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MULTIVARIATE NORMALITY

Harold L. Crutcher and Lee W. Falls

George C. Marshall Space Flight Center

Marshall Space Flight Center, Ala. 35812

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16. ABSTRACT

Sets of experimentally determined or routinely observed data provide information about the past, present and, hopefully, future sets of similarly produced data.

An infinite set of statistical models exists which may be used to describe the data sets. Some are better than others. The normal distribution is one model. If it serves at all, it serves well. If a data set, or a transformation of the set, representative of a larger population can be described by the normal distribution, then valid statistical inferences can be drawn.

There are several tests which may be applied to a data set to determine whether the univariate normal model adequately describes the set. The chi-square test based on Pearson's work in the late nineteenth and early twentieth centuries is often used. Like all tests, it has some weaknesses which are discussed in elementary texts.

This report provides extension of the chi-square test to the multivariate normal model. Tables and graphs permit easier application of the test in the higher dimensions. Several examples, using recorded data, illustrate the procedures. Tests of maximum absolute differences, mean sum of squares of residuals, runs and changes of sign are included in these tests. Dimensions one through five with selected sample sizes 11 to 101 are used to illustrate the statistical tests developed in this report.

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LIST OF SYMBOLS

Symbol	Definition
a	above, greater than
b	below, less than
c	constant, summation of α and β
d.f.	degrees of freedom
f	degrees of freedom
i	subscripts, superscript, also indicates inverse as superscript
j	subscripts, superscript, also indicates inverse as superscript
k	number, degrees of freedom
l	subscript
ln	natural logarithm
log	common logarithm
m	mean
n	number of data, n-tant
p	probability, number of fitted parameters
pdf	probability density function
r	sample correlation
s	sample standard deviation (Std. Dev.)
s^2	sample variance
u	longitudinal component of wind
v	lateral component of wind

LIST OF SYMBOLS (Continued)

Symbol	Definition
w	vertical component of wind
x	variate, deviation from mean
y	variate, deviation from mean
C	constant, correlation matrix
D	determinant of correlation matrix
E	expectation
I.D.	identification number
K	constant
K-S	Kolmogorov-Smirnov test
MAD	maximum absolute difference
MSSR	mean sum squares of residuals
NID	normally and independently distributed
Q	a chi-square variate
R(u)	random uniform variate
RAL	runs above the line
RBL	runs below the line
RAM	runs above the median
RBM	runs below the median
T^2	Hotellings "T^2"
X	measured value of variate
X^2	sum, square of X, square of deviation from expected divided by the expected

LIST OF SYMBOLS (Continued)

Symbol	Definition
Y	measured value of a variate
Y^2	sum, square of Y
α	rejection level probability, parameter, constant
β	power of the test, probability, parameter, constant
Γ	gamma function
χ	chi or CHI
χ^2	chi-square
χ_0^2	computed chi-square from quadratic
λ	power (exponent)
μ	population mean
ν	nu, dimensions, number of variates, degrees of freedom
ρ	population correlation
σ	population standard deviation
σ^2	population variance
Σ	summation
ζ	variable
-	overbar, average
[ ]	matrix
′	transpose, also deviation from mean
|[ ]|	determinant of matrix
!	factorial

LIST OF SYMBOLS (Concluded)

Symbol	Definition
∞	infinity
$\sqrt{\ }$	square root
+, -	algebraic signs, quadrant signs
degrees of freedom	k, v, f, d.f.

MULTIVARIATE NORMALITY

I. INTRODUCTION

One or more theoretical functions may be used to describe a set of experimental or routinely observed data. It is necessary that, wherever possible, the selected function satisfies the physical bounds or restraints of the data. Some sets of data contain information concerning only one variable, while others contain information about two or more simultaneously observed variables. The data may be homogeneous or heterogeneous, i.e., a single type or a mixed type.

Edgeworth (1916) and Geary (1947) provide interesting discussions on the applicability of the normal distribution. Edgeworth represents an earlier stage of statistical development than does Geary. Edgeworth considers the normal law of error to be a first approximation to the actual shape of a distribution. Corrections or modifications to this form a second approximation. Presumably, as more empirical evidence is gathered, further approximations might be made. Perhaps this is the way the procedure really works. Models are used until they are found to be deficient; then a new model is proposed. Geary indicates this in a rather interesting chapter on the fluctuations in the attitudes of statisticians concerning the question of the occurrence of the normal frequency distribution. This is still a relevant point. Up to approximately the end of the last century, a main current in the thinking was a favorable inclination toward the hypothesis of universal normality; i.e., in a sense, everything was distributed normally and departures from normality were only the result of sampling.

Geary (1947) indicates that near the first of the present century, with the development of the theory of moments, concepts changed from that of universal normality to that of universal nonnormality. This feeling was engendered by the use of moments and the Pearsonian curve system. In this system the normal distribution function is only one of the many functions. Another shift in thinking occurred approximately 25 years later when Fisher showed that, when universal normality could be assumed, inferences of wide practical usefulness could be drawn from samples of any size. Geary states, "Normality is a myth; there never was, and never will be, a normal distribution." As Geary further states, "This is an over statement from the practical point of view, but it represents a safer initial attitude than any in fashion during the past two decades." This was prior to 1947. This led to Geary's hope that he had created a "prima facie" case for the importance of testing for normality. It is this testing for normality in the multivariate sense that is the basis for this report. The reader is referred to Fisher (1924, 1950).

Tests are usually made without regard to alternatives, according to Cochran (1954) who suggests that some alternatives should be considered. Cochran's 1952 paper also is excellent reading. Tests made without regard to alternatives are made without regard to the Type I and Type II errors. A Type I error is the rejection of a null hypothesis when it is true, whereas a Type II error is the nonrejection of the null hypothesis when it is false. The probability of the Type I error is equal to the probability level selected for testing, usually designated as α. The probability of the Type II error, β, is equal to one minus the power of the test, usually written as $(1 - \beta)$. The more powerful the test, the less is the Type II error. Cochran (1954) indicates that most uses of the chi-square (χ^2) test may be strengthened by: (1) use of small expectations in computing χ^2, (2) use of a single degree of freedom or a group of degrees of freedom from the total χ^2, and (3) use of alternative tests.

Where multivariate multimodal distributions are treated, techniques to separate the data into unimodal homogeneous groups or clusters may be used prior to detailed study. Among the several clustering techniques available, discriminant function, principal component, or factor analysis may be used to effect the separation.

In this report only the multivariate single mode data set, which may be described by the normal law of errors, is considered. This is restrictive but with the knowledge that the population from which the data set has been selected may be considered to be multivariate normal, valid inferences can be drawn and probabilistic statements can be made.

There is a need for tests to determine whether a data set is multivariate normal. As Andrews, Gnanadesikan, and Warner (1973) indicate and discuss, there are procedures to help meet this need. These are: (1) likelihood ratio tests associated with transformations to enhance joint normality; (2) goodness-of-fit tests such as the chi-square (χ^2) and Kolmogorov-Smirnov (K-S), tests based on local densities, nearest distance tests; and (3) informal graphical methods associated with radii and angles representation of the data. The informal graphical methods are of particular interest in the last paper and this paper also deals with graphical testing procedures.

Mauchly (1940a) proposes a sphericity test which is often used. In the two-dimensional (1940b) (bivariate) sense he proposes an ellipticity test. Crutcher (1957) applies the latter test to upper wind distributions. Votaw (1948) discusses the testing of component symmetry in a multivariate normal distribution. Hald (1952a) uses the square of the radii for the bivariate case. Kac, Kiefer, and Wolfowitz (1955), Weiss (1958), and Anderson (1966) discuss tests of normality based on density and distance methods. Healy (1968) and Kessel and Fukunaga (1972) have suggested procedures based on the squared radii. Andrews, Gnanadesikan and Warner (1973) describe graphical procedures based on radii and angles. Considerable work is currently under way on this problem. More recent tests and articles dealing with these problems may be familiar to the reader but are not apparent to the authors. An apology is tendered to any author whose work has inadvertently been missed.

Examination of the marginal distributions, including all the univariate and the bivariate combinations within the multivariate set, provides considerable insight. Sometimes this is quite tedious.

In this report the Monte Carlo or random sampling technique provides the sampling used for the multivariate normal distributions. In the multivariate sense, each vector has a magnitude and a direction. In the sense of scaled residuals, these are magnitudes and directions from the centroid. In the multivariate normal and independent case in large samples, the direction and magnitudes are not correlated and the sample swarm is spherical in shape, hence, Mauchly's sphericity test. The squares of the magnitudes are distributed approximately as chi-square with degrees of freedom equal to the number of variables and exactly as a constant multiple of a beta distributed variable, Gnanadesikan and Kettenring (1972). A chi-square probability plot of the squared radii in the bivariate case versus the uniform probability plot of the angles should, for the null hypothesis, produce a reasonably linear plot, Hald (1952a,b) and Andrews, Gnanadesikan and Warner (1973); as shown by this group, a plot of the squares of the radii versus the angles for the null hypothesis produces a random scatter on the unit square. Thus, visually, the investigator can make the decision as to whether the data set is multivariate normal.

Anderson (1958) shows that sampling from a multivariate normal distribution developed from one-dimensional normally and independently distributed variables; i.e., NID $(0, \sigma^2)$ will produce a multivariate Wishart distribution (1928, 1948) with NID (μ, σ^2) marginals, Smith and Hocking (1972). This includes the univariate marginals. That is, sampling from a known multivariate normal set with a zero mean and a dispersion matrix σ^2 will produce sets which have means different from zero with the same dispersion matrix. As Kendall and Stuart (1968) indicate, this may be regarded as a generalization of the χ^2 distribution in the multivariate sense.

The Wishart distribution and its ramifications are so complicated and difficult that it has not found much use. However, for small samples, in sampling from the Wishart central multivariate normal distribution, the sampling follows the Wishart distribution. The sample means may not be zero and the sampling distributions do not follow the χ^2 for large samples.

Tests for multivariate normality also may be made from the viewpoint of the roots of the dispersion matrices. Though these were examined in the preparation of this report, they are not used and are not discussed further. The reader may refer to Kendall and Stuart (1968) and the references they provide.

II. THE MULTIVARIATE NORMAL DISTRIBUTION

The normal distribution law in one dimension is due to De Moivre (1733). Usually, one associates it with the names of Gauss and LaPlace. More than a century after De Moivre, Bravais (1846) developed and published his study of normal frequency distributions in two and more variables. Contributions to the study of this problem were made by Maxwell (1859), Bertrand (1888a,b), Czuber (1891), Pearson (1900), Kluyver

(1906), Student (1908, 1925), Strutt (1919), and Hotelling (1951). The distribution of vector magnitudes in two and three dimensions, when the vector means and correlations are zero and the variances are equal, often are referred to as Rayleigh and Maxwellian, respectively, although sometimes these are interchanged. Where the means are not zero, they are referred to as the generalized noncentral Rayleigh and Maxwellian distributions. Sometimes only the projection of the multivariate distribution onto one axis is referred to as Rayleigh or Maxwellian.

The probability density function (pdf) of the multivariate normal distribution may be written as

$$p(x_1, x_2, \ldots, x_\nu) = Ce^{-1/2(Q)} \tag{1}$$

as indicated by Pearson (1900) for ν correlated random variates, $(x_1, x_2, \ldots, x_\nu)$. The main feature of the ν-dimensional normal distribution is that all of its properties are deducible or determined from the information contained in the means and the covariances. Cochran (1952) and Elderton and Johnson (1969) provide excellent discussions of Pearson's work.

Denoting E as the expectation, Σ as a summation, and n as the number of data, the following may be defined for use here, where the subscripts i, j range from 1 through ν:

$$\mu_i = E(X_i) \approx \frac{\Sigma X_i}{n} = \bar{X} \tag{2a}$$

$$\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)] \approx \Sigma\Sigma(X_i - \mu_i)(X_j - \mu_j)/n \tag{2b}$$

$$\sigma_{ii} = E[(X_i - \mu_i)^2] = \sigma_i^2 \approx \Sigma(X_i - \mu_i)^2/n = s_i^2 \tag{2c}$$

$$x_i = (X_i - \mu_i) \approx (X_i - \bar{X}) \quad , \tag{2d}$$

where $\bar{X}_i$ and s_i^2 are the sample estimates of the mean and variance.

Pearson (1900) showed that the quadratic form, Q of equation (1), which is a positive definite matrix, is distributed as χ^2 (chi-square) with ν degrees of freedom.

$$\chi^2 = Q = \Sigma[\sigma^{ij}](X_i - \mu_i)(X_j - \mu_j) \quad (3a)$$

$$= [x][\sigma^{ij}][x'] \quad (3b)$$

$$= [x][\rho^{ij}][x'] \quad , \quad (3c)$$

where the brackets [ ] indicate a matrix, $[x']$ is the transpose of $[x]$ where $[x] = [x_1, x_2, x_3, \ldots, x_\nu]$, and $[\sigma^{ij}]$ and $[\rho^{ij}]$ are the inverses of the covariance and correlation matrices $[\sigma_{ij}]$ and $[\rho_{ij}]$, respectively. In equation (3c), x is a standardized variable.

C is a constant for a distribution and may be written as

$$C = (2\pi)^{-(\nu/2)} |[\sigma^{ij}]|^{1/2} \quad (4a)$$

or

$$C = (2\pi)^{-(\nu/2)} |[\rho^{ij}]|^{1/2} \quad (4b)$$

where $|[\sigma^{ij}]|$ is the determinant of the covariance matrix. In equation (4b) standardized variables are used. If standardized variables are not used, then equation (4b) must be divided by the product of the appropriate standard deviations. An element of the inverse matrix σ^{ij} is equal to the determinant of the cofactor (minor) of the element of the matrix, σ_{ij}, divided by the determinant of the covariance matrix. The determinant of the cofactor of σ_{ij} is the determinant of order $(\nu$-1) obtained by striking the ith row and jth column of the covariance matrix multiplied by $(-1)^{i+j}$, as shown by Anderson and Bancroft (1952) and Bendat and Piersol (1966) among many others.

All marginal and conditional distributions of a multivariate normal distribution are normal. The converse is not true as indicated by many writers. Cramer (1946) proves this in a theorem. As Feller (1966) and Kowalski (1970) indicate, many an investigator assumes multivariate normality if the marginal distributions are normal. If any tests are rejected, the rejected distributions are transformed to normal and the assumption of multivariate normality then is made. Transformations are discussed by Box and Cox (1964) where they use the power from $(X + \xi)^\lambda$. Transformations of the total distribution in terms of the dispersion matrix also are discussed by Joshi (1970) and Andrews, Gnanadesikan and Warner (1973). In the succeeding sections of this report, a geometric illustration of this feature will be shown for the two-dimensional case. A few standard statistical texts provide analogous illustrations. Extension to higher dimensions hopefully will not be difficult for the reader.

A test of multivariate normality is tantamount to a test of normality of the total distribution as well as all its marginal and conditional distributions.

The special cases of the uni-, bi-, and tri-variate cases now are discussed in greater detail for better understanding.

A. The Univariate (One-Dimensional) Distribution

From equation (1) the pdf of the univariate normal distribution may be written

$$p(x) = Ce^{-1/2\chi^2} \tag{5a}$$

where

$$C = (1/((2\pi)^{1/2}\,\sigma_{X_1})) = (1/((2\pi)^{1/2}\,s_{X_1})) \quad , \tag{5b}$$

where σ_{X_1} is estimated by its sample estimate s_{X_1}.

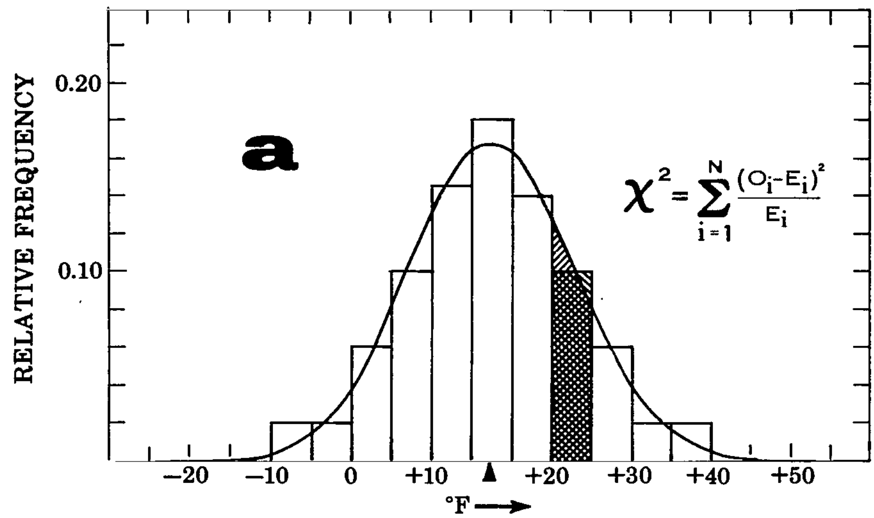
Chi-square, χ^2, in this case may be written as

$$\chi^2 = (X_1 - \mu_{X_1})^2/\sigma_{X_1}{}^2 = ((X_1 - \mu_{X_1})/\sigma_{X_1})^2 \tag{5c}$$

$$\approx (X_1 - X_1)^2/s_{X_1}{}^2 = ((X_1 - X_1)/s_{X_1})^2 \quad , \tag{5d}$$

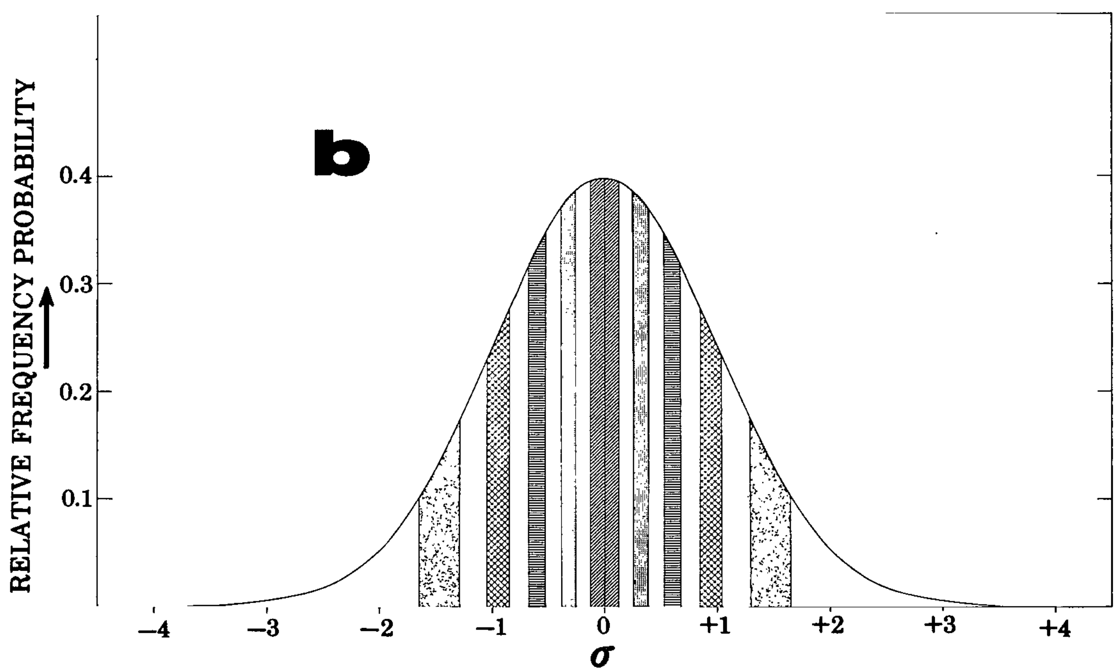
where μ_{X_1} is estimated by its sample estimate X_1. Chi then can be visualized as simply the standardized variable $((X_1 - \mu_{X_1})/\sigma_{X_1})$. This is a scaled residual.

Figure 1a shows a univariate normal distribution curve overlying a histogram. The area of the histogram and the area bounded by the normal curve and the variate axis can be made equal and set to one. This figure will be referred to later with respect to the hachured areas. Figure 1b also shows a normal distribution curve of which the total area has been sectioned into twenty equiprobability areas. The symmetrical portions are either similarly hachured or blank. These also will be discussed later.



a. Overlaid on a histogram.

Observed, O   Expected, E   N=Number of class intervals



b. Twenty equiprobability areas, each equal to 0.05 (similar area markings show symmetric areas).

Figure 1. Schematics of univariate normal distribution curves.

Figures 2a, b, c, and d also represent a normal distribution in one dimension. These illustrate how some common inferential statements can be made. The captions and indicated probability areas are believed to be self-explanatory. The distribution is determined completely by the mean and standard deviation. The latter is the square root of the variance.

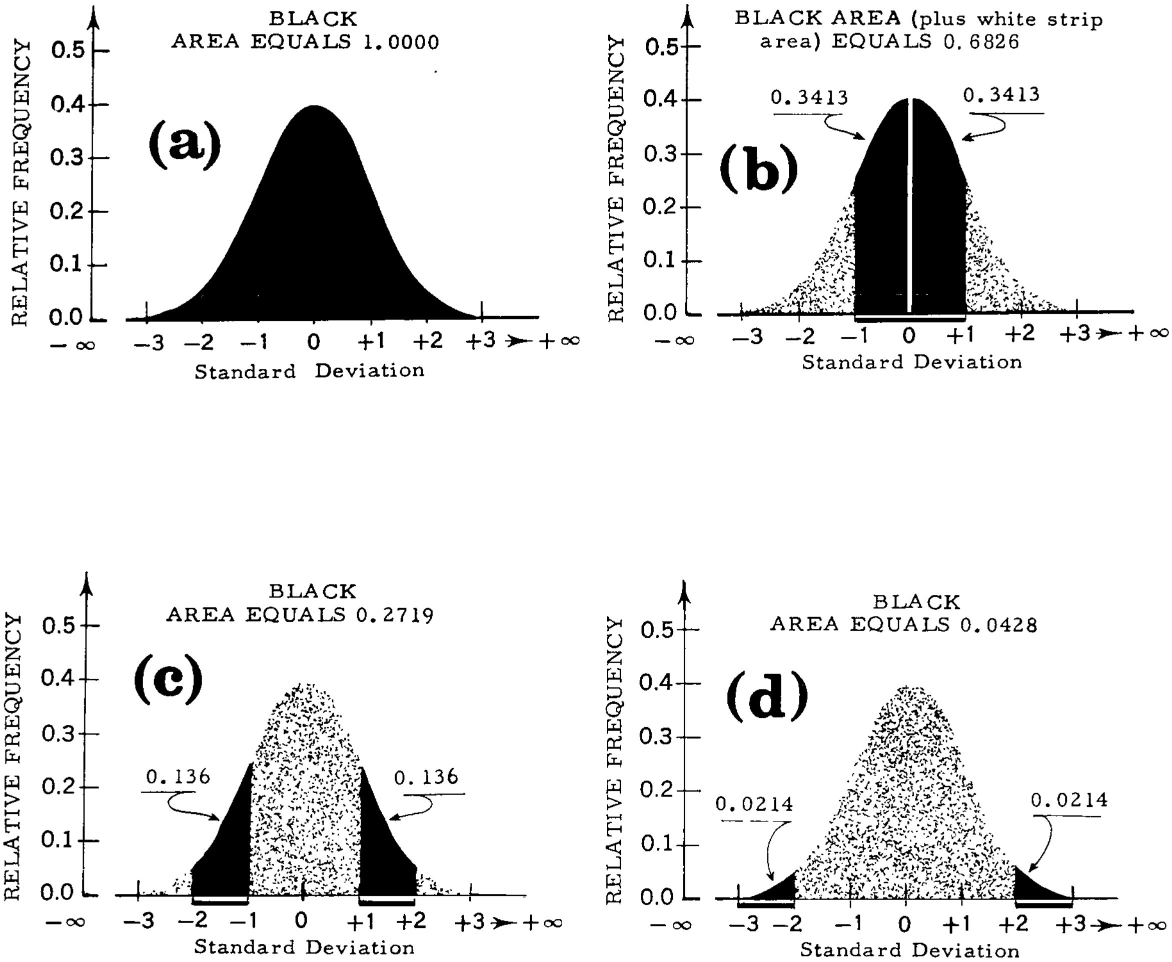


Figure 2. Normal distribution curve areas for selected intervals of the standard deviation.

B. The Bivariate Distribution

The pdf of the bivariate or two-dimensional normal distribution may be written as

$$p(x_1, x_2) = Ce^{-1/2\,\chi^2} \quad . \tag{6a}$$

In terms of the sample correlation statistics the estimate of C may be written as

$$C \approx (2\pi)^{-1}\, s_{X_1}^{-1}\, s_{X_2}^{-1}\, (1 - r^2_{X_1 X_2})^{-1/2} \quad , \tag{6b}$$

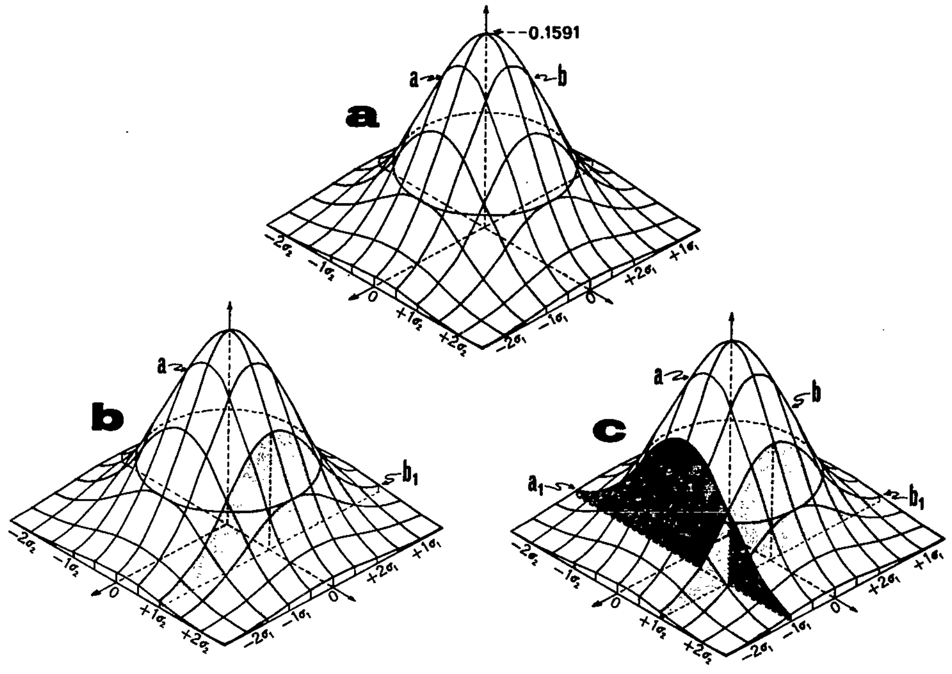
with the χ^2 estimate as

$$\chi^2 \approx (1/(1 - r^2_{X_1 X_2}))\left\{\left(\frac{X_1 - \bar{X}_1}{s_{X_1}}\right)^2 + \left(\frac{X_2 - \bar{X}_2}{s_{X_2}}\right)^2 - 2r_{X_1 X_2}\left(\frac{X_1 - \bar{X}_1}{s_{X_1}}\right)\left(\frac{X_2 - \bar{X}_2}{s_{X_2}}\right)\right\} \quad . \tag{6c}$$

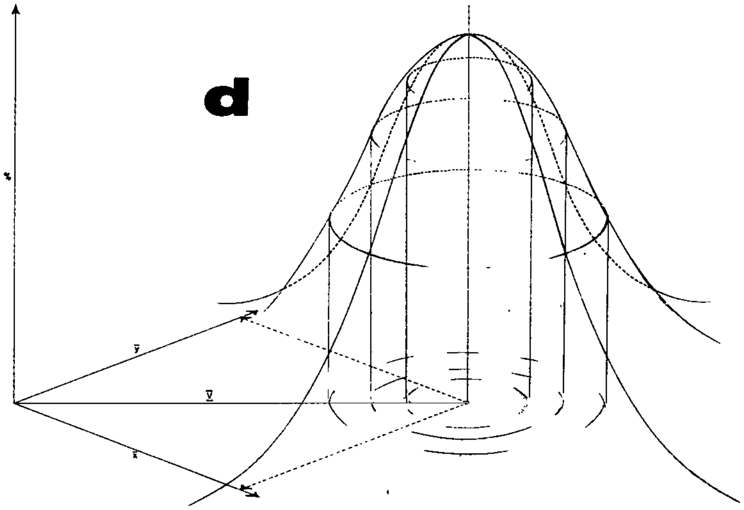
Here $(1 - r^2_{X_1 X_2})$ is the determinant of the correlation matrix, $|[r_{12}]|$, and r, the sample estimate, replaces ρ, the population correlation.

Figures 3a, b, and c show a bivariate normal frequency distribution where conditional univariate normal distributions are indicated by the shaded univariate normal distribution planes (a), (b), and (a and b), respectively. Each distribution plane is perpendicular to or parallel to the variate axes and perpendicular to the bivariate plane. The units are in terms of standard deviations.

Figure 3d is another representation of the bivariate normal frequency (volume) under the bivariate normal frequency surface. The representation is more in a polar coordinate form rather than a Cartesian form. Here, the vector radii or χ centered on the vector mean sweep out circular (elliptic) cylinders bounded by the bivariate frequency surface and the bivariate plane. The projection of the cylinders onto the bivariate plane is visualized as circles or ellipses. Certain volumes, interpretable as probabilities, are contained within each cylinder. The volumes swept out are dependent on the magnitudes of the radii. For example, a vector radius dependent upon the two variate standard deviations will sweep out the central 0.40 probability core, while if the vector radius is equal to one vector standard deviation (the square root of the sum of the squares of the individual variate standard deviation), $(\sigma_{X_1}^{\ 2} + \sigma_{X_2}^{\ 2})^{1/2}$ will sweep out the central 0.63 probability circular (elliptic) cylinder. If the distribution is circular, the vector standard deviation multiplied by $[\ln (1/(1-p))]^{1/2}$ provides the vector radius of the p probability ellipse. If the distribution is elliptical, the eigenvector-eigenvalue matrices provide the



Selected areas indicate conditional univariate normal distributions (Modified from Hald, 1952).



Volume under a bivariate normal surface (The elliptic cylinders cut out specified probabilities with specified vector radii centered at the centroid at the terminus of V.) (Crutcher, 1960)

Figure 3. Bivariate normal frequency distributions.

orientation of and variances along the major and minor axes. When the square roots of the eigenvalues are multiplied by [2(ln (1/(1-p)))], the lengths of the semimajor and semiminor axes for the p probability ellipse are obtained, Crutcher (1957, 1962). Bertrand (1888a,b) provides limited tables for these. Brooks and Carruthers (1953) apply these concepts to the specialized field of upper air wind distributions. Tables of χ^2 with 2 degrees of freedom (Table 1) can be used to determine the magnitude of the vector radius (in terms of the vector standard deviation) to core out central elliptical (circular) cylinders containing specified portions of the volume representing specified probabilities. Groenewoud, Hoaglin, Vitalis, and Crutcher (1967) provide extensive and detailed tables in two volumes for general use and some applications to problems in geophysics in the third volume. These permit elliptical (circular) coring of specified probability cores or the probabilities of specified elliptical (circular) cylinders.

Extensions of these concepts to multivariate quality control can be made. Thomas and Crigler (1974) discuss tolerance limits for the multivariate radial error distribution.

Bates (1966) provides an excellent discussion of applications of the chi-square goodness of fit for a bivariate distribution. The use of Bertrand's limited table and the detailed table mentioned above or any other similar tables is dependent upon the nonrejection of the null hypothesis concerning the bivariate normality of the data.

C. The Marginal and Conditional Distributions

Normality of the multivariate distributions implies normality of all marginal and conditional distributions. The converse is not true, Cramér (1946), Joshi (1970), Anderson (1958), Freund (1962), Kendall and Stuart (1968), Kowalski (1973), etc. Kowalski presents an excellent review of the situation.

Figure 4a is another representation of a bivariate normal distribution under the bivariate normal frequency surface. This discussion embellishes Anderson's (1958) presentation. It appears here as a limiting form of mean zero and variance one around the centroid. Note the marginal distributions shown as a projection against the planes perpendicular to the bivariate plane and as planes passing through the coordinate pair (0, 0) or origin of axes in the bivariate plane. Note also the conditional univariate normal distribution perpendicular to the bivariate plane and passing through the centroid of the distribution. Units are shown in terms of the standard deviations. Note the shaded squares in the bivariate plane located between plus and minus one standard deviation along the variate axes. These will be referred to later.

Figures 4b, c, and d, in conjunction with Figure 4a, demonstrate this lack of double implication. Figure 4a presents a bivariate normal distribution with the two marginal or univariate distributions shown perpendicular to the bivariate plane, parallel and perpendicular to the variate axes, and each one some distance from the centroid. Figure 4a shows four shaded square areas in the bivariate plane, each between one and

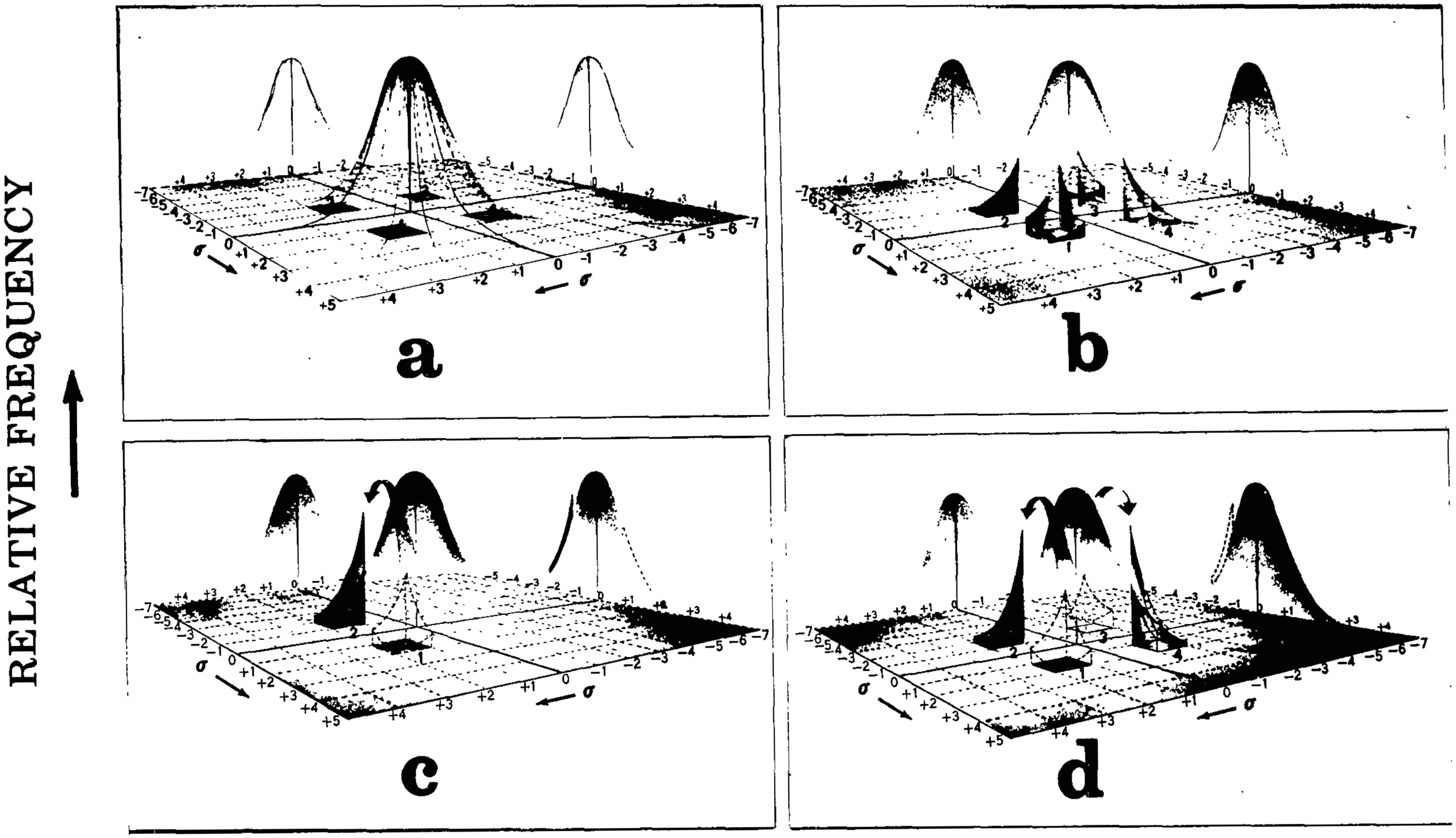


Figure 4. Bivariate distributions under a bivariate frequency surface. (a,b) Normal with marginal normals modified to: (c) Nonnormal with one marginal normal and one marginal nonnormal modified to: (d) Nonnormal with marginal normals.

two standard deviations, plus and minus, from the variate axes. These are repeated in Figures 4b, c, and d. For reference these are numbered 1, 2, 3, and 4. In Figure 4b volumes (equivalent to probabilities) are shown under the bivariate frequency surface and above the shaded areas, 1, 2, 3, and 4.

In Figure 4c the volume labeled "1" is cored from the distribution by means of a square corer or cookie cutter with sides equal to one standard deviation by a vertical downward stroke above the area labeled "1." The core is moved upwards and sideways parallel to one variate axis and perpendicular to the other variate axis to the left towards "2" and is superposed above the bivariate frequency surface above "2." The volumes under the bivariate frequency surface above the bivariate plane areas "1" and "2" are equal and symmetrically located by deliberate choice.

The bivariate frequency surface is discontinuous in the regions of "1" and "2." A hollow is evident at "1" and a spike is evident at "2" in Figure 4c. Note that the marginal distribution on the left is not changed but that the marginal distribution on the right shows a hump on its left and a dip on its right. These are located between one and two standard deviations from the mean. It is pointed out that the vertical scales are not the same for the bivariate as for the marginal distributions. The relative heights for distributions are the constant C in the equations given previously. For the marginal or univariate distributions these are $(2\pi)^{-1/2}$, for the bivariate distributions $(2\pi)^{-1}$, and for the trivariate distributions $(2\pi)^{-3/2}$. These are, respectively, 0.3989, 0.1591, and 0.0635. These represent the relative maxima of the various standardized frequency surfaces.

Figure 4d follows and is a modification of Figure 4c. Here, the volume over "3" is cored, lifted out, moved to the right, and superposed over "4." Please note that the marginal distribution on the left remains the same, while the marginal distribution on the right returns to its orginal form as the areas moved to the left and right, respectively, are equal and at the same distance from the centroid. However, note that the bivariate distribution is severely modified. It has two hollow spaces and two spikes which obviously deform the surface. In fact, the conditional distributions between one and two standard deviations from the centroid parallel to the left axis are discontinuous. It is obvious that the illustration is simplified by moving equally shaped cores symmetrically, yet this is sufficient for demonstration. Since a two-dimensional distribution is a marginal of a three-dimensional, the analogy can be projected from any higher dimensions to lower dimensions. Therefore, a simple test for multivariate normality resulting in nonrejection of the null hypothesis automatically assures nonrejection of the null hypothesis for any lower marginal or conditional distribution. This is the thrust of this report. Pearson's (1900) chi-square test is used.

D. The Trivariate Distribution

The pdf of the three-dimensional normal distribution can be written in covariance form rather than the correlation form so as to show the similarity or sameness with the correlation form illustrated previously,

$$p(x_1, x_2, x_3) = Ce^{-1/2(\chi^2)} \quad , \tag{7a}$$

where

$$C = (2\pi)^{-3/2} \, |[\sigma^{ij}]|^{1/2} \approx (2\pi)^{-3/2} \, |[s^{ij}]|^{1/2} \tag{7b}$$

and

$$\chi_0{}^2 = \Sigma s^{ii} z_i{}^2 + 2\Sigma s^{ij} z_i z_j \quad ; \quad i,j = 1,2,3 \tag{7c}$$

adapted from Pearson (1900), and $z_i = (X_i - \bar{X}_i)/s_{X_i}$, the standardized variate, or

$$\begin{aligned} \chi_0{}^2 = \Big\{ & s^{11}((X_1 - \bar{X}_1)/s_{X_1})^2 + s^{22}((X_2 - \bar{X}_2)/s_{X_2})^2 + s^{33}((X_3 - \bar{X}_3)/s_{X_3})^2 \\ & + 2s^{12}((X_1 - \bar{X}_1)/s_{X_1})((X_2 - \bar{X}_2)/s_{X_2}) \\ & + 2s^{13}((X_1 - \bar{X}_1)/s_{X_1})((X_3 - \bar{X}_3)/s_{X_3}) \\ & + 2s^{23}((X_2 - \bar{X}_2)/s_{X_2})((X_3 - \bar{X}_3)/s_{X_3}) \Big\} \quad . \end{aligned}$$

Symmetry of $s^{ij} = s^{ji}$ permits the use of the coefficient 2 as

$$s^{12}((X_1 - \bar{X}_1)/s_{X_1})((X_2 - \bar{X}_2)/s_{X_2}) = s^{21}((X_2 - \bar{X}_2)/s_{X_2})((X_1 - \bar{X}_1)/s_{X_1}) \quad .$$

Figure 5 schematically represents a trivariate normal distribution centered on the centroid with mean zero and variances one with covariances zero.

Three schematic spheres or shells are shown. Certain probabilities will be contained within spheres. A sphere (ellipsoid) centered on the centroid with a vector radius equal in magnitude to the variate standard deviation will contain a central sphere (ellipsoid) core representing 0.20 probability. An ellipsoid core swept or cut out of the distribution and centered on the centroid with a vector radius of 1.0 vector standard deviation will represent 0.608 probability. The vector standard deviation,

$$\sigma_V = \sqrt{\sigma_{X_1}{}^2 + \sigma_{X_2}{}^2 + \sigma_{X_3}{}^2} \quad ,$$

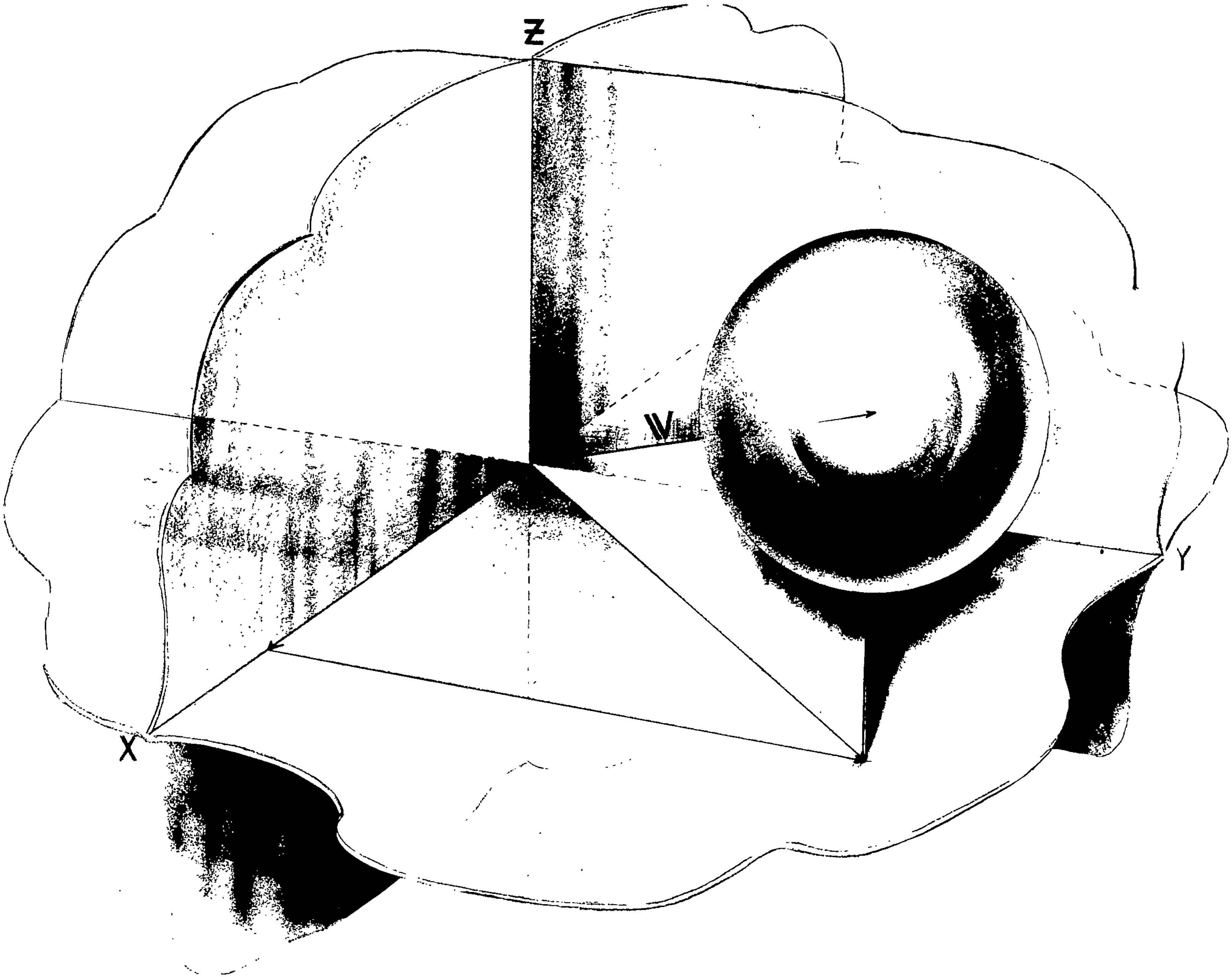


Figure 5. Trivariate distribution: variances equal and covariances zero.

is invariant and is equal to the square root of the trace of the covariance matrix; the trace is the vector variance.

Other probability ellipsoids can be selected. The magnitude of the vector radius required to core specified probabilities around the centroid can be determined easily by use of χ^2 tables with 3 degrees of freedom. No immediately available tables permit ellipsoidal coring within the trivariate frequency surface for off-center specified probabilities or specified ellipsoids, as are available for the two-dimensional case. The vector variance or trace of the covariance matrix is

$$\sigma_V{}^2 = \sigma_{X_1}{}^2 + \sigma_{X_2}{}^2 + \ldots + \sigma_{X_\nu}{}^2$$

or after rotation and determination of the eigenvalue matrix

$$\sigma_V{}^2 = \sigma_a{}^2 + \sigma_b{}^2 + \ldots + \sigma_\nu{}^2 \quad ,$$

where the letters a, b, . . . , ν refer to variances along the principal axes, where the variances are usually ordered as to magnitude. The mathematics and computer routines are known to be available but cost prohibits suitable tabular preparation and publication.

III. THE χ^2 DISTRIBUTION

The following paragraphs follow Cochran's (1952) cogent discussion of Pearson's (1900) paper on the χ^2 goodness of fit.

> "In the standard applications of the test, the n observations in a random sample from a population are classified into k mutually exclusive classes. There is some theory or null hypothesis which gives the probability, p_i, that an observation falls into the ith class (i = 1, 2, 3, . . . k). Sometimes the p_i are specified completely by theory as known numbers and sometimes they are less completely specified as known functions of one or more parameters, α_1, α_2 . . . whose actual values are unknown. The quantiles $m_i = np_i$ are called the expected numbers, where

$$\text{(a)} \quad \sum_{i=1}^{k} p_i = 1 \qquad \text{(b)} \quad \sum_{i=1}^{k} m_i = n \quad . \qquad (8)$$

"The starting point in the theory is the joint frequency distribution of the observed numbers x_i falling in the respective classes. If the theory is correct, these observed numbers follow a multinomial distribution with the p_i as probabilities. The joint distribution of the x_i is therefore specified by the probabilities

$$\frac{n!}{x_1!x_2!\ldots x_k!} \quad p_1^{x_1} \, p_2^{x_2} \ldots p_k^{x_k} \quad . \tag{9}$$

"As a test criterion for the null hypothesis that the theory is correct, Karl Pearson (1900) proposed the quantity

$$X^2 = \sum_{i=1}^{k} (x_i - m_i)^2/m_i = \sum_{i=1}^{k} (x_i^2/m_i) - n \quad ." \tag{10}$$

If the probabilities p_i remain fixed as $n \to \infty$, the limiting distribution of X^2 when the null hypothesis is true is the familiar χ^2 distribution,

$$\left\{1/((2^{\nu/2})\left\{((\nu/2)-1)\right\}!)\right\} (\chi^2)^{(\nu/2)-1} \, e^{-1/2(\chi^2)} \, d(\chi^2) \quad , \tag{11}$$

where ν is the number of degrees of freedom or the dimensions in χ^2. This particular distribution also is known to be that followed by the quantity shown in equation (3) and by the quantity

$$y_1^2 + y_2^2 + \ldots + y_\nu^2 \tag{12}$$

shown by many authors such as Cochran (1952) and Hald (1952a), and where the y_i are the standardized uncorrelated variates and are distributed normally and independently with zero means and unit variances.

Now, χ^2, as a constant, is the equation of a generalized "ellipsoid" over which the frequency of the deviations or errors is constant. Chi (χ), as does χ^2, ranges from zero to infinity. Chi is the vector radius or ray which sweeps out the ellipsoid in ν-dimensional space. If the ellipsoid is viewed with respect to its principal axes rather than its variate axes, it may be massaged, i.e., compressed or stretched along one axis at a time until it forms a sphere. Effectively, this is accomplished by standardizing the components along the axes by dividing them by the respective standard deviations, i.e., scaling the residuals. The principal axes are located as eigen, latent, or characteristic vectors. The respective eigenvalues, latent or characteristic roots, are the variances of the transformed variates along the respective principal axes. The covariances and correlations off the diagonal of the respective matrices are zero; i.e., these are uncorrelated. Bartlett (1934) discusses the vector representation of a sample. There the individual vector, of course, would be a "χ" as indicated previously.

Values of χ^2 through the first 15 degrees of freedom are given in Table 1. These values are abridged from Pearson's Tables (1900) with the permission of the publishers and trustees of the estate and from A. Hald and S. A. Sinkbaek (1950) with the permission of the authors and publishers. These same tabular values can be computed and printed by the electronic computer and its peripheral equipment. Such computation is done in the application of the chi-square tests in this paper. Many other texts serve as references for the χ^2 distribution. Cramér (1946) also offers a good discussion on the χ^2 distribution. Lancaster's (1969) text on the χ^2 distribution is superlative.

IV. TECHNIQUES OF THE TESTS

The tests considered here are:

1. X^2

2. χ^2 (Graph)

3. MSSR (Mean Sums of Squares of Residuals)

4. MAD (K-S) – Maximum Absolute Difference (Kolmogorov-Smirnov)

5. Signs Test

6. Runs above and below the line

7. Runs above and below the median.

Hald (1952a) suggests two procedures for the examination of an observed two-dimensional distribution for normality. These are the first two mentioned.

1. The computation and evaluation of X^2 per equation (6) for the ellipsoidal or circular shells.

2. The visual comparison of computed and ordered χ^2 values against the appropriate theoretical cumulative χ^2 distribution (graphs).

These will be discussed separately and in some detail for the multivariate case which includes the univariate case as a special case.

A. The Computation and Evaluation of X^2

In the previous discussion it may not be apparent that the test depends upon the application of "χ^2" in two forms. The quadratic form (χ^2) provides a form where each observation as a χ^2 value of which χ, the square root of χ^2, is the deviation of the observational vector measured from the centroid. Thus, if there are 90 observations, there are 90 χ^2 values or 90 vectors, each of which has direction and magnitude. Therefore, for each data set there is a set of vectors with a χ value representing the magnitude of each vector. Each χ value necessarily has been computed by means of the proposed model using computed statistics, estimates of the distribution parameter(s). The test now proceeds to check whether the vectors follow the proposed form.

Considerable discussion prevails in the literature as to how best to categorize the data or establish class intervals. Following the arguments of Pearson (1900), suggesting at least one vector or observation in each category, and the arguments of Cochran (1952) and Mann and Wald (1942), use as few in each category as possible and use a relatively large number of categories to ensure a power of the test of about one-half and thereby reduce the Type II error. Cochran also argues for allowing the number in an interval to dip as low as one. Most investigators use a less conservative approach and suggest 5 as a minimum number allowable in a category while others, still less conservative, suggest 10. Bates (1966), in providing a good discussion of the test for a bivariate distribution, provides a survey on these approaches. The less conservative the approach is, the greater is the Type II error.

The question arises as to whether the class intervals may be equal or unequal in terms of variate scale or probability. Pearson (1900), Mann and Wald (1942), Cochran (1952), and Hald (1952a) suggest the use of equiprobability class intervals. Vessereau (1958), Kempthorne (1967), and Roscoe and Byars (1971) discuss these problems. Figures 1a and 1b, respectively, schematically illustrate equal intervals of scale and probability. In future work the use of equiprobability intervals will be investigated, but in this report equal class intervals of scale will be used except where "pooling" is necessary to obtain at least five observations in each interval.

Mann and Wald recommend the following formula to determine the number of categories to use for the univariate distribution. This formula is

$$k_N = 4 \sqrt[5]{2(N-1)^2/c^2} \quad , \qquad (13a)$$

where "k_N" is the number of intervals based on "N", the number of observations or data points, and "c" is a constant. "c" is determined so that $(1/\sqrt{2\pi})\int_c^\infty e^{-x^2/2}dx$ is equal to a preselected critical region. For a rejection level of 0.05, c = 1.64; i.e., it is the normal variate value that satisfies the above integral.

Another formula is proposed by Sturges (1926). It is

$$k_s = 1 + 3.3 \log_{10} N \quad . \tag{13b}$$

Dahiya and Gurland (1973) provide further insight into the rather tenuous problem. They state, "Even for this case there is no unique answer and the best choice of 'k' depends on the alternative distribution." Their results do provide a range of choices of 'k' for several different alternatives in testing for normality. Their formula maximizes the power, thereby decreasing the Type II error. According to Mann and Wald (1942), if the number of intervals used is $k = k_N$, the power of the test is equal to or greater than one-half for all the alternatives. Therefore, the probability of a Type II error is equal to or less than 0.50. If k is not equal to k_N, there is at least one alternative where the Type II error is less than 0.50.

Some investigators suggest that the number of intervals can be one-half of k_N without seriously impairing the test. Hamdan (1963) implies that the use of a large number of intervals (categories) as obtained by the formula of Mann and Wald perhaps is not as good as a lesser number, for example 10 to 20.

Tate and Hyer (1973) indicate that the power of the chi-square probabilities is weakened when the expectations are small. They also indicate that increasing k, the number of class intervals, does not increase the power of the test. This argument parallels and supports that of Mann and Wald and Hamdan.

Another formula, commonly used and incorporated in the computer program in this study, is given by Mills (1955), Brooks and Carruthers (1953), and Panofsky and Brier (1968). It is

$$K_N = 5 \log_{10} N \quad . \tag{14}$$

For a given N, $K_N < k_N$.

If the multinomial distribution or the multivariate distribution, both in terms of hypercubes, is used and the dimensions are high, the number of cells or hypercubes is great; the probability for each hypercube, especially in terms of equal probability cubes, must be low. This requires a large number of observations for resolution. Techniques to compute the probability for the two-dimensional cases are adapted from the work of several investigators. Tables to do this are available in the Bureau of Standards publication (1959) and from Owen (1956, 1962). Crutcher (1962) provides some examples of the use of the tables. Milton (1970) provides techniques and procedures for the multivariate case.

Spherical shells, circular annular rings or area bands, or even rectangular or square shells or cubical shells are special cases of the ellipsoidal shell, also called the hypercube. Figure 1b shows the univariate distribution where the matched hachured areas are added. Here, the normal distribution is divided into 10 shell areas where the matched hachured area represents one shell representing a probability of 0.10. In the usual sense, as often employed in making the χ^2 test, each matched hachured area would be separated into its two parts, providing 20 smaller equiprobability sections, each of 0.05 probability; the two matched hachured areas would be a shell in one dimension.

Cochran's (1952) extensive experience provides us with ideas of what to do for attribute and continuous data. For continuous data, such as will be used in examples in this report, Cochran (1954), following Williams (1950) for the one-dimensional case, proposes an expected value of 12 per shell for n = 200, 20 per shell for n = 400, and 30 per shell for n = 1000. At the tails, pool (if necessary) so that the minimum expectation is five. Since the number of data may be small, the major operation will be to broaden the shells, especially the outer shell, such that each shell has an expectation of at least five.

For a given number of shells, the degrees of freedom available decrease with increasing dimensions. The available degrees of freedom are k-p-1 where k is the number of shells (categories), p is the number of parameters fitted, and the "1" represents the normalizing of the volume (area) to equal that under or within the normal surface. The number of parameters fitted is the number of means and covariances estimated. For the one-, two-, and three-dimensional cases, there are 2, 5, and 9 means and covariances, respectively, providing k-3, k-6, and k-10 degrees of freedom. Thus, 10 equiprobability shells for a one-dimensional case provide 7 degrees of freedom for testing. The general formula for calculation of the degrees of freedom is $k-((\nu+1)(\nu+2)/2)$.

In the case of two dimensions, the shells (or categories) may be elliptical cylinders bounded on the bottom by the bivariate plane and on the top by the bivariate frequency surface. Figure 3d may be viewed for concept. It does show equiprobability concentric cylindrical shells. In the case of three or higher dimensions, the shells (hyper-shells) are concentric around the centroid. Again, for concept, examine Figure 5. It, too, does not necessarily show equiprobability shells.

If 10, 20, or 25 equiprobability intervals are considered that allow for equiprobability intervals of 0.10, 0.05, or 0.04, respectively, then all that is required is simply to compute the χ or χ^2 values serving as limits of the equiprobability levels and distribute the computed χ or χ^2 values of the data set. However, it is not required that the intervals be equal in either scale or probability. If graphical procedures and visual counting are used, it is the fractional multiples of the χ which are used as vector radii of the ellipses or circles. Squares, cubes, rectangles, or hypercubical or rectangular shells can be used. The shapes need not be specified, but the user must be able to compute the expected and observed values within each category. Ordinarily, the circles (ellipses) or squares are the easiest to use. However, with the computer, the appropriate χ or χ^2 values and their inverse values for any number of sets or intervals may be computed and used. The differences between the expected and observed number of χ or χ^2 values in an interval are squared and this is divided by the expected. Here, χ^2 values are computed and used, which bypasses the necessity to compute the χ values and the necessary square roots of the χ^2 values used as boundaries of its equiprobability shells. As indicated, graphical procedures require the χ values. These results then are summed as indicated in equation (10) and checked against the appropriate degrees of freedom in Table 1 at the preselected probability level for testing. If a computer is used, this can be included in the program. Here, α, the probability level selected for rejection, is 0.05. The theoretical value of α for the X^2 criterion is computed and printed.

Application of the usual χ^2 test to the X^2 statistic is fraught with difficulties. A small number of observations and a large number of dimensions (variates) decrease the chance of each hypercube (sphere shell) containing observations as the number of hypercubes increases. The expected frequency in a hypercube cannot be less than zero, so for a starter the number of observations should at least equal the number of hypercubes. The minimum count expected in each hypercube would be one. The resulting χ^2 test would be very conservative.

With the degrees of freedom (d.f.) equal to $(k\text{-}((\nu+1)\ (\nu+2)/2))$ and with the dimensions ν known, k, the number of shells, may be determined. For any test the d.f. must be one or greater. If k is set equal to $5 \log_{10} n$, for a degree of freedom equal to one; n must be greater than 7, 25, 159, 1585, and 25, 119 for dimensions 1, 2, 3, 4, and 5, respectively. For higher degrees of freedom, the number of observations is greater. For 2 degrees of freedom the numbers required are, respectively, 10, 40, 252, 2512, and 39, 811. With X^2 then computed and with the d.f. known, the significance level can be computed. The investigator then can make the decision to reject or not to reject on the basis of a previously selected significance level. For example, if $\alpha = 0.05$, any probability level less than 0.05 automatically calls for rejection. The program permits the option of selecting the level of rejection, α, prior to computation so that the decision making process stays as honest as possible. Both the computed significance level and the prior selected level are printed.

Only the χ^2 distribution is utilized so there is no test of symmetry. These χ^2 tests are deficient in this respect. To this extent these tests may be considered necessary but, perhaps, not sufficient. However, the use of the χ^2 test for goodness of fit is well established. Its insufficiency in this regard does not negate its general worth as a goodness of fit test. Later sections provide for other tests.

B. Comparison of Empirical Chi-Squared Values Against the Cumulative χ^2 Distribution

Pearson (1900) provides theoretical cumulative distribution curves. Hald (1952b) gives tabular presentation of both the χ^2 variate and its inverse form. The discussion now follows that of Hald (1952a) for the two-dimensional form. Extension is made to the multivariate form.

On semilogarithmic paper the theoretical χ^2 values may be plotted against the theoretical probabilities provided by the χ^2 model. Manually, these values are obtained most easily by inverse χ^2 tables such as Table 1. With electronic data computer program modules and with appropriate peripheral equipment, the χ^2 values and their respective theoretical probabilities may be computed and graphed. Andrews, Gnanadesikan, and Warner (1973) show a graphing procedure for the χ^2 values.

The distribution function of χ^2 depends only on the degrees of freedom, f. The probability that χ^2 belongs to the interval $(\chi^2, \chi^2 + d(\chi^2))$ as shown by Hald (1952a) is

$$p\left\{\chi^2\right\} d(\chi^2) = 2^{-f/2} (\Gamma(f/2))^{-1} (\chi^2)^{(f/2)-1} e^{-(\chi^2/2)} d(\chi^2) \quad , \quad 0 \leqslant \chi^2 < \infty \quad . \tag{15}$$

As indicated and shown by Hald (1952a) for the two-dimensional distribution (ν=2), the theoretical line passes through the point (0,1) (chi-square versus logarithm of probability with a slope of -0.217 for common logarithm and -0.5 for natural logarithm). For χ^2 equal to zero, 1 is absolute certainty or a probability of 1.00. Figure 6 shows the theoretical lines for ν = 1,2,3, . . . ,12 that respective ν-dimensional distributions may be expected to follow. Hald's example (1952a) is an example of a two-dimensional distribution. The ordered calculated χ^2 values plotted against the empirical probabilities obtained from 1.00 - ((i-(c))/n), where c is set to 0.5, shown by Hald show a pattern apparently randomly scattered about the straight line. This formulation will be discussed in more detail in a later section.

