$$\dot{\underset{\sim}{w}}_B = \dot{\underset{\sim}{w}}_{LB} + \underset{\sim}{\Omega}_B \times (\underset{\sim}{r}_B + \underset{\sim}{u}_B) + \dot{\underset{\sim}{u}}_B = \dot{\underset{\sim}{w}}_{LB} - (\underset{\sim}{r}_B + \underset{\sim}{u}_B) \times \underset{\sim}{\Omega}_B + \dot{\underset{\sim}{u}}_B \tag{43}$$

where $\underset{\sim}{r}_B$ is the radius vector from B to the point in question when the blade is undeformed, $\underset{\sim}{u}_B$ is the displacement vector of the point and, $\dot{\underset{\sim}{u}}_B$ is the elastic velocity vector of the point relative to $x_B y_B z_B$. We note that $\dot{\underset{\sim}{w}}_{LB}$ is given in terms of components along axes $x_L y_L z_L$ whereas the remaining two terms are in terms of components along $x_B y_B z_B$. Following the established pattern, we can write the velocity vector $\dot{\underset{\sim}{w}}_B$ in terms of components along axes $x_B y_B z_B$ in the matrix form

$$\{\dot{w}_B\} = [T_{BL}]\{\dot{w}_{LB}\} - [r_B + u_B]\{\Omega_B\} + \{\dot{u}_B\} \tag{44}$$

where $[r_B + u_B]$ is the skew symmetric matrix associated with $\underset{\sim}{r}_B + \underset{\sim}{u}_B$.

It remains for us to define the elastic motion of the blade relative to axes $x_B y_B z_B$. To this end, we can write the position of any point in the deformed blade relative to axes $x_B y_B z_B$ in the matrix form (Fig. 7; see also Ref. 3)

$$\{r_B + u_B\} = \begin{Bmatrix} x + u \\ e_0 + v \\ w \end{Bmatrix} + [T_{DB}]^T \begin{Bmatrix} -\lambda(\phi_0 + \phi)' \\ \eta \\ \zeta \end{Bmatrix} \tag{45}$$

where u, v, w are elastic displacements along axes x_B, y_B, z_B respectively, e_0 is the distance at the blade root between the pitch axis x_B and the elastic axis, measured in the $x_B y_B$ plane and positive in the direction of the leading edge, $\lambda(\phi_0 + \phi)'$ represents an axial position where $\lambda = \lambda(\eta,\zeta)$ is the warp function (note: $\lambda(0,0) = 0$), and η, ζ are cross sectional coordinates along local principal axes at a distance x from B, in which η is along the chord pointing toward the leading edge and ζ is normal to the blade mid-plane. Moreover, $[T_{DB}]$ is the transformation matrix from the undeformed blade coordinates to the deformed coordinates.

When deriving the rotor blade kinetic and potential energy, it becomes necessary to omit certain small terms arising from the blade elastic deformation in order to keep from unduly complicating our equations. These terms are generally known as "higher-order" terms. At this point, we wish to introduce the idea of an ordering scheme which will enable us to compare the relative magnitudes of terms in a systematic manner, thus permitting us to determine which terms to retain and which terms to omit from the formulation. To this end, let us define the nondimensional parameter ε, which is taken to be the same throughout this report. The approximate magnitude of ε is taken to be less than one tenth and a term of the same order of magnitude as ε is said to be of order ε and denoted $O(\varepsilon)$. We can compare the relative magnitudes of terms in an equation by first nondimensionalizing that equation and then comparing the resulting nondimensional terms directly with the parameter ε. Hence, to compare terms in Eq. (45), we divide Eq. (45) by the length of a rotor blade L_B and compare each resulting term to ε. As in Ref. 3, we shall adopt the following ordering scheme for the blades

$$\frac{u}{L_B} = O(\varepsilon^2) \qquad \frac{v}{L_B} = O(\varepsilon) \qquad \frac{w}{L_B} = O(\varepsilon) \qquad \frac{\eta}{L_B} = O(\varepsilon) \qquad \frac{\zeta}{L_B} = O(\varepsilon)$$

$$\frac{\lambda}{L_B^2} = O(\varepsilon^2) \qquad \phi = O(\varepsilon) \qquad v' = O(\varepsilon) \qquad w' = O(\varepsilon) \qquad \frac{x}{L_B} = O(1) \tag{46}$$

$$\frac{\partial\lambda/\partial\eta}{L_B} = O(\varepsilon) \qquad \frac{\partial\lambda/\partial\zeta}{L_B} = O(\varepsilon) \qquad \phi_0 = O(1)$$

Note that primes denote differentiations with respect to the spatial variable x, and $\phi_0 = \phi_0(x)$, $\phi = \phi(x,t)$ are blade pretwist and blade torsional displacement about the deformed elastic axis, respectively, where the pretwist is considered as a static torsional displacement. The angles ϕ_0 and ϕ are defined here differently than in Ref. 3 where the sum $\phi_0 + \phi$ was taken to be the "total torsional bending" about an axis parallel to axis x_B. We shall see in the next section that it is necessary to obtain the vector $\{r_B + u_B\}$ given by Eq. (45) to $O(\varepsilon^3)$.

In view of Eqs. (46), the largest terms in the vector multiplying $[T_{DB}]^T$ in Eq. (45) are $O(\varepsilon)$ terms so that it is necessary to express $[T_{DB}]$ in terms of the elastic displacements to $O(\varepsilon^2)$ only. Hence, the matrix $[T_{DB}]$ can be written in the form (see Ref. 3)

$$[T_{DB}] = \begin{bmatrix} 1 - \frac{v'^2}{2} - \frac{w'^2}{2} & v' & w' \\ -[v'\cos(\phi_0+\phi)+w'\sin(\phi_0+\phi)] & (1 - \frac{v'^2}{2})\cos(\phi_0+\phi+v'w') & (1 - \frac{w'^2}{2})\sin(\phi_0+\phi) \\ [v'\sin(\phi_0+\phi)-w'\cos(\phi_0+\phi)] & (1 - \frac{v'^2}{2})\sin(\phi_0+\phi+v'w') & (1 - \frac{w'^2}{2})\cos(\phi_0+\phi) \end{bmatrix} \tag{47}$$

in which it is assumed that the pretwist and torsional displacement are both taken to be zero at the blade root. It is understood that consistent order series expansions are used for sine and cosine terms. Introducing Eq. (47) into Eq. (45) and expressing η and ζ in terms of components along y_B and z_B, we obtain

$$\{r_B+u_B\} = \begin{Bmatrix} x+u-\lambda(\phi_0+\phi)'-(v'+w'\phi)(\eta\cos\phi_0-\zeta\sin\phi_0) - (w'-v'\phi)(\eta\sin\phi_0+\zeta\cos\phi_0) \\ e_0+v+(1 - \frac{v'^2}{2})(\eta\cos\phi_0-\zeta\sin\phi_0) - (\phi+v'w')(\eta\sin\phi_0+\zeta\cos\phi_0) \\ w+(1 - \frac{w'^2}{2})(\eta\sin\phi_0+\zeta\cos\phi_0) + \phi(\eta\cos\phi_0-\zeta\sin\phi_0) \end{Bmatrix} \tag{48}$$

where $O(\varepsilon^4)$ terms and smaller have been neglected. Equations (48) contain nonlinear terms which we desire to retain. This is in contrast to the airframe and shaft, whose deformations are described strictly in terms of linear elasticity.

Finally, it will prove convenient to obtain an expression for the angular velocity of the deformed blade axes $x_D y_D z_D$ at any point on the elastic axis a distance x from the blade root, where y_D and z_D are local principal axes of the cross section with origin on the elastic axis, in which y_D is along the chord pointing toward the leading edge and z_D is normal to the blade mid-plane, and where x_D is tangent to the deformed elastic axis positive toward the free blade end. Note that η and ζ are measured along axes y_D and z_D, respectively. An expression for the angular velocity of axes $x_D y_D z_D$ relative to axes $x_B y_B z_B$ can be deduced from the bending curvature expressions given in Ref. 3 where care must be exercized in distinguishing between the torsional displacement as defined herein and the elastic kinematical pitch angle of Ref. 3 (see Ref. 4). Such an expression can be written in the form

$$\{\omega_D\} = \left\{\begin{matrix} w'\dot{v}'+\dot{\phi} \\ (\dot{v}'-\dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2)\sin\phi_0-(\dot{w}'-\dot{v}'v'w'-\dot{w}'w'^2 - \frac{1}{2}\dot{w}'v'^2)\cos\phi_0 \\ (\dot{v}'-\dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2)\cos\phi_0+(\dot{w}'-\dot{v}'v'w'-\dot{w}'w'^2 - \frac{1}{2}\dot{w}'v'^2)\sin\phi_0 \end{matrix}\right.$$

$$\left.\begin{matrix} \\ +\dot{v}'(\phi\cos\phi_0 - \frac{1}{2}\phi^2\sin\phi_0)+\dot{w}'(\phi\sin\phi_0 + \frac{1}{2}\phi^2\cos\phi_0) \\ -\dot{v}'(\phi\sin\phi_0 + \frac{1}{2}\phi^2\cos\phi_0)+\dot{w}'(\phi\cos\phi_0 - \frac{1}{2}\phi^2\sin\phi_0) \end{matrix}\right\} \qquad (49)$$

where $O(\varepsilon^4)$ terms and smaller have been neglected. Hence, the angular velocity of axes $x_D y_D z_D$ is

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$$\{\Omega_D\} = [T_{DB}]\{\Omega_B\} + \{\omega_D\} \tag{50}$$

where $[T_{DB}]$ is simply Eq. (47) evaluated at the point D in question.

A summary of some important equations in this section can be found in Appendix A.

3. The Kinetic and Potential Energy

To derive Lagrange's equations of motion it is necessary to produce expressions for the kinetic energy, the potential energy, and the nonconservative virtual work. In this section we shall consider the kinetic and potential energy for the system and in the next section we shall present the virtual work due to aerodynamic forces.

The kinetic energy of the airframe can be written in the general form

$$T_A = \frac{1}{2}\int_{m_A} \{\dot{w}_A\}^T\{\dot{w}_A\}dm_A \tag{51}$$

where m_A is the mass of the airframe. Introducing Eq. (8) into Eq. (51), we obtain

$$\begin{aligned} T_A = {} & \frac{1}{2} m_A\{\dot{w}_{0A}\}^T\{\dot{w}_{0A}\} + \frac{1}{2}\{\Omega_A\}^T[J_A]\{\Omega_A\} + \frac{1}{2}\int_{m_A} \{\dot{u}_A\}^T\{\dot{u}_A\}dm_A \\ & + \{\dot{w}_{0A}\}^T[T_{A0}]^T\{\dot{\bar{u}}_A\} - \{\dot{w}_{0A}\}^T[T_{A0}]^T[\bar{r}_A]\{\Omega_A\} - \{\bar{H}_A\}^T\{\Omega_A\} \end{aligned} \tag{52}$$

where

$$m_A = \int_{m_A} dm_A \tag{53a}$$

$$[\bar{r}_A] = \int_{m_A} [r_A + u_A]dm_A \tag{53b}$$

$$\{\dot{\bar{u}}_A\} = \int_{m_A} \{\dot{u}_A\}dm_A \tag{53c}$$

$$[J_A] = \int_{m_A} [r_A + u_A]^T[r_A + u_A]dm_A \tag{53d}$$

$$\{\bar{H}_A\} = \int_{m_A} [r_A + u_A]^T\{\dot{u}_A\}dm_A \tag{53e}$$

will be referred to as "mass integrals", where m_A is the total mass of the airframe, $[J_A]$ is the matrix of moments of inertia of the airframe in deformed configuration about axes $x_A y_A z_A$, and $\{\bar{H}_A\}$ is an angular momentum vector of the airframe due to elastic velocities. Note that Eq. (52) takes into account the fact that $[T_{A0}]$ is an orthonormal matrix.

Next, let us consider the shaft kinetic energy and write it in the general form

$$T_S = \frac{1}{2}\int_{m_S} \{\dot{w}_S\}^T\{\dot{w}_S\}dm_S \tag{54}$$

where $\{\dot{w}_S\}$ is given by Eq. (19). Introducing Eq. (19) into Eq. (54), we obtain

$$T_S = \frac{1}{2} m_S\{\dot{w}_{AS}\}^T\{\dot{w}_{AS}\} + \frac{1}{2}\{\Omega_S\}^T[J_S]\{\Omega_S\} + \frac{1}{2}\int_{m_S}\{\dot{u}_S\}^T\{\dot{u}_S\}dm_S$$

$$+ \{\dot{w}_{AS}\}^T[T_{SA}]^T\{\dot{\bar{u}}_S\} - \{\dot{w}_{AS}\}^T[T_{SA}]^T[\bar{r}_S]\{\Omega_S\} - \{\bar{H}_S\}^T\{\Omega_S\} \tag{55}$$

where

$$m_S = \int_{m_S} dm_S \tag{56a}$$

$$[\bar{r}_S] = \int_{m_S} [r_S + u_S]dm_S \tag{56b}$$

$$\{\dot{\bar{u}}_S\} = \int_{m_S} \{\dot{u}_S\}dm_S \tag{56c}$$

$$[J_S] = \int_{m_S} [r_S + u_S]^T[r_S + u_S]dm_S \tag{56d}$$

$$\{\bar{H}_S\} = \int_{m_S} [r_S + u_S]^T\{\dot{u}_S\}dm_S \tag{56e}$$

are certain mass integrals and $[T_{SA}]$ is an orthonormal transformation. Equation (55) looks deceivingly simple and is in fact quite complicated. For example, the vectors $\{\dot{w}_{AS}\}$ and $\{\Omega_S\}$ depend not only on the rigid body motions of the airframe but on the airframe elastic motion as well. Moreover, $[T_{SA}]$ involves the elastic rotation of the airframe at S and the angles between axes $x_{ES}y_{ES}z_{ES}$ and $x_S y_S z_S$, which vary with time. In particular the translational velocity and angular velocity expressions, Eqs. (10) and (17) respectively, must be substituted into Eq. (55). Later in this section, we will simplify Eq. (55) by ignoring certain higher order terms.

Assuming that the point T coincides with the mass center of the rigid tail rotor, the kinetic energy of the tail rotor can be written as

$$T_T = \frac{1}{2} m_T\{\dot{w}_{AT}\}^T\{\dot{w}_{AT}\} + \frac{1}{2}\{\Omega_T\}^T[J_T]\{\Omega_T\} \tag{57}$$

where m_T is the mass and $[J_T]$ is the matrix of moments of inertia about axes $x_T y_T z_T$ of the tail rotor. Similarly, assuming that the point H coincides with the mass center of the rigid hub, we can write the hub kinetic energy as

$$T_H = \frac{1}{2} m_H\{\dot{w}_{SH}\}^T\{\dot{w}_{SH}\} + \frac{1}{2}\{\Omega_H\}^T[J_H]\{\Omega_H\} \tag{58}$$

where m_H is the mass and $[J_H]$ is the matrix of moments of inertia about axes $x_H y_H z_H$ of the hub. Although Eqs. (57) and (58) are considerably simpler than Eq. (55), the velocities $\{\dot{w}_{AT}\}$ and $\{\Omega_T\}$ for the tail rotor and $\{\dot{w}_{SH}\}$ and $\{\Omega_H\}$ for the hub still depend on the motion of the airframe. In addition, $\{\dot{w}_{SH}\}$ and $\{\Omega_H\}$ also include the motion of the transmission shaft.

Finally, we can write the kinetic energy for each rotor blade in the general form

$$T_B = \frac{1}{2}\int_{m_B} \{\dot{w}_B\}^T\{\dot{w}_B\}dm_B \tag{59}$$

where m_B is the mass of the blade. Note that the subscript i, identifying a particular blade, has been ignored temporarily. Introducing Eq. (44) into Eq. (59), we obtain

$$T_B = \frac{1}{2} m_B \{\dot{w}_{LB}\}^T\{\dot{w}_{LB}\} + \frac{1}{2} \{\Omega_B\}^T[J_B]\{\Omega_B\} + \frac{1}{2} \int_{m_B} \{\dot{u}_B\}^T\{\dot{u}_B\}dm_B$$

$$+ \{\dot{w}_{LB}\}^T[T_{BL}]^T\{\dot{\bar{u}}_B\} - \{\dot{w}_{LB}\}^T[T_{BL}]^T[\bar{r}_B]\{\Omega_B\} - \{\bar{H}_B\}^T\{\Omega_B\} \tag{60}$$

where

$$m_B = \int_{m_B} dm_B \tag{61a}$$

$$[\bar{r}_B] = \int_{m_B} [r_B + u_B]dm_B \tag{61b}$$

$$\{\dot{\bar{u}}_B\} = \int_{m_B} \{\dot{u}_B\}dm_B \tag{61c}$$

$$[J_B] = \int_{m_B} [r_B + u_B]^T[r_B + u_B]dm_B \tag{61d}$$

$$\{\bar{H}_B\} = \int_{m_B} [r_B + u_B]^T\{\dot{u}_B\}dm_B \tag{61e}$$

The various quantities in Eqs. (60) and (61) are analogous to those pertaining to the airframe and the shaft. Of all the component kinetic energies, Eq. (60) is the most complex. In general, the velocity $\{\dot{w}_{LB}\}$ and angular velocity $\{\Omega_B\}$ take into account the rigid body translation and rotation of the airframe, the airframe elastic motion, the rigid body rotation of the transmission shaft relative to the airframe, the transmission shaft elastic motion, the rigid body rotation of the flap link relative to the hub, and the rigid body rotation of the lag link relative to the flap link. A method of simplifying these coupling effects will be given later in this report. The blade ordering scheme of Sec. 2 will be used in this section to obtain

$O(\varepsilon^3)$ expansions of Eqs. (61).

Equations (52), (55), (57), (58), and (60) give the kinetic energy for each component. The system kinetic energy is formed by simply summing up these equations, where it is now necessary to introduce the subscript i in Eq. (60) and sum over the number of blades. This yields

$$T = T_A + T_T + T_S + T_H + \sum_{i=1}^{M} T_{B_i} \tag{62}$$

Substituting Eqs. (52), (55), (57), (58), and (60) into Eq. (62) and grouping together terms of the same type, we can write

$$T = \sum_{j=1}^{6} T_j \tag{63}$$

in which

$$T_1 = \frac{1}{2} m_A \{\dot{w}_{OA}\}^T\{\dot{w}_{OA}\} + \frac{1}{2} m_T\{\dot{w}_{AT}\}^T\{\dot{w}_{AT}\} + \frac{1}{2} m_S \{\dot{w}_{AS}\}^T\{\dot{w}_{AS}\} + \frac{1}{2} m_H \{\dot{w}_{SH}\}^T\{\dot{w}_{SH}\} + \frac{1}{2} m_B \sum_{i=1}^{M} \{\dot{w}_{LB}\}_i^T\{\dot{w}_{LB}\}_i \tag{64a}$$

$$T_2 = -\{\dot{w}_{OA}\}^T[T_{AO}]^T[\bar{r}_A]\{\Omega_A\} - \{\dot{w}_{AS}\}^T[T_{SA}]^T[\bar{r}_S]\{\Omega_S\} - \sum_{i=1}^{M} \{\dot{w}_{LB}\}_i^T[T_{BL}]_i^T[\bar{r}_B]_i\{\Omega_B\}_i \tag{64b}$$

$$T_3 = \{\dot{w}_{OA}\}^T[T_{AO}]^T\{\dot{\bar{u}}_A\} + \{\dot{w}_{AS}\}^T[T_{SA}]^T\{\dot{\bar{u}}_S\} + \sum_{i=1}^{M} \{\dot{w}_{LB}\}_i^T[T_{BL}]_i^T\{\dot{\bar{u}}_B\}_i \tag{64c}$$

$$T_4 = \frac{1}{2} \{\Omega_A\}^T[J_A]\{\Omega_A\} + \frac{1}{2} \{\Omega_T\}^T[J_T]\{\Omega_T\} + \frac{1}{2} \{\Omega_S\}^T[J_S]\{\Omega_S\} + \frac{1}{2} \{\Omega_H\}^T[J_H]\{\Omega_H\} + \frac{1}{2} \sum_{i=1}^{M} \{\Omega_B\}_i^T[J_B]_i\{\Omega_B\}_i \tag{64d}$$

$$T_5 = -\{\bar{H}_A\}^T\{\Omega_A\} - \{\bar{H}_S\}^T\{\Omega_S\} - \sum_{i=1}^{M} \{\bar{H}_B\}_i^T\{\Omega_B\}_i \tag{64e}$$

$$T_6 = \frac{1}{2} \int_{m_A} \{\dot{u}_A\}^T\{\dot{u}_A\}dm_A + \frac{1}{2} \int_{m_S} \{\dot{u}_S\}^T\{\dot{u}_S\}dm_S + \frac{1}{2} \sum_{i=1}^{M} \int_{m_{Bi}} \{\dot{u}_B\}_i^T\{\dot{u}_B\}_i dm_{Bi} \tag{64f}$$

Equations (64) possess all of the complexities mentioned earlier. A direct approach to evaluating the system kinetic energy explicitly is to substitute explicit expressions for Eqs. (53), (56), and (61) as well as explicit expressions for the velocities and angular velocities given by Eqs. (A28)-(A32) and (A12)-(A16) into Eqs. (64). The extreme tediousness involved forces us to seek explicit forms for some terms at one time and for other terms at a different time. This is motivated by the different treatments necessary for the two different types of quantities, those which depend on the spatial position and time and those which depend on time alone. One possibility is to substitute explicit expressions for Eqs. (53), (56), and (61) into Eqs. (64) while retaining all other quantities in implicit form, i.e., to write the components of the velocity and angular velocity vectors of each set of axes implicitly. This approach is convenient when the elastic displacements and velocities are of interest. It is the approach of this section. Another possibility is to substitute Eqs. (A28)-(A32) and (A12)-(A16) into Eqs. (64) while retaining the matrix quantities defined by Eqs. (53), (56), and (61) implicitly. Such an approach is more suitable when the discrete coordinates are being considered and the approach will be discussed in more detail later in this report.

