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COMPUTATION OF SCALAR FAR-FIELD PATTERNS OF LARGE-APERTURE ANTENNAS

Thomas A. O'Malley

Lewis Research Center

Cleveland, Ohio 44135

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16. Abstract

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COMPUTATION OF SCALAR FAR-FIELD PATTERNS OF

LARGE-APERTURE ANTENNAS

by Thomas A. O'Malley

Lewis Research Center

SUMMARY

In computer programs used for evaluating the performance of high-gain antennas, efficient numerical methods for calculating the far-field patterns must be used since the majority of computer time and storage requirements may be attributed to this phase of the program. The numerical method most frequently used is the Fast Fourier Transform (FFT), which computes the far field as the Fourier transform of the field distribution in the antenna aperture. This report describes a new numerical method that in many applications is superior to the FFT in terms of reducing computer time and storage requirements.

In the new method, the arbitrarily specified field in the aperture is represented by a Fourier series expansion. The far field is obtained by evaluating a double integral where the integrand contains the aperture function. When the Fourier series expression is substituted into the integrand, the integral can be evaluated in closed form.

The new method requires less computer storage than the FFT method. Which of the two methods requires less computer time depends on the number and location of far-field points. If the application requires far-field points only in the principal planes of the pattern, the new method is generally faster. If the application requires a full rectangular array of far-field points, the FFT is generally faster. A significant advantage of the new method is the complete flexibility in choosing the number and location of far-field points. If the calculation of the field in the aperture is time consuming, the number of aperture grid points required becomes an important consideration. The number of aperture grid points required is approximately the same for the two methods.

INTRODUCTION

Reflector antennas and lens antennas whose dimensions are large compared to a wavelength are used extensively in applications requiring high gain. To determine the performance characteristics of these antennas, accurate calculation of the antenna pattern in the far field is required. One method of calculating the far-field pattern of such antennas is the aperture-field method (ref. 1). In this method, the field vectors $\overline{E}$ and $\overline{H}$ are specified in a planar aperture in the vicinity of the antenna. The far field can be obtained from vector diffraction equations involving integration over the aperture. These vector equations are complex and require considerable computational effort. In many antennas the field over the aperture is almost completely linearly polarized. Only a small fraction of the energy is in the cross-polarization component of the field. By neglecting the cross-polarization components, the problem can be reduced to a scalar diffraction problem, which requires considerably less computational effort.

Scalar diffraction theory can be used to show that the far field is the two-dimensional Fourier transform of the field distribution in the aperture of the antenna (ref. 1). In computer programs used for evaluating antenna performance, efficient numerical methods for calculating the two-dimensional Fourier transform must be used since the majority of computer time and storage requirements may be attributed to this phase of the program. The most widely used method for calculating the far field is the Fast Fourier Transform (FFT), which, as its name implies, is considerably faster than previously used methods (ref. 2).

This report presents a new numerical method for calculating from scalar diffraction theory the far-field patterns of high-gain antennas. The method evaluates the far-field pattern as the two-dimensional Fourier transform of an arbitrarily specified field distribution in the antenna aperture. It is shown to be superior to the Fast Fourier Transform in many applications.

The new numerical method is described in detail. A brief discussion of the Fast Fourier Transform follows. A comparison of the two numerical methods for three common aperture distributions is then given. Criteria for determining which of the two numerical methods is superior in a given application are developed.

DESCRIPTION OF NEW NUMERICAL METHOD

In this new numerical method, we assume that the antenna aperture lies in the x-y plane (fig. 1). In nearly all cases of interest, the aperture is either elliptical or circular. Let $\overline{R}$ be the vector from the center of the aperture to a far-field point. Let

α be the angle between the z-axis and the projection of $\overline{R}$ into the x-z plane, and let β be the angle between the z-axis and the projection of $\overline{R}$ into the y-z plane. Let the coordinates of a point in the aperture be x, y. The field over the aperture will be designated by F(x, y). A(x, y) will be the amplitude distribution, and φ(x, y) will be the phase distribution, so that

$$F(x, y) = A(x, y)e^{j\varphi(x, y)} \tag{1}$$

The far field will be designated by E(α, β). From reference 3, the far field is given by

$$E(\alpha, \beta) = \iint_A F(x, y)e^{j(2\pi/\lambda)(x \sin \alpha + y \sin \beta)} \, dx \, dy \tag{2}$$

where the integration is over the aperture A and λ is the wavelength. Let us define

$$k_x = \frac{-2\pi}{\lambda} \sin \alpha \tag{3}$$

$$k_y = \frac{-2\pi}{\lambda} \sin \beta \tag{4}$$

Equation (2) then becomes

$$E(k_x, k_y) = \iint_A F(x, y)e^{-j(k_x x + k_y y)} \, dx \, dy \tag{5}$$

Let us define the function u(x, y) as

$$u(x, y) = F(x, y) \quad \text{inside A} \tag{6}$$

$$u(x, y) = 0 \quad \text{outside A} \tag{7}$$

Then equation (5) becomes

$$E(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y)e^{-j(k_x x + k_y y)} \, dx \, dy \tag{8}$$

The far field expressed in this form is the two-dimensional Fourier transform of the aperture function.