If the empirical plot appears to be distributed more or less randomly about the theoretical line, the investigator may reasonably infer that the data from which the sample was obtained are distributed normally. However, this satisfies only an intuitive feeling based on experience. Although such decisions are valid, it is comforting if some helpful numerical or graphical decision process can be evolved and developed. The example that Hald employs appears to serve the decision to not reject the null hypothesis. No quantitative test is provided.

It is the purpose of this section to develop tables and graphs which may be used to help make decisions as to the ν-variate normality of distributions.

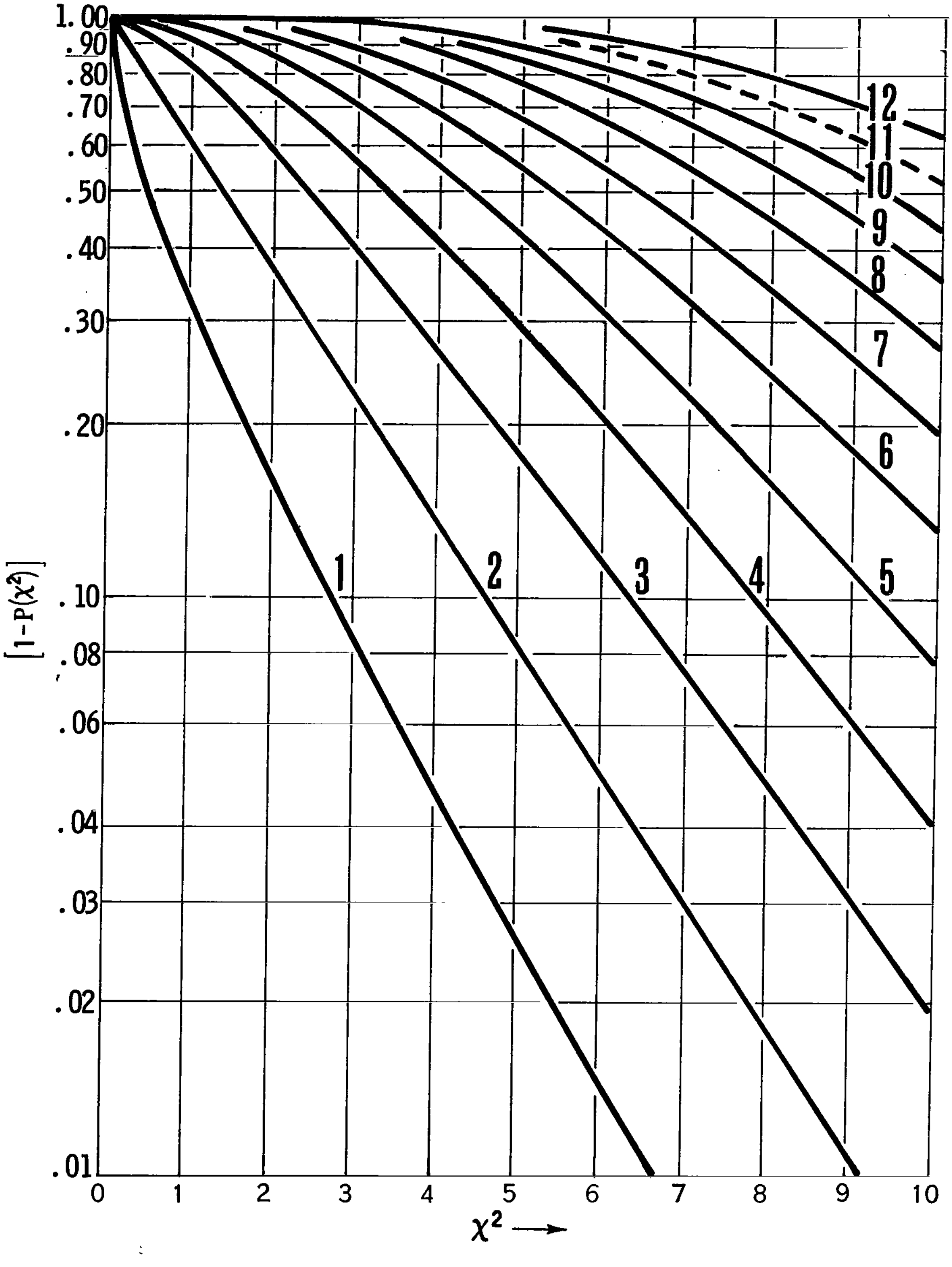


Figure 6. Plot of χ^2 versus $[1\text{-}p(\chi^2)]$ prepared from data given in Table 1 (Note straight line for f=2 with intercept (0,1) and slope -0.5; slope is -0.217 if common logs are used; f is degrees of freedom for chi-square and ν for dimensions).

Sets of vectors, in terms of components, are generated randomly from a random number generator with presumably no correlation between components. It is not important whether there is correlation. It is important that there be no autocorrelation. One hundred sets of such vectors in subsets of 5, 7, 11, 15, 21, 31, 51, and 101 are computed for ν = 1, 2, 3, 4, and 5. Odd numbers in the samples permit easier selection of the medians. Observed χ^2 values are computed for each vector and ordered and plotted against their empirical probabilities for comparison against the ν-dimensional theoretical χ^2 and theoretical probability values.

C. Random Number Generators

No random number generator is truly random. Each is a pseudorandom generator. Any reference in this paper to a random number generator implies a pseudorandom generator. The idea is to produce a sequence of integers or numbers which, in spite of being produced by a fixed procedure, will serve as random variables in computer simulations.

As fixed procedures are used, the sequence of numbers produced by any random number generator will repeat. The length of interval over which nonrepetition of any pattern occurs is in a sense a measure of the goodness of the generator. Given an initialization number, the sequence generated exists in some hyperplane, hypercube, or hypersphere of space. Changing of the initialization numbers is, in a sense, a change of the hyperspace. If a series of random number generators can be so perturbed, or perturbed in other ways, then the final output may be expected to exist in a more truly random fashion in hyperspace.

The random number generator used here by permission is the Univac-Stat-Pack in the 1108 Computation Library of the Marshall Space Flight Center (MSFC). The generator first generates a random number over the interval 0 to 1; i.e., R(U) is the uniform variate $(0.00 < R(U) \leqslant 1.0)$. From a sequence of R(U) numbers a random normal number R(N) is generated as $R(N) = [\Sigma R(U) - (N/2)]$, where the summation runs from 1 to N. If N is chosen as 12, the distribution of the output has an expected mean of zero and a variance of 1. Doubling of the summation sequence from 12 to 24 simply doubles the variance or increases the standard deviation by the square root of 2. It was found during the course of this work that this generator has the unhappy facility of truncation; i.e., an insufficient number of large deviations is produced. However, after much work with the rather extensive output, it was decided to not rerun the problem with a new generator. The truncation problem, though unresolved, did not create an untenable situation.

Following are a few brief comments on other random number generators. One is selected which will be used if this work is to be extended.

Box and Muller (1958), Hull and Dobell (1962), Knuth (1969), Marsaglia (1972), and Marsaglia, Anathanarayanan, and Paul (1972) are among the many who have worked on the problems associated with random number generators. Marsaglia's work is found to be good. He indicates that congruential generators are not suitable for precision Monte Carlo use. Some means should be used to perturb the generator so as to destroy its gross lattice structure in ν-dimensions. Through the years Marsaglia and his colleagues have developed such a perturbed generator. Many universities now utilize this program. This is known as "The McGill Random Number Package 'Super Duper'." It has been tested by the senior author and by Mr. R. L. Joiner of the National Climatic Center (NCC) and will better fulfill our future requirements.

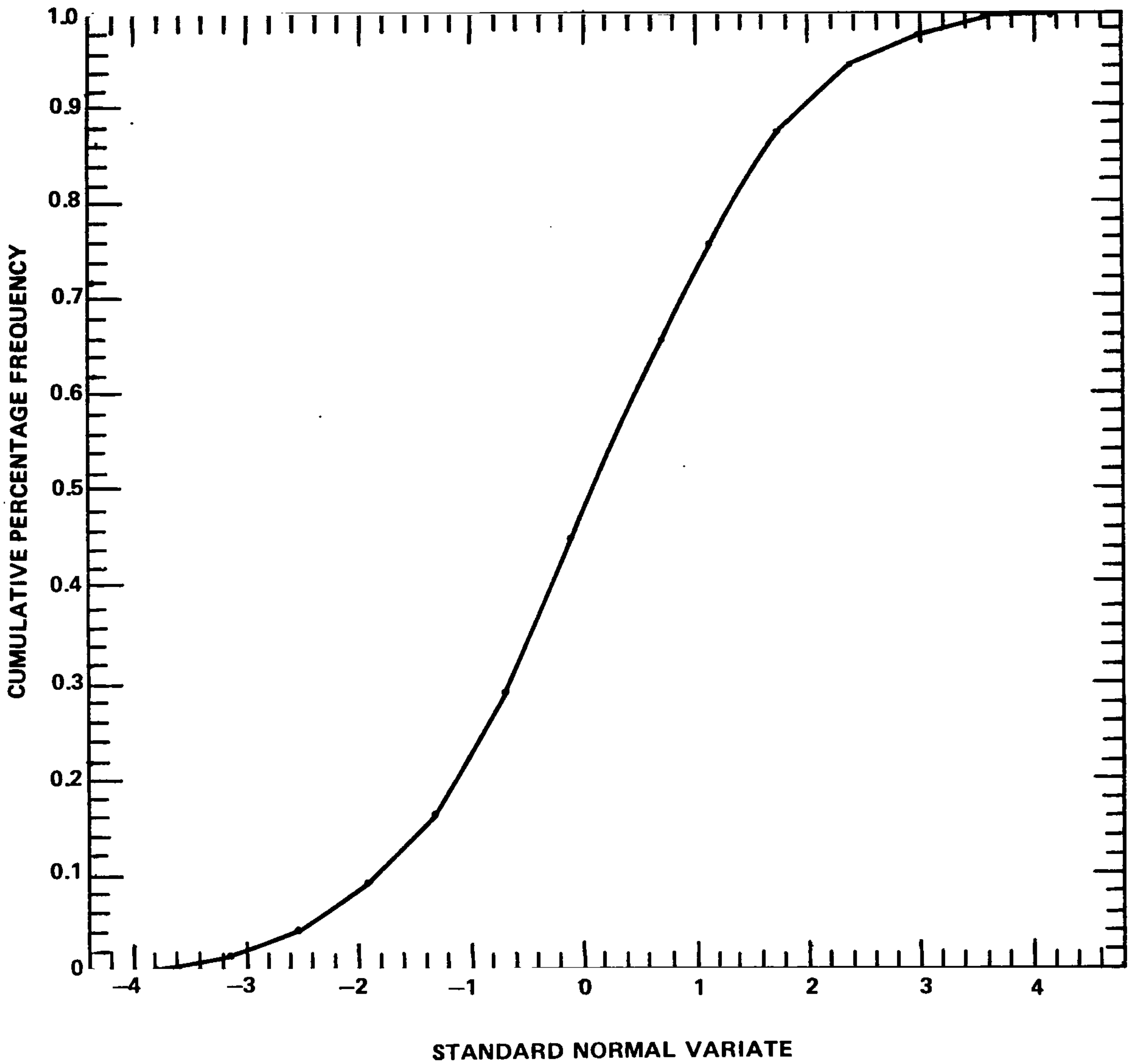
Figures 7a, b, and c show processing output of a sequence of 1000 random numbers generated by the Univac-Stat-Pack generator. This is a congruential random number generator. Some perturbing was done by using different initializing numbers for each subsample. It was modified to permit extension of the range from ±6 to ±12 standard deviations. Figure 7a shows the distribution of normal deviates versus their percentage frequency. Figure 7b shows the lag correlation graph correlogram of data out to 100. Figure 7c shows a spectrum analysis of the same set of data with confidence bands. The spectrum is white; i.e., no pasteling is evident.

Figure 8 further illustrates the output of the random number generator. Pairs of uniform random numbers are plotted pairwise. Each dot represents a pair. No pattern is evident to the authors. This does not imply that there is no pattern. Comparison of this plot to a plot of the "Super Duper" output at the NCC indicated that the output of the Univac-Stat-Pack package would serve our purposes. Obviously, to do otherwise would require an extensive and expensive rerun of the problem. A further check using the Box-Muller technique supported this decision.

Figure 9 further illustrates the output of the random number generator where the paired values are normal. The distribution appears to be a zero mean cluster or swarm with an expected high density near the center. There are 10,000 data points. For purposes of this study, except for the slight truncation error, the output of the Univac-Stat-Pack normal random number generator is considered to be sufficient. Preference in future work will be given to the McGill University "Super Duper" program.

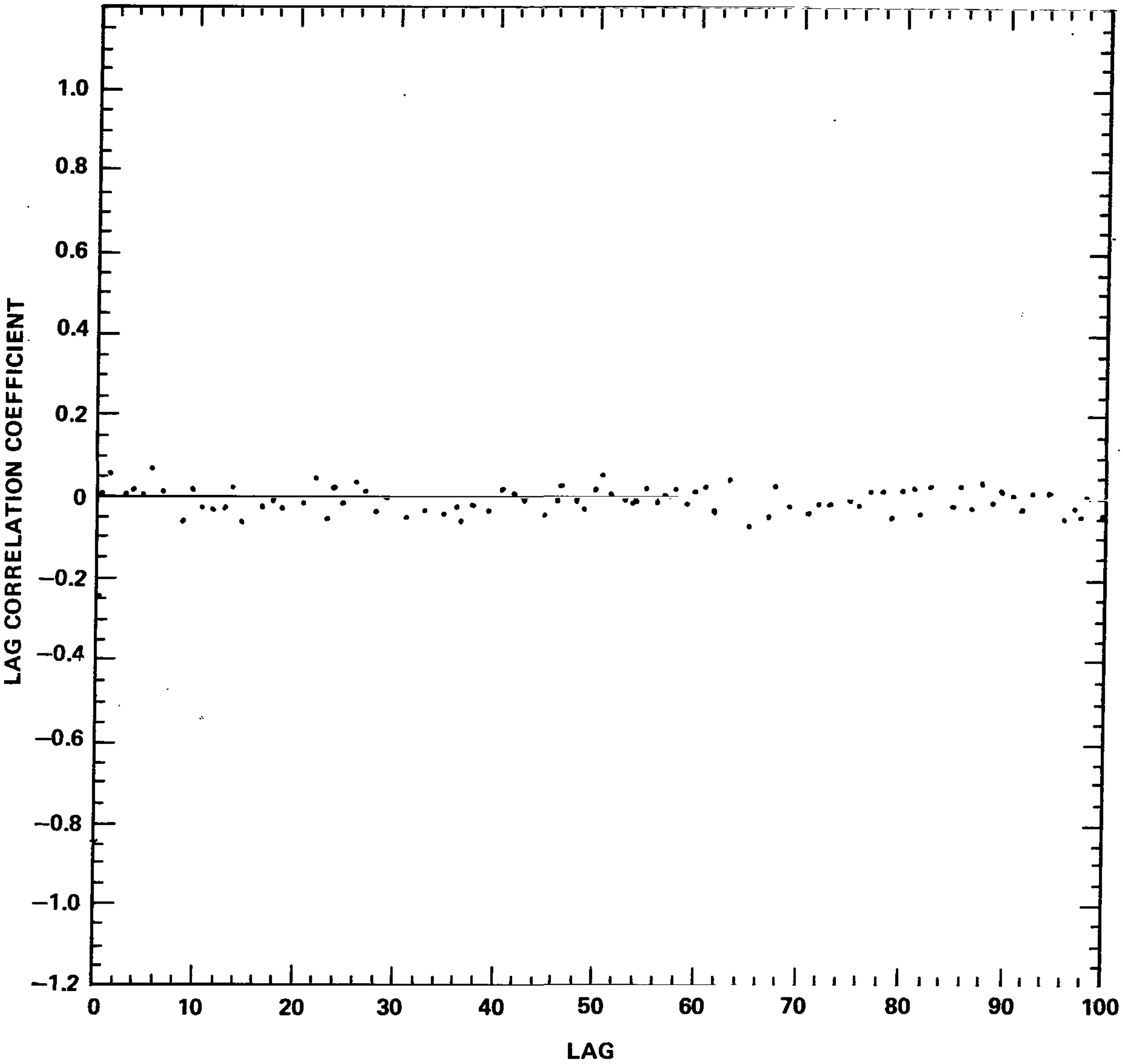
D. Selection and Processing of Samples with Zero Means and Zero Covariances

The random number generator described previously produces a sample from a univariate normal distribution. Pairs of these numbers produce a sample of a bivariate normal distribution. Triplets, quadruplets, and quintuplets form trivariate, quadrivariate, and pentavariate distributions. Combinations of still more will produce the respective multivariate distribution. This report treats distributions only through the pentavariate.



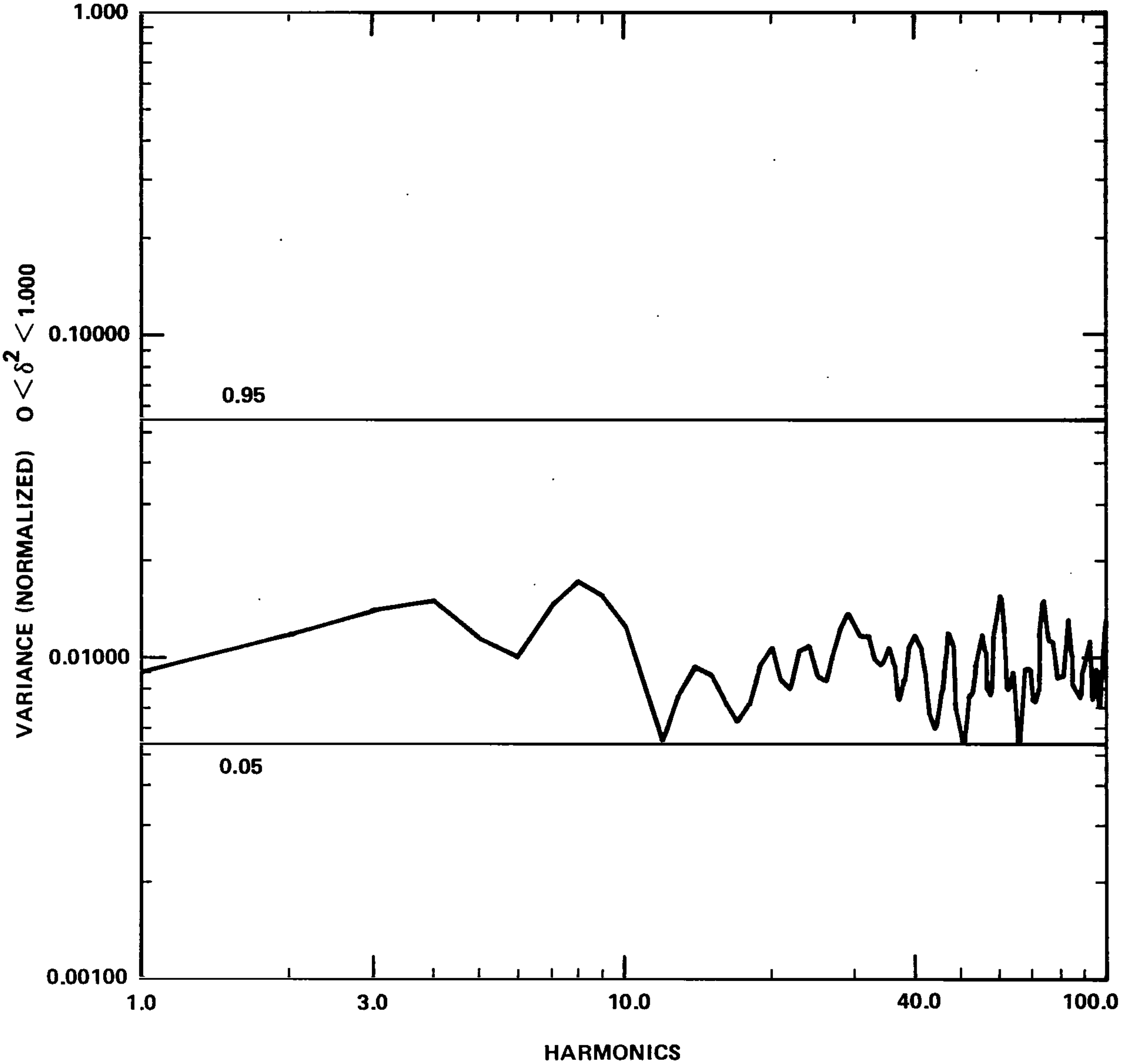
a. Cumulative distribution of 1000 univariate normal data points.

Figure 7. 1000 data points generated by the Univac Stat-Pack random number generator.



b. Lag correlation coefficients for a sequence of 1000 normal data points

Figure 7. (Continued).



c. Spectral density plot for 1000 normal data points.

Figure 7. (Concluded).

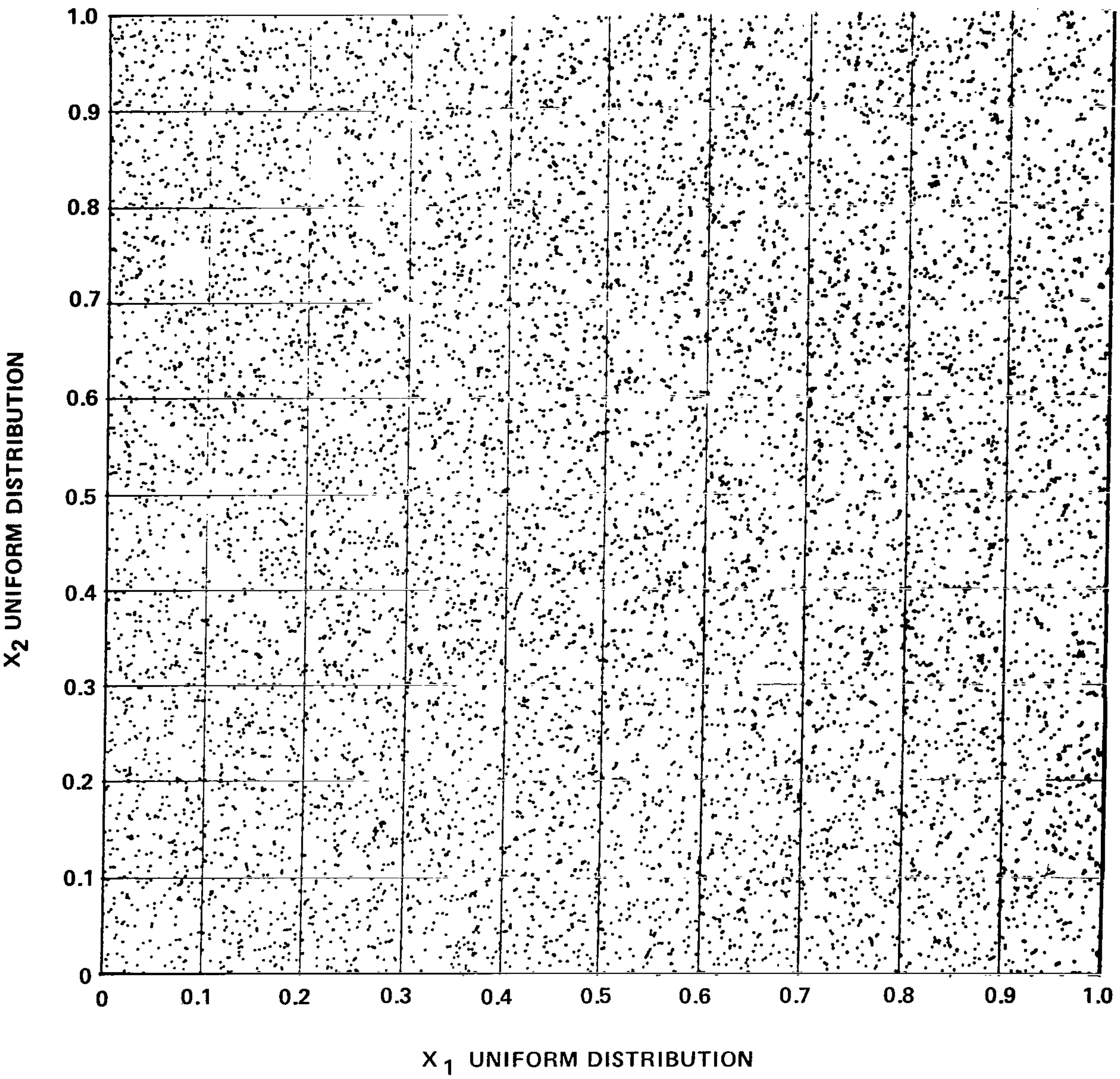


Figure 8. Two-way plot of pairs of uniform random numbers generated by the pseudorandom number generator of the Univac Stat-Pack (10 000 points).

Since the mean of the distribution in each case from which the sampling is made is zero, the medians of the samples should follow the curve of chi-square values given in Table 1 and illustrated in Figure 6. In actuality, only with large samples will this be an exact condition. A calculated mean of zero, diagonalized unit (identity) covariance, or correlation matrix will be that exception. It is important to discuss this feature before proceeding to sample means and sample covariances in application. The assumption is

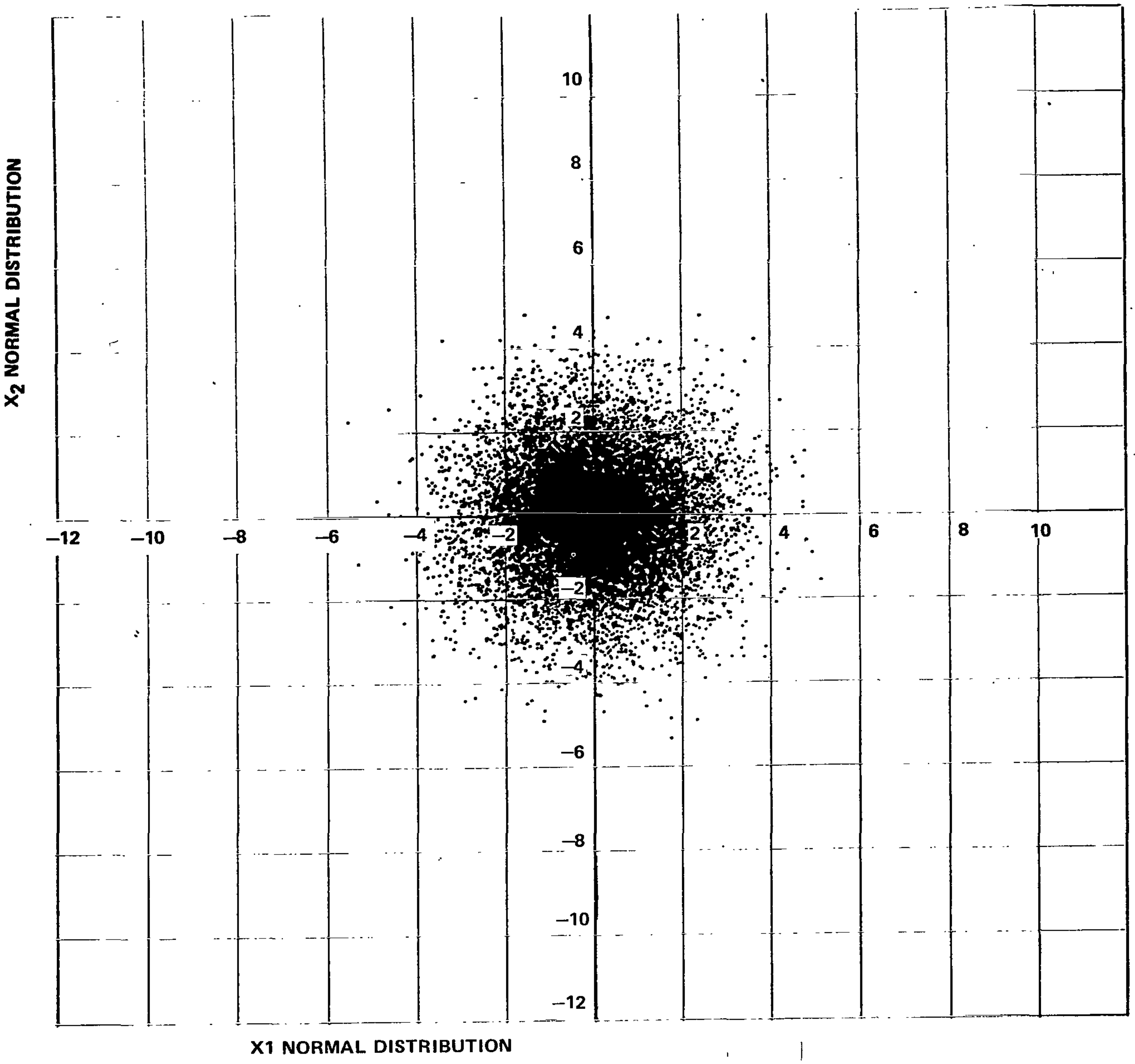


Figure 9. Two-way plot of bivariate normal distribution (NID(0, 1)) generated by the Univac Stat-Pack random number generator (10 000 points; range of the components is 24 standard deviations).

made, for purposes of the following discussion, that any sample in ν-dimensions obtained by means of a random number generator will have a mean of zero and that the actual correlation is zero; i.e., the distributions are central spherical. Confidence bounds are empirically determined by the computation of medians of percentile values of computed chi-squares. The chi-squares are the squares of the vector radii. Various sample sizes of 5, 7, 9, 11, 15, 21, 25, 31, 51, and 101 were computed for dimensions 2, 3, 4, and 5. Only two of these tables are shown here. One hundred samples of each sample size were determined, and selected percentile values were obtained. This procedure was repeated 100 times and the median value selected and printed.

Table 2 provides the median values of selected percentiles of 10 100 random chi-square values of the multivariate distribution with zero means and zero covariances (N1D(0,1,0)) and their probability plotting positions, p = (1-((i-0.5)/n)) for three dimensions and a sample size of 7. In Table 2 and subsequent tables ν is dimension and n is sample size. The chi-square values are computed as $\chi_0^2 = \Sigma x_i^2$, i = 1, 2, 3, where x_i is a random number from a random normal generator, with a mean of zero, a variance of 3, and with covariances equal to zero. In other words, sample variances, sample covariances, and sample means are not computed and used in this procedure.

The values shown in Table 2 were obtained as follows for a sample size of 7:

1. A sample of seven chi-square values was obtained.

2. The sample values were ordered from low to high.

3. This sample was set aside.

4. Steps 1, 2, and 3 were repeated to obtain 100 samples of size 7, each ordered from low to high. A two-dimensional matrix of 7 columns and 100 rows then was available.

5. The values in each column then were ordered from low to high so that selected percentiles in each column would be selected. The second number in each column would be the second percentile.

6. Steps 1 through 5 were repeated to obtain 101 similarly formed two-dimensional 7 by 100 matrices.

7. A selected percentile value in the ordered column, for example the second percentile value in the first ordered column, then was available for each layer of a stack of 101 two-dimensional matrices. The number 101 and the odd numbered sample sizes 7, 11, and others were used to permit easy selection of median values. The stack of 101 second percentile values was ordered and the median values were selected. It is this median value which appears in Table 2 and other similar tables. It is this value which is selected and plotted as an example in Figure 10.

The various percentile values shown in Table 2 may be used to establish confidence bands. For example, the 2nd and 98th percentile values establish the central 0.96 probability confidence band. A set of seven ordered chi-square values computed from such a three-dimensional distribution as described previously, when viewed against the probability plotting points, should fall within the limits. If this is true, then the null hypothesis that the population from which the sample was taken is not different from the three-dimensional multivariate normal distribution need not be rejected. The level of significance is 1.00 - 0.96, or 0.04.

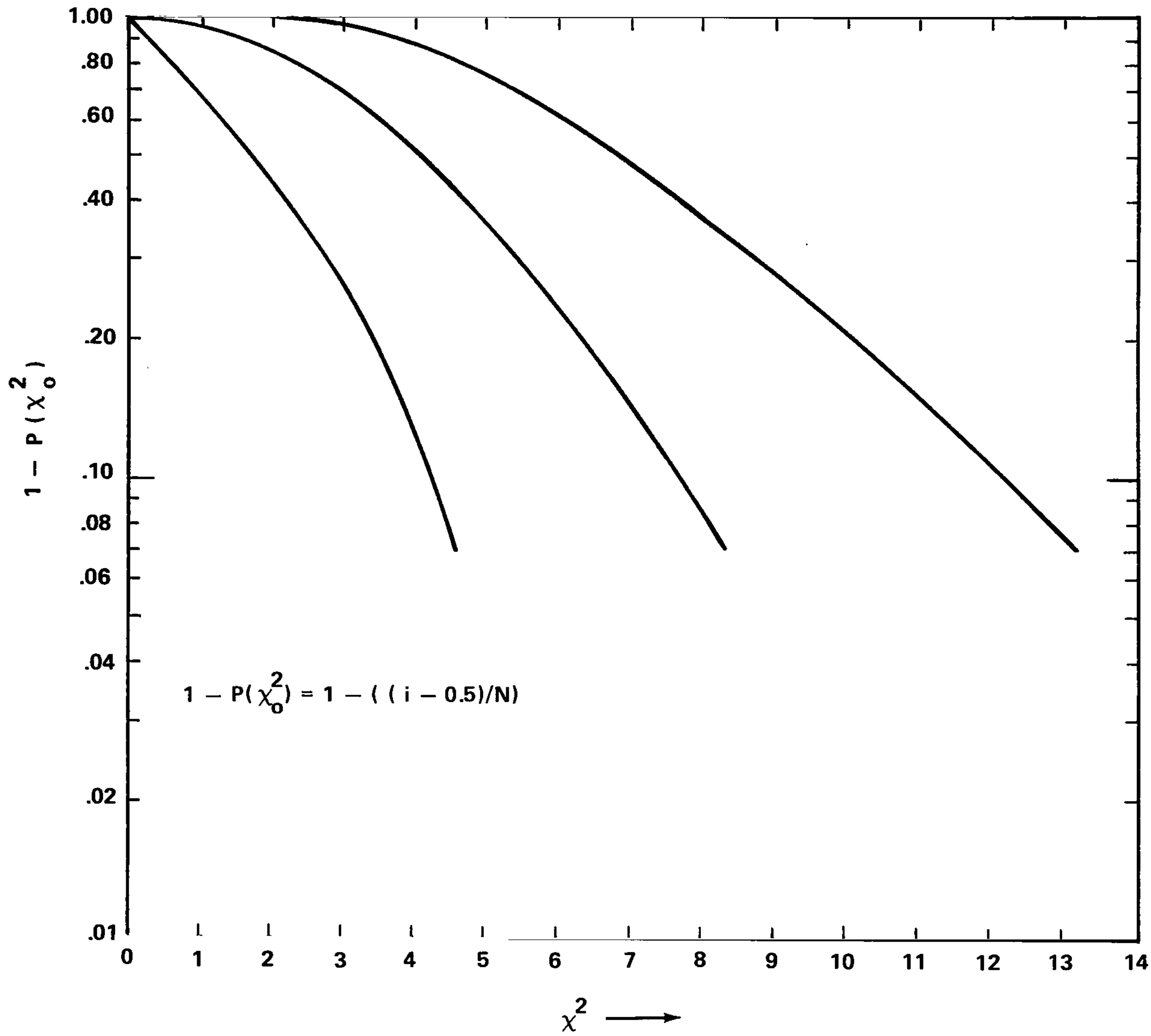


Figure 10. Three-dimensional sample size 7 median values of the 2nd, 50th, and 98th percentile values of 10 100 random chi-square values of the multivariate normal distribution with zero mean, variance one and covariance zero (These provide the 96 percent central confidence band).

Figure 10 presents the 0.02, 0.50, and 0.98 probability levels, giving the expected central curve and the central 0.96 confidence band. Table 2 and Figure 11 illustrate the same concept with the larger sample size of 31. Here, some symmetry is evident in the flaring of the confidence band. A difference between the central median curves and the corresponding curves in Table 1 or Figure 6 signifies the departure of the distribution from the normal distribution even though the mean is specified as zero. Even more marked will be the departure (the Wishart distribution) from the normal for small samples.

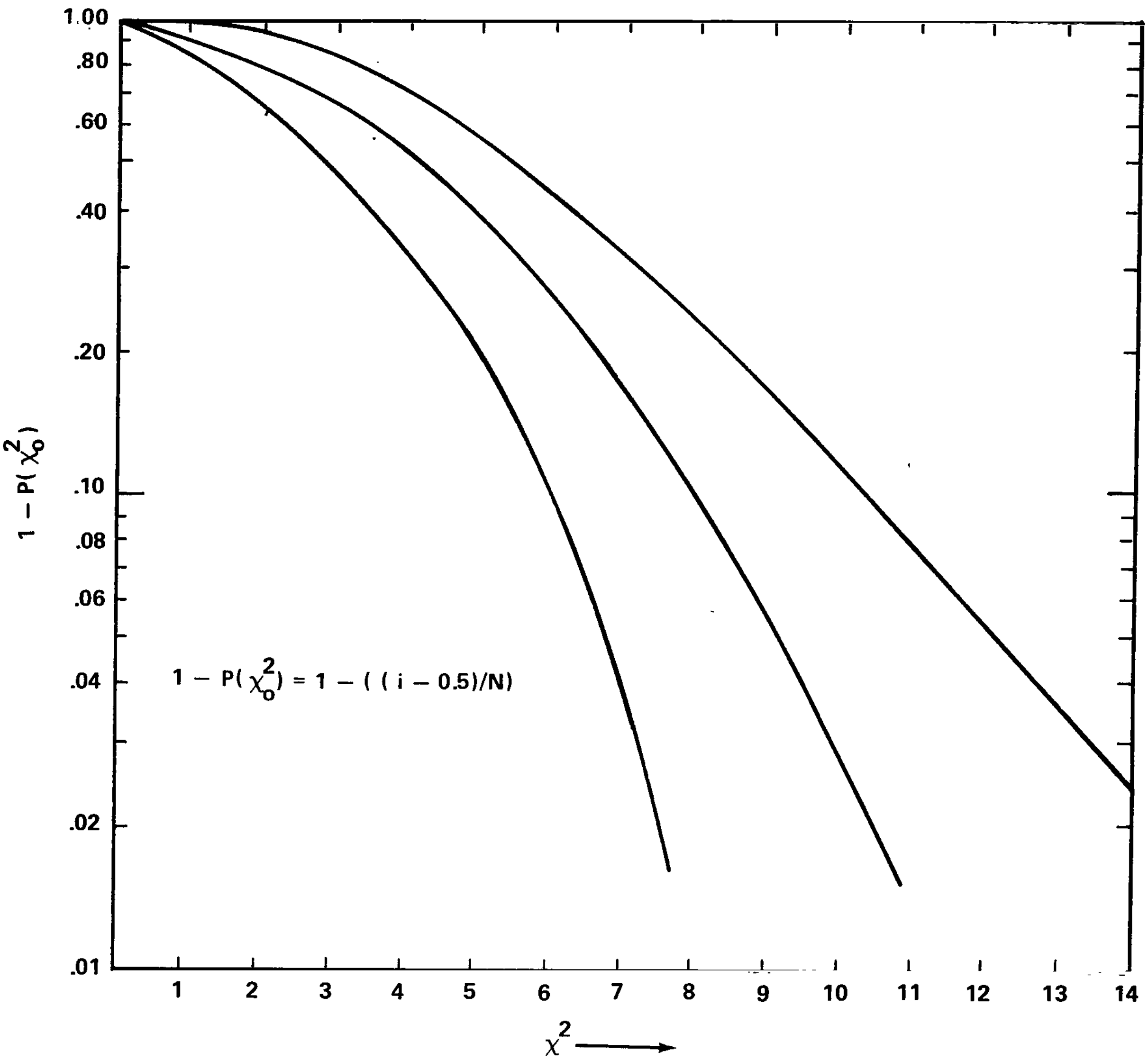


Figure 11. Three-dimensional sample size 31 median values of the 2nd, 50th, and 98th percentile values of 10 100 random chi-square values of the multivariate normal distribution with zero mean, variance one and covariance zero (These provide the 96 percent central confidence band).

The problem faced by most researchers, including the authors, deals with small samples and populations for which the means, variances, and covariances are unknown. It was these problems and the lack of suitable techniques to adequately answer questions dealing with these problems that led the authors to develop and prepare this report. Hopefully, it will be useful to others.

E. Selection and Processing of Sample Data (General)

Following are the final instructions to the electronic computer programmer after viewing the results of several generators.

1. Select 100 samples each of 5, 7, 11, 15, 21, 25, 31, 51, and 101 sets of random normal variables using a random normal variate generator with zero mean, variance 1, and covariance zero with a range of ±12 s. It is important here to note the difference in approach in subsection D and this subsection. Subsection D assumes and uses no sample correlations. This subsection assumes and uses the sample correlations though the correlations presumably are not different from zero, having been obtained from uncorrelated and independent random numbers. However, in the real world of sampling, there may or may not be correlation among the variates. In this study these correlations are used even though a Type II error may be committed. That is, we may be using a non-zero correlation even though correlations should be zero. Also, in the real world of sampling, non-zero means may be obtained. Here, it is assumed that the data will be distributed about the sample mean in the functional form of the basic underlying distribution. Of course, Hotelling's (1931) T^2 test could be used to test the difference of the mean from zero. However, this test is not performed since it is believed that the empirical tables developed here will assist in decision making. In this report we are testing for multivariate normality. For small samples, the distribution is the multivariate Wishart. The data sets selected from the random normal generator will be dimensional one through five. Retain the random normal variables in the order in which they are selected. For example, prepare sets for the two-dimensional distribution by taking sequential numbers, i.e., (first, second), (third, fourth), (fifth, sixth), etc. There will be 100 samples of 5 pairs, 100 samples of 7 pairs, . . . , 100 samples of 101 pairs. Assign each of the 100 samples an I.D. number ranging from 1 to 100.

Include in this part 1 options to select one variate, two variates (pairs), three variates (triplets), four variates, and five variates. Five variates are the maximum for this study. The first paragraph of part 1 describes the procedure for generating 100 samples each of 5, 7, 11, 15, 21, 31, 51, and 101 pairs (two variates) of random normal variables. For triplets (three variables), the random normal variates again are selected in the sequential order that they were generated, i.e., (first, second, third), (fourth, fifth, sixth), (seventh, eighth, ninth), etc. Follow this same procedure for four variates and five variates. There are five options in part 1 which are:

Option 1: One-dimensional case (one variate) designated as X_1.

Option 2: Two-dimensional case (two variates or "pairs") designated as (X_1,X_2).

Option 3: Three-dimensional case (three variates or "triplets") designated as (X_1,X_2,X_3).

Option 4: Four-dimensional case (four variates) designated as (X_1, X_2, X_3, X_4).

Option 5: Five-dimensional case (five variates) designated as $(X_1, X_2, X_3, X_4, X_5)$.

For example, follow the procedure below for the 100 samples of 11 pairs (option 2, two dimensions, n = 11). Use this same procedure for the other options.

2. For each of the 100 samples of 11 pairs compute the following parameter estimates where n = 11:

$$\bar{X}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i \ , \quad \bar{X}_2 = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{16}$$

$$s^2_{X_1} = \frac{n \sum\limits_{i=1}^{n} X_i^2 - \left(\sum\limits_{i=1}^{n} X_i \right)^2}{n(n-1)} \ , \quad s^2_{X_2} = \frac{n \sum\limits_{i=1}^{n} X_i^2 - \left(\sum\limits_{i=1}^{n} X_i \right)^2}{n(n-1)} \tag{17}$$

$$s_{X_1} = \sqrt{s^2_{X_1}} \ , \quad s_{X_2} = \sqrt{s^2_{X_2}} \tag{18}$$

$$r_{X_1 X_2} = \frac{n \sum\limits_{i=1}^{n} (X_1)_i (X_2)_i - \sum\limits_{i=1}^{n} (X_1)_i \sum\limits_{i=1}^{n} (X_2)_i}{n(n-1)\, s_{X_1}\, s_{X_2}} \ . \tag{19}$$

Equations (16) through (19) provide estimates of the population parameters, μ_{X_1}, μ_{X_2}, $\sigma^2_{X_2}$, $\sigma^2_{X_2}$, and $\rho_{X_1 X_2}$.

Equation (19) gives the correlation coefficient $r_{X_1 X_2}$ between the variables X_1 and X_2. The general equation for correlations is

$$r_{X_\ell, X_j} = \frac{n \sum\limits_{i=1}^{n} (X_\ell)_i (X_j)_i - \sum\limits_{i=1}^{n} (X_\ell)_i \sum\limits_{i=1}^{n} (X_j)_i}{n(n-1)\, s_{X_\ell}\, s_{X_j}} \ . \tag{20}$$

When ℓ = j, then $r_{X_\ell X_\ell}$ or $r_{X_j X_j}$ = 1; i.e., in the following correlation matrix, all diagonal elements equal 1. (Note ℓ and j = 1, 2, 3, 4, 5 to include all options.) Five dimensions are the maximum used in this paper. Using equation (20), form the correlation matrix C as follows:

Bivariate Case

$$C = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} \end{bmatrix} \qquad (21)$$

Note that this matrix is a symmetric matrix with $r_{12} = r_{21}$, $r_{13} = r_{31}$, etc. Consequently, our notation will be confined to the upper right triangle above and including the diagonal elements of the correlation matrix. Solve for the determinant of C; call this determinant D.

Now, the general equation for χ_0^2 is

$$\chi_0^2 = \frac{1}{D}\Bigg[R_{X_1,X_1}\left(\frac{X_1-\bar{X}_1}{s_{X_1}}\right)^2 + R_{X_2,X_2}\left(\frac{X_2-\bar{X}_2}{s_{X_2}}\right)^2 + \ldots R_{X_5,X_5}\left(\frac{X_5-\bar{X}_5}{s_{X_5}}\right)^2$$

$$+ 2R_{X_1,X_2}\left(\frac{X_1-\bar{X}_1}{s_{X_1}}\right)\left(\frac{X_2-\bar{X}_2}{s_{X_2}}\right)$$

$$+ 2R_{X_1,X_3}\left(\frac{X_1-\bar{X}_1}{s_{X_1}}\right)\left(\frac{X_3-\bar{X}_3}{s_{X_3}}\right) + \ldots 2R_{X_1,X_5}\left(\frac{X_1-\bar{X}_1}{s_{X_1}}\right)\left(\frac{X_5-\bar{X}_5}{s_{X_5}}\right)$$

$$+ 2R_{X_2,X_3}\left(\frac{X_2-\bar{X}_2}{s_{X_2}}\right)\left(\frac{X_3-\bar{X}_3}{s_{X_3}}\right) + \ldots 2R_{X_2,X_5}\left(\frac{X_2-\bar{X}_2}{s_{X_2}}\right)\left(\frac{X_5-\bar{X}_5}{s_{X_5}}\right)$$

$$+ 2R_{X_3,X_4}\left(\frac{X_3-\bar{X}_3}{s_{X_3}}\right)\left(\frac{X_4-\bar{X}_4}{s_{X_4}}\right) + \ldots 2R_{X_4,X_5}\left(\frac{X_4-\bar{X}_4}{s_{X_4}}\right)\left(\frac{X_5-\bar{X}_5}{s_{X_5}}\right)\Bigg] , \qquad (22)$$

where $R_{X_iX_j}$ is the cofactor of r_{ij} (the elements of correlation matrix C), Whittaker and Robinson (1954).

3.a. The option chosen in part 1 of this subsection will determine the dimensions to be used in the general equations. As before, we will use option 2 (two dimensions, 100 samples of 11 pairs) for illustration.

b. For n = 11, compute $\bar{X}_1$, $\bar{X}_2$, s_{X_1}, s_{X_2}, r_{11}, r_{12}, r_{21}, and r_{22} using equations (16), (17), (18), and (20). Note that $r_{X_1,X_2} = r_{12}$, $r_{X_1,X_3} = r_{13}$, $r_{X_1,X_4} = r_{14}$, $r_{X_2,X_3} = r_{23}$, $r_{X_2,X_4} = r_{24}$, etc., in matrix (21).

c. Form the correlation matrix C. For the two-dimensional case (bivariate) this will include elements r_{11}, r_{12}, r_{21}, and r_{22} only. Solve for D, the determinant of matrix C.

d. Introducing the 11 observations along with the parameters computed in 3.b and 3.c into equation (22), we obtain 11 values of χ_0^2. Retain the random order of the χ_0^2 values for future use in the analysis of runs.

e. Order the 11 values of χ_0^2 obtained in part 3.d in order of increasing magnitude. Compute the empirical probability of exceeding χ_0^2 corresponding to the n = 11 ordered χ_0^2 values using

$$[1 - p(\chi_0^2)] = \frac{n - i - c + 1}{n - 2c + 1} \rightarrow \frac{12 - i - c}{12 - 2c} , \tag{23}$$

for i = 1, 2, 3, . . . , 11. "c" equals $[1 - (\nu/4) - (4/n(\ln n))]$, where ν is the dimensions; i.e., if ν is 2, the dimensions are 2. ln is the natural logarithm.

f. Prepare Table A. This table will be the 11 ordered χ_0^2 values versus $[1 - p(\chi_0^2)]$. Note there will be 100 table A's corresponding to the 100 samples of size 11. Print out only the first 11 samples of the 100 samples in all cases and options. (The others must be computed and retained, however.)

g. Plot the ordered χ_0^2 values versus $[1 - p(\chi_0^2)]$ on linear versus $\log_{10}$ coordinates. Plot the theoretical fit to these points using the Univac 1108 Electronic Computer subroutine 12.2, CHI, x, Univac Stat-Pack. Use the appropriate degrees of freedom (d.f.); i.e., for the two-dimensional case, d.f. = 2 and for the three-dimensional case, d.f. = 3, etc. Plot only the first 11 samples of the 100 samples in all cases and options.

h. Repeat 3.b through 3.g for each of the 100 sets of 11 pairs. Establish a matrix 11 by 100; i.e., prepare a table of 11 columns with 100 items in a column.

i. Order from low values to high values the 100 values of χ_0^2 available in each column. The smallest χ_0^2 value will be in the upper left diagonal corner of the array.

j. Repeat 3.a through 3.i for each of 101 similarly arranged arrays.

k. Arrange these arrays so that in essence a modular form, 11 X 100 X 101, is obtained. In general form this would be an n X 100 X 101 module.

l. Order from low values to high values the 101 values of χ_0^2 available in each vertical column. The smallest χ_0^2 value will be in one corner of the three-dimensional array, while the largest value will be in the opposite end of the array diagonal.

m. Various statistics may be selected from these arrays. The first selection is the median of the vertical columns, which is the 51st value. The medians are chosen without interpolation for the 2nd, 5th, 10th, 50th, 90th, 95th, and 98th percentiles. These are the respective positions in each of the 11 vertical columns of the 11 X 100 X 101 array in the final diagonalized form. Print these values by percentile versus "i," i = 1, 2, . . . , 11.

n. Repeat these procedures for each sample size and dimension selected, n = 5, 7, 11, 15, 21, 25, 31, 51, and 101 and ν = 1, 2, 3, 4, and 5.

o. Maximum Absolute Difference (MAD). Compute the theoretical cumulative probabilities $[1 - p(\chi^2)]$ for the n = 11 ordered χ_0^2 values using the Univac 1108 subroutine 12.2, CHI. Add these values of $[1 - p(\chi^2)]$ to Table A. Compute the MAD between $[1 - p(\chi_0^2)]$ and $[1 - p(\chi^2)]$. Note there will be 100 values of MAD corresponding to the 100 samples of size n - 11.

p. Compile Table B. This will be a table of the 100 values of MAD corresponding to the 100 samples of size n = 11. Order these 100 values of MAD in order of increasing magnitude in the table. Give the corresponding sample number in the table, i.e., sample number 1, 2, 3, . . . , 100.

q. Using the Univac 1108 subroutine 13.4, CHIN, for the n = 11 values of $[1 - p(\chi_0^2)]$ in Table A, calculate the corresponding values of χ^2 located on the theoretical straight line. Add these values of χ^2 to Table A.

r. Compute the mean sum of squares of the residuals (MSSR) from the theoretical line using

$$\text{MSSR} = \frac{\sum_{i=1}^{N} (\chi_0^2 - \chi^2)^2}{N} \quad . \tag{24}$$

This is the general equation for MSSR. For the two-dimensional case we are using for illustration (n = 11), equation (24) will be

$$\mathrm{MSSR} = \frac{\sum_{i=1}^{11} (\chi_0^2 - \chi^2)^2}{11} . \tag{25}$$

There will be 100 values of MSSR corresponding to the 100 samples of size n = 11. Add these values of MSSR to Table B corresponding to their respective sample number.

4. Analysis of Runs.

a. The random order of χ_0^2 values retained in part 3.d is used here for analysis of runs. Also, the paired random χ_0^2 and corresponding χ^2 values from Table A are to be retained for this analysis.

b. Runs Above and Below the Theoretical Line. (In the case of the two-dimensional distribution the line is straight with a slope of -0.5 when plotted on semilogarithm paper where the logarithm is the natural logarithm.) Examine the paired random χ_0^2 and χ^2 values described in part 4.a. If $\chi_0^2 < \chi^2$, this is an element below the line and will be denoted as "b." If $\chi_0^2 > \chi^2$, this is an element above the line and will be denoted as "a."

We now have a series of a's and b's, the total number of elements being 11. (We have 100 of these series of 11 elements corresponding to the 100 samples of size n = 11.) An example might be:

aa	b	aa	b	aaa	bb
1	2	3	4	5	6

A run is defined as a sequence of identical observations that are followed or preceded by a different observation or no observation at all. In this example there are six runs in the sequence of n = 11 observations. In Table B record the number of runs above (a) the line and the number of runs below (b) the line for the 100 samples. For this example there are three runs above the line and three runs below the line.

c. Runs Above and Below the Median. For each sample of n = 11 compute the median of the χ_0^2 values in Table A as follows. Arrange the χ_0^2 values in order of increasing magnitude. If n is odd, the median is the middle item. If n is even, the median is the mean of the values of the two middle items. Examine the original random set of

χ_0^2 values described in part 3.d. Let "a" denote those values of χ_0^2 greater than the median. Let "b" denote those values of χ_0^2 less than the median. Again, we have a series of a's and b's as in part 4.b.

In Table B record the number of runs above (a) the median and the number of runs below (b) the median for the 100 samples of n = 11.

d. Runs of Quadrant Signs. Retain the random order of χ_0^2 observations described in part 3.d. Assign to each χ_0^2 value the corresponding quadrant signs. These are the signs of $(X_1 - \bar{X}_1)$ and $(X_2 - \bar{X}_2)$ in equation (22). For the two-dimensional case an example would be (n = 11),

$$\left.\begin{array}{c} ++ \\ ++ \end{array}\right\} 1$$

$$\left.\begin{array}{c} +- \\ +- \end{array}\right\} 2$$

$$\left.\begin{array}{c} -+ \\ -+ \\ -+ \end{array}\right\} 3$$

$$\left.\begin{array}{c} -- \\ -- \end{array}\right\} 4$$

$$\left.\begin{array}{c} ++ \end{array}\right\} 5$$

$$\left.\begin{array}{c} +- \end{array}\right\} 6$$

(Note there will be three signs for the three-dimensional case, four signs for the four-dimensional case, etc.)

Now, a run is defined as in part 4.b. In this example there are six runs in the sequence of n = 11 observations.

All 100 samples of n = 11 groups of quadrant signs will be combined. Form a frequency distribution of the number of runs of quadrant signs; i.e., the random variable is the number of runs per sample of n = 11. For a single sample of n = 11, there could be at most 11 runs and at least 1 run. The total frequency for this distribution will be $\geqslant$ 100. Label this frequency distribution Table C.

Also, print out as Table D the first 11 sets of signs (of the total 100 samples) for sample size n = 11. This will be an 11 X 11 table. (See the suggested printout format.)

5. Cumulative Probabilities. Prepare Table E as follows.

Using the data from Table B, order the following in order of increasing magnitude:

a. MSSR

b. Number of runs above theoretical line

c. Number of runs below theoretical line

d. Number of runs above the median

e. Number of runs below the median.

6. Repeat parts 2 through 5 of subsection IV.E for the 5, 7, 15, 21, 25, 31, 51, and 101 paired samples (Option 2).

7. Repeat parts 2 through 5 of subsection IV.E for:

a. Option 3: three-dimensional case

b. Option 4: four-dimensional case

c. Option 5: five-dimensional case.

8. Combine Tables A through E for Option 2 for all sample sizes n = 5, 7, 11, 15, 21, 25, 31, 51, and 101.

9. Repeat part 8 for Options 3, 4, and 5.

10. Suggested Printout Format:

Table A: 2 dimensions n = 11

χ_0^2	$1 - p(\chi_0^2)$	χ^2	$1 - p(\chi^2)$

Table B: 2 dimensions n = 11

Sample No.	MAD	MSSR	Runs Above Line	Runs Below Line	Runs Above Median	Runs Below Median

Table C: Runs of quadrant signs

2 dimensions n = 11

Number Runs	F_i
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

Table D: 2 dimensions n = 11

++	- -		+ -
++	+ -	(11 columns)	- +
+ -	- +		- -
+ -	- +		++
- -	++		- +
- -	++		++
- +	- -		- +
+ -	+ -		+ -
++	- +		++
+ -	- -		++
++	+ -		- +

Table E: 2 dimensions n = 11 Ordered Values

MSSR	Runs Above Line	Runs Below Line	Runs Above Median	Runs Below Median

F. Comparison of Ordered Random Chi-Square Values with the Theoretical Chi-Square Values

The calculation and ordering of the chi-square (χ^2) values is the first step in the visual examination of multivariate (ν-variate) distributions for normality. The familiar χ^2 test is used though it offers no test for asymmetry. The graphs, either of the original data in histogram form or in the cumulative percentage graphical techniques discussed here, do offer qualitative assessment of asymmetry. If the null hypothesis is not rejected, then symmetry is assumed in the assumption of normality. Other tests provided for in this report do permit symmetry tests.

Equation (3) permits the calculation of χ^2 values by use of the estimates of the population parameters, i.e., the sample statistics. These empirical data are ordered and processed as in subsection IV.E. The difference between the data treatments in subsections IV.E and IV.F is that in the former the means and covariances were zero.

There remains the difficult and somewhat controversial problem of the determination of the plotting points for the empirical data. Blom (1958) discusses the calculation of empirical probabilities for the univariate distribution. The general formula is $p = ((i - \alpha)/(n - \alpha - \beta + 1))$. Simplifying this where $\alpha = \beta$ and setting $\alpha = \beta = c$, $p = ((i - c)/(n - 2c + 1))$. Letting c equal 0, $p = i/(n + 1)$, while letting c equal 0.5, $p = (i - 0.5)/n$. Earlier workers use c equal to 0.25 or 0.33. Sarhan and Greenberg (1962) discuss this formula. Gringorten (1963) prefers to use c equal to 0.44.

Undoubtedly, the more general formula, Blom (1958), would be generalized still more when higher dimensions are used. Apparently, the more general formula would need to consider the sample size n, the ith ordered position, and the dimension ν. Considerable work by the authors reveals no simple solution. The plotting graph paper serves for comparison only of sample data against the random generator or Monte Carlo results. Therefore, the simple formula $p = (i - c)/(n - 2c + 1)$ will be used for all sample sizes and all dimensions. From plots of the deviation of sample data from the theoretical curves indicated in Figure 6, a formula to determine an appropriate value of c was obtained. This formula is $c = 1 - (\nu/4) - (4/n) \ln n$, where ν is the dimension and ln is the natural logarithm.

Table 3 provides values of "c" derived from the formula $c = 1 - (\nu/4) - (4/n) \ln n$. Table 4 provides probability plotting positions determined from the formula $(1 - p) = 1 - ((i-c)/(n - 2c + 1))$. These are shown for sample sizes 5, 7, 9, 11, 15, 21, 25, 31, 35, 41, 45, 51, and 101 for one through five dimensions.