Let us now examine the airframe kinetic energy, Eq. (52), more closely. Assuming that the angular velocity vector $\{\Omega_A\}$ and the elastic displacement vector $\{u_A\}$ are sufficiently small that their product can be ignored, we can make the following simplifications in Eqs. (53b, d, and e)

$$[\bar{r}_A] = \int_{m_A} [r_A]dm_A \tag{65a}$$

$$[J_A] = \int_{m_A} [r_A]^T[r_A]dm_A \tag{65b}$$

$$\{\bar{H}_A\} = \int_{m_A} [r_A]^T\{\dot{u}_A\}dm_A \tag{65c}$$

Note that this is equivalent to ignoring the cross product $\underset{\sim}{\Omega}_A \times \underset{\sim}{u}_A$ in Eq. (7). Equations (65) have simple physical meaning. Equation (65a) is the skew symmetric matrix associated with the vector from A to the airframe mass center in undeformed configuration, Eq. (65b) is the inertia matrix of the undeformed airframe about axes $x_A y_A z_A$, and Eq. (65c) is an angular momentum vector due to the airframe elastic velocities.

Thus far, we have been concerned with the airframe elastic displacements and velocities in a general way only. Now we wish to explore the possibility of representing these displacements in terms of airframe modes. Recognizing that these modes are three-dimensional, we can represent them by 3 x 1 column matrices $\{\phi_{Ai}(x_A, y_A, z_A)\}$ and write

$$\{u_A(x_A, y_A, z_A, t)\} = \sum_{i=1}^{P} \{\phi_{Ai}(x_A, y_A, z_A)\}\eta_{Ai}(t) \qquad (66)$$

where $\eta_{Ai}(t)$ are generalized coordinates associated with these modes. Note that the mode $\{\phi_{Ai}\}$ gives the three displacement components at every mass point of the airframe. Introducing the 3 x P airframe modal matrix

$$[\phi_A] = [\{\phi_{A1}\}\ \{\phi_{A2}\}\ .\ .\ .\ \{\phi_{AP}\}] \qquad (67)$$

and the airframe elastic generalized coordinate vector

$$\{\eta_A\} = \{\eta_{A1}\ \eta_{A2}\ .\ .\ .\ \eta_{AP}\}^T \qquad (68)$$

Eq. (66) can be written in the compact form

$$\{u_A\} = [\phi_A]\{\eta_A\} \qquad (69)$$

Substituting Eq. (69) into Eqs. (53) and taking into account Eqs. (65) we can write the airframe kinetic energy as

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$$T_A = \frac{1}{2} m_A \{\dot{w}_{OA}\}^T\{\dot{w}_{OA}\} + \frac{1}{2}\{\Omega_A\}^T[J_A]\{\Omega_A\} + \frac{1}{2}\{\dot{\eta}_A\}^T[M_A]\{\dot{\eta}_A\}$$

$$+ \{\dot{w}_{OA}\}^T[T_{AO}]^T[\bar{\Phi}_A]\{\dot{\eta}_A\} - \{\dot{w}_{OA}\}^T[T_{AO}]^T[\tilde{\bar{r}}_A]\{\Omega_A\}$$

$$- \{\dot{\eta}_A\}^T[I_A]^T\{\Omega_A\} \tag{70}$$

in which we have introduced the definitions

$$[M_A] = \int_{m_A} [\Phi_A]^T[\Phi_A]dm_A \tag{71a}$$

$$[\bar{\Phi}_A] = \int_{m_A} [\Phi_A]dm_A \tag{71b}$$

$$[I_A] = \int_{m_A} [\tilde{r}_A]^T[\Phi_A]dm_A \tag{71c}$$

Substantial simplification can be achieved if the point A is chosen to coincide with the center of mass of the undeformed airframe. Then, if the elastic modes are orthogonal to the rigid body modes, we can write

$$\int_{m_A} [\tilde{r}_A]dm_A = \int_{m_A} [\Phi_A]dm_A = \int_{m_A} [\tilde{r}_A]^T[\Phi_A]dm_A = [0] \tag{72}$$

so that

$$[\tilde{\bar{r}}_A] = [\bar{\Phi}_A] = [I_A] = [0] \tag{73}$$

The requirement that $x_Ay_Az_A$ be principal axes of the airframe does not yield significant savings, so that it need not be made. Inserting Eqs. (73) into Eq. (70), the kinetic energy reduces to

$$T_A = \frac{1}{2} m_A \{\dot{w}_{OA}\}^T\{\dot{w}_{OA}\} + \frac{1}{2}\{\Omega_A\}^T[J_A]\{\Omega_A\} + \frac{1}{2}\{\dot{\eta}_A\}^T[M_A]\{\dot{\eta}_A\} \tag{74}$$

where $[J_A]$ is now the matrix of moments of inertia of the airframe in undeformed configuration about axes $x_Ay_Az_A$ and $[M_A]$ is the diagonal matrix of "generalized masses".

The above development is predicated on the availability of free-free airframe modes, which cannot be taken for granted. Moreover, it may not be the most advantageous way to describe the airframe motion. Hence, we wish to consider the possibility of using a different type of airframe modes.

As mentioned earlier, an alternative to the use of airframe free-free modes is the use of cantilever modes, obtained by regarding the airframe as being fixed at the transmission base. We recall that this point was denoted by S in the preceding section. This alternative has the advantage that it eliminates the need of considering the intermediate axes $x_{ES}y_{ES}z_{ES}$, and indeed one can assume that $x_Ay_Az_A$ take their place. On the other hand, the cantilever modes are not orthogonal to the rigid body modes, so that additional terms appear in the kinetic energy which do not appear in Eq. (74).

In view of the above, let us assume that axes $x_Ay_Az_A$ have the origin at point S and that axis z_A coincides with the direction of the shaft axis in undeformed state. Axes x_A and y_A are attached to the airframe and are normal to z_A. Then Eqs. (65a), (65b), and (71) remain valid except that their interpretation is different. Of course, the difference comes from the fact that the matrix $[\Phi_A]$ now represents cantilever modes and not free-free modes. Likewise, the vector $\{r_A\}$ is now measured from point S, which affects $[\tilde{r}_A]$, $[I_A]$, and $[J_A]$. In particular, $[J_A]$ is now the inertia matrix of the airframe in undeformed configuration about axes $x_Ay_Az_A$ with the origin at S. Clearly, simplifications (72) are no longer possible and the kinetic energy remains in the form (70).

In contrast to the airframe where all components of $\{\Omega_A\}$ are small and vary with time, the third component of the shaft angular velocity vector $\{\Omega_S\}$ contains a large constant part. Giving special consideration to this angular velocity component, in the sense that quantities containing this component may not necessarily be small, we substitute Eq. (20) into Eqs. (56), integrate over the length of the shaft, and obtain

$$[\bar{r}_S] = \begin{bmatrix} 0 & -\bar{r}_{Sz} & \bar{r}_{Sy} \\ \bar{r}_{Sz} & 0 & -\bar{r}_{Sx} \\ -\bar{r}_{Sy} & \bar{r}_{Sx} & 0 \end{bmatrix} \tag{75a}$$

$$\{\dot{\bar{u}}_S\} = \begin{Bmatrix} \dot{\bar{u}}_{Sx} \\ \dot{\bar{u}}_{Sy} \\ \dot{\bar{u}}_{Sz} \end{Bmatrix} \tag{75b}$$

$$[J_S] = \begin{bmatrix} J_{Sxx} & -J_{Sxy} & -J_{Sxz} \\ -J_{Sxy} & J_{Syy} & -J_{Syz} \\ -J_{Sxz} & -J_{Syz} & J_{Szz} \end{bmatrix} \tag{75c}$$

$$\{\bar{H}_S\} = \begin{Bmatrix} \bar{H}_{Sx} \\ \bar{H}_{Sy} \\ \bar{H}_{Sz} \end{Bmatrix} \tag{75d}$$

where

$$\bar{r}_{Sx} = \int_0^{L_S} \rho_S u_S dz_S \qquad \bar{r}_{Sy} = \int_0^{L_S} \rho_S v_S dz_S \qquad \bar{r}_{Sz} = \frac{1}{2} m_S L_S \tag{76a}$$

$$\dot{\bar{u}}_{Sx} = \int_0^{L_S} \rho_S \dot{u}_S dz_S \qquad \dot{\bar{u}}_{Sy} = \int_0^{L_S} \rho_S \dot{v}_S dz_S \qquad \dot{\bar{u}}_{Sz} = 0 \tag{76b}$$

$$\begin{gathered} J_{Sxx} = J_{Syy} = \frac{1}{3} m_S L_S^2 \qquad J_{Szz} = \int_0^{L_S} \rho_S (u_S^2 + v_S^2) dz_S \\ J_{Sxy} = 0 \qquad J_{Sxz} = \int_0^{L_S} \rho_S u_S z_S dz_S \qquad J_{Syz} = \int_0^{L_S} \rho_S v_S z_S dz_S \end{gathered} \tag{76c}$$

$$\bar{H}_{Sx} = \int_0^{L_S} \rho_S \dot{v}_S z_S dz_S \qquad \bar{H}_{Sy} = -\int_0^{L_S} \rho_S \dot{u}_S z_S dz_S$$

$$\bar{H}_{Sz} = \int_0^{L_S} \rho_S (\dot{u}_S v_S - \dot{v}_S u_S) dz_S \tag{76d}$$

in which ρ_S is the mass per unit length, L_S is the length, and m_S is the total mass of the uniform shaft. In addition, we can write the third term in Eq. (55) as

$$\frac{1}{2} \int_{m_S} \{\dot{u}_S\}^T \{\dot{u}_S\} dm_S = \frac{1}{2} \int_0^{L_S} \rho_S (\dot{u}_S^2 + \dot{v}_S^2) dz_S \tag{77}$$

Next, we wish to discretize Eqs. (76) and (77) and use the result to write a discretized shaft kinetic energy. Thus, let us assume that the shaft elastic displacements can be written as linear combinations of space dependent functions multiplied by time dependent generalized coordinates in the form

$$u_S = \sum_{i=1}^{S_u} \psi_{Si}(z_S) \eta_{ui}(t)$$

$$v_S = \sum_{i=1}^{S_u} \psi_{Si}(z_S) \eta_{vi}(t) \tag{78}$$

$$\phi_S = \sum_{i=1}^{S_\phi} \Theta_{Si}(z_S) \eta_{\phi i}(t)$$

Note that we are simulating the elastic shaft by $2S_u + S_\phi$ degrees of freedom. We can write $\{\Omega_S\} = [\Omega_{Sx} \;\; \Omega_{Sy} \;\; \Omega_{Sz}]^T$ and $[T_{SA}]\{\dot{w}_{AS}\} = [\dot{w}'_{ASx} \; \dot{w}'_{ASy} \; \dot{w}'_{ASz}]^T$ where $\dot{w}'_{ASx}$, $\dot{w}'_{ASy}$, $\dot{w}'_{ASz}$ are components of the vector $\dot{\underline{w}}_{AS}$ along axes $x_S y_S z_S$ in contrast to $\dot{w}_{ASx}$, $\dot{w}_{ASy}$, $\dot{w}_{ASz}$ which are the components along axes $x_A y_A z_A$. Substituting Eqs. (78) into Eqs. (76) and (77), we can write the shaft kinetic energy, Eq. (55), in the discrete form

$$T_S = \frac{1}{2} m_S(\dot{w}'^2_{ASx} + \dot{w}'^2_{ASy} + \dot{w}'^2_{ASz}) + \frac{1}{6} m_S L_S^2 (\Omega^2_{Sx} + \Omega^2_{Sy})$$

$$+ \frac{1}{2} \Omega^2_{Sz} \sum_{i=1}^{S_u} \sum_{j=1}^{S_u} M^{\psi\psi}_{ij} (\eta_{ui}\eta_{uj} + \eta_{vi}\eta_{vj}) - \Omega_{Sx}\Omega_{Sz} \sum_{i=1}^{S_u} M^{\psi z}_{i} \eta_{ui}$$

$$- \Omega_{Sy}\Omega_{Sz} \sum_{i=1}^{S_u} M^{\psi z}_{i} \eta_{vi} + \frac{1}{2} \sum_{i=1}^{S_u} \sum_{j=1}^{S_u} M^{\psi\psi}_{ij}(\dot{\eta}_{ui}\dot{\eta}_{uj} + \dot{\eta}_{vi}\dot{\eta}_{vj})$$

$$+ \dot{w}'_{ASx} \sum_{i=1}^{S_u} M^{\psi}_{i}\dot{\eta}_{ui} + \dot{w}'_{ASy} \sum_{i=1}^{S_u} M^{\psi}_{i}\dot{\eta}_{vi} + (\dot{w}'_{ASy}\Omega_{Sz} - \dot{w}'_{ASz}\Omega_{Sy}) \sum_{i=1}^{S_u} M^{\psi}_{i}\eta_{ui}$$

$$+ (\dot{w}'_{ASz}\Omega_{Sx} - \dot{w}'_{ASx}\Omega_{Sz}) \sum_{i=1}^{S_u} M^{\psi}_{i}\eta_{vi} + \frac{1}{2} m_S L_S(\dot{w}_{ASx}\Omega_{Sy} - \dot{w}_{ASy}\Omega_{Sx})$$

$$- \Omega_{Sx} \sum_{i=1}^{S_u} M^{\psi z}_{i}\dot{\eta}_{vi} + \Omega_{Sy} \sum_{i=1}^{S_u} M^{\psi z}_{i}\dot{\eta}_{ui} - \Omega_{Sz} \sum_{i=1}^{S_u} \sum_{j=1}^{S_u} M^{\psi\psi}_{ij} (\dot{\eta}_{ui}\eta_{vj} - \dot{\eta}_{vi}\eta_{uj}) \tag{79}$$

where we introduced the definitions

$$M^{\psi\psi}_{ij} = \int_0^{L_S} \rho_S \psi_{Si} \psi_{Sj} dz_S$$

$$M^{\psi}_{i} = \int_0^{L_S} \rho_S \psi_{Si} dz_S \tag{80}$$

$$M^{\psi z}_{i} = \int_0^{L_S} \rho_S \psi_{Si} z_S dz_S$$

Before we can write explicit expressions for the rotor blade "mass integrals" Eqs. (61), we must apply the ordering scheme (46) to the blade kinetic energy Eq. (60). To this end, it is desirable to compare nondimensional quantities directly with the parameter ε. Let us divide Eq. (60) by the quantity $m_B\Omega^2 L_B^2$ which has units of energy, where m_B is the mass and L_B is the length of a rotor blade, and Ω is the constant angular velocity imparted to the shaft by the engine. Note that this division is compatible with the ordering scheme (46).

We can now examine the rationale for retaining $O(\varepsilon^3)$ terms in the blade kinetic energy. This criterion is based on the derivation of Ref. 3 which is concerned solely with rotor blades rotating uniformly about an axis fixed in inertial space. Hence, the derivation of Ref. 3 contains all of the third term and parts of the second and fifth terms in the blade kinetic energy, Eq. (60). Terms of order ε^3 were retained while $O(\varepsilon^4)$ terms were ignored. By examination of the fourth and fifth terms in Eq. (60), it is readily seen that $O(\varepsilon^3)$ terms in $\{r_B + u_B\}$ and $\{\dot{u}_B\}$ can be of the same order of magnitude as the $O(\varepsilon^3)$ terms of Ref. 3, depending on the magnitude of $\{\dot{w}_{LB}\}$. These terms are a direct result of the translational velocity of the blade root, i.e., of the point B, represented by the vector $\dot{\underline{w}}_{LB}$ in Eq. (43), and we note that in high speed flight $\{\dot{w}_{LB}\}$ can have large components, so that the fourth and fifth terms in Eq. (60) must include $O(\varepsilon^3)$ terms in $\{\dot{u}_B\}$ and $\{r_B + u_B\}$, respectively.

Substituting Eq. (48) into Eqs. (61), integrating over the undeformed blade, and retaining terms through $O(\varepsilon^3)$, we can write the components of Eqs. (61) in a form analogous to Eqs. (76) for the shaft, as

$$\bar{r}_{Bx} = \int_0^{L_B} \rho[x + u - e_m(v'+w'\phi)\cos\phi_0 - e_m(w'-v'\phi)\sin\phi_0]dx_B$$

$$\bar{r}_{By} = \int_0^{L_B} \rho[e_0 + v + e_m(1 - \frac{1}{2}\phi^2 - \frac{1}{2}v'^2)\cos\phi_0 - e_m(\phi+v'w')\sin\phi_0]dx_B \quad (81a)$$

$$\bar{r}_{Bz} = \int_0^{L_B} \rho[w + e_m(1 - \frac{1}{2}\phi^2 - \frac{1}{2}w'^2)\sin\phi_0 + e_m\phi\cos\phi_0]dx_B$$

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$$\dot{\bar{u}}_{Bx} = \int_0^{L_B} \rho[\dot{u} - e_m(\dot{v}'+\dot{w}'\phi+w'\dot{\phi})\cos\phi_0 - e_m(\dot{w}'-\dot{v}'\phi-v'\dot{\phi})\sin\phi_0]dx_B$$

$$\dot{\bar{u}}_{By} = \int_0^{L_B} \rho[\dot{v} - e_m(\phi\dot{\phi}+v'\dot{v}')\cos\phi_0 - e_m(\dot{\phi}+\dot{v}'w'+v'\dot{w}')\sin\phi_0]dx_B \tag{81b}$$

$$\dot{\bar{u}}_{Bz} = \int_0^{L_B} \rho[\dot{w} - e_m(\phi\dot{\phi}+w'\dot{w}')\sin\phi_0 + e_m\dot{\phi}\cos\phi_0]dx_B$$

$$\begin{aligned}J_{Bxx} = \int_0^{L_B} &[\rho(e_0^2+v^2+w^2+2e_0v) + 2\rho e_m(e_0\cos\phi_0 - e_0\phi\sin\phi_0 + v\cos\phi_0 \\ &- v\phi\sin\phi_0 + w\sin\phi_0 + w\phi\cos\phi_0) + J_p]dx_B\end{aligned}$$

$$\begin{aligned}J_{Byy} = \int_0^{L_B} &[\rho(x^2+2xu+w^2) + 2\rho e_m(-xv'\cos\phi_0+xv'\phi\sin\phi_0 - xw'\sin\phi_0 \\ &- xw'\phi\cos\phi_0 + w\sin\phi_0 + w\phi\cos\phi_0) + J_2(\sin^2\phi_0 + 2\phi\sin\phi_0\cos\phi_0 \\ &+ \underline{\phi^2\cos^2\phi_0} - \underline{\phi^2\sin^2\phi_0}) + J_1(\cos^2\phi_0 - 2\phi\sin\phi_0\cos\phi_0 \\ &+ \underline{\phi^2\sin^2\phi_0} - \underline{\phi^2\cos^2\phi_0})]dx_B\end{aligned}$$

$$\begin{aligned}J_{Bzz} = \int_0^{L_B} &[\rho(x^2+2xu+e_0^2+v^2+2e_0v) + 2\rho e_m(-xv'\cos\phi_0 + xv'\phi\sin\phi_0 - xw'\sin\phi_0 \\ &- xw'\phi\cos\phi_0 + e_0\cos\phi_0 - e_0\phi\sin\phi_0 + v\cos\phi_0 - v\phi\sin\phi_0) \\ &+ J_2(\cos^2\phi_0 - 2\phi\sin\phi_0\cos\phi_0 - \underline{\phi^2\cos^2\phi_0} + \underline{\phi^2\sin^2\phi_0}) \\ &+ J_1(\sin^2\phi_0 + 2\phi\sin\phi_0\cos\phi_0 - \underline{\phi^2\sin^2\phi_0} + \underline{\phi^2\cos^2\phi_0})]dx_B\end{aligned}$$

$$J_{Bxy} = \int_0^{L_B} [\rho(xe_0+xv+ue_0+uv) + \rho e_m(x\cos\phi_0 - x\phi\sin\phi_0 - \frac{1}{2}x\phi^2\cos\phi_0$$

$$- \frac{1}{2}xv'^2\cos\phi_0 - xv'w'\sin\phi_0 - e_0v'\cos\phi_0 - e_0w'\sin\phi_0$$

$$- vv'\cos\phi_0 - vw'\sin\phi_0 + u\cos\phi_0) + \rho C_m\phi_0'\sin\phi_0 + J_2(-v'\cos^2\phi_0 \tag{81c}$$

$$- w'\sin\phi_0\cos\phi_0) + J_1(w'\sin\phi_0\cos\phi_0 - v'\sin^2\phi_0)]dx_B$$