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Fourier Series Representation of Aperture Illumination

In the numerical method developed herein, the aperture illumination function $F(x, y)$ is represented by a Fourier series expansion. When the Fourier series expression is substituted into equation (2), the integral for the far field can be evaluated in closed form.

The aperture is assumed to be elliptical in shape. The elliptical aperture includes the circular aperture as a special case. Let us consider an extension of the aperture illumination function to a rectangular region enclosing the elliptical region, as in figure 2. The enclosing rectangle will be called the Fourier rectangle. The semimajor and semiminor axes of the elliptical aperture are a and b, respectively. The semilength and semiwidth of the Fourier rectangle are c and d, respectively. On this rectangle the aperture illumination can be approximated by a Fourier series $S_M(x, y)$ of the form

$$F(x, y) \approx S_M(x, y) \equiv \sum_{r=-M}^{M} \sum_{s=-M}^{M} c_{rs} e^{j(r\pi x/c)} e^{j(s\pi y/d)} \tag{9}$$

The Fourier coefficients c_{rs} are given by

$$c_{rs} = \frac{1}{4cd} \int_{-c}^{c} \int_{-d}^{d} F(x, y) e^{-j(r\pi x/c)} e^{-j(s\pi y/d)} \, dy \, dx$$

$$-M \leq r \leq M; \ -M \leq s \leq M \tag{10}$$

We assume that the aperture illumination is specified on a set of grid points within the Fourier rectangle. We further assume that the grid points form a $(2L + 1)$-by-$(2L + 1)$ array of equally spaced points, where L is some integer. The grid points (x_i, y_j) are given by

$$x_i = -c + \frac{2ci}{2L + 1} \qquad i = 0, \ldots, 2L \tag{11}$$

$$y_j = -d + \frac{2dj}{2L + 1} \qquad j = 0, \ldots, 2L \tag{12}$$

To evaluate the Fourier coefficients in equation (10), we replace the double integral by the following double summation:

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$$c_{rs} = \left(\frac{1}{2L+1}\right)^2 \sum_{i=0}^{2L} \sum_{j=0}^{2L} F(x_i, y_j) e^{-j(r\pi x_i/c)} e^{-j(s\pi y_j/d)}$$

$$-M \le r \le M; \quad -M \le s \le M \qquad (13)$$

To calculate the c_{rs} in equation (13), one may, of course, evaluate the summations directly. Instead of doing this, we use a more efficient algorithm devised by Goetzel. For details of the algorithm, see reference 4.

The values of c and d and the method of extending the aperture illumination function outside the elliptical aperture are chosen so as to obtain a good Fourier series approximation within the elliptical aperture. For grid points outside the elliptical aperture, a good approximation is not required because the far-field pattern is obtained by evaluating an integral over the aperture only.

The Fourier series approximation $S_M(x, y)$ has the periodicity property that $S_M(c, y) = S_M(-c, y)$ for all y and $S_M(x, d) = S_M(x, -d)$ for all x. In general, the aperture illumination function does not have periodicity properties. Thus, if c = a and d = b, the Fourier series approximation may be poor in the vicinity of the points (a, 0), (-a, 0), (0, b), and (0, -b). As an example of this poor approximation, consider figure 3, which is a plot of the imaginary part of the aperture function and its Fourier series representation as a function of x, with y set equal to 0. The aperture function is

$$F(x, y) = \left\{ 0.3 + 0.7 \left[1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \right] \right\} e^{j(\pi/2)(x/a)^3} \qquad (14)$$

In figure 3, a = b = c = d = 1, and M = 4. It is clear from figure 3 that the approximation is poor at x = ±a. Figure 4 is the same plot but with c = 1.1 a and d = 1.1 b. The Fourier series approximation at x = ±a is significantly better. Furthermore, the approximation is significantly better throughout the entire interval. Numerical tests suggest that the ratio of c to a and the ratio of d to b should be between 1.05 and 1.10 to obtain good approximation throughout the entire aperture.

The method of extending the aperture function outside the aperture should be done such that the function has the same periodicity properties as its Fourier series representation; that is, F(c, y) = F(-c, y) for all y and F(x, d) = F(x, -d) for all x. The method used is illustrated in figure 5. Let y be fixed at $y = y_j$ and consider the row of grid points (x_i, y_j), i = 0, . . ., 2L + 1. For some value of i, let us say i = i1, the grid point (x_{i1}, y_j) lies just inside the aperture in the second quadrant. For some value of i, let us say i = i2, the grid point (x_{i2}, y_j) lies just inside the aperture in the

first quadrant. The extension of the function $F(x, y)$ is such that

$$F(x_0, y_j) = F(x_{2L+1}, y_j) = \frac{1}{2}\left[F(x_{i1}, y_j) + F(x_{i2}, y_j)\right] \tag{15}$$

The function $F(x, y_j)$ is extended by two linear segments: one extending from (x_0, y_j) to (x_{i1}, y_j), and the other extending from (x_{i2}, y_j) to (x_{2L+1}, y_j), as shown in figure 5. These extensions are performed for all such rows of grid points. The same procedure is then used to extend all columns of grid points.