Before proceeding further it is important to discuss the problem of small samples. With large samples, 500 for example, the curves or plots obtained will approach the theoretical in the sense portrayed in Figure 6 and figures of the type subsequent to and similar to Figure 12. The confidence bands will widen and flare rapidly at the lower

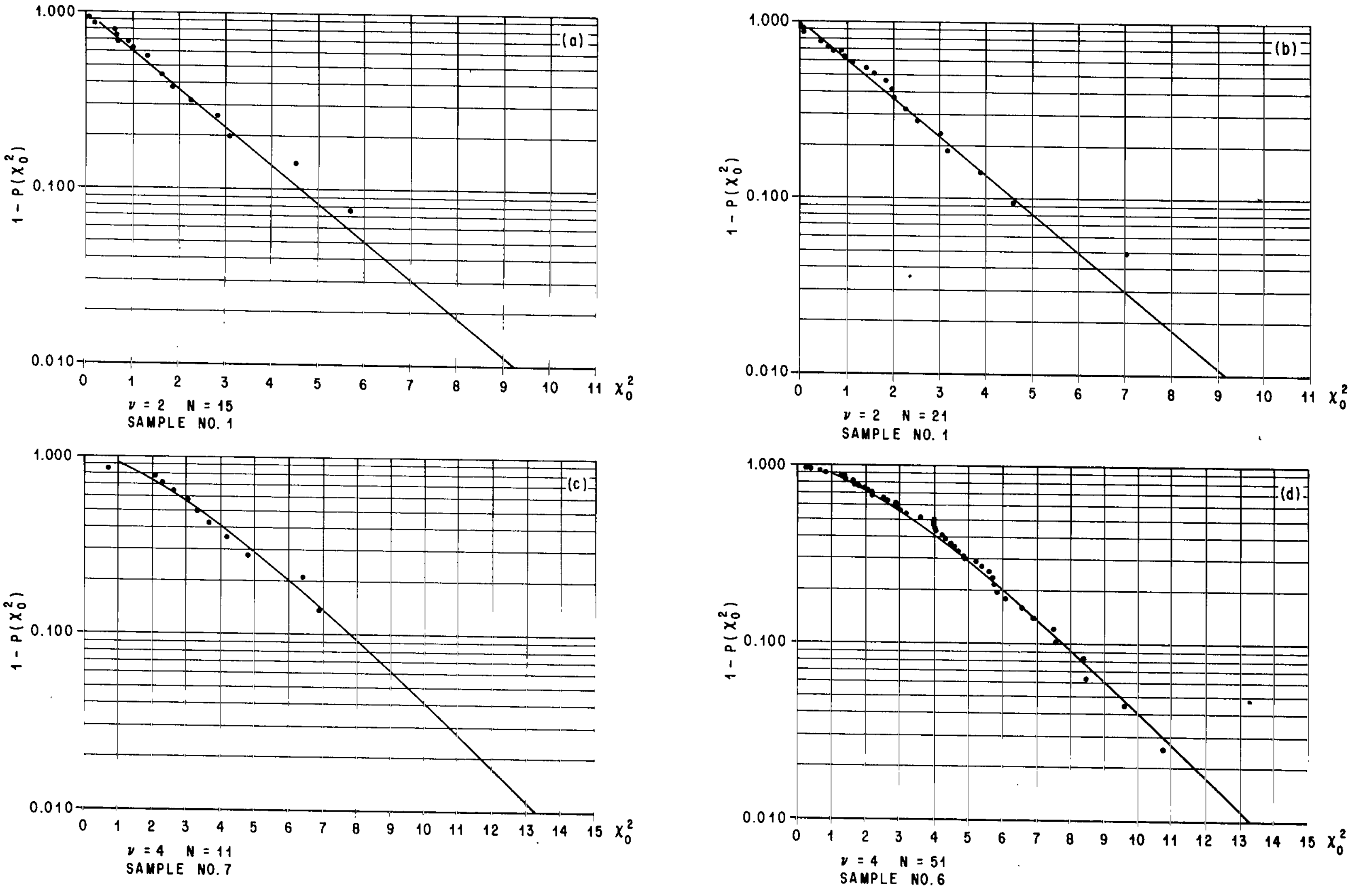


Figure 12. Examples of "Good Fit" of randomly generated normal variates for differing dimensions (ν) and sample sizes (n).

probability levels of (1-p). Thomas and Crigler (1974) discuss, in general, tolerance limits in the multidimensional radial error distribution. The assumption is made that the multivariate mean is zero and the multivariate dispersion is circular in form; i.e., the vector variance is the square root of the number of dimensions. This assumption is not usable for small samples where sample estimates of the mean and variance are used. This fact caused many difficulties in assessing the situation. These difficulties are inherent in small sampling problems. The problems of Student (1925) are revisited, and the reader is reminded that this is the problem of the multivariate Wishart distribution previously discussed. The decision was made to develop the tables and graphs using the multivariate normal form of distribution.

The crux of the problem is that the values of maximum chi-square (χ^2) versus (1 - p) converge to some value. Therefore, the confidence bands converge. This is not apparent in the tables and graphs because the median values of the sample percentiles are used. The convergence is more noticeable as the sample size gets smaller. For example, see Figure 18. Briefly, the maximum possible chi-square (χ^2) is $(n-1)^2/n$ when the unbiased estimate of the population variance in each dimension is used. If the biased estimate of the population variance is used, the maximum chi-square (χ^2) is (n-1). Thus, the computed maximum chi-square values to be obtained in a sample of 5, no matter what the dimensions may be, will be 3.2 and 4.0, respectively, for the case of the unbiased and biased variances. Therefore, the confidence bands beginning at a probability of 1 diverge from χ^2 equal zero and then converge toward the values given previously. If computed maximum chi-square values are greater than the limiting values given, there is an error in computation. If the error is small, it may be due to numerical rounding errors. Actual convergence is to a number less than the maximum given. It is a function of both sample size and dimension.

As the number in the sample increases, the probability of reaching a maximum chi-square diminishes and the confidence bands converge more slowly. As the dimensions increase, the probability of reaching a maximum chi-square increases. As both the sample size and dimensions increase, the possibility of increasing magnitude of chi-square increases. With infinite n and infinite dimension, the chi-square limit is infinite.

G. Selected Tables of Randomly Generated Chi-Square Values

Subsection IV.D gives the procedures to generate random normal variates and calculate, select, and order chi-square values. Subsection IV.E discusses the comparison problems.

Table 5 provides median values of selected percentiles of 10 100 random chi-square values of the multivariate normal distribution. Their probability plotting positions are indicated. The order and the plotting probabilities are given.

Figure 6 provides the background of theoretical curves against which ordered chi-square values of a random sample may be plotted. Curves appear for dimensions one through twelve or 12 degrees of freedom. Curves of only the first 5 degrees of freedom are used here. Figures 12a through 12d show some selected values of what visually appear to be good fits. These data are taken directly from the computer output to microfilm where 11 samples of each dimension one through five and sample sizes of 5, 7, 11, 15, 21, 31, 51, and 101 were prepared. Visually appearing bad fits also are obtained. Figures 13a through 13d are examples of what some of the bad fits might look like in future sampling. It must be remembered that when data are ordered, there is a correlation involved in the sequential ordering. Therefore, the usual change in a curve from point to point is relatively slow. Ordinarily, there are no abrupt changes or breaks.

Comparison of the good fits with the bad fits illustrates how well the ordered random chi-square values shown as dots approach the expected theoretical line as indicated by the solid central curve. Figures 14a through 14d show Figures 12a through 12d with confidence bounds of 0.02 and 0.98 probability. Figures 15a through 15d show Figures 13a through 13d with the same confidence bounds. Figures 14a through 15d indicate that the visual inspection was correct. This would be expected for these figures are selected from those that were considered to be the best and the worst fits. The confidence bounds are taken from tables prepared as designed under subsection IV.E.

If α percent or more of the empirical data fall outside the confidence band, the distribution can be said to be significantly different from the respective k-variate normal distribution. (α is the rejection level.) The usual Type I or Type II error may be made. This procedure is mechanistic, yet it does serve to provide objective guidelines for decision.

Figures 16, 17, and 18 show several features of the tabulated results of this study. These figures are for one dimension with a sample size of 5, two dimensions with a sample size of 51, and five dimensions with a sample size of 7, respectively. Figure 16 shows the squeezing in of the curves near the probability level of 0.50. This is characteristic of many of the data ensembles. Figure 17 illustrates that with a sufficiently large number of data, the confidence bands tend to flare as shown in Figures 10 and 11. Figure 18 shows marked characteristics of the small samples. Remember that the maximum possible chi-square is equal to $(n-1)^2/n$ or $n-2+(1/n)$ for the unbiased sample and n-1 for the biased sample. Therefore, the upper bound for the empirical data sample must at times be less than that implied by the data curve plotted from the usual chi-square tables.

H. Selected Tables of Randomly Generated Chi-Square Values Including Tests for Symmetry

In subsection IV.E the instructions to the computer also resulted in tabular preparation of:

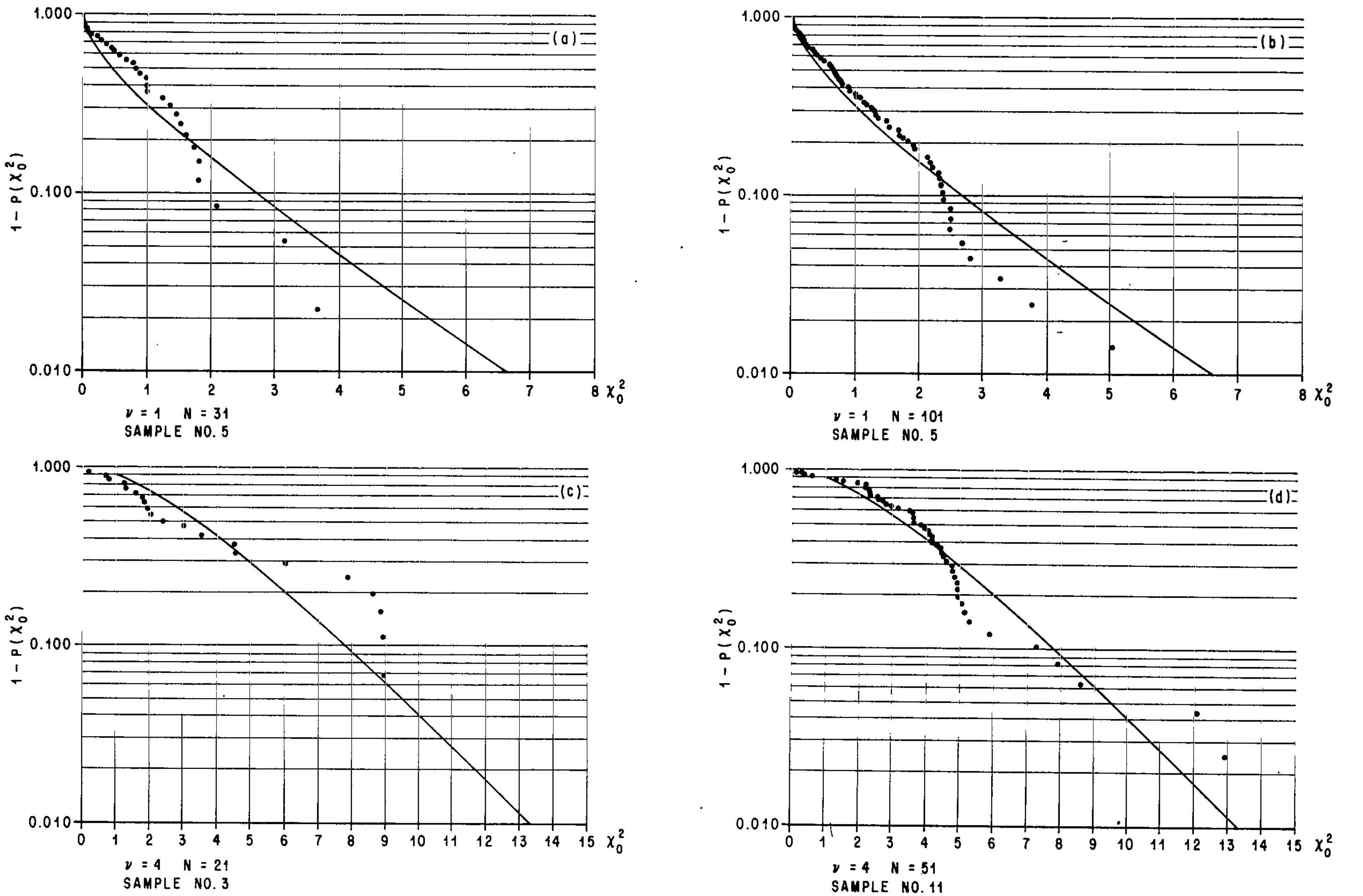


Figure 13. Examples of "Bad Fit" of randomly generated normal variates for differing dimensions (ν) and sample sizes (n).

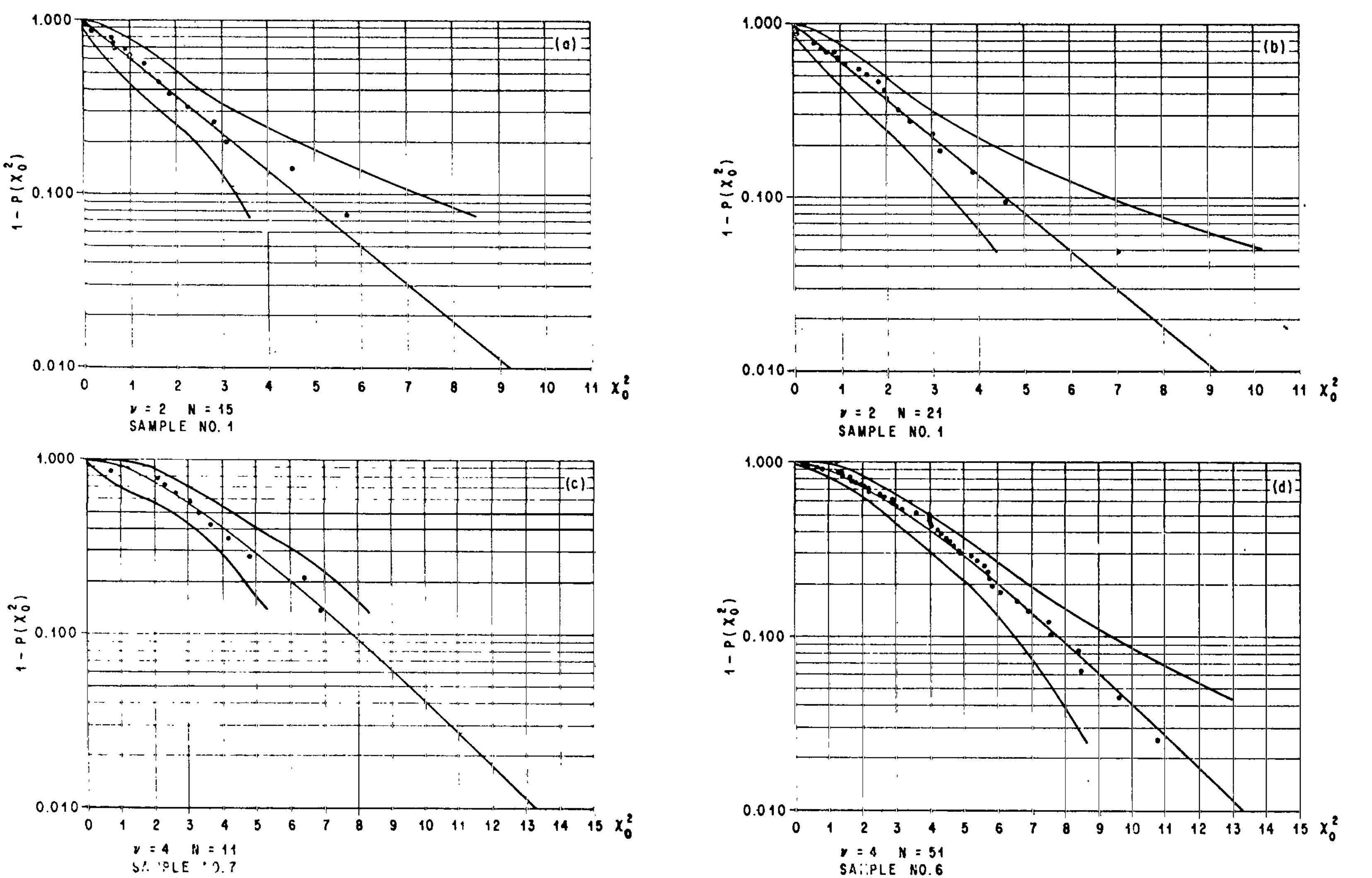


Figure 14. Examples of "Good Fit" of randomly generated normal variates for differing dimensions (ν) and sample sizes (n) with the central 0.96 confidence band.

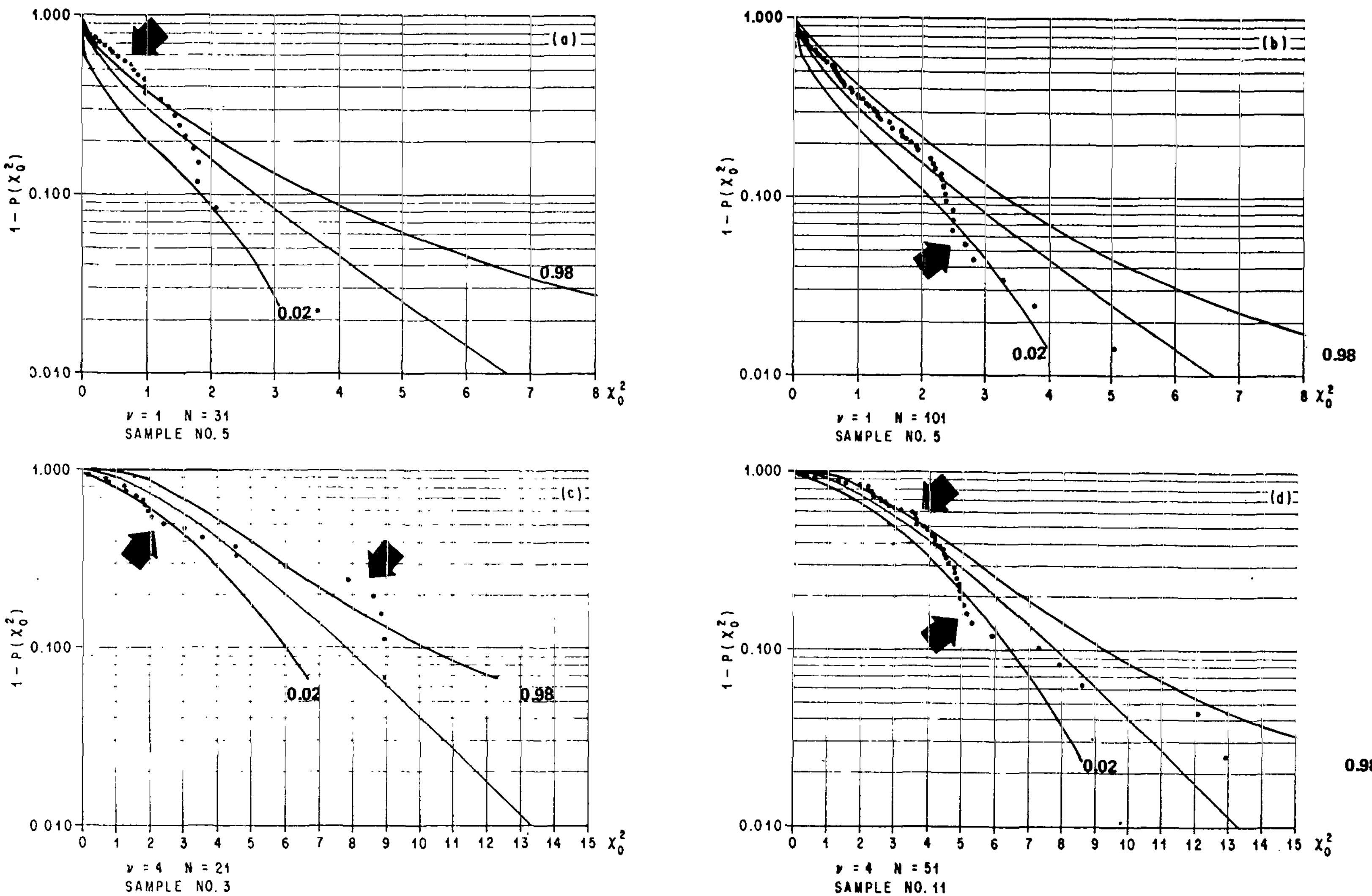


Figure 15. Examples of "Bad Fit" of randomly generated normal variates for differing dimensions (ν) and sample sizes (n) with the central 0.96 confidence band.

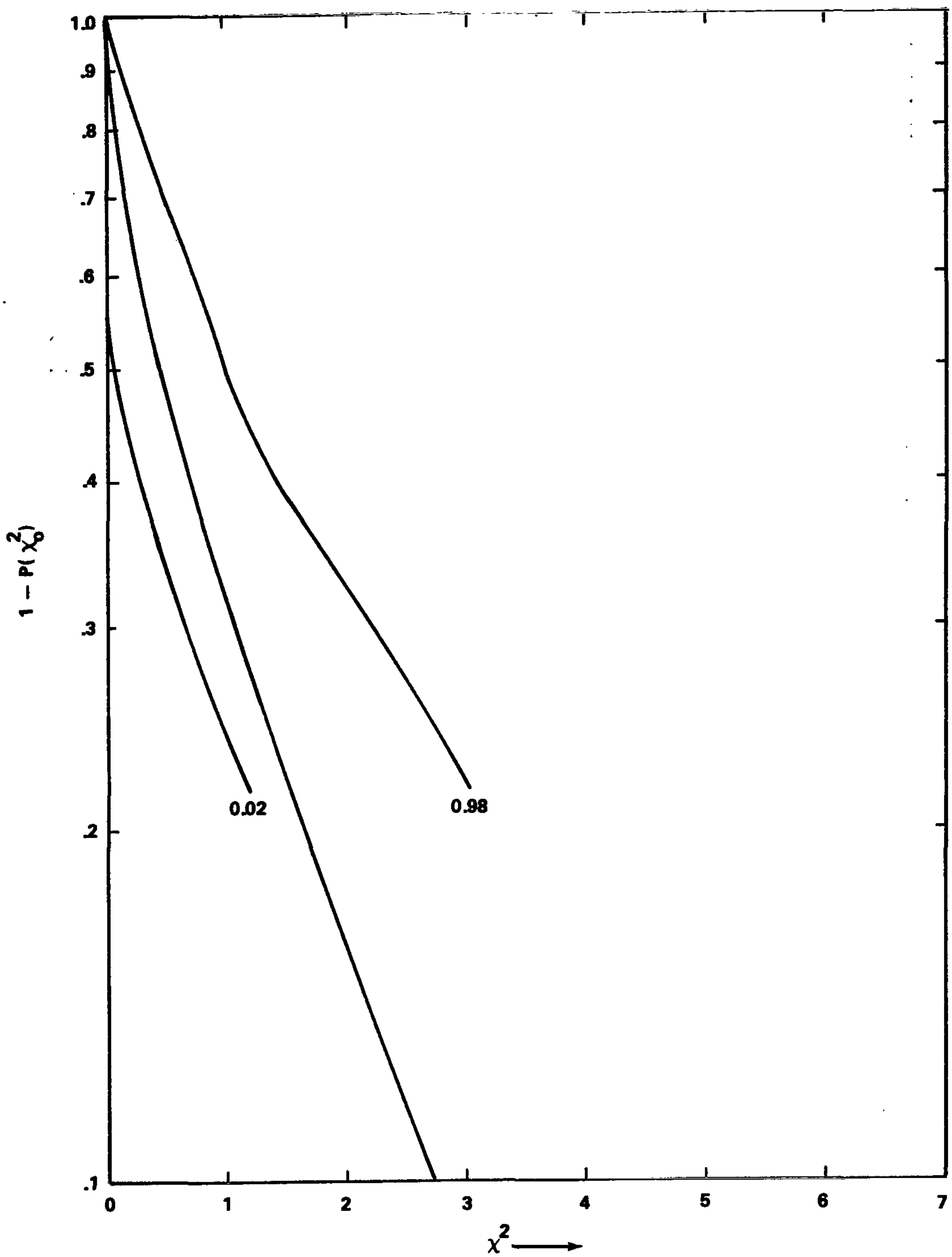


Figure 16. Plot of data from Table 5 (These are the median values of the 2nd and 98th percentile values of 10 100 random chi-square values of the multivariate one-dimensional distribution for a sample size of 5. The central line is the median value for infinite n).

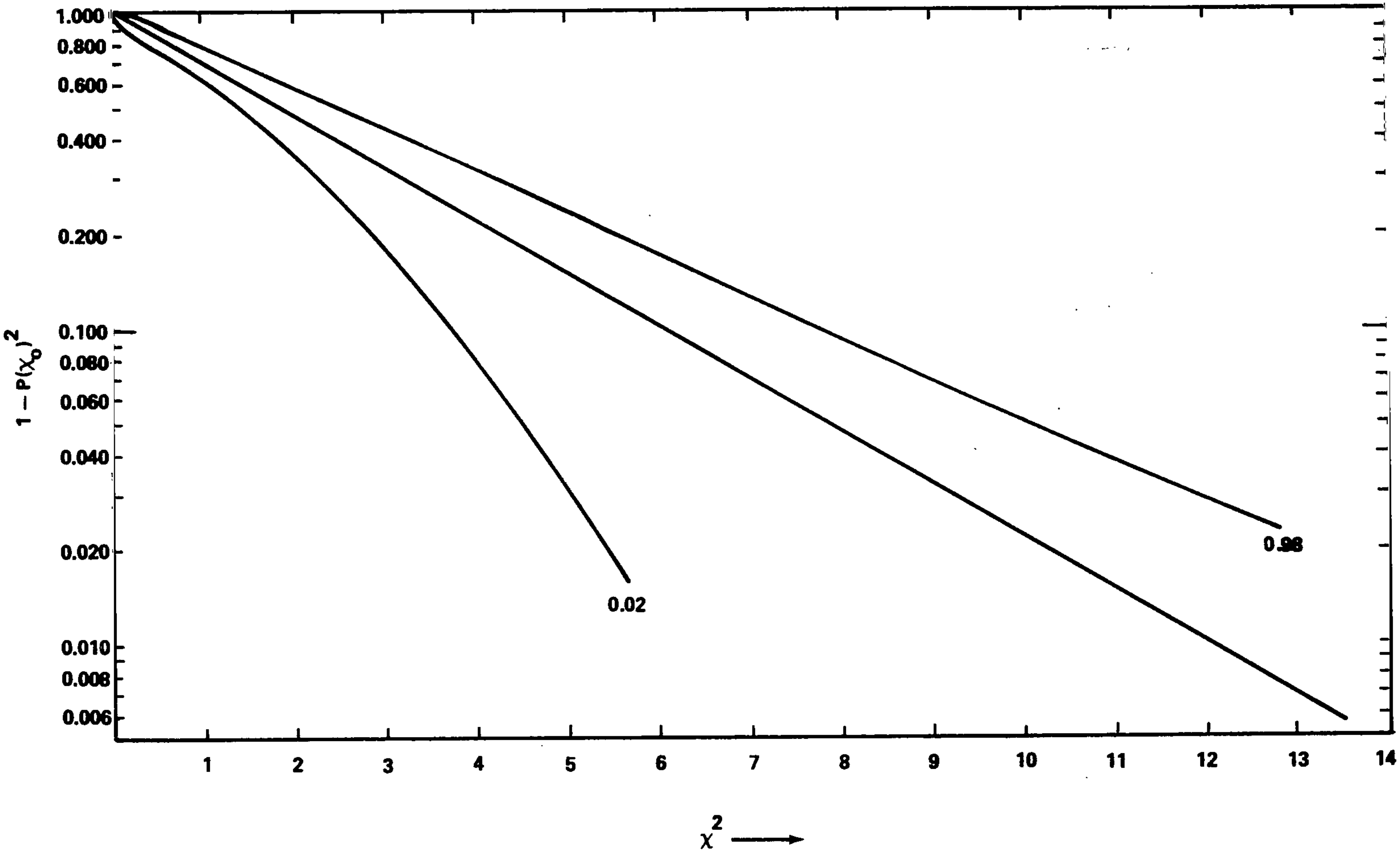


Figure 17. Plot of data from Table 5 (These are the median values of the 2nd and 98th percentile values of 10 100 random chi-square values of the multivariate two-dimensional distribution for a sample size of 51. The central line is the median value for infinite n).

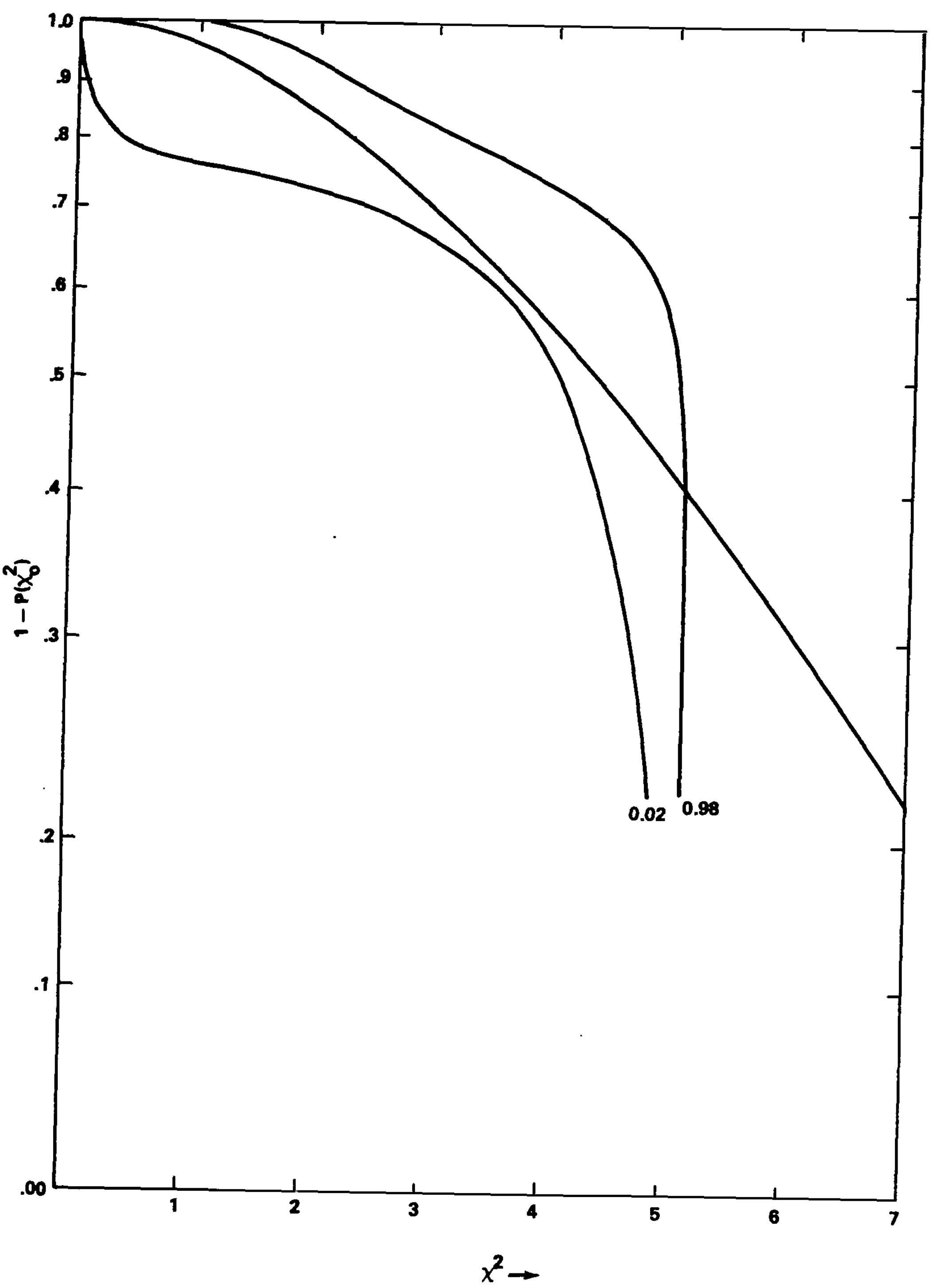


Figure 18. Plot of data from Table 5 (These are the median values of the 2nd and 98th percentile values of 10 100 random chi-square values of the multivariate five-dimensional distribution for a sample size of 7. The central line is the median value for infinite n).

1. The maximum absolute differences (MAD) between the theoretical and the empirical curves. In one dimension this is similar to the Kolmogorov-Smirnov tables, Beyer (1966). In multivariate form these are similar to the work of Malkovich and Afifi (1973).

2. The mean sums of squares of the residuals (MSSR) between the theoretical and empirical curves.

3. Runs of the empirical data above (RAL) and below (RBL) the theoretical lines of best fit.

4. Runs above (RAM) and below (RBM) the median value. These values are similar in formation to the one-dimensional tables of Swed and Eisenhardt (1943) and those found in many handbooks such as Beyers (1966) and Owens (1962).

5. Runs of n-tant signs. In the computer instructions quadrant is correct for the two-dimensional case. Octant will be correct for the three-dimensional case.

Table 6 provides for selected empirical probability levels from 0.01 to 0.99, the corresponding values of MAD, MSSR, runs above or below the theoretical curve, and runs above or below the median. Table 6 includes dimensions 1 through 5 at five selected sample sizes 11 through 101.

Table 7 is derived from Table 6. For the same selected probability levels in Table 6, the maximum absolute differences are presented for dimensions 1 through 5 by sample size.

Table 8 is still another presentation of the MAD shown in Table 6. Table 8 presents for one variate the MAD values for selected sample sizes.

Tables 9 and 10 show presentations analogous to Tables 7 and 8 of the MSSR taken from Table 6.

Tables 11 and 12 show presentations analogous to Tables 7 and 8 of the runs above or below the line of chi-square versus the probability.

Tables 13 and 14 show presentations analogous to Tables 7 and 8 of the runs above or below the median chi-square.

Table 15 further identified by dimensions 2, 3, 4, and 5 and prepared for sample sizes within each dimension 11, 15, 21, 25, 31, 51, and 101 shows the number of runs to be expected within each n-tant. Table 15 also gives the number of runs within quadrants that might be expected on a random sample basis. Though 100 samples of each sample size were used to provide a grand sample, only 13 to 14 grand samples were used to provide the basis for the maximum, mean, and minimum values shown in the tables. Thus, if 100 samples are taken, the values given in the tables will be appropriate. If only one sample is taken, the tabular values should be divided by 100 and rounded to the nearest integer. For example, in a sample of 11 in two dimensions there could be a maximum of 10 quadrant sign changes; i.e., each succeeding vector would lie in a different quadrant. From the 14 grand samples to the nearest integer there will be a maximum of 8, a mean of 7, and a minimum of 6 occurrences where the sequential vectors would shift to another quadrant after the first occurrence. There will be a maximum of 9, an average of 8, and a minimum of 7 occurrences where one or two sequential vectors lie in a quadrant followed by a third vector in another quadrant.

This, in essence, is a test of homogeneity, a test of randomness. This test or the values given do not provide any insight if only two n-tants are involved. If this were the case, the data set would be oscillatory and would not be homogeneous but would be heterogeneous. A visual perusal of a sign combination output in the data presented does not reveal any heterogeneity. However, this does not imply that a future set of data might not be oscillatory. Hopefully, the authors can investigate this problem in the future.

The cumulative sum across all runs is not necessarily a constant due to varying sample sizes and truncation in preparing the tables.

V. APPLICATIONS TO REAL DATA

A. Temperature and Winds

Cramer (1970) supplied data for this portion of the study. This is a four-dimensional problem. Computer listings were keyed on magnetic tape of u', v', w', and T' for three two-level trials (runs 32, 33, and 34 of Cramer's study on the Round Hill 40 meter tower). The records for each trial cover a 1 hour period and comprise sequential time series of the discrete values of the four variables at 1.2 second intervals. There are 3000 data groups in each run at each level. The units of the wind are meters per second and the temperature unit is degrees Celsius. The primes indicate deviations from the run average. The east-west, north-south, and vertical wind components are, respectively, u, v, and w. These components are positive from the west toward the east, from the south toward the north, and upwards from below. The temperature symbol is T. The heights of the two levels were 16 and 40 meters. Cramer (1966) provides the raw statistics and other information of his data in Volume III of the ECOM-65-G10 Report. Run 32 at 16 meters above the surface was selected for use. The run began at 0834 and ran to 0934 EST on October 5, 1961.

Figures 19 through 22 illustrate a data subset from run 32. These are separate sequential plots of u', v', w', and T'. These figures serve to show the variability of the types of data. Plotted are the 300 points 2701 to 3000 for run 32 at the height 16

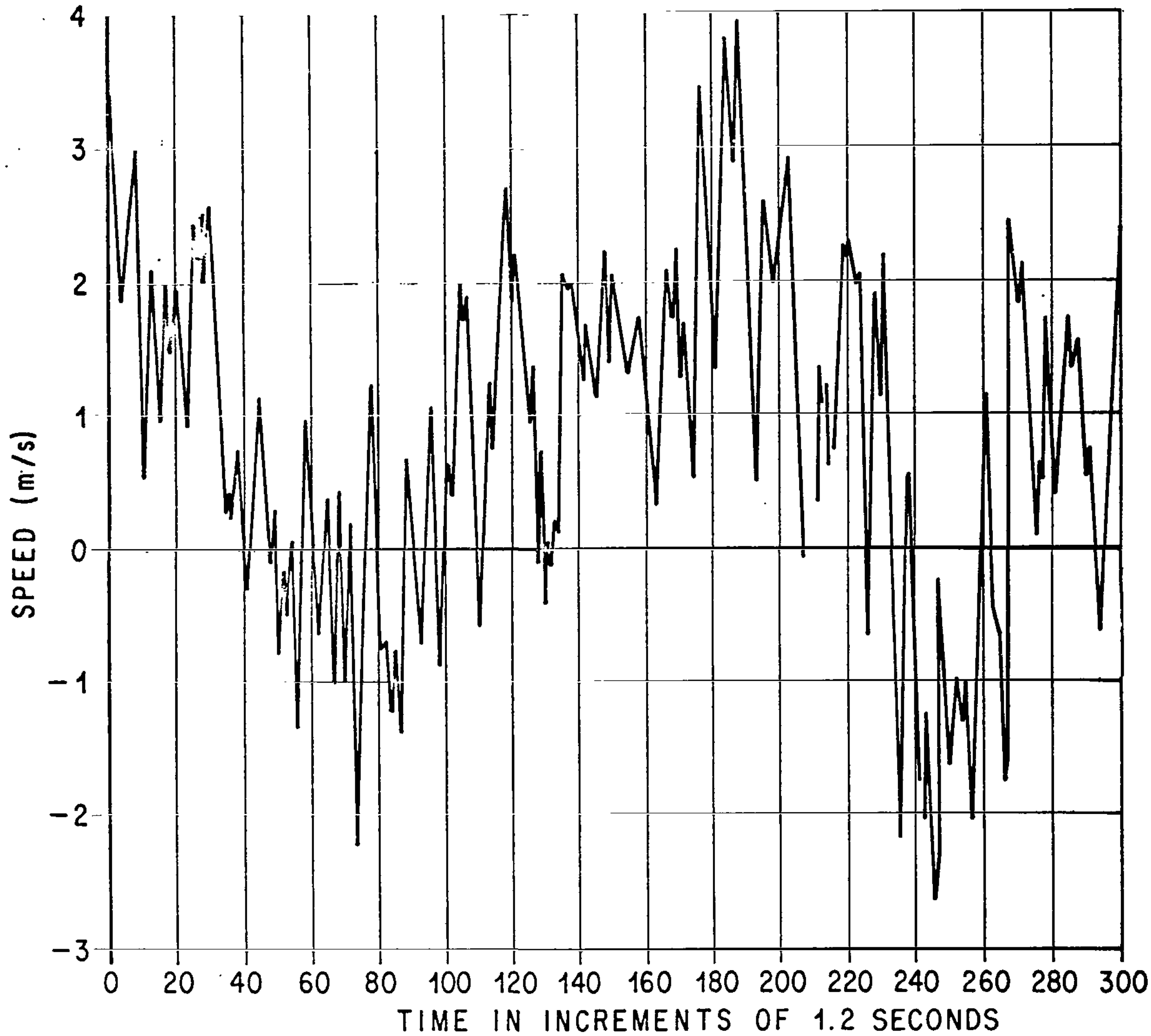


Figure 19. Zonal deviations of the wind (m/s) during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0929-0934 EST. (The components are positive from the west. The deviations are from the zonal mean for the hour 0834-0934 EST. The elevation is 16 meters.) (Cramer, 1970)

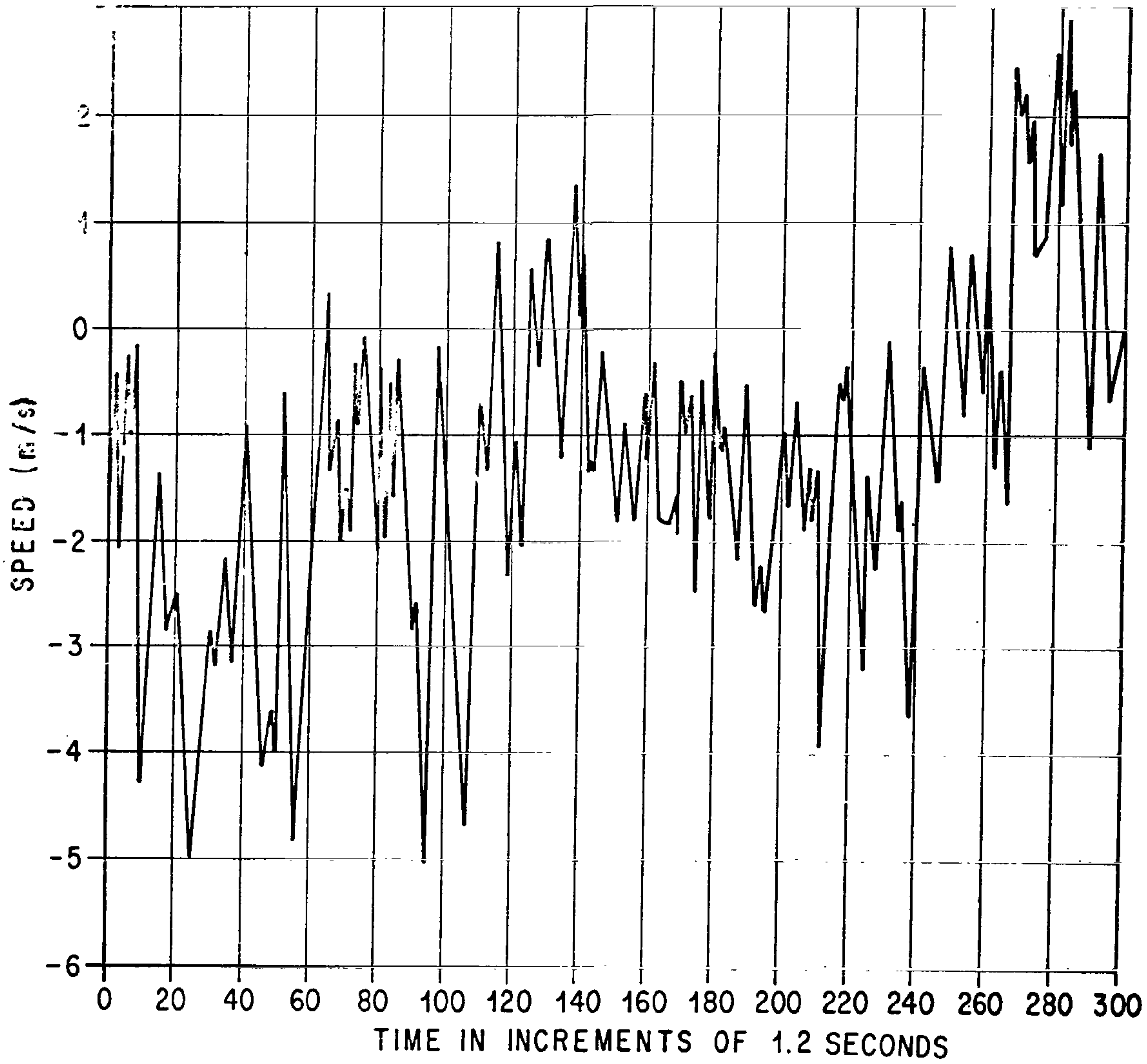


Figure 20. Meridional deviations of the wind (m/s) during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0929-0934 EST. (The components are positive from the south. The deviations are from the meridional mean for the hour 0834-0934 EST. The elevation is 16 meters.) (Cramer, 1970)

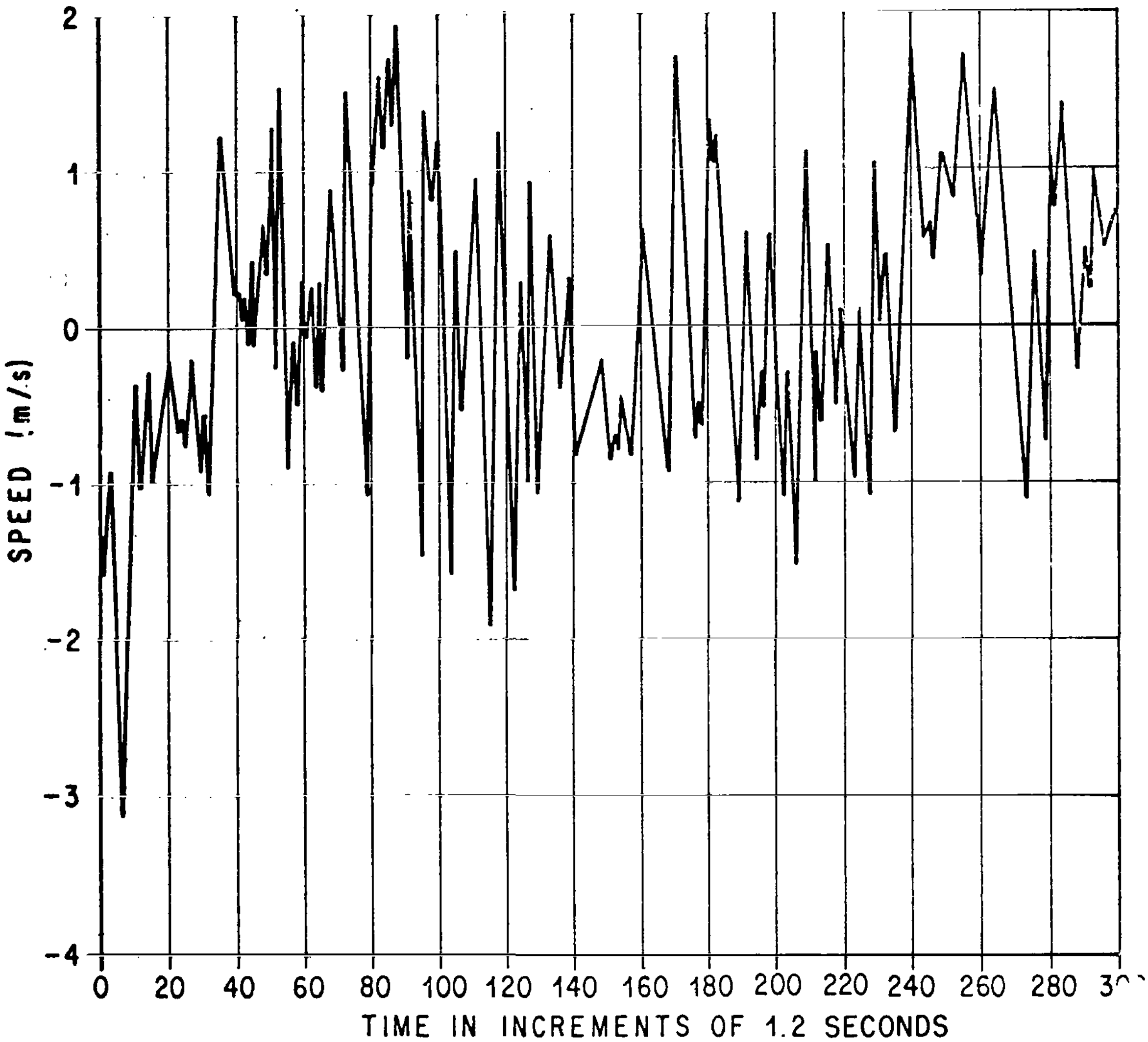


Figure 21. Vertical deviations of the wind (m/s) during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0929-0934 EST. (The components are positive from below. The deviations are from the vertical mean for the hour, 0834-0934 EST. The elevation is 16 meters.) (Cramer, 1970)

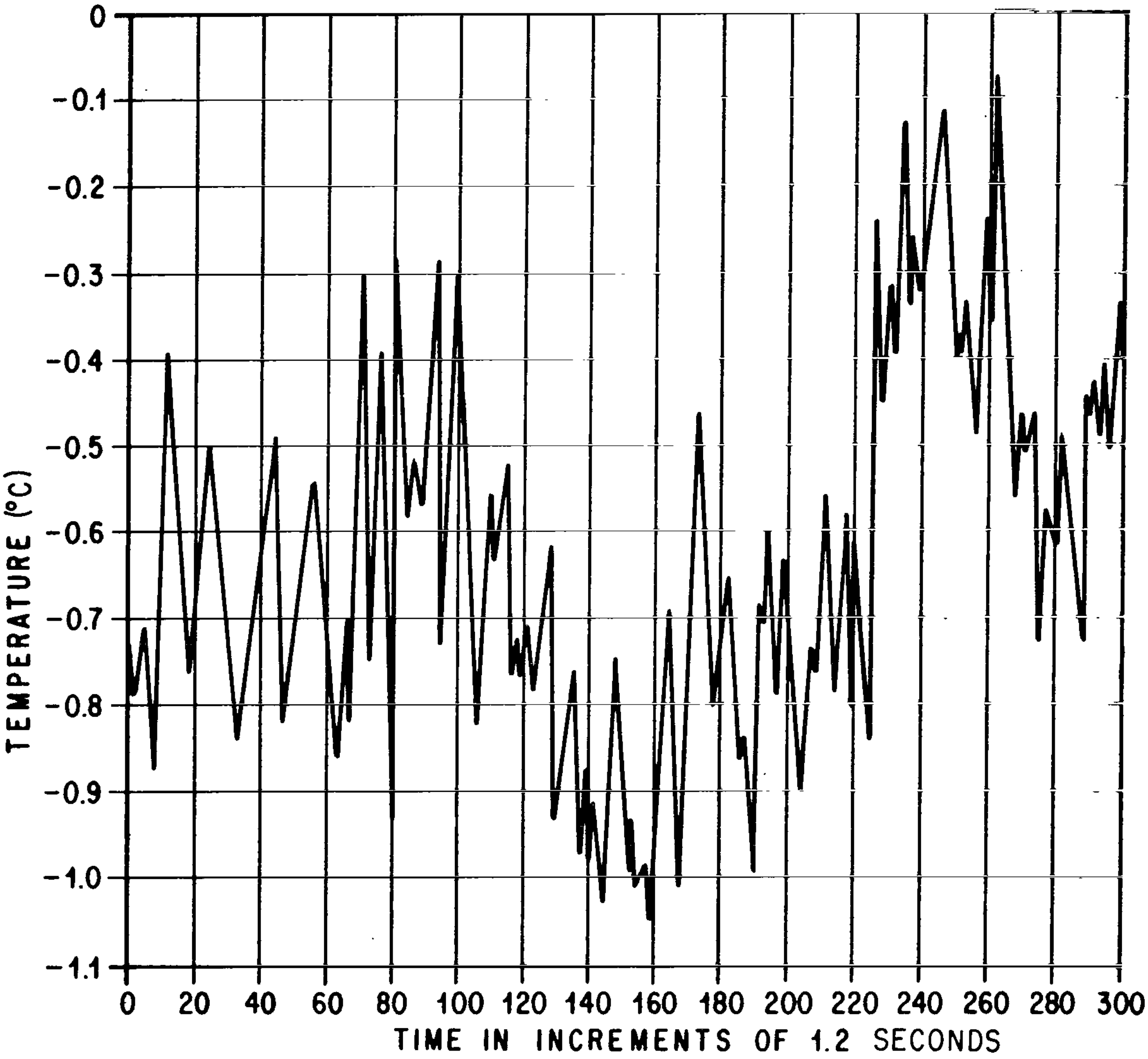


Figure 22. Temperature deviations in degrees Celsius during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0929-0934 EST. (The deviations are from the mean for the hour, 0834-0934 EST. The elevation is 16 meters.) (Cramer, 1970)

meters on October 5, 1961, 0929-0934 EST. The primes indicate deviations of the data from the 1 hour mean. For example, note that the temperature deviations shown in Figure 22 are all negative. There has been some smoothing in the drafting of these figures. Figure 23 is a graphical output of the four-dimensional chi-square ordered distribution of the same 300 points of run 32. The central line is the expected line of best fit. This line may be drawn either from a set of chi-square tables such as Table 1 or as illustrated in Figure 6. The chi-squares are computed from the data groups per equation (3).

Figure 24 superposes 0.96 probability confidence bands for a sample size of 101 on Figure 23, although there are 300 data groups or 300 four-dimensional points involved. The band actually would be somewhat narrower for the larger sample size of 300. Since none of the chi-square values fall outside the limits shown, the null hypothesis that the observed distribution is not different from the theoretical multivariate normal distribution is not rejected. If confidence bands were available for the 300 sample size, it is recognized that the hypothesis might be rejected. However, the data set appears to be well within bounds. The bounds are constructed from data given in Table 2.

Figures 25 and 26 present a similar example for the data subset, points 301 to 400, of the same set used above. Here the sample size is 100 rather than 300. Note that near the 0.19 probability level the empirical curve does exceed the limits. However, as the curve draws back within bounds, the multivariate normality hypothesis is not rejected.

Autocorrelations presumably are inherent in the data sets. These autocorrelations will not invalidate the tests. In fact, autocorrelations have the tendency to cause rejection of the null hypotheses. The determination of the first zero crossing for the autocorrelation and the use of this lag to determine a sampling rate from the data sets will minimize the deleterious autocorrelation effects. If the correlation goes from positive to negative at the sixth lag or reaches a positive minimum, the number 6 provides the sampling rate. That is, use every sixth observation. Since the hypotheses were not invalidated here, the autocorrelation effects were not considered at this point.

The nonrejection of the null hypothesis in the two cases above permits the inference that the distribution of these data, the Round Hill turbulence measurements at this point in time and space, may be described by the four-dimensional normal distribution. This inference permits the further inference that each of the four three-dimensional, the six two-dimensional, and the four one-dimensional distributions may be described by the respective multivariate normal distributions.

The MAD, MSSR, RAL, RBL, RAM, RBM, and runs of n-tant signs tests are not discussed for these data.

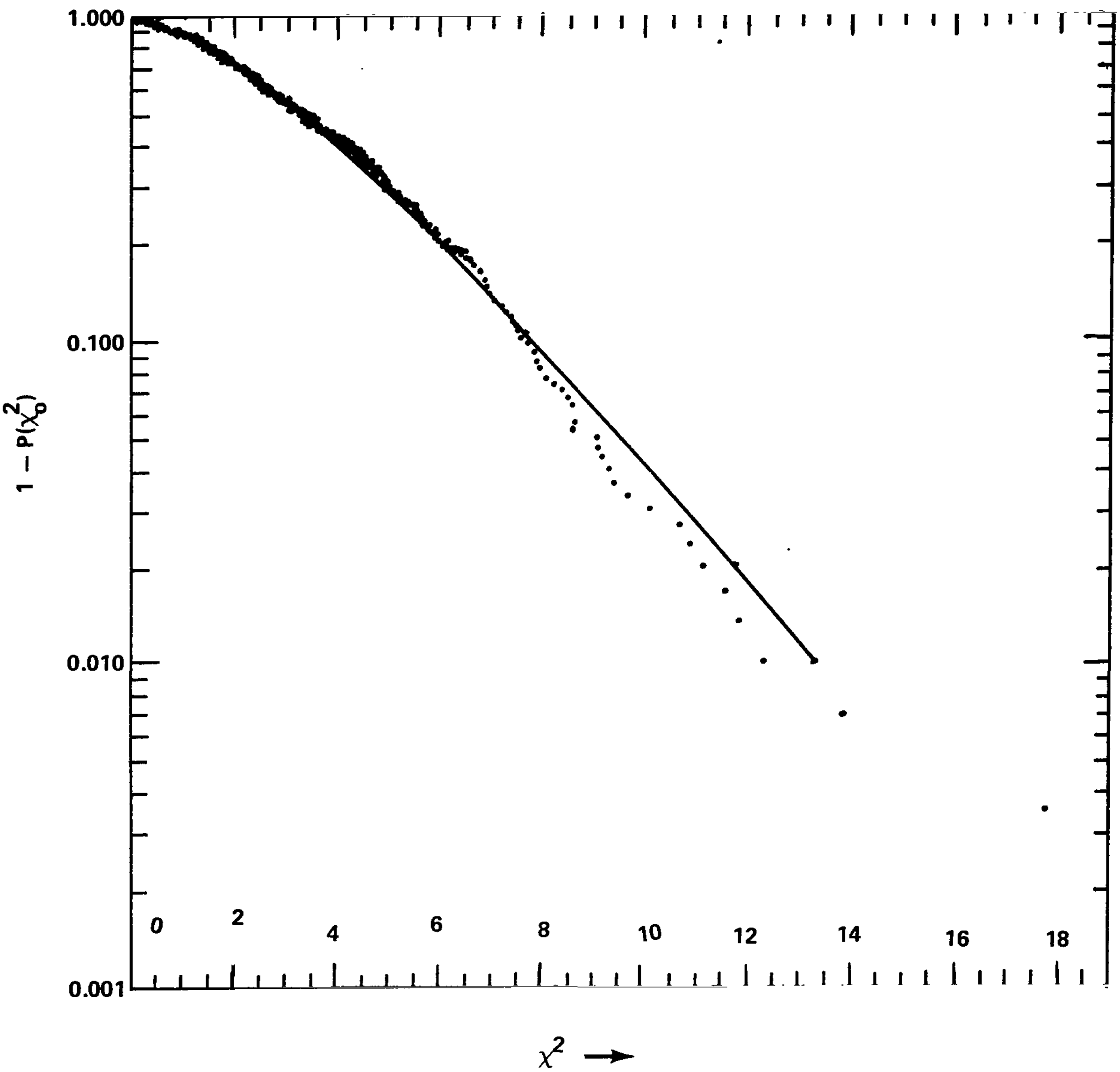


Figure 23. Four-dimensional chi-square ordered distribution of 300 consecutive data points of wind component deviations (m/s) u′, v′ and w′, and temperature deviations in degrees Celsius during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0929-0934 EST. (The elevation is 16 meters.) (Cramer, 1970)

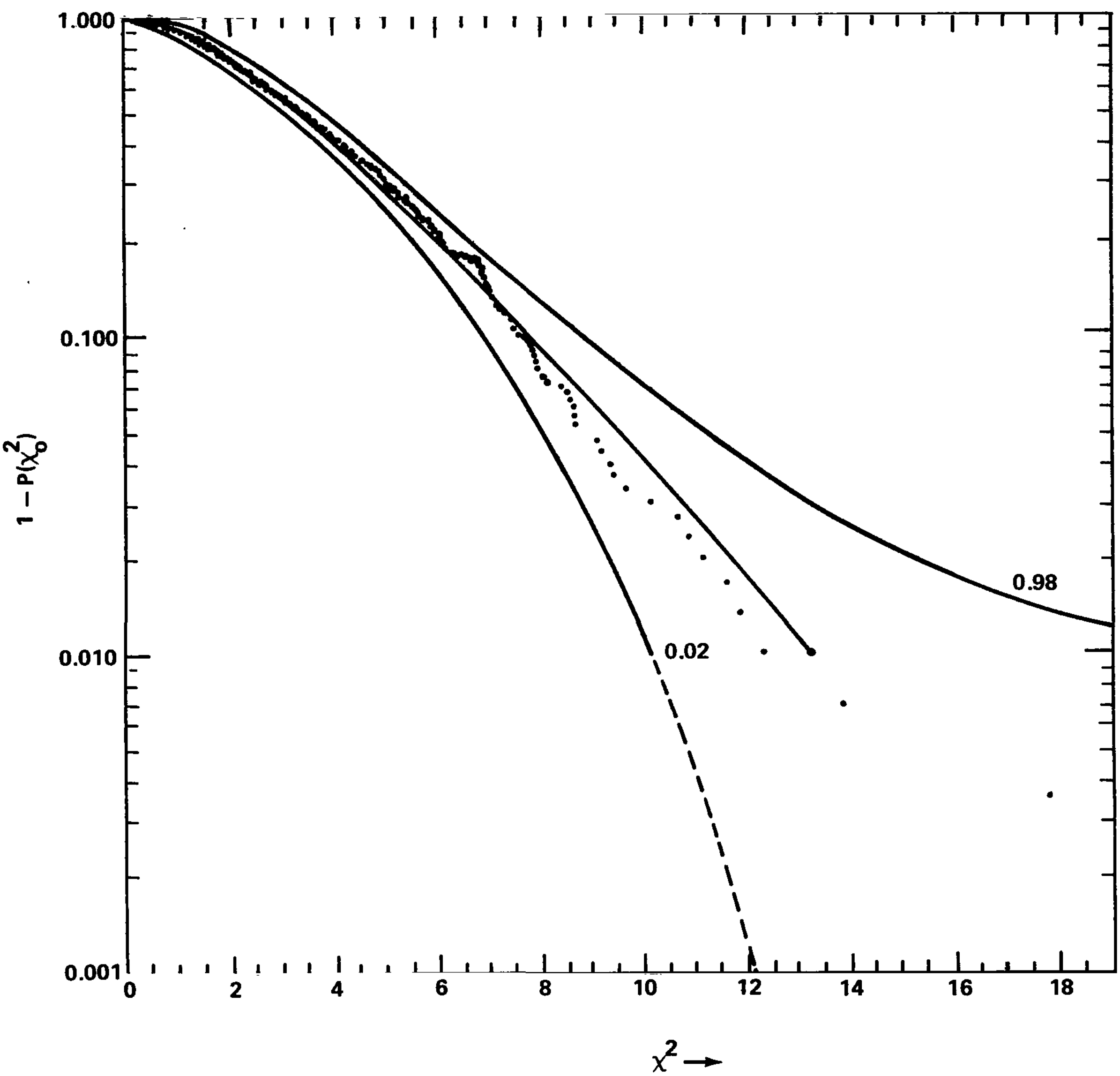


Figure 24. Four-dimensional chi-square ordered distribution of 300 consecutive data points of wind component deviations (m/s), u′, v′ and w′, and temperature deviations in degrees Celsius during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0929-0934 EST with the central 0.96 confidence band. (The elevation is 16 meters.) (Cramer, 1970)

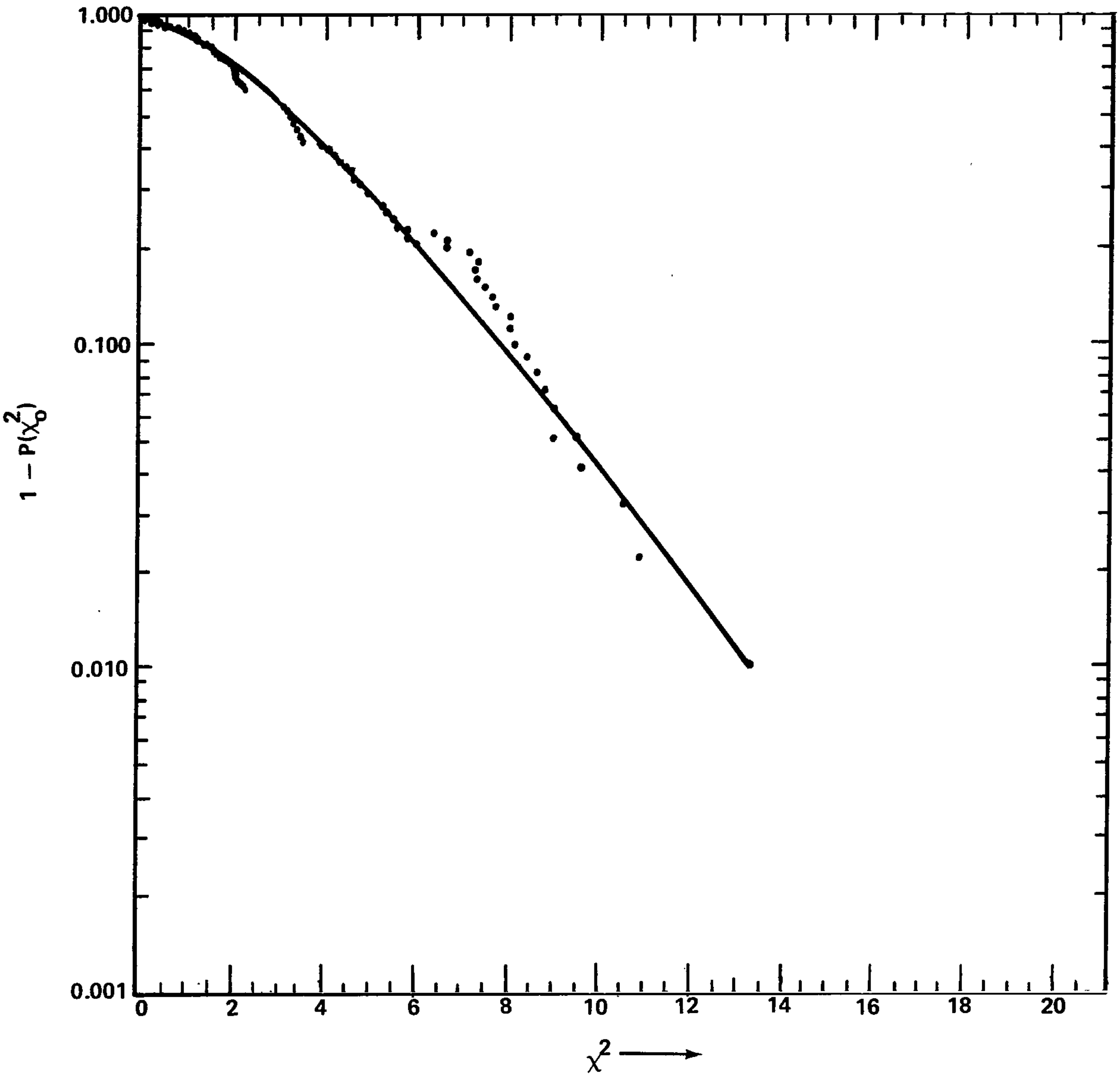


Figure 25. Four-dimensional chi-square ordered distribution of 100 consecutive data points of wind component deviations (m/s), u′, v′ and w′, and temperature deviations in degrees Celsius during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0840-0842 EST. (The elevation is 16 meters.) (Cramer, 1970)

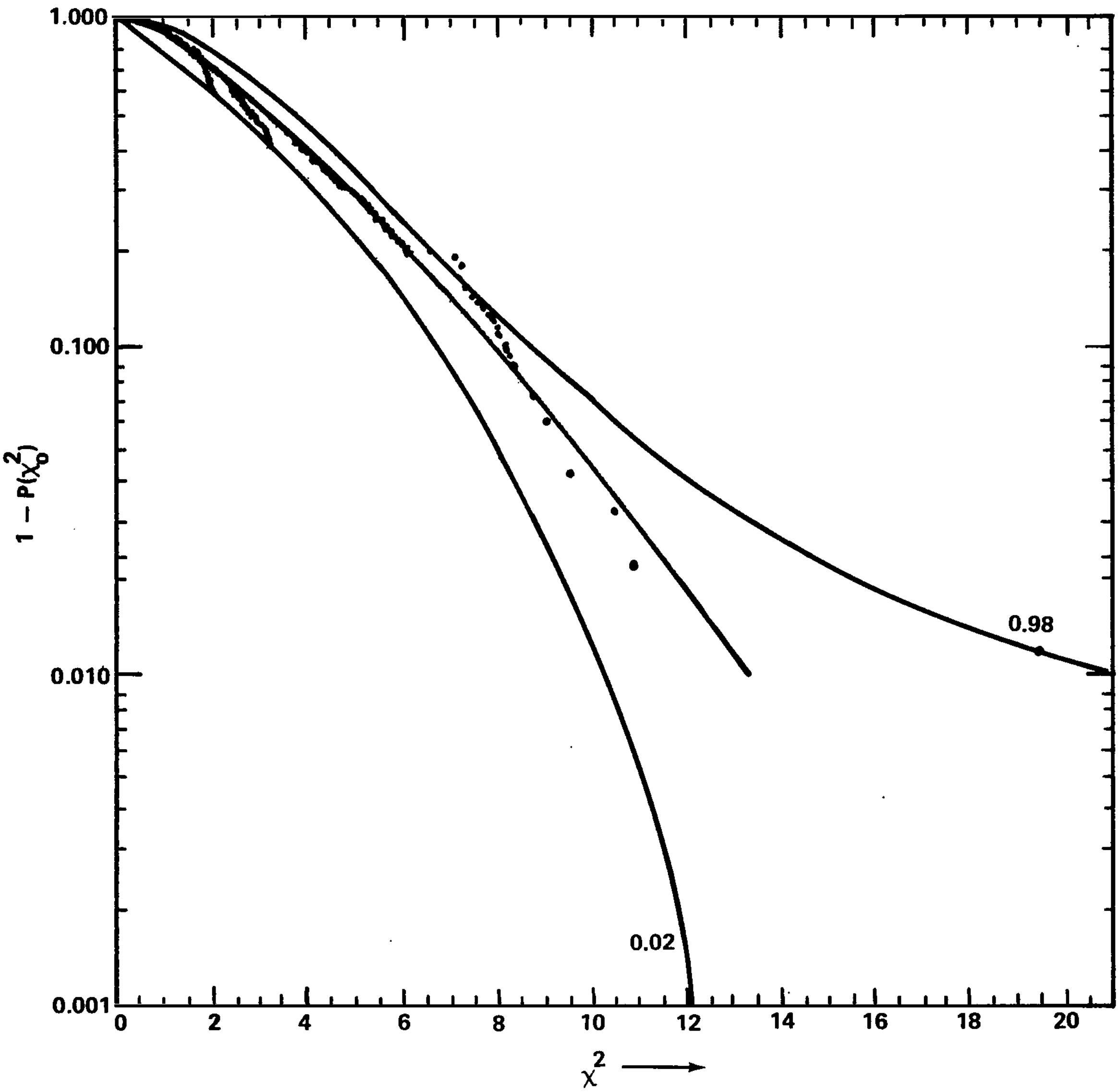


Figure 26. Four-dimensional chi-square ordered distribution of 100 consecutive data points of wind component deviations (m/s), u′, v′ and w′, and temperature deviations in degrees Celsius during the Round Hill Turbulence Measurements Program at Round Hill, MA, October 5, 1961, 0840-0842 EST with the central 0.96 confidence bands. (The elevation is 16 meters. The central line is the theoretical line of best fit. The outside bounding lines give the central 0.96 probability confidence bands for a sample size of 101.) (Cramer, 1970)

B. Wind – Surface

Data for examples of trivariate distributions were supplied by Adelfang (1970). Wind gust data were obtained a few meters above the desert at Palmdale, California using a Vector Vane, Adelfang (1970). The data were furnished as run 19, March 3, 1969, beginning at 1330 PST. The 1 hour period provides 3600 data points in three-dimensional space. The wind components are longitudinal, lateral, and vertical. The units are in ft/s. The chi-squares computed are dimensionless in terms of units. Two samples of 300 data points each were selected.