$$J_{Bxz} = \int_0^{L_B} [\rho(xw+uw) + \rho e_m(x\sin\phi_0 + x\phi\cos\phi_0 - \frac{1}{2}x\phi^2\sin\phi_0 - \frac{1}{2}xw'^2\sin\phi_0$$

$$+ u\sin\phi_0 - wv'\cos\phi_0 - ww'\sin\phi_0) - \rho C_m\phi_0'\cos\phi_0$$

$$+ J_2(-v'\sin\phi_0\cos\phi_0 - w'\sin^2\phi_0) + J_1(v'\sin\phi_0\cos\phi_0$$

$$- w'\cos^2\phi_0)]dx_B$$

$$J_{Byz} = \int_0^{L_B} [\rho(e_0w+vw) + \rho e_m(e_0\sin\phi_0 + e_0\phi\cos\phi + v\sin\phi_0 + v\phi\cos\phi_0$$

$$+ w\cos\phi_0 - w\phi\sin\phi_0) + J_2(\sin\phi_0\cos\phi_0 - \phi\sin^2\phi_0$$

$$+ \phi\cos^2\phi_0) + J_1(-\sin\phi_0\cos\phi_0 + \phi\sin^2\phi_0 - \phi\cos^2\phi_0)]dx_B$$

$$\overline{H}_{Bx} = \int_0^{L_B} [\rho(w\dot{v}-e_0\dot{w}-v\dot{w}) - J_p\dot{\phi} + \rho e_m(\dot{v}\sin\phi_0 + \dot{v}\phi\cos\phi_0 - \dot{w}\cos\phi_0$$

$$+ \dot{w}\phi\sin\phi_0 - w\dot{\phi}\sin\phi_0 - e_0\dot{\phi}\cos\phi_0 - v\dot{\phi}\cos\phi_0)]dx_B$$

$$\overline{H}_{By} = \int_0^{L_B} [\rho(x\dot{w}+u\dot{w}-w\dot{u}) + \rho e_m(x\dot{\phi}\cos\phi_0 - x\phi\dot{\phi}\sin\phi_0 - w'\dot{w}'\sin\phi_0 + w\dot{v}'\cos\phi_0$$

$$+ w\dot{w}'\sin\phi_0 - \dot{u}\sin\phi_0 - v'\dot{w}\cos\phi_0 - w'\dot{w}\sin\phi_0) + J_2(\dot{v}'\sin\phi_0\cos\phi_0$$

$$+ \dot{w}'\sin^2\phi_0) + J_1(\dot{w}'\cos^2\phi_0 - \dot{v}'\sin\phi_0\cos\phi_0)]dx_B \tag{81d}$$

$$\overline{H}_{Bz} = \int_0^{L_B} [\rho(e_0\dot{u}+v\dot{u}-x\dot{v}-u\dot{v}) + \rho e_m(x\dot{\phi}\sin\phi_0 + x\phi\dot{\phi}\cos\phi_0 - e_0\dot{v}'\cos\phi_0$$

$$- e_0\dot{w}'\sin\phi_0 - v\dot{v}'\cos\phi_0 - v\dot{w}'\sin\phi_0 + \dot{u}\cos\phi_0 + xv'\dot{v}'\cos\phi_0$$

$$+ x\dot{v}'w'\sin\phi_0 + v'\dot{v}\cos\phi_0 + w'\dot{v}\sin\phi_0) + J_2(-\dot{v}'\cos^2\phi_0$$

$$- \dot{w}'\sin\phi_0\cos\phi_0) + J_1(\dot{w}'\sin\phi_0\cos\phi_0 - \dot{v}'\sin^2\phi_0)]dx_B$$

in which we introduced the definitions

$$J_1 = \iint_A \gamma\zeta^2 d\eta d\zeta \qquad J_2 = \iint_A \gamma\eta^2 d\eta d\zeta \qquad \rho = \iint_A \gamma d\eta d\zeta$$

$$J_p = J_1 + J_2 = \iint_A \gamma(\eta^2 + \zeta^2)d\eta d\zeta \quad \rho e_m = \iint_A \gamma\eta d\eta d\zeta \tag{82}$$

$$\rho C_m = \iint_A \gamma\lambda\zeta d\eta d\zeta$$

where $\gamma = \gamma(x, \eta, \zeta)$ is the mass density of the blade and the symbol A denotes here integration over the blade cross-sectional area. Due to assumptions of cross-sectional symmetry about the η axis and anti-symmetry of the warp function λ, we also used the relations

$$\iint_A \gamma\zeta d\eta d\zeta = \iint_A \gamma\eta\zeta d\eta d\zeta = \iint_A \gamma\lambda d\eta d\zeta = \iint_A \gamma\lambda\eta d\eta d\zeta = 0 \tag{83}$$

In addition, we can write the third term in Eq. (60) in the form

$$\frac{1}{2}\int_{m_B} \{\dot{u}_B\}^T\{\dot{u}_B\}dm_B = \frac{1}{2}\int_0^{L_B} [\rho(\dot{v}^2+\dot{w}^2) + \underline{J_p\dot{\phi}^2} + 2\rho e_m(-\dot{v}\dot{\phi}\sin\phi_0$$

$$+ \dot{w}\dot{\phi}\cos\phi_0)]dx_B \tag{84}$$

Equations (81) and (84) contain several $O(\epsilon^4)$ torsion terms which have been underlined. These terms are formally of higher-order and could be ignored. These terms were shown in Ref. 3, however, to be important for low torsional natural frequencies, and hence, have been retained here for completeness.

In actuality, to obtain Eqs. (81) and (84) we should integrate over the deformed blade or, equivalently, correct the integration over the undeformed blade by introducing into the integrand the absolute value of the Jacobian associated with the transformation from the position of a point in the undeformed blade to the position of the same point in the deformed blade. The Jacobian in question differs from unity by a quantity of order ε^2, so that its introduction would not change our results substantially. Hence, the added complication appears unwarranted and will be left out.

Next, we wish to discretize Eqs. (81) and use the result to write the blade kinetic energy in a discrete form. To this end, let us consider series expansions

$$\begin{aligned}
u &= \sum_{i=1}^{N_u} \phi_{ui}(x) q_{ui}(t) \\
v &= \sum_{i=1}^{N_v} \phi_{vi}(x) q_{vi}(t) \\
w &= \sum_{i=1}^{N_w} \phi_{wi}(x) q_{wi}(t) \\
\phi &= \sum_{i=1}^{N_\phi} \phi_{\phi i}(x) q_{\phi i}(t)
\end{aligned} \tag{85}$$

where ϕ_{ui}, ϕ_{vi}, ϕ_{wi}, $\phi_{\phi i}$ are space dependent admissible functions and q_{ui}, q_{vi}, q_{wi}, $q_{\phi i}$ are associated generalized coordinates. Note that we are representing the elastic motion of each blade by $N_u + N_v + N_w + N_\phi$ degrees of freedom. Substituting Eqs. (85) into Eqs. (81) and (84) we can write the discretized blade kinetic energy in a form resembling Eq. (79), where this form contains a very large number of integrals similar to the mass integrals

(80). The discretized blade kinetic energy is extremely lengthy and its presentation here is omitted for brevity.

As mentioned earlier, a derivation of Lagrange's equations of motion requires also a derivation of the system potential energy. The potential energy is composed of two distinctly different parts; namely, the potential energy due to gravity, and the potential energy due to the elastic effects. The following discussion is directed towards finding expressions for the potential energy of each component and ultimately the potential energy of the system.

To calculate the gravitational potential energy of the airframe, we assume that the gravitational field is uniform. Hence, the potential energy of any mass point on the airframe is the product of the weight of the mass point multiplied by the height above the earth's surface, i.e., the vertical distance of the mass point from O. This distance can be written in the form

$$h_A = w_{OAZ} + \{e_3\}^T[T_{AO}]^T\{r_A + u_A\} \tag{86}$$

so that the gravitational potential is

$$\begin{aligned} V_{GA} &= g\int_{m_A} (w_{OAZ} + \{e_3\}^T[T_{AO}]^T\{r_A + u_A\})dm_A \\ &= m_A g(w_{OAZ} + \frac{1}{m_A}\{e_3\}^T[T_{AO}]^T\int_{m_A}\{r_A + u_A\}dm_A) \end{aligned} \tag{87}$$

where g is the local acceleration due to gravity. The last term of Eq. (87) is recognized as the vertical distance from A to the mass center of the deformed airframe. When the elastic deformations are small, they can be ignored in Eq. (87) and one can replace the last term by the vertical distance from A to the mass center of the undeformed airframe.

The airframe elastic potential energy can be written in the general form

$$V_{EA} = \frac{1}{2}\{\eta_A\}^T[K_A]\{\eta_A\} \tag{88}$$

where

$$[K_A] = [M_A][\Lambda_A^2] \tag{89}$$

in which $[\Lambda_A^2]$ is the diagonal matrix of natural frequencies squared, where the natural frequencies correspond to the elastic modes of the airframe. Of course, the matrix $[K_A]$ depends on whether free-free or cantilevered modes are used. Similarly, the matrix $[M_A]$ of generalized masses and the matrix $[\Lambda_A^2]$ of natural frequencies squared are different in each case.

By analogy with Eq. (87), the gravitational potential energy of the shaft is

$$V_{GS} = m_S g(w_{OAZ} + \{e_3\}^T[T_{AO}]^T\{r_{AS} + u_{AS}\}) + g\{e_3\}^T[T_{AO}]^T[T_{SA}]^T \int_{m_S} \{r_S + u_S\}dm_S \tag{90}$$

Assuming that the shaft undergoes normal stress along axis z_S as well as shearing stress, so that letting EI_S be the flexural rigidity and GJ_S the torsional stiffness of the shaft, the elastic potential energy of the shaft is

$$V_{ES} = \frac{1}{2}\int_0^{L_S} EI_S\left[\left(\frac{\partial^2 u_S}{\partial z_S^2}\right)^2 + \left(\frac{\partial^2 v_S}{\partial z_S^2}\right)^2\right]dz_S + \frac{1}{2}\int_0^{L_S} GJ_S\left(\frac{\partial \phi_S}{\partial z_S}\right)^2 dz_S \tag{91}$$

Substituting Eqs. (78) into Eq. (91), the elastic potential energy can be written in the discrete form

$$V_{ES} = \frac{1}{2}\sum_{i=1}^{S_u}\sum_{j=1}^{S_u} K_{ij}^{\psi''\psi''}(\eta_{ui}\eta_{uj} + \eta_{vi}\eta_{vj}) + \frac{1}{2}\sum_{i=1}^{S_\phi}\sum_{j=1}^{S_\phi} K_{ij}^{\Theta'\Theta'}\eta_{\phi i}\eta_{\phi j} \tag{92}$$

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$$K_{ij}^{\psi''\psi''} = \int_0^{L_S} EI_S \psi''_{Si} \psi''_{Sj} dz_S \tag{93a}$$

$$K_{ij}^{\Theta'\Theta'} = \int_0^{L_S} GJ_S \Theta'_{Si} \Theta'_{Sj} dz_S \tag{93b}$$

where primes designate derivatives with respect to z_S.

Assuming that the tail rotor is a rigid fan with its mass center at point T, the gravitational potential energy for the tail rotor can be written in the form

$$V_{GT} = m_T g(w_{OAZ} + \{e_3\}^T[T_{AO}]^T\{r_{AT} + u_{AT}\}) \tag{94}$$

while the tail rotor elastic potential energy is zero by virtue of the rigidity assumption. Similarly, the gravitational potential energy of the rigid hub with mass center at H can be written in the form

$$V_{GH} = m_H g(w_{OAZ} + \{e_3\}^T[T_{AO}]^T\{r_{AS} + u_{AS}\} + \{e_3\}[T_{AO}]^T[T_{SA}]^T\{r_{SH} + u_{SH}\}) \tag{95}$$

By analogy with Eqs. (87) and (90), the gravitational potential energy of a typical rotor blade can be written in the form

$$\begin{aligned} V_{GB} = m_B g(w_{OAZ} &+ \{e_3\}^T[T_{AO}]^T\{r_{AS} + u_{AS}\} + \{e_3\}[T_{AO}]^T[T_{SA}]^T\{r_{SH} + u_{SH}\} \\ &+ \{e_3\}^T[T_{AO}]^T[T_{SA}]^T[T_{HS}]^T\{r_{HF}\} + L_F\{e_3\}^T[T_{AO}]^T[T_{SA}]^T[T_{HS}]^T[T_{FH}]^T\{e_1\} \\ &+ L_L\{e_3\}^T[T_{AO}]^T[T_{SA}]^T[T_{HS}]^T[T_{FH}]^T[T_{LF}]^T\{e_1\} \\ &+ \{e_3\}^T[T_{AO}]^T[T_{SA}]^T[T_{HS}]^T[T_{FH}]^T[T_{LF}]^T[T_{BL}]^T \frac{1}{m_B}\int_{m_B}\{r_B + u_B\}dm_B) \end{aligned} \tag{96}$$

The elastic potential energy for a rotor blade is identical to that of Ref. 5 before the axial displacement u was eliminated. Assuming that the blade undergoes normal and shearing stresses the classical nonlinear strain-displacement relations to $O(\varepsilon^3)$ are (see Ref. 3)

$$\epsilon_{xx} = u' + \frac{1}{2}(v'^2 + w'^2) - \lambda\phi'' + (\eta^2 + \zeta^2)(\phi_0'\phi' + \frac{1}{2}\phi'^2)$$

$$- v''[\eta\cos(\phi_0 + \phi) - \zeta\sin(\phi_0 + \phi)] - w''[\eta\sin(\phi_0 + \phi) + \zeta\cos(\phi_0 + \phi)] \quad (97)$$

$$\epsilon_{x\eta} = -(\zeta + \frac{\partial\lambda}{\partial\eta})\phi' \quad , \qquad \epsilon_{x\zeta} = (\eta - \frac{\partial\lambda}{\partial\zeta})\phi'$$

For a linear stress-strain law, the blade elastic potential energy is simply

$$V_{EB} = \frac{1}{2}\int_{Vcl}[E\epsilon_{xx}^2 + G(\epsilon_{x\eta}^2 + \epsilon_{x\zeta}^2)]d\eta d\zeta dx \quad (98)$$

where E is the modulus of elasticity and G is the shear modulus. Substituting Eqs. (97) into Eq. (98), integrating over the blade cross-section, and retaining terms as in Ref. 5 yields

$$\begin{aligned} V_{EB} = \frac{1}{2}\int_0^{L_B} \Big\langle & E\{[u' + \frac{1}{2}(v'^2 + w'^2)]^2 A + (\phi_0'\phi' + \frac{1}{2}\phi'^2)^2 B_1 + \phi''^2 C_1 \\ & + v''^2[I_2\cos^2\phi_0 + I_1\sin^2\phi_0 - 2\phi\sin\phi_0\cos\phi_0(I_2 - I_1)] \\ & + w''^2[I_2\sin^2\phi_0 + I_1\cos^2\phi_0 + 2\phi\sin\phi_0\cos\phi_0(I_2 - I_1)] \\ & + 2[u' + \frac{1}{2}(v'^2 + w'^2)](\phi_0'\phi' + \frac{1}{2}\phi'^2)Ak_A^2 \\ & - 2[u' + \frac{1}{2}(v'^2 + w'^2)][v''\cos\phi_0 + w''\sin\phi_0 - 2\phi\sin\phi_0\cos\phi_0(v''-w'')]Ae_A \\ & - 2(\phi_0'\phi' + \frac{1}{2}\phi'^2)[v''\cos\phi_0 + w''\sin\phi_0 - 2\phi\sin\phi_0\cos\phi_0(v''-w'')]B_2 \\ & + 2v''w''[\sin\phi_0\cos\phi_0 + \phi(\cos^2\phi_0 - \sin^2\phi_0)](I_2 - I_1) \\ & - 2v''\phi''(\sin\phi_0 + \phi\cos\phi_0)C_2 + 2w''\phi''(\cos\phi_0 - \phi\sin\phi_0)C_2\} + GI_p\phi'^2\Big\rangle dx \end{aligned} \quad (99)$$

where A is the cross-sectional area and

$$B_1 = \iint_A (\eta^2 + \zeta^2)^2 \, d\eta d\zeta \qquad B_2 = \iint_A \eta(\eta^2 + \zeta^2) d\eta d\zeta$$

$$I_1 = \iint_A \zeta^2 d\eta d\zeta \qquad I_2 = \iint_A \eta^2 d\eta d\zeta \qquad Ae_A = \iint_A \eta d\eta d\zeta$$

$$Ak_A^2 = \iint_A (\eta^2 + \zeta^2) d\eta d\zeta \qquad I_p = \iint_A \left[\left(\eta - \frac{\partial\lambda}{\partial\zeta}\right)^2 + \left(\zeta + \frac{\partial\lambda}{\partial\eta}\right)^2\right] d\eta d\zeta \tag{100a}$$

$$C_1 = \iint_A \lambda^2 d\eta d\zeta \qquad C_2 = \iint_A \zeta\lambda d\eta d\zeta$$

are certain cross-section integrals. Note that I_1 and I_2 are flapwise and chordwise area moments of inertia, respectively, Ak_A^2 is the area polar moment of inertia, I_p is the torsional constant including cross-sectional warping and e_A is the tension offset from the elastic axis. We note that because of symmetry of the cross-section about the η axis and anti-symmetry of the warp function, we have taken into account in Eq. (99) the fact that

$$\iint_A \zeta d\eta d\zeta = \iint_A \eta\zeta d\eta d\zeta = \iint_A \zeta(\eta^2 + \zeta^2) d\eta d\zeta = \iint_A \lambda d\eta d\zeta = \iint_A \lambda\eta d\eta d\zeta = 0 \tag{100b}$$

Equation (99) can be discretized by using the expansions (85). Such a discrete expression for the blade elastic potential energy is very lengthy and will be omitted here for brevity.

Equations (87), (88), (90), (91), (94), (95), (96), and (99) define the gravitational and elastic potential energy for all the system components. In addition the shaft can undergo rigid body rotation relative to the air-frame, the flap link can rotate relative to the hub, and the lag link can rotate relative to the flap link. For generality, we shall assume that torsional springs are present to counteract these rotations.

The potential energies due to these springs can be written as

$$V_{KSx} = \frac{1}{2} k_{Sx}\psi_x^2 \tag{101a}$$

$$V_{KSy} = \frac{1}{2} k_{Sy}\psi_y^2 \tag{101b}$$

$$V_{K\alpha i} = \frac{1}{2} k_\alpha \alpha_i^2 \tag{101c}$$

$$V_{K\beta i} = \frac{1}{2} k_\beta \beta_i^2 \tag{101d}$$

where k_{Sx}, k_{Sy}, k_α, k_β are the torsional spring constants. Thus, the total system potential energy can be written in the form

$$V = V_{GA} + V_{EA} + V_{GT} + V_{GS} + V_{ES} + V_{GH} + V_{KSx} + V_{KSy} + \sum_{i=1}^{M} (V_{K\alpha i} + V_{K\beta i} + V_{GBi} + V_{EBi}) \tag{102}$$

4. The Aerodynamic Loading and Virtual Work

As indicated in Section 3, Lagrange's equations of motion require the expression for the nonconservative virtual work. In this section, we shall produce this expression for the main rotor, tail rotor, and airframe.

We begin with the formulation of the main rotor aerodynamic loads. A strip theory or blade element approach is adopted because the results are in the form of loads per unit span, and thus are easily integrable over the total area of each blade, and because the theory allows a high level of flexibility in regard to the flow field assumptions.

Before beginning the detailed analysis, a brief outline of assumptions and definitions is in order. The lift and pitching moment at a typical main rotor blade station are obtained via the Theodorsen expressions for oscillating airfoils. The velocity used in these expressions is the local two-dimensional relative wind due to both nonoscillatory motions of the airfoil and induced flow, hereafter referred to as the nonoscillatory relative wind. In contrast, the blade element drag is assumed to depend upon and act along the total two-dimensional or oscillatory relative wind, which is defined as the velocity due to all blade motions and induced flow. In all phases of the aerodynamic formulations, effects due to reversed flow, stall, and shed wakes are ignored; compressibility is treated by use of the Prandtl-Glauert factor.

In the induced flow analysis, it is convenient to work in terms of the hub $x_H y_H$ plane rather than the average tip path plane so that the complications of determining the latter are avoided. For the flight regimes to be studied, the orientations of the two planes should differ by only a few degrees; this justification for using the hub plane remains to be verified.

The mean induced velocity, as determined from simple momentum theory (see Ref. 6) is

$$W = T_R/(2\pi L_B{}^2 \gamma_A B^2 V') \tag{103}$$

where T_R is the rotor thrust along the z_H axis, L_B the rotor radius, γ_A the the air density, B the tip loss factor, and V' the relative wind at the hub. The tip loss factor is given by

$$B = 1 - \frac{\sqrt{2C_T}}{M} \tag{104}$$

where M is the number of rotor blades and

$$C_T = T_R/(\pi\gamma_A \Omega^2 L_B{}^4) \tag{105}$$

The relative wind at the hub has the expression

$$V' = [(W-\dot{w}_{OAX} \sin \alpha_R)^2 + (\dot{w}_{OAX} \cos \alpha_R)^2]^{1/2} \tag{106}$$

where α_R is the angle between the velocity of the hub and the plane normal to the hub axis z_R.