Evaluation of Diffraction Integral

Substituting the Fourier series representation of $F(x, y)$ into equation (2), we obtain

$$E(\alpha, \beta) = \iint\limits_{A} \sum_{r=-M}^{M} \sum_{s=-M}^{M} c_{rs} e^{j(r\pi x/c)} e^{j(s\pi y/a)} e^{j(2\pi/\lambda)(x \sin \alpha + y \sin \beta)} \, dy \, dx \tag{16}$$

In the appendix, the following integral is evaluated:

$$\iint\limits_{A} e^{j(\gamma x + \delta y)} \, dy \, dx = \frac{2\pi ab J_1\left[\sqrt{(\gamma a)^2 + (\delta b)^2}\right]}{\sqrt{(\gamma a)^2 + (\delta b)^2}} \tag{17}$$

Combining equations (16) and (17), we obtain

$$E(\alpha, \beta) = 2\pi ab \sum_{r=-M}^{M} \sum_{s=-M}^{M} c_{rs} \frac{J_1(z_{rs})}{z_{rs}} \tag{18}$$

where z_{rs}^2 is given by

$$z_{rs}^2 = \left(\frac{r\pi a}{c} + \frac{2\pi a \sin \alpha}{\lambda}\right)^2 + \left(\frac{s\pi b}{d} + \frac{2\pi b \sin \beta}{\lambda}\right)^2 \tag{19}$$

The computer time required to evaluate equation (18) is reduced by prestoring the

function $f(x) = J_1(\sqrt{x})/\sqrt{x}$ for a set of values of x. After z_{rs}^2 is calculated by equation (19), the expression $J_1(z_{rs})/z_{rs}$ is evaluated by linear interpolation on the table. Since the far field is expressed as a sum of Bessel functions, this numerical method will be called the Bessel Function method, or simply the BF method.

FAST FOURIER TRANSFORM

The Fast Fourier Transform (FFT), as its name implies, is a numerical method for calculating Fourier transforms that is computationally faster than previously used methods. Since the far field can be expressed as the two-dimensional Fourier transform of the aperture illumination, the FFT can be used to perform the computation. For details of the FFT algorithm, see reference 5. We discuss briefly here only those features of the FFT needed to make a comparison between the FFT method and the BF method.

The FFT requires that the aperture illumination be specified on a rectangular array of N-by-N points. As in the discussion of the BF method, we call this rectangle the Fourier rectangle. From figure 6, the Fourier rectangle has a semilength of c and a semiwidth of d. Unlike the BF method, the FFT method requires that the aperture illumination be set to zero for those points that are outside the aperture. Also unlike the BF method, the Fourier rectangle is large compared to the aperture. The reason for this is that the ratio of c to a (or d to b) determines the number of points per side lobe calculated by the FFT. Approximately c/a (or d/b) points per side lobe will be calculated.

The number of grid points lying within the aperture determines the number of lobes calculated by the FFT. If the number of grid points on the x-axis, extending from $x = -a$ to $x = a$, is k, the FFT calculates approximately k lobes. Also the number of grid points lying within the aperture determines the accuracy of the far-field calculation. The accuracy increases as k increases.

The Fourier-rectangle array of N-by-N points has the restriction that N must be a power of 2. For most applications, N will be either 64, 128, or 256. Another restriction is that the number of far-field (output) points must also form an N-by-N array, where N is the same as for the Fourier-rectangle (input) array.

The computer time for calculating the FFT is proportional to $N^2 \log_2 N$. Previously used methods had computer times proportional to N^3. For large values of N, the savings in computer time by using the FFT is substantial.

COMPARISON OF BF METHOD WITH FFT METHOD

For either the FFT method or the BF method, the error at a far-field point is defined as the difference between the computed and true values at that point, divided by the peak value of the true pattern. By true value, we mean the exact value of the scalar diffraction integral of equation (2). The error criterion used in this report is that the error cannot exceed 0.01 at any far-field point. This criterion will not yield good accuracy on a decibel scale for side lobes far removed from the main lobe. However, far-out side lobes are not accurately calculated from the scalar diffraction integral (ref. 6). Vector diffraction integration is required. The error criterion of 0.01 is acceptable for those portions of the far field that are accurately described by the scalar diffraction integral.

For the FFT method, the accuracy of computing the scalar diffraction integral increases as the number of lobes included in the calculation increases. To satisfy the accuracy criterion of 0.01, approximately 16 lobes have to be included. And to obtain visual resolution of the side lobes, approximately 8 points per side lobe are needed. Thus, a 128-by-128 array of points is required.

For the BF method, the upper limit M in the far-field summation has to be 4 to satisfy the error criterion of 0.01. For some aperture illuminations, a smaller value of M would suffice, as discussed later in this section. For calculating the Fourier coefficients c_{rs}, a 17-by-17 array of points is sufficient for the Fourier rectangle. In the sample aperture illuminations to follow, $c = 1.05\,a$ and $d = 1.05\,b$ for the BF method.

The first aperture illumination to be considered is given by

$$\left. \begin{array}{l} A(x, y) = 0.3 + 0.7 \left(1 - \dfrac{x^2 + y^2}{a^2} \right) \\[3em] \varphi(x, y) = 0 \end{array} \right\} \qquad (20)$$

This illumination is typical for large-aperture antennas. We assume that the aperture is circular and thus $a = b$. For the BF method, $M = 2$ is sufficient to satisfy the error criterion. Figure 7 is a plot of the relative power $|E(\alpha, 0)/E(0, 0)|^2$ in decibels as a function of α. The FFT method, the BF method with $M = 2$, and the BF method with $M = 8$ are shown. The BF method with $M = 8$ can be considered as the exact evaluation of the scalar diffraction integral since larger values of M do not produce significant changes.