Figure 27 shows the three-dimensional chi-square order distribution of 300 consecutive data points of wind components in ft/s, u, v, and w in gust research a few meters above the desert. Longitudinal, lateral, and vertical components are u, v, and w, respectively. The bounding lines define the central 0.96 confidence region for a sample size of 101. The region will be a little narrower for the actual sample size of 300. The theoretical line of best fit is the central heavy line which may be obtained from Table 1 or Figure 6. The null hypothesis of multivariate normality is not rejected.

Figure 28 shows a second subset of the gust data. Again, the subset remains well within the 0.96 central confidence band. Therefore, the null hypothesis is not rejected and the usual probability inferences may be drawn for this multivariate distribution.

If the assumption of multivariate normality is valid and if 100 separate uncorrelated runs of this type were analyzed, four of the 100 would be expected to indicate rejection of the null hypothesis.

The MAD, MSSR, RAL, RBL, RAM, RBM, and runs of n-tant signs tests are not discussed for these data.

C. Wind – Upper Air

The assumption of normality of the two-dimensional distribution of upper air winds has been made by Brooks et al. (1946), Brooks et al. (1950), and Crutcher (1957). Groenewoud et al. (1967) provide several examples of applications to geophysical data and provide extensive tables to provide easy calculation of probabilities over specified regions.

Figures 29, 30, and 31 show chi-square distributions of upper winds in the two-dimensional distribution of the zonal and meridional components and shears. The data are taken from records at the National Climatic Center prepared and placed on magnetic tape for the National Aeronautics and Space Administration, Marshall Space Flight Center, Huntsville, Alabama 35812. The data are for Cape Kennedy, Florida. The altitudes selected are 8, 12, and 16 km above the ground. Figure 29 shows the 8 km

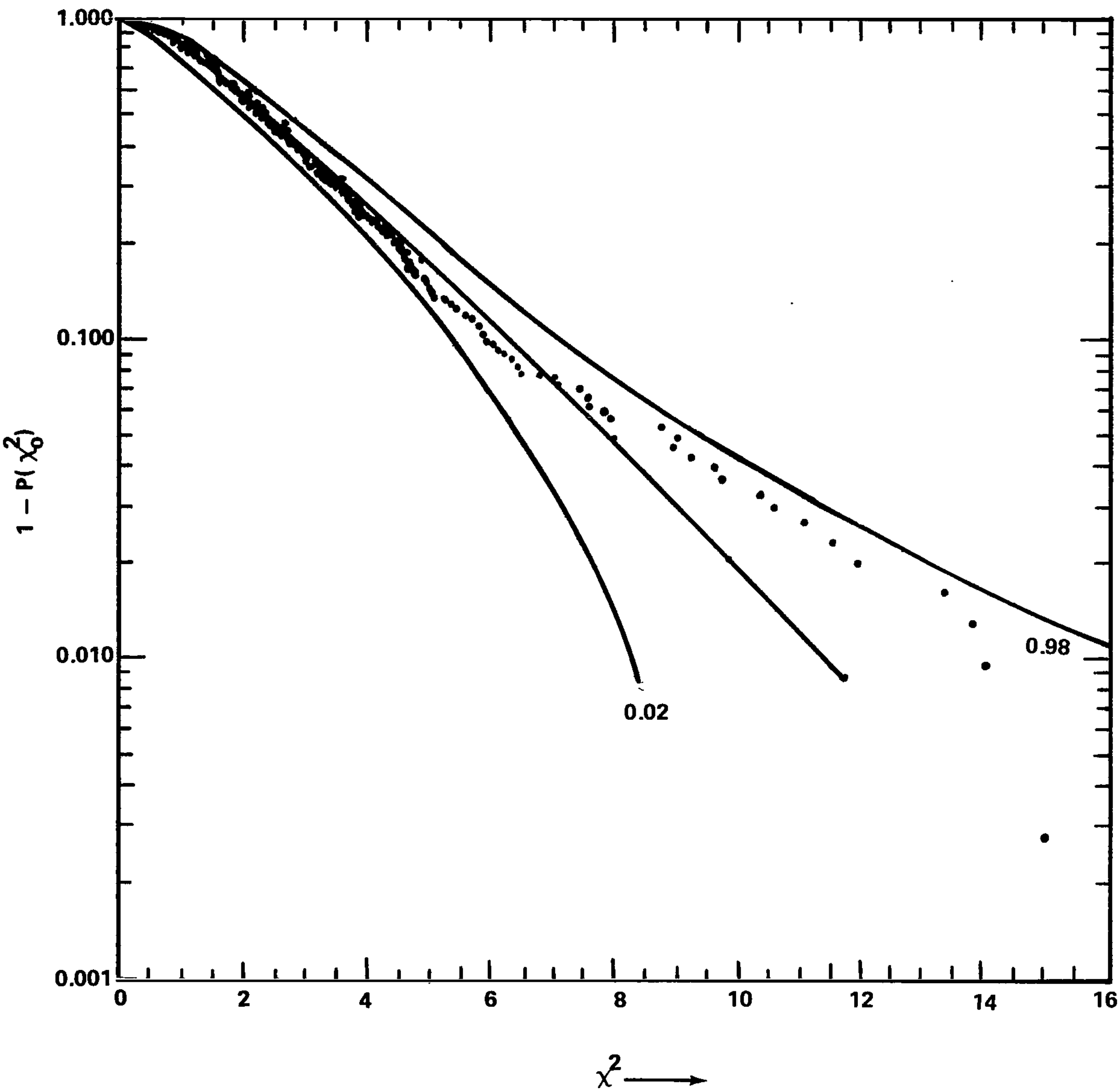


Figure 27. Three-dimensional chi-square ordered distribution of 300 consecutive data points of wind components, u, v, and w (m/s) in gust research measured 4.7 meters above the desert at Palmdale, California using a Vector Vane, March 21, 1969, 1330 PST. (Adelfang, 1970).

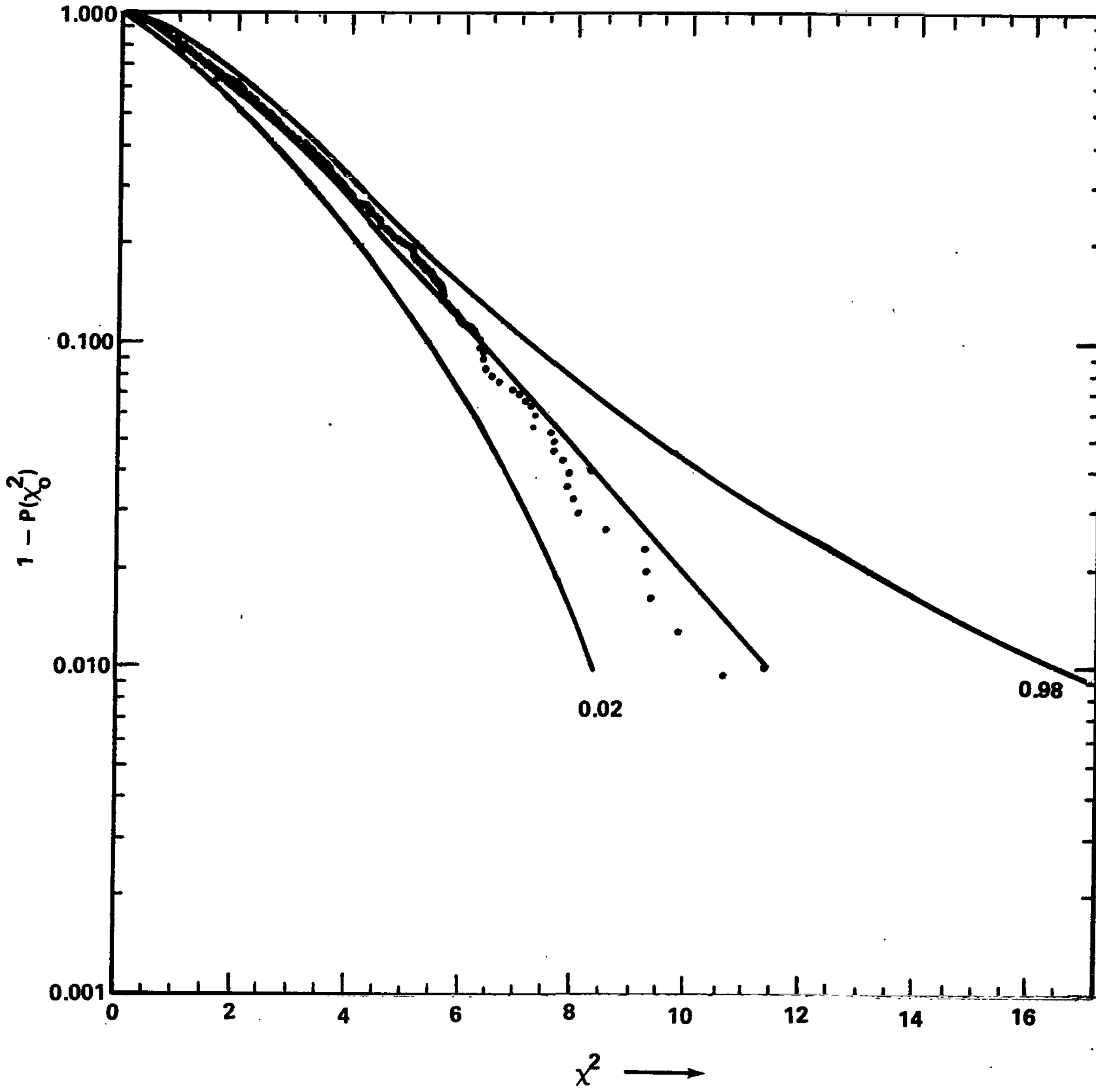


Figure 28. Three-dimensional chi-square ordered distribution of 300 consecutive data points of wind components, u, v, and w (m/s) in gust research measured 4.7 meters above the desert at Palmdale, California using a Vector Vane, March 3, 1969, 1330 PST. (Adelfang, 1970).

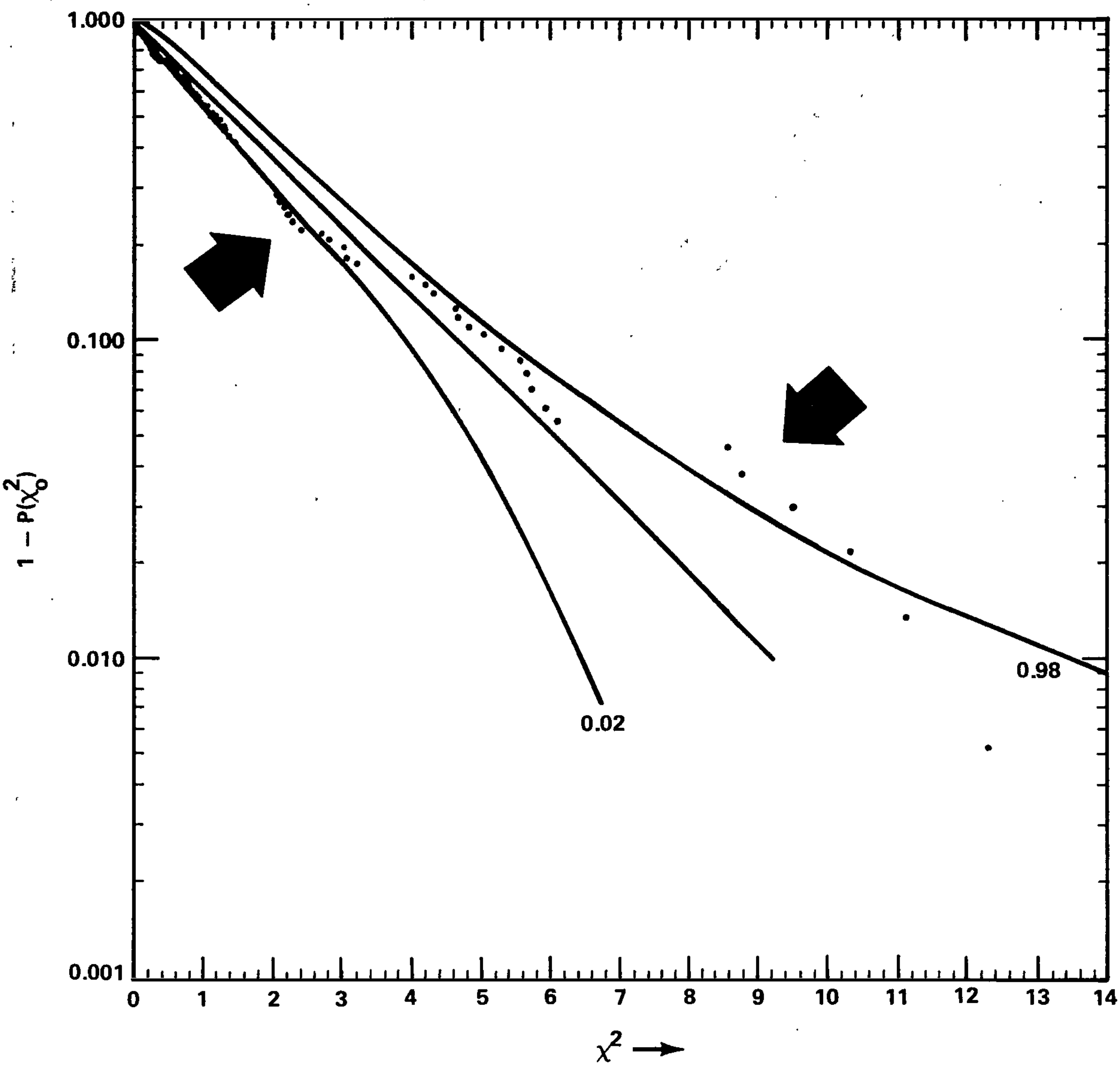


Figure 29. Two-dimensional chi-square ordered distribution of 124 data points of Cape Kennedy, FL, upper wind zonal and meridional components u and v (m/s) at 8 km during the month of January; January 1, 1956 to December 31, 1967. (The central 0.96 confidence band is shown.)

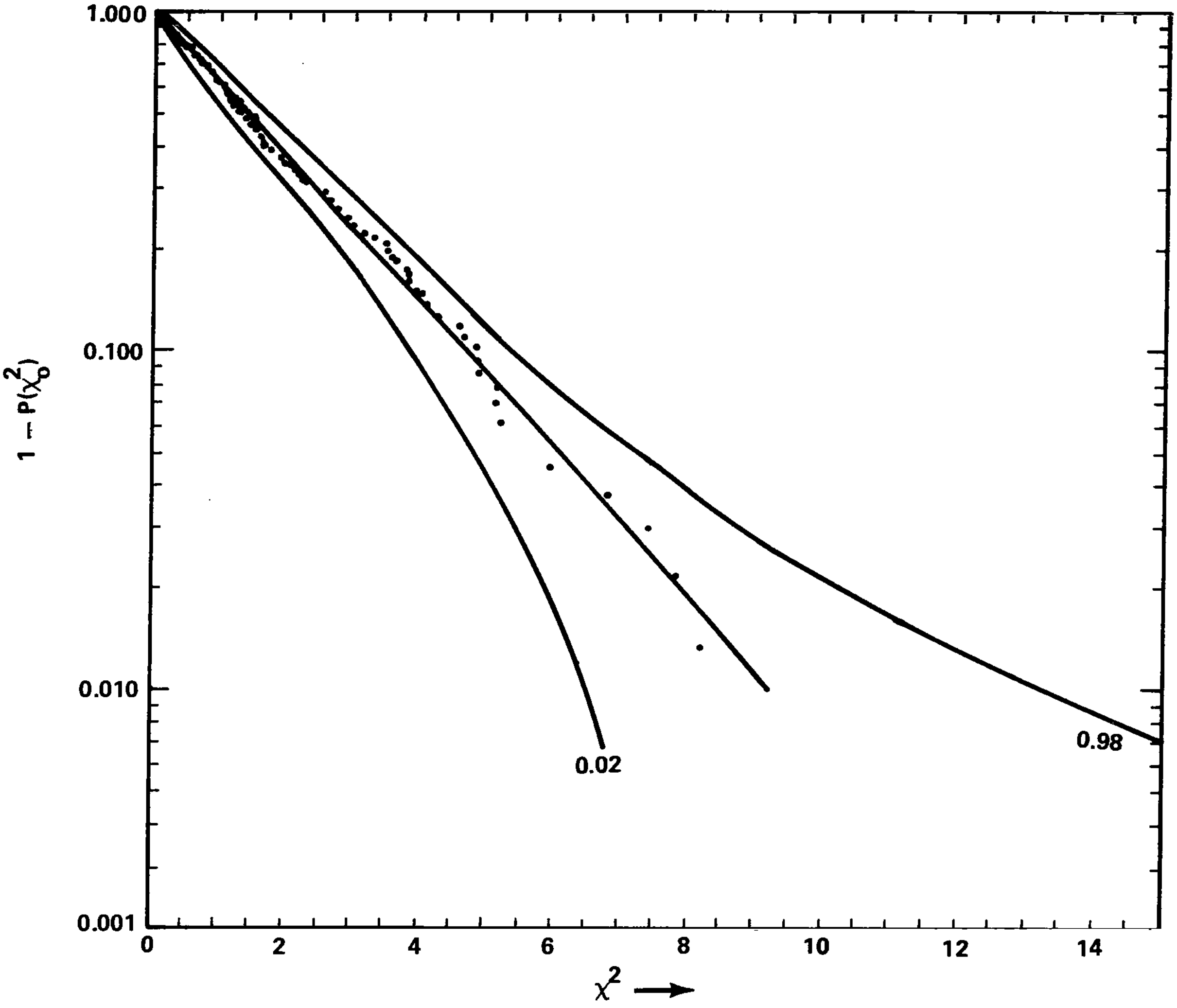


Figure 30. Two-dimensional chi-square ordered distribution of 124 data points of Cape Kennedy, FL, upper wind components u and v (m/s) at 16 km during the month of January; January 1, 1956 to December 31, 1967. (The central 0.96 confidence band is shown.)

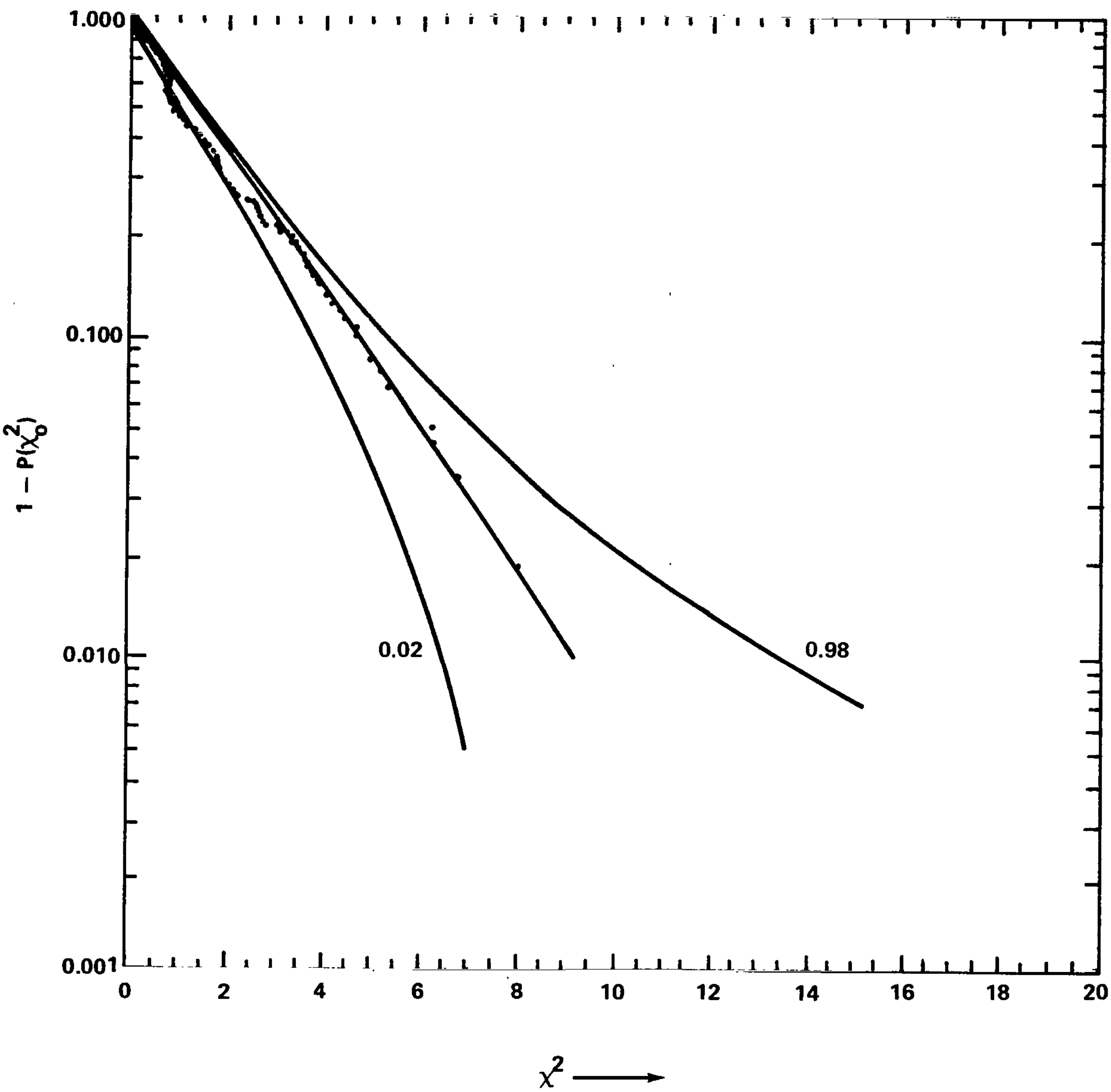


Figure 31. Two-dimensional chi-square ordered distribution of 124 data points of Cape Kennedy, FL, upper wind zonal and meridional component shears (m/s) between altitudes 12 km and 8 km, for the month of January. (The central 0.96 confidence band is shown.)

distribution while Figure 30 shows the 16 km distribution. Figure 31 shows the distribution of differences (shears) between the 12 km and 8 km winds. The central 0.96 probability confidence band for a sample size of 101 is shown. Tables have not been prepared for the sample size used here, 124. Figure 29 is included to provide an example of rejection of the null hypothesis of two-dimensional or bivariate normality. Arrows point to the regions suggesting rejection. It is possible that the central 0.99 probability confidence band, i.e., an alpha (α) level of rejection of 0.01, would not call for rejection. Figure 30 illustrates nonrejection of the null hypothesis. Sums and differences of normally distributed variables also are distributed normally. Figure 31 is included to show an example of nonrejection of wind shears between 8 and 12 km above the ground. A similar plot of 12 km component winds (not shown here) indicates nonrejection of the null hypothesis at the 0.04 rejection level, i.e., with the central 0.96 confidence band. Although the plot of 8 km winds in Figure 29 implies rejection for that level, the shears between 8 and 12 km (Fig. 31) are assumed to be described by the bivariate normal distribution.

An examination of Figure 29 indicates that four chi-square values between 8 and 11 are one reason for the rejection. Examination of these winds from a quality assurance point of view might indicate some error. Any change in these winds would change the value of all chi-squares as there would be a shift in the vector mean. Such a change or changes more than likely would also materially change the relative magnitude of the smaller vectors. Thus, the other rejection region near a chi-square value of 2 to 3 might also be changed. Such a procedure is only suggested if plots such as these are used for quality assurance or edit checks of data. It may also be pointed out that Figure 29 as a single presentation also might exemplify a mixture of two bivariate normal distributions due to the S-shape of the empirical plot. If this is true, its effect is not great because the shears shown in Figure 31 do not support this hypothesis. The inference that upper air component winds and shears are distributed normally is not rejected here.

The MAD, MSSR, RAL, RBL, RAM, RBM, and runs of n-tant signs tests are not discussed for these data.

D. Hurricane Motions

Hurricanes are a part of the general circulation of the atmosphere. As such, they appear to move under the complete control of the general circulation. At other times they do not appear to be under such control and seem to move as an entity. These movements in distances in stated time periods appear to have some consistency in direction and speed. Hurricane movement statistics are presented in many forms.

Hurricanes develop during the warm season of the year and usually in unison with the general circulation pattern. The occurrence and frequency of hurricanes vary with the region. Crutcher (1971), for the North Atlantic, Caribbean, and Gulf regions, showed that sets of hurricane movements could be described by the two-dimensional normal distribution. The variates were the components along two sets of orthogonal axes imposed over the usual latitude-longitude coordinates. The geographical areas selected were bounded by the 5° longitude and latitude meridians. The squares thus located were designated by a four digit number. The first two digits gave the southern latitude boundary of the square, while the second two digits multiplied by five gave the western boundary. Thus, the identifier 2014 would be a quadrangle bounded on the south by the 20°N latitude line and on the west by the 70°W longitude line.

Sets of the same hurricane data base have been selected to illustrate the tests developed for the present report. These tests include the χ^2, MAD, MSSR, RAL, RBL, RAM, RBM, and the quadrant signs (QS) tests at the 0.04 level of rejection. The null hypothesis being tested is that the hurricane distributions are not significantly different from the two-dimensional normal distribution.

The tests are made on the squares of the standardized vector deviations from the centroid of the distribution. Only these chi-squares are shown in the tables and graphs.

The data are September-October 24 hour hurricane movements for the 5° quadrangles in the North Atlantic, Caribbean, and Gulf regions, 2514, 2515, 2014, 2516, 3015, and 2015. The sample sizes are 11, 21, and 31. The movements were taken in the chronological order of their occurrence over the time period.

The tests are summarized in Tables 16 and 17. The first column in the tables gives the order number of the sample positions and the identification of the test. The second and last columns give the 0.02 and 0.98 empirical probability levels for the tests. The remaining columns provide the results of the tests for the quadrangle identified by the latitude and longitude heading the column. An asterisk indicates where a bound has been exceeded requiring rejection of the null hypotheses.

Table 16 provides results of tests for quadrangles 2514, 2515, and 2014 for sample sizes 11, 21, and 31, respectively. Table 17 provides similar results for quadrangles 2516, 3015, and 2015.

The χ^2 test devised in this paper implied rejection of the null hypothesis for the data sets of quadrangles 2014 and 2015. This prompted a look at the data since quadrangle 2015 lies just west of quadrangle 2014.

Prior screening had removed all tropical depressions from the tropical storm data. Also, only one 24 hour movement for any storm was retained for the data set. However, the same storm could appear in two or more data sets for different quadrangles. Examination of the observations indicated one or two seeming outliers in each set. In one case this occurred in quadrangles 2014 and 2015. These were extraordinarily fast moving hurricanes. Also, the hurricanes occurred in the earlier period of the record and because each sample built on the prior chronologically, a rejection in the first sample would be expected to cause rejection in succeeding samples until a statistically representative sample was reached. For example, Table 17 shows that a sample size of 31 does not imply the rejection that its subsets, sample sizes 11 and 21, indicate. Sample sizes 11 and 21 showing rejection were included to illustrate the technique. Ten quadrangles were tested of which only six are illustrated. Two of the six (2014 and 2015) exhibit rejection which was caused in one quadrangle pair by the same storm.

The MAD test, which is a Kolmogorov-Smirnov test, implied rejection of the null hypothesis for only quadrangle 2014 at the 0.04 rejection level. The value of 0.1876 is not significant at the 0.01 rejection level.

The MAD test supports the χ^2 rejection for quadrangle 2014.

The MSSR tests imply rejection of the null hypothesis only for squares 2014 and 2015. This test substantiates the χ^2 test for rejection.

The RAL and RBL imply rejection of square 2514 with a sample size 31 and square 2014 with sample sizes 21 and 31. The χ^2 test previously made shows that rejection for square 2014 is substantiated. This test rejects the null hypothesis for the three quadrangles 2516, 3015, 2015. There are too few runs above and below the median line. This implies a mixture of two distributions insofar as magnitude of distance traveled is concerned. Direction is not considered here. However, these premises are not examined here.

The RAM and RBM imply no rejection of any of the null hypothesis. This implies a certain homogeneity in the data that is not implied in the RAL and RBL tests. There does not seem to be a clustering insofar as time is concerned. However, direction is not a consideration.

None of the tests involve any more than tests for the magnitude of the vectors in the multivariate distribution. The last test shown in Tables 16 and 17, the quadrant sign (QS) test, considers direction of hurricane movement. The QS test is deficient because it considers only the direction of the vector radiating from the centroid and not the direction and magnitude simultaneously. However, it does provide additional information as one of the necessary, but not sufficient, tests supplementing those shown previously.

The QS test is based on a grand sample of 14 sets, each set comprised of 100 samples each for sample sizes of 11, 15, 21, 25, 31, 51, and 101. A summary is presented in Table 15. The portions of the table are identified by the dimensions 2, 3, 4, and 5. Only that portion of Table 15 for two dimensions is used for the hurricane movement examples discussed here.

Referring to Tables 16 and 17, the minimum and maximum QS values observed in each run are placed under the 0.02 and 0.98 probability columns. The QS values shown in Tables 16 and 17 are those for the runs of length two which is the second column of the two-dimensional part of Table 15. The empirical bounds given in Table 15 (for two dimensions) are exceeded in 4 cases out of the 18 examples shown.

The changes of the deviations from one quadrant to another around the centroid seem to be quasi-random. The 11 random changes in a sample of 11 (Table 17) show that in no case was a deviation in one quadrant followed by another in the same quadrant. In a one-dimensional problem this would imply a highly oscillatory process. This may be the case here, but it would require switching from one quadrant to another and back again on a rotating vector. This is possible but not very likely. The fact that it occurred in only one of the six cases for a sample size of 11 implies that it does not occur often. When 10 more sequential data were added, the centroid shifted and the quadrant sign changes fell within bounds. This implies that it was a small sample problem. Therefore, the rejection of the null hypothesis in this case is discounted. Insofar as directional changes are concerned, the null hypothesis that the data set is not different from a bivariate normal distribution is not rejected.

Figures 32 through 37 illustrate the hurricane data shown in Tables 16 and 17. The 0.02 and 0.98 empirical confidence bands are shown in each figure. Each figure contains the data for three quadrangles for sample size 11, 21, or 31. Each quadrangle is represented by one symbol, an X, box, or asterisk. If α percent (α is the rejection level) or more of the sequential symbols fall outside the 0.02 and 0.98 bounds, the null hypothesis of bivariate normality is rejected. The decisions reached regarding the null hypotheses in the discussion of the tables is graphically supported.

Another test for direction not illustrated here is based on the fact that the multivariate normal distribution is spherical in its standardized and eigenvalue-eigenvector form. All directions are equally possible and the distribution of the directions from the centroid is the uniform or rectangular. In higher dimensions this would be hypercubical. Andrews et al. (1973), as pointed out previously, discuss this in more detail in Multivariate Analysis, Krishnaiah (1972). This test is a necessary but insufficient test since it considers only the direction.

The appendix presents graphs for plotting the 0.02, 0.50, and 0.98 probability levels given in Table 4. These graphs permit plotting of empirical chi-square (squared radii) values and subsequent testing for multivariate normality for the dimensions and sample sizes shown. The rejection level, α, is 0.04.

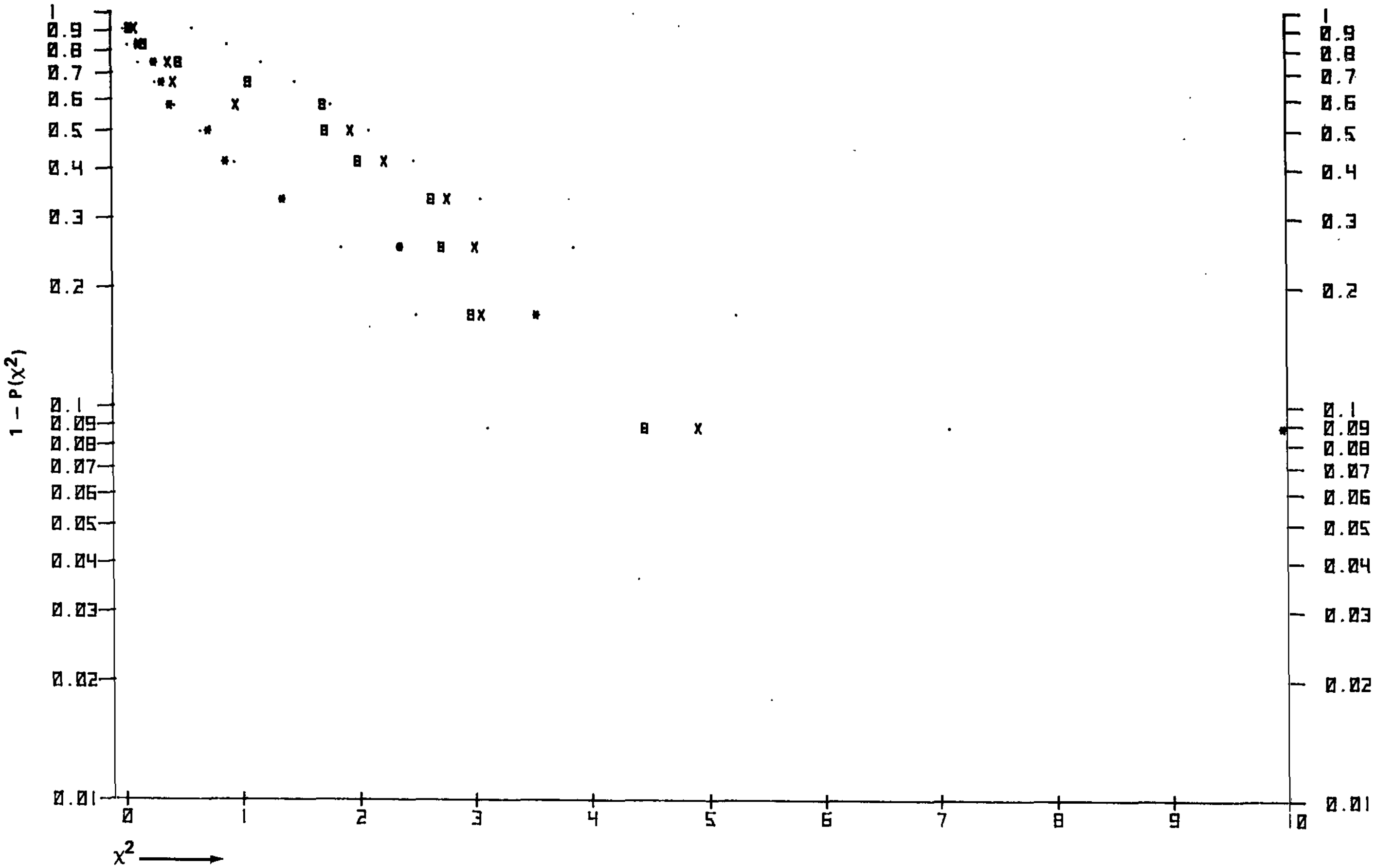


Figure 32. Chi-square (vector deviation square) values of September-October 24 hour hurricane movements versus the empirical $(1\text{-}p(\chi^2))$ value of occurrence (n=11). (The 0.02 and 0.98 probability lines provide the central 0.96 confidence band. The symbols represent data from one 5 degree longitude-latitude-quadrangle.) (See Table 16.)

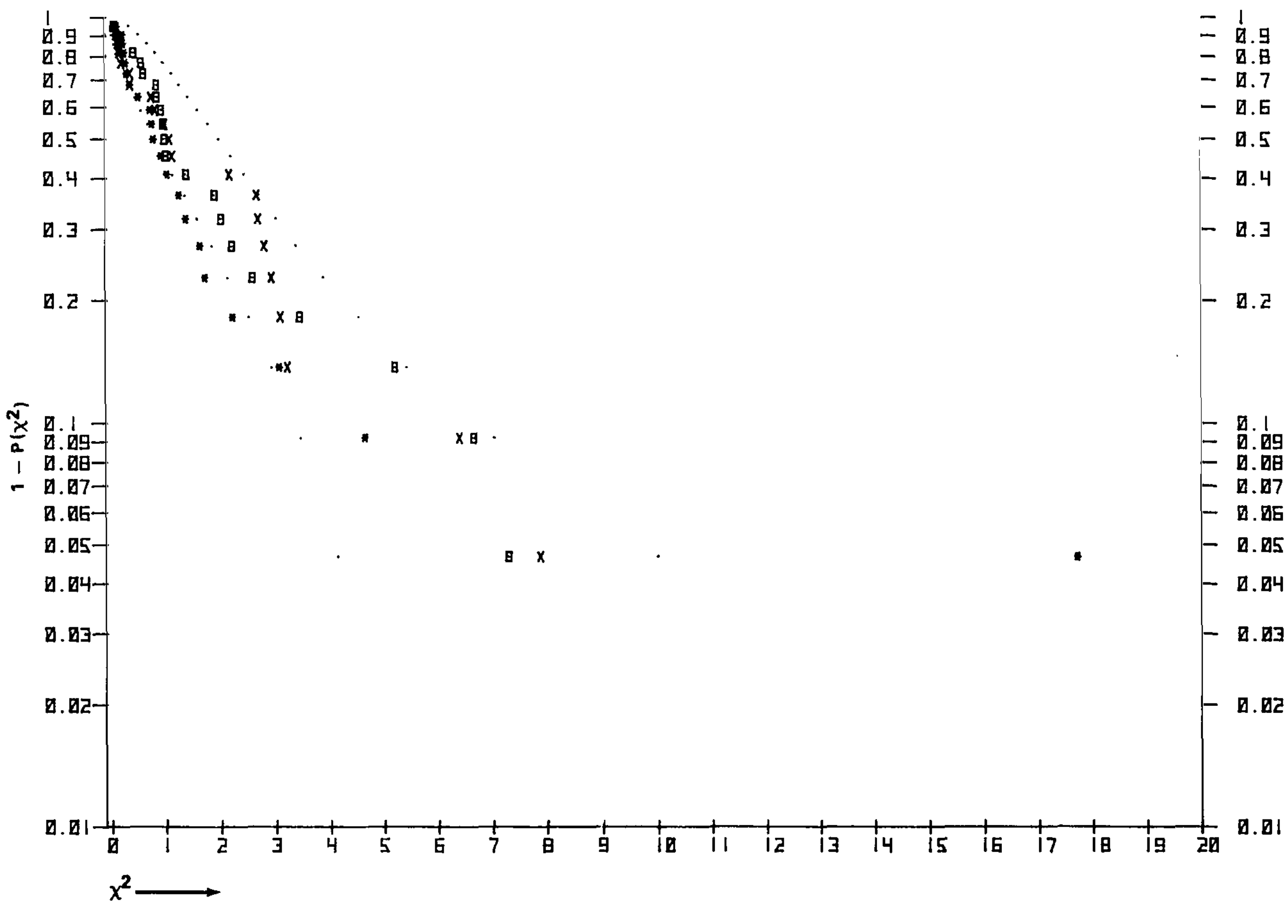


Figure 33. Chi-square (vector deviation square) values of September-October 24 hour hurricane movements versus the empirical (1-p(χ^2)) value of occurrence (n=21). (The 0.02 and 0.98 probability lines provide the central 0.96 confidence band. The symbols represent data from one 5 degree longitude-latitude-quadrangle.) (See Table 16.)

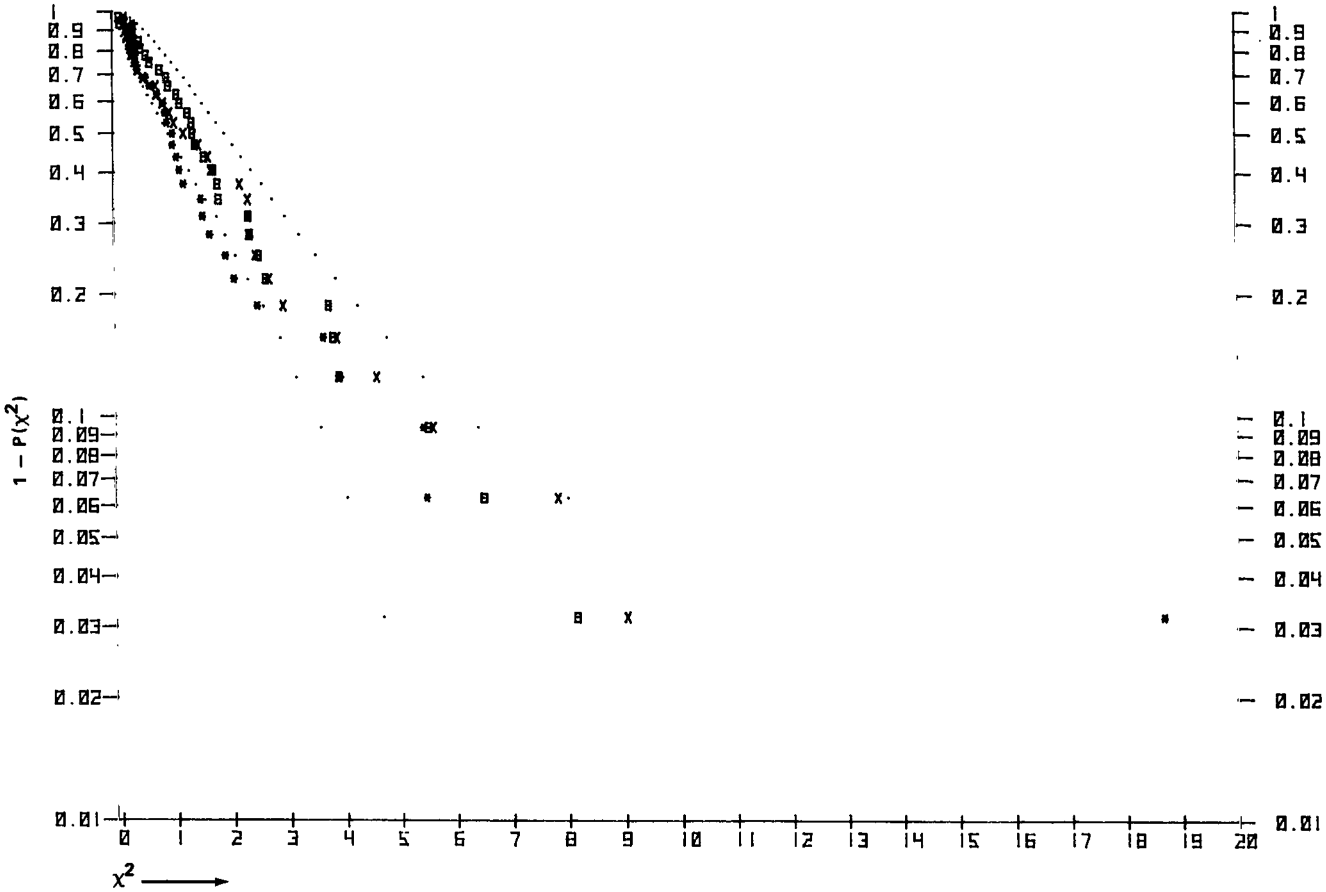


Figure 34. Chi-square (vector deviation square) values of September-October 24 hour hurricane movements versus the empirical $(1\text{-}p(\chi^2))$ value of occurrence (n=31). (The 0.02 and 0.98 probability lines provide the central 0.96 confidence band. The symbols represent data from one 5 degree longitude-latitude-quadrangle.) (See Table 16.)

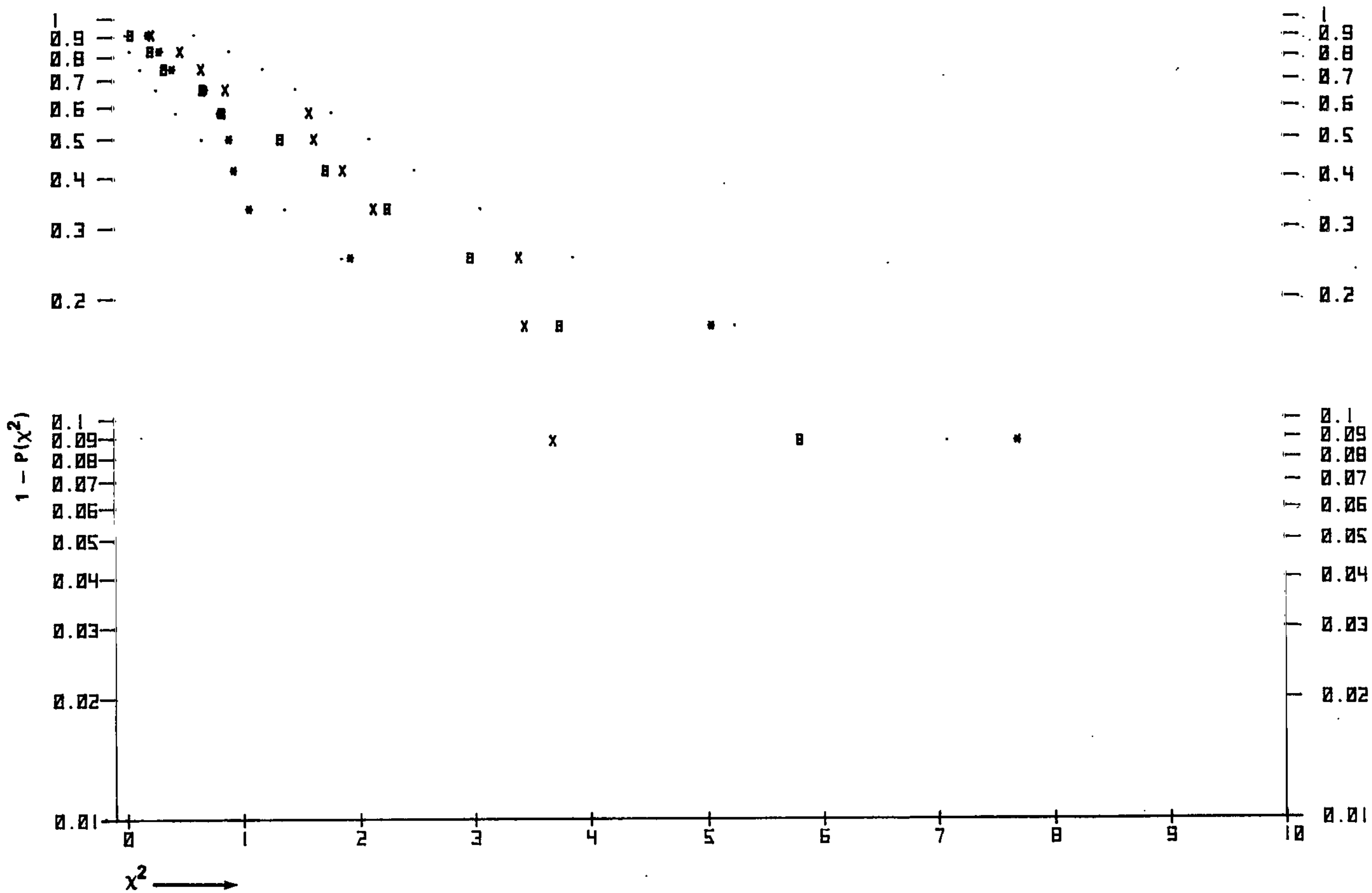


Figure 35. Chi-square (vector deviation square) values of September-October 24 hour hurricane movements versus the empirical (1-p(χ^2)) value of occurrence (n=11). (The 0.02 and 0.98 probability lines provide the central 0.96 confidence band. The symbols represent data from one 5 degree longitude-latitude-quadrangle.) (See Table 17.)

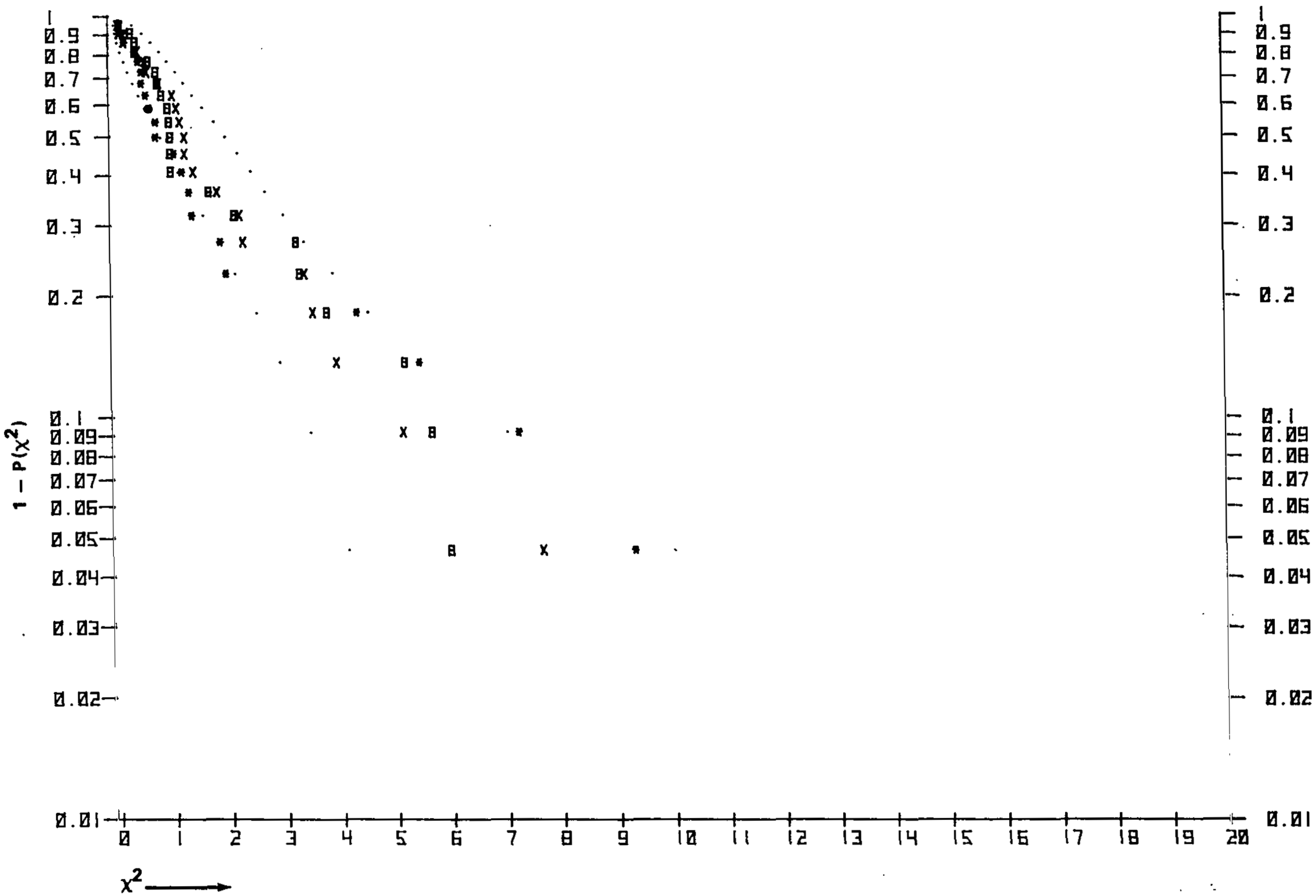


Figure 36. Chi-square (vector deviation square) values of September-October 24 hour hurricane movements versus the empirical (1-p(χ^2)) value of occurrence (n=21). (The 0.02 and 0.98 probability lines provide the central 0.96 confidence band. The symbols represent data from one 5 degree longitude-latitude-quadrangle.) (See Table 17.)

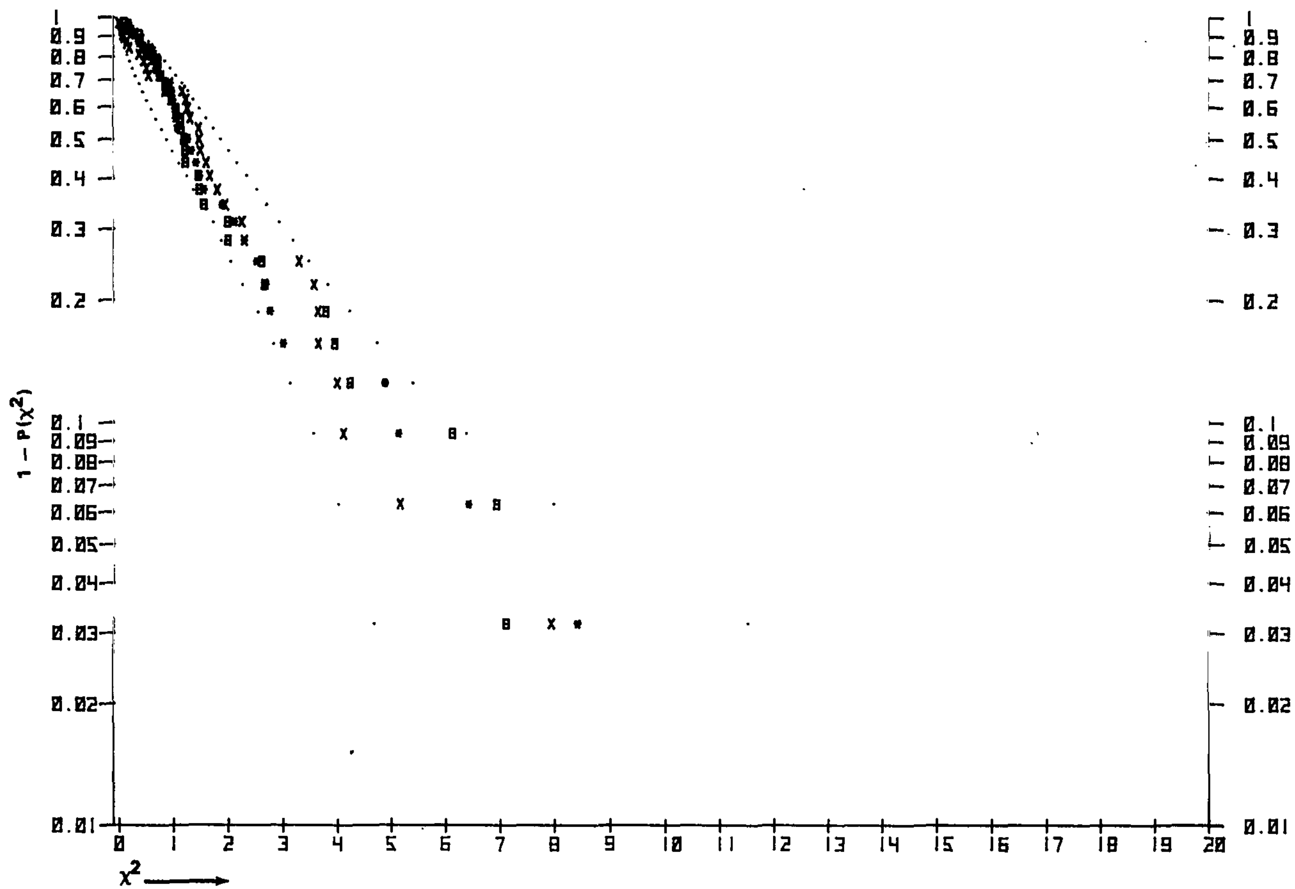


Figure 37. Chi-square (vector deviation square) values of September-October 24 hour hurricane movements versus the empirical $(1\text{-}p(\chi^2))$ value of occurrence (n=31). (The 0.02 and 0.98 probability lines provide the central 0.96 confidence band. The symbols represent data from one 5 degree longitude-latitude-quadrangle.) (See Table 17.)