For hover, $\dot{w}_{OAX} \equiv 0$ and the induced velocity is

$$W_H = [T_R/2\pi\gamma_A B^2 L_B{}^2]^{1/2} \tag{107}$$

For high forward speeds, $W << \dot{w}_{OAX}$, $V' \doteq \dot{w}_{OAX}$, and

$$W_{HS} = T_R/(2\pi\gamma_A B^2 L_B{}^2 \dot{w}_{OAX}) \tag{108}$$

For moderate forward flight, Eqs. (103) and (106) are solved simultaneously for W, resulting in the quartic

$$W^4 - (2\dot{w}_{OAX} \sin \alpha_R)W^3 + (\dot{w}_{OAX})^2 W^2 - T_R{}^2/(4\pi^2 L_B{}^4 \gamma_A{}^2 B^4) = 0 \tag{109}$$

The mean induced flow is then corrected by Glauert's expression

$$W_{in} = W(1 - \frac{r}{L_B} K_v \cos \psi_Z) \tag{110}$$

where r is the radial distance from the hub center and K_v, according to Payne (Ref. 7), has the form

$$K_v = \frac{4}{3} \frac{\mu}{\gamma} (1.2 + \frac{\mu}{\lambda}) \tag{111}$$

in which

$$\mu = \frac{\dot{w}_{OAX} \cos \alpha_R}{\Omega L_B} \tag{112}$$

is the advance ratio and

$$\lambda = \frac{W - \dot{w}_{OAX} \sin \alpha_R}{\Omega L_B} \tag{113}$$

is the rotor inflow ratio.

The induced velocity W_{in}, which is assumed to be in the $-z_H$ direction, is then expressed in terms of components along the deformed blade section coordinate system $x_D y_D z_D$ as

$$\left\{W_{in}\right\} = [T_{DH}] \begin{Bmatrix} 0 \\ 0 \\ -W_{in} \end{Bmatrix} \tag{114}$$

where $[T_{DH}]$ is the matrix product $[T_{DB}]\,[T_{BL}]\,[T_{LF}]\,[T_{FH}]$ (see Appendix A).

The inertial velocity of a typical blade element is calculated by utilizing the previously introduced coordinate systems and velocity expressions. Specifically, the velocity of any point on the elastic axis of blade i*, in terms of components along $x_D y_D z_D$, is given by Eq. (A33) of Appendix A with η and ζ set equal to zero. If this velocity is denoted by $\{\dot{w}_B'\}_E$, where the subscript E refers to the elastic axis, then the oscillatory relative wind {U} is given by

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \{W_{in}\} - \{\dot{w}_B'\}_E \tag{115}$$

Since the profile drag is assumed to depend upon the two-dimensional oscillatory relative wind, we can use Eq. (115) to write its magnitude as

$$D = \frac{1}{2} C_{DO} \gamma_A U^2 c \tag{116}$$

where $U^2 = U_2^{\,2} + U_3^{\,2}$, c is the local blade chord length, and C_{DO} is the local profile drag coefficient (see Fig. 8).

As mentioned at the beginning of this section, Theodorsen's equations call for the use of the nonoscillatory blade motion, which is defined as the sum of all motions except those which contribute to the unsteady velocity of the airfoil in the direction perpendicular to the nonoscillatory relative wind. This nonoscillatory motion is obtained by setting the flap angle rate $\dot{\beta}$ and the blade bending rate $\dot{w}$ (along with η and ζ) equal to zero in the right side of Eq. (A33) of Appendix A. If the resulting velocity is termed $\{\dot{w}_B'\}_{EN}$, where the subscript EN refers to nonoscillatory motion of the elastic axis, then the nonoscillatory relative wind is given by

$$\{V\} = \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \{W_{in}\} - \{\dot{w}_B'\}_{EN} \tag{117}$$

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* As mentioned in Section 2, it is understood that the subscript i on variables associated with a particular rotor blade is deleted for clarity.

Theodorsen's expressions for the unsteady lift and pitching moment about the elastic axis are (see Ref. 8)

$$L = C_{\ell\alpha}\gamma_A VbC(k)[V\alpha_A-\dot{z}+b(\frac{1}{2} - a)\ \dot{\alpha}_A] + \pi\gamma_A b^2(V\dot{\alpha}_A-\ddot{z}-ba\ddot{\alpha}_A) \tag{118}$$

and

$$M_{xD} = C_{\ell\alpha}\gamma_A Vb^2(a + \frac{1}{2})C(k)[V\alpha_A-\dot{z} + b(\frac{1}{2} - a)\ \dot{\alpha}_A]$$

$$- \pi\gamma_A b^2\ [(\frac{1}{2} - a)\ Vb\dot{\alpha}_A + b^2\ (\frac{1}{8} + a^2)\ \ddot{\alpha}_A + ba\ddot{z}] \tag{119}$$

In these expressions, $C_{\ell\alpha}$ is the local blade element lift coefficient per unit length, $V = (V_2{}^2 + V_3{}^2)^{1/2}$, b is the blade semichord, a is the distance from the midchord to the elastic axis in percent of semichord b, measured positive towards the trailing edge, C(k) is the complex lift deficiency function, α_A is the instantaneous inclination of the chord to the nonoscillatory relative wind, and $\dot{z}$ is an upward (perpendicular to the nonoscillatory relative wind) translational velocity of the elastic axis. The angle of attack α_A can be written as (see Fig. 8)

$$\alpha_A = \tan^{-1}\left[\frac{V_3}{V_2}\right] \tag{120}$$

The terms $\dot{\alpha}_A$ and $\dot{z}$ are given by

$$\dot{\alpha}_A = \dot{\phi} + \dot{\theta}_{cy} + \dot{\theta}_{cp} \tag{121}$$

$$\dot{z} = \frac{1}{\cos\alpha_A}\ (\dot{\beta}B_3 + T_{DB31}\ \dot{u} + T_{DB32}\ \dot{v} + T_{DB33}\ \dot{w}) \tag{122}$$

where B_3 is the third component of the 3 × 1 matrix $([T_{DF}][r_{FL}] + [T_{DL}] \times [r_{LB}][T_{LF}] + [T_{DB}][r_B+u_B][T_{BF}])\{e_2\}$, which is recognized as the coefficient of $\dot{\beta}$ in Eq. (A33) and T_{DB31}, T_{DB32}, and T_{DB33} are elements of the last row of matrix $[T_{DB}]$. The terms $\ddot{z}$ and $\ddot{\alpha}_A$ in Eqs. (118) and (119) will be neglected (see Ref. 8).

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The complex lift deficiency function C(k) is given by

$$C(k) = F + iG \tag{123}$$

where

$$F = \frac{J_1(J_1+Y_0) + Y_1 (Y_1-J_0)}{(J_1+Y_0)^2 + (Y_1-J_0)^2} \tag{124}$$

and

$$G = -\frac{Y_1Y_0 + J_1J_0}{(J_1+Y_0)^2 + (Y_1-J_0)^2} \tag{125}$$

The Bessel functions J_1, J_0, Y_1, and Y_0 depend on the reduced frequency

$$k = \frac{\omega b}{V} \tag{126}$$

where ω, the frequency of oscillation is set equal to Ω for 1/rev oscillations; more generally, ω is the frequency which best represents the oscillatory motion of the main rotor blades. Because this frequency is not known in advance, an iterative process may be required, so that the value of ω assumed for the force computation is verified to agree with that resulting from the solution of the blade equations of motion. Note that C(k) introduces both a reduction in magnitude of the unsteady loads and a time lag of

$$t_{lag} = \frac{1}{\omega} \tan^{-1}\left|\frac{G}{F}\right| \tag{127}$$

In the case of the quasi-steady assumption, C(k) = 1 and $t_{lag} = 0$.

At time t, the components of the aerodynamic forces in the y_Dz_D coordinate system are

$$F_{yD} = \mathrm{Re}\left\{L(t - t_{lag})\right\} \ \sin \ \alpha_A - D(t) \cos\left[\tan^{-1}\left(\frac{U_3}{-U_2}\right)\right] \tag{128}$$

$$F_{zD} = \mathrm{Re}\left\{L(t - t_{lag})\right\} \ \cos \ \alpha_A + D(t) \sin\left[\tan^{-1}\left(\frac{U_3}{-U_2}\right)\right] \tag{129}$$

and the pitching moment is

$$M_{xD}(t) = \mathrm{Re}\left\{M_{xD}(t - t_{lag})\right\} \tag{130}$$

Now that the aerodynamic forces and moment acting on a typical blade section have been derived in terms of components along $x_D y_D z_D$, we proceed to formulate the nonconservative virtual work for the main rotor subsystem. The virtual work expression for the ith blade is

$$\delta W = \int_0^{L_B} \left(F_{yD}\delta w_{Dy} + F_{zD}\delta w_{Dz} + M_{xD}\delta\Theta_{Dx}\right) dx \tag{131}$$

where δw_{Dy} and δw_{Dz} are virtual displacements in the y_D and z_D directions and $\delta\Theta_{xD}$ is a virtual angular displacement about the x_D axis. Explicit expressions for these quantities are listed in Appendix A as Eqs. (A39), (A40), and (A36).

The fuselage aerodynamic characteristics are required for an eventual trim solution and for the airframe virtual work. The tail rotor exerts an aerodynamic force T_R on the airframe at point T. This force, which is assumed to act in the z_T direction (perpendicular to the tail rotor fan plane) and is calculated as part of the trim procedure, counteracts any airframe rotational tendency due to drag on the main rotor blades or other aerodynamic effects. Also acting on the airframe at its center of mass are the overall airframe lift L_A acting normal to the distant free-stream velocity, the airframe drag D_A acting along the distant free-stream velocity, and the airframe pitching moment M_A about the y_A axis. These three airframe loads are given by

$$L_A = \frac{1}{2} C_{L\alpha A} \gamma_A V_{AF}{}^2 S_{ref} \alpha_{AF} \tag{132}$$

$$D_A = \frac{1}{2} C_{DA} \gamma_A V_{AF}{}^2 S_{ref} \tag{133}$$

$$M_A = \frac{1}{2} C_{M\alpha A} \gamma_A V_{AF}{}^2 S_{ref} L_{SM} \alpha_{AF} \tag{134}$$

where $C_{L\alpha A}$, C_{DA}, and $C_{M\alpha A}$ are the airframe lift, drag and pitching moment coefficients based on the reference area S_{ref}, L_{SM} is the static margin, α_{AF} is the airframe angle of attack, and $V_{AF}{}^2 = (\dot{w}_{OAX}{}^2 + \dot{w}_{OAZ}{}^2)$. Due to the nature of the flight regimes to be studied, side forces are not considered.

The virtual work due to the three airframe loads and the tail rotor thrust is (see Fig. 9)

$$\begin{aligned} \delta W_A = {} & (L_A \sin \alpha_{AF} - D_A \cos \alpha_{AF})\, \delta w_{OAX} + \\ & (L_A \cos \alpha_{AF} + D_A \sin \alpha_{AF})\, \delta w_{OAZ} + M_A \delta\lambda_y \\ & + T_{TR} \delta w_{ATz} \end{aligned} \tag{135}$$

where δw_{ATz} is given by Eq. (A41) of Appendix A.

5. Lagrange's Equations of Motion in General Form

The kinetic energy, potential energy, and nonconservative virtual work have been discussed in detail in Secs. 2-4. In this section we shall present a general form of Lagrange's equations for the system.

Before we can proceed with Lagrange's equations, it is necessary to identify the functional dependence of T and V on the various generalized coordinates. The functional dependence presented here is the most general possible and not restricted to the linear case. First, we recognize that the airframe is described by three rigid-body translations w_{OAX}, w_{OAY}, w_{OAZ} three rigid-body rotations λ_x, λ_y, λ_z, and P elastic generalized coordinates η_{Ai} (i = 1,2,...,P). Then, considering Eqs. (65a,b), (69), (70), (71), (87), and (88) in conjunction with Eqs. (3) and (4) as well as the definition of $\{w_{OA}\}$, we conclude that the functional dependence of T_A, V_{GA}, and V_{EA} is as follows:

$$T_A = T_A(\dot{w}_{OAX},\dot{w}_{OAY}\ \dot{w}_{OAZ},\dot{\lambda}_x,\dot{\lambda}_y,\dot{\lambda}_z,\dot{\eta}_{Ai},\lambda_x,\lambda_y,\lambda_z)\ ,\quad i = 1,2,\ldots,P \qquad (136a)$$

$$V_{GA} = V_{GA}(w_{OAZ},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai})\ ,\quad i = 1,2,\ldots,P \qquad (136b)$$

$$V_{EA} = V_{EA}(\eta_{Ai})\quad ,\quad i = 1,2,\ldots,P \qquad (136c)$$

The transmission shaft is described by three rigid-body rotations ψ_x, ψ_y, and ψ_z relative to the deformed airframe and $2S_u + S_\phi$ elastic generalized coordinates η_{uj}, η_{vj}, $\eta_{\phi k}$ ($j = 1,2,\ldots,S_u$; $k = 1,2,\ldots,S_\phi$). Note that the rotation $\psi_z = \int \Omega dt$ is a specified function of time. Considering Eqs. (79), (90), (92), and (101a,b) for the shaft kinetic energy, gravitational potential energy, elastic potential energy, and potential energies due to torsional springs at the base, respectively, together with Eqs. (3), (4), (11), (12), (20), (69), (78), (A14), and (A30), we can write the functional dependence of T_S, V_{GS}, V_{ES}, V_{KSx} and V_{KSy} as follows:

$$T_S = T_S(\dot{w}_{OAX},\dot{w}_{OAY},\dot{w}_{OAZ},\dot{\lambda}_x,\dot{\lambda}_y,\dot{\lambda}_z,\dot{\eta}_{Ai},\dot{\psi}_x,\dot{\psi}_y,\dot{\eta}_{uj},\dot{\eta}_{vj},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai},\psi_x,\psi_y,\eta_{uj},\eta_{vj}),$$
$$i = 1,2,\ldots,P;\ j = 1,2,\ldots,S_u \qquad (137a)$$

$$V_{GS} = V_{GS}(w_{OAZ},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai},\psi_x,\psi_y,\eta_{uj},\eta_{vj})$$
$$i = 1,2,\ldots,P;\ j = 1,2,\ldots,S_u \qquad (137b)$$

$$V_{ES} = V_{ES}(\eta_{uj},\eta_{vj},\eta_{\phi k}) \qquad j = 1,2,\ldots,S_u;\ k = 1,2,\ldots,S_\phi \qquad (137c)$$

$$V_{KSx} = V_{KSx}(\psi_x) \qquad (137d)$$

$$V_{KSy} = V_{KSy}(\psi_y) \qquad (137e)$$

The tail rotor is described by one specified rotation $\int \Omega_T\, dt$ relative to the deformed airframe. The kinetic energy of the tail rotor is given by Eq. (57), so that considering Eqs. (3), (4), (26), (27), (28), and (66) the functional dependence of T_T can be written as

$$T_T = T_T(\dot{w}_{OAX},\dot{w}_{OAY},\dot{w}_{OAZ},\dot{\lambda}_x,\dot{\lambda}_y,\dot{\lambda}_z,\dot{\eta}_{Ai},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai}),\ i = 1,2,\ldots,) \qquad (138)$$

On the other hand, the tail rotor potential energy is given by Eq. (94), and it is readily seen that

$$V_{GT} = V_{GT}(w_{OAZ},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai})\ ,\ i = 1,2,\ldots,P \qquad (139)$$

Following the same pattern, the hub is fixed to the end of the transmission shaft and moves together with it, so that considering Eqs. (58) and (95) in conjunction with Eqs. (3), (4), (11), (12), (20), (21), (69), (78), (A15), and (A31), we conclude that the functional dependence of T_H and V_{GH} is

$$T_H = T_H(\dot{w}_{OAX},\dot{w}_{OAY},\dot{w}_{OAZ},\dot{\lambda}_x,\dot{\lambda}_y,\dot{\lambda}_z,\dot{\eta}_{Ai},\dot{\psi}_x,\dot{\psi}_y,\dot{\eta}_{uj},\dot{\eta}_{vj},\dot{\eta}_{\phi K},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai},$$
$$\psi_x,\psi_y,\eta_{uj},\eta_{vj},\eta_{\phi k})\ ,\ i = 1,2,\ldots,P;\ j = 1,2,\ldots,S_u;$$
$$k = 1,2,\ldots,S_\phi \qquad (140a)$$

$$V_{GH} = V_{GH}(w_{OAZ},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai},\psi_x,\psi_y,\eta_{uj},\eta_{vj}) \;,\; i = 1,2,\ldots,P;\; j = 1,2,\ldots,S_u \tag{140b}$$

Finally, the rth rotor blade is described by the three rotations α_r (lead-lag), β_r (flapping), and θ_r (pitch) as well as the $N_u + N_v + N_w + N_\phi$ elastic generalized coordinates $q_{ur\ell}$, q_{vrm}, q_{wrn}, $q_{\phi rs}$ ($\ell = 1,2,\ldots,N_u$; $m = 1,2,\ldots,N_v$; $n = 1,2,\ldots,N_w$; $s = 1,2,\ldots,N_\phi$). As mentioned in Sec. 2, of the three rotations only α_r and β_r are generalized coordinates. The kinetic energy is given in general form by Eq. (60). Considering Eqs. (81), (84), and (85) along with Eqs. (3), (4), (11), (12), (20), (21), (30), (31), (36), (40), (69), (78), (A16), and (A32), the functional dependence of T_{Br} ($r = 1,2,\ldots,M$) is deduced to be

$$\begin{aligned} T_{Br} = T_{Br}(&\dot{w}_{OAX},\dot{w}_{OAY},\dot{w}_{OAZ},\dot{\lambda}_x,\dot{\lambda}_y,\dot{\lambda}_z,\dot{\eta}_{Ai},\dot{\psi}_x,\dot{\psi}_y,\dot{\eta}_{uj},\dot{\eta}_{vj},\dot{\eta}_{\phi k},\dot{\alpha}_r,\dot{\beta}_r,\dot{q}_{ur\ell},\dot{q}_{vrm},\\ &\dot{q}_{wrn},\dot{q}_{\phi rs},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai},\psi_x,\psi_y,\psi_z,\eta_{uj},\eta_{vj},\eta_{\phi k},\alpha_r,\beta_r,q_{ur\ell},q_{vrm},\\ &q_{wrn},q_{\phi rs}) \;,\; i = 1,2,\ldots,P;\; j = 1,2,\ldots,S_u;\; k = 1,2,\ldots,S_\phi;\\ &\ell = 1,2,\ldots,N_u;\; m = 1,2,\ldots,N_v;\; n = 1,2,\ldots,N_w;\; s = 1,2,\ldots,N_\phi \end{aligned} \tag{141}$$

The gravitational and elastic potential energies V_{GBr} and V_{EBr} are given by Eqs. (96) and (99) respectively, whereas the potential energies $V_{K\alpha r}$ and $V_{K\beta r}$ are given by Eqs. (101c) and (101d). Their functional dependence is readily seen to be

$$\begin{aligned} V_{GBr} = V_{GBr}(&w_{OAZ},\lambda_x,\lambda_y,\lambda_z,\eta_{Ai},\psi_x,\psi_y,\eta_{uj},\eta_{vj},\eta_{\phi k},\alpha_r,\beta_r,q_{ur\ell},q_{vrm},q_{wrn},q_{\phi rs}) \;,\\ &i = 1,2,\ldots,P;\; j = 1,2,\ldots,S_u;\; k = 1,2,\ldots,S_\phi;\; \ell = 1,2,\ldots,N_u;\\ &m = 1,2,\ldots,N_v;\; n = 1,2,\ldots,N_w;\; s = 1,2,\ldots,N_\phi \end{aligned} \tag{142a}$$

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$$V_{EBr} = V_{EBr}(q_{ur\ell}, q_{vrm}, q_{wrn}, q_{\phi rs}) \;,\; \ell = 1,2,\ldots,N_u;\; m = 1,2,\ldots,N_v;$$

$$n = 1,2,\ldots,N_w;\; s = 1,2,\ldots,N_\phi \qquad (142b)$$

$$V_{K\alpha r} = V_{K\alpha r}(\alpha_r) \qquad (142c)$$

$$V_{K\beta r} = V_{K\beta r}(\beta_r) \qquad (142d)$$

Considering Eqs. (62) and (102), the system Lagrangian is

$$L = T\text{-}V = T_A + T_T + T_S + T_H - V_{GA} - V_{EA} - V_{GT} - V_{GS} - V_{ES} - V_{GH} - V_{KSx} - V_{KSy} + \sum_{r=1}^{M} (T_{Br} - V_{GBr} - V_{EBr} - V_{K\alpha r} - V_{K\beta r}) \qquad (143)$$