We now consider an aperture illumination having a quadratic, or square-law,

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phase function. This phase function occurs in large-aperture antennas when the feed is defocused along the axis of symmetry. For the BF method, $M = 4$ is required to satisfy the error criterion. The aperture illumination is given by

$$\left.\begin{aligned} A(x, y) &= 0.3 + 0.7\left(1 - \frac{x^2 + y^2}{a^2}\right) \\[2em] \varphi(x, y) &= \frac{\pi}{2}\left(\frac{x^2 + y^2}{a^2}\right) \end{aligned}\right\} \tag{21}$$

Figure 8 shows the FFT method, the BF method with $M = 4$, and the BF method with $M = 8$. Again, the BF method with $M = 8$ can be considered as the exact evaluation of the scalar diffraction integral.

Finally, we consider an aperture illumination having a cubic phase function. This phase function occurs in large-aperture antennas when the feed is moved off axis to tilt the beam. For the BF method, $M = 4$ is required to satisfy the error criterion. The aperture illumination is given by

$$\left.\begin{aligned} A(x, y) &= 0.3 + 0.7\left(1 - \frac{x^2 + y^2}{a^2}\right) \\[2em] \varphi(x, y) &= \frac{\pi}{2}\left(\frac{x}{a}\right)^3 \end{aligned}\right\} \tag{22}$$

Figure 9 shows the FFT method, the BF method with $M = 4$, and the BF method with $M = 8$. The latter can be considered as the exact evaluation of the scalar diffraction integral.

In comparing the merits of the two methods, one should consider the computer storage and computer time requirements. The FFT requires an N-by-N array of complex numbers for input. Thus, $2N^2$ real numbers must be stored. The FFT uses the same storage space for calculating the N-by-N array of complex outputs. For $N = 128$, the storage requirement is approximately 32 000 words. The BF method requires 2000 words of storage for storing the table of the function $J_1(\sqrt{x})/\sqrt{x}$. For a K-by-K array of aperture points, an additional storage of $2K^2$ is required. For typical values of K, this storage requirement will not exceed 1000 words. Finally, the output points must be stored. The maximum storage requirement would occur when a full rectangular array of N-by-N output points is desired. Since only the amplitude of

the far field is of interest, the storage requirement is N^2 words. The total storage requirement for the BF method is $2K^2 + N^2 + 2000$ words. For any reasonable values of N and K, the BF method requires less storage than the $2N^2$ words required by the FFT.

In discussing computer time requirements, there are two major parts of the computation to be considered. One part is the computation of the aperture illumination at the aperture grid points. For idealized aperture illuminations, such as those considered in this report, this computer time is insignificant. But for illuminations calculated from feed horn patterns, the computer time may be significant. For the two methods, the computer time is approximately proportional to the number of grid points within the aperture. In the examples used in this report, a 16-by-16 array of aperture points was required for the FFT method, and a 17-by-17 array was required for the BF method. It may be concluded that the computer time for calculating aperture illuminations is comparable for the two methods.

The second major part of the computation is the calculation of the far field at the output points. For an N-by-N array of output points, the computer time requirement for the FFT method is proportional to $N^2 \log_2 N$. Since $\log_2 N$ will be either 6, 7, or 8 in most applications, we may conclude that the computer time is approximately proportional to N^2. For the BF method, the computer time is proportional to $M^2 J$, where M is the upper limit in the double summation for the far field and J is the number of output points.

To determine the computer time for calculating the far field at the output points, we use results obtained from the IBM 360/67 time-sharing system. These results may not be the same for another computer and, therefore, should be taken only as guidelines. For the FFT method, the computer time was $1.2 N^2$ milliseconds. For the BF method, the computer time was $1.6 M^2 J$ milliseconds.

In many cases, the far field needs to be known only in the two principal planes of the pattern. Suppose that N points per plane are required. Then the BF method requires 2N output points and the FFT method requires N^2 output points. Of course, in the FFT method, we obtain more information than we need because a full rectangular array of output points is obtained. We compare the computer time needed for both methods for N of 64, 128, and 256, assuming $M = 4$ for the BF method, in the following table:

Method	Value of N		
	64	128	256
	Computing time, sec		
FFT	$4\frac{1}{2}$	21	96
BF	3	6	13

For such cases, it is clear that the BF method is faster than the FFT.

For some applications, the full rectangular array of output points is required. Then N^2 output points are required for both methods. For this case the computer time for the BF method divided by the computer time for the FFT method is 1.3 M^2. Clearly, the FFT method is faster for this case.

A significant advantage of the BF method is the complete flexibility in choosing the number and location of output points. Recall that the FFT method is restricted to having the number of output points equal to the number of input points and that this number must be a power of 2. The BF method has no such restrictions. Also short-cuts can be taken with the BF method by taking advantage of symmetries in the pattern. For a circularly symmetric pattern, for example, output points are needed only for one-half of one principal plane of the pattern.