VI. SUMMARY

Work at the National Aeronatuics and Space Administration, Marshall Space Flight Center at Huntsville, Alabama, and at the National Oceanic and Atmospheric Administration, Environmental Data Service, National Climatic Center (for NASA and the U.S. Navy) required some assessment of the distributions of multivariate or vector-form data sets.

One of the basic statistical distributions is the multivariate normal. When only one dimension or variate is considered, it usually is called the normal distribution. Adequate and well-known tests for testing in the normal higher dimensional forms, such as the bivariate and trivariate, are not readily available.

We realize that the studies in this report are incomplete, that substantiation by others is required, and that extensions should be made. We also realize that the number of samples used in the small sample studies should be increased so that more stable configurations would be obtained. Also, a larger range of sample sizes should be processed and the results made available.

The work discussed in this document was initiated approximately 6 years ago to provide us with statistical tools to test multivariate distributions for normality. We are reasonably confident that these tools will be useful to others.

George C. Marshall Space Flight Center
National Aeronatuics and Space Administration
Marshall Space Flight Center, Alabama 35812, December 1975

TABLE 1. FRACTILES (QUANTILES) OF THE χ^2 DISTRIBUTION

P / ν	Probability in Per Cent 0·05	0·1	0·5	1·0	2·5	5·0	10·0	20·0	30·0	40·0
1	$\cdot 0^{5}393$	$\cdot 0^{5}157$	$\cdot 0^{4}393$	$\cdot 0^{3}157$	$\cdot 0^{3}982$	$\cdot 0^{2}393$	·0158	·0642	·148	·275
2	$\cdot 0^{2}100$	$\cdot 0^{2}200$	·0100	·0201	·0506	·103	·211	·446	·713	1·02
3	·0153	·0243	·0717	·115	·216	·352	·584	1·00	1·42	1·87
4	·0639	·0908	·207	·297	·484	·711	1·06	1·65	2·19	2·75
5	·158	·210	·412	·554	·831	1·15	1·61	2·34	3·00	3·66
6	·299	·381	·676	·872	1·24	1·64	2·20	3·07	3·83	4·57
7	·485	·598	·989	1·24	1·69	2·17	2·83	3·82	4·67	5·49
8	·710	·857	1·34	1·65	2·18	2·73	3·49	4·59	5·53	6·42
9	·972	1·15	1·73	2·09	2·70	3·33	4·17	5·38	6·39	7·36
10	1·26	1·48	2·16	2·56	3·25	3·94	4·87	6·18	7·27	8·30
11	1·59	1·83	2·60	3·05	3·82	4·57	5·58	6·99	8·15	9·24
12	1·93	2·21	3·07	3·57	4·40	5·23	6·30	7·81	9·03	10·2
13	2·31	2·62	3·57	4·11	5·01	5·89	7·04	8·63	9·93	11·1
14	2·70	3·04	4·07	4·66	5·63	6·57	7·79	9·47	10·8	12·1
15	3·11	3·48	4·60	5·23	6·26	7·26	8·55	10·3	11·7	13·0

Probability in Per Cent 50·0	60·0	70·0	80·0	90·0	95·0	97·5	99·0	99·5	99·9	99·95	P / ν
·455	·708	1·07	1·64	2·71	3·84	5·02	6·63	7·88	10·8	12·1	1
1·39	1·83	2·41	3·22	4·61	5·99	7·38	9·21	10·6	13·8	15·2	2
2·37	2·95	3·67	4·64	6·25	7·81	9·35	11·3	12·8	16·3	17·7	3
3·36	4·04	4·88	5·99	7·78	9·49	11·1	13·3	14·9	18·5	20·0	4
4·35	5·13	6·06	7·29	9·24	11·1	12·8	15·1	16·7	20·5	22·1	5
5·35	6·21	7·23	8·56	10·6	12·6	14·4	16·8	18·5	22·5	24·1	6
6·35	7·28	8·38	9·80	12·0	14·1	16·0	18·5	20·3	24·3	26·0	7
7·34	8·35	9·52	11·0	13·4	15·5	17·5	20·1	22·0	26·1	27·9	8
8·34	9·41	10·7	12·2	14·7	16·9	19·0	21·7	23·6	27·9	29·7	9
9·34	10·5	11·8	13·4	16·0	18·3	20·5	23·2	25·2	29·6	31·4	10
10·3	11·5	12·9	14·6	17·3	19·7	21·9	24·7	26·8	31·3	33·1	11
11·3	12·6	14·0	15·8	18·5	21·0	23·3	26·2	28·3	32·9	34·8	12
12·3	13·6	15·1	17·0	19·8	22·4	24·7	27·7	29·8	34·5	36·5	13
13·3	14·7	16·2	18·2	21·1	23·7	26·1	29·1	31·3	36·1	38·1	14
14·3	15·7	17·3	19·3	22·3	25·0	27·5	30·6	32·8	37·7	39·7	15

Abridged with permission of:

1. Professors A. Hald and A. Sinkbaek and their publishers, Skandinavisk Aktuerietidskrift, as shown in Statistical Tables and Formulas published by John Wiley and Sons, Table V, 1952.
2. Professor E. S. Pearson and to the Biometrika Trustees from the χ^2 tables (Tables 7 and 8) presented in Biometrika Tables for Statisticians, Vol. 1, 1956.

TABLE 2. MEDIANS OF SELECTED PERCENTILES OF 10 100 RANDOM CHI-SQUARE VALUES OF THE MULTIVARIATE NORMAL DISTRIBUTION WITH ZERO MEANS AND COVARIANCES (NID (0,1,0)) AND THEIR PROBABILITY PLOTTING POSITIONS p = (1 - ((i - 0.5)/n)) FOR 3 DIMENSIONS

n = 7

Probability	0.929	0.786	0.643	0.500	0.357	0.214	0.071
				i			
Percentile	1	2	3	4	5	6	7
2	.1104	.6143	1.1836	1.6272	2.5878	3.3996	4.5556
5	.2314	.8786	1.5023	2.2215	3.0202	3.9535	5.2391
10	.3801	1.1350	1.8788	2.6418	3.4505	4.4632	5.8456
50	1.2748	2.3006	3.2372	4.1415	5.1009	6.3326	8.2563
90	2.6101	3.7817	4.7914	5.8407	7.0022	8.5381	11.3032
95	3.0306	4.2132	5.2928	6.3367	7.5058	9.1838	12.2894
98	3.4971	4.7349	5.7669	6.8765	8.1216	9.8958	13.1932

n = 31

Probability	0.984	0.952	0.920	0.887	0.855	0.823	0.790	0.758	0.723
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0400	.2060	.4084	.6117	.8164	1.0313	1.2246	1.4230	1.6059
5	.0804	.2998	.5425	.7638	.9904	1.1881	1.4086	1.5918	1.7865
10	.1393	.4076	.6674	.8954	1.1431	1.3551	1.5841	1.7881	1.9850
50	.4906	.8762	1.1845	1.4668	1.7387	1.9883	2.2096	2.4235	2.6533
90	1.0456	1.4679	1.8270	2.1052	2.3915	2.6463	2.9058	3.1442	3.3589
95	1.2089	1.6763	2.0241	2.3269	2.5962	2.8373	3.0916	3.3199	3.5734
98	1.4113	1.8432	2.2114	2.5249	2.7755	3.0436	3.2960	3.5350	3.7779

TABLE 2. (Continued)

Probability	0.693	0.661	0.629	0.597	0.565	0.532	0.500	0.468	0.435
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	1.7752	1.9798	2.1800	2.3679	2.5388	2.7254	2.9408	3.1409	3.3556
5	2.0002	2.1947	2.4024	2.5953	2.7739	2.9787	3.1818	3.3767	3.5971
10	2.1820	2.3733	2.5950	2.7805	2.9782	3.1795	3.3932	3.5873	3.8111
50	2.8638	3.0822	3.2674	3.4939	3.7092	3.9223	4.1422	4.3711	4.5959
90	3.5788	3.8086	4.0202	4.2600	4.4863	4.6905	4.9174	5.1627	5.3962
95	3.7924	4.0209	4.2564	4.4795	4.6964	4.9174	5.1465	5.4068	5.6106
98	4.0373	4.2118	4.4615	4.6654	4.9073	5.1546	5.4012	5.6740	5.8791

Probability	0.403	0.371	0.339	0.307	0.277	0.242	0.210	0.177	0.145
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	3.5401	3.7882	3.9046	4.1767	4.4399	4.6579	4.9150	5.2103	5.5207
5	3.7867	4.0084	4.2544	4.4649	4.7103	4.9607	5.2520	5.5545	5.8988
10	4.0307	4.2488	4.4640	4.7025	4.9754	5.2531	5.5533	5.8841	6.2251
50	4.8232	5.0660	5.3147	5.5923	5.8871	6.1932	6.5322	6.8931	7.3332
90	5.6616	5.9613	6.2350	6.5306	6.8679	7.2096	7.6148	8.0291	8.5487
95	5.8902	6.1862	6.5246	6.7933	7.1436	7.4972	7.9071	8.3522	8.8911
98	6.1518	6.4698	6.7777	7.0770	7.4097	7.8382	8.2286	8.7019	9.2488

TABLE 2. (Concluded)

Probability	0.113	0.081	0.048	0.016
		i		
Percentile	28	29	30	31
2	5.9038	6.3420	6.9085	7.7307
5	6.2985	6.7704	7.3517	8.2260
10	6.6280	7.1068	7.7466	8.7292
50	7.8087	8.4499	9.2524	10.7251
90	9.1481	9.9266	11.0975	13.3151
95	9.5535	10.3608	11.6467	14.1185
98	9.9906	10.8254	12.1919	15.0132

TABLE 3. VALUES OF "c" DERIVED FROM FORMULA $c = 1 - (\nu/4) - (4 \ln n/n)$ WHERE ν IS THE NUMBER OF DIMENSIONS, n IS THE NUMBER OF DATA, AND ln IS THE NATURAL LOGARITHM

ν \ n	11	15	21	25	31	51	101
1	-0.12196	+0.02785	+0.17009	+0.23498	+0.30690	+0.44162	+0.56722
2	-0.31796	-0.22215	-0.07991	-0.01502	+0.05690	+0.19162	+0.31722
3	-0.62196	-0.47215	-0.32991	-0.26502	-0.19310	-0.05838	+0.06722
4	-0.87196	-0.72215	-0.57991	-0.51502	-0.44310	-0.30838	-0.18278
5	-1.12196	-0.97215	-0.82991	-0.76502	-0.69310	-0.55838	-0.43278

TABLE 4. PROBABILITY PLOTTING POSITIONS BASED ON EQUATION $1 - p = 1 - (i - c)/(n - 2c + 1)$ WHERE $c = 1 - (\nu/4) - (4/n)(\ln n)$. ν IS DIMENSION AND i VARIES FROM 1 TO n.

n = 5

i	ν = 1	2	3	4	5
1	0.783	0.764	0.748	0.733	0.720
2	0.641	0.632	0.624	0.617	0.610
3	0.500	0.500	0.500	0.500	0.500
4	0.359	0.368	0.376	0.383	0.390
5	0.217	0.236	0.252	0.267	0.280

n = 7

i	ν = 1	2	3	4	5
1	0.844	0.825	0.809	0.793	0.780
2	0.729	0.717	0.706	0.696	0.686
3	0.615	0.608	0.603	0.598	0.593
4	0.500	0.500	0.500	0.500	0.500
5	0.385	0.392	0.397	0.402	0.407
6	0.271	0.283	0.294	0.304	0.314
7	0.156	0.175	0.191	0.207	0.220

n = 9

i	ν = 1	2	3	4	5
1	0.883	0.865	0.849	0.835	0.821
2	0.787	0.774	0.762	0.751	0.741
3	0.691	0.683	0.675	0.667	0.661
4	0.596	0.591	0.587	0.584	0.580
5	0.500	0.500	0.500	0.500	0.500
6	0.404	0.409	0.413	0.416	0.420
7	0.309	0.317	0.325	0.333	0.339
8	0.213	0.226	0.238	0.249	0.259
9	0.117	0.135	0.151	0.165	0.179

n = 11

i	ν = 1	2	3	4	5
1	0.908	0.892	0.878	0.864	0.851
2	0.827	0.814	0.802	0.791	0.781
3	0.745	0.735	0.727	0.718	0.711
4	0.663	0.657	0.651	0.646	0.640
5	0.582	0.578	0.576	0.573	0.570
6	0.500	0.500	0.500	0.500	0.500
7	0.418	0.422	0.424	0.427	0.430
8	0.337	0.343	0.349	0.354	0.360
9	0.255	0.265	0.273	0.282	0.289
10	0.173	0.186	0.198	0.209	0.219
11	0.092	0.108	0.122	0.136	0.149

TABLE 4. (Continued)

n = 15

i	ν = 1	2	3	4	5
1	0.939	0.926	0.913	0.901	0.890
2	0.876	0.865	0.854	0.844	0.834
3	0.814	0.804	0.795	0.787	0.779
4	0.751	0.743	0.736	0.729	0.723
5	0.688	0.682	0.677	0.672	0.667
6	0.625	0.622	0.618	0.615	0.611
7	0.563	0.561	0.559	0.557	0.556
8	0.500	0.500	0.500	0.500	0.500
9	0.437	0.439	0.441	0.443	0.444
10	0.375	0.378	0.382	0.385	0.389
11	0.312	0.318	0.323	0.328	0.333
12	0.249	0.257	0.264	0.271	0.277
13	0.186	0.196	0.205	0.213	0.221
14	0.124	0.135	0.146	0.156	0.166
15	0.061	0.074	0.087	0.099	0.110

n = 21

i	ν = 1	2	3	4	5
1	0.962	0.951	0.941	0.932	0.923
2	0.916	0.906	0.897	0.889	0.880
3	0.869	0.861	0.853	0.845	0.838
4	0.823	0.816	0.809	0.802	0.796
5	0.777	0.771	0.765	0.759	0.754
6	0.731	0.726	0.721	0.716	0.711
7	0.685	0.681	0.677	0.673	0.669
8	0.639	0.635	0.632	0.630	0.627
9	0.592	0.590	0.588	0.586	0.585
10	0.546	0.545	0.544	0.543	0.542
11	0.500	0.500	0.500	0.500	0.500
12	0.454	0.455	0.456	0.457	0.458
13	0.408	0.410	0.412	0.414	0.415
14	0.361	0.365	0.368	0.370	0.373
15	0.315	0.319	0.323	0.327	0.331
16	0.269	0.274	0.279	0.284	0.289
17	0.223	0.229	0.235	0.241	0.246
18	0.177	0.184	0.191	0.198	0.204
19	0.131	0.139	0.147	0.155	0.162
20	0.084	0.094	0.103	0.111	0.120
21	0.038	0.049	0.059	0.068	0.077

TABLE 4. (Continued)

n = 25

i	ν = 1	2	3	4	5
1	0.970	0.961	0.952	0.944	0.936
2	0.931	0.923	0.915	0.907	0.900
3	0.892	0.884	0.877	0.870	0.863
4	0.853	0.846	0.839	0.833	0.827
5	0.813	0.807	0.802	0.796	0.791
6	0.774	0.769	0.764	0.759	0.754
7	0.735	0.731	0.726	0.722	0.718
8	0.696	0.692	0.688	0.685	0.682
9	0.657	0.654	0.651	0.648	0.645
10	0.618	0.615	0.613	0.611	0.609
11	0.578	0.577	0.575	0.574	0.573
12	0.539	0.538	0.538	0.537	0.536
13	0.500	0.500	0.500	0.500	0.500
14	0.461	0.462	0.462	0.463	0.464
15	0.422	0.423	0.425	0.426	0.427
16	0.382	0.385	0.387	0.389	0.391
17	0.343	0.346	0.349	0.352	0.355
18	0.304	0.308	0.312	0.315	0.318
19	0.265	0.269	0.274	0.278	0.282
20	0.226	0.231	0.236	0.241	0.246
21	0.187	0.193	0.198	0.204	0.209
22	0.147	0.154	0.161	0.167	0.173
23	0.108	0.116	0.123	0.130	0.137
24	0.069	0.077	0.085	0.093	0.100
25	0.030	0.039	0.048	0.056	0.064

n = 31

i	ν = 1	2	3	4	5
1	0.978	0.970	0.963	0.956	0.949
2	0.946	0.939	0.932	0.926	0.919
3	0.914	0.908	0.901	0.895	0.889
4	0.882	0.876	0.871	0.865	0.859
5	0.850	0.845	0.840	0.834	0.829
6	0.819	0.814	0.809	0.804	0.800
7	0.787	0.782	0.778	0.774	0.770
8	0.755	0.751	0.747	0.743	0.740
9	0.723	0.720	0.716	0.713	0.710
10	0.691	0.688	0.685	0.682	0.680
11	0.659	0.657	0.654	0.652	0.650
12	0.627	0.625	0.624	0.622	0.620
13	0.596	0.594	0.593	0.591	0.590
14	0.564	0.563	0.562	0.561	0.560
15	0.532	0.531	0.531	0.530	0.530
16	0.500	0.500	0.500	0.500	0.500
17	0.468	0.469	0.469	0.470	0.470
18	0.436	0.437	0.438	0.439	0.440
19	0.404	0.406	0.407	0.409	0.410
20	0.373	0.375	0.376	0.378	0.380
21	0.341	0.343	0.346	0.348	0.350
22	0.309	0.312	0.315	0.318	0.320
23	0.277	0.280	0.284	0.287	0.290
24	0.245	0.249	0.253	0.257	0.260
25	0.213	0.218	0.222	0.226	0.230
26	0.181	0.186	0.191	0.196	0.200
27	0.150	0.155	0.160	0.166	0.171
28	0.118	0.124	0.129	0.135	0.141
29	0.086	0.092	0.099	0.105	0.111
30	0.054	0.061	0.068	0.074	0.081
31	0.022	0.030	0.037	0.044	0.051

TABLE 4. (Continued)

n = 35

i	ν = 1	2	3	4	5
1	0.981	0.975	0.968	0.962	0.956
2	0.953	0.947	0.941	0.935	0.929
3	0.925	0.919	0.913	0.907	0.902
4	0.896	0.891	0.886	0.880	0.875
5	0.868	0.863	0.858	0.853	0.848
6	0.840	0.835	0.830	0.826	0.822
7	0.812	0.807	0.803	0.799	0.795
8	0.783	0.779	0.775	0.772	0.768
9	0.755	0.751	0.748	0.744	0.741
10	0.727	0.723	0.720	0.717	0.714
11	0.698	0.695	0.693	0.690	0.688
12	0.670	0.668	0.665	0.663	0.661
13	0.642	0.640	0.638	0.636	0.634
14	0.613	0.612	0.610	0.609	0.607
15	0.585	0.584	0.583	0.581	0.580
16	0.557	0.556	0.555	0.554	0.554
17	0.528	0.528	0.528	0.527	0.527
18	0.500	0.500	0.500	0.500	0.500
19	0.472	0.472	0.472	0.473	0.473
20	0.443	0.444	0.445	0.446	0.446
21	0.415	0.416	0.417	0.419	0.420
22	0.387	0.388	0.390	0.391	0.393
23	0.358	0.360	0.362	0.364	0.366
24	0.330	0.332	0.335	0.337	0.339
25	0.302	0.305	0.307	0.310	0.312
26	0.273	0.277	0.280	0.283	0.286
27	0.245	0.249	0.252	0.256	0.259
28	0.217	0.221	0.225	0.228	0.232
29	0.188	0.193	0.197	0.201	0.205
30	0.160	0.165	0.170	0.174	0.178
31	0.132	0.137	0.142	0.147	0.152
32	0.104	0.109	0.114	0.120	0.125
33	0.075	0.081	0.087	0.093	0.098
34	0.047	0.053	0.059	0.065	0.071
35	0.019	0.025	0.032	0.038	0.044

TABLE 4. (Continued)

n = 41

i	ν = 1	2	3	4	5
1	0.985	0.979	0.974	0.968	0.963
2	0.961	0.955	0.950	0.945	0.940
3	0.937	0.931	0.926	0.921	0.916
4	0.912	0.907	0.903	0.898	0.893
5	0.888	0.883	0.879	0.874	0.870
6	0.864	0.860	0.855	0.851	0.847
7	0.840	0.836	0.832	0.828	0.824
8	0.815	0.812	0.808	0.804	0.801
9	0.791	0.788	0.784	0.781	0.778
10	0.767	0.764	0.761	0.757	0.754
11	0.743	0.740	0.737	0.734	0.731
12	0.718	0.716	0.713	0.711	0.708
13	0.694	0.692	0.689	0.687	0.685
14	0.670	0.668	0.666	0.664	0.662
15	0.646	0.644	0.642	0.640	0.639
16	0.621	0.620	0.618	0.617	0.616
17	0.597	0.596	0.595	0.594	0.593
18	0.573	0.572	0.571	0.570	0.569
19	0.549	0.548	0.547	0.547	0.546
20	0.524	0.524	0.524	0.523	0.523
21	0.500	0.500	0.500	0.500	0.500
22	0.476	0.476	0.476	0.477	0.477
23	0.451	0.452	0.453	0.453	0.454
24	0.427	0.428	0.429	0.430	0.431
25	0.403	0.404	0.405	0.406	0.407
26	0.379	0.380	0.382	0.383	0.384
27	0.354	0.356	0.358	0.360	0.361
28	0.330	0.332	0.334	0.336	0.338
29	0.306	0.308	0.311	0.313	0.315
30	0.282	0.284	0.287	0.289	0.292
31	0.257	0.260	0.263	0.266	0.269
32	0.233	0.236	0.239	0.243	0.246
33	0.209	0.212	0.216	0.219	0.222
34	0.185	0.188	0.192	0.196	0.199
35	0.160	0.164	0.168	0.172	0.176
36	0.136	0.140	0.145	0.149	0.153
37	0.112	0.117	0.121	0.126	0.130
38	0.088	0.093	0.097	0.102	0.107
39	0.063	0.069	0.074	0.079	0.084
40	0.039	0.045	0.050	0.055	0.060
41	0.015	0.021	0.026	0.032	0.037

TABLE 4. (Continued)

n = 45

i	ν = 1	2	3	4	5
1	0.987	0.982	0.976	0.971	0.966
2	0.965	0.960	0.955	0.950	0.945
3	0.943	0.938	0.933	0.928	0.924
4	0.921	0.916	0.911	0.907	0.903
5	0.898	0.894	0.890	0.886	0.882
6	0.876	0.872	0.868	0.864	0.860
7	0.854	0.850	0.846	0.843	0.839
8	0.832	0.828	0.825	0.821	0.818
9	0.810	0.807	0.803	0.800	0.797
10	0.788	0.785	0.782	0.779	0.776
11	0.766	0.763	0.760	0.757	0.754
12	0.743	0.741	0.738	0.736	0.733
13	0.721	0.719	0.717	0.714	0.712
14	0.699	0.697	0.695	0.693	0.691
15	0.677	0.675	0.673	0.671	0.670
16	0.655	0.653	0.652	0.650	0.648
17	0.633	0.631	0.630	0.629	0.627
18	0.611	0.609	0.608	0.607	0.606
19	0.589	0.588	0.587	0.586	0.585
20	0.566	0.566	0.565	0.564	0.564
21	0.544	0.544	0.543	0.543	0.542
22	0.522	0.522	0.522	0.521	0.521
23	0.500	0.500	0.500	0.500	0.500
24	0.478	0.478	0.478	0.479	0.479
25	0.456	0.456	0.457	0.457	0.458
26	0.434	0.434	0.435	0.436	0.436
27	0.411	0.412	0.413	0.414	0.415
28	0.389	0.391	0.392	0.393	0.394
29	0.367	0.369	0.370	0.371	0.373
30	0.345	0.347	0.348	0.350	0.352
31	0.323	0.325	0.327	[illegible]	0.330
32	0.301	0.303	0.305	[illegible]	0.309
33	0.279	0.281	0.283	0.286	0.288
34	0.257	0.259	0.262	0.264	0.267
35	0.234	0.237	0.240	0.243	0.246
36	0.212	0.215	0.218	0.221	0.224
37	0.190	0.193	0.197	0.200	0.203
38	0.168	0.172	0.175	0.179	0.182
39	0.146	0.150	0.154	0.157	0.161
40	0.124	0.128	0.132	0.136	0.140
41	0.102	0.106	0.110	0.114	0.118
42	0.079	0.084	0.089	0.093	0.097
43	0.057	0.062	0.067	0.072	0.076
44	0.035	0.040	0.045	0.050	0.055
45	0.013	0.018	0.024	0.029	0.034

TABLE 4. (Continued)

n = 51

i	ν = 1	2	3	4	5
1	0.989	0.984	0.980	0.975	0.971
2	0.970	0.965	0.961	0.956	0.952
3	0.950	0.946	0.941	0.937	0.933
4	0.930	0.926	0.922	0.918	0.914
5	0.911	0.907	0.903	0.899	0.895
6	0.891	0.887	0.884	0.880	0.877
7	0.872	0.868	0.865	0.861	0.858
8	0.852	0.849	0.845	0.842	0.839
9	0.833	0.829	0.826	0.823	0.820
10	0.813	0.810	0.807	0.804	0.801
11	0.793	0.791	0.788	0.785	0.782
12	0.774	0.771	0.769	0.766	0.764
13	0.754	0.752	0.749	0.747	0.745
14	0.735	0.732	0.730	0.728	0.726
15	0.715	0.713	0.711	0.709	0.707
16	0.696	0.694	0.692	0.690	0.688
17	0.676	0.674	0.673	0.671	0.669
18	0.657	0.655	0.654	0.652	0.651
19	0.637	0.636	0.634	0.633	0.632
20	0.617	0.616	0.615	0.614	0.613
21	0.598	0.597	0.596	0.595	0.594
22	0.578	0.577	0.577	0.576	0.575
23	0.559	0.558	0.558	0.557	0.556
24	0.539	0.539	0.538	0.538	0.538
25	0.520	0.519	0.519	0.519	0.519
26	0.500	0.500	0.500	0.500	0.500
27	0.480	0.481	0.481	0.481	0.481
28	0.461	0.461	0.462	0.462	0.462
29	0.441	0.442	0.442	0.443	0.444
30	0.422	0.423	0.423	0.424	0.425
31	0.402	0.403	0.404	0.405	0.406
32	0.383	0.384	0.385	0.386	0.387
33	0.363	0.364	0.366	0.367	0.368
34	0.343	0.345	0.346	0.348	0.349
35	0.324	0.326	0.327	0.329	0.331
36	0.304	0.306	0.308	0.310	0.312
37	0.285	0.287	0.289	0.291	0.293
38	0.265	0.268	0.270	0.272	0.274
39	0.246	0.248	0.251	0.253	0.255
40	0.226	0.229	0.231	0.234	0.236
41	0.207	0.209	0.212	0.215	0.218
42	0.187	0.190	0.193	0.196	0.199
43	0.167	0.171	0.174	0.177	0.180
44	0.148	0.151	0.155	0.158	0.161
45	0.128	0.132	0.135	0.139	0.142
46	0.109	0.113	0.116	0.120	0.123
47	0.089	0.093	0.097	0.101	0.105
48	0.070	0.074	0.078	0.082	0.086
49	0.050	0.054	0.059	0.063	0.067
50	0.030	0.035	0.039	0.044	0.048
51	0.011	0.016	0.020	0.025	0.029

TABLE 4. (Concluded)

n = 101

ν

	1	2	3	4	5
1	0.996	0.993	0.991	0.988	0.986
2	0.986	0.983	0.981	0.979	0.976
3	0.976	0.974	0.971	0.969	0.967
4	0.966	0.964	0.961	0.959	0.957
5	0.956	0.954	0.952	0.949	0.947
6	0.946	0.944	0.942	0.940	0.937
7	0.936	0.934	0.932	0.930	0.928
8	0.926	0.924	0.922	0.920	0.918
9	0.916	0.914	0.912	0.910	0.908
10	0.906	0.904	0.902	0.901	0.899
11	0.897	0.895	0.893	0.891	0.889
12	0.887	0.885	0.883	0.881	0.879
13	0.877	0.875	0.873	0.871	0.869
14	0.867	0.865	0.863	0.861	0.860
15	0.857	0.855	0.853	0.852	0.850
16	0.847	0.845	0.844	0.842	0.840
17	0.837	0.835	0.834	0.832	0.831
18	0.827	0.826	0.824	0.822	0.821
19	0.817	0.816	0.814	0.813	0.811
20	0.807	0.806	0.804	0.803	0.801
21	0.797	0.796	0.795	0.793	0.792
22	0.788	0.786	0.785	0.783	0.782
23	0.778	0.776	0.775	0.774	0.772
24	0.768	0.766	0.765	0.764	0.762
25	0.758	0.756	0.755	0.754	0.753
26	0.748	0.747	0.745	0.744	0.743
27	0.738	0.737	0.736	0.734	0.733
28	0.728	0.727	0.726	0.725	0.724
29	0.718	0.717	0.716	0.715	0.714
30	0.708	0.707	0.706	0.705	0.704
31	0.698	0.697	0.696	0.695	0.694
32	0.688	0.687	0.687	0.686	0.685
33	0.678	0.678	0.677	0.676	0.675
34	0.669	0.668	0.667	0.666	0.665
35	0.659	0.658	0.657	0.656	0.656
36	0.649	0.648	0.647	0.647	0.646
37	0.639	0.638	0.637	0.637	0.636
38	0.629	0.628	0.628	0.627	0.626
39	0.619	0.618	0.618	0.617	0.617
40	0.609	0.609	0.608	0.607	0.607
41	0.599	0.599	0.598	0.598	0.597
42	0.589	0.589	0.588	0.588	0.587
43	0.579	0.579	0.579	0.578	0.578
44	0.569	0.569	0.569	0.568	0.568
45	0.559	0.559	0.559	0.559	0.558
46	0.550	0.549	0.549	0.549	0.549
47	0.540	0.539	0.539	0.539	0.539
48	0.530	0.530	0.529	0.529	0.529
49	0.520	0.520	0.520	0.520	0.519
50	0.510	0.510	0.510	0.510	0.510
51	0.500	0.500	0.500	0.500	0.500
52	0.490	0.490	0.490	0.490	0.490
53	0.480	0.480	0.480	0.480	0.481
54	0.470	0.470	0.471	0.471	0.471
55	0.460	0.461	0.461	0.461	0.461
56	0.450	0.451	0.451	0.451	0.451
57	0.441	0.441	0.441	0.441	0.442
58	0.431	0.431	0.431	0.432	0.432
59	0.421	0.421	0.421	0.422	0.422
60	0.411	0.411	0.412	0.412	0.413
61	0.401	0.401	0.402	0.402	0.403
62	0.391	0.391	0.392	0.393	0.393
63	0.381	0.382	0.382	0.383	0.383
64	0.371	0.372	0.372	0.373	0.374
65	0.361	0.362	0.363	0.363	0.364
66	0.351	0.352	0.353	0.353	0.354
67	0.341	0.342	0.343	0.344	0.344
68	0.331	0.332	0.333	0.334	0.335
69	0.322	0.322	0.323	0.324	0.325
70	0.312	0.313	0.313	0.314	0.315
71	0.302	0.303	0.304	0.305	0.306
72	0.292	0.293	0.294	0.295	0.296
73	0.282	0.283	0.284	0.285	0.286
74	0.272	0.273	0.274	0.275	0.276
75	0.262	0.263	0.264	0.266	0.267
76	0.252	0.253	0.255	0.256	0.257
77	0.242	0.244	0.245	0.246	0.247
78	0.232	0.234	0.235	0.236	0.238
79	0.222	0.224	0.225	0.226	0.228
80	0.212	0.214	0.215	0.217	0.218
81	0.203	0.204	0.205	0.207	0.208
82	0.193	0.194	0.196	0.197	0.199
83	0.183	0.184	0.186	0.187	0.189
84	0.173	0.174	0.176	0.178	0.179
85	0.163	0.165	0.166	0.168	0.169
86	0.153	0.155	0.156	0.158	0.160
87	0.143	0.145	0.147	0.148	0.150
88	0.133	0.135	0.137	0.139	0.140
89	0.123	0.125	0.127	0.129	0.131
90	0.113	0.115	0.117	0.119	0.121
91	0.103	0.105	0.107	0.109	0.111
92	0.094	0.096	0.098	0.099	0.101
93	0.084	0.086	0.088	0.090	0.092
94	0.074	0.076	0.078	0.080	0.082
95	0.064	0.066	0.068	0.070	0.072
96	0.054	0.056	0.058	0.060	0.063
97	0.044	0.046	0.048	0.051	0.053
98	0.034	0.036	0.039	0.041	0.043
99	0.024	0.026	0.029	0.031	0.033
100	0.014	0.017	0.019	0.021	0.024
101	0.004	0.007	0.009	0.012	0.014

TABLE 5. MEDIAN VALUES OF SELECTED PERCENTILE VALUES OF 10 100 RANDOM CHI-SQUARE VALUES OF THE MULTIVARIATE NORMAL DISTRIBUTION AND THEIR PROBABILITY PLOTTING POSITIONS FOR DIMENSION (ν) WITH A SAMPLE SIZE (n).

$\nu = 1$ n = 5

Probability	0.783	0.641	0.500	0.359	0.217
			i		
Percentile	1	2	3	4	5
2	.0000	.0043	.0502	.4361	1.2120
5	.0002	.0119	.0959	.5334	1.3152
10	.0007	.0265	.1476	.6548	1.4324
50	.0286	.1859	.4523	1.0886	2.0683
90	.1835	.5251	.8340	1.5487	2.7982
95	.2493	.6084	.9188	1.6601	2.9343
98	.3187	.6941	.9750	1.7762	3.0365

$\nu = 1$ n = 7

Probability	0.844	0.729	0.615	0.500	0.385	0.271	0.156
				i			
Percentile	1	2	3	4	5	6	7
2	.0000	.0019	.0147	.0679	.2094	.6400	1.4942
5	.0001	.0049	.0337	.1189	.3088	.8047	1.6465
10	.0004	.0112	.0614	.1766	.4070	.9490	1.8043
50	.0142	.0920	.2416	.4956	.8632	1.4630	2.5553
90	.1105	.3079	.5362	.8597	1.2671	2.0361	3.7323
95	.1603	.3819	.6160	.9573	1.3611	2.1983	3.9965
98	.2189	.4659	.6990	1.0392	1.4508	2.3691	4.3132

TABLE 5. (Continued)

$\nu = 1$ n = 11

Probability	0.908	0.827	0.745	0.663	0.582	0.500	0.418	0.337	0.255
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0000	.0006	.0050	.0187	.0465	.0999	.1938	.3415	.5891
5	.0000	.0020	.0107	.0311	.0759	.1487	.2669	.4680	.7625
10	.0001	.0043	.0192	.0500	.1094	.2013	.3509	.5731	.9097
50	.0059	.0354	.0915	.1782	.3043	.4712	.6926	.9915	1.4062
90	.0504	.1412	.2580	.4005	.5822	.7846	1.0502	1.4002	1.9017
95	.0789	.1848	.3167	.4697	.6564	.8666	1.1462	1.4980	2.0347
98	.1138	.2386	.3856	.5471	.7427	.9528	1.2438	1.6094	2.1632

Probability	0.173	0.092
	i	
Percentile	10	11
2	1.1157	1.9410
5	1.3060	2.1274
10	1.4714	2.3541
50	2.0687	3.3666
90	2.8175	4.9138
95	3.0372	5.4303
98	3.2875	5.9654

TABLE 5. (Continued)

$\nu = 1$ n = 15

Probability	0.939	0.876	0.814	0.751	0.688	0.625	0.563	0.500	0.437
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0000	.0003	.0026	.0094	.0235	.0450	.0803	.1333	.2067
5	.0000	.0011	.0055	.0158	.0369	.0663	.1136	.1848	.2727
10	.0001	.0023	.0094	.0251	.0526	.0929	.1526	.2360	.3435
50	.0032	.0191	.0500	.0957	.1583	.2391	.3386	.4667	.6280
90	.0296	.0810	.1505	.2335	.3377	.4498	.5884	.7529	.9384
95	.0447	.1095	.1914	.2879	.3985	.5133	.6596	.8325	1.0239
98	.0652	.1435	.2363	.3466	.4549	.5805	.7336	.9089	1.1036

Probability	0.375	0.312	0.249	0.186	0.124	0.061
				i		
Percentile	10	11	12	13	14	15
2	.3116	.4397	.6983	1.0064	1.4983	2.2787
5	.4057	.5850	.8383	1.1729	1.7077	2.4859
10	.4877	.6898	.9652	1.3193	1.8732	2.7247
50	.8225	1.0737	1.3987	1.8353	2.5152	3.8443
90	1.1678	1.4534	1.8346	2.3919	3.3594	5.6407
95	1.2581	1.5503	1.9609	2.5647	3.6269	6.2559
98	1.3453	1.6486	2.1069	2.7375	3.9581	6.9673

TABLE 5. (Continued)

$\nu = 1 \qquad n = 21$

Probability	0.962	0.916	0.869	0.823	0.777	0.731	0.685	0.639	0.592
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0000	.0001	.0011	.0037	.0106	.0194	.0335	.0516	.0810
5	.0000	.0004	.0025	.0078	.0172	.0303	.0487	.0743	.1130
10	.0000	.0010	.0047	.0124	.0250	.0433	.0676	.0992	.1413
50	.0017	.0099	.0256	.0478	.0773	.1158	.1633	.2229	.2931
90	.0161	.0451	.0822	.1258	.1804	.2429	.3153	.3967	.4898
95	.0245	.0611	.1060	.1545	.2198	.2877	.3597	.4466	.5447
98	.0382	.0833	.1331	.1889	.2609	.3313	.4138	.5088	.6147

Probability	0.546	0.500	0.454	0.408	0.361	0.315	0.269	0.223	0.177
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.1164	.1573	.2268	.2845	.3883	.5230	.6526	.8385	1.1080
5	.1551	.2092	.2780	.3649	.4759	.6108	.7626	.9913	1.2540
10	.1937	.2544	.3355	.4319	.5497	.6861	.8689	1.1019	1.3884
50	.3695	.4660	.5749	.7035	.8534	1.0287	1.2529	1.5225	1.8659
90	.5936	.7085	.8437	.9919	1.1585	1.3679	1.6226	1.9323	2.3419
95	.6546	.7739	.9124	1.0674	1.2410	1.4549	1.7124	2.0552	2.5087
98	.7203	.8435	.9901	1.1480	1.3227	1.5512	1.8017	2.1867	2.6769

TABLE 5. (Continued)

Probability	0.131	0.084	0.038
		i	
Percentile	19	20	21
2	1.4201	1.9394	2.6386
5	1.6224	2.1334	2.8924
10	1.7792	2.3002	3.1324
50	2.3295	3.0259	4.4030
90	2.9655	4.0309	6.4939
95	3.1805	4.3283	7.2760
98	3.3760	4.6725	8.0819

$\nu = 1 \qquad n = 31$

Probability	0.978	0.973	0.914	0.882	0.850	0.819	0.787	0.755	0.723
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0000	.0001	.0005	.0018	.0041	.0080	.0130	.0221	.0320
5	.0000	.0002	.0011	.0031	.0070	.0125	.0206	.0319	.0455
10	.0000	.0004	.0020	.0053	.0109	.0184	.0296	.0421	.0590
50	.0007	.0046	.0120	.0221	.0362	.0535	.0745	.1001	.1288
90	.0072	.0210	.0395	.0611	.0881	.1183	.1536	.1896	.2324
95	.0121	.0298	.0515	.0784	.1108	.1447	.1770	.2206	.2651
98	.0180	.0402	.0694	.0969	.1319	.1662	.2038	.2531	.3020

TABLE 5. (Continued)

Probability	0.691	0.659	0.627	0.596	0.564	0.532	0.500	0.468	0.436
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.0449	.0622	.0816	.1047	.1297	.1647	.1984	.2451	.3029
5	.0621	.0821	.1059	.1333	.1667	.2010	.2495	.2959	.3537
10	.0786	.1037	.1297	.1623	.1987	.2399	.2866	.3418	.4059
50	.1605	.1970	.2397	.2894	.3418	.3984	.4663	.5344	.6187
90	.2780	.3291	.3829	.4467	.5121	.5845	.6606	.7433	.8403
95	.3121	.3691	.4271	.4918	.5622	.6351	.7110	.8033	.9073
98	.3500	.4145	.4768	.5401	.6081	.6834	.7618	.8573	.9655

Probability	0.404	0.373	0.341	0.309	0.277	0.245	0.213	0.181	0.150
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	.3446	.4145	.4897	.5919	.7041	.8238	.9829	1.1394	1.3715
5	.4109	.4906	.5743	.6844	.7974	.9254	1.0970	1.2823	1.5359
10	.4729	.5595	.6569	.7600	.8788	1.0386	1.1950	1.3994	1.6519
50	.7061	.8060	.9215	1.0502	1.1941	1.3688	1.5648	1.7898	2.0807
90	.9432	1.0582	1.1864	1.3357	1.5023	1.6904	1.9087	2.1968	2.5458
95	1.0063	1.1245	1.2549	1.4090	1.5830	1.7708	2.0131	2.3093	2.6969
98	1.0786	1.1908	1.3270	1.4849	1.6668	1.8657	2.1164	2.4392	2.8575

TABLE 5. (Continued)

Probability	0.118	0.086	0.054	0.023
		i		
Percentile	28	29	30	31
2	1.6588	1.9837	2.4391	3.0930
5	1.7989	2.1625	2.6486	3.3991
10	1.9413	2.3211	2.8361	3.6938
50	2.4410	2.9300	3.6650	5.0676
90	3.0146	3.6869	4.8173	7.3689
95	3.1917	3.9109	5.2021	8.2211
98	3.3909	4.1388	5.6107	9.2171

$\nu = 1 \qquad n = 51$

Probability	0.989	0.970	0.950	0.930	0.911	0.891	0.872	0.852	0.833
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0000	.0000	.0002	.0006	.0014	.0027	.0044	.0070	.0108
5	.0000	.0001	.0004	.0011	.0025	.0042	.0069	.0101	.0149
10	.0000	.0002	.0007	.0019	.0037	.0063	.0097	.0142	.0198
50	.0003	.0017	.0043	.0082	.0133	.0195	.0272	.0361	.0461
90	.0028	.0080	.0156	.0244	.0346	.0463	.0603	.0736	.0905
95	.0049	.0115	.0208	.0312	.0427	.0577	.0729	.0885	.1071
98	.0075	.0168	.0277	.0387	.0539	.0702	.0864	.1073	.1240

TABLE 5. (Continued)

Probability	0.813	0.793	0.774	0.754	0.735	0.715	0.696	0.676	0.657
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.0143	.0192	.0297	.0309	.0391	.0477	.0575	.0700	.0826
5	.0199	.0265	.0339	.0419	.0506	.0613	.0721	.0881	.1031
10	.0261	.0336	.0422	.0518	.0621	.0743	.0881	.1044	.1215
50	.0576	.0707	.0847	.1006	.1178	.1364	.1559	.1788	.2010
90	.1084	.1273	.1488	.1714	.1945	.2204	.2485	.2734	.3042
95	.1267	.1466	.1707	.1945	.2184	.2457	.2764	.3034	.3356
98	.1476	.1695	.1947	.2215	.2480	.2761	.3070	.3370	.3690

Probability	0.637	0.617	0.598	0.578	0.559	0.539	0.520	0.500	0.480
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	.0963	.1123	.1308	.1502	.1694	.1945	.2186	.2439	.2745
5	.1200	.1368	.1585	.1813	.2003	.2268	.2546	.2811	.3145
10	.1412	.1603	.1804	.2037	.2280	.2578	.2880	.3188	.3542
50	.2268	.2528	.2819	.3128	.3459	.3841	.4207	.4605	.5027
90	.3393	.3698	.4070	.4415	.4826	.5297	.5703	.6189	.6663
95	.3666	.4047	.4415	.4799	.5206	.5670	.6126	.6625	.7144
98	.4037	.4408	.4731	.5145	.5604	.6068	.6571	.7068	.7560

TABLE 5. (Continued)

Probability	0.461	0.441	0.422	0.402	0.383	0.363	0.343	0.324	0.304
					i				
Percentile	28	29	30	31	32	33	34	35	36
2	.3084	.3449	.3817	.4225	.4613	.5106	.5634	.6175	.6832
5	.3515	.3891	.4299	.4751	.5303	.5753	.6372	.6943	.7096
10	.3940	.4352	.4817	.5278	.5824	.6345	.6973	.7642	.8291
50	.5532	.6045	.6556	.7084	.7707	.8417	.9093	.9848	1.0672
90	.7187	.7791	.8387	.9015	.9650	1.0412	1.1169	1.2058	1.2971
95	.7669	.8253	.8844	.9471	1.0201	1.0917	1.1817	1.2663	1.3588
98	.8127	.8709	.9362	1.0098	1.0757	1.1583	1.2418	1.3257	1.4838

Probability	0.285	0.265	0.246	0.226	0.207	0.187	0.167	0.148	0.128
					i				
Percentile	37	38	39	40	41	42	43	44	45
2	.7541	.8304	.9195	1.0122	1.1189	1.2417	1.3584	1.5192	1.6709
5	.8387	.9166	1.0063	1.1056	1.2203	1.3376	1.4733	1.6212	1.7987
10	.9149	.9931	1.0852	1.1908	1.3050	1.4252	1.5597	1.7240	1.8987
50	1.1509	1.2461	1.3512	1.4669	1.5911	1.7380	1.8934	2.0742	2.2811
90	1.3931	1.4922	1.6154	1.7397	1.8839	2.0542	2.2360	2.4586	2.7036
95	1.4560	1.5641	1.6841	1.8092	1.9627	2.1404	2.3472	2.5529	2.8341
98	1.5242	1.6272	1.7476	1.8982	2.0555	2.2320	2.4390	2.6689	2.9538

TABLE 5. (Continued)

Probability	0.109	0.089	0.070	0.050	0.030	0.011
			i			
Percentile	46	47	48	49	50	51
2	1.8718	2.0926	2.3477	2.6680	3.0925	3.7021
5	1.9995	2.2380	2.5165	2.8513	3.3254	4.0509
10	2.1178	2.3555	2.6501	3.0246	3.5433	4.3488
50	2.5357	2.8388	3.1968	3.7205	4.4853	5.9592
90	3.0023	3.3976	3.8860	4.6137	5.8202	8.5327
95	3.1783	3.5691	4.1079	4.9267	6.2516	9.4304
98	3.3202	3.7726	4.3521	5.2872	6.7889	10.5413

$\nu = 1 \qquad n = 101$

Probability	0.996	0.976	0.956	0.936	0.916	0.897	0.877	0.857	0.837
					i				
Percentile	1	3	5	7	9	11	13	15	17
2	.0000	.0000	.0003	.0010	.0024	.0045	.0074	.0109	.0158
5	.0000	.0001	.0006	.0017	.0035	.0061	.0096	.0143	.0197
10	.0000	.0002	.0009	.0024	.0048	.0080	.0122	.0177	.0234
50	.0001	.0011	.0033	.0068	.0113	.0177	.0249	.0335	.0430
90	.0008	.0042	.0090	.0158	.0235	.0338	.0448	.0580	.0719
95	.0013	.0056	.0117	.0192	.0291	.0403	.0524	.0670	.0813
98	.0020	.0073	.0149	.0236	.0342	.0474	.0607	.0772	.0930

TABLE 5. (Continued)

Probability	0.817	0.797	0.778	0.758	0.738	0.718	0.698	0.678	0.659
					i				
Percentile	19	21	23	25	27	29	31	33	35
2	.0213	.0282	.0347	.0425	.0539	.0659	.0793	.0918	.1069
5	.0262	.0336	.0429	.0523	.0648	.0776	.0908	.1078	.1236
10	.0314	.0403	.0506	.0611	.0747	.0883	.1045	.1202	.1385
50	.0544	.0673	.0819	.0974	.1140	.1327	.1538	.1742	.2001
90	.0882	.1050	.1230	.1437	.1664	.1876	.2122	.2388	.2682
95	.0988	.1163	.1374	.1590	.1813	.2064	.2320	.2593	.2874
98	.1106	.1319	.1522	.1747	.1986	.2243	.2512	.2791	.3081

Probability	0.639	0.619	0.599	0.579	0.559	0.540	0.520	0.500	0.480
					i				
Percentile	37	39	41	43	45	47	49	51	53
2	.1251	.1440	.1652	.1877	.2135	.2387	.2677	.2980	.3328
5	.1425	.1645	.1863	.2113	.2371	.2670	.2951	.3280	.3644
10	.1585	.1809	.2053	.2315	.2586	.2894	.3214	.3564	.3950
50	.2229	.2499	.2800	.3122	.3445	.3806	.4181	.4572	.5043
90	.2979	.3298	.3626	.3990	.4389	.4798	.5227	.5685	.6178
95	.3205	.3531	.3875	.4265	.4676	.5087	.5557	.6022	.6496
98	.3435	.3778	.4159	.4499	.4943	.5397	.5827	.6311	.6880

TABLE 5. (Continued)

Probability	0.460	0.441	0.441	0.421	0.401	0.381	0.361	0.341	0.322
					i				
Percentile	55	57	59	61	63	65	67	69	71
2	.3699	.4048	.4471	.4932	.5434	.6045	.6605	.7293	.7995
5	.4015	.4434	.4888	.5358	.5883	.6466	.7081	.7814	.8555
10	.4344	.4776	.5232	.5772	.6307	.6894	.7548	.8283	.9004
50	.5502	.5983	.6526	.7102	.7725	.8389	.9113	.9865	1.0694
90	.6716	.7248	.7865	.8495	.9127	.9862	1.0636	1.1478	1.2346
95	.7048	.7611	.8253	.8855	.9551	1.0220	1.1047	1.1901	1.2813
98	.7417	.7954	.8589	.9288	.9933	1.0622	1.1452	1.2396	1.3299

Probability	0.282	0.262	0.242	0.222	0.203	0.183	0.163	0.143	0.123
					i				
Percentile	73	75	77	79	81	83	85	87	89
2	.8721	.9585	1.0509	1.1506	1.2743	1.3869	1.5429	1.7158	1.8913
5	.9351	1.0216	1.1155	1.2292	1.3368	1.4751	1.6165	1.7932	1.9779
10	.9871	1.0769	1.1724	1.2879	1.4038	1.5324	1.6857	1.8641	2.0646
50	1.1583	1.2617	1.3658	1.4888	1.6195	1.7652	1.9341	2.1281	2.3524
90	1.3337	1.4391	1.5589	1.6867	1.8367	2.0067	2.1891	2.4082	2.6729
95	1.3792	1.4882	1.6127	1.7457	1.8974	2.0667	2.2618	2.4846	2.7531
98	1.4323	1.5414	1.6689	1.8026	1.9636	2.1297	2.3525	2.5694	2.8554

TABLE 5. (Continued)

Probability	0.103	0.084	0.064	0.044	0.024	0.004
			i			
Percentile	91	93	95	97	99	101
2	2.1110	2.3580	2.6836	3.0882	3.6098	4.5764
5	2.2030	2.4810	2.8013	3.2335	3.8040	4.9862
10	2.2992	2.5721	2.9163	3.3810	4.0134	5.3912
50	2.6154	2.9482	3.3640	3.9351	4.8586	7.1110
90	2.9745	3.3713	3.8787	4.6598	5.9743	9.9689
95	3.0735	3.4970	4.0367	4.8723	6.3168	10.9848
98	3.1866	3.6247	4.2129	5.1104	6.7353	12.1836

$\nu = 2 \qquad n = 5$

Probability	0.764	0.632	0.500	0.368	0.236
			i		
Percentile	1	2	3	4	5
2	.0108	.1746	.9105	1.5599	2.0830
5	.0308	.2949	1.0438	1.6852	2.2118
10	.0657	.4245	1.1816	1.8232	2.3379
50	.3682	.9140	1.5915	2.2699	2.8266
90	.8447	1.3586	1.9979	2.7655	3.1309
95	.9632	1.4716	2.1209	2.8817	3.1636
98	1.0548	1.5981	2.2239	2.9733	3.1833

TABLE 5. (Continued)

$\nu = 2 \qquad n = 7$

Probability	0.825	0.717	0.608	0.500	0.392	0.283	0.175
				i			
Percentile	1	2	3	4	5	6	7
2	.0065	.0795	.2788	.6176	1.2532	1.9390	2.5208
5	.0170	.1467	.3954	.7969	1.4028	2.0794	2.6990
10	.0375	.2142	.5238	.9744	1.5637	2.2262	2.8567
50	.2318	.5941	1.0148	1.4792	2.0683	2.7646	3.6641
90	.6227	1.0522	1.4531	1.9325	2.5759	3.4944	4.4924
95	.7606	1.1610	1.5687	2.0622	2.7081	3.7383	4.6802
98	.9002	1.2768	1.6695	2.1704	2.8604	3.9878	4.8180

$\nu = 2 \qquad n = 11$

Probability	0.892	0.814	0.735	0.657	0.578	0.500	0.422	0.343	0.265
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0038	.0434	.1357	.2748	.4453	.6673	.9598	1.3795	1.8699
5	.0104	.0776	.1954	.3585	.5745	.8449	1.1444	1.5656	2.0851
10	.0216	.1179	.2663	.4470	.6834	.9798	1.2912	1.7224	2.2286
50	.1334	.3472	.5767	.8273	1.1212	1.4374	1.8181	2.2550	2.8040
90	.4070	.6956	.9728	1.2476	1.5502	1.8843	2.2637	2.7632	3.4219
95	.4990	.7861	1.0678	1.3666	1.6528	1.9988	2.3880	2.9057	3.6509
98	.5964	.8957	1.1878	1.4686	1.7761	2.1073	2.4964	3.0618	3.8671

TABLE 5. (Continued)

Probability	0.186	0.108
	i	
Percentile	10	11
2	2.5097	3.1223
5	2.6841	3.3818
10	2.8576	3.6299
50	3.5564	4.7520
90	4.5189	6.1970
95	4.8943	6.6230
98	5.2501	7.0900

$\nu = 2$ $n = 15$

Probability	0.926	0.865	0.804	0.743	0.682	0.622	0.561	0.500	0.439
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0020	.0320	.0901	.1741	.2718	.4160	.5562	.7443	.9789
5	.0070	.0530	.1290	.2332	.3561	.5065	.6826	.8793	1.1231
10	.0136	.0818	.1737	.2957	.4290	.6093	.7804	1.0044	1.2609
50	.0986	.2494	.4044	.5754	.7639	.9581	1.1812	1.4248	1.6956
90	.2984	.5115	.7113	.9110	1.1191	1.3333	1.5762	1.8337	2.1228
95	.3647	.5963	.8070	1.0175	1.2087	1.4390	1.6754	1.9383	2.2248
98	.4439	.6791	.8974	1.1050	1.3154	1.5355	1.7826	2.0345	2.3450

TABLE 5. (Continued)

Probability	0.378	0.318	0.257	0.196	0.135	0.074
			i			
Percentile	10	11	12	13	14	15
2	1.1977	1.5395	1.9199	2.4024	2.9527	3.5661
5	1.3996	1.7192	2.1110	2.5706	3.1646	3.8758
10	1.5371	1.8723	2.2667	2.7484	3.3512	4.1499
50	2.0093	2.3721	2.8051	3.3621	4.1444	5.4021
90	2.4610	2.8686	3.3660	4.1012	5.2263	7.3268
95	2.5738	3.0049	3.5250	4.3450	5.6033	7.9399
98	2.7094	3.1545	3.7297	4.5676	6.0180	8.4897

$\nu = 2 \qquad n = 21$

Probability	0.951	0.906	0.861	0.816	0.771	0.726	0.681	0.635	0.590
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0019	.0213	.0616	.1123	.1843	.2543	.3478	.4476	.5556
5	.0047	.0338	.0878	.1554	.2309	.3178	.4140	.5276	.6547
10	.0098	.0530	.1167	.1964	.2802	.3795	.4803	.6181	.7431
50	.0683	.1712	.2790	.3946	.5091	.6457	.7744	.9180	1.0799
90	.2165	.3620	.5117	.6549	.7933	.9387	1.0807	1.2388	1.4033
95	.2635	.4288	.5829	.7248	.8585	1.0100	1.1706	1.3304	1.4931
98	.3285	.5026	.6648	.8128	.9529	1.1040	1.2439	1.4068	1.5809

TABLE 5. (Continued)

Probability	0.545	0.500	0.455	0.410	0.365	0.319	0.274	0.229	0.184
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.6865	.8204	.9642	1.1393	1.3559	1.5815	1.8562	2.1404	2.5244
5	.7846	.9450	1.0972	1.2831	1.4991	1.7431	2.0147	2.3284	2.7136
10	.8895	1.0481	1.2059	1.4137	1.6280	1.8742	2.1534	2.4769	2.8568
50	1.2384	1.4188	1.6073	1.8242	2.0593	2.3265	2.6372	2.9753	3.4200
90	1.5829	1.7794	1.9967	2.2119	2.4677	2.7564	3.1326	3.5480	4.1041
95	1.6709	1.8764	2.0948	2.3201	2.5807	2.8773	3.2527	3.7167	4.3166
98	1.7734	1.9724	2.1936	2.4309	2.6905	3.0152	3.3822	3.8829	4.5287

Probability	0.139	0.094	0.049
		i	
Percentile	19	20	21
2	2.9358	3.4594	4.1453
5	3.1288	3.6668	4.4266
10	3.3079	3.8736	4.7382
50	4.0046	4.7925	6.1702
90	4.8517	6.0625	8.3238
95	5.1274	6.4608	9.1264
98	5.4153	7.0227	10.0008

TABLE 5. (Continued)

$\nu = 2 \qquad n = 31$

Probability	0.970	0.939	0.908	0.876	0.845	0.814	0.784	0.751	0.720
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0011	.0135	.0374	.0735	.1109	.1562	.2122	.2616	.3165
5	.0032	.0222	.0591	.0976	.1439	.1980	.2529	.3174	.3858
10	.0069	.0341	.0775	.1265	.1811	.2382	.3044	.3675	.4393
50	.0457	.1132	.1847	.2548	.3312	.4075	.4923	.5768	.6670
90	.1470	.2485	.3388	.4365	.5326	.6269	.7212	.8134	.9166
95	.1873	.2946	.3959	.4890	.5893	.6839	.7826	.8756	.9746
98	.2298	.3465	.4459	.5492	.6544	.7426	.8464	.9462	1.0557

Probability	0.688	0.657	0.625	0.594	0.563	0.531	0.500	0.469	0.437
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.3873	.4551	.5268	.6126	.7055	.7970	.8884	.9950	1.1286
5	.4567	.5285	.6129	.6964	.7938	.8922	1.0032	1.1206	1.2410
10	.5157	.5944	.6871	.7764	.8732	.9795	1.0951	1.2067	1.3391
50	.7568	.8495	.9459	1.0512	1.1590	1.2808	1.4042	1.5338	1.6763
90	1.0191	1.1226	1.2209	1.3415	1.4572	1.5806	1.7129	1.8467	2.0010
95	1.0836	1.1968	1.3063	1.4165	1.5368	1.6679	1.7942	1.9343	2.0818
98	1.1684	1.2765	1.3798	1.4936	1.6093	1.7366	1.8732	2.0201	2.1723

TABLE 5. (Continued)

Probability	0.406	0.375	0.343	0.312	0.280	0.249	0.218	0.186	0.155
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	1.2599	1.3902	1.5476	1.7408	1.8924	2.0703	2.2944	2.5674	2.8517
5	1.3892	1.5220	1.6798	1.8561	2.0394	2.2451	2.4653	2.7337	3.0252
10	1.4742	1.6290	1.7956	1.9582	2.1643	2.3670	2.5937	2.8784	3.1694
50	1.8278	1.9918	2.1692	2.3578	2.5718	2.7955	3.0497	3.3588	3.7165
90	2.1593	2.3464	2.5258	2.7348	2.9682	3.2331	3.5372	3.8894	4.3468
95	2.2524	2.4333	2.6179	2.8419	3.0731	3.3515	3.6872	4.0500	4.5279
98	2.3486	2.5411	2.7154	2.9443	3.1920	3.4906	3.8412	4.2423	4.7526

Probability	0.124	0.092	0.061	0.030
		i		
Percentile	28	29	30	31
2	3.1373	3.5742	4.0384	4.6812
5	3.3513	3.7610	4.2960	5.0960
10	3.5160	3.9796	4.5383	5.4352
50	4.1689	4.7384	5.5653	7.0698
90	4.9078	5.7424	6.9634	9.5566
95	5.1353	6.0653	7.5163	10.4307
98	5.4034	6.3860	7.9955	11.5191

TABLE 5. (Continued)