Noting the functional dependencies (136) - (142), Lagrange's equations for the three rigid-body translations can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{w}_{OAX}} - \frac{\partial L}{\partial w_{OAX}} = \frac{d}{dt}\frac{\partial}{\partial \dot{w}_{OAX}}(T_A + T_T + T_S + T_H + \sum_{r=1}^{M} T_{Br}) = F_{AOX} \qquad (144a)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{w}_{OAY}} - \frac{\partial L}{\partial w_{OAY}} = \frac{d}{dt}\frac{\partial}{\partial \dot{w}_{OAY}}(T_A + T_T + T_S + T_H + \sum_{r=1}^{M} T_{Br}) = F_{AOY} \qquad (144b)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{w}_{OAZ}} - \frac{\partial L}{\partial w_{OAZ}} = \frac{d}{dt}\frac{\partial}{\partial \dot{w}_{OAZ}}(T_A + T_T + T_S + T_H + \sum_{r=1}^{M} T_{Br}) + \frac{\partial}{\partial w_{OAZ}}(V_{GA} + V_{GT} + V_{GS} + V_{GH} + \sum_{r=1}^{M} V_{GBr}) = F_{AOZ} \qquad (144c)$$

where F_{OAX}, F_{OAY}, F_{OAZ} are the nonconservative generalized forces associated with w_{OAX}, w_{OAY}, w_{OAZ} respectively. Similarly, we can write the rotation equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}_x} - \frac{\partial L}{\partial \lambda_x} = \frac{d}{dt}\frac{\partial}{\partial \dot{\lambda}_x}(T_A + T_T + T_S + T_H + \sum_{r=1}^{M} T_{Br}) - \frac{\partial}{\partial \lambda_x}[T_A + T_T + T_S + T_H - V_{GA} - V_{GT} - V_{GS} - V_{GH} + \sum_{r=1}^{M} (T_{Br} - V_{GBr})] = F_{\lambda x} \qquad (145)$$

where analogous equations can be written for λ_y and λ_z and $F_{\lambda x}$, $F_{\lambda y}$, $F_{\lambda z}$ are the nonconservative generalized forces associated with λ_x, λ_y, λ_z, respectively. In the same manner, we can write the airframe elastic deformation equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_{Ai}} - \frac{\partial L}{\partial \eta_{Ai}} = \frac{d}{dt}\frac{\partial}{\partial \dot{\eta}_{Ai}}\left(T_A + T_T + T_S + T_H + \sum_{r=1}^{M} T_{Br}\right)$$

$$- \frac{\partial}{\partial \eta_{Ai}}\Big[T_A + T_T + T_S + T_H - V_{GA} - V_{GT} - V_{GS} - V_{GH}$$

$$- V_{EA} + \sum_{r=1}^{M}(T_{Br} - V_{GBr})\Big] = F_{\eta Ai}\ ,\ i = 1,2,\ldots,P \tag{146}$$

the transmission shaft rotation equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}_x} - \frac{\partial L}{\partial \psi_x} = \frac{d}{dt}\frac{\partial}{\partial \dot{\psi}_x}\left(T_S + T_H + \sum_{r=1}^{M} T_{Br}\right) - \frac{\partial}{\partial \psi_x}\Big[T_S + T_H - V_{GS}$$

$$- V_{KSx} + \sum_{r=1}^{M}(T_{Br} - V_{GBr})\Big] = F_{\psi x} \tag{147}$$

where an analogous equation can be written for ψ_y, the transmission shaft elastic deformation equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_{uj}} - \frac{\partial L}{\partial \eta_{uj}} = \frac{d}{dt}\frac{\partial}{\partial \dot{\eta}_{uj}}\left(T_S + T_H + \sum_{r=1}^{M} T_{Br}\right) - \frac{\partial}{\partial \eta_{uj}}\Big[T_S + T_H - V_{GS}$$

$$- V_{GH} + \sum_{r=1}^{M}(T_{Br} - V_{GBr}) - V_{ES}\Big] = F_{\eta uj} \tag{148a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_{vj}} - \frac{\partial L}{\partial \eta_{vj}} = \frac{d}{dt}\frac{\partial}{\partial \dot{\eta}_{vj}}\left(T_S + T_H + \sum_{r=1}^{M} T_{Br}\right) - \frac{\partial}{\partial \eta_{vj}}\Big[T_S + T_H - V_{GS}$$

$$- V_{GH} + \sum_{r=1}^{M}(T_{Br} - V_{GBr}) - V_{ES}\Big] = F_{\eta vj} \tag{148b}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_{\phi k}} - \frac{\partial L}{\partial \eta_{\phi k}} = \frac{d}{dt}\frac{\partial}{\partial \dot{\eta}_{\phi k}}(T_H + \sum_{r=1}^{M} T_{Br}) - \frac{\partial}{\partial \eta_{\phi k}}[T_H + \sum_{r=1}^{M}(T_{Br} - V_{GBr})$$

$$- V_{ES}] = F_{\eta\phi k} \quad j = 1,2,\ldots,S_u;\ k = 1,2,\ldots,S_\phi \tag{148c}$$

the rotor blade rotation equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_r} - \frac{\partial L}{\partial \alpha_r} = \frac{d}{dt}\frac{\partial T_{Br}}{\partial \dot{\alpha}_r} + \frac{\partial V_{GBr}}{\partial \alpha_r} + \frac{\partial V_{K\alpha r}}{\partial \alpha_r} = F_{\alpha r},\ r = 1,2,\ldots,M \tag{149a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_r} - \frac{\partial L}{\partial \beta_r} = \frac{d}{dt}\frac{\partial T_{Br}}{\partial \dot{\beta}_r} + \frac{\partial V_{GBr}}{\partial \beta_r} + \frac{\partial V_{KBr}}{\partial \beta_r} = F_{\beta r},\ r = 1,2,\ldots,M \tag{149b}$$

and the rotor blade elastic deformation equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{ur\ell}} - \frac{\partial L}{\partial q_{ur\ell}} = \frac{d}{dt}\frac{\partial T_{Br}}{\partial \dot{q}_{ur\ell}} - \frac{\partial T_{Br}}{\partial q_{ur\ell}} + \frac{\partial V_{GBr}}{\partial q_{ur\ell}} + \frac{\partial V_{zBr}}{\partial q_{ur\ell}} = F_{qur\ell} \tag{150a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{urm}} - \frac{\partial L}{\partial q_{urm}} = \frac{d}{dt}\frac{\partial T_{Br}}{\partial \dot{q}_{urm}} - \frac{\partial T_{Br}}{\partial q_{urm}} + \frac{\partial V_{GBr}}{\partial q_{urm}} + \frac{\partial V_{EBr}}{\partial q_{urm}} = F_{qurm} \tag{150b}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{wrn}} - \frac{\partial L}{\partial q_{wrn}} = \frac{d}{dt}\frac{\partial T_{Br}}{\partial \dot{q}_{wrn}} - \frac{\partial T_{Br}}{\partial q_{wrn}} + \frac{\partial V_{GBr}}{\partial q_{wrn}} + \frac{\partial V_{EBr}}{\partial q_{wrn}} = F_{qwrn} \tag{150c}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{\phi rs}} - \frac{\partial L}{\partial q_{\phi rs}} = \frac{d}{dt}\frac{\partial T_{Br}}{\partial \dot{q}_{\phi rs}} - \frac{\partial T_{Br}}{\partial q_{\phi rs}} + \frac{\partial V_{GBr}}{\partial q_{\phi rs}} + \frac{\partial V_{EBr}}{\partial q_{\phi rs}} = F_{q\phi rs} \tag{150d}$$

$\ell = 1,2,\ldots,N_u;\ m = 1,2,\ldots,N_v;\ n = 1,2,\ldots,N_w;\ s = 1,2,\ldots,N_\phi$

$r = 1,2,\ldots,M$

where the meaning of $F_{\eta Ai}$, $F_{\psi x}$, $F_{\psi y}$, $F_{\eta uj}$, $F_{\eta vj}$, $F_{\eta\phi k}$, $F_{\alpha r}$, $F_{\beta r}$, $F_{qur\ell}$, F_{qvrm}, F_{qwrn}, and $F_{q\phi rs}$ ($i = 1,2,\ldots,P$; $j = 1,2,\ldots,S_u$; $k = 1,2,\ldots,S_\phi$; $\ell = 1,2,\ldots,N_u$; $m = 1,2,\ldots,N_v$; $n = 1,2,\ldots,N_w$; $s = 1,2,\ldots,N_\phi$; $r = 1,2,\ldots,M$) is obvious.

Equations (144) - (150) represent a set of $6 + P + 2 + 2S_u + S_\phi + (N_u$

$+ N_v + N_w + N_\phi)M$ coupled nonlinear equations which must be solved simultaneously. Before we proceed with the actual solution, we must first render the equations in a more explicit form by replacing the various kinetic energy components, potential energy components, and generalized nonconservative forces by their specific matrix expressions. Then, we derive the variational equations by expanding the nonlinear equations about trim solutions. A general form of the trim solution and variational equations will be discussed in the next section.

Lagrange's equations, Eqs. (144) - (150) involve an extremely large number of matrix multiplications and differentiations both with respect to generalized coordinates and velocities and time. The magnitude of the task demands a more automated approach, so that in Sec. 7 we present a procedure for the derivation of Lagrange's equations by computer manipulation.

6. The Perturbation Equations

Let us assume that, following discretization, the helicopter can be represented by an n-degree-of-freedom system, so that its motion is described by n second-order Lagrange's equations or 2n first-order Hamilton's equations. The latter set can be written in the form

$$\dot{x}_i = X_i(x_1, x_2, \ldots, x_{2n}, t) \quad , \quad i = 1,2,\ldots,2n \tag{151}$$

where X_i are generally nonlinear functions of the variables x_i $(i = 1,2,\ldots, 2n)$ and of the time t. Note that n of the variables x_i represent generalized displacements and the remaining n represent generalized velocities, or generalized momenta. The 2n variables $x_i(t)$ define the state of the system at any time t.

Next, let us consider a special solution of Eqs. (151), namely, a trim solution. In general, there are many such solutions, but solutions of particular interest are those corresponding to hovering and to forward flight. Denoting a particular trim solution by $\phi_i(t)$, and recognizing that $\phi_i(t)$ must satisfy Eqs. (151), we can write

$$\dot{\phi}_i(t) = X_i(\phi_1, \phi_2, \ldots, \phi_{2n}, t) \quad , \quad i = 1,2,\ldots,2n \tag{152}$$

In general, such solutions are periodic, $\phi_i(t) = \phi_i(t + T)$. We shall refer to ϕ_i as the <u>unperturbed motion</u>.

Letting $y_i(t)$ be <u>perturbations</u> about a given trim solution $\phi_i(t)$, the general perturbed motion can be written in the form

$$x_i(t) = \phi_i(t) + y_i(t) \quad , \quad i = 1,2,\ldots,2n \tag{153}$$

so that, introducing Eqs. (153) into Eqs. (151), we obtain

$$\dot{\phi}_i + \dot{y}_i = X_i(\phi_1 + y_1, \phi_2 + y_2, \ldots, \phi_{2n} + y_{2n}, t) \; , \; i = 1,2,\ldots,2n \tag{154}$$

Considering Eqs. (152), Eqs. (154) reduced to

$$\dot{y}_i = X_i(\phi_1 + y_1, \phi_2 + y_2, \ldots, \phi_{2n} + y_{2n}, t) - X_i(\phi_1, \phi_2, \ldots, \phi_{2n}, t)$$

$$i = 1,2,\ldots,2n \tag{155}$$

which are referred to as the <u>differential equations of the perturbed motion</u>. Equations (155) can be expressed in a different form. To this end, let us expand the first term on the right side of Eqs. (155) in the Taylor's series about the trim solution

$$X_i(\phi_1 + y_1, \phi_2 + y_2, \ldots, \phi_{2n} + y_{2n}, t) = X_i(\phi_1, \phi_2, \ldots, \phi_{2n}, t)$$

$$+ \sum_{j=1}^{2n} \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}} y_i + 0_i(\underset{\sim}{y}^2) \; , \; i = 1,2,\ldots,2n \tag{156}$$

where $\underset{\sim}{x}$, $\underset{\sim}{\phi}$, and $\underset{\sim}{y}$ are 2n-dimensional vectors associated with x_i, ϕ_i, and y_i, respectively, and $0_i(\underset{\sim}{y}^2)$ denotes terms of second order in y_i. Introducing the notation

$$a_{ij}(t) = \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}} \; , \; i,j = 1,2,\ldots,2n \tag{157}$$

and considering Eqs. (156), Eqs. (155) can be rewritten in the form

$$\dot{y}_i = \sum_{j=1}^{2n} a_{ij}(t) y_j + 0_i(\underset{\sim}{y}^2) \quad , \; i = 1,2,\ldots,2n \tag{158}$$

where in general the coefficients a_{ij} are periodic, $a_{ij}(t) = a_{ij}(t + T)$. Note that Eqs. (158) are nonlinear because of the terms $0_i(\underset{\sim}{y}^2)$.

A case of particular interest is that in which the perturbations $y_i(t)$ are small. In this case, we can neglect the second-order terms in Eqs. (158) and obtain the set of <u>linearized equations</u>

$$\dot{y}_i = \sum_{j=1}^{2n} a_{ij}(t)\, y_j \quad , \; i = 1,2,\ldots,2n \tag{159}$$

which are referred to as the <u>variational equations</u>.

The perturbation equations, Eqs. (158), or the linearized version, Eqs. (159), were derived on the assumption that the solution $\phi_i(t)$ represents an actual trim solution, i.e., they represent a solution of the original equations (151). Trim solutions, however, are difficult to obtain and at times one may wish to assume an approximate solution and derive a set of perturbation equations about the "assumed trim". The question arises naturally as to the behavior of these equations. To answer this question, let us denote the <u>actual trim</u> by $\phi_i(t)$ and the <u>assumed trim</u> by $\phi_i^*(t)$. Assuming that the two solutions differ to some extent, we can write

$$\phi_i^*(t) = \phi_i(t) + \delta_i(t) \quad , \quad i = 1,2,\ldots,2n \tag{160}$$

where $\delta_i(t)$ $(i = 1,2,\ldots,2n)$ represents the difference between the two solutions. Then, the perturbed motion can be written in the form

$$x_i(t) = \phi_i^*(t) + y_i^*(t) \quad , \; i = 1,2,\ldots,2n \tag{161}$$

where $y_i^*(t)$ are perturbations from the assumed trim. Inserting Eqs. (161) into Eqs. (151), we obtain

$$\dot{\phi}_i^*(t) + \dot{y}_i^*(t) = X_i(\phi_1^* + y_1^*, \phi_2^* + y_2^*, \ldots, \phi_{2n}^* + y_{2n}^*, t)$$
$$i = 1,2,\ldots,2n \tag{162}$$

so that, expanding X_i about the assumed trim ϕ_i^*, we can write Eqs. (162) in the form

$$\dot{y}_i^*(t) = X_i(\phi_1^*, \phi_2^*, \ldots, \phi_{2n}^*, t) - \dot{\phi}_i^*(t) + \sum_{j=1}^{2n} \left.\frac{\partial X_i}{\partial x_j}\right|_{\underline{x}=\underline{\phi}^*} y_i^* + O_i(\underline{y}^{*2}) \; ,$$
$$i = 1,2,\ldots,2n \tag{163}$$

Unlike the case in which the expansion was about the actual trim, however, the first two terms on the right side of Eqs. (165) do not cancel out, because the assumed trim does not solve Eqs. (151). It will prove of interest to examine these terms a little closer.

Assuming that the difference between the actual and the assumed trim is relatively small, we can write the expansion about the actual trim

$$X_i(\phi_1^*,\phi_2^*,\ldots,\phi_{2n}^*,t) \approx X_i(\phi_1,\phi_2,\ldots,\phi_{2n},t) + \sum_{k=1}^{2n} b_{ik}(t)\delta_k(t)$$

$$i = 1,2,\ldots,2n \tag{164}$$

where

$$b_{ik}(t) = \left.\frac{\partial X_i}{\partial x_k}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}} \tag{165}$$

are the actual coefficients, which are <u>generally not known</u>. Considering Eqs. (161) and (164), as well as Eqs. (152), we obtain

$$X_i(\phi_1^*,\phi_2^*,\ldots,\phi_{2n}^*,t) - \dot{\phi}_i^*(t) \approx \sum_{k=1}^{2n} b_{ik}(t)\delta_k(t) - \dot{\delta}_i(t) \tag{166}$$

Moreover, introducing the notation

$$\varepsilon_i(t) = \sum_{k=1}^{2n} b_{ik}(t)\delta_k(t) - \dot{\delta}_i(t) \quad , \quad i = 1,2,\ldots,2n$$

$$a_{ij}^*(t) = \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}^*} \quad , \tag{167}$$

Eqs. (163) reduce to

$$\dot{y}_i^*(t) = \sum_{j=1}^{2n} a_{ij}^*(t)y_j^*(t) + \varepsilon_i(t) + 0_i(\underset{\sim}{y}^{*2}) \tag{168}$$

Hence, $\varepsilon_i(t)$ play the role of <u>unknown extraneous forces</u> introduced by the process of using an approximate trim instead of an actual one. Although these forces are not generally known, one may be able to estimate them.

The solution y_i^* represents perturbations from the assumed trim instead of from an actual trim. The question can be asked as to how they compare with the perturbations y_i from the actual trim. For small deviations δ_i from the actual trim, the response y_i^* should not differ very much from y_i, but this cannot be taken for granted. The relation between y_i^* and y_i depends of course on ε_i. Methods for estimating bounds for $\underset{\sim}{y}^* - \underset{\sim}{y}$ for given bounds for $\underset{\sim}{\phi}^* - \underset{\sim}{\phi}$ appear highly desirable.

7. Algebraic Computer Manipulation

In Sec. 3 we mentioned two approaches for expanding the system kinetic energy, Eq. (63). One approach is to obtain explicit expressions for the mass integrals given by Eqs. (53), (56), and (61) while retaining the translational velocity and angular velocity of each set of axes in implicit form. Such an approach is convenient when the elastic displacements are of interest and was the approach used in Sec. 3. The other possibility is to substitute explicit expressions for the translational velocity and angular velocity of each set of axes given by Eqs. (A28)-(A32) and (A12)-(A17) into Eqs. (64) while retaining the mass integrals (53), (56), and (61) in implicit form. This approach is convenient when the rotational coordinates as well as the terms due to coupling between bodies are of interest, and it must be adopted if one wishes to derive explicit expressions for Lagrange's equations. Examining Eqs. (A28)-(A32) and (A12)-(A17) one concludes immediately that working with explicit expressions for the velocities and angular velocities of each set of axes involves the calculation of very lengthy matrix products. Moreover, these matrix products involve quantities of different orders of magnitude and/or importance. Many of these terms are insignificantly small and can be ignored. Because of the complexity associated with algebraic multiplication of a large number of matrices and because of the high probability of human error in performing these multiplications by hand, it seems highly desirable to computerize these algebraic operations. It should be pointed out that algebraic computer manipulation need not be restricted to matrix multiplication. Indeed, many operations involved in the derivation of Lagrange's equations, including differentiation, can be performed by computer.

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In this section, we shall outline an algebraic computer procedure which can be used to expand the system kinetic energy explicitly in terms of the translational and angular velocity of each set of axes in such a way that small terms are ignored automatically. In addition, the same method can be applied to the gravitational potential energies, Eqs. (81), (90), (94), (95), and (96). Furthermore, the system kinetic energy and gravitational potential energy expressions obtained by the computer method are in a form which can be easily differentiated algebraically by the computer, so that ultimately Lagrange's equations, Eqs. (144)-(150), can be obtained explicitly with a minimum of human effort.

The ideas used in implementing algebraic manipulation on a computer, are best introduced via an explicit example. Let us consider the product

$$-0.5\ \dot{\beta}\ s\theta\ c\alpha(-\dot{\beta}\ J_{Bxz}\ s\alpha - \dot{\beta}\ J_{Byz}\ c\theta\ c\alpha - \dot{\beta}\ J_{Bzz}\ s\theta\ c\alpha) \qquad (169)$$

which appears in the expansion of $\frac{1}{2}\{\Omega_B\}^T[J_B]\{\Omega_B\}$, where the latter was encountered in Sec. 3. To calculate this product on the computer, we shall associate numbers with the different groups of characters that represent quantities to be manipulated algebraically, i.e., $\dot{\beta}$, J_{Bxz}, J_{Byz}, J_{Bzz}, $s\theta$, $c\theta$, $s\alpha$, ar $c\alpha$. These groups of characters constitute what will be referred to as symbols. The association of numbers with symbols allows us to substitute manipulation of numbers for manipulation of symbols and is accomplished by the formation of a symbol table, Table I. The symbol table contains two entries per line. These entries give the character representation of a symbol and a weight assigned to the symbol, where the weight is determined by the analyst according to his knowledge of the symbol's magnitude or his desire to retain its effects. The weight need not be a fixed quantity and can be changed at will. A high numerical value of the weight implies a high-order term, i.e., a less significant term. Note that here we assigned J_{Bzz}, $s\theta$, $c\theta$, and $c\alpha$ a weight of zero, J_{Bxz}, $s\alpha$, and $\dot{\beta}$ a weight of unity, and J_{Byz} a weight of two, where the weights represent anticipated magnitudes of these symbols. The number associated with a particular symbol is the line number in the symbol table. In this case, one is associated with $\dot{\beta}$, two with J_{Bxz}, three with J_{Byz}, etc.