SUMMARY OF RESULTS

In computer programs used for evaluating the performance of high-gain antennas, efficient numerical methods for calculating the far-field patterns must be used since the majority of computer time and storage requirements may be attributed to this phase of the program. The numerical method most frequently used is the Fast Fourier Transform (FFT), which computes the far field as the Fourier transform of the field distribution in the antenna aperture. This report describes a new numerical method that in many applications is superior to the FFT in terms of reducing computer time and storage requirements.

The new method requires less computer storage in all applications. Which of the two methods requires less computer time depends on the number and location of the far-field points. If the application requires far-field points only in the principal planes of the pattern, the new method is generally faster. If the application requires a full rectangular array of far-field points, the FFT method is generally faster. A signifi-

cant advantage of the new method is the complete flexibility in choosing the number and location of far-field points. If the calculation of the field in the aperture is time consuming, the number of aperture grid points required becomes an important consideration. The number of aperture grid points required is approximately the same for the two methods.

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APPENDIX – EVALUATION OF INTEGRAL

The following integral is to be evaluated:

$$I = \int_{-a}^{a} \int_{-y(x)}^{y(x)} e^{j(\gamma x + \delta y)} \, dy \, dx \tag{A1}$$

$$y(x) = b\sqrt{1 - \left(\frac{x}{a}\right)^2} \tag{A2}$$

The integration over y is readily done to obtain

$$I = \frac{2}{\delta} \int_{-a}^{a} e^{j\gamma x} \sin\left[\delta b \sqrt{1 - \left(\frac{x}{a}\right)^2}\right] dx \tag{A3}$$

Since the imaginary part of the integrand of equation (A3) is an odd function of x, the imaginary part of I is zero. The trigonometric part of the integrand can be expressed as

$$\sin\left[\delta b \sqrt{1 - \left(\frac{x}{a}\right)^2}\right] = \frac{e^{j\delta b\sqrt{1-(x/a)^2}} - e^{-j\delta b\sqrt{1-(x/a)^2}}}{2j} \tag{A4}$$

Substituting equation (A4) into (A3), we obtain

$$I = \frac{1}{j\delta} \left\{ \int_{-a}^{a} e^{j\left[\gamma x + \delta b\sqrt{1-(x/a)^2}\right]} dx - \int_{-a}^{a} e^{j\left[\gamma x - \delta b\sqrt{1-(x/a)^2}\right]} dx \right\} \tag{A5}$$

Making the substitution

$$\left. \begin{array}{l} x = a \sin\theta \\ dx = a \cos\theta \, d\theta \end{array} \right\} \tag{A6}$$

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we obtain

$$I = \frac{a}{j\delta} \left[\int_{-\pi/2}^{\pi/2} e^{j(\gamma a \sin \theta + \delta b \cos \theta)} \cos \theta \, d\theta - \int_{-\pi/2}^{\pi/2} e^{j(\gamma a \sin \theta - \delta b \cos \theta)} \cos \theta \, d\theta \right]$$

(A7)

The following substitutions are made:

$$\gamma a \sin \theta + \delta b \cos \theta = A \sin(\theta + \sigma_1)$$

(A8)

$$\gamma a \sin \theta - \delta b \cos \theta = A \sin(\theta + \sigma_2)$$

(A9)

where A, σ_1, and σ_2 are given by

$$A = \sqrt{(\gamma a)^2 + (\delta b)^2}$$

(A10)

$$\sigma_1 = \tan^{-1} \left(\frac{\delta b}{\gamma a} \right)$$

(A11)

$$\sigma_2 = \tan^{-1} \left(\frac{-\delta b}{\gamma a} \right)$$

(A12)

The integral I now becomes

$$I = \frac{a}{j\delta} \left[\int_{-\pi/2}^{\pi/2} e^{jA \sin(\theta + \sigma_1)} \cos \theta \, d\theta - \int_{-\pi/2}^{\pi/2} e^{jA \sin(\theta + \sigma_2)} \cos \theta \, d\theta \right]$$

(A13)

Since I is real and $a/j\delta$ is purely imaginary, the real parts of the two integrals in equation (A13) are zero. These two integrals, to be called I_1 and I_2, are evaluated in the following equations:

$$I_i = \int_{-\pi/2}^{\pi/2} e^{jA \sin(\theta + \sigma_i)} \cos \theta \, d\theta \quad i = 1, 2$$

(A14)

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$$I_i = \int_{-(\pi/2)+\sigma_i}^{(\pi/2)+\sigma_i} e^{jA \sin \epsilon} \cos(\epsilon - \sigma_i) d\epsilon \qquad i = 1, 2 \qquad (A15)$$

$$I_i = \cos \sigma_i \int_{-(\pi/2)+\sigma_i}^{(\pi/2)+\sigma_i} e^{jA \sin \epsilon} \cos \epsilon \, d\epsilon$$

$$+ \sin \sigma_i \int_{-(\pi/2)+\sigma_i}^{(\pi/2)+\sigma_i} e^{jA \sin \epsilon} \sin \epsilon \, d\epsilon \qquad i = 1, 2 \qquad (A16)$$