$\nu = 2$ n = 51

Probability	0.984	0.965	0.946	0.926	0.907	0.887	0.868	0.849	0.829
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0008	.0080	.0234	.0410	.0649	.0905	.1190	.1452	.1755
5	.0019	.0145	.0338	.0589	.0840	.1141	.1453	.1761	.2143
10	.0041	.0213	.0452	.0726	.1034	.1356	.1714	.2077	.2468
50	.0274	.0683	.1081	.1512	.1953	.2404	.2847	.3321	.3776
90	.0894	.1544	.2121	.2638	.3176	.3721	.4288	.4829	.5345
95	.1143	.1856	.2391	.3005	.3563	.4106	.4715	.5278	.5889
98	.1440	.2166	.2774	.3454	.3981	.4564	.5185	.5773	.6317

Probability	0.810	0.791	0.771	0.752	0.732	0.713	0.694	0.674	0.655
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.2139	.2499	.2849	.3189	.3607	.4015	.4435	.4834	.5427
5	.2507	.2896	.3285	.3695	.4072	.4565	.4995	.5488	.6024
10	.2855	.3247	.3670	.4131	.4553	.5036	.5534	.6055	.6580
50	.4288	.4771	.5281	.5763	.6295	.6860	.7390	.8023	.8619
90	.5879	.6502	.7066	.7650	.8205	.8826	.9451	1.0025	1.0691
95	.6416	.7007	.7581	.8138	.8762	.9347	1.0011	1.0680	1.1267
98	.6934	.7569	.8099	.8727	.9280	.9907	1.0534	1.1275	1.1937

TABLE 5. (Continued)

Probability	0.636	0.616	0.597	0.577	0.558	0.539	0.519	0.500	0.481
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	.5955	.6452	.6947	.7440	.8131	.8776	.9422	1.0129	1.0878
5	.6600	.7183	.7672	.8269	.8877	.9528	1.0203	1.0835	1.1592
10	.7181	.7714	.8300	.8900	.9511	1.0138	1.0820	1.1548	1.2230
50	.9247	.9804	1.0461	1.1118	1.1795	1.2523	1.3270	1.4003	1.4796
90	1.1371	1.2030	1.2767	1.3459	1.4196	1.4979	1.5754	1.6566	1.7408
95	1.1986	1.2630	1.3352	1.4123	1.4865	1.5596	1.6344	1.7155	1.8019
98	1.2600	1.3259	1.4067	1.4788	1.5521	1.6282	1.7027	1.7829	1.8620

Probability	0.461	0.442	0.423	0.403	0.384	0.364	0.345	0.326	0.306
					i				
Percentile	28	29	30	31	32	33	34	35	36
2	1.1549	1.2224	1.3084	1.3911	1.4759	1.5709	1.6605	1.7658	1.8820
5	1.2321	1.3144	1.3948	1.4850	1.5708	1.6642	1.7660	1.8670	1.9829
10	1.3106	1.3841	1.4695	1.5550	1.6468	1.7465	1.8516	1.9554	2.0693
50	1.5656	1.6540	1.7452	1.8344	1.9337	2.0364	2.1435	2.2572	2.3795
90	1.8272	1.9132	2.0025	2.1006	2.1991	2.3201	2.4324	2.5581	2.6927
95	1.8918	1.9850	2.0690	2.1768	2.2740	2.3936	2.5110	2.6395	2.7790
98	1.9605	2.0583	2.1471	2.2543	2.3574	2.4621	2.5999	2.7157	2.8580

TABLE 5. (Continued)

Probability	0.287	0.268	0.248	0.229	0.209	0.190	0.171	0.151	0.132
					i				
Percentile	37	38	39	40	41	42	43	44	45
2	1.9959	2.1015	2.2348	2.3682	2.5119	2.6815	2.8359	3.0240	3.2436
5	2.0931	2.2188	2.3466	2.4909	2.6427	2.8008	2.9755	3.1759	3.3958
10	2.1894	2.3124	2.4408	2.5848	2.7472	2.9181	3.0964	3.2924	3.5307
50	2.5068	2.6426	2.7899	2.9475	3.1224	3.3169	3.5225	3.7480	4.0190
90	2.8314	2.9745	3.1374	3.3126	3.5081	3.7247	3.9722	4.2372	4.5781
95	2.9128	3.0623	3.2376	3.4314	3.6188	3.8466	4.0989	4.3946	4.7393
98	3.0099	3.1545	3.3376	3.5217	3.7299	3.9698	4.2468	4.5452	4.9300

Probability	0.113	0.093	0.074	0.054	0.035	0.016
			i			
Percentile	46	47	48	49	50	51
2	3.4841	3.7466	4.0181	4.3589	4.8428	5.5143
5	3.6261	3.9056	4.2477	4.6136	5.1202	5.9228
10	3.7659	4.0651	4.4143	4.8294	5.3964	6.3085
50	4.3295	4.6875	5.1258	5.7089	6.5511	8.0794
90	4.9729	5.4333	6.0245	6.8670	8.1091	10.8130
95	5.1646	5.6459	6.2825	7.2279	8.6231	11.8215
98	5.3690	5.8904	6.6023	7.5673	9.2162	12.8615

TABLE 5. (Continued)

$\nu = 2$ $\quad$ $n = 101$

Probability	0.993	0.974	0.954	0.934	0.914	0.895	0.875	0.855	0.835
					i				
Percentile	1	3	5	7	9	11	13	15	17
2	.0003	.0105	.0301	.0550	.0803	.1107	.1442	.1783	.2144
5	.0010	.0162	.0390	.0678	.0979	.1310	.1652	.2052	.2430
10	.0020	.0222	.0503	.0814	.1145	.1505	.1872	.2271	.2680
50	.0137	.0542	.0961	.1383	.1813	.2266	.2692	.3168	.3649
90	.0462	.1054	.1603	.2131	.2661	.3148	.3687	.4193	.4736
95	.0581	.1217	.1800	.2377	.2912	.3436	.3974	.4499	.5038
98	.0725	.1414	.2044	.2601	.3173	.3728	.4272	.4812	.5376

Probability	0.816	0.796	0.776	0.756	0.737	0.717	0.697	0.678	0.658
					i				
Percentile	19	21	23	25	27	29	31	33	35
2	.2524	.2905	.3323	.3757	.4195	.4631	.5109	.5648	.6125
5	.2829	.3238	.3646	.4132	.4567	.5070	.5526	.6055	.6617
10	.3119	.3535	.4000	.4466	.4939	.5412	.5929	.6435	.7017
50	.4149	.4628	.5147	.5658	.6165	.6736	.7300	.7861	.8448
90	.5259	.5820	.6362	.6986	.7522	.8106	.8683	.9362	.9960
95	.5595	.6152	.6730	.7349	.7842	.8481	.9070	.9741	1.0365
98	.5927	.6505	.7075	.7679	.8283	.8842	.9543	1.0169	1.0822

TABLE 5. (Continued)

Probability	0.638	0.618	0.599	0.579	0.559	0.539	0.520	0.500	0.480
					i				
Percentile	37	39	41	43	45	47	49	51	53
2	.6625	.7161	.7805	.8359	.8990	.9623	1.0281	1.1019	1.1662
5	.7146	.7697	.8323	.8879	.9582	1.0230	1.0915	1.1644	1.2391
10	.7564	.8147	.8721	.9327	1.0036	1.0722	1.1415	1.2174	1.2927
50	.9062	.9703	1.0346	1.1029	1.1734	1.2439	1.3203	1.3960	1.4769
90	1.0620	1.1295	1.1958	1.2679	1.3398	1.4170	1.4932	1.5739	1.6612
95	1.1015	1.1720	1.2456	1.3091	1.3885	1.4666	1.5392	1.6269	1.7106
98	1.1509	1.2158	1.2889	1.3571	1.4312	1.5085	1.5905	1.6705	1.7626

Probability	0.460	0.441	0.421	0.401	0.382	0.362	0.342	0.322	0.303
					i				
Percentile	55	57	59	61	63	65	67	69	71
2	1.2431	1.3330	1.4154	1.4998	1.5960	1.6861	1.7987	1.9009	2.0298
5	1.3190	1.3969	1.4791	1.5683	1.6646	1.7650	1.8695	1.9852	2.1074
10	1.3717	1.4501	1.5407	1.6314	1.7311	1.8312	1.9400	2.0492	2.1720
50	1.5617	1.6496	1.7403	1.8368	1.9375	2.0446	2.1591	2.2703	2.3974
90	1.7522	1.8356	1.9330	2.0312	2.1417	2.2463	2.3635	2.4949	2.6235
95	1.7966	1.8897	1.9840	2.0839	2.1973	2.3011	2.4233	2.5556	2.6881
98	1.8538	1.9464	2.0385	2.1458	2.2504	2.3577	2.4773	2.6111	2.7502

TABLE 5. (Continued)

Probability	0.283	0.263	0.244	0.224	0.204	0.184	0.165	0.145	0.125
					i				
Percentile	73	75	77	79	81	83	85	87	89
2	2.1447	2.2786	2.4031	2.5482	2.7108	2.8829	3.0673	3.2935	3.5352
5	2.2306	2.3547	2.4948	2.6452	2.8080	2.9841	3.1809	3.3943	3.6433
10	2.2982	2.4247	2.5671	2.7181	2.8873	3.0721	3.2662	3.4935	3.7415
50	2.5315	2.6717	2.8223	2.9937	3.1763	3.3705	3.5931	3.8376	4.1199
90	2.7673	2.9155	3.0855	3.2618	3.4585	3.6805	3.9201	4.2046	4.5325
95	2.8279	2.9911	3.1599	3.3397	3.5409	3.7705	4.0159	4.3092	4.6599
98	2.8898	3.0548	3.2246	3.4284	3.6254	3.8483	4.1248	4.4228	4.7944

Probability	0.105	0.086	0.066	0.046	0.026	0.007
			i			
Percentile	91	93	95	97	99	101
2	3.7803	4.0957	4.4648	4.9092	5.5550	6.7349
5	3.9174	4.2222	4.6172	5.1102	5.8280	7.1887
10	4.0161	4.3612	4.7659	5.2960	6.0619	7.6049
50	4.4498	4.8480	5.3533	6.0315	7.0859	9.5396
90	4.9214	5.4031	6.0467	6.9222	8.3732	12.6119
95	5.0554	5.5844	6.2297	7.1649	8.8023	13.8253
98	5.2014	5.7695	6.4413	7.5072	9.3122	15.1859

TABLE 5. (Continued)

$\nu = 3 \quad n = 5$

Probability	0.748	0.624	0.500	0.376	0.252
			i		
Percentile	1	2	3	4	5
2	.1479	1.3683	2.1842	2.4523	2.8357
5	.2681	1.4976	2.2556	2.5519	2.9200
10	.4147	1.6295	2.3455	2.6487	2.9894
50	1.1481	2.0961	2.7209	2.9964	3.1687
90	1.7576	2.5002	3.0366	3.1696	3.1990
95	1.8857	2.6063	3.0891	3.1845	3.1997
98	1.9562	2.7029	3.1363	3.1934	3.1999

$\nu = 3 \quad n = 7$

Probability	0.809	0.706	0.603	0.500	0.397	0.294	0.191
				i			
Percentile	1	2	3	4	5	6	7
2	.0101	.4607	1.0144	1.7405	2.3725	2.8733	3.3451
5	.1465	.6202	1.2127	1.9339	2.5193	3.0243	3.5302
10	.2496	.7882	1.4160	2.0782	2.6413	3.1630	3.6959
50	.7867	1.4295	2.0067	2.5597	3.1170	3.7042	4.3474
90	1.4319	1.9544	2.4675	3.0228	3.7083	4.3328	4.8971
95	1.5796	2.0980	2.5889	3.1557	3.8772	4.4932	4.9852
98	1.7274	2.2131	2.7025	3.2718	4.0540	4.6349	5.0433

TABLE 5. (Continued)

$\nu = 3 \qquad n = 11$

Probability	0.878	0.802	0.727	0.651	0.576	0.500	0.424	0.349	0.273
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0459	.2655	.5432	.8489	1.2552	1.6556	2.0850	2.5308	3.0102
5	.0965	.3780	.7112	1.0537	1.4283	1.8564	2.2793	2.7229	3.1743
10	.1648	.4902	.8486	1.2131	1.6010	2.0110	2.4296	2.8714	3.3300
50	.5296	.9759	1.3737	1.7504	2.1310	2.5180	2.9239	3.3769	3.9055
90	1.0647	1.5091	1.8771	2.2511	2.6006	2.9816	3.4133	3.9320	4.6206
95	1.2133	1.6700	1.9982	2.3674	2.7179	3.1080	3.5380	4.0967	4.8450
98	1.3694	1.8053	2.1256	2.4792	2.8207	3.2236	3.6847	4.2745	5.0858

Probability	0.198	0.122
	i	
Percentile	10	11
2	3.5081	4.0498
5	3.6988	4.3195
10	3.8526	4.5531
50	4.5951	5.6114
90	5.5440	6.9004
95	5.8420	7.2747
98	6.1603	7.6230

TABLE 5. (Continued)

$\nu = 3 \qquad n = 15$

Probability	0.913	0.854	0.795	0.736	0.677	0.618	0.559	0.500	0.441
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0246	.1964	.3912	.6444	.8892	1.1434	1.4246	1.7272	2.0290
5	.0717	.2828	.5067	.7731	1.0411	1.3242	1.5890	1.8981	2.2144
10	.1224	.3715	.6261	.9068	1.1740	1.4561	1.7316	2.0470	2.3690
50	.4104	.7694	1.0811	1.3717	1.6687	1.9327	2.2128	2.5232	2.8364
90	.8202	1.2331	1.5498	1.8364	2.0975	2.3698	2.6544	2.9455	3.2868
95	.9952	1.3619	1.6650	1.9589	2.2092	2.4771	2.7567	3.0628	3.3835
98	1.1357	1.4965	1.8106	2.0606	2.3457	2.5996	2.8782	3.1721	3.5073

Probability	0.382	0.323	0.254	0.205	0.146	0.087
			i			
Percentile	10	11	12	13	14	15
2	2.3817	2.7444	3.1258	3.5267	3.9760	4.5749
5	2.5551	2.8996	3.3048	3.7174	4.1950	4.8511
10	2.6925	3.0472	3.4385	3.8936	4.3930	5.1355
50	3.1736	3.5516	3.9785	4.5113	5.2001	6.3528
90	3.6489	4.0901	4.6159	5.2718	6.2585	8.0016
95	3.7960	4.2469	4.7939	5.5377	6.6097	8.5107
98	3.9284	4.3939	4.9872	5.7667	6.9894	9.0045

TABLE 5. (Continued)

$\nu = 3$ n = 21

Probability	0.941	0.897	0.853	0.809	0.765	0.721	0.677	0.632	0.588
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0331	.1560	.2919	.4620	.6412	.8075	1.0263	1.2113	1.4065
5	.0600	.2218	.3964	.5890	.7703	.9552	1.1541	1.3573	1.5407
10	.0959	.2893	.4910	.6837	.8744	1.0713	1.2671	1.4799	1.6744
50	.3254	.6047	.8437	1.0638	1.2635	1.4781	1.6850	1.8879	2.0981
90	.6930	.9921	1.2396	1.4652	1.6744	1.8851	2.0870	2.2811	2.4849
95	.8106	1.1019	1.3582	1.5759	1.7828	1.9857	2.1827	2.3916	2.6052
98	.9340	1.2264	1.4813	1.6897	1.8944	2.0982	2.2786	2.4997	2.6988

Probability	0.544	0.500	0.456	0.412	0.368	0.323	0.279	0.235	0.191
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	1.6408	1.8391	2.0537	2.3016	2.5596	2.7954	3.0978	3.4063	3.7345
5	1.7786	1.9969	2.1968	2.4342	2.7044	2.9648	3.2394	3.5596	3.8957
10	1.8912	2.1134	2.3360	2.5716	2.8355	3.0979	3.3889	3.6979	4.0484
50	2.3115	2.5327	2.7588	3.0008	3.2683	3.5458	3.8655	4.2211	4.6331
90	2.7033	2.9244	3.1539	3.4158	3.6949	4.0320	4.3765	4.8102	5.3131
95	2.8032	3.0195	3.2639	3.5296	3.8156	4.1641	4.5377	4.9756	5.5049
98	2.9075	3.1209	3.3815	3.6593	3.9501	4.3008	4.6827	5.1550	5.7392

TABLE 5. (Continued)

Probability	0.147	0.103	0.059
		i	
Percentile	19	20	21
2	4.0978	4.4888	5.0732
5	4.2925	4.7450	5.3865
10	4.4352	4.9530	5.6974
50	5.1358	5.8328	7.0195
90	6.0141	7.0077	8.9299
95	6.2816	7.3985	9.5586
98	6.5524	7.8060	10.2951

$\nu = 3$ $n = 31$

Probability	0.963	0.932	0.901	0.871	0.840	0.809	0.778	0.747	0.716
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0179	.0904	.1851	.2831	.3847	.4986	.5950	.7079	.8285
5	.0352	.1338	.2483	.3572	.4656	.5833	.6959	.8110	.9318
10	.0571	.1809	.2994	.4226	.5400	.6580	.7819	.8989	1.0291
50	.2053	.3784	.5417	.6853	.8227	.9573	1.0940	1.2309	1.3672
90	.4548	.6659	.8282	.9891	1.1382	1.2889	1.4267	1.5605	1.7148
95	.5408	.7415	.9119	1.0710	1.2271	1.3756	1.5176	1.6591	1.8049
98	.6250	.8334	1.0036	1.1683	1.3056	1.4652	1.6152	1.7652	1.9067

TABLE 5. (Continued)

Probability	0.685	0.654	0.624	0.593	0.562	0.531	0.500	0.469	0.438
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.9425	1.0884	1.2103	1.3321	1.4683	1.6106	1.7773	1.9249	2.0983
5	1.0619	1.1889	1.3159	1.4397	1.5870	1.7466	1.8926	2.0544	2.2323
10	1.1496	1.2871	1.4125	1.5619	1.7038	1.8478	2.0076	2.1770	2.3493
50	1.5030	1.6380	1.7824	1.9293	2.0791	2.2411	2.3999	2.5739	2.7508
90	1.8563	2.0007	2.1451	2.3007	2.4452	2.6085	2.7745	2.9551	3.1303
95	1.9613	2.0935	2.2448	2.3911	2.5430	2.7029	2.8805	3.0551	3.2265
98	2.0532	2.1910	2.3272	2.4890	2.6516	2.7999	2.9655	3.1451	3.3482

Probability	0.407	0.376	0.346	0.315	0.284	0.253	0.222	0.191	0.160
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	2.2632	2.4486	2.6233	2.8396	3.0644	3.3175	3.5792	2.8450	4.1492
5	2.4021	2.5835	2.7849	3.0078	3.2303	3.4623	3.7383	4.0157	4.3424
10	2.5283	2.7152	2.9212	3.1354	3.3564	3.5912	3.8726	4.1797	4.5123
50	2.9373	3.1416	3.3500	3.5719	3.8199	4.0879	4.3884	4.7501	5.1584
90	3.3289	3.5413	3.7696	4.0191	4.3012	4.6072	4.9816	5.4013	5.9108
95	3.4375	3.6551	3.8941	4.1483	4.4398	4.7637	5.1254	5.6090	6.1224
98	3.5370	3.7782	4.0170	4.2603	4.5592	4.9340	5.3240	5.8065	6.3816

TABLE 5. (Continued)

Probability	0.129	0.099	0.068	0.037
		i		
Percentile	28	29	30	31
2	4.4904	4.9051	5.4188	6.1690
5	4.7138	5.1620	5.6801	6.5468
10	4.9063	5.3634	5.9738	6.9044
50	5.6326	6.2631	7.1256	8.6727
90	6.5338	7.3592	8.6508	11.1334
95	6.8071	7.7366	9.1474	12.0331
98	7.1212	8.1195	9.6680	12.8834

$\nu = 3$ $n = 51$

Probability	0.980	0.961	0.944	0.923	0.903	0.884	0.865	0.845	0.826
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0128	.0623	.1312	.1911	.2573	.3178	.3878	.4509	.5303
5	.0268	.3906	.1656	.2381	.3075	.3759	.4567	.5331	.6009
10	.0400	.1203	.2046	.2772	.3575	.4310	.5145	.5898	.6625
50	.1467	.2655	.3724	.4674	.5616	.6471	.7312	.8146	.8966
90	.3237	.4724	.5852	.6973	.7907	.8832	.9723	1.0611	1.1481
95	.3864	.5294	.6437	.7611	.8537	.9512	1.0472	1.1383	1.2215
98	.4565	.5998	.7230	.8345	.9361	1.0209	1.1264	1.2219	1.3021

TABLE 5. (Continued)

Probability	0.807	0.788	0.769	0.749	0.730	0.711	0.692	0.673	0.654
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	.6048	.6732	.7431	.8116	.8843	.9512	1.0244	1.1087	1.1915
5	.6747	.7409	.8151	.8989	.9710	1.0439	1.1239	1.1996	1.2787
10	.7296	.8098	.8892	.9641	1.0441	1.1207	1.1982	1.2812	1.3632
50	.9771	1.0608	1.1456	1.2299	1.3099	1.3918	1.4754	1.5576	1.6456
90	1.2363	1.3254	1.4106	1.4982	1.5783	1.6673	1.7446	1.8367	1.9288
95	1.3132	1.3965	1.4894	1.5687	1.6640	1.7405	1.8216	1.9153	2.0071
98	1.3944	1.4743	1.5651	1.6467	1.7338	1.8158	1.9055	1.9940	2.0908

Probability	0.634	0.615	0.596	0.571	0.558	0.538	0.519	0.500	0.481
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	1.2596	1.3435	1.4258	1.5036	1.5932	1.7005	1.7840	1.8692	1.9664
5	1.3604	1.4486	1.5227	1.6115	1.7086	1.8035	1.8930	1.9852	2.0875
10	1.4449	1.5254	1.6110	1.7018	1.7972	1.8925	1.9815	2.0763	2.1750
50	1.7305	1.8195	1.9087	1.9982	2.1007	2.1922	2.2927	2.3931	2.5017
90	2.0143	2.1115	2.2039	2.2900	2.3906	2.4879	2.5923	2.6925	2.8097
95	2.0995	2.1845	2.2742	2.3764	2.4710	2.5678	2.6716	2.7796	2.8840
98	2.1772	2.2738	2.3624	2.4537	2.5519	2.6601	2.7454	2.8626	2.9656

TABLE 5. (Continued)

Probability	0.462	0.442	0.423	0.404	0.385	0.366	0.346	0.327	0.308
					i				
Percentile	28	29	30	31	32	33	34	35	36
2	2.0678	2.1599	2.2744	2.4920	2.5003	2.6266	2.7338	2.8827	3.0038
5	2.1794	2.2877	2.3901	2.5920	2.6157	2.7471	2.8707	3.0000	3.1321
10	2.2785	2.3807	2.4908	2.6016	2.7271	2.8494	2.9784	3.0998	3.2398
50	2.6046	2.7147	2.8326	2.9488	3.0707	3.1974	3.3305	3.4721	3.6154
90	2.9123	3.0301	3.1583	3.2737	3.4025	3.5358	3.6810	3.8245	3.9812
95	2.9959	3.1231	3.2305	3.3608	3.4908	3.6231	3.7721	3.9280	4.0895
98	3.0877	3.2090	3.3265	3.4573	3.5770	3.7287	3.8767	4.0298	4.1897

Probability	0.289	0.270	0.251	0.231	0.212	0.193	0.174	0.155	0.135
					i				
Percentile	37	38	39	40	41	42	43	44	45
2	3.1457	3.3043	3.4573	3.6327	3.7890	3.9660	4.1458	4.3820	4.5850
5	3.2726	3.4217	3.5859	3.7567	3.9325	4.1222	4.3095	4.5341	4.7633
10	3.3883	3.5322	3.7141	3.8780	4.0481	4.2445	4.4446	4.6886	4.9240
50	3.7755	3.9376	4.1055	4.2807	4.4889	4.7034	4.9425	5.2121	5.5051
90	4.1484	4.3229	4.5451	4.7322	4.9548	5.2177	5.5044	5.8123	6.1846
95	4.2510	4.4400	4.6447	4.6567	5.1099	5.3623	5.6624	5.9847	6.3565
98	4.3686	4.5768	4.7824	4.9896	5.2606	5.5434	5.8431	6.1856	6.5824

TABLE 5. (Continued)

Probability	0.116	0.097	0.078	0.059	0.039	0.020
			i			
Percentile	46	47	48	49	50	51
2	4.8396	5.1449	5.4643	5.8622	6.3607	7.1004
5	5.0378	5.3338	5.6952	6.1518	6.6884	7.5662
10	5.2038	5.5290	5.9059	6.3866	7.0211	7.9817
50	5.8480	6.2443	6.7487	7.3898	8.2948	9.9261
90	6.5895	7.1153	7.7568	8.6679	9.9853	12.8678
95	6.8247	7.3712	8.0951	9.0620	10.6058	13.8407
98	7.0523	7.6576	8.4479	9.5102	11.2605	14.8702

$\nu = 3$ $n = 101$

Probability	0.991	0.971	0.952	0.932	0.912	0.893	0.873	0.853	0.834
					i				
Percentile	1	3	5	7	9	11	13	15	17
2	.0077	.0768	.1545	.2325	.3091	.3787	.4518	.5285	.6027
5	.0155	.1006	.1868	.2711	.3453	.4254	.5016	.5792	.6518
10	.0265	.1243	.2189	.3035	.3851	.4692	.5441	.6258	.7030
50	.0928	.2288	.3378	.4360	.5278	.6154	.7027	.7874	.8723
90	.2015	.3613	.4821	.5882	.6898	.7822	.8726	.9619	1.0446
95	.2377	.4014	.5217	.6367	.7370	.8310	.9189	1.0076	1.0992
98	.2762	.4430	.5660	.6872	.7844	.8812	.9729	1.0609	1.1522

TABLE 5. (Continued)

Probability	0.814	0.795	0.775	0.755	0.736	0.716	0.696	0.677	0.657
					i				
Percentile	19	21	23	25	27	29	31	33	35
2	.6753	.7493	.8197	.8976	.9731	1.0443	1.1241	1.2111	1.2913
5	.7311	.8054	.8804	.9536	1.0296	1.1136	1.1937	1.2763	1.3591
10	.7751	.8556	.9339	1.0075	1.0880	1.1718	1.2520	1.3360	1.4206
50	.9548	1.0381	1.1196	1.2093	1.2869	1.3721	1.4545	1.5392	1.6259
90	1.1332	1.2231	1.3076	1.4902	1.4759	1.5689	1.6513	1.7439	1.8323
95	1.1863	1.2747	1.3564	1.4409	1.5259	1.6169	1.7068	1.7921	1.8894
98	1.2329	1.3286	1.4072	1.4955	1.5835	1.6688	1.7640	1.8533	1.9404

	Probability	0.637	0.618	0.598	0.579	0.559	0.539	0.520	0.500	0.480
						i				
	Percentile	37	39	41	43	45	47	49	51	53
	2	1.3755	1.4645	1.5454	1.6343	1.7184	1.8120	1.9005	1.9928	2.1016
	5	1.4520	1.5295	1.6129	1.7044	1.7960	1.8884	1.9870	2.0808	2.1853
	10	1.5022	1.5911	1.6799	1.7704	1.8617	1.9585	2.0546	2.1522	2.2569
50	50	1.7146	1.8064	1.8986	1.9901	2.0879	2.1828	2.2827	2.3828	2.4899
	90	1.9179	2.0113	2.1059	2.1992	2.3018	2.4013	2.5029	2.6038	2.7110
	95	1.9761	2.0664	2.1574	2.2551	2.3555	2.4637	2.5632	2.6700	2.7819
	98	2.0344	2.1167	2.2184	2.3197	2.4205	2.5249	2.6203	2.7379	2.8491

TABLE 5. (Continued)

Probability	0.461	0.441	0.421	0.402	0.382	0.363	0.343	0.323	0.304
					i				
Percentile	55	57	59	61	63	65	67	69	71
2	2.2003	2.3038	2.4313	2.5445	2.6652	2.7803	2.9269	3.0629	3.2095
5	2.2862	2.3994	2.5173	2.6329	2.7590	2.8751	3.0162	3.1511	3.2920
10	2.3618	2.4706	2.5821	2.7089	2.8249	2.9548	3.0883	3.2259	3.3715
50	2.5949	2.7096	2.8317	2.9539	3.0778	3.2084	3.3501	3.4943	3.6440
90	2.8295	2.9396	3.0647	3.1924	3.3253	3.4584	3.6009	3.7587	3.9153
95	2.8893	3.0052	3.1277	3.2591	3.3916	3.5273	3.6733	3.8320	3.9861
98	2.9594	3.0701	3.1987	3.3203	3.4537	3.5984	3.7450	3.9008	4.0725

Probability	0.284	0.264	0.245	0.225	0.205	0.186	0.166	0.147	0.127
					i				
Percentile	73	75	77	79	81	83	85	87	89
2	3.3502	3.4978	3.6758	3.8493	4.0299	4.2309	4.4568	4.6915	4.9550
5	3.4440	3.6060	3.7648	3.9476	4.1250	4.3472	4.5762	4.8236	5.0969
10	3.5290	3.6914	3.8478	4.0370	4.2232	4.4457	4.6742	4.9312	5.2123
50	3.8041	3.9780	4.1586	4.3549	4.5654	4.8015	5.0527	5.3392	5.6586
90	4.0899	4.2748	4.4758	4.6855	4.9191	5.1761	5.4587	5.7765	6.1513
95	4.1899	4.3525	4.5639	4.7772	5.0271	5.2677	5.5868	5.9043	6.3137
98	4.2531	4.4561	4.6540	4.8870	5.1199	5.4028	5.7169	6.0370	6.4666

TABLE 5. (Continued)

Probability	0.107	0.088	0.068	0.048	0.029	0.009
			i			
Percentile	91	93	95	97	99	101
2	5.2471	5.5980	5.9910	6.4957	7.2023	8.4081
5	5.3933	5.7494	6.1782	6.7345	7.4666	8.9158
10	5.5373	5.9009	6.3701	6.9482	7.7621	9.4285
50	6.0346	6.4837	7.0292	7.7778	8.8828	11.5020
90	6.6002	7.1128	7.8283	8.7897	10.3687	14.6263
95	6.7750	7.3270	8.0656	9.0942	10.8869	15.7611
98	6.9395	7.5614	8.3065	9.4403	11.4272	17.1230

$\nu = 4 \qquad n = 7$

Probability	0.793	0.696	0.508	0.500	0.402	0.304	0.207
				i			
Percentile	1	2	3	4	5	6	7
2	.2579	1.0410	2.1762	2.9226	3.4160	3.8139	4.1969
5	.4293	1.3443	2.3831	3.0584	3.5433	3.9427	4.3585
10	.6200	1.5772	2.5414	3.1826	3.6672	4.0680	4.4931
50	1.4551	2.3425	3.0666	3.6420	4.1356	4.5463	4.9066
90	2.2410	2.9007	3.5632	4.1615	4.6102	4.9141	5.0997
95	2.4031	3.0112	3.6892	4.3064	4.7179	4.9717	5.1198
98	2.5665	3.1248	3.8367	4.4247	4.8159	5.0382	5.1318

TABLE 5. (Continued)

$\nu = 4$ $n = 11$

Probability	0.864	0.791	0.718	0.646	0.573	0.500	0.427	0.354	0.282
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.1449	.5657	1.0485	1.4669	1.9648	2.4972	3.0532	3.5879	4.0734
5	.2627	.7414	1.2128	1.7119	2.2150	2.7320	3.2556	3.7759	4.2539
10	.3733	.9143	1.4273	1.9063	2.4259	2.9204	3.4254	3.9426	4.4251
50	.9534	1.5775	2.0992	2.5752	3.0241	3.4911	3.9840	4.5008	5.0856
90	1.6779	2.2249	2.6913	3.1240	3.5616	4.0195	4.5509	5.1741	5.8759
95	1.8821	2.3974	2.8558	3.2579	3.6916	4.1541	4.7093	5.3781	6.1019
98	2.0365	2.5738	3.0021	3.3765	3.8096	4.2927	4.8675	5.5821	6.3399

Probability	0.209	0.136
	i	
Percentile	10	11
2	4.5809	5.1796
5	4.7942	5.4589
10	4.9844	5.7086
50	5.7847	6.7221
90	6.7129	7.7950
95	6.9643	8.0430
98	7.2579	8.2843

TABLE 5. (Continued)

$\nu = 4$ n = 15

Probability	0.901	0.844	0.787	0.729	0.672	0.615	0.557	0.500	0.443
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.1151	.4329	.7533	1.0778	1.4333	1.7654	2.1227	2.4828	2.9092
5	.1971	.5627	.9408	1.2952	1.6218	1.9655	2.3144	2.7066	3.0970
10	.2871	.7013	1.0850	1.4443	1.7990	2.1359	2.4978	2.8762	3.2819
50	.7542	1.2629	1.6598	2.0288	2.3827	2.7331	3.0860	3.4485	3.8408
90	1.3649	1.8398	2.2345	2.5851	2.9191	3.2499	3.5958	3.9661	4.3689
95	1.5329	2.0190	2.3937	2.7332	3.0475	3.3620	3.7333	4.1052	4.5244
98	1.7299	2.1688	2.5285	2.8482	3.1742	3.5085	3.8577	4.2507	4.6777

Probability	0.385	0.328	0.271	0.213	0.156	0.099
			i			
Percentile	10	11	12	13	14	15
2	3.3521	3.7622	4.2099	4.6670	5.1696	5.8486
5	3.5205	3.9324	4.3981	4.8834	5.4268	6.1732
10	3.6932	4.1164	4.5739	5.0661	5.6519	6.4956
50	4.2540	4.6973	5.2206	5.8182	6.6017	7.7876
90	4.8338	5.3520	5.9769	6.7703	7.7791	9.4530
95	4.9881	5.5143	6.1889	7.0621	8.1357	9.8619
98	5.1791	5.7523	6.4319	7.3901	8.5200	10.3852

TABLE 5. (Continued)

$\nu = 4$ n = 21

Probability	0.932	0.889	0.845	0.802	0.759	0.716	0.673	0.630	0.586
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0850	.3436	.5771	.8112	1.0540	1.3015	1.5290	1.7841	2.0482
5	.1487	.4477	.7153	.9761	1.2131	1.4544	1.7145	1.9768	2.2207
10	.2254	.5528	.8342	1.0946	1.3582	1.6005	1.8597	2.1101	2.3631
50	.6031	1.0025	1.3119	1.5959	1.8619	2.1266	2.3828	2.6391	2.9032
90	1.1259	1.5197	1.8306	2.0995	2.3691	2.6161	2.8516	3.1155	3.3838
95	1.2695	1.6664	1.9685	2.2397	2.4934	2.7488	2.9943	3.2400	3.5080
98	1.4233	1.7986	2.0977	2.3703	2.6182	2.8779	3.1209	3.3499	3.6363

Probability	0.543	0.500	0.457	0.414	0.370	0.327	0.284	0.241	0.198
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	2.2812	2.5592	2.8642	3.1606	3.4681	3.7939	4.1483	4.4771	4.9157
5	2.4717	2.7492	3.0279	3.3371	3.6422	3.9737	4.3152	4.7046	5.1418
10	2.6322	2.8988	3.1881	3.4894	3.7901	4.1167	4.4807	4.8651	5.3103
50	3.1599	3.4368	3.7129	4.0072	4.3236	4.6739	5.0580	5.4918	6.0176
90	3.6391	3.9031	4.2011	4.5085	4.8687	5.2534	5.7132	6.2496	6.8811
95	3.7510	4.0289	4.3177	4.6575	5.0031	5.4017	5.9042	6.4482	7.1285
98	3.4892	4.1657	4.4684	4.7975	5.1640	5.5975	6.0849	6.6816	7.4332

TABLE 5. (Continued)

Probability	0.155	0.111	0.068
		i	
Percentile	19	20	21
2	5.3709	5.8676	6.6351
5	5.5859	6.1606	6.9574
10	5.7999	6.4316	7.2866
50	6.6507	7.5095	8.8874
90	7.7252	8.9088	11.0005
95	8.0688	9.3003	11.6164
98	8.4500	9.7815	12.2477

$\nu = 4$ $n = 31$

Probability	0.956	0.926	0.895	0.865	0.834	0.804	0.774	0.743	0.713
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0726	.2627	.4327	.6370	.7824	.9677	1.1275	1.2860	1.4463
5	.1238	.3426	.5455	.7315	.8995	1.0933	1.2438	1.4291	1.5846
10	.1800	.4216	.6409	.8349	1.0122	1.1824	1.3674	1.5393	1.7108
50	.4771	.7819	1.0171	1.2257	1.4184	1.6110	1.7823	1.9653	2.1441
90	.8896	1.2014	1.4385	1.6435	1.8440	2.0191	2.2030	2.3924	2.5696
95	1.0011	1.3196	1.5537	1.7605	1.9495	2.1337	2.3242	2.4977	2.6676
98	1.1262	1.4187	1.6582	1.8749	2.0688	2.2337	2.4394	2.5851	2.7814

TABLE 5. (Continued)

Probability	0.682	0.652	0.622	0.591	0.561	0.530	0.500	0.470	0.439
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	1.6321	1.7765	1.9374	2.1311	2.3060	2.4772	2.6790	2.8512	3.0495
5	1.7600	1.9207	2.0976	2.2658	2.4656	2.6504	2.8257	3.0081	3.2162
10	1.8800	2.0517	2.2251	2.3993	2.5910	2.7830	2.9677	3.1499	3.3466
50	2.3226	2.4892	2.6588	2.8373	3.0224	3.2090	3.4053	3.6032	3.8109
90	2.7386	2.9087	3.0768	3.2650	3.4417	3.6259	3.8242	4.0218	4.1511
95	2.8512	3.0180	3.1974	3.3665	3.5526	3.7475	3.9382	4.1346	4.3612
98	2.9595	3.1399	3.3099	3.4729	3.6523	3.8539	4.0570	4.2608	4.4729

Probability	0.409	0.378	0.348	0.318	0.287	0.257	0.226	0.196	0.166
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	3.2531	3.4815	3.7016	3.9153	4.1665	4.4596	4.7308	5.0289	5.3526
5	3.4227	3.6243	3.8731	4.0963	4.3476	4.6125	4.9084	5.2149	5.5656
10	3.5470	3.7734	4.0084	4.2442	4.4996	4.7795	5.0644	5.3977	5.7514
50	4.0217	4.2487	4.4863	4.7505	5.0264	5.3283	5.6562	6.0305	6.4828
90	4.4762	4.7135	4.9842	5.2557	5.5628	5.9166	6.3195	6.7496	7.2985
95	4.5992	4.8406	5.1252	5.3838	5.7208	6.0717	6.5006	6.9732	7.5317
98	4.6994	4.9771	5.2536	5.5640	5.8891	6.2512	6.6992	7.1717	7.7934

TABLE 5. (Continued)

Probability	0.135	0.105	0.074	0.044
	i			
Percentile	28	29	30	31
2	5.7223	6.1795	6.7082	7.5147
5	5.9676	6.4270	7.0410	7.9252
10	6.1658	6.6917	7.3523	8.2934
50	6.9918	7.6601	8.5613	10.1070
90	7.9568	8.8309	10.1473	12.7392
95	8.2588	9.2446	10.6703	13.5752
98	8.5626	9.6955	11.2737	14.4730

$\nu = 4 \qquad n = 51$

Probability	0.975	0.956	0.937	0.918	0.899	0.880	0.861	0.842	0.823
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.0552	.1940	.3230	.4424	.5596	.6911	.7882	.9023	.9871
5	.0883	.2603	.4053	.5257	.6526	.7799	.8903	.9960	1.1039
10	.1350	.3156	.4704	.6121	.7414	.8587	.9742	1.0855	1.1977
50	.3608	.5883	.7516	.8964	1.0323	1.1592	1.2825	1.3993	1.5174
90	.6622	.8894	1.0616	1.2297	1.3573	1.4930	1.6121	1.7371	1.8452
95	.7567	.9868	1.1550	1.3139	1.4589	1.5747	1.7036	1.8208	1.9366
98	.8581	1.0728	1.2430	1.4141	1.5588	1.6827	1.7831	1.9133	2.0131

TABLE 5. (Continued)

Probability	0.804	0.785	0.766	0.747	0.728	0.709	0.690	0.671	0.652
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	1.0987	1.2133	1.3046	1.4095	1.5138	1.6110	1.7190	1.8204	1.9092
5	1.2049	1.3118	1.4229	1.5271	1.6262	1.7219	1.8331	1.9364	2.0394
10	1.3029	1.4014	1.5168	1.6149	1.7209	1.8300	1.9289	2.0242	2.1325
50	1.6263	1.7383	1.8410	1.9504	2.0533	2.1638	2.2641	2.3809	2.4825
90	1.9621	2.0709	2.1839	2.2904	2.3935	2.4979	2.6058	2.7124	2.8148
95	2.0557	2.1632	2.2714	2.3839	2.4809	2.5828	2.6975	2.7932	2.9133
98	2.1278	2.2636	2.3563	2.4690	2.5653	2.6755	2.7812	2.8938	2.9872

Probability	0.633	0.614	0.595	0.576	0.557	0.538	0.519	0.500	0.481
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	2.0165	2.1245	2.2470	2.3491	2.4563	2.5658	2.6850	2.8007	2.9073
5	2.1385	2.2576	2.3665	2.4754	2.5827	2.6969	2.8196	2.9220	3.0420
10	2.2399	2.3543	2.4602	2.5724	2.6844	2.7973	2.9200	3.0324	3.1522
50	2.5877	2.7027	2.8114	2.9300	3.0343	3.1474	3.2694	3.3940	3.5111
90	2.9303	3.0381	3.1455	3.2594	3.3746	3.4915	3.6154	3.7294	3.8556
95	3.0245	3.1255	3.2381	3.3455	3.4695	3.5847	3.6941	3.8304	3.9538
98	3.1149	3.2210	3.3265	3.4270	3.5608	3.6850	3.7998	3.9186	4.0659

TABLE 5. (Continued)

Probability	0.462	0.443	0.424	0.405	0.386	0.367	0.348	0.329	0.310
					i				
Percentile	28	29	30	31	32	33	34	35	36
2	3.0287	3.1516	3.2721	3.4089	3.5452	3.6931	3.8321	3.9995	4.1404
5	3.1685	3.2773	3.4152	3.5474	3.6955	3.8328	3.9734	4.1245	4.2661
10	3.2655	3.3915	3.5246	3.6582	3.7931	3.9459	4.0843	4.2327	4.3850
50	3.6335	3.7668	3.8984	4.0331	4.1834	4.3189	4.4801	4.6430	4.8060
90	3.9927	4.1282	4.2540	4.4095	4.5539	4.7062	4.8764	5.0415	5.2274
95	4.0892	4.2181	4.3613	4.5014	4.6564	4.8114	4.9837	5.1519	5.3374
98	4.1771	4.3055	4.4577	4.6137	4.7563	4.9231	5.0919	5.2788	5.4514

Probability	0.291	0.272	0.253	0.234	0.215	0.196	0.177	0.158	0.139
					i				
Percentile	37	38	39	40	41	42	43	44	45
2	4.3031	4.4525	4.6360	4.8127	4.9989	5.1926	5.4101	5.6488	5.8897
5	4.4457	4.5981	4.7743	4.9631	5.1791	5.3569	5.5892	5.8272	6.0851
10	4.5585	5.7219	4.9069	5.0978	5.2974	5.5070	5.7339	5.9883	6.2551
50	4.9797	5.1683	5.3582	5.5643	5.7804	6.0196	6.2844	6.5772	6.9031
90	5.4154	5.6211	5.8388	6.0608	6.3089	6.5928	6.9105	7.2397	7.6594
95	5.5370	5.7463	5.9757	6.2063	6.4566	6.7408	7.0711	7.4584	7.8683
98	5.6776	5.8736	6.1045	6.3642	6.6335	6.9352	7.2746	7.6630	8.1124

TABLE 5. (Continued)

Probability	0.120	0.101	0.082	0.063	0.044	0.025
			i			
Percentile	46	47	48	49	50	51
2	6.1632	6.4687	6.8317	7.2214	7.7723	8.5786
5	6.3838	6.7078	7.0920	7.5380	8.1283	9.0191
10	6.5507	6.8929	7.3250	7.7910	8.4959	9.5007
50	7.2718	7.6914	8.2380	8.9064	9.8663	11.5328
90	8.1054	8.6723	9.3705	10.2782	11.6686	14.5028
95	8.3598	8.9354	9.7687	10.7016	12.2827	15.5050
98	8.6194	9.2328	10.1347	11.1283	12.9519	16.7038

$\nu = 4$ $n = 101$

Probability	0.988	0.969	0.949	0.930	0.910	0.891	0.871	0.852	0.832
					i				
Percentile	1	3	5	7	9	11	13	15	17
2	.0394	.2233	.3838	.5205	.6491	.7643	.8875	1.0089	1.1021
5	.0640	.2698	.4436	.5800	.7216	.8464	.9567	1.0735	1.1826
10	.0949	.3195	.4923	.6406	.7783	.9033	1.0188	1.1325	1.2469
50	.2505	.5104	.6929	.8519	.9936	1.1230	1.2476	1.3664	1.4810
90	.4642	.7332	.9185	1.0781	1.2195	1.3513	1.4731	1.5924	1.7142
95	.5288	.7969	.9827	1.1392	1.2780	1.4224	1.5394	1.6580	1.7776
98	.5942	.8590	1.0441	1.2083	1.3453	1.4752	1.6014	1.7238	1.8359

TABLE 5. (Continued)

Probability	0.813	0.793	0.774	0.754	0.734	0.715	0.695	0.676	0.656
					i				
Percentile	19	21	23	25	27	29	31	33	35
2	1.2126	1.3130	1.4169	1.5208	1.6231	1.7265	1.8324	1.9490	2.0483
5	1.2907	1.3938	1.5051	1.6038	1.7120	1.8253	1.9260	2.0270	2.1409
10	1.3593	1.4707	1.5737	1.6754	1.7816	1.8924	1.9972	2.1048	2.2096
50	1.5909	1.6996	1.8129	1.9205	2.0242	3.1333	2.2406	2.3524	2.4589
90	1.8219	1.9379	2.0476	2.1564	2.2664	2.3783	2.4879	2.5882	2.6926
95	1.8933	2.0042	2.1162	2.2200	2.3346	2.4417	2.5486	2.6588	2.7648
98	1.9658	2.0701	2.1731	2.2810	2.3967	2.5028	2.6158	2.7272	2.8352

Probability	0.637	0.617	0.598	0.578	0.559	0.539	0.520	0.500	0.480
					i				
Percentile	37	39	41	43	45	47	49	51	53
2	2.1683	2.2734	2.3749	2.4847	2.6024	2.7098	2.8247	2.9354	3.0654
5	2.2438	2.3581	2.4587	2.5720	2.6891	2.8019	2.9253	3.0387	3.1631
10	2.3164	2.4255	2.5384	2.6486	2.7626	2.8786	2.9938	3.1195	3.2409
50	2.5689	2.6797	2.7936	2.9043	3.0226	3.1397	3.2581	3.3832	3.5091
90	2.8153	2.9233	3.0371	3.1537	3.2766	3.3939	3.5131	3.6325	3.7602
95	2.8814	2.9885	3.1045	3.2229	3.3386	3.4582	3.5785	3.7077	3.8242
98	2.9364	3.0550	3.1698	3.2842	3.4087	3.5204	3.6448	3.7729	3.8980

TABLE 5. (Continued)

Probability	0.461	0.441	0.422	0.402	0.383	0.363	0.344	0.324	0.305
					i				
Percentile	55	57	59	61	63	65	67	69	71
2	3.1917	3.3046	3.4446	3.5689	3.7242	3.8694	3.0046	4.1704	4.3150
5	3.2869	3.4063	3.5395	3.6848	3.8204	3.9784	4.1235	4.2675	4.4321
10	3.3637	3.4911	3.6227	3.7653	3.9063	4.0592	4.2069	4.3523	4.5176
50	3.6371	3.7671	3.9069	4.0469	4.1934	4.3420	4.5033	4.6654	4.8407
90	3.8986	4.0334	4.1706	4.3120	4.4760	4.6277	4.7922	4.9708	5.1460
95	3.9692	4.1019	4.2452	4.3982	4.5456	4.6979	4.8657	5.0526	5.2309
98	4.0334	4.1653	4.3157	4.4632	4.6249	4.7800	4.9528	5.1303	5.3164

Probability	0.285	0.266	0.246	0.226	0.207	0.187	0.168	0.148	0.129
					i				
Percentile	73	75	77	79	81	83	85	87	89
2	4.5039	4.6799	4.8606	5.0644	5.2691	5.4929	5.7181	6.0119	6.3010
5	4.6037	4.7920	4.9740	5.1937	5.3900	5.6288	5.8738	6.1463	6.4507
10	4.6922	4.8823	5.0755	5.2885	5.5038	5.7134	5.9920	6.2769	6.5933
50	5.0232	5.2186	5.4202	5.6503	5.8827	6.1469	6.4210	6.7361	7.0937
90	5.3500	5.5508	5.7746	6.0162	6.2773	6.5557	6.8729	7.2308	7.6443
95	5.4375	5.6442	5.8872	6.1378	6.3887	6.6782	6.9995	7.3819	7.7956
98	5.5368	5.7574	5.9768	6.2368	6.4968	6.7991	7.1400	7.5373	7.9816

TABLE 5. (Continued)

Probability	0.109	0.090	0.070	0.051	0.031	0.012
			i			
Percentile	91	93	95	97	99	101
2	6.5832	6.9889	7.4272	7.9869	8.7251	10.0342
5	6.7863	7.1637	7.6403	8.2125	9.0332	10.5852
10	6.9291	7.3390	7.8249	8.4565	9.3356	11.0882
50	7.5059	7.9771	8.5817	9.3721	10.5790	13.3698
90	8.1224	8.7277	9.4507	10.4612	12.1743	16.7198
95	8.3023	8.9397	9.7020	10.7906	12.6368	17.8425
98	8.5260	9.1668	9.9624	11.1430	13.1559	19.3843

$\nu = 5$ $n = 7$

Probability	0.780	0.686	0.593	0.500	0.407	0.314	0.220
				i			
Percentile	1	2	3	4	5	6	7
2	.7027	2.6928	3.6509	4.0501	4.4032	4.6336	4.8819
5	1.0464	2.9076	3.7519	4.1632	4.5110	4.7488	4.9693
10	1.4117	3.1039	3.8615	4.2625	4.6050	4.8369	5.0265
50	2.5329	3.6586	4.2764	4.6392	4.8990	5.0502	5.1270
90	3.2990	4.1674	4.7270	4.9541	5.0794	5.1304	5.1424
95	3.4397	4.2943	4.8121	5.0149	5.1043	5.1367	5.1427
98	3.5411	4.4222	4.9068	5.0555	5.1186	5.1397	5.1428

TABLE 5. (Continued)

$\nu = 5 \qquad n = 11$

Probability	0.851	0.781	0.711	0.640	0.570	0.500	0.430	0.360	0.289
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.3532	1.0725	1.7047	2.3496	2.9804	3.5956	4.1258	4.6201	5.0789
5	.5522	1.3049	1.9747	2.6304	3.2251	3.8024	4.3235	4.8005	5.2557
10	.7505	1.5480	2.2176	2.8276	3.4307	3.9872	4.4825	4.9567	5.4290
50	1.5785	2.3642	3.0022	3.5471	4.0499	4.5346	5.0267	5.5363	6.0775
90	2.4834	3.1333	3.6519	4.1199	4.5875	5.0817	5.6260	6.2170	6.8339
95	2.6839	3.3003	3.7951	4.2572	4.7208	5.2456	5.8147	6.4274	7.0470
98	2.8983	3.4608	3.9397	4.3849	4.8642	5.4002	6.0032	6.6451	7.2541

Probability	0.219	0.149
	i	
Percentile	10	11
2	5.5617	6.1058
5	5.7554	6.3583
10	5.9305	6.6017
50	6.6879	7.4850
90	7.5048	8.3299
95	7.7186	8.4975
98	7.9546	8.6779

TABLE 5. (Continued)

$\nu = 5 \qquad n = 15$

Probability	0.890	0.834	0.779	0.723	0.667	0.611	0.556	0.500	0.444
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.2876	.8227	1.3197	1.7516	2.1863	2.6193	3.0216	3.4012	3.8815
5	.4296	1.0128	1.5346	1.9823	2.4138	2.8498	3.2759	3.6936	4.1212
10	.5860	1.1980	1.7242	2.1741	2.6158	3.0334	3.4556	3.8733	4.3113
50	1.2561	1.9271	2.4288	2.8607	3.2780	3.6748	4.0703	4.4785	4.8774
90	2.0686	2.6249	3.1033	3.4838	3.8559	4.2493	4.6204	5.0085	5.4450
95	2.2718	2.8149	3.2681	3.6574	4.0103	4.3877	4.7548	5.1502	5.5880
98	2.4445	3.0176	3.4290	3.7967	4.1609	4.5341	4.9101	5.2864	5.7579

Probability	0.389	0.333	0.277	0.221	0.166	0.110
				i		
Percentile	10	11	12	13	14	15
2	4.3540	4.8146	5.2445	5.7065	6.2196	6.8528
5	4.5751	5.0172	5.4571	5.9153	6.4761	7.2046
10	4.7436	5.1688	5.6389	6.1246	6.6977	7.5350
50	5.3126	5.7915	6.3016	6.9018	7.6754	8.8017
90	5.9141	6.4803	7.1107	7.8506	8.8450	10.2645
95	6.0966	6.7013	7.3658	8.1570	9.1650	10.7476
98	6.2824	6.9151	7.6355	8.4759	9.5611	11.1266

TABLE 5. (Continued)

$\nu = 5 \qquad n = 21$

Probability	0.923	0.880	0.838	0.796	0.754	0.711	0.669	0.627	0.585
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.2313	.6486	1.0321	1.3313	1.6813	1.9433	2.2643	2.5564	2.8698
5	.3547	.8158	1.1900	1.5423	1.8847	2.1457	2.4841	2.7894	3.0943
10	.4752	.9599	1.3586	1.7158	2.2084	2.3685	2.6591	2.9689	3.2623
50	1.0591	1.5668	1.9688	2.3194	2.6419	2.9412	3.2444	3.5329	3.8390
90	1.7149	2.2123	2.5740	2.9145	3.2143	3.5117	3.8061	4.0757	4.3526
95	1.9190	2.3846	2.7440	3.0792	3.3722	3.6442	3.9410	4.2143	4.4770
98	2.0854	2.5608	2.8997	3.2378	3.5031	3.7705	4.0883	4.3360	4.6135

Probability	0.542	0.500	0.458	0.415	0.373	0.331	0.289	0.246	0.204
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	3.1914	3.4929	3.8250	4.1930	4.4977	4.8519	5.1987	5.6170	6.0362
5	3.3877	3.6920	4.0199	4.3690	4.7062	5.0588	5.4014	5.7936	6.2268
10	3.5565	3.8717	4.1874	4.5105	4.8597	5.2182	5.5787	5.9916	6.4152
50	4.1311	4.4277	4.7416	5.0766	5.4226	5.7957	6.2053	6.6556	7.1928
90	4.6416	4.9414	5.2720	5.6104	6.0085	6.4333	6.8910	7.4482	8.1344
95	4.7766	5.0894	5.4212	5.7696	6.1763	6.6212	7.1064	7.7108	8.4086
98	4.8965	5.2080	5.5652	5.9323	6.3500	6.8039	7.3408	7.9403	8.7029

TABLE 5. (Continued)

Probability	0.162	0.120	0.077
		i	
Percentile	19	20	21
2	6.4675	7.0260	7.7390
5	6.7192	7.3087	8.0972
10	6.9171	7.5596	8.4883
50	7.8307	8.7083	10.1093
90	8.9701	10.1251	12.0978
95	9.3008	10.5160	12.6600
98	9.6678	11.0111	13.2939

$\nu = 5$ $n = 31$

Probability	0.949	0.919	0.889	0.859	0.829	0.800	0.770	0.740	0.710
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.1789	.4995	.7904	1.0534	1.2823	1.4975	1.7039	1.9150	2.1266
5	.2683	.6308	.9340	1.2117	1.4474	1.6728	1.8825	2.0753	2.3022
10	.3767	.7619	1.0743	1.3444	1.5911	1.8168	2.0167	2.2416	2.4518
50	.8418	1.2632	1.5778	1.8462	2.0878	2.3252	2.5386	2.7550	2.9599
90	1.4023	1.7907	2.0973	2.3639	2.5785	2.8129	3.0359	3.2336	3.4439
95	1.5487	1.9427	2.2382	2.4928	2.7155	2.9527	3.1626	3.3705	3.5604
98	1.7191	2.0829	2.4023	2.6469	2.8602	3.0889	3.2769	3.5044	3.6957

TABLE 5. (Continued)

Probability	0.680	0.650	0.620	0.590	0.560	0.530	0.500	0.470	0.440
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	2.3324	2.5487	2.7395	2.9482	3.1656	3.3808	3.5943	3.8130	4.0561
5	2.5107	2.7050	2.9210	3.1449	3.3360	3.5376	3.7585	3.9939	4.2205
10	2.6659	2.8629	3.0646	3.2793	3.4870	3.6980	3.9013	4.1262	4.3580
50	3.1616	3.3756	3.5795	3.7728	3.9811	4.2005	4.4186	4.6284	4.8593
90	3.6397	3.8445	4.0511	4.2369	4.4381	4.6620	4.8739	5.0991	5.3356
95	3.7651	3.9630	4.1557	4.3581	4.5652	4.7812	4.9927	5.2270	5.4656
98	3.9013	4.0904	4.2709	4.4842	4.6858	4.9015	5.1178	5.3526	5.5921

Probability	0.410	0.380	0.350	0.320	0.290	0.260	0.230	0.200	0.171
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	4.2676	4.5127	4.7593	5.0166	5.2974	5.5681	5.8641	6.1729	6.5656
5	4.4327	4.6825	4.9409	5.2023	5.4866	5.7700	6.0608	6.3793	6.7470
10	4.5862	4.8349	5.0915	5.3451	5.6238	5.9189	6.2318	6.5646	6.9488
50	5.1037	5.3434	5.6043	5.8822	6.1826	6.5009	6.8598	7.2577	7.7235
90	5.5811	5.8561	6.1356	6.4285	6.7581	7.1512	7.5566	8.0517	8.6277
95	5.7236	5.9824	6.2766	6.5852	6.9205	7.3214	7.7683	8.2926	8.9054
98	5.8610	6.1354	6.4385	6.7487	7.1247	7.5506	7.9659	8.5198	9.1672

TABLE 5. (Continued)

Probability	0.141	0.111	0.081	0.051
		i		
Percentile	28	29	30	31
2	6.9085	7.3453	7.9361	8.7318
5	7.1441	7.6118	8.2420	9.1631
10	7.3866	7.9174	8.5567	9.6074
50	8.2690	8.9591	9.8739	11.4460
90	9.3595	10.2529	11.5567	13.9791
95	9.6739	10.6111	12.1099	14.9425
98	10.0425	11.0984	12.7178	15.8378

$\nu = 5$ n = 51

Probability	0.971	0.958	0.933	0.914	0.895	0.877	0.858	0.839	0.820
					i				
Percentile	1	2	3	4	5	6	7	8	9
2	.1385	.4015	.6124	.8026	.9399	1.1144	1.2771	1.4396	1.5767
5	.2171	.4989	.7293	.9130	1.0887	1.2632	1.4123	1.5593	1.6973
10	.2967	.6040	.8305	1.0210	1.2013	1.3727	1.5143	1.6646	1.8105
50	.6534	.9803	1.2178	1.4222	1.6020	1.7665	1.9212	2.0611	2.1972
90	1.0939	1.4088	1.6423	1.8304	1.9973	2.1631	2.3161	2.4588	2.5954
95	1.2110	1.5229	1.7630	1.9355	2.1153	2.2824	2.4237	2.5624	2.6962
98	1.3483	1.6610	1.8836	2.0602	2.2316	2.3841	2.5316	2.6841	2.8091

TABLE 5. (Continued)

Probability	0.801	0.782	0.764	0.745	0.726	0.707	0.688	0.669	0.651
					i				
Percentile	10	11	12	13	14	15	16	17	18
2	1.6932	1.8214	1.9601	2.0787	2.1971	2.3309	2.4647	2.5817	2.7098
5	1.8263	1.9527	2.0832	2.2120	2.3476	2.4681	2.5945	2.7303	2.8512
10	1.9377	2.0650	2.1899	2.3302	2.4617	2.5861	2.7084	2.8341	2.9598
50	2.3333	2.4740	2.6027	2.7255	2.8547	2.9846	3.1092	3.2319	3.3599
90	2.7328	2.8618	2.9851	3.1145	3.2275	3.3700	3.4875	3.6101	3.7406
95	2.8360	2.9631	3.0893	3.2035	3.3298	3.4563	3.5909	3.7206	3.8367
98	2.9475	3.0622	3.1865	3.3133	3.4484	3.5640	3.7000	3.8182	3.9419

Probability	0.632	0.613	0.594	0.575	0.556	0.538	0.519	0.500	0.481
					i				
Percentile	19	20	21	22	23	24	25	26	27
2	2.8404	2.9448	3.0608	3.2063	3.3226	3.4656	3.6010	3.7151	3.8542
5	2.9613	3.0927	3.2158	3.3557	3.4693	3.6070	3.7408	3.8578	4.0037
10	3.0840	3.2128	3.3402	3.4624	3.5824	3.7173	3.8475	3.9956	4.1372
50	3.4836	3.6148	3.7353	3.8616	3.9908	4.1242	4.2524	4.3962	4.5324
90	3.8628	3.9858	4.1083	4.2438	4.3738	4.5154	4.6431	4.7880	4.9276
95	3.9640	4.0821	4.2110	4.3440	4.4701	4.6158	4.7413	4.8842	5.0373
98	4.0625	4.1808	4.3241	4.4437	4.5704	4.7052	4.8506	4.9868	5.1290

TABLE 5. (Continued)

Probability	0.462	0.444	0.425	0.406	0.387	0.368	0.349	0.331	0.312
					i				
Percentile	28	29	30	31	32	33	34	35	36
2	3.9927	4.1365	4.2810	4.4366	4.5841	4.7226	4.9025	5.0484	5.2247
5	4.1436	4.2876	4.4273	4.5761	4.7371	4.9012	5.0567	5.1974	5.3813
10	4.2609	4.4040	4.5550	4.6999	4.8606	5.0189	5.1753	5.3373	5.5071
50	4.6797	4.8157	4.9621	5.1139	5.2654	5.4315	5.5999	5.7804	5.9545
90	5.0684	5.2188	5.3634	5.5196	5.6744	5.8508	6.0394	6.2124	6.4065
95	5.1717	5.3300	5.4659	5.6306	5.7842	5.9656	6.1553	6.3295	6.5384
98	5.2602	5.4216	5.5586	5.7422	5.8973	6.0792	6.2690	6.4801	6.6722

Probability	0.298	0.274	0.255	0.236	0.218	0.189	0.180	0.161	0.142
					i				
Percentile	37	38	39	40	41	42	43	44	45
2	5.3878	5.5506	5.7598	5.9473	6.1616	6.3952	6.5803	6.8448	7.1099
5	5.5458	5.7257	5.9271	6.1146	6.3565	6.5675	6.7824	7.0532	7.3196
10	5.6821	5.8726	6.0662	6.2586	6.4853	6.7142	6.9544	7.2121	7.5200
50	6.1416	6.3437	6.5471	6.7732	7.0081	7.2729	7.5534	7.8731	8.2092
90	6.6271	6.8422	7.0801	7.3334	7.6138	7.9187	8.2322	8.5916	9.0178
95	6.7465	6.9872	6.2329	7.4756	7.7727	8.1076	8.4169	8.8247	9.2644
98	6.8839	7.1364	7.3625	7.6585	7.9387	8.2732	8.6165	9.0283	9.5292

TABLE 5. (Continued)

Probability	0.123	0.105	0.086	0.067	0.048	0.029
			i			
Percentile	46	47	48	49	50	51
2	7.3985	7.7457	8.1588	8.5566	9.0740	9.9692
5	7.6512	7.9915	8.3677	8.8362	9.4324	10.4095
10	7.8330	8.1915	8.6133	9.1430	9.8351	10.8510
50	8.6128	9.0696	9.6122	10.2944	11.2788	13.0004
90	9.5165	10.0895	10.8880	11.7474	13.2284	16.0655
95	9.7771	10.3976	11.1788	12.2448	13.7944	17.1642
98	10.1198	10.7200	11.5841	12.7129	14.5153	18.2737

$\nu = 5$ $n = 101$

Probability	0.986	0.967	0.947	0.928	0.908	0.889	0.869	0.850	0.831
					i				
Percentile	1	3	5	7	9	11	13	15	17
2	.1049	.4287	.6796	.8923	1.0606	1.2381	1.3778	1.5310	1.6798
5	.1568	.5171	.7671	.9690	1.1587	1.3243	1.4815	1.6232	1.7774
10	.2171	.5913	.8471	1.0455	1.2310	1.4002	1.5643	1.7115	1.8558
50	.4835	.8708	1.1310	1.3405	1.5197	1.6883	1.8431	1.9896	2.1353
90	.7916	1.1736	1.4256	1.6314	1.8105	1.9742	2.1225	2.2733	2.4209
95	.8938	1.2644	1.5005	1.7136	1.8927	2.0465	2.2053	2.3490	2.4899
98	.9854	1.3493	1.5772	1.7964	1.9701	2.1249	2.2831	2.4337	2.5686

TABLE 5. (Continued)

Probability	0.811	0.792	0.772	0.753	0.733	0.714	0.694	0.675	0.656
					i				
Percentile	19	21	23	25	27	29	31	33	35
2	1.8202	1.9547	2.0879	2.2058	2.3342	2.4656	2.6051	2.7130	2.8472
5	1.9056	2.0449	2.1750	2.3084	2.4360	2.5658	2.6887	2.8208	2.9413
10	1.9927	2.1239	2.2618	2.3882	2.5195	2.6456	2.7656	2.9025	3.0287
50	2.2790	2.4122	2.5422	2.6780	2.8029	2.9328	3.0641	3.1843	3.3172
90	2.5598	2.6919	2.8249	2.9565	3.0832	3.2132	3.3408	3.4669	3.5936
95	2.6339	2.7686	2.9049	3.0375	3.1585	3.2848	3.4141	3.5329	3.6641
98	2.7093	2.8373	2.9734	3.1078	3.2342	3.3614	3.4802	3.6215	3.7407

Probability	0.636	0.617	0.597	0.578	0.558	0.539	0.519	0.500	0.481
					i				
Percentile	37	39	41	43	45	47	49	51	53
2	2.9744	3.1028	3.2287	3.3571	3.4866	3.6273	3.7642	3.8936	4.0367
5	3.0719	3.2002	3.3290	3.4611	3.5951	3.7244	3.8543	3.9931	4.1341
10	3.1518	3.2868	3.4149	3.5437	3.6775	3.8166	3.9480	4.0850	4.2295
50	3.4494	3.5754	3.7105	3.8403	3.9724	4.1034	4.2443	4.3838	4.5222
90	3.7249	3.8476	3.9801	4.1148	4.2576	4.3891	4.5305	4.6671	4.8082
95	3.7943	3.9220	4.0618	4.1958	4.3332	4.4646	4.6165	4.7507	4.8864
98	3.8650	3.9996	4.1373	4.2761	4.4079	4.5473	4.6885	4.8197	4.9706

TABLE 5. (Continued)

Probability	0.461	0.442	0.422	0.403	0.383	0.364	0.344	0.325	0.306
					i				
Percentile	55	57	59	61	63	65	67	69	71
2	4.1712	4.3228	4.4690	4.6294	4.7851	4.9334	5.1097	5.2837	5.4617
5	4.2746	4.4269	4.5748	4.7342	4.8972	5.0429	5.2293	5.3947	5.5767
10	4.3692	4.5173	4.6674	4.8227	4.9790	5.1461	5.3104	5.4828	5.6713
50	4.6664	4.8169	4.9756	5.1306	5.2959	5.4636	5.6396	5.8294	6.0214
90	4.9557	5.1111	5.2688	5.4384	5.6039	5.7725	5.9629	6.1590	6.3630
95	5.0287	5.1823	5.3545	5.5165	5.6818	5.8577	6.0533	6.2446	6.4634
98	5.1151	5.2676	5.4273	5.6007	5.7697	5.9522	6.1385	6.3376	6.5619

Probability	0.286	0.267	0.247	0.228	0.208	0.189	0.169	0.150	0.131
					i				
Percentile	73	75	77	79	81	83	85	87	89
2	5.6456	5.8440	6.0240	6.2416	6.4744	6.7340	6.9918	7.2639	7.5841
5	5.7653	5.9556	6.1673	6.3844	6.6322	6.8790	7.1502	7.4187	7.7563
10	5.8632	6.0605	6.2807	6.5014	6.7501	7.0045	7.2720	7.5586	7.9137
50	6.2166	6.4288	6.6649	6.9030	7.1599	7.4316	7.7455	8.0818	8.4615
90	6.5765	6.7957	7.0520	7.3107	7.5902	7.8823	8.2220	8.6338	9.0475
95	6.6750	6.8911	7.1601	7.4356	7.7041	8.0119	8.3637	8.7758	9.2327
98	6.7844	7.0226	7.2647	7.5628	7.8192	8.1492	8.5273	8.9497	9.4292

TABLE 5. (Concluded)

Probability	0.111	0.092	0.073	0.053	0.033	0.014
			i			
Percentile	91	93	95	97	99	101
2	7.9329	8.3301	8.8038	9.3655	10.2155	11.5337
5	8.1359	8.5389	9.0412	9.6756	10.4997	12.1101
10	8.3022	8.7090	9.2551	9.9231	10.8147	12.6454
50	8.9077	9.4101	10.0375	10.9013	12.1630	14.9807
90	9.5837	10.1899	11.0004	12.0587	13.7662	18.4422
95	9.7979	10.4497	11.2754	12.3882	14.1558	19.6405
98	10.0162	10.6897	11.5586	12.7606	14.7997	21.0941

TABLE 6. SELECTED PERCENTILE VALUES OF SELECTED STATISTICS FOR TESTING NORMALITY

From the expected line of best fit these are:

1. The maximum absolute difference.
2. The mean sum of squares of residuals.
3. The number of runs above and below the line.