Examining Eq. (169), we see that it is necessary to form algebraic products of symbols, for example $- 0.5\ \dot{\beta}\ s\theta\ c\alpha$. To this end, we define <u>terms</u>. A term consists of a signed numerical <u>coefficient</u>, a <u>pattern</u> consisting of the numbers associated with each symbol appearing in the product, and a <u>weight</u> which is the sum of the weights of each individual symbol appearing in the term. Hence, in view of the number-symbol associations and the weights of Table I, we represent $-0.5\ \dot{\beta}\ s\theta\ c\alpha$ as a term having a coefficient of -0.5, a pattern of 1, 5, 8, and a weight of 1. All terms, i.e., all coefficients, weights, and patterns are stored in numbered storage stacks. The coefficient and weight of any term are always single numbers and are stored side-by-side in the <u>coefficients and weights free storage stack</u>, exhibited in the form of Table II. On the other hand, the pattern of a term may differ from the pattern of other terms and must be able to represent the product of any number of symbols. Because of the different lengths of different patterns, all patterns are stored in the separate <u>patterns free storage stack</u>, labeled as Table III. Note that the coefficient and weight of $-0.5\ \dot{\beta}\ s\theta\ c\alpha$ are stored in line 3 of Table II and the pattern is stored in lines 5, 6, and 7 of Table III.

It is also necessary to form algebraic sums of terms, such as $-\dot{\beta}\ J_{Bxz}\ s\alpha - \dot{\beta}\ J_{Byz}\ c\theta\ c\alpha - \dot{\beta}\ J_{Bzz}\ s\theta\ c\alpha$, which we shall call <u>series</u>. Each series is given a distinct <u>series name</u>. As examples, we shall call the series consisting of the single term $-0.5\ \dot{\beta}\ s\theta\ c\alpha$ by the name λ, the series $-\dot{\beta}\ J_{Bxz}\ s\alpha$

$- \dot{\beta}\, J_{Byz}\, c\theta\, c\alpha - \dot{\beta}\, J_{Bzz}\, s\theta\, c\alpha$ by the name Y and the series resulting from the product of X and Y by the name Z. A series is described by a sequence of terms with coefficients and weights stored sequentially in the coefficients and weights free storage stack and with patterns stored sequentially in the patterns free storage stack. The summation of terms in the sequence is understood. Hence, the series Y is described by the coefficients and weights stored in lines 8, 9, and 10 of Table II and the patterns stored in lines 12-14, 16-19, and 21-24 of Table III. To distinguish sequences of terms forming series, each series name is assigned a number corresponding to a line in the series definition table, Table IV. Each line of this table contains three entries giving the line number of the coefficient and weight in the coefficients and weights free storage stack of the first term in the series, the line number of the beginning of the pattern in the patterns free storage stack of the first term in the series, and the number of terms in the series. Assigning the number 5 to the series name Y, the fifth line of Table IV contains the entries 8 and 12 giving the storage locations of the series and the entry 3 designating that there are three terms in the series.

Let us now consider the multiplication of two series, namely, the multiplication of Y by X, which can be performed term by term. The product of two terms yields a new term. If the total weight of the new term, given by adding up the weights of the two terms in the product, is greater than a specified value, for example 3, then the new term is deleted. Otherwise, the coefficient of the new term is the product of coefficients and the pattern of the new term is the concatenation of the patterns of the two terms in the product. Multiplying the first two terms in Y by the single term in X, the total weight of each resulting new term is 4 which is greater than 3, so that these terms are deleted. Multiplying the last term in Y by the single term in X, the resulting new term has a total weight of 2, a coefficient of 0.5, and a pattern of 1, 1, 4, 5, 5, 8, 8. Assigning the number 6 to the series name Z, this new term which is the product of X and Y is stored according to the information in line 6 of Table IV.

The method outlined can be programmed easily in Fortran IV and is appealing because of its simplicity. We have discussed only mulitplication of series. Clearly, considering each entry in a matrix to be a series, algebraic multiplication of matrices is accomplished by multiplying and adding series. In addition it is not hard to see that differentiation is simply a matter of looking for the occurrence of particular symbols in the pattern of each term. In future work, we shall present detailed documentation of a computer program which performs algebraic manipulation and we shall use the program to obtain explicit expressions for Lagrange's equations.

	Symbol	Weight
1	$\dot{\beta}$	1
2	J_{Bxz}	1
3	J_{Byz}	2
4	J_{Bzz}	0
5	$s\theta$	0
6	$c\theta$	0
7	$s\alpha$	1
8	$c\alpha$	0
20	-	-

Table I - Symbol Table

		Coefficient of Term	Weight of Term
	1	-	-
	2	-	-
X	3	-0.5	1
	4	-	-
	5	-	-
	6	-	-
	7	-	-
Y	8	-1.0	3
	9	-1.0	3
	10	-1.0	1
	11	-	-
	12	-	-
	13	-	-
	14	0.5	2
	15	-	-
	250	-	-

Table II - Coefficients and Weights Free Storage Stack

	1	-	Y	12	1		23	5		34	5
	2	-		13	2		24	8		35	5
	3	-		14	7		25	-1		36	8
	4	-		15	0		26	-		37	8
X	5	1		16	1		27	-		38	-1
	6	5		17	3		28	-		39	-
	7	8		18	6		29	-		40	-
	8	-1		19	8		30	-		41	-
	9	-		20	0	Z	31	1		42	-
	10	-		21	1		32	1		43	-
	11	-		22	4		33	4			
										2000	-

0 denotes the end of a term

-1 denotes the end of a series

Table III - Pattern Free Storage Stack

		Location of Coefficients	Location of Patterns	Number of Terms
	1	-	-	-
X = 2	2	3	5	1
	3	-	-	-
	4	-	-	-
Y = 5	5	8	12	3
Z = 6	6	14	31	1
	50	-	-	-

Table IV - Series Definition Table

8. Summary and Future Plans

This report presents a formulation of the equations of motion of a helicopter. The method of approach is a variation of the component-mode synthesis in the sense that it regards the aircraft as an assemblage of interconnected substructures. The substructures identified are the airframe, the transmission shaft, the tail rotor, and the main rotor. The rotor blades are assumed to be articulated with the flap-lag-pitch configuration. The equations of motion are derived in general form by means of the Lagrangian formulation in conjunction with an orderly kinematical procedure that takes into account the superposition of motion of various substructures, thus circumventing constraint problems.

Because of the complexity of the problem, the derivation of explicit equations of motion is sure to be extremely tedious and time consuming. Moreover, the probability of error in deriving the equations is large indeed. Fortunately, a number of assumptions can be made to simplify these equations. In particular, one can ignore certain higher-order quantities in the equations for the rotor blades. However, this task is also sure to be tedious and time consuming. Hence, a procedure for the automation of the derivation of the equations of motion is unavoidable if time and effort are to be minimized. Such a procedure consists of a computer program capable of performing the many matrix multiplications involved, certain differentiations, and elimination of higher-order terms. This latter task can be made easier by adopting an ordering scheme. In a computer manipulation the ordering scheme can be altered, thus producing sets of equations corresponding to different sets of assumptions.

The next phase of the investigation is concerned with the derivation of the equations of motion in explicit form. To this end, a method for the derivation of the perturbation equations (which are generally nonlinear) by means of computer manipulation will be developed along the lines of Sec. 7.

Implicit in the derivation of the explicit equations of motion is the truncation problem. Truncation will be done first on the substructure level and then on the complete aircraft level. An important question is that of the airframe modes and how they can be used to evaluate certain "mass integrals." In this regard, it may prove advantageous to look into the possibility of using "admissible vectors" to represent the motion of the airframe instead of using actual airframe modes.

Another problem that needs to be answered is the effect of nonlinear terms in the perturbation equations for the blade motion. If such terms cannot be ignored, considerable difficulty is likely to be encountered in the determination of the dynamic characteristics of the aircraft. Intimately related is the question of the axial displacement of the helicopter blade, as the axial displacement introduces nonlinear terms. The question is whether one should treat this axial displacement as an independent distributed coordinate or attempt to express it in terms of the bending displacements. Of course, if the axial displacement can be expressed in terms of the bending displacements, then this fact in itself implies a reduction in the number of degrees of freedom of the simulation, as the differential equation for the axial displacement is eliminated for every blade. Under certain circumstances, it may be possible to eliminate also the differential equation for the torsional motion in a similar fashion.

It appears desirable, if at all possible, to derive a set of linear ordinary differential equations with constant coefficients for the system, as such a set leads to an eigenvalue problem likely to yield useful information concerning the helicopter dynamic characteristics, such as natural frequencies and natural modes of vibration. Then, it may be possible to treat nonlinear effects as perturbations on the linear case. If a set of linear differential equations with constant coefficients can be obtained for the system, then it must by necessity correspond to the case of hovering. In the absence of aerodynamic forces, the set of equations is bound to be of gyroscopic type. The solution of the eigenvalue problem for the gyroscopic system can be obtained by the method of Ref. 9. The possibility of using the natural modes of the gyroscopic system to truncate the overall problem will be explored.

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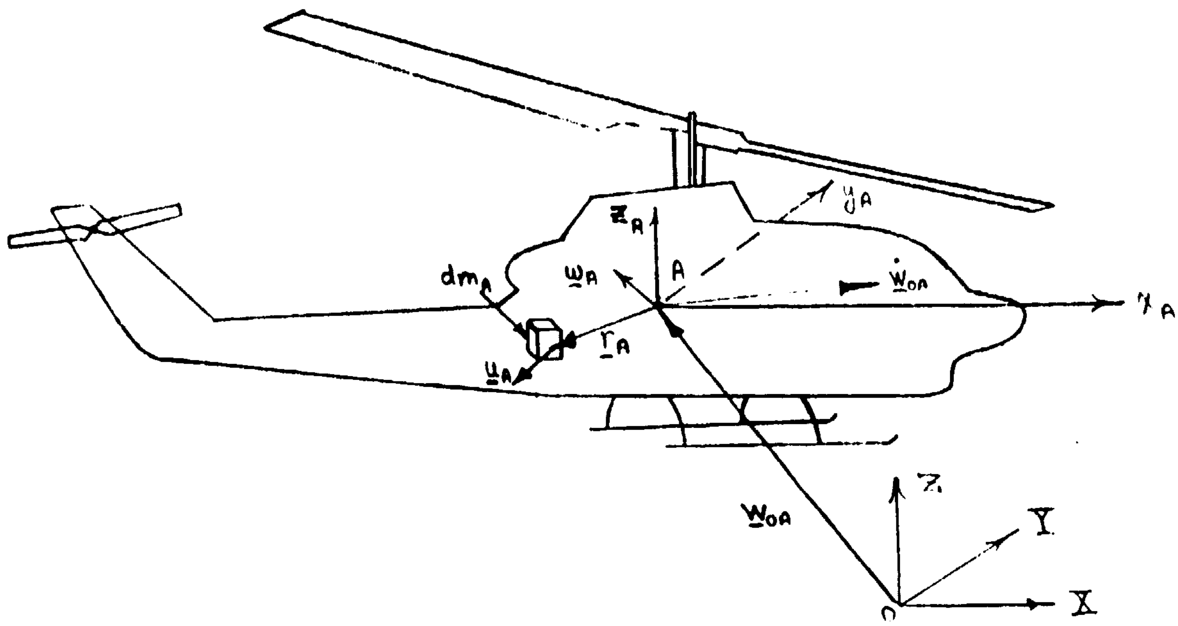

Figure 1. The Fuselage Generalized Coordinates

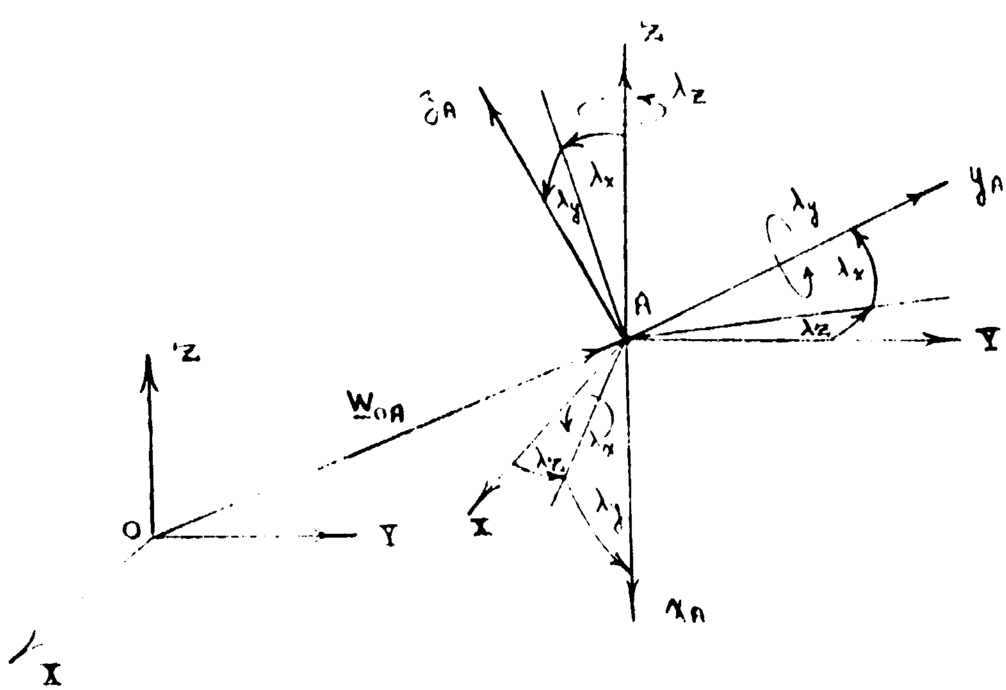

Figure 2. The Fuselage Coordinate System

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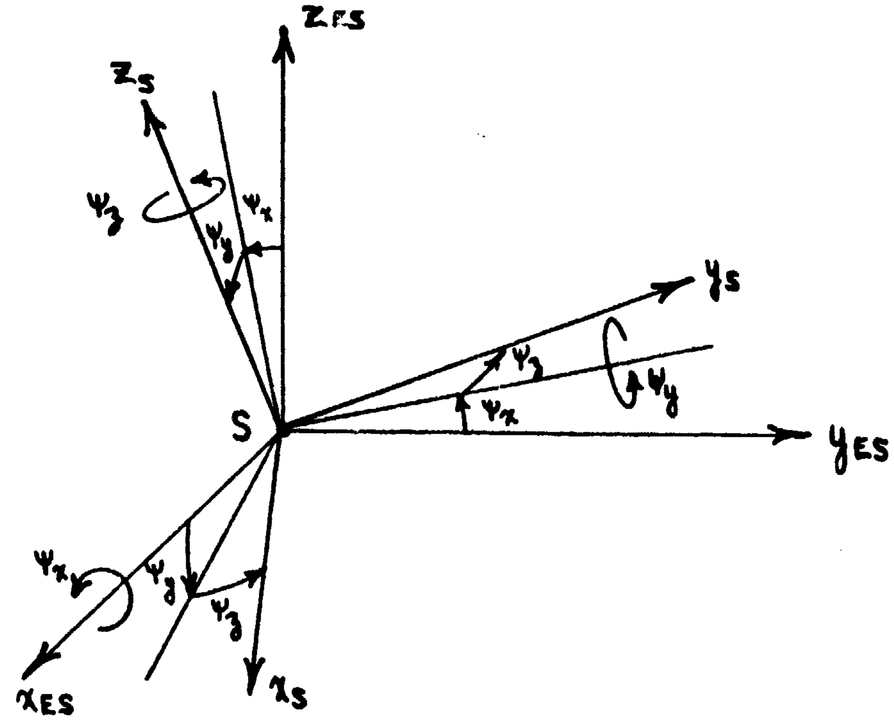