Since the real parts of I_i are zero, we obtain

$$I_i = j \cos \sigma_i \int_{-(\pi/2)+\sigma_i}^{(\pi/2)+\sigma_i} \sin(A \sin \epsilon) \cos \epsilon \, d\epsilon$$

$$+ j \sin \sigma_i \int_{-(\pi/2)+\sigma_i}^{(\pi/2)+\sigma_i} \sin(A \sin \epsilon) \sin \epsilon \, d\epsilon \qquad (A17)$$

The integrands in equation (A17) are periodic, with period π, so we can change the limits of integration to range from $-(1/2)\pi$ to $(1/2)\pi$. The first integrand in equation (A17) is odd, and the second integrand is even. We therefore have

$$I_i = 2j \sin \sigma_i \int_0^{\pi/2} \sin(A \sin \epsilon) \sin \epsilon \, d\epsilon \qquad i = 1, 2 \qquad (A18)$$

From reference 7, we obtain

$$I_i = j\pi \sin \sigma_i J_1(A) \qquad i = 1, 2 \qquad (A19)$$

Returning now to the evaluation of I, we have

$$I = \frac{a}{j\delta} (I_1 - I_2) = \frac{a\pi J_1(A)}{\delta} (\sin \sigma_1 - \sin \sigma_2) \qquad (A20)$$

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From equations (A11) and (A12), it follows that

$$\sin \sigma_1 = \frac{\delta b}{\sqrt{(\gamma a)^2 + (\delta b)^2}} \tag{A21}$$

$$\sin \sigma_2 = \frac{-\delta b}{\sqrt{(\gamma a)^2 + (\delta b)^2}} \tag{A22}$$

Using equations (A10), (A20), (A21), and (A22), we finally obtain

$$I = \frac{2\pi ab J_1\left[\sqrt{(\gamma a)^2 + (\delta b)^2}\right]}{\sqrt{(\gamma a)^2 + (\delta b)^2}} \tag{A23}$$