From the computed Chi-square values in each sample, these are:

4. The runs above and below the median X^2 value.

$\nu = 1$ n = 11

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0648	0.0241	1	1
0.02	0.0693	0.0266	1	1
0.05	0.0834	0.0347	1	2
0.10	0.0936	0.0450	2	2
0.20	0.1104	0.0598	2	2
0.30	0.1263	0.0750	2	3
0.40	0.1413	0.0968	2	3
0.50	0.1544	0.1252	3	3
0.60	0.1694	0.1576	3	3
0.70	0.1868	0.2222	3	3
0.80	0.2037	0.3426	3	4
0.90	0.2302	0.5619	4	4
0.95	0.2567	0.8082	4	4
0.98	0.2731	0.9987	5	4
0.99	0.2890	1.2486	5	4

TABLE 6. (Continued)

$\nu = 1 \qquad n = 21$

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0519	0.0150	2	3
0.02	0.0605	0.0191	2	3
0.05	0.0736	0.0250	3	4
0.10	0.0788	0.0348	3	4
0.20	0.0953	0.0482	4	5
0.30	0.1054	0.0648	4	5
0.40	0.1140	0.0823	5	5
0.50	0.1233	0.1014	5	5
0.60	0.1333	0.1243	5	6
0.70	0.1444	0.1469	6	6
0.80	0.1620	0.2062	6	6
0.90	0.1839	0.3846	7	7
0.95	0.2087	0.6208	7	7
0.98	0.2350	0.8860	7	8
0.99	0.2485	1.0038	8	8

TABLE 6. (Continued)

$\nu = 1$ $\quad$ $n = 31$

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0487	0.0153	3	5
0.02	0.0539	0.0196	3	5
0.05	0.0624	0.0251	4	6
0.10	0.0703	0.0342	4	6
0.20	0.0809	0.0464	5	7
0.30	0.0899	0.0594	6	7
0.40	0.1004	0.0777	6	8
0.50	0.1080	0.0941	7	8
0.60	0.1168	0.1119	7	8
0.70	0.1263	0.1403	8	9
0.80	0.1429	0.1973	8	9
0.90	0.1660	0.3297	9	10
0.95	0.1828	0.4999	10	10
0.98	0.1980	0.7333	10	11
0.99	0.2138	0.9393	10	11

TABLE 6. (Continued)

$\nu = 1$ n = 51

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0448	0.0133	5	8
0.02	0.0481	0.0154	6	9
0.05	0.0533	0.0210	7	10
0.10	0.0598	0.0292	8	11
0.20	0.0676	0.0388	9	12
0.30	0.0747	0.0495	10	12
0.40	0.0821	0.0610	11	13
0.50	0.0900	0.0746	11	13
0.60	0.0951	0.0903	12	13
0.70	0.1074	0.1122	13	14
0.80	0.1193	0.1423	14	15
0.90	0.1357	0.1991	15	15
0.95	0.1485	0.2582	15	16
0.98	0.1639	0.4267	16	17
0.99	0.1796	0.5136	16	17

TABLE 6. (Continued)

$\nu = 1$ n = 101

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0320	0.0078	8	19
0.02	0.0355	0.0124	10	20
0.05	0.0407	0.0156	12	21
0.10	0.0460	0.0216	15	22
0.20	0.0512	0.0312	18	23
0.30	0.0551	0.0400	20	24
0.40	0.0599	0.0469	21	25
0.50	0.0648	0.0601	23	26
0.60	0.0708	0.0719	24	26
0.70	0.0778	0.0893	25	27
0.80	0.0860	0.1140	26	28
0.90	0.0947	0.1521	27	29
0.95	0.1050	0.2042	28	30
0.98	0.1186	0.2689	29	30
0.99	0.1229	0.3314	30	31

TABLE 6. (Continued)

$\nu = 2$ $n = 11$

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0606	0.0298	1	1
0.02	0.0686	0.0448	1	1
0.05	0.0806	0.0589	2	2
0.10	0.0897	0.0744	2	2
0.20	0.1026	0.0999	2	2
0.30	0.1120	0.1215	2	3
0.40	0.1230	0.1462	3	3
0.50	0.1353	0.1718	3	3
0.60	0.1490	0.2131	3	3
0.70	0.1646	0.2679	3	3
0.80	0.1793	0.3508	4	4
0.90	0.2068	0.5175	4	4
0.95	0.2260	0.6752	4	4
0.98	0.2545	0.9456	5	4
0.99	0.2700	1.1528	5	5

TABLE 6. (Continued)

$\nu = 2$ n = 21

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0532	0.0226	2	3
0.02	0.0568	0.0313	3	3
0.05	0.0669	0.0465	3	4
0.10	0.0766	0.0589	4	4
0.20	0.0876	0.0798	4	5
0.30	0.0960	0.1094	4	5
0.40	0.1062	0.1368	5	5
0.50	0.1141	0.1636	5	6
0.60	0.1232	0.1941	5	6
0.70	0.1367	0.2433	6	6
0.80	0.1527	0.3140	6	6
0.90	0.1761	0.4711	7	7
0.95	0.1976	0.7116	7	7
0.98	0.2166	1.0265	7	8
0.99	0.2317	1.1209	8	8

TABLE 6. (Continued)

$\nu = 2$ n = 31

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0485	0.0280	3	5
0.02	0.0539	0.0308	4	5
0.05	0.0625	0.0435	5	6
0.10	0.0692	0.0554	6	6
0.20	0.0796	0.0764	6	7
0.30	0.0871	0.0986	7	7
0.40	0.0958	0.1184	7	8
0.50	0.1033	0.1409	7	8
0.60	0.1127	0.1672	8	8
0.70	0.1245	0.1945	8	9
0.80	0.1359	0.2544	9	9
0.90	0.1524	0.3727	10	10
0.95	0.1676	0.5813	10	10
0.98	0.1860	0.9884	10	11
0.99	0.1902	1.0850	11	11

TABLE 6. (Continued)

$\nu = 2$ n = 51

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0447	0.0176	6	9
0.02	0.0478	0.0301	7	10
0.05	0.0525	0.0363	8	10
0.10	0.0573	0.0455	9	11
0.20	0.0662	0.0599	10	12
0.30	0.0717	0.0773	11	12
0.40	0.0774	0.0924	11	13
0.50	0.0848	0.1161	12	13
0.60	0.0901	0.1367	12	13
0.70	0.0976	0.1693	13	14
0.80	0.1089	0.2074	14	15
0.90	0.1249	0.3509	15	15
0.95	0.1395	0.4701	15	16
0.98	0.1587	0.6287	16	17
0.99	0.1700	0.7902	16	17

TABLE 6. (Continued)

$\nu = 2$ n = 101

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0324	0.0185	11	19
0.02	0.0339	0.0230	13	20
0.05	0.0391	0.0273	15	21
0.10	0.0435	0.0344	18	22
0.20	0.0493	0.0452	20	23
0.30	0.0525	0.0554	21	24
0.40	0.0575	0.0682	23	25
0.50	0.0623	0.0809	24	26
0.60	0.0669	0.0977	24	26
0.70	0.0726	0.1171	25	27
0.80	0.0803	0.1456	26	28
0.90	0.0924	0.2065	28	29
0.95	0.0995	0.2865	28	30
0.98	0.1067	0.3833	29	30
0.99	0.1151	0.4421	30	31

TABLE 6. (Continued)

$\nu = 3$ n = 11

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0576	0.0392	1	1
0.02	0.0667	0.0493	2	1
0.05	0.0773	0.0725	2	2
0.10	0.0877	0.0932	2	2
0.20	0.1000	0.1261	2	2
0.30	0.1108	0.1561	3	3
0.40	0.1189	0.1804	3	3
0.50	0.1272	0.2156	3	3
0.60	0.1374	0.2602	3	3
0.70	0.1504	0.3185	4	3
0.80	0.1641	0.3876	4	4
0.90	0.1874	0.5340	4	4
0.95	0.2107	0.6700	4	4
0.98	0.2292	0.9200	5	5
0.99	0.2444	1.0202	5	5

TABLE 6. (Continued)

$\nu = 3 \qquad n = 21$

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0526	0.0406	2	3
0.02	0.0590	0.0508	3	3
0.05	0.0668	0.0684	3	4
0.10	0.0709	0.0833	4	4
0.20	0:0827	0.1158	4	5
0.30	0.0926	0.1373	5	5
0.40	0.1035	0.1696	5	5
0.50	0.1113	0.2097	5	6
0.60	0.1186	0.2612	6	6
0.70	0.1284	0.3438	6	6
0.80	0.1439	0.4411	6	6
0.90	0.1637	0.6489	7	7
0.95	0.1812	0.7800	7	7
0.98	0.2004	1.1000	7	8
0.99	0.2187	1.2000	8	8

TABLE 6. (Continued)

$\nu = 3$ $n = 31$

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0482	0.0323	4	5
0.02	0.0530	0.0499	5	5
0.05	0.0594	0.0661	5	6
0.10	0.0657	0.0832	6	6
0.20	0.0759	0.1039	6	7
0.30	0.0839	0.1311	7	7
0.40	0.0925	0.1534	7	8
0.50	0.0983	0.1894	8	8
0.60	0.1057	0.2344	8	8
0.70	0.1148	0.2920	9	9
0.80	0.1255	0.3674	9	9
0.90	0.1445	0.5384	10	10
0.95	0.1569	0.7285	10	10
0.98	0.1784	1.0554	10	11
0.99	0.1836	1.1781	11	11

TABLE 6. (Continued)

$\nu = 3$ n = 51

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0446	0.0283	7	9
0.02	0.0466	0.0365	8	9
0.05	0.0512	0.0514	9	10
0.10	0.0572	0.0637	10	11
0.20	0.0648	0.0821	10	12
0.30	0.0703	0.1023	11	12
0.40	0.0758	0.1266	12	13
0.50	0.0827	0.1477	13	13
0.60	0.0882	0.1764	13	13
0.70	0.0941	0.2118	14	14
0.80	0.1049	0.2733	14	15
0.90	0.1172	0.3887	15	15
0.95	0.1316	0.5992	16	16
0.98	0.1517	0.7507	16	17
0.99	0.1618	0.8411	17	17

TABLE 6. (Continued)

$\nu = 3$ n = 101

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0329	0.0276	12	19
0.02	0.0339	0.0306	15	20
0.05	0.0378	0.0365	18	21
0.10	0.0438	0.0440	19	22
0.20	0.0488	0.0561	21	23
0.30	0.0530	0.0723	22	24
0.40	0.0573	0.0867	23	25
0.50	0.0618	0.1021	24	26
0.60	0.0666	0.1216	24	26
0.70	0.0717	0.1580	25	27
0.80	0.0790	0.2048	26	28
0.90	0.0893	0.3041	28	29
0.95	0.0963	0.3876	28	30
0.98	0.1052	0.6191	29	30
0.99	0.1104	0.7258	30	31

TABLE 6. (Continued)

$\nu = 4 \qquad n = 11$

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0520	0.0485	1	1
0.02	0.0648	0.0602	2	1
0.05	0.0741	0.0810	2	2
0.10	0.0787	0.1087	2	2
0.20	0.0905	0.1497	2	2
0.30	0.1029	0.1783	3	3
0.40	0.1122	0.2250	3	3
0.50	0.1231	0.2684	3	3
0.60	0.1325	0.3208	3	3
0.70	0.1446	0.3761	4	3
0.80	0.1608	0.4500	4	4
0.90	0.1809	0.5964	4	4
0.95	0.1989	0.7376	4	4
0.98	0.2208	0.9293	5	5
0.99	0.2303	1.0078	5	5

TABLE 6. (Continued)

$\nu = 4$ n = 21

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0488	0.0444	3	3
0.02	0.0561	0.0708	3	3
0.05	0.0651	0.0953	4	4
0.10	0.0746	0.1164	4	4
0.20	0.0842	0.1540	5	5
0.30	0.0939	0.1930	5	5
0.40	0.1012	0.2256	5	5
0.50	0.1106	0.2798	5	6
0.60	0.1186	0.3278	6	6
0.70	0.1297	0.3994	6	6
0.80	0.1408	0.4948	7	6
0.90	0.1596	0.6788	7	7
0.95	0.1757	0.8309	7	7
0.98	0.1982	1.2311	8	8
0.99	0.2108	1.3964	8	8

TABLE 6. (Continued)

$\nu = 4$ n = 31

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0480	0.0550	4	5
0.02	0.0518	0.0614	5	5
0.05	0.0584	0.0797	5	6
0.10	0.0654	0.1074	6	6
0.20	0.0750	0.1355	6	7
0.30	0.0829	0.1669	7	7
0.40	0.0894	0.1987	7	8
0.50	0.0970	0.2346	8	8
0.60	0.1044	0.2817	8	8
0.70	0.1120	0.3554	9	9
0.80	0.1227	0.4456	9	9
0.90	0.1416	0.6983	10	10
0.95	0.1568	0.9180	10	10
0.98	0.1654	1.1949	10	11
0.99	0.1762	1.3856	11	11

TABLE 6. (Continued)

$\nu = 4$ n = 51

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0418	0.0378	7	9
0.02	0.0453	0.0466	8	9
0.05	0.0494	0.0645	9	10
0.10	0.0544	0.0786	10	11
0.20	0.0627	0.1018	11	12
0.30	0.0692	0.1251	11	12
0.40	0.0748	0.1515	12	13
0.50	0.0791	0.1790	13	13
0.60	0.0858	0.2187	13	13
0.70	0.0941	0.2648	14	14
0.80	0.1028	0.3522	14	15
0.90	0.1164	0.5517	15	15
0.95	0.1289	0.7519	16	16
0.98	0.1500	1.0198	16	17
0.99	0.1581	1.0927	17	17

TABLE 6. (Continued)

$\nu = 4$ n = 101

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0330	0.0298	14	19
0.02	0.0342	0.0365	16	20
0.05	0.0390	0.0430	18	21
0.10	0.0422	0.0529	20	22
0.20	0.0476	0.0694	22	23
0.30	0.0525	0.0874	23	24
0.40	0.0576	0.1038	24	25
0.50	0.0618	0.1276	24	26
0.60	0.0667	0.1512	25	26
0.70	0.0719	0.1840	26	27
0.80	0.0774	0.2328	27	27
0.90	0.0852	0.3529	28	28
0.95	0.0924	0.4650	29	29
0.98	0.1022	0.6746	30	30
0.99	0.1061	0.7690	30	31

TABLE 6. (Continued)

$\nu = 5$ n = 11

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0500	0.0685	1	1
0.02	0.0582	0.0873	2	1
0.05	0.0703	0.1204	2	2
0.10	0.0780	0.1434	2	2
0.20	0.0898	0.1941	2	2
0.30	0.1003	0.2382	3	3
0.40	0.1090	0.2791	3	3
0.50	0.1184	0.3166	3	3
0.60	0.1298	0.3794	3	3
0.70	0.1379	0.4497	4	3
0.80	0.1504	0.5678	4	4
0.90	0.1711	0.6882	4	4
0.95	0.1876	0.8638	5	4
0.98	0.2060	1.0428	5	5
0.99	0.2114	1.1367	5	5

TABLE 6. (Continued)

$\nu = 5$ n = 21

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0523	0.0714	3	3
0.02	0.0564	0.0904	3	3
0.05	0.0647	0.1068	4	4
0.10	0.0701	0.1408	4	4
0.20	0.0799	0.1813	5	5
0.30	0.0873	0.2121	5	5
0.40	0.0946	0.2500	5	5
0.50	0.1031	0.2880	5	6
0.60	0.1120	0.3537	6	6
0.70	0.1219	0.4238	6	6
0.80	0.1364	0.5676	7	6
0.90	0.1514	0.7500	7	7
0.95	0.1671	0.9515	7	7
0.98	0.1788	1.2848	8	8
0.99	0.1846	1.4530	8	8

TABLE 6. (Continued)

$\nu = 5$ n = 31

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0474	0.0569	4	5
0.02	0.0516	0.0775	5	5
0.05	0.0580	0.0971	5	6
0.10	0.0653	0.1262	6	6
0.20	0.0725	0.1554	7	7
0.30	0.0793	0.1936	7	7
0.40	0.0854	0.2325	8	8
0.50	0.0930	0.2721	8	8
0.60	0.0996	0.3397	8	8
0.70	0.1075	0.3982	9	9
0.80	0.1183	0.5254	9	9
0.90	0.1326	0.7851	10	10
0.95	0.1451	1.0283	10	10
0.98	0.1604	1.5767	11	11
0.99	0.1708	1.8290	11	11

TABLE 6. (Continued)

$\nu = 5 \qquad n = 51$

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0390	0.0537	7	9
0.02	0.0451	0.0635	8	9
0.05	0.0497	0.0758	9	10
0.10	0.0545	0.1002	10	11
0.20	0.0619	0.1294	11	12
0.30	0.0674	0.1617	12	12
0.40	0.0735	0.1903	12	13
0.50	0.0798	0.2253	13	13
0.60	0.0859	0.2734	13	13
0.70	0.0923	0.3401	14	14
0.80	0.1021	0.4719	14	15
0.90	0.1131	0.6605	15	15
0.95	0.1227	0.8283	16	16
0.98	0.1363	1.1094	16	17
0.99	0.1429	1.3017	17	17

TABLE 6. (Concluded)

$\nu = 5$ n = 101

Prob.	MAD	MSSR	Runs Above or Below Line	Runs Above or Below Median
0.01	0.0322	0.0375	16	19
0.02	0.0342	0.0434	17	20
0.05	0.0385	0.0543	19	21
0.10	0.0414	0.0669	20	22
0.20	0.0472	0.0878	22	23
0.30	0.0529	0.1106	23	24
0.40	0.0566	0.1325	24	25
0.50	0.0609	0.1541	24	26
0.60	0.0654	0.1832	25	26
0.70	0.0708	0.2312	26	27
0.80	0.0760	0.2989	27	27
0.90	0.0860	0.3960	29	28
0.95	0.0914	0.5609	29	29
0.98	0.1011	0.6627	30	30
0.99	0.1047	0.7989	30	31

TABLE 7. MAXIMUM ABSOLUTE DIFFERENCE ARRAYED BY DIMENSIONS

n = 11

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0648	0.0606	0.0576	0.0520	0.0500
0.02	0.0693	0.0686	0.0667	0.0648	0.0582
0.05	0.0834	0.0806	0.0773	0.0741	0.0703
0.10	0.0936	0.0897	0.0877	0.0787	0.0780
0.20	0.1104	0.1026	0.1000	0.0905	0.0895
0.30	0.1263	0.1120	0.1108	0.1029	0.1003
0.40	0.1413	0.1230	0.1189	0.1122	0.1090
0.50	0.1544	0.1353	0.1272	0.1231	0.1184
0.60	0.1694	0.1490	0.1374	0.1325	0.1298
0.70	0.1868	0.1646	0.1504	0.1446	0.1379
0.80	0.2037	0.1793	0.1641	0.1608	0.1594
0.90	0.2302	0.2068	0.1874	0.1809	0.1711
0.95	0.2567	0.2260	0.2107	0.1989	0.1876
0.98	0.2731	0.2545	0.2292	0.2208	0.2060
0.99	0.2890	0.2700	0.2444	0.2303	0.2114

TABLE 7. (Continued)

n = 21

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0519	0.0532	0.0526	0.0488	0.0523
0.02	0.0605	0.0568	0.0590	0.0561	0.0564
0.05	0.0736	0.0669	0.0668	0.0651	0.0647
0.10	0.0788	0.0766	0.0709	0.0746	0.0701
0.20	0.0953	0.0876	0.0827	0.0842	0.0799
0.30	0.1054	0.0960	0.0926	0.0939	0.0873
0.40	0.1140	0.1062	0.1035	0.1012	0.0946
0.50	0.1233	0.1141	0.1113	0.1106	0.1031
0.60	0.1333	0.1232	0.1186	0.1186	0.1120
0.70	0.1444	0.1367	0.1284	0.1297	0.1219
0.80	0.1620	0.1527	0.1439	0.1408	0.1364
0.90	0.1839	0.1761	0.1637	0.1596	0.1514
0.95	0.2087	0.1976	0.1812	0.1757	0.1671
0.98	0.2350	0.2166	0.2004	0.1982	0.1788
0.99	0.2485	0.2317	0.2187	0.2108	0.1846

TABLE 7. (Continued)

n = 31

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0487	0.0485	0.0482	0.0480	0.0474
0.02	0.0539	0.0539	0.0530	0.0518	0.0516
0.05	0.0624	0.0625	0.0594	0.0584	0.0580
0.10	0.0703	0.0692	0.0657	0.0654	0.0653
0.20	0.0809	0.0796	0.0759	0.0750	0.0725
0.30	0.0899	0.0871	0.0839	0.0829	0.0793
0.40	0.1004	0.0958	0.0925	0.0894	0.0854
0.50	0.1080	0.1033	0.0983	0.0970	0.0930
0.60	0.1168	0.1127	0.1057	0.1044	0.0996
0.70	0.1263	0.1245	0.1148	0.1120	0.1075
0.80	0.1429	0.1359	0.1255	0.1227	0.1163
0.90	0.1660	0.1524	0.1445	0.1416	0.1326
0.95	0.1828	0.1676	0.1569	0.1568	0.1451
0.98	0.1980	0.1860	0.1784	0.1654	0.1604
0.99	0.2138	0.1902	0.1836	0.1762	0.1703

TABLE 7. (Continued)

n = 51

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0448	0.0447	0.0446	0.0418	0.0390
0.02	0.0481	0.0478	0.0466	0.0453	0.0451
0.05	0.0533	0.0525	0.0512	0.0494	0.0497
0.10	0.0598	0.0573	0.0572	0.0544	0.0545
0.20	0.0676	0.0662	0.0648	0.0627	0.0619
0.30	0.0747	0.0717	0.0703	0.0692	0.0674
0.40	0.0821	0.0774	0.0758	0.0748	0.0735
0.50	0.0900	0.0848	0.0827	0.0791	0.0798
0.60	0.0951	0.0901	0.0882	0.0858	0.0859
0.70	0.1074	0.0976	0.0941	0.0941	0.0923
0.80	0.1193	0.1089	0.1049	0.1028	0.1021
0.90	0.1357	0.1249	0.1172	0.1164	0.1131
0.95	0.1485	0.1395	0.1316	0.1289	0.1227
0.98	0.1639	0.1587	0.1517	0.1500	0.1363
0.99	0.1796	0.1700	0.1618	0.1581	0.1429

TABLE 7. (Concluded)

n = 101

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0320	0.0324	0.0329	0.0330	0.0322
0.02	0.0355	0.0339	0.0339	0.0342	0.0342
0.05	0.0407	0.0391	0.0378	0.0390	0.0385
0.10	0.0460	0.0435	0.0438	0.0422	0.0414
0.20	0.0512	0.0493	0.0488	0.0476	0.0472
0.30	0.0551	0.0525	0.0530	0.0525	0.0529
0.40	0.0599	0.0575	0.0573	0.0576	0.0566
0.50	0.0648	0.0623	0.0618	0.0618	0.0609
0.60	0.0708	0.0669	0.0666	0.0667	0.0654
0.70	0.0778	0.0726	0.0717	0.0719	0.0708
0.80	0.0860	0.0803	0.0790	0.0774	0.0760
0.90	0.0947	0.0924	0.0893	0.0852	0.0860
0.95	0.1050	0.0995	0.0963	0.0924	0.0914
0.98	0.1186	0.1067	0.1052	0.1022	0.1011
0.99	0.1229	0.1151	0.1104	0.1061	0.1047

TABLE 8. MAXIMUM ABSOLUTE DIFFERENCE ARRAYED BY SAMPLE SIZE

$\nu = 1$

	n				
Prob.	11	21	31	51	101
0.01	0.0648	0.0519	0.0487	0.0448	0.0320
0.02	0.0693	0.0605	0.0539	0.0481	0.0355
0.05	0.0834	0.0736	0.0624	0.0533	0.0407
0.10	0.0936	0.0788	0.0703	0.0598	0.0460
0.20	0.1104	0.0953	0.0809	0.0676	0.0512
0.30	0.1263	0.1054	0.0899	0.0747	0.0551
0.40	0.1413	0.1140	0.1004	0.0821	0.0599
0.50	0.1544	0.1233	0.1080	0.0900	0.0648
0.60	0.1694	0.1333	0.1168	0.0951	0.0708
0.70	0.1868	0.1444	0.1263	0.1074	0.0778
0.80	0.2037	0.1620	0.1429	0.1193	0.0860
0.90	0.2302	0.1839	0.1660	0.1357	0.0947
0.95	0.2567	0.2087	0.1828	0.1485	0.1050
0.98	0.2731	0.2350	0.1980	0.1639	0.1186
0.99	0.2890	0.2485	0.2138	0.1796	0.1229

TABLE 8. (Continued)

$\nu = 2$

n

Prob.	11	21	31	51	101
0.01	0.0606	0.0532	0.0485	0.0447	0.0324
0.02	0.0686	0.0568	0.0539	0.0478	0.0339
0.05	0.0806	0.0669	0.0625	0.0525	0.0391
0.10	0.0897	0.0766	0.0692	0.0573	0.0435
0.20	0.1026	0.0876	0.0796	0.0662	0.0493
0.30	0.1120	0.0960	0.0871	0.0717	0.0525
0.40	0.1230	0.1062	0.0958	0.0774	0.0575
0.50	0.1353	0.1141	0.1033	0.0848	0.0623
0.60	0.1490	0.1232	0.1127	0.0901	0.0669
0.70	0.1646	0.1367	0.1245	0.0976	0.0726
0.80	0.1793	0.1527	0.1359	0.1089	0.0803
0.90	0.2068	0.1761	0.1524	0.1249	0.0924
0.95	0.2260	0.1976	0.1676	0.1395	0.0995
0.98	0.2545	0.2166	0.1860	0.1587	0.1067
0.99	0.2700	0.2317	0.1902	0.1700	0.1151

TABLE 8. (Continued)

$\nu = 3$

n

Prob.	11	21	31	51	101
0.01	0.0576	0.0526	0.0482	0.0446	0.0329
0.02	0.0667	0.0590	0.0530	0.0466	0.0339
0.05	0.0773	0.0668	0.0594	0.0512	0.0378
0.10	0.0877	0.0709	0.0657	0.0572	0.0438
0.20	0.1000	0.0827	0.0759	0.0648	0.0488
0.30	0.1108	0.0926	0.0839	0.0703	0.0530
0.40	0.1189	0.1035	0.0925	0.0758	0.0573
0.50	0.1272	0.1113	0.0983	0.0827	0.0618
0.60	0.1374	0.1186	0.1057	0.0882	0.0666
0.70	0.1504	0.1284	0.1148	0.0941	0.0717
0.80	0.1641	0.1439	0.1255	0.1049	0.0790
0.90	0.1874	0.1637	0.1445	0.1172	0.0893
0.95	0.2107	0.1812	0.1569	0.1316	0.0963
0.98	0.2292	0.2004	0.1784	0.1517	0.1052
0.99	0.2444	0.2187	0.1836	0.1618	0.1104

TABLE 8. (Continued)

$\nu = 4$

n

Prob.	11	21	31	51	101
0.01	0.0520	0.0488	0.0480	0.0418	0.0330
0.02	0.0648	0.0561	0.0518	0.0453	0.0342
0.05	0.0741	0.0651	0.0584	0.0494	0.0390
0.10	0.0787	0.0746	0.0654	0.0544	0.0422
0.20	0.0905	0.0842	0.0750	0.0627	0.0476
0.30	0.1029	0.0939	0.0829	0.0692	0.0525
0.40	0.1122	0.1012	0.0894	0.0748	0.0576
0.50	0.1231	0.1106	0.0970	0.0791	0.0618
0.60	0.1325	0.1186	0.1044	0.0858	0.0667
0.70	0.1446	0.1297	0.1120	0.0941	0.0719
0.80	0.1608	0.1408	0.1227	0.1028	0.0774
0.90	0.1809	0.1596	0.1416	0.1164	0.0852
0.95	0.1989	0.1757	0.1568	0.1289	0.0924
0.98	0.2208	0.1982	0.1654	0.1500	0.1022
0.99	0.2303	0.2108	0.1762	0.1581	0.1061

TABLE 8. (Concluded)

$\nu = 5$

n

Prob.	11	21	31	51	101
0.01	0.0500	0.0523	0.0474	0.0390	0.0322
0.02	0.0582	0.0564	0.0516	0.0451	0.0342
0.05	0.0703	0.0647	0.0580	0.0497	0.0385
0.10	0.0780	0.0701	0.0653	0.0545	0.0414
0.20	0.0898	0.0799	0.0725	0.0619	0.0472
0.30	0.1003	0.0873	0.0793	0.0674	0.0529
0.40	0.1090	0.0946	0.0854	0.0735	0.0566
0.50	0.1184	0.1031	0.0930	0.0798	0.0609
0.60	0.1298	0.1120	0.0996	0.0859	0.0654
0.70	0.1379	0.1219	0.1075	0.0923	0.0708
0.80	0.1504	0.1364	0.1183	0.1021	0.0760
0.90	0.1711	0.1514	0.1326	0.1131	0.0860
0.95	0.1876	0.1671	0.1451	0.1227	0.0914
0.98	0.2060	0.1788	0.1604	0.1363	0.1011
0.99	0.2114	0.1846	0.1708	0.1429	0.1047

TABLE 9. MEAN SUM SQUARES RESIDUALS ARRAYED BY DIMENSIONS

n = 11

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0241	0.0298	0.0392	0.0485	0.0685
0.02	0.0266	0.0448	0.0493	0.0602	0.0873
0.05	0.0347	0.0589	0.0725	0.0810	0.1204
0.10	0.0450	0.0744	0.0932	0.1087	0.1434
0.20	0.0598	0.0999	0.1261	0.1497	0.1941
0.30	0.0750	0.1215	0.1561	0.1783	0.2382
0.40	0.0968	0.1462	0.1804	0.2250	0.2791
0.50	0.1252	0.1718	0.2156	0.2684	0.3166
0.60	0.1576	0.2131	0.2602	0.3208	0.3794
0.70	0.2222	0.2679	0.3185	0.3761	0.4497
0.80	0.3426	0.3508	0.3876	0.4500	0.5678
0.90	0.5619	0.5175	0.5340	0.5964	0.6882
0.95	0.8082	0.6752	0.6700	0.7376	0.8638
0.98	0.9987	0.9456	0.9200	0.9293	1.0428
0.99	1.2486	1.1528	1.0202	1.0078	1.1367

TABLE 9. (Continued)

n = 21

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0150	0.0226	0.0406	0.0444	0.0714
0.02	0.0191	0.0313	0.0508	0.0708	0.0904
0.05	0.0250	0.0465	0.0684	0.0953	0.1068
0.10	0.0348	0.0589	0.0833	0.1164	0.1408
0.20	0.0482	0.0798	0.1158	0.1540	0.1813
0.30	0.0648	0.1094	0.1373	0.1930	0.2121
0.40	0.0823	0.1368	0.1696	0.2256	0.2500
0.50	0.1014	0.1636	0.2097	0.2798	0.2880
0.60	0.1243	0.1941	0.2612	0.3278	0.3537
0.70	0.1469	0.2433	0.3438	0.3994	0.4238
0.80	0.2062	0.3140	0.4411	0.4948	0.5676
0.90	0.3846	0.4711	0.6489	0.6788	0.7500
0.95	0.6208	0.7116	0.7800	0.8309	0.9515
0.98	0.8860	1.0265	1.1000	1.2311	1.2848
0.99	1.0038	1.1209	1.2000	1.3964	1.4530

TABLE 9. (Continued)

n = 31

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0153	0.0280	0.0323	0.0550	0.0569
0.02	0.0196	0.0308	0.0499	0.0614	0.0775
0.05	0.0251	0.0435	0.0661	0.0797	0.0971
0.10	0.0342	0.0554	0.0832	0.1074	0.1262
0.20	0.0464	0.0764	0.1039	0.1355	0.1554
0.30	0.0594	0.0986	0.1311	0.1669	0.1936
0.40	0.0777	0.1184	0.1534	0.1987	0.2325
0.50	0.0941	0.1409	0.1894	0.2346	0.2721
0.60	0.1119	0.1672	0.2344	0.2817	0.3397
0.70	0.1403	0.1945	0.2920	0.3554	0.3982
0.80	0.1973	0.2544	0.3674	0.4456	0.5254
0.90	0.3297	0.3727	0.5384	0.6983	0.7851
0.95	0.4999	0.5813	0.7285	0.9180	1.0283
0.98	0.7333	0.9884	1.0554	1.1949	1.5767
0.99	0.9393	1.0850	1.1781	1.3856	1.8290

TABLE 9. (Continued)

n = 51

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0133	0.0176	0.0283	0.0378	0.0537
0.02	0.0154	0.0301	0.0365	0.0466	0.0635
0.05	0.0210	0.0363	0.0514	0.0645	0.0758
0.10	0.0292	0.0455	0.0637	0.0786	0.1002
0.20	0.0388	0.0599	0.0821	0.1018	0.1294
0.30	0.0495	0.0773	0.1023	0.1251	0.1617
0.40	0.0610	0.0924	0.1266	0.1515	0.1903
0.50	0.0746	0.1161	0.1477	0.1790	0.2253
0.60	0.0903	0.1367	0.1764	0.2187	0.2734
0.70	0.1122	0.1693	0.2118	0.2648	0.3401
0.80	0.1423	0.2074	0.2733	0.3522	0.4719
0.90	0.1991	0.3509	0.3887	0.5517	0.6605
0.95	0.2582	0.4701	0.5992	0.7519	0.8283
0.98	0.4267	0.6287	0.7507	1.0198	1.1094
0.99	0.5136	0.7902	0.8411	1.0927	1.3017

TABLE 9. (Concluded)

n = 101

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	0.0078	0.0185	0.0276	0.0298	0.0375
0.02	0.0124	0.0230	0.0306	0.0365	0.0434
0.05	0.0156	0.0273	0.0365	0.0430	0.0543
0.10	0.0216	0.0344	0.0440	0.0529	0.0669
0.20	0.0312	0.0452	0.0561	0.0694	0.0878
0.30	0.0400	0.0554	0.0723	0.0874	0.1106
0.40	0.0469	0.0682	0.0867	0.1038	0.1325
0.50	0.0601	0.0809	0.1021	0.1276	0.1541
0.60	0.0719	0.0977	0.1216	0.1512	0.1832
0.70	0.0893	0.1171	0.1580	0.1840	0.2312
0.80	0.1140	0.1456	0.2048	0.2328	0.2989
0.90	0.1521	0.2065	0.3041	0.3529	0.3960
0.95	0.2042	0.2865	0.3876	0.4650	0.5609
0.98	0.2689	0.3833	0.6191	0.6746	0.6627
0.99	0.3314	0.4421	0.7258	0.7690	0.7989

TABLE 10. MEAN SUM SQUARES RESIDUALS ARRAYED BY SAMPLE SIZE

$\nu = 1$

n

Prob.	11	21	31	51	101
0.01	0.0241	0.0150	0.0153	0.0133	0.0078
0.02	0.0266	0.0191	0.0196	0.0154	0.0124
0.05	0.0347	0.0250	0.0251	0.0210	0.0156
0.10	0.0450	0.0348	0.0342	0.0292	0.0216
0.20	0.0598	0.0482	0.0464	0.0388	0.0312
0.30	0.0750	0.0648	0.0594	0.0495	0.0400
0.40	0.0968	0.0823	0.0777	0.0610	0.0469
0.50	0.1252	0.1014	0.0941	0.0746	0.0601
0.60	0.1576	0.1243	0.1119	0.0903	0.0719
0.70	0.2222	0.1469	0.1403	0.1122	0.0893
0.80	0.3426	0.2062	0.1973	0.1423	0.1140
0.90	0.5619	0.3846	0.3297	0.1991	0.1521
0.95	0.8082	0.6208	0.4999	0.2582	0.2042
0.98	0.9987	0.8860	0.7333	0.4267	0.2689
0.99	1.2486	1.0038	0.9393	0.5136	0.3314

TABLE 10. (Continued)

$\nu = 2$

n

Prob.	11	21	31	51	101
0.01	0.0298	0.0226	0.0280	0.0176	0.0185
0.02	0.0448	0.0313	0.0308	0.0301	0.0230
0.05	0.0589	0.0465	0.0435	0.0363	0.0273
0.10	0.0744	0.0589	0.0554	0.0455	0.0344
0.20	0.0999	0.0798	0.0764	0.0599	0.0452
0.30	0.1215	0.1094	0.0986	0.0773	0.0554
0.40	0.1462	0.1368	0.1184	0.0924	0.0682
0.50	0.1718	0.1636	0.1409	0.1161	0.0809
0.60	0.2131	0.1941	0.1672	0.1367	0.0977
0.70	0.2679	0.2433	0.1945	0.1693	0.1171
0.80	0.3508	0.3140	0.2544	0.2074	0.1456
0.90	0.5175	0.4711	0.3727	0.3509	0.2065
0.95	0.6752	0.7116	0.5813	0.4701	0.2865
0.98	0.9456	1.0265	0.9884	0.6287	0.3833
0.99	1.1528	1.1209	1.0850	0.7902	0.4421

TABLE 10. (Continued)

$\nu = 3$

n

Prob.	11	21	31	51	101
0.01	0.0392	0.0406	0.0323	0.0283	0.0276
0.02	0.0493	0.0508	0.0499	0.0365	0.0306
0.05	0.0725	0.0684	0.0661	0.0514	0.0365
0.10	0.0932	0.0833	0.0832	0.0637	0.0440
0.20	0.1261	0.1158	0.1039	0.0821	0.0561
0.30	0.1561	0.1373	0.1311	0.1023	0.0723
0.40	0.1804	0.1696	0.1534	0.1266	0.0867
0.50	0.2156	0.2097	0.1894	0.1477	0.1021
0.60	0.2602	0.2612	0.2344	0.1764	0.1216
0.70	0.3185	0.3438	0.2920	0.2118	0.1580
0.80	0.3876	0.4411	0.3674	0.2733	0.2048
0.90	0.5340	0.6489	0.5384	0.3887	0.3041
0.95	0.6700	0.7800	0.7285	0.5992	0.3876
0.98	0.9200	1.1000	1.0554	0.7507	0.6191
0.99	1.0202	1.2000	1.1781	0.8411	0.7258

TABLE 10. (Continued)

$\nu = 4$

n

Prob.	11	21	31	51	101
0.01	0.0485	0.0444	0.0550	0.0378	0.0298
0.02	0.0602	0.0708	0.0614	0.0466	0.0365
0.05	0.0810	0.0953	0.0797	0.0645	0.0430
0.10	0.1087	0.1164	0.1074	0.0786	0.0529
0.20	0.1497	0.1540	0.1355	0.1018	0.0694
0.30	0.1783	0.1930	0.1669	0.1251	0.0874
0.40	0.2250	0.2256	0.1987	0.1515	0.1038
0.50	0.2684	0.2798	0.2346	0.1790	0.1276
0.60	0.3208	0.3278	0.2817	0.2187	0.1512
0.70	0.3761	0.3994	0.3554	0.2648	0.1840
0.80	0.4500	0.4948	0.4456	0.3522	0.2328
0.90	0.5964	0.6788	0.6983	0.5517	0.3529
0.95	0.7376	0.8309	0.9180	0.7519	0.4650
0.98	0.9293	1.2311	1.1949	1.0198	0.6746
0.99	1.0078	1.3964	1.3856	1.0927	0.7690

TABLE 10. (Concluded)

$\nu = 5$

n

Prob.	11	21	31	51	101
0.01	0.0685	0.0714	0.0569	0.0537	0.0375
0.02	0.0873	0.0904	0.0775	0.0635	0.0434
0.05	0.1204	0.1068	0.0971	0.0758	0.0543
0.10	0.1434	0.1408	0.1262	0.1002	0.0669
0.20	0.1941	0.1813	0.1554	0.1294	0.0878
0.30	0.2382	0.2121	0.1936	0.1617	0.1106
0.40	0.2791	0.2500	0.2325	0.1903	0.1325
0.50	0.3166	0.2880	0.2721	0.2253	0.1541
0.60	0.3794	0.3537	0.3397	0.2734	0.1832
0.70	0.4497	0.4238	0.3982	0.3401	0.2312
0.80	0.5678	0.5676	0.5254	0.4719	0.2989
0.90	0.6882	0.7500	0.7851	0.6605	0.3960
0.95	0.8638	0.9515	1.0283	0.8283	0.5609
0.98	1.0428	1.2848	1.5767	1.1094	0.6627
0.99	1.1367	1.4530	1.8290	1.3017	0.7989

TABLE 11. RUNS ABOVE OR BELOW LINE OF CHI-SQUARE VERSUS PROBABILITY (1-p) ARRAYED BY DIMENSIONS

n = 11

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	1	1	1	1	1
0.02	1	1	2	2	2
0.05	1	2	2	2	2
0.10	2	2	2	2	2
0.20	2	2	2	2	2
0.30	2	2	3	3	3
0.40	2	3	3	3	3
0.50	3	3	3	3	3
0.60	3	3	3	3	3
0.70	3	3	4	4	4
0.80	3	4	4	4	4
0.90	4	4	4	4	4
0.95	4	4	4	4	5
0.98	5	5	5	5	5
0.99	5	5	5	5	5

n = 21

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	2	2	2	3	3
0.02	2	3	3	3	3
0.05	3	3	3	4	4
0.10	3	4	4	4	4
0.20	4	4	4	5	5
0.30	4	4	5	5	5
0.40	5	5	5	5	5
0.50	5	5	5	5	5
0.60	5	5	6	6	6
0.70	6	6	6	6	6
0.80	6	6	6	7	7
0.90	7	7	7	7	7
0.95	7	7	7	7	7
0.98	7	7	7	8	8
0.99	8	8	8	8	8

TABLE 11. (Continued)

n = 31

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	3	3	4	4	4
0.02	3	4	5	5	5
0.05	4	5	5	5	5
0.10	4	6	6	6	6
0.20	5	6	6	6	7
0.30	6	7	7	7	7
0.40	6	7	7	7	8
0.50	7	7	8	8	8
0.60	7	8	8	8	8
0.70	8	8	9	9	9
0.80	8	9	9	9	9
0.90	9	10	10	10	10
0.95	10	10	10	10	10
0.98	10	10	10	10	11
0.99	10	11	11	11	11

n = 51

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	5	6	7	7	7
0.02	6	7	8	8	8
0.05	7	8	9	9	9
0.10	8	9	10	10	10
0.20	9	10	10	11	11
0.30	10	11	11	11	12
0.40	11	11	12	12	12
0.50	11	12	13	13	13
0.60	12	12	13	13	13
0.70	13	13	14	14	14
0.80	14	14	14	14	14
0.90	15	15	15	15	15
0.95	15	15	16	16	16
0.98	16	16	16	16	16
0.99	16	16	17	17	17

TABLE 11. (Concluded)

n = 101

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	8	11	12	14	16
0.02	10	13	15	16	17
0.05	12	15	18	18	19
0.10	15	18	19	20	20
0.20	18	20	21	22	22
0.30	20	21	22	23	23
0.40	21	23	23	24	24
0.50	23	24	24	24	24
0.60	24	24	24	25	25
0.70	25	25	25	26	26
0.80	26	26	26	27	27
0.90	27	28	28	28	29
0.95	28	28	28	29	29
0.98	29	29	29	30	30
0.99	30	30	30	30	30

TABLE 12. RUNS ABOVE OR BELOW LINE OF CHI-SQUARE VERSUS PROBABILITY (1-p) ARRAYED BY SAMPLE SIZE

	$\nu = 1$						$\nu = 2$				
	n						n				
Prob.	11	21	31	51	101	Prob.	11	21	31	51	101
0.01	1	2	3	5	8	0.01	1	2	3	6	11
0.02	1	2	3	6	10	0.02	1	3	4	7	13
0.05	1	3	4	7	12	0.05	2	3	5	8	15
0.10	2	3	4	8	15	0.10	2	4	6	9	18
0.20	2	4	5	9	18	0.20	2	4	6	10	20
0.30	2	4	6	10	20	0.30	2	4	7	11	21
0.40	2	5	6	11	21	0.40	3	5	7	11	23
0.50	3	5	7	11	23	0.50	3	5	7	12	24
0.60	3	5	7	12	24	0.60	3	5	8	12	24
0.70	3	6	8	13	25	0.70	3	6	8	13	25
0.80	3	6	8	14	26	0.80	4	6	9	14	26
0.90	4	7	9	15	27	0.90	4	7	10	15	28
0.95	4	7	10	15	28	0.95	4	7	10	15	28
0.98	5	7	10	16	29	0.98	5	7	10	16	29
0.99	5	8	10	16	30	0.99	5	8	11	16	30

TABLE 12. (Continued)

	$\nu = 3$						$\nu = 4$				
	n						n				
Prob.	11	21	31	51	101	Prob.	11	21	31	51	101
0.01	1	2	4	7	12	0.01	1	3	4	7	14
0.02	2	3	5	8	15	0.02	2	3	5	8	16
0.05	2	3	5	9	18	0.05	2	4	5	9	18
0.10	2	4	6	10	19	0.10	2	4	6	10	20
0.20	2	4	6	10	21	0.20	2	5	6	11	22
0.30	3	5	7	11	22	0.30	3	5	7	11	23
0.40	3	5	7	12	23	0.40	3	5	7	12	24
0.50	3	5	8	13	24	0.50	3	5	8	13	24
0.60	3	6	8	13	24	0.60	3	6	8	13	25
0.70	4	6	9	14	25	0.70	4	6	9	14	26
0.80	4	6	9	14	26	0.80	4	7	9	14	27
0.90	4	7	10	15	28	0.90	4	7	10	15	28
0.95	4	7	10	16	28	0.95	4	7	10	16	29
0.98	5	7	10	16	29	0.98	5	8	10	16	30
0.99	5	8	11	17	30	0.99	5	8	11	17	30

TABLE 12. (Concluded)

$\nu = 5$

n

Prob.	11	21	31	51	101
0.01	1	3	4	7	16
0.02	2	3	5	8	17
0.05	2	4	5	9	19
0.10	2	4	6	10	20
0.20	2	5	7	11	22
0.30	3	5	7	12	23
0.40	3	5	8	12	24
0.50	3	5	8	13	24
0.60	3	6	8	13	25
0.70	4	6	9	14	26
0.80	4	7	9	14	27
0.90	4	7	10	15	29
0.95	5	7	10	16	29
0.98	5	8	11	16	30
0.99	5	8	11	17	30

TABLE 13. RUNS ABOVE OR BELOW MEDIAN CHI-SQUARE ARRAYED BY DIMENSIONS

n = 11

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	1	1	1	1	1
0.02	1	1	1	1	1
0.05	2	2	2	2	2
0.10	2	2	2	2	2
0.20	2	2	2	2	2
0.30	3	3	3	3	3
0.40	3	3	3	3	3
0.50	3	3	3	3	3
0.60	3	3	3	3	3
0.70	3	3	3	3	3
0.80	4	4	4	4	4
0.90	4	4	4	4	4
0.95	4	4	4	4	4
0.98	4	4	5	5	5
0.99	4	5	[illegible]	5	5

n = 21

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	3	3	3	3	3
0.02	3	3	3	3	3
0.05	4	4	4	4	4
0.10	4	4	4	4	4
0.20	5	5	5	5	5
0.30	5	5	5	5	5
0.40	5	5	5	5	5
0.50	5	6	6	6	6
0.60	6	6	6	6	6
0.70	6	6	6	6	6
0.80	6	6	6	6	6
0.90	7	7	7	7	7
0.95	7	7	7	7	7
0.98	8	8	8	8	8
0.99	8	8	8	8	8

TABLE 13. (Continued)

n = 31						n = 51					
Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.	Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	5	5	5	5	5	0.01	8	9	9	9	9
0.02	5	5	5	5	5	0.02	9	10	9	9	9
0.05	6	6	6	6	6	0.05	10	10	10	10	10
0.10	6	6	6	6	6	0.10	11	11	11	11	11
0.20	7	7	7	7	7	0.20	12	12	12	12	12
0.30	7	7	7	7	7	0.30	12	12	12	12	12
0.40	8	8	8	8	8	0.40	13	13	13	13	13
0.50	8	8	8	8	8	0.50	13	13	13	13	13
0.60	8	8	8	8	8	0.60	13	13	13	13	13
0.70	9	9	9	9	9	0.70	14	14	14	14	14
0.80	9	9	9	9	9	0.80	15	15	15	15	15
0.90	10	10	10	10	10	0.90	15	15	15	15	15
0.95	10	10	10	10	10	0.95	16	16	16	16	16
0.98	11	11	11	11	11	0.98	17	17	17	17	17
0.99	11	11	11	11	11	0.99	17	17	17	17	17

TABLE 13. (Concluded)

n = 101

Prob.	1 Dim.	2 Dim.	3 Dim.	4 Dim.	5 Dim.
0.01	19	19	19	19	19
0.02	20	20	20	20	20
0.05	21	21	21	21	21
0.10	22	22	22	22	22
0.20	23	23	23	23	23
0.30	24	24	24	24	24
0.40	25	25	25	25	25
0.50	26	26	26	26	26
0.60	26	26	26	26	26
0.70	27	27	27	27	27
0.80	28	28	28	27	27
0.90	29	29	29	28	28
0.95	30	30	30	29	29
0.98	30	30	30	30	30
0.99	31	31	31	31	31

TABLE 14. RUNS ABOVE OR BELOW MEDIAN CHI-SQUARE ARRAYED BY SAMPLE SIZE

	$\nu = 1$						$\nu = 2$				
	n						n				
Prob.	11	21	31	51	101	Prob.	11	21	31	51	101
0.01	1	3	5	8	19	0.01	1	3	5	9	19
0.02	1	3	5	9	20	0.02	1	3	5	10	20
0.05	2	4	6	10	21	0.05	2	4	6	10	21
0.10	2	4	6	11	22	0.10	2	4	6	11	22
0.20	2	5	7	12	23	0.20	2	5	7	12	23
0.30	3	5	7	12	24	0.30	3	5	7	12	24
0.40	3	5	8	13	25	0.40	3	5	8	13	25
0.50	3	5	8	13	26	0.50	3	6	8	13	26
0.60	3	6	8	13	26	0.60	3	6	8	13	26
0.70	3	6	9	14	27	0.70	3	6	9	14	27
0.80	4	6	9	15	28	0.80	4	6	9	15	28
0.90	4	7	10	15	29	0.90	4	7	10	15	29
0.95	4	7	10	16	30	0.95	4	7	10	16	30
0.98	4	8	11	17	30	0.98	4	8	11	17	30
0.99	4	8	11	17	31	0.99	5	8	11	17	31

TABLE 14. (Continued)

$\nu = 3$

	n				
Prob.	11	21	31	51	101
0.01	1	3	5	9	19
0.02	1	3	5	9	20
0.05	2	4	6	10	21
0.10	2	4	6	11	22
0.20	2	5	7	12	23
0.30	3	5	7	12	24
0.40	3	5	8	13	25
0.50	3	6	8	13	26
0.60	3	6	8	13	26
0.70	3	6	9	14	27
0.80	4	6	9	15	28
0.90	4	7	10	15	29
0.95	4	7	10	16	30
0.98	5	8	11	17	30
0.99	5	8	11	17	31

$\nu = 4$

	n				
Prob.	11	21	31	51	101
0.01	1	3	5	9	19
0.02	1	3	5	9	20
0.05	2	4	6	10	21
0.10	2	4	6	11	22
0.20	2	5	7	12	23
0.30	3	5	7	12	24
0.40	3	5	8	13	25
0.50	3	6	8	13	26
0.60	3	6	8	13	26
0.70	3	6	9	14	27
0.80	4	6	9	15	27
0.90	4	7	10	15	28
0.95	4	7	10	16	29
0.98	5	8	11	17	30
0.99	5	8	11	17	31

TABLE 14. (Concluded)

$\nu = 5$

n

Prob.	11	21	31	51	101
0.01	1	3	5	9	19
0.02	1	3	5	9	20
0.05	2	4	6	10	21
0.10	2	4	6	11	22
0.20	2	5	7	12	23
0.30	3	5	7	12	24
0.40	3	5	8	13	25
0.50	3	6	8	13	26
0.60	3	6	8	13	26
0.70	3	6	9	14	27
0.80	4	6	9	15	27
0.90	4	7	10	15	28
0.95	4	7	10	16	29
0.98	5	8	11	17	30
0.99	5	8	11	17	31

TABLE 15. NUMBER OF n-tant RUNS TO BE EXPECTED FOR THE χ OR χ^2 VECTOR FROM MULTIVARIATE NORMAL (Wishart) DISTRIBUTIONS.
(The number of grand samples is 14. The number of samples in each grand sample is 100. The number in each sample is n. Mx is the maximum noted. Mn is the minimum noted. R is the length of runs. M is the mean. ν is dimension. Percentage of total possible is cumulative.)