Figure 3. The Main Rotor Transmission Shaft Coordinate System

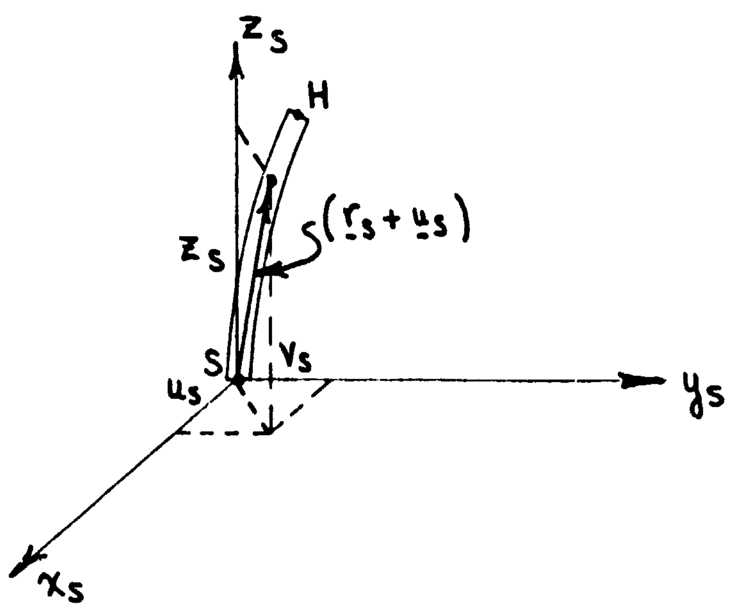

Figure 4. The Deformed Main Rotor Transmission Shaft

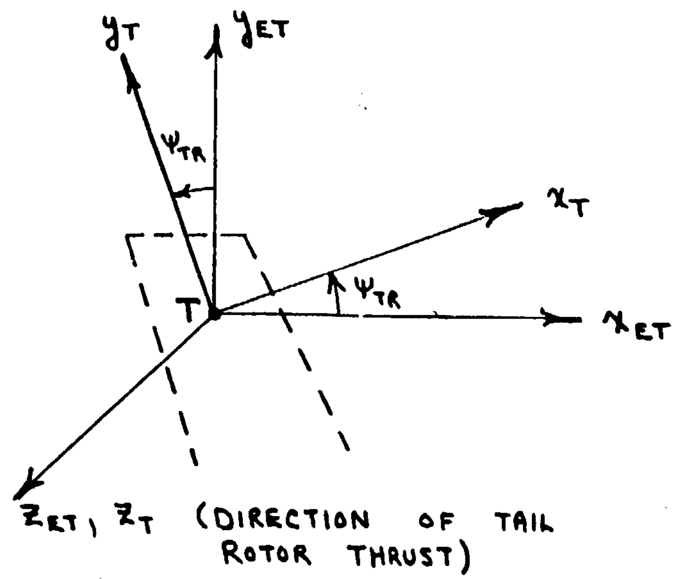

Figure 5. The Tail Rotor Coordinate System

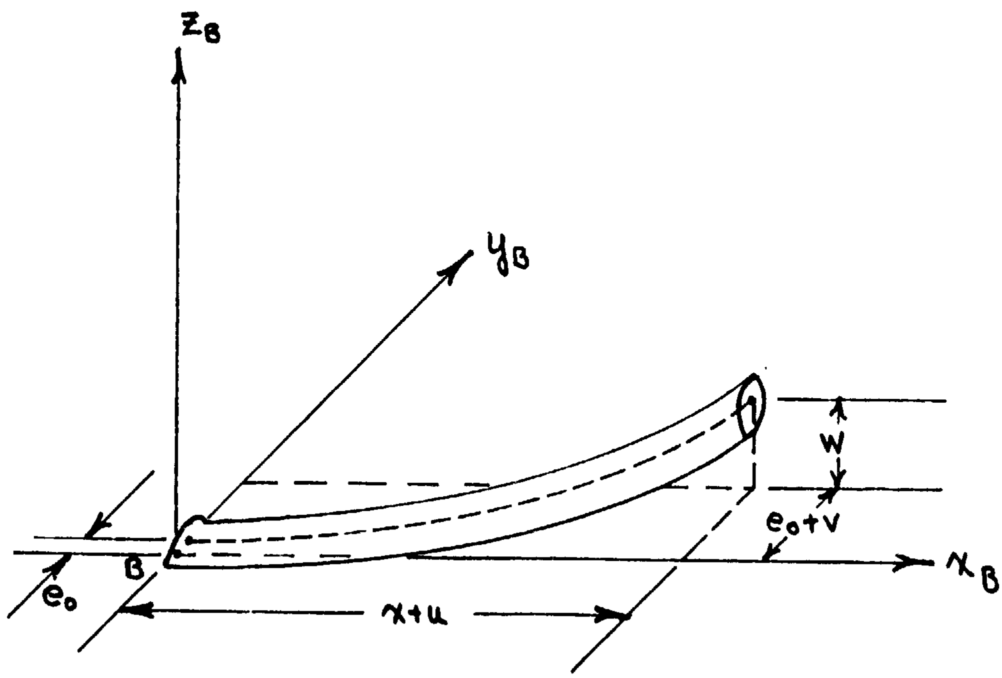

Figure 7. The Deformed Main Rotor Blade

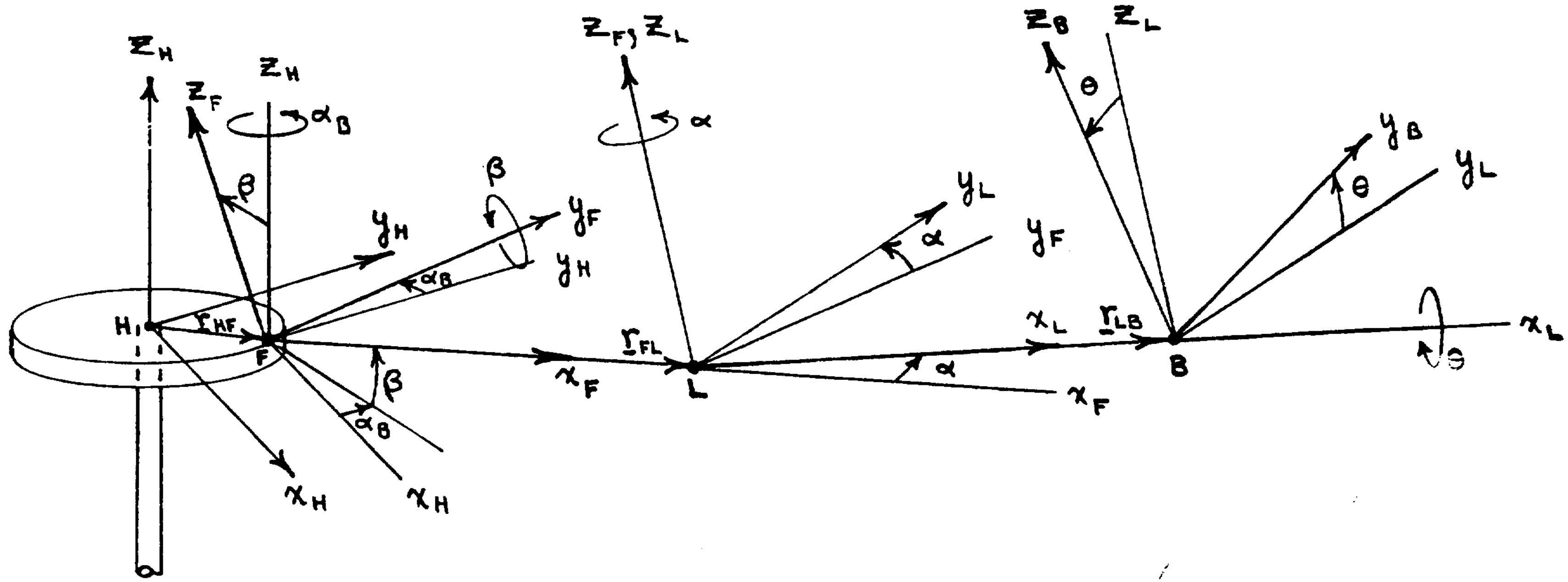

Figure 6a. The Main Rotor Hub and Flap Coordinate Systems

6b. The Lag Coordinate System

6c. The Undeformed Blade Coordinate System

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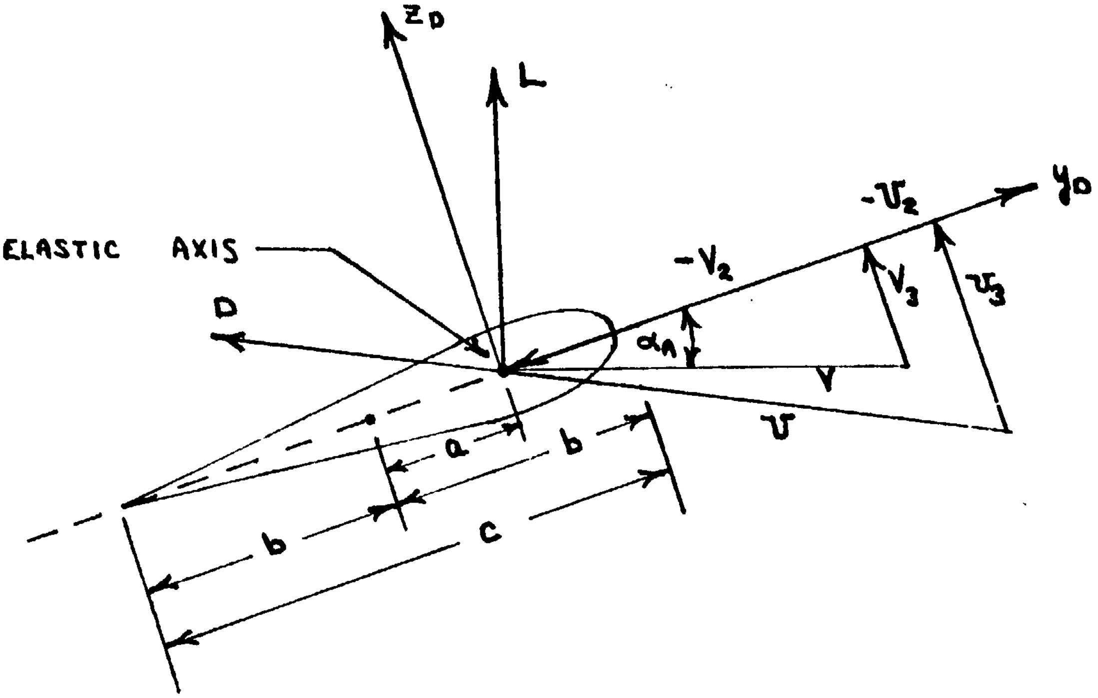

Figure 8. Relative Winds and Aerodynamic Loadings at a Typical Main Rotor Blade Element

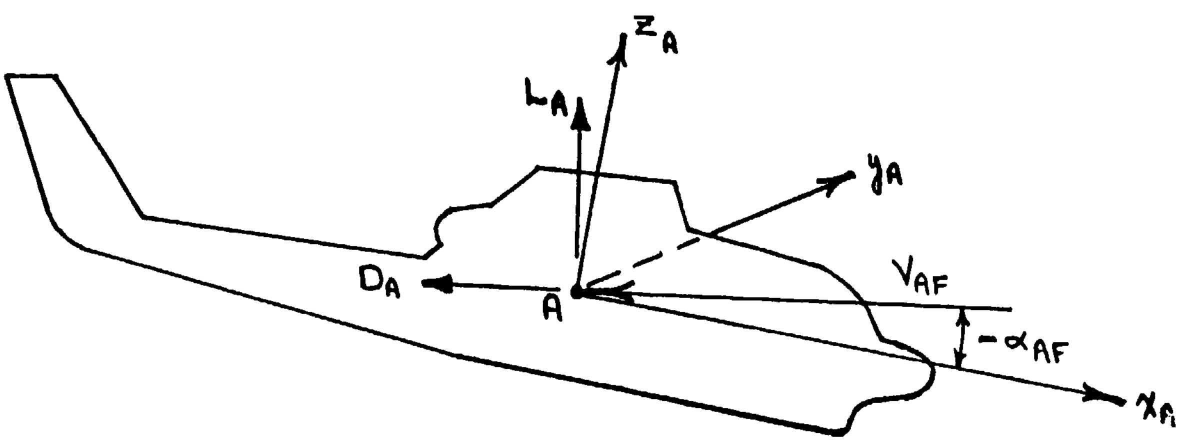

Figure 9. The Fuselage Aerodynamics

Appendix A

Contained herein is a summary of the key translational and angular velocity relationships developed in Sec. 2. Also included, when needed elsewhere in the text, are the full expansions of these expressions in terms of generalized coordinates, control variables, and elastic displacements. Important virtual displacement expressions are also listed. As mentioned in Sec. 2, it is understood that the subscript i on variables associated with a particular main rotor blade is deleted for clarity. Because many of the matrix products appear repeatedly, the following compact notation is developed:

$$[T_{DL}] = [T_{DB}][T_{BL}] \tag{A1}$$

$$[T_{DF}] = [T_{DB}][T_{BL}][T_{LF}] \quad , \text{etc.} \tag{A2}$$

The important angular velocity expressions in Sec. 2 are:

$$\{\Omega_A\} = \dot{\lambda}_z[\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[\lambda_y]\{e_1\} + \dot{\lambda}_y\{e_2\} \tag{A3}$$

$$\{\Omega_T\} = [T_{TA}]\{\Omega_A\} + [T_{TA}][\nabla]([\ell_{GT}]\{\dot{u}_{AT}\}) + \Omega_{TR}\{e_3\} \tag{A4}$$

$$\{\Omega_S\} = [T_{SA}]\{\Omega_A\} + [T_{SA}][\nabla]([\ell_{GS}]\{\dot{u}_{AS}\}) + \dot{\psi}_x[\psi_z][\psi_y]\{e_1\} + \dot{\psi}_y[\psi_z]\{e_2\} + \Omega\{e_3\} \tag{A5}$$

$$\{\Omega_H\} = [T_{HS}]\{\Omega_S\} + [T_{HS}]\begin{Bmatrix} -\dot{v}_S'\ (L_S,t) \\ \dot{u}_S'\ (L_S,t) \\ \dot{\phi}_S\ (L_S,t) \end{Bmatrix} \tag{A6}$$

$$\{\Omega_F\} = [T_{FH}]\{\Omega_H\} - \dot{\beta}\{e_2\} \tag{A7}$$

$$\{\Omega_L\} = [T_{LF}]\{\Omega_F\} + \dot{\alpha}\{e_3\} \tag{A8}$$

$$\{\Omega_B\} = [T_{BL}]\{\Omega_L\} + \dot{\theta}\{e_1\} \tag{A9}$$

$$\{\Omega_D\} = [T_{DB}]\{\Omega_B\} + \{\omega_D\} \tag{A10}$$

$$\{\omega_D\} = \begin{Bmatrix} w'\dot{v}' + \dot{\phi} \\ (\dot{v}' - \dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2)\sin\phi_0 - (\dot{w}' - \dot{v}'v'w' - \dot{w}'w'^2 - \frac{1}{2}\dot{w}'v'^2)\cos\phi_0 \\ (\dot{v}' - \dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2)\cos\phi_0 + (\dot{w}' - \dot{v}'v'w' - \dot{w}'w'^2 - \frac{1}{2}\dot{w}'v'^2)\sin\phi_0 \\ + \dot{v}'(\phi\cos\phi_0 - \frac{1}{2}\phi^2\sin\phi_0) + \dot{w}'(\phi\sin\phi_0 + \frac{1}{2}\phi^2\cos\phi_0) \\ - \dot{v}'(\phi\sin\phi_0 + \frac{1}{2}\phi^2\cos\phi_0) + \dot{w}'(\phi\cos\phi_0 - \frac{1}{2}\phi^2\sin\phi_0) \end{Bmatrix} \tag{A11}$$

Substitution of Eq. (A3) into Eq. (A4) yields a fully expanded expression for $\{\Omega_T\}$, substitution of Eq. (A3) into Eq. (A5) yields a fully expanded expression for $\{\Omega_S\}$, etc. Repeating Eq. (A3), this process yields the following equations:

$$\{\Omega_A\} = \dot{\lambda}_z[\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[\lambda_y]\{e_1\} + \dot{\lambda}_y\{e_2\} \tag{A12}$$

$$\begin{aligned}\{\Omega_T\} = {} & \dot{\lambda}_z[T_{TA}][\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[T_{TA}][\lambda_y]\{e_1\} + \dot{\lambda}_y[T_{TA}]\{e_2\} \\ & + [T_{TA}][\nabla]([\ell_{GT}]\{\dot{u}_{AT}\}) + \Omega_{TR}\{e_3\}\end{aligned} \tag{A13}$$

$$\begin{aligned}\{\Omega_S\} = {} & \dot{\lambda}_z[T_{SA}][\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[T_{SA}][\lambda_y]\{e_1\} + \dot{\lambda}_y[T_{SA}]\{e_2\} \\ & + [T_{SA}][\nabla]([\ell_{GS}]\{\dot{u}_{AS}\}) + \dot{\psi}_x[\psi_z][\psi_y]\{e_1\} + \dot{\psi}_y[\psi_z]\{e_z\} + \Omega\{e_3\}\end{aligned} \tag{A14}$$

$$\begin{aligned}\{\Omega_H\} = {} & \dot{\lambda}_z[T_{HA}][\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[T_{HA}][\lambda_y]\{e_1\} + \dot{\lambda}_y[T_{HA}]\{e_2\} \\ & + [T_{HA}][\nabla][\ell_{GS}]\{\dot{u}_{AS}\}) + \dot{\psi}_x[T_{HS}][\psi_z][\psi_y]\{e_1\} + \dot{\psi}_y[T_{HS}][\psi_z]\{e_2\} \\ & + \Omega[T_{HS}]\{e_3\} + [T_{HS}]\begin{Bmatrix} -\dot{v}_S' \ (L_S,t) \\ \dot{u}_S' \ (L_S,t) \\ \dot{\phi}_S \ (L_S,t) \end{Bmatrix}\end{aligned} \tag{A15}$$

$$\{\Omega_B\} = \dot{\lambda}_z[T_{BA}][\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[T_{BA}][\lambda_y]\{e_1\} + \dot{\lambda}_y[T_{BA}]\{e_2\}$$

$$+ [T_{BA}][\nabla]([\ell_{GS}]\{\dot{u}_{AS}\}) + \dot{\psi}_x[T_{BS}][\psi_z][\psi_y]\{e_1\} + \dot{\psi}_y[T_{BS}][\psi_z]\{e_2\}$$

$$+ \Omega[T_{BS}]\{e_3\} + [T_{BS}]\begin{Bmatrix} -\dot{v}_S' \ (L_S,t) \\ \dot{u}_S' \ (L_S,t) \\ \dot{\phi}_S \ (L_S,t) \end{Bmatrix} - \dot{\beta}[T_{BF}]\{e_2\} + \dot{\alpha}[T_{BL}]\{e_3\}$$

$$+ \dot{\theta}\{e_1\} \tag{A16}$$

$$\{\Omega_D\} = \dot{\lambda}_z[T_{DA}][\lambda_y][\lambda_x]\{e_3\} + \dot{\lambda}_x[T_{DA}][\lambda_y]\{e_1\} + \dot{\lambda}_y[T_{DA}]\{e_2\}$$

$$+ [T_{DA}][\nabla]([\ell_{GS}]\{\dot{u}_{AS}\}) + \dot{\psi}_x[T_{DS}][\psi_z][\psi_y]\{e_1\} + \dot{\psi}_y[T_{DS}][\psi_z]\{e_2\}$$

$$+ \Omega[T_{DS}]\{e_3\} + [T_{DS}]\begin{Bmatrix} -\dot{v}_S' \ (L_S,t) \\ \dot{u}_S' \ (L_S,t) \\ \dot{\phi}_S \ (L_S,t) \end{Bmatrix} - \dot{\beta}[T_{DF}]\{e_2\} + \dot{\alpha}[T_{DL}]\{e_3\}$$

$$+ \dot{\theta}[T_{DB}]\{e_1\} + \begin{Bmatrix} (\dot{v}' - \dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2)\sin\phi_0 - (\dot{w}' - \dot{v}'v'w' - \dot{w}'w'^2 - \frac{1}{2}\dot{w}'v'^2)\cos\phi_0 + v'(\phi\cos\phi_0 - \frac{1}{2}\phi^2\sin\phi_0) + \dot{w}'(\phi\sin\phi_0 + \frac{1}{2}\phi^2\cos\phi_0) \\ (\dot{v}' - \dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2)\cos\phi_0 - (\dot{w}' - \dot{v}'v'w' - \dot{w}'w'^2 - \frac{1}{2}\dot{w}'v'^2)\sin\phi_0 - v'(\phi\sin\phi_0 + \frac{1}{2}\phi^2\cos\phi_0) + \dot{w}'(\phi\cos\phi_0 - \frac{1}{2}\phi^2\sin\phi_0) \\ w'\dot{v}' + \dot{\phi} \end{Bmatrix} \tag{A17}$$

The important translational velocity expressions in Sec. 2 are:

$$\{\dot{w}_{OA}\} = [\dot{w}_{OAX} \quad \dot{w}_{OAY} \quad \dot{w}_{OAZ}]^T \tag{A18}$$

$$\{\dot{w}_A\} = [T_{AO}]\{\dot{w}_{OA}\} - [r_A + u_A]\{\Omega_A\} + \{\dot{u}_A\} \tag{A19}$$

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$$\{\dot{w}_{AT}\} = [T_{AO}]\{\dot{w}_{OA}\} - [r_{AT} + u_{AT}]\{\Omega_A\} + \{\dot{u}_{AT}\} \tag{A20}$$

$$\{\dot{w}_{AS}\} = [T_{AO}]\{\dot{w}_{OA}\} - [r_{AS} + u_{AS}]\{\Omega_A\} + \{\dot{u}_{AS}\} \tag{A21}$$

$$\{\dot{w}_{S}\} = [T_{SA}]\{\dot{w}_{AS}\} - [r_{S} + u_{S}]\{\Omega_S\} + \{\dot{u}_{S}\} \tag{A22}$$

$$\{\dot{w}_{SH}\} = [T_{SA}]\{\dot{w}_{AS}\} - [r_{SH} + u_{SH}]\{\Omega_S\} + \{\dot{u}_{SH}\} \tag{A23}$$

$$\{\dot{w}_{HF}\} = [T_{HS}]\{\dot{w}_{SH}\} - [r_{HF}]\{\Omega_H\} \tag{A24}$$

$$\{\dot{w}_{FL}\} = [T_{FH}]\{\dot{w}_{HF}\} - [r_{FL}]\{\Omega_F\} \tag{A25}$$

$$\{\dot{w}_{LB}\} = [T_{LF}]\{\dot{w}_{FL}\} - [r_{LB}]\{\Omega_L\} \tag{A26}$$

$$\{\dot{w}_{B}\} = [T_{BL}]\{\dot{w}_{LB}\} - [r_{B} + u_{B}]\{\Omega_B\} + \{\dot{u}_{B}\} \tag{A27}$$

Substitution of (A18) into (A21) yields a fully expanded expression for $\{\dot{w}_{AS}\}$, substitution of the results into Eq. (A23) yields a fully expanded expression for $\{\dot{w}_{SH}\}$, etc. Note that the use of Eqs. (A3)-(A9) is necessary. Repeating Eq. (18), this process gives the following selected equations:

$$\{\dot{w}_{OA}\} = [\dot{w}_{OAX} \quad \dot{w}_{OAY} \quad \dot{w}_{OAZ}]^T \tag{A28}$$

$$\begin{aligned}\{\dot{w}_{AT}\} = {} & [T_{AO}][\dot{w}_{OAX} \quad \dot{w}_{OAY} \quad \dot{w}_{OAZ}]^T - \dot{\lambda}_z[r_{AT} + u_{AT}][\lambda_y][\lambda_x]\{e_3\} \\ & - \dot{\lambda}_x[r_{AT} + u_{AT}][\lambda_y]\{e_1\} - \dot{\lambda}_y[r_{AT} + u_{AT}]\{e_2\} + \{\dot{u}_{AT}\}\end{aligned} \tag{A29}$$

$$\begin{aligned}\{\dot{w}_{AS}\} = {} & [T_{AO}][\dot{w}_{OAX} \quad \dot{w}_{OAY} \quad \dot{w}_{OAZ}]^T - \dot{\lambda}_z[r_{AS} + u_{AS}][\lambda_y][\lambda_x]\{e_3\} \\ & - \dot{\lambda}_x[r_{AS} + u_{AS}][\lambda_y]\{e_1\} - \dot{\lambda}_y[r_{AS} + u_{AS}]\{e_2\} + \{\dot{u}_{AS}\}\end{aligned} \tag{A30}$$

$$\begin{aligned}\{\dot{w}_{SH}\} = {} & [T_{SO}][\dot{w}_{OAX} \quad \dot{w}_{OAY} \quad \dot{w}_{OAZ}]^T - \dot{\lambda}_z([T_{SA}][r_{AS} + u_{AS}][\lambda_y][\lambda_x] \\ & + [r_{SH} + u_{SH}][T_{SA}][\lambda_y][\lambda_x])\{e_3\} - \dot{\lambda}_x([T_{SA}][r_{AS} + u_{AS}][\lambda_y] \\ & + [r_{SH} + u_{SH}][T_{SA}][\lambda_y])\{e_1\} - \dot{\lambda}_y([T_{SA}][r_{AS} + u_{AS}]\end{aligned}$$

$$+ [r_{SH} + u_{SH}][T_{SA}])\{e_2\} + [T_{SA}]\{\dot{u}_{AS}\} - [r_{SH} + u_{SH}][T_{SA}][\nabla]([\ell_{GS}]\{u_{AS}\})$$

$$- \dot{\psi}_x[r_{SH} + u_{SH}][\psi_z][\psi_y]\{e_1\} - \dot{\psi}_y[r_{SH} + u_{SH}][\psi_z]\{e_2\}$$

$$- \Omega[r_{SH} + u_{SH}]\{e_3\} + \{\dot{u}_{SH}\} \qquad \text{(A31)}$$

$$\{\dot{w}_{LB}\} = [T_{LO}][\dot{w}_{OAX} \quad \dot{w}_{OAY} \quad \dot{w}_{OAZ}]^T - \dot{\lambda}_z([T_{LA}][r_{AS} + u_{AS}][\lambda_y][\lambda_x]$$

$$+ [T_{LS}][r_{SH} + u_{SH}][T_{SA}][\lambda_y][\lambda_x] + [T_{LH}][r_{HF}][T_{HA}][\lambda_y][\lambda_x]$$

$$+ [T_{LF}][r_{FL}][T_{FA}][\lambda_y][\lambda_x] + [r_{LB}][T_{LA}][\lambda_y][\lambda_x])\{e_3\}$$

$$- \dot{\lambda}_x([T_{LA}][r_{AS} + u_{AS}][\lambda_y] + [T_{LS}][r_{SH} + u_{SH}][T_{SA}][\lambda_y]$$