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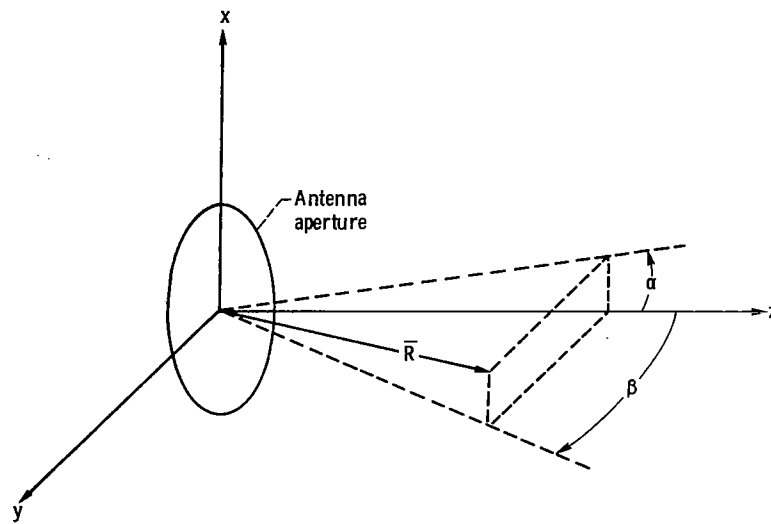
Figure 1. - Coordinate system.
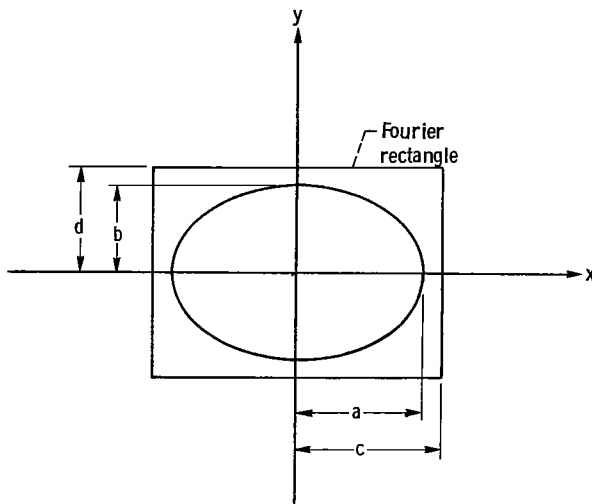
Figure 2. - Coordinate system for antenna aperture and Fourier rectangle.
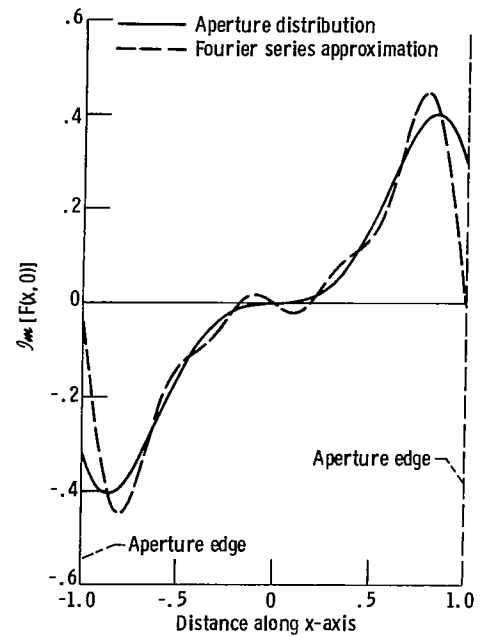
Figure 3. - Fourier series approximation to imaginary part of aperture distribution for a cut along x-axis: $a = b = c = d = 1$.
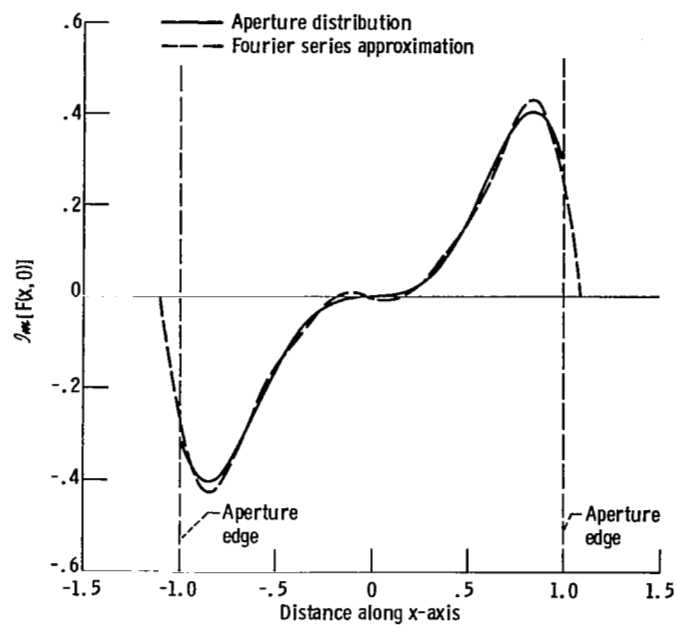
Figure 4. - Fourier series approximation to imaginary part of aperture distribution for a cut along x-axis: $a = b = 1$; $c = d = 1.1$.
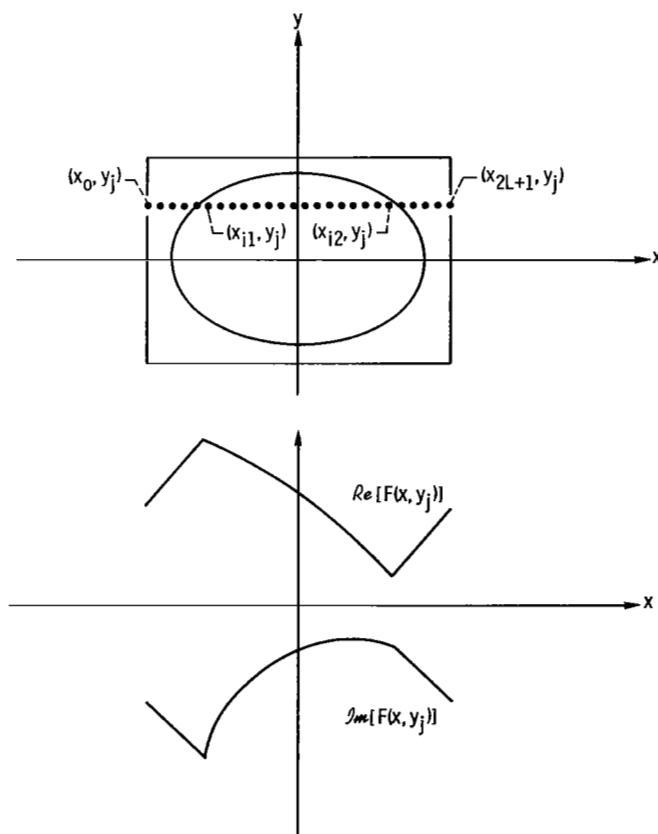

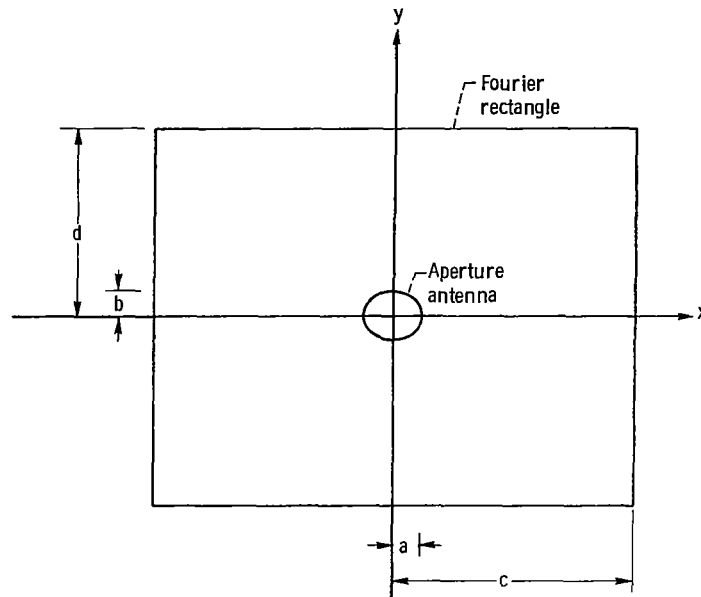Figure 5. - Extension of aperture function to Fourier rectangle - BF method.