$\nu = 2$

R	1	2	3	4	5
n	11				
Mx	8	9	10	10	10
M	7	8	9	9	9
Mn	6	7	7	7	7
n	15				
Mx	10	13	14	14	14
M	9	12	12	12	12
Mn	8	9	10	10	10
n	21				
Mx	13	18	19	20	20
M	12	15	16	16	16
Mn	11	13	13	13	13
n	25				
Mx	16	21	22	22	23
M	14	18	19	19	19
Mn	13	16	16	16	16
n	31				
Mx	21	27	29	30	31
M	18	22	23	23	23
Mn	16	19	19	19	19
n	51				
Mx	33	43	45	46	47
M	30	36	38	38	38
Mn	26	31	31	31	31
n	101				
Mx	60	80	86	88	89
M	56	71	74	75	75
Mn	46	48	49	49	49

TABLE 15. (Continued)

$\nu = 3$

R	1	2	3	4	5
n	11				
Mx	9	11	11	11	11
M	9	10	10	10	10
Mn	8	9	9	9	9
n	15				
Mx	12	14	15	15	15
M	12	13	13	13	13
Mn	11	11	11	11	11
n	21				
Mx	17	20	20	21	21
M	16	18	18	18	18
Mn	15	16	16	16	16
n	25				
Mx	20	23	24	24	24
M	19	21	22	22	22
Mn	19	20	20	20	20
n	31				
Mx	25	28	29	29	29
M	24	26	26	26	26
Mn	22	23	23	23	23
n	51				
Mx	41	47	48	49	49
M	39	43	44	44	44
Mn	34	39	39	39	39
n	101				
Mx	82	95	97	98	98
M	76	85	86	86	86
Mn	71	75	75	75	75

TABLE 15. (Continued)

$\nu = 4$

R	1	2	3	4	5
n	11				
Mx	10	11	11	11	11
M	10	10	10	10	10
Mn	9	9	9	9	9
n	15				
Mx	14	15	15	15	15
M	13	14	14	14	14
Mn	13	12	12	12	12
n	21				
Mx	19	20	21	21	21
M	18	19	19	19	19
Mn	17	18	18	18	18
n	25				
Mx	23	24	24	24	24
M	22	23	23	23	23
Mn	21	21	21	21	21
n	31				
Mx	27	29	30	30	30
M	27	28	28	28	28
Mn	26	27	27	27	27
n	51				
Mx	45	51	52	52	52
M	43	46	46	46	46
Mn	41	42	42	42	42
n	101				
Mx	88	97	99	99	99
M	85	91	91	91	91
Mn	82	85	85	85	85

TABLE 15. (Concluded)

$\nu = 5$

R	1	2	3	4	5
n	11				
Mx	10	11	11	11	11
M	10	10	10	10	10
Mn	10	10	10	10	10
n	15				
Mx	14	15	15	15	15
M	14	14	14	14	14
Mn	13	13	13	13	13
n	21				
Mx	19	20	20	20	20
M	19	19	20	20	20
Mn	19	19	19	19	19
n	25				
Mx	23	25	25	25	25
M	23	23	23	23	23
Mn	21	21	22	22	22
n	31				
Mx	28	31	31	31	31
M	28	29	29	29	29
Mn	26	27	27	27	27
n	51				
Mx	47	50	51	51	51
M	46	47	47	47	47
Mn	44	44	44	44	44
n	101				
Mx	93	99	99	99	99
M	91	93	93	93	93
Mn	87	88	88	88	88

TABLE 16. TESTS OF BIVARIATE NORMALITY OF 24 HOUR HURRICANE MOVEMENTS FOR SELECTED 5 DEGREE LATITUDE-LONGITUDE QUADRANGLES. (The null hypothesis is that the data set is not different from the bivariate normal. The probability rejection level is 0.04. This provides the central 0.96 probability confidence band. The latitude is the southern boundary while the longitude is the western boundary of the quadrangle.)

n	Prob. 0.02	25 N 70 W	25 N 75 W	20 N 70 W	Prob. 0.98
1	0.0038	0.1000	0.0500	0.0700	0.5964
2	0.0434	0.1400	0.1800	0.1300	0.8957
3	0.1357	0.3900	0.4800	0.2700	1.1878
4	0.2748	0.4300	1.0700	0.3300	1.4686
5	0.4453	0.9700	1.7100	0.4000*	1.7761
6	0.6673	1.9500	1.7300	0.7300	2.1073
7	0.9598	2.2400	2.0100	0.8800*	2.4964
8	1.3795	2.7800	2.6400	1.3600*	3.0618
9	1.8699	3.0200	2.7300	2.3700	3.8671
10	2.5097	3.0700	2.9800	3.5400	5.2501
11	3.1223	4.9200	4.4600	9.9600*	7.0900
MAD	0.0686	0.1420	0.1570	0.2364	0.2545
MSSR	0.0448	0.1170	0.1040	2.7642*	0.9456
RAL	1	2	1	1	5
RBL	1	2	2	1	5
RAM	1	3	4	1	4
RBM	1	3	4	2	4
X^2	0.7500	8.0000	14.0000*	10.0000	13.3900
(5 d.f.)	0.0200	0.1562	0.0156*	0.0752	0.9800
	MN				MX
QS	7	9	8	8	9

TABLE 16. (Continued)

n	Prob. 0.02	25 N 70 W	25 N 75 W	20 N 70 W	Prob. 0.98
1	0.0019	0.0700	0.0500	0.1100	0.3285
2	0.0213	0.1100	0.1000	0.2200	0.5026
3	0.0616	0.1600	0.1400	0.2300	0.6648
4	0.1123	0.1600	0.4200	0.2500	0.8128
5	0.1843	0.1900	0.5500	0.2700	0.9529
6	0.2543	0.3500	0.5900	0.2900	1.1048
7	0.3478	0.3500	0.8300	0.3500	1.2439
8	0.4476	0.7400	0.8400	0.5000	1.4068
9	0.5556	0.8100	0.9200	0.7200	1.5809
10	0.6865	0.9800	0.9600	0.7500	1.7734
11	0.8204	1.0800	0.9800	0.7800*	1.9724
12	0.9642	1.1300	1.0100	0.9200*	2.1938
13	1.1393	2.1700	1.3800	1.0300*	2.4309
14	1.3559	2.6600	1.8900	1.2400*	2.6905
15	1.5815	2.6900	2.0100	1.3700*	3.0152
16	1.8562	2.8000	2.2100	1.6300*	3.3822
17	2.1404	2.9400	2.5900	1.7200*	3.8829
18	2.5244	3.0900	3.4400	2.2200*	4.5287
19	2.9358	3.2200	5.2000	3.0700	5.4153
20	3.4594	6.3900	6.6500	4.6500	7.0227
21	4.1453	7.8700	7.2900	17.7100*	10.0008
MAD	0.0568	0.1582	0.1488	0.1951	0.2166
MSSR	0.0313	0.3696	0.3563	6.7677*	1.0265
RAL	3	3	3	2*	7
RBL	3	2*	3	1*	7
RAM	3	6	6	5	8
RBM	3	5	5	5	8
X^2	0.7800	13.5000*	4.5000	13.0000	13.3900
(5 d.f.)	0.0200	0.0191*	0.4799	0.0233	0.9800
QS	MN 13	15	15	11*	MX 18

TABLE 16. (Concluded)

n	Prob. 0.02	25 N 70 W	25 N 75 W	20 N 70 W	Prob. 0.98
1	0.0011	0.1100	0.0100	0.1200	0.2298
2	0.0135	0.1300	0.0300	0.2700	0.3465
3	0.0374	0.1400	0.1600	0.2800	0.4459
4	0.0735	0.1600	0.1900	0.2800	0.5492
5	0.1109	0.2000	0.3700	0.2900	0.6544
6	0.1562	0.2200	0.4000	0.3000	0.7426
7	0.2122	0.2400	0.4900	0.3100	0.8464
8	0.2616	0.3000	0.5600	0.3200	0.9462
9	0.3165	0.3500	0.7400	0.3400	1.0557
10	0.3873	0.4500	0.8600	0.4900	1.1684
11	0.4551	0.6700	0.8900	0.5600	1.2765
12	0.5268	0.6800	1.0400	0.7000	1.3798
13	0.6126	0.7900	1.0900	0.8000	1.4936
14	0.7055	0.9100	1.2300	0.8300	1.6093
15	0.7973	0.9900	1.3000	0.8600	1.7366
16	0.8884	1.1600	1.3200	0.9600	1.8732
17	0.9950	1.4200	1.3600	0.9600*	2.0201
18	1.1286	1.5900	1.5200	1.0300*	2.1723
19	1.2599	1.6400	1.6800	1.0800*	2.3486
20	1.3902	2.1500	1.7500	1.1500*	2.5411
21	1.5476	2.2900	1.7700	1.4600*	2.7154
22	1.7408	2.2900	2.2900	1.4900*	2.9443
23	1.8924	2.3100	2.3200	1.6100*	3.1920
24	2.0703	2.4200	2.4800	1.8900*	3.4906
25	2.2944	2.6600	2.6000	2.0400*	3.8412
26	2.5674	2.9100	3.7200	2.4500*	4.2423
27	2.8517	3.8600	3.7900	3.6200	4.7526
28	3.1373	4.5800	3.8900	3.9200	5.4034
29	3.5742	5.5900	5.5000	5.4100	6.3863
30	4.0364	7.8200	6.4900	5.4700	7.9955
31	4.6812	9.0400	8.1500	18.6700*	11.5191
MAD	0.0539	0.1210	0.0688	0.1876*	0.1860
MSSR	0.0308	0.3653	0.1140	4.6122*	0.9884
RAL	4	3*	4	3*	10
RBL	4	2*	4	2*	10
RAM	5	8	8	6	11
RBM	5	7	8	7	11
X^2	0.7500	9.3300	3.3300	15.3300*	13.3900
(5 d.f.)	0.0200	0.0965	0.6487	0.0090*	0.9800
	MN				MX
QS	19	23	22	22	27

TABLE 17. TESTS OF BIVARIATE NORMALITY OF 24 HOUR HURRICANE MOVEMENTS FOR SELECTED 5 DEGREE LATITUDE-LONGITUDE QUADRANGLES. (The null hypothesis is that the data set is not different from the bivariate normal. The probability rejection level is 0.04. This provides the central 0.96 probability confidence band. The latitude is the southern boundary while the longitude is the western boundary of the quadrangle.)

n	Prob. 0.02	25 N 80 W	30 N 75 W	20 N 75 W	Prob. 0.98
1	0.0038	0.2300	0.0500	0.2000	0.5964
2	0.0434	0.4800	0.2200	0.3000	0.8957
3	0.1357	0.6600	0.3400	0.4100	1.1878
4	0.2748	0.8700	0.6700	0.6900	1.4686
5	0.4453	1.5900	0.8400	0.8200	1.7761
6	0.6673	1.6300	1.3400	0.9000	2.1073
7	0.9598	1.8800	1.7300	0.9400*	2.4964
8	1.3795	2.1400	2.2600	1.0700*	3.0618
9	1.8699	3.4000	2.9800	1.9400	3.8671
10	2.5097	3.4500	3.7500	5.0500	5.2501
11	3.1223	3.6900	5.8100	7.6900*	7.0900
MAD	0.0686	0.1307	0.0968	0.2503	0.2545
MSSR	0.0448	0.1640	0.1262	1.2925*	0.9456
RAL	1	2	1	2	5
RBL	1	2	1	1	5
RAM	1	3	4	3	4
RBM	1	2	3	3	4
X^2	0.7500	10.0000	4.0000	12.0000	13.3900
(5 d.f.)	0.0200	0.0752	0.5444	0.0348	0.9800
	MN				MX
QS	7	11*	7	9	9

TABLE 17. (Continued)

n	Prob. 0.02	25 N 80 W	30 N 75 W	20 N 75 W	Prob. 0.98
1	0.0019	0.0700	0.1000	0.0900	0.3285
2	0.0213	0.0900	0.3000	0.1600	0.5026
3	0.0616	0.1800	0.3800	0.1900	0.6648
4	0.1123	0.4200	0.3800	0.3700	0.8128
5	0.1843	0.5500	0.6100	0.4300	0.9529
6	0.2543	0.5900	0.7500	0.4900	1.1048
7	0.3478	0.8000	0.7800	0.4900	1.2439
8	0.4476	1.0500	0.8500	0.5700	1.4068
9	0.5556	1.1200	0.9600	0.6300	1.5809
10	0.6865	1.1700	0.9700	0.7300	1.7734
11	0.8204	1.2500	0.9900	0.7300*	1.9724
12	0.9642	1.2500	0.9900	1.0600	2.1938
13	1.1393	1.4100	1.0000*	1.2000	2.4309
14	1.3559	1.8300	1.6800	1.3300*	2.6905
15	1.5815	2.2300	2.1300	1.3800*	3.0152
16	1.8562	2.2900	3.2400	1.8800	3.3822
17	2.1404	3.3800	3.2900	1.9700*	3.8829
18	2.5244	3.5200	3.7700	4.3200	4.5287
19	2.9358	3.9400	5.1800	5.4500*	5.4153
20	3.4594	5.1400	5.6600	7.2300*	7.0227
21	4.1453	7.6600	5.9900	9.3000	10.0008
MAD	0.0568	0.0848	0.1972	0.1942	0.2166
MSSR	0.0313	0.1464	0.2056	1.1197*	1.0265
RAL	3	5	3	2*	7
RBL	3	4	3	1*	7
RAM	3	5	7	4	8
RBM	3	4	6	4	8
X^2	0.7500	7.0000	11.5000	10.5000	13.3900
(5 d.f.)	0.0200	0.2206	0.0423	0.0622	0.9800
	MN				MX
QS	13	11*	12*	16	18

TABLE 17. (Concluded)

n	Prob. 0.02	25 N 80 W	30 N 75 W	20 N 75 W	Prob. 0.98
1	0.0011	0.0100	0.1100	0.1900	0.2298
2	0.0135	0.0700	0.1500	0.2400	0.3465
3	0.0374	0.0900	0.3900	0.3000	0.4459
4	0.0735	0.1600	0.4000	0.3800	0.5492
5	0.1109	0.2000	0.4500	0.5900	0.6544
6	0.1562	0.3800	0.5200	0.6500	0.7426
7	0.2122	0.4700	0.7200	0.6500	0.8464
8	0.2616	0.5200	0.7200	0.6900	0.9462
9	0.3165	0.5600	0.7700	0.7600	1.0557
10	0.3873	0.9500	0.8600	0.9500	1.1684
11	0.4551	1.1800	0.8700	0.9800	1.2765
12	0.5268	1.2500	0.9800	1.0000	1.3798
13	0.6126	1.2800	1.0700	1.0500	1.4936
14	0.7055	1.3300	1.1500	1.0600	1.6093
15	0.7973	1.4800	1.1500	1.0900	1.7366
16	0.8884	1.4900	1.2100	1.2800	1.8732
17	0.9950	1.5100	1.2300	1.3500	2.0201
18	1.1286	1.6200	1.2400	1.4400	2.1723
19	1.2599	1.6800	1.4700	1.4900	2.3486
20	1.3902	1.8200	1.4900	1.5700	2.5411
21	1.5476	1.9600	1.5700	1.9100	2.7154
22	1.7408	2.2600	2.0000	2.1300	2.9443
23	1.8924	2.3100	2.0100	2.3100	3.1920
24	2.0703	3.3200	2.6300	2.5400	3.4906
25	2.2944	3.5900	2.6900	2.7100	3.8412
26	2.5674	3.6600	3.8000	2.7800	4.2423
27	2.8517	3.6700	3.9700	3.0300	4.7526
28	3.1373	4.0000	4.2400	4.8900	5.4034
29	3.5742	4.1300	6.1400	5.1500	6.3863
30	4.0364	5.1800	6.9500	6.4400	7.9955
31	4.6812	7.9400	7.1100	8.4100	11.5191
MAD	0.0539	0.1017	0.1122	0.0988	0.1860
MSSR	0.0308	0.0827	0.1940	0.1666	0.9884
RAL	4	3*	2*	2*	10
RBL	4	3*	1*	1*	10
RAM	5	8	8	7	11
RBM	5	7	8	8	11
X^2	0.7500	10.0000	10.0000	6.6700	13.3900
(5 d.f.)	0.0200	0.0752	0.0752	0.2466	0.9800
	MN				MX
QS	19	24	19	22	27

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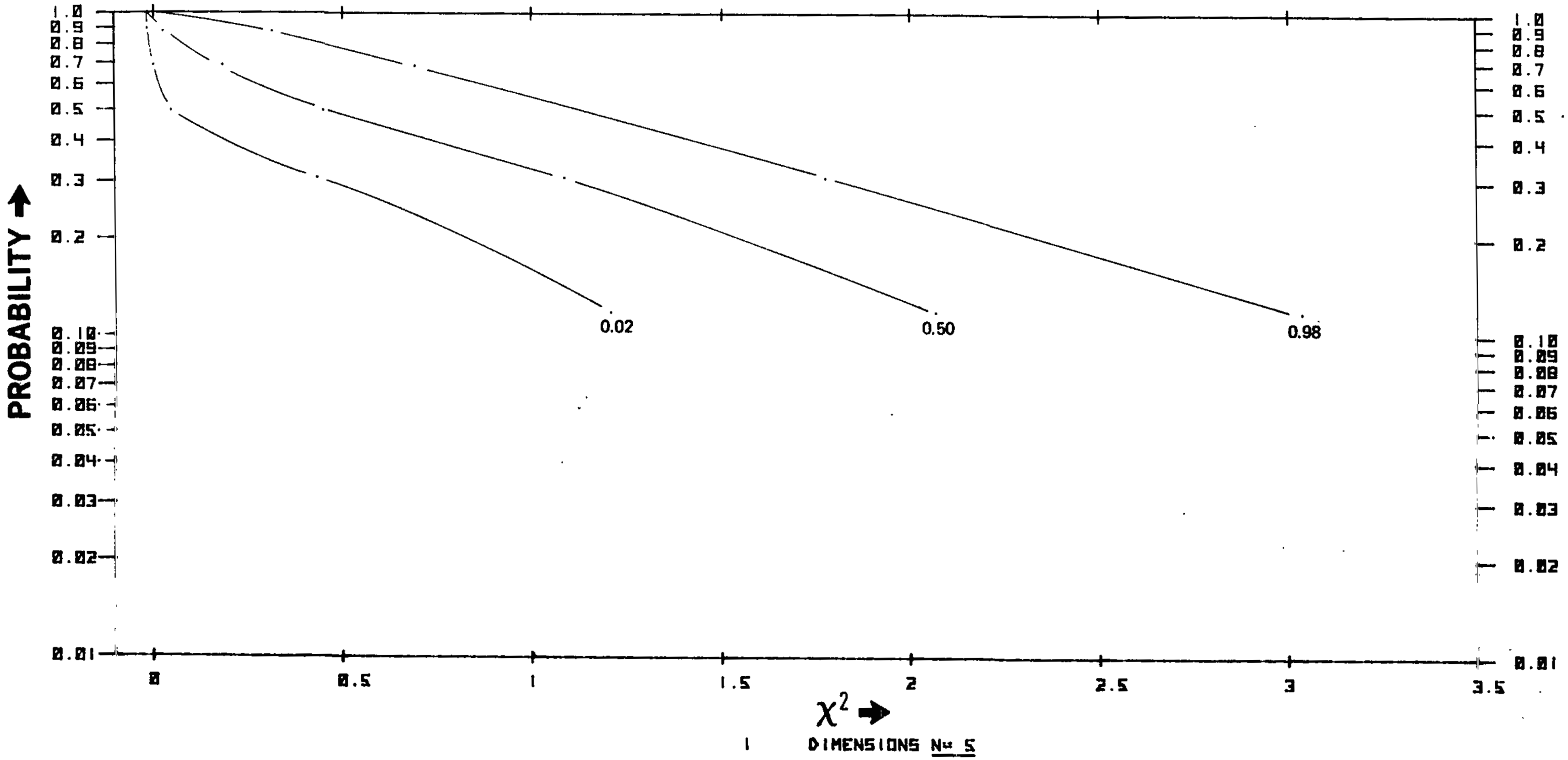
APPENDIX

PLOTTING DIAGRAMS WITH CONFIDENCE BAND (CENTRAL 0.96 PROBABILITY) FOR MULTIVARIATE NORMAL DISTRIBUTIONS FOR VARIOUS DIMENSION AND SAMPLE SIZE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0.8
0.7
0.6
0.5
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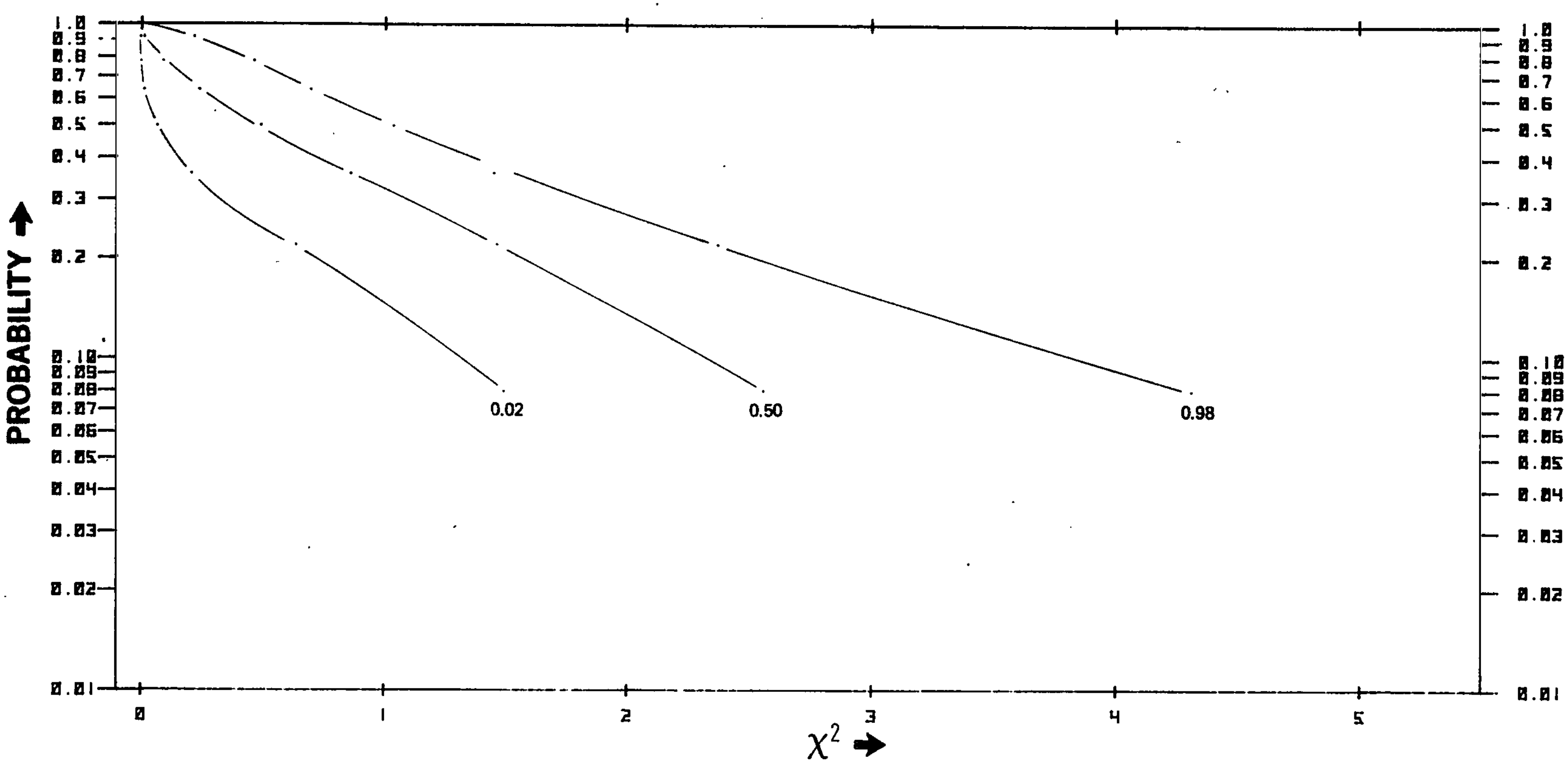
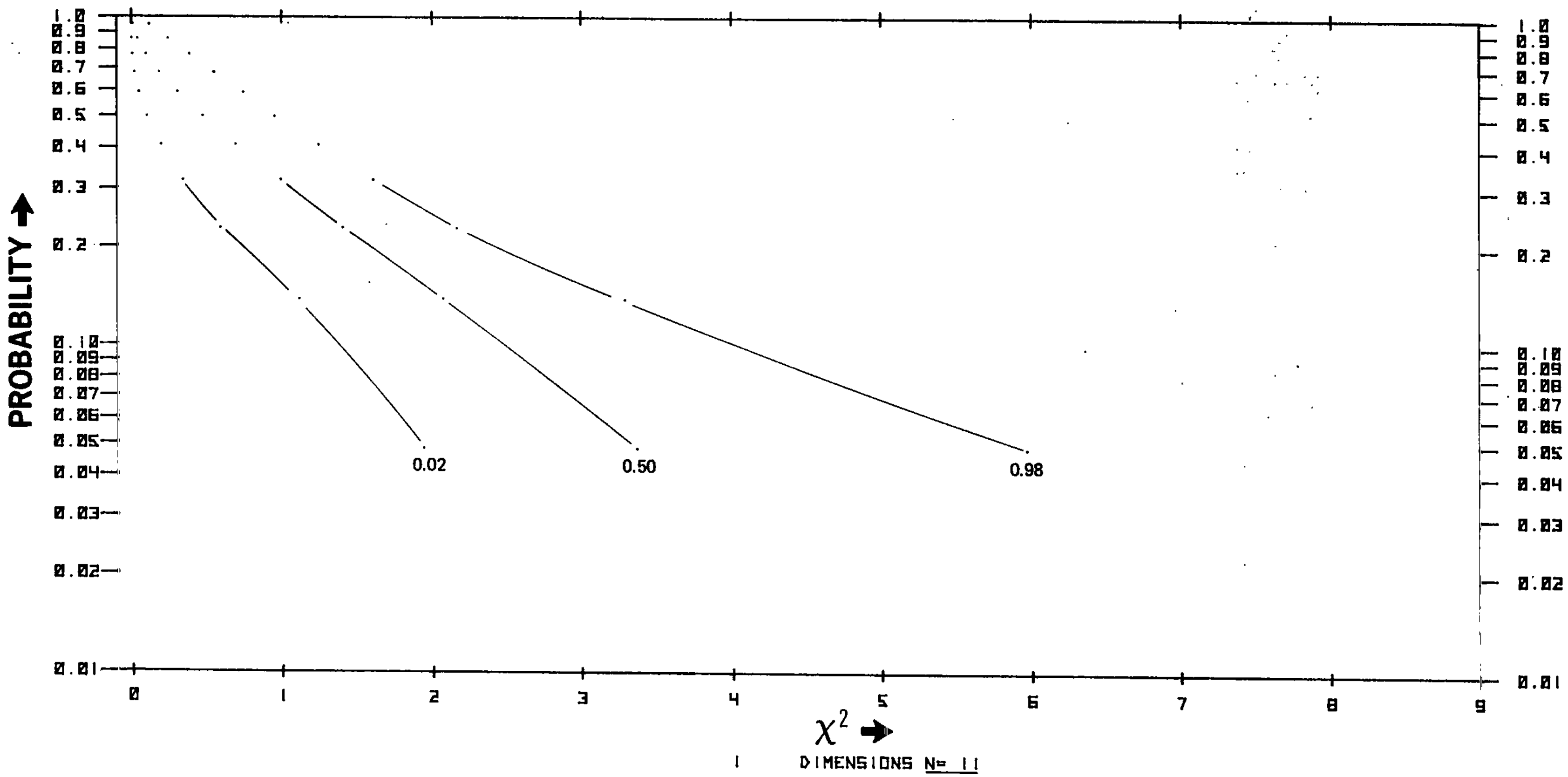
0.3
0.2
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.02
0.50
0.98
0
0.5
1
1.5
2
2.5
3
3.5
χ²
DIMENSIONS N= 5



I DIMENSIONS N= 7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χ²
DIMENSIONS N= 11
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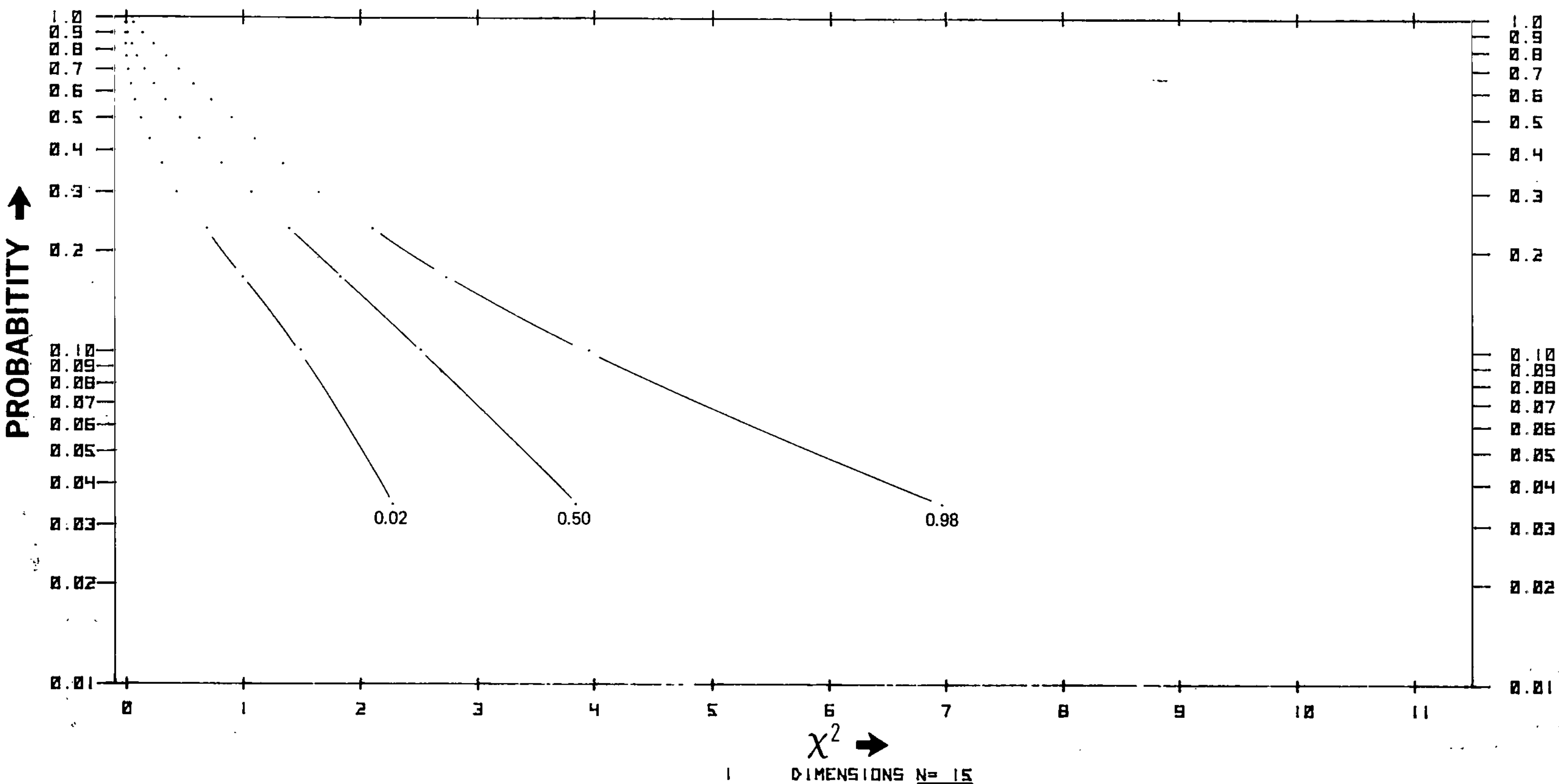
0.50
0.98


PROBABITITY
χ^2
0.02
0.50
0.98
1.0
0.9
0.8
0.7
0.6
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0.01
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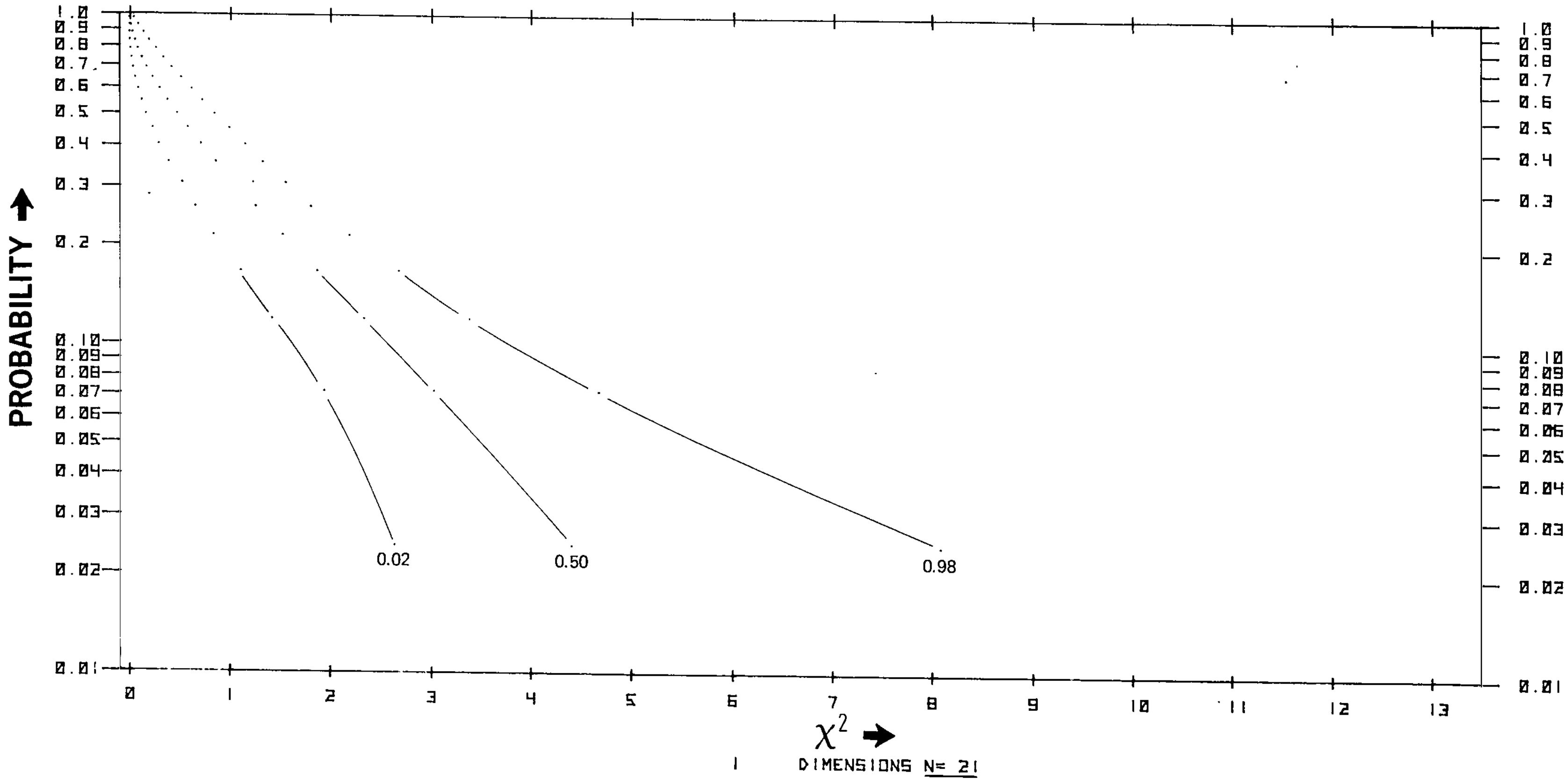
7
8
9
10
11
DIMENSIONS N= 15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0.03
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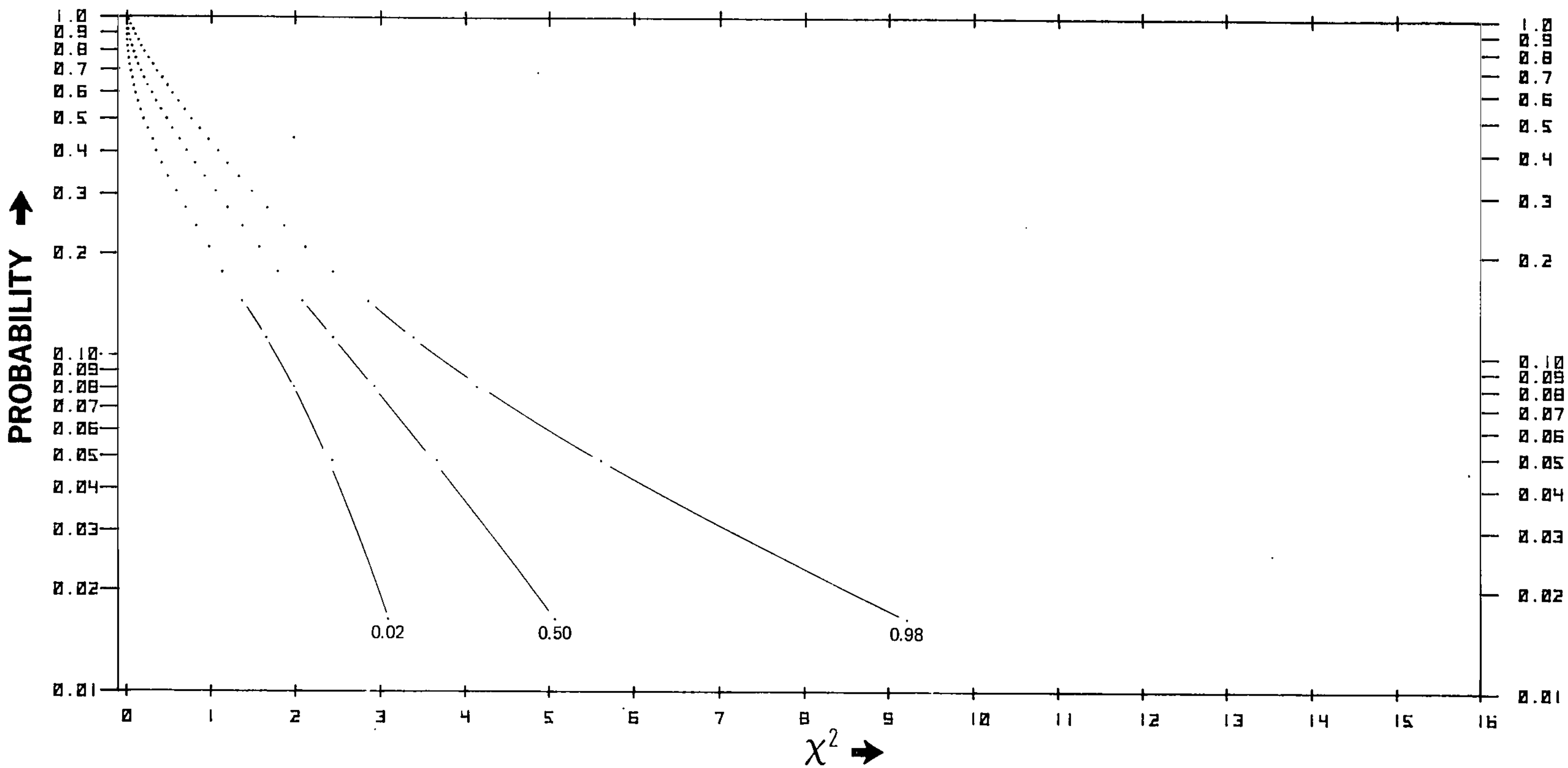
0.01
0.02
0.50
0.98
0 1 2 3 4 5 6 7 8 9 10 11 12 13
χ²
1 DIMENSIONS N= 21



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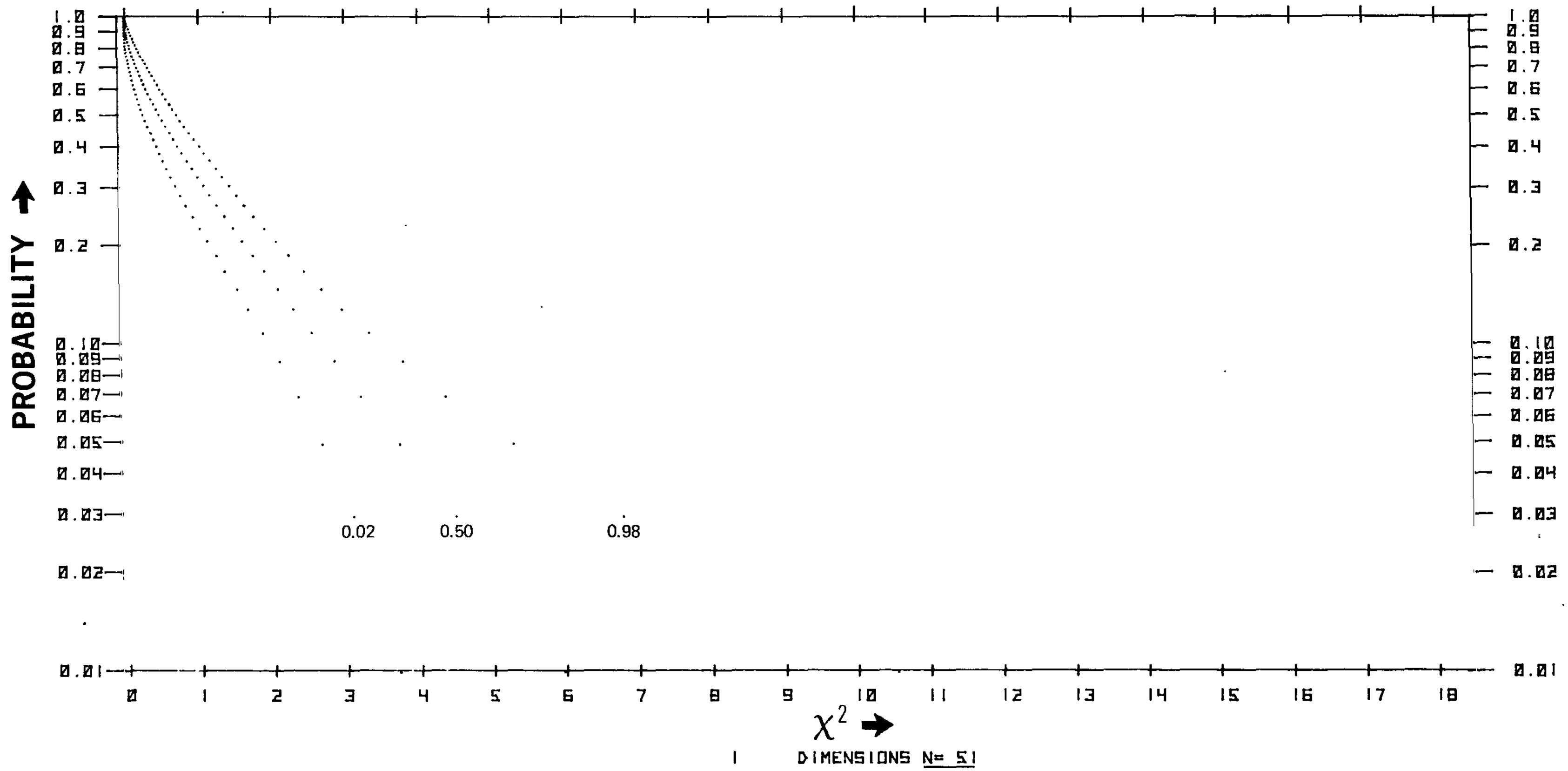
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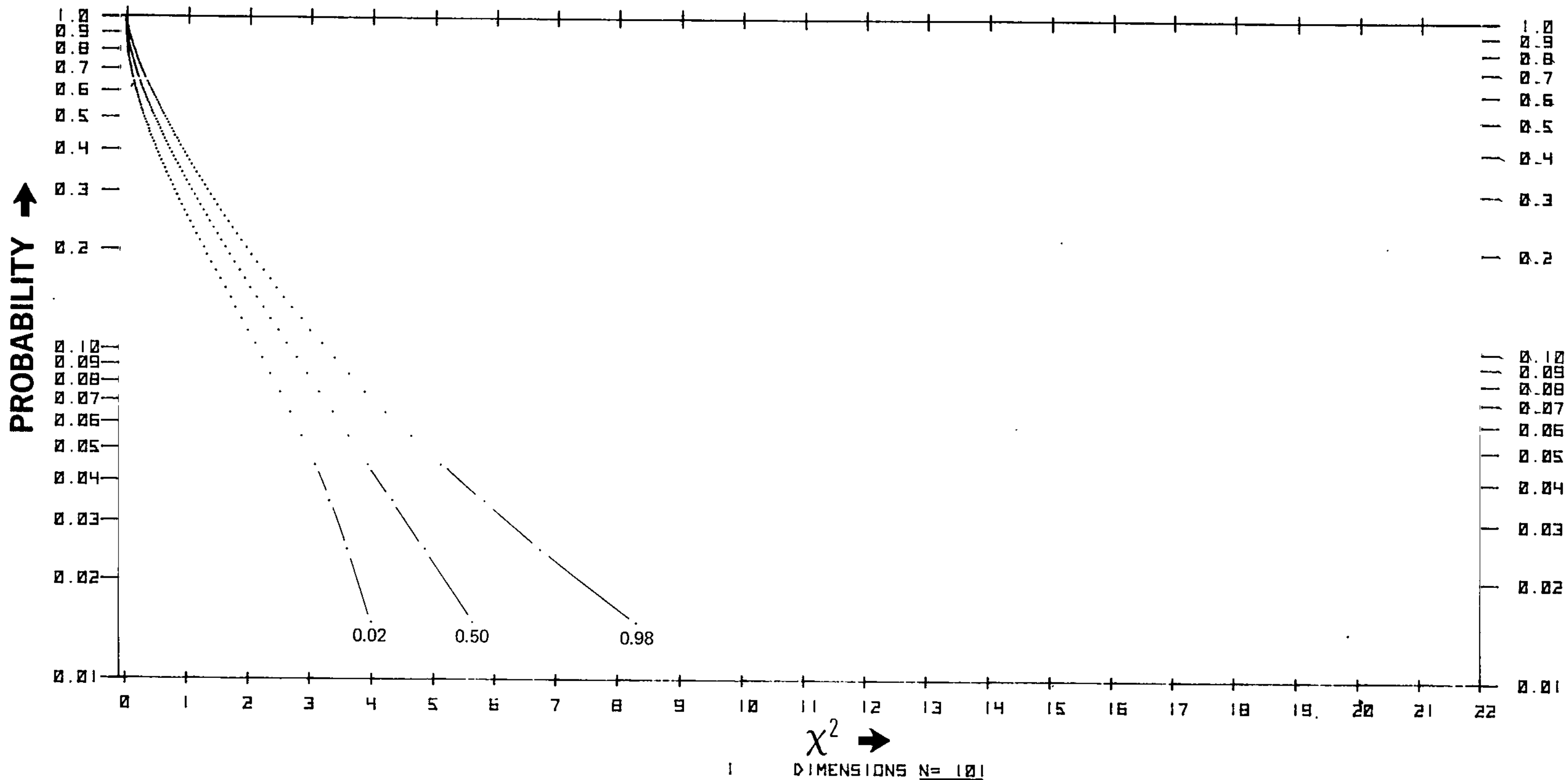
0.50
0.98
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
χ²
1 DIMENSIONS N= 51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0.02
0.01
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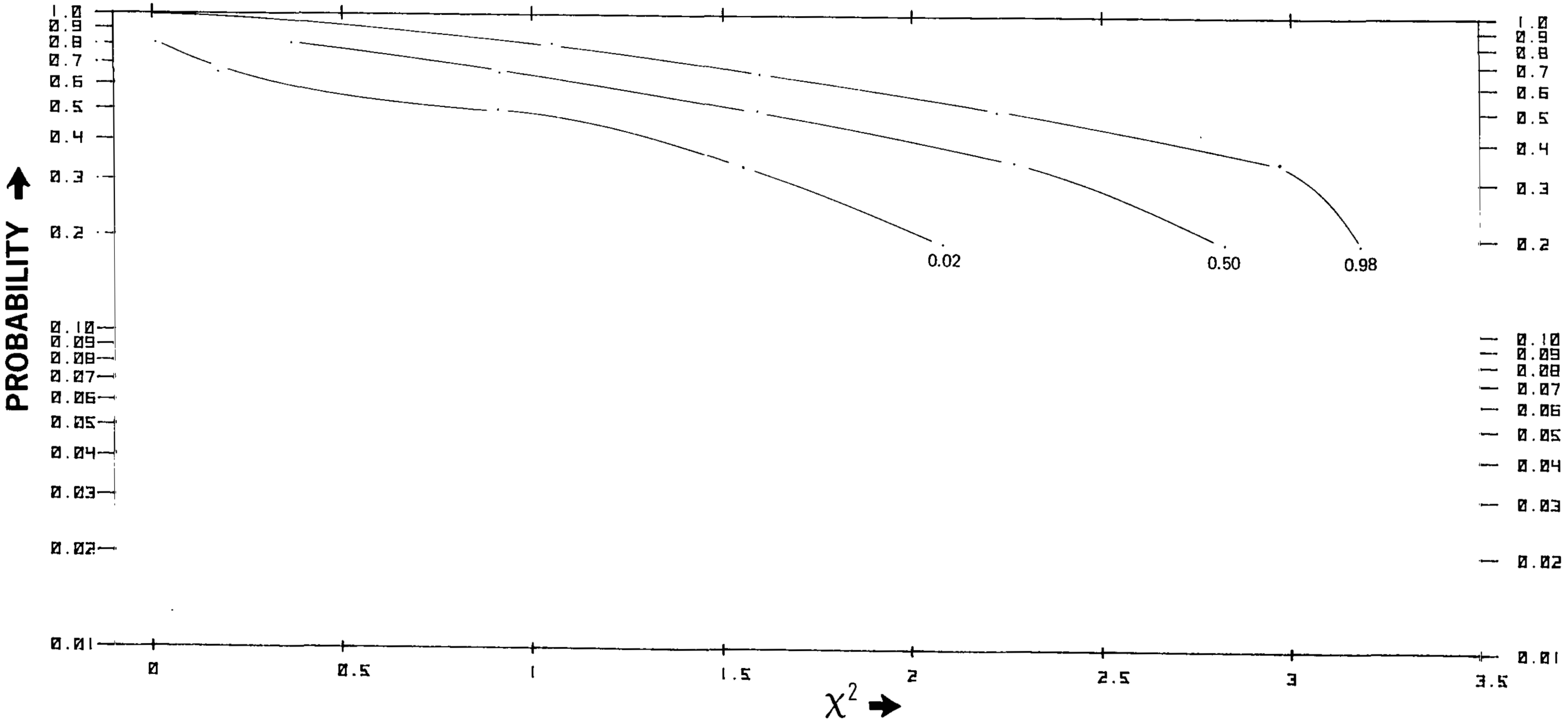
0.02
0.50
0.98
χ^2
DIMENSIONS N= 101



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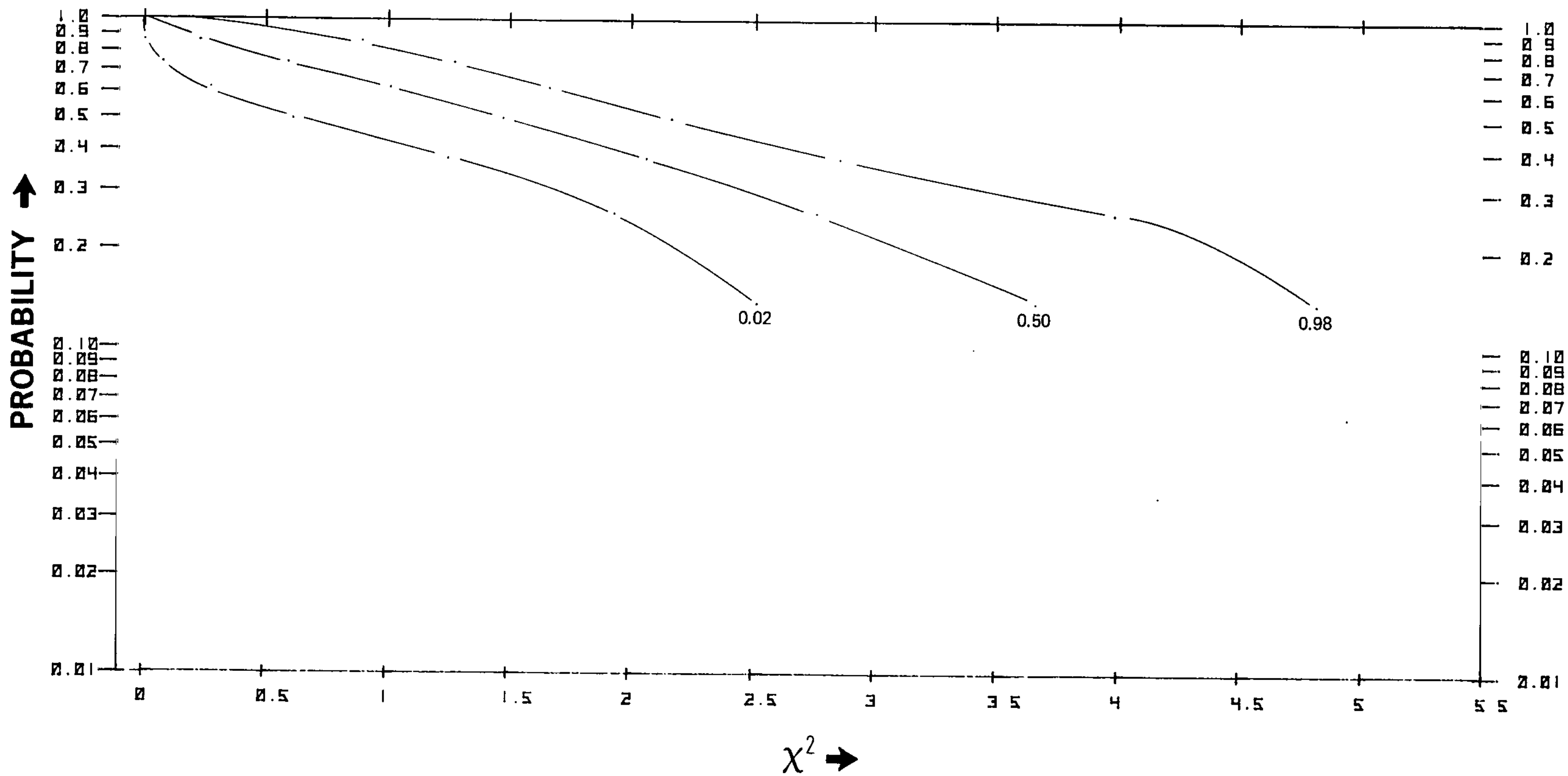

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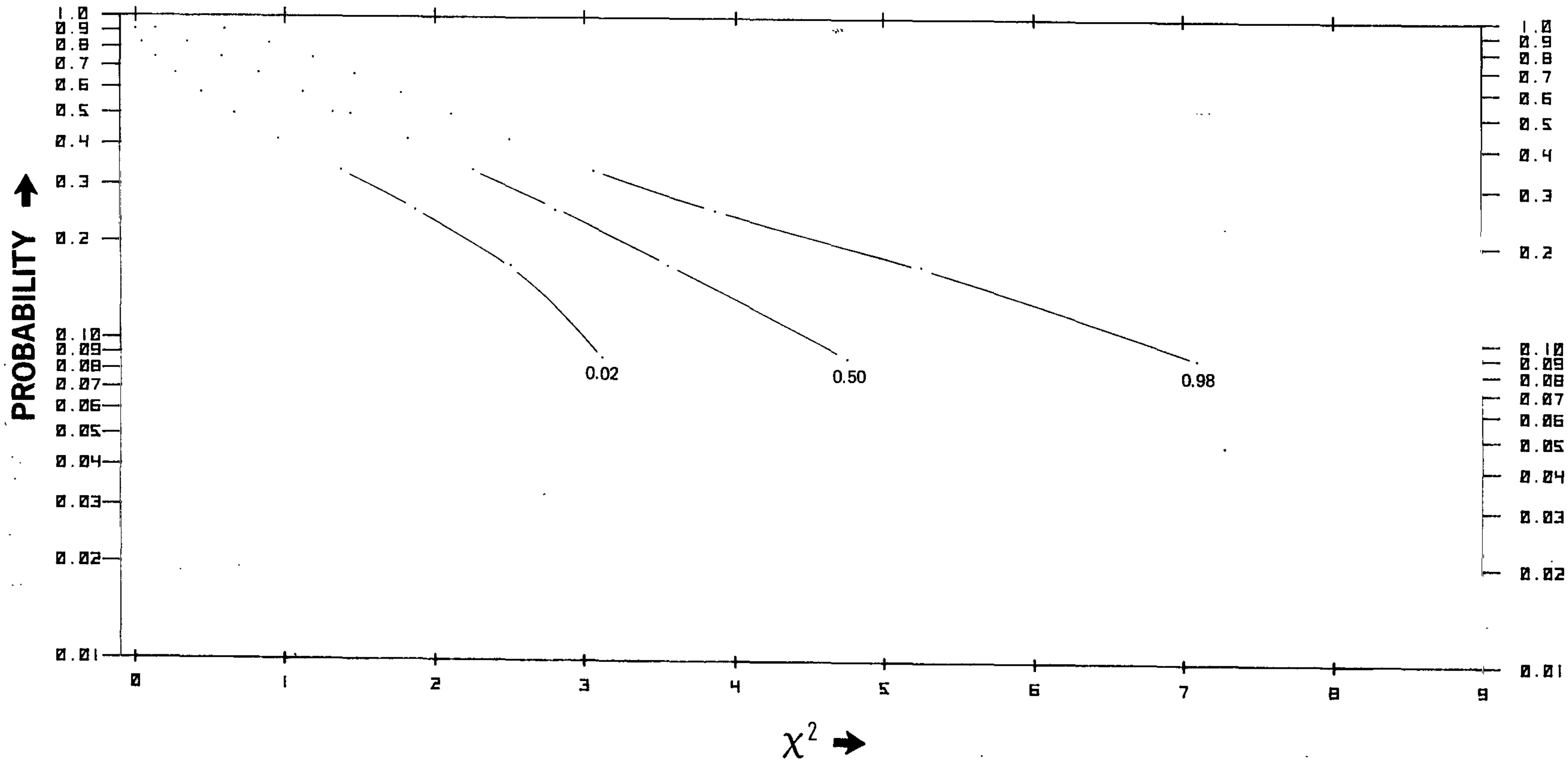

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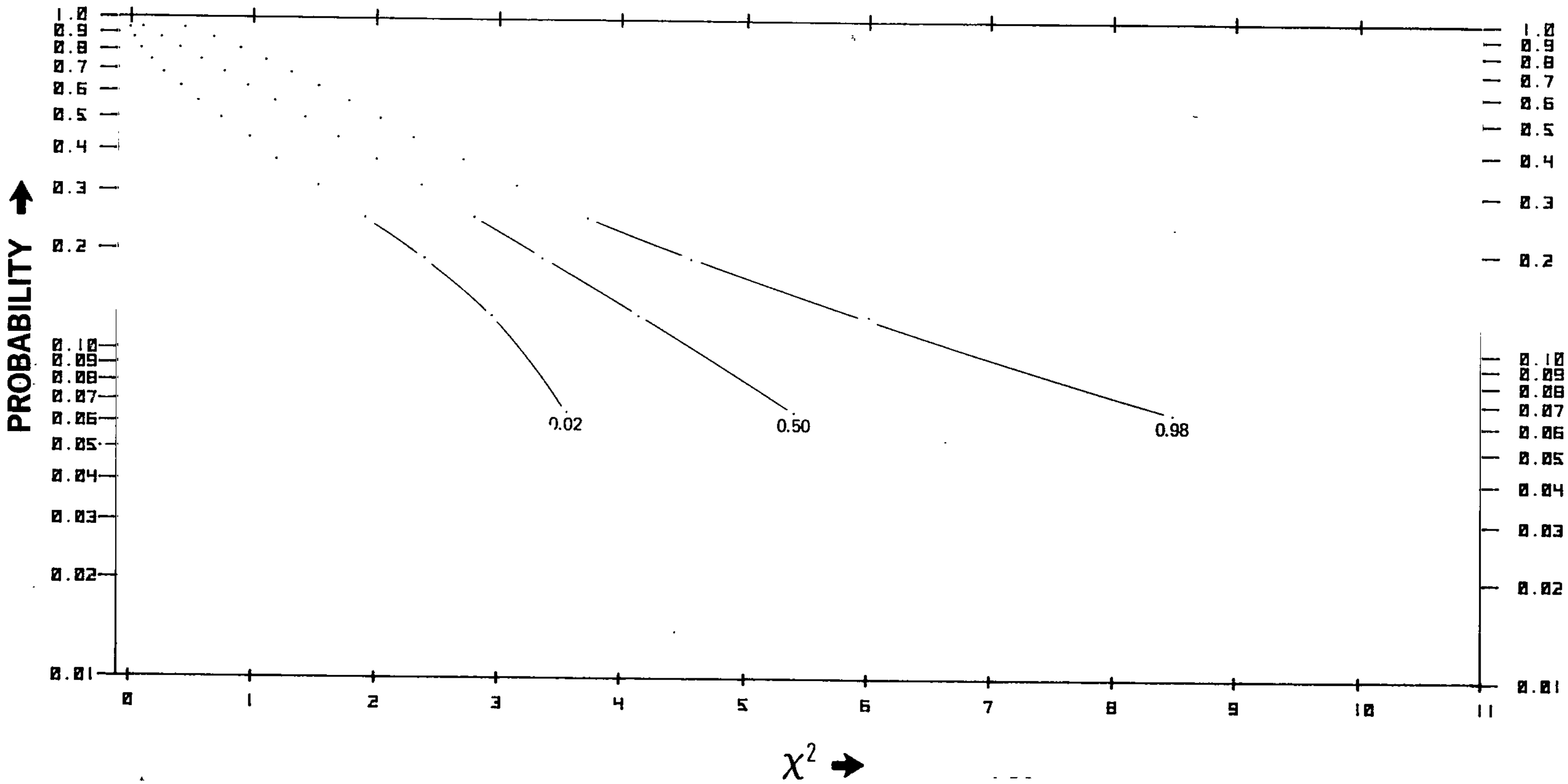

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