$$+ [T_{LH}][r_{HF}][T_{HA}][\lambda_y] + [T_{LF}][r_{FL}][T_{FA}][\lambda_y] + [r_{LB}][T_{LA}][\lambda_y])\{e_1\}$$

$$- \dot{\lambda}_y([T_{LA}][r_{AS} + u_{AS}] + [T_{LS}][r_{SH} + u_{SH}][T_{SA}] + [T_{LH}][r_{HF}][T_{HA}]$$

$$+ [T_{LF}][r_{FL}][T_{FA}] + [r_{LB}][T_{LA}])\{e_2\} + [T_{LA}]\{\dot{u}_{AS}\}$$

$$- ([T_{LS}][r_{SH} + u_{SH}][T_{SA}] + [T_{LH}][r_{HF}][T_{HA}] + [T_{LF}][r_{FL}][T_{FA}]$$

$$+ [r_{LB}][T_{LA}])[\nabla]([\ell_{GS}]\{\dot{u}_{AS}\}) - \dot{\psi}_x([T_{LS}][r_{SH} + u_{SH}][\psi_z][\psi_y]$$

$$+ [T_{LH}][r_{HF}][T_{HS}][\psi_z][\psi_y] + [T_{LF}][r_{FL}][T_{FS}][\psi_z][\psi_y]$$

$$+ [r_{LB}][T_{LS}][\psi_z][\psi_y])\{e_1\} - \dot{\psi}_y([T_{LS}][r_{SH} + u_{SH}][\psi_z]$$

$$+ [T_{LH}][r_{HF}][T_{HS}][\psi_z] + [T_{LF}][r_{FL}][T_{FS}][\psi_z] + [r_{LB}][T_{LS}][\psi_z])\{e_2\}$$

$$- \Omega([T_{LS}][r_{SH} + u_{SH}] + [T_{LH}][r_{HF}][T_{HS}] + [T_{LF}][r_{FL}][T_{FS}]$$

$$+ [r_{LB}][T_{LS}][\psi_z])\{e_3\} - ([T_{LH}][r_{HF}][T_{HA}] + [T_{LF}][r_{FL}][T_{FA}]$$

$$+ [r_{LB}][T_{LA}]) \begin{Bmatrix} -\dot{v}_S' \ (L_S, t) \\ \dot{u}_S' \ (L_S, t) \\ \dot{\phi}_S \ (L_S, t) \end{Bmatrix} + [T_{LS}]\{\dot{u}_{SH}\} + \dot{\beta}([T_{LF}][r_{FL}]$$

$$+ [r_{LB}][T_{LF}])\{e_2\} - \dot{\alpha}[r_{LB}]\{e_3\} \qquad \text{(A32)}$$

From Eqs. (A27), (A32), and the fact that $\{\dot{w}_B'\} = [T_{DB}]\{\dot{w}_B\}$, we can write the following fully expanded expression for the inertial translational velocity of an arbitrary point on a main rotor blade in terms of components along $x_D y_D z_D$ axes:

$$
\begin{aligned}
\{\dot{w}_B'\} = {} & [T_{DO}][\dot{w}_{OAX} \quad \dot{w}_{OAY} \quad \dot{w}_{OAZ}]^T - \dot{\lambda}_z([T_{DA}][r_{AS} + u_{AS}][\lambda_y][\lambda_x] \\
& + [T_{DS}][r_{SH} + u_{SH}][T_{SA}][\lambda_y][\lambda_x] + [T_{DH}][r_{HF}][T_{HA}][\lambda_y][\lambda_x] \\
& + [T_{DF}][r_{FL}][T_{FA}][\lambda_y][\lambda_x] + [T_{DL}][r_{LB}][T_{LA}][\lambda_y][\lambda_x] \\
& + [T_{DB}][r_B + u_B][T_{BA}][\lambda_y][\lambda_x])\{e_3\} - \dot{\lambda}_x([T_{DA}][r_{AS} + u_{AS}][\lambda_y] \\
& + [T_{DS}][r_{SH} + u_{SH}][T_{SA}][\lambda_y] + [T_{DH}][r_{HF}][T_{HA}][\lambda_y] \\
& + [T_{DF}][r_{FL}][T_{FA}][\lambda_y] + [T_{DL}][r_{LB}][T_{LA}][\lambda_y] \\
& + [T_{DB}][r_B + u_B][T_{BA}][\lambda_y])\{e_1\} - \dot{\lambda}_y([T_{DA}][r_{AS} + u_{AS}] \\
& + [T_{DS}][r_{SH} + u_{SH}][T_{SA}] + [T_{DH}][r_{HF}][T_{HA}] + [T_{DF}][r_{FL}][T_{FA}] \\
& + [T_{DL}][r_{LB}][T_{LA}] + [T_{DB}][r_B + u_B][T_{BA}])\{e_2\} + [T_{DA}]\{\dot{u}_{AS}\} \\
& - ([T_{DS}][r_{SH} + u_{SH}][T_{SA}] + [T_{DH}][r_{HF}][T_{HA}] + [T_{DF}][r_{FL}][T_{FA}] \\
& + [T_{DL}][r_{LB}][T_{LA}] + [T_{DB}][r_B + u_B][T_{BA}])[\nabla]([\ell_{GS}]\{\dot{u}_{AS}\}) \\
& - \dot{\psi}_x([T_{DS}][r_{SH} + u_{SH}][\psi_z][\psi_y] + [T_{DH}][r_{HF}][T_{HS}][\psi_z][\psi_y] \\
& + [T_{DF}][r_{FL}][T_{FS}][\psi_z][\psi_y] + [T_{DL}][r_{LB}][T_{LS}][\psi_z][\psi_y] \\
& + [T_{DB}][r_B + u_B][T_{BS}][\psi_z][\psi_y])\{e_1\} - \dot{\psi}_y([T_{DS}][r_{SH} + u_{SH}][\psi_z] \\
& + [T_{DH}][r_{HF}][T_{HS}][\psi_z] + [T_{DF}][r_{FL}][T_{FS}][\psi_z] + [T_{DL}][r_{LB}][T_{LS}][\psi_z] \\
& + [T_{DB}][r_B + u_B][T_{BS}][\psi_z])\{e_2\} - \Omega([T_{DS}][r_{SH} + u_{SH}] + [T_{DH}][r_{HF}][T_{HS}] \\
& + [T_{DF}][r_{FL}][T_{FS}] + [T_{DL}][r_{LB}][T_{LS}] + [T_{DB}][r_B + u_B][T_{BS}])\{e_3\} \\
& - ([T_{DH}][r_{HF}][T_{HS}] + [T_{DF}][r_{FL}][T_{FS}] + [T_{DL}][r_{LB}][T_{LS}] \\
& + [T_{DB}][r_B + u_B][T_{BS}])[-\dot{v}_S'(L_S,t) \quad \dot{u}_S'(L_S,t) \quad \dot{\phi}_S(L_S,t)]^T \\
& + [T_{DS}]\{\dot{u}_{SH}\} + \dot{\beta}([T_{DF}][r_{FL}] + [T_{DL}][r_{LB}][T_{LF}]
\end{aligned}
$$

$$+ [T_{DB}][r_B + u_B][T_{BF}])\{e_2\} - \dot{\alpha}([T_{DL}][r_{LB}] + [T_{DB}][r_B + u_B][T_{BL}])\{e_3\}$$

$$- \dot{\theta}([T_{DB}][r_B + u_B])\{e_1\} + [T_{DB}]\{\dot{u}_B\} \qquad \text{(A33)}$$

The virtual displacement vector associated with Eq. (A17) is

$$\{\delta\Theta_D\} = \delta\lambda_z[T_{DA}][\lambda_y][\lambda_x]\{e_3\} + \delta\lambda_x[T_{DA}][\lambda_y]\{e_1\} + \delta\lambda_y[T_{DA}]\{e_2\}$$

$$+ [T_{DA}][\nabla]([\ell_{GS}]\{\delta u_{AS}\}) + \delta\psi_x[T_{DS}][\psi_z][\psi_y]\{e_1\} + \delta\psi_y[T_{DS}][\psi_z]\{e_2\}$$

$$+ [T_{DS}]\begin{Bmatrix} -\delta v_S'\ (L_S,t) \\ \delta u_S'\ (L_S,t) \\ \delta\phi_S\ (L_S,t) \end{Bmatrix} - \delta\beta[T_{DF}]\{e_2\} + \delta\alpha[T_{DL}]\{e_3\} + \delta\theta_{cp}[T_{DB}]\{e_1\}$$

$$+ \delta\phi\{e_1\} + \delta v' \begin{Bmatrix} w' \\ (1 - v'^2 - \frac{1}{2} w'^2)\sin\phi_0 + v'w'\cos\phi_0 \\ (1 - v'^2 - \frac{1}{2} w'^2)\cos\phi_0 + v'w'\sin\phi_0 \end{Bmatrix}$$

$$+ \delta w' \begin{Bmatrix} 0 \\ (-1 + w'^2 + \frac{1}{2} v'^2)\cos\phi_0 + \phi\sin\phi_0 + \frac{1}{2}\phi^2\cos\phi_0 \\ (-1 + w'^2 + \frac{1}{2} v'^2)\sin\phi_0 + \phi\cos\phi_0 - \frac{1}{2}\phi^2\sin\phi_0 \end{Bmatrix} \qquad \text{(A34)}$$

Discretization of the continuous variables in Eq. (A34) via the methods of Sec. 3 yields

$$\{\delta\Theta_D\} = \delta\lambda_z[T_{DA}][\lambda_y][\lambda_x]\{e_3\} + \delta\lambda_x[T_{DA}][\lambda_y]\{e_1\} + \delta\lambda_y[T_{DA}]\{e_2\}$$

$$+ [T_{DA}][\nabla]([\ell_{GS}][\Phi_{AS}])\{\delta\eta_A\} + \delta\psi_x[T_{DS}][\psi_z][\psi_y]\{e_1\} + \delta\psi_y[T_{DS}][\psi_z]\{e_2\}$$

$$+ [T_{DS}]\begin{Bmatrix} -\sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{vi} \\ \sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{ui} \\ \sum_{i=1}^{S_\phi} \Theta_{Si}(z_S)\delta\eta_{\phi i} \end{Bmatrix} - \delta\beta[T_{DF}]\{e_2\}$$

$$+ \delta\alpha[T_{DL}]\{e_3\} + \delta\theta_{cp}[T_{DB}]\{e_1\} + \sum_{i=1}^{N_\phi} \phi_{\phi i}(x)\delta q_{\phi i}\{e_1\}$$

$$+ \sum_{i=1}^{N_v} \phi'_{vi}(x)\delta q_{vi} \begin{Bmatrix} w' \\ (1 - v'^2 - \frac{1}{2} w'^2)\sin \phi_0 + v'w'\cos \phi_0 \\ (1 - v'^2 - \frac{1}{2} w'^2)\cos \phi_0 + v'w'\sin \phi_0 \end{Bmatrix}$$

$$+ \sum_{i=1}^{N_w} \phi'_{wi}(x)\delta q_{wi} \begin{Bmatrix} 0 \\ (-1 + w'^2 + \frac{1}{2} v'^2)\cos \phi_0 + \phi\sin \phi_0 + \frac{1}{2}\phi^2 \cos \phi_0 \\ (-1 + w'^2 + \frac{1}{2} v'^2)\sin \phi_0 + \phi\cos \phi_0 - \frac{1}{2}\phi^2 \sin \phi_0 \end{Bmatrix} \quad \text{(A35)}$$

where $[\Phi_{AS}]$ is the matrix $[\Phi_A]$ evaluated at the point S of the airframe.

The first component $\delta\Theta_{Dx}$ of $\{\delta\Theta_D\}$ is needed in the main rotor virtual work expression of Sec. 4. It can be written from Eq. (A35) as

$$\begin{aligned} \delta\Theta_{Dx} = {} & A_{\lambda z1}\delta\lambda_z + A_{\lambda x1}\delta\lambda_x + T_{DA12}\delta\lambda_y + \sum_{j=1}^{P} a_{ij}\delta\eta_{Aj} + A_{\psi x1}\delta\psi_x \\ & + A_{\psi y1}\delta\psi_y - T_{DS11} \sum_{i=1}^{S_u} \psi'_{Si}(z_S)\delta\eta_{vi} + T_{DS12} \sum_{i=1}^{S_u} \psi'_{Si}(z_S)\delta\eta_{ui} \\ & + T_{DS13} \sum_{i=1}^{S_\phi} \Theta_{Si}(z_S)\delta\eta_{\phi i} - T_{DF12}\,\delta\beta + T_{DL13}\,\delta\alpha + T_{DB11}\,\delta\Theta_{cp} \\ & + \sum_{i=1}^{N_\phi} \phi_{\phi i}(x)\delta q_{\phi i} + w' \sum_{i=1}^{N_v} \phi'_{vi}(x)\delta q_{vi} \end{aligned} \quad \text{(A36)}$$

where $A_{\lambda z1}$ is the first component of the 3×1 matrix $[T_{DA}][\lambda_y][\lambda_x]\{e_3\}$, $A_{\lambda x1}$ is the first component of the 3×1 matrix $[T_{DA}][\lambda_y]\{e_1\}$, $A_{\psi x1}$ is the first component of the 3×1 matrix $[T_{DS}][\psi_z][\psi_y]\{e_1\}$, $A_{\psi y1}$ is the first component of the 3×1 matrix $[T_{DS}][\psi_z]\{e_2\}$, a_{ij} are the first row components of $[T_{DA}][\nabla]([\ell_{GS}][\Phi_{AS}])$, T_{DA12} is the first row, second column element of $[T_{DA}]$, T_{DS11} is the first row, first column element of $[T_{DS}]$, etc.

The virtual displacement vector associated with Eq. (A33) is

$$\{\delta w_B'\} = [T_{DO}][\delta w_{OAX} \quad \delta w_{OAY} \quad \delta w_{OAZ}]^T - \{T_{\lambda z}\}\delta\lambda_z - \{T_{\lambda x}\}\delta\lambda_x - \{T_{\lambda y}\}\delta\lambda_y$$

$$+ [T_{DA}]\{\delta u_{AS}\} - [T_{uAS}][\nabla]([\ell_{GS}]\{\delta u_{AS}\}) - \{T_{\psi x}\}\delta\psi_x - \{T_{\psi y}\}\delta\psi_y$$

$$- [T_{uS}]\begin{Bmatrix} -\delta v_S' \ (L_S,t) \\ \delta u_S' \ (L_S,t) \\ \delta\phi_S \ (L_S,t) \end{Bmatrix} + [T_{DS}]\{\delta u_{SH}\} + \{T_\beta\}\delta\beta - \{T_\alpha\}\delta\alpha$$

$$- \{T_{\theta cp}\}\delta\theta_{cp} + [T_{DB}]\{\delta u_B\} \qquad \text{(A37)}$$

where the matrices $\{T_{\lambda z}\}$, $\{T_{\lambda x}\}$, etc. are the respective coefficients of $-\dot{\lambda}_z$, $-\dot{\lambda}_x$, etc. in Eq. (A33).

Discretization of the continuous variables in Eq. (A37) via the methods of Sec. 3 yields

$$\{\delta w_B'\} = [T_{DO}][\delta w_{OAX} \quad \delta w_{OAY} \quad \delta w_{OAZ}]^T - \{T_{\lambda z}\}\delta\lambda_z - \{T_{\lambda x}\}\{\delta\lambda_x\} - [T_{\lambda y}]\delta\lambda_y$$

$$+ [T_{DA}][\Phi_{AS}]\{\delta\eta_A\} - [T_{uAS}][\nabla]([\ell_{GS}][\Phi_{AS}])\{\delta\eta_A\} - \{T_{\psi x}\}\delta\psi_x$$

$$- \{T_{\psi y}\}\delta\psi_y - [T_{uS}]\begin{Bmatrix} -\sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{vi} \\ \sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{ui} \\ \sum_{i=1}^{S_\phi} \Theta_{Si}(z_S)\delta\eta_{\phi i} \end{Bmatrix} + [T_{DS}][\Phi_{SH}]\{\delta\eta_S\}$$

$$+ \{T_\beta\}\delta\beta - \{T_\alpha\}\delta\alpha - \{T_{\theta cp}\}\delta\theta_{cp} + [T_{DB}]\begin{Bmatrix} \sum_{i=1}^{N_u} \phi_{ui}(x)\delta q_{ui} \\ \sum_{i=1}^{N_v} \phi_{vi}(x)\delta q_{vi} \\ \sum_{i=1}^{N_w} \phi_{wi}(x)\delta q_{wi} \end{Bmatrix} \qquad \text{(A38)}$$

where $[\Phi_{SH}]$ is the matrix $[\Phi_S]$ evaluated at point H.

In the last term of Eq. (A38), η and ζ have been set equal to zero in antipation of the aerodynamic virtual work requirements of Sec. 4.

The latter two components of $\{\delta w_B'\}$, δw_{Dy} and δw_{Dz}, are needed in the main rotor virtual work expression. These two quantities can be written from Eq. (A38) as

$$\begin{aligned}
\delta w_{Dy} = {} & T_{DO21}\delta w_{OAX} + T_{DO22}\delta w_{OAY} + T_{DO23}\delta w_{CAZ} - T_{\lambda z2}\delta\lambda_z - T_{\lambda x2}\delta\lambda_x - T_{\lambda y2}\delta\lambda_y \\
& + \sum_{j=1}^{P} b_{2j}\delta\eta_{Aj} - \sum_{j=1}^{P} c_{2j}\delta\eta_{Aj} - T_{\psi x2}\delta\psi_x - T_{\psi y2}\delta\psi_y + T_{uS21}\sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{vi} \\
& - T_{uS22}\sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{ui} - T_{uS23}\sum_{i=1}^{S_\phi} \Theta_{Si}(z_S)\delta\eta_{\phi i} + \sum_{j=1}^{S_u} d_{2j}\delta\eta_{Sj} \\
& + T_{\beta 2}\delta\beta - T_{\alpha 2}\delta\alpha - T_{\theta cp2}\delta\theta_{cp} + T_{DB21}\sum_{i=1}^{N_u} \phi_{ui}(x)\delta q_{ui} \\
& + T_{DB22}\sum_{i=1}^{N_v} \phi_{vi}(x)\delta q_{vi} + T_{DB23}\sum_{i=1}^{N_w} \phi_{wi}(x)\delta q_{wi}
\end{aligned} \tag{A39}$$

and

$$\begin{aligned}
\delta w_{Dz} = {} & T_{DO31}\delta w_{OAX} + T_{DO32}\delta w_{OAY} + T_{DO33}\delta w_{OAZ} - T_{\lambda z3}\delta\lambda_z - T_{\lambda x3}\delta\lambda_x \\
& - T_{\lambda y3}\delta\lambda_y + \sum_{j=1}^{P} b_{3j}\delta\eta_{Aj} - \sum_{j=1}^{P} c_{3j}\delta\eta_{Aj} - T_{\psi x3}\delta\psi_x - T_{\psi y3}\delta\psi_y \\
& + T_{uS31}\sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{vi} - T_{uS32}\sum_{i=1}^{S_u} \psi_{Si}'(z_S)\delta\eta_{ui} \\
& + T_{uS33}\sum_{i=1}^{S_\phi} \Theta_{Si}(z_S)\delta\eta_{\phi i} + \sum_{j=1}^{S_u} d_{3j}\delta\eta_{Sj} + T_{\beta 3}\delta\beta - T_{\alpha 3}\delta\alpha \\
& + T_{\theta cp3}\delta\theta_{cp} + T_{DB31}\sum_{i=1}^{N_u} \phi_{ui}(x)\delta q_{ui} + T_{DB32}\sum_{i=1}^{N_v} \phi_{vi}(x)\delta q_{vi}
\end{aligned}$$

$$+ T_{B33} \sum_{i=1}^{N_w} \phi_{wi}(x)\delta q_{wi} \tag{A40}$$

In Eqs. (A39) and (A40), the b_{ij} are elements of the matrix $[T_{DA}][\Phi_{AS}]$, the c_{ij} are elements of the matrix $[T_{uAS}][\nabla]([\ell_{GS}][\Phi_{AS}])$, and the d_{ij} are elements of the matrix $[T_{DS}][\Phi_{SH}]$.

Finally, the third component of virtual displacement vector associated with Eq. (A29) is

$$\delta w_{ATz} = T_{A031}\delta w_{OAX} + T_{A032}\delta w_{OAy} + T_{A033}\delta w_{OAz} - C_{\lambda z3}\delta\lambda_z - C_{\lambda x3}\delta\lambda_x$$

$$- C_{\lambda y3}\delta\lambda_y + \sum_{j=1}^{P} \Phi_{AT3j}\delta\eta_{Aj} \tag{A41}$$

where $C_{\lambda z3}$ is the third component of the 3 x 1 matrix $[r_{AT} + u_{AT}][\lambda_y][\lambda_x]\{e_3\}$, $C_{\lambda x3}$ is the third component of the 3 x 1 matrix $[r_{AT} + u_{AT}][\lambda_y]\{e_1\}$, $C_{\lambda y3}$ is the third component of the 3 x 1 matrix $[r_{AT} + u_{AT}]\{e_2\}$, and the Φ_{AT3j} are the third row elements of the matrix $[\Phi_{AT}]$.

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