Figure 6. - Coordinate system for antenna aperture and Fourier rectangle - FFT method.
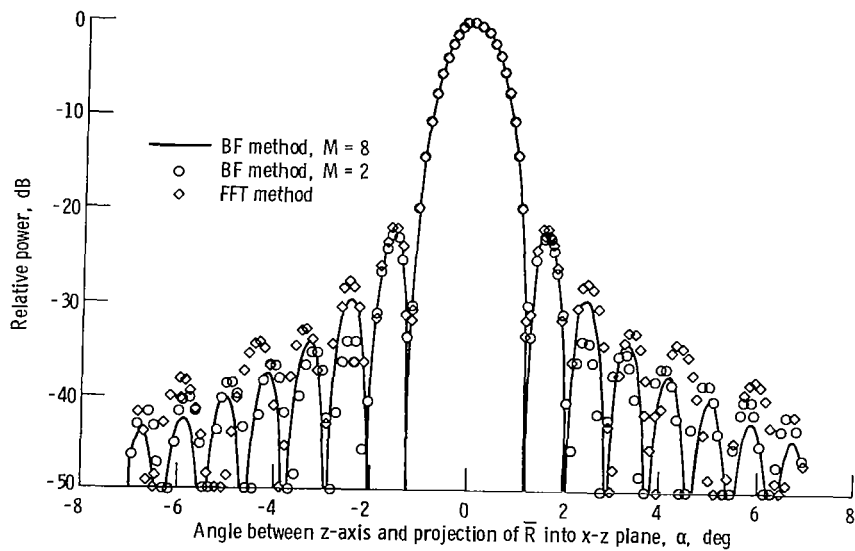
Figure 7. - Far-field pattern for aperture distribution 1: $A(x,y) = 0.3 + 0.7 [1 - (x^2 + y^2)/a^2]$; $\varphi(x,y) = 0$.
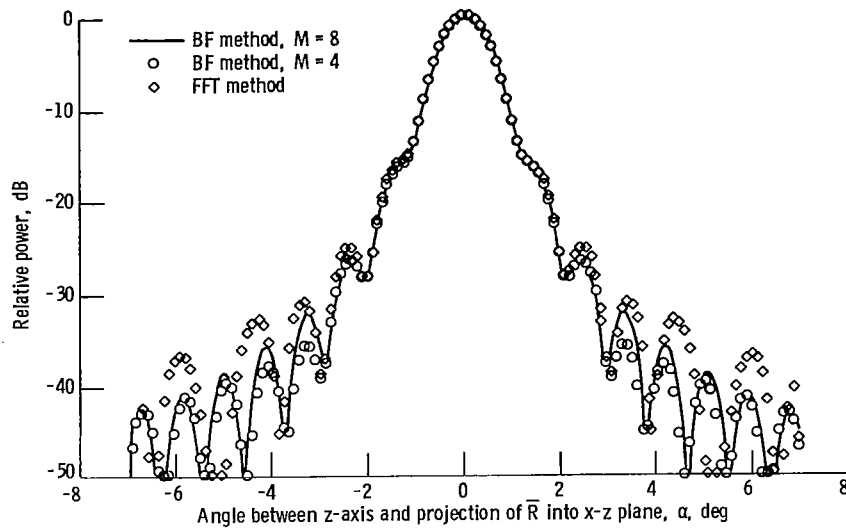
Figure 8. - Far-field pattern for aperture distribution 2: $A(x,y) = 0.3 + 0.7 [1 - (x^2 + y^2)/a^2]$; $\varphi(x,y) = (\pi/2)[(x^2 + y^2)/a^2]$.
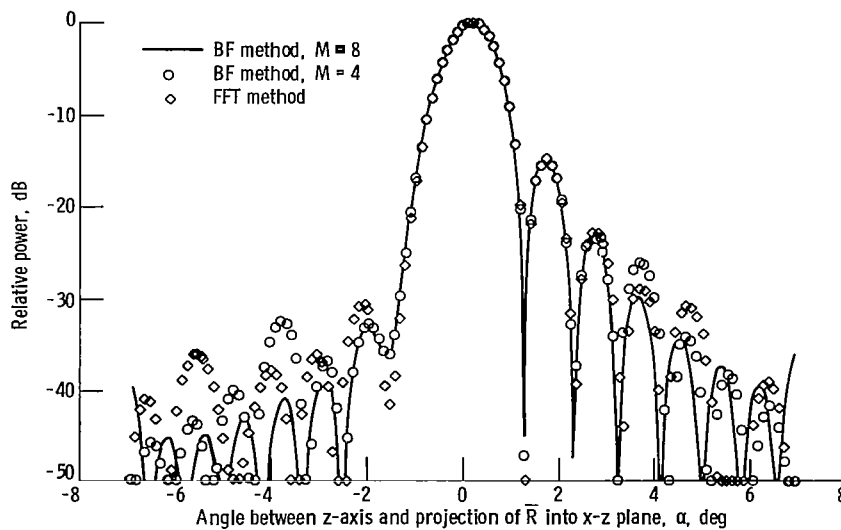

Figure 9. - Far-field pattern for aperture distribution 3: $A(x,y) = 0.3 + 0.7 [1 - (x^2 + y^2)/a^2]$; $\varphi(x,y) = (\pi/2)(x/a)^3$.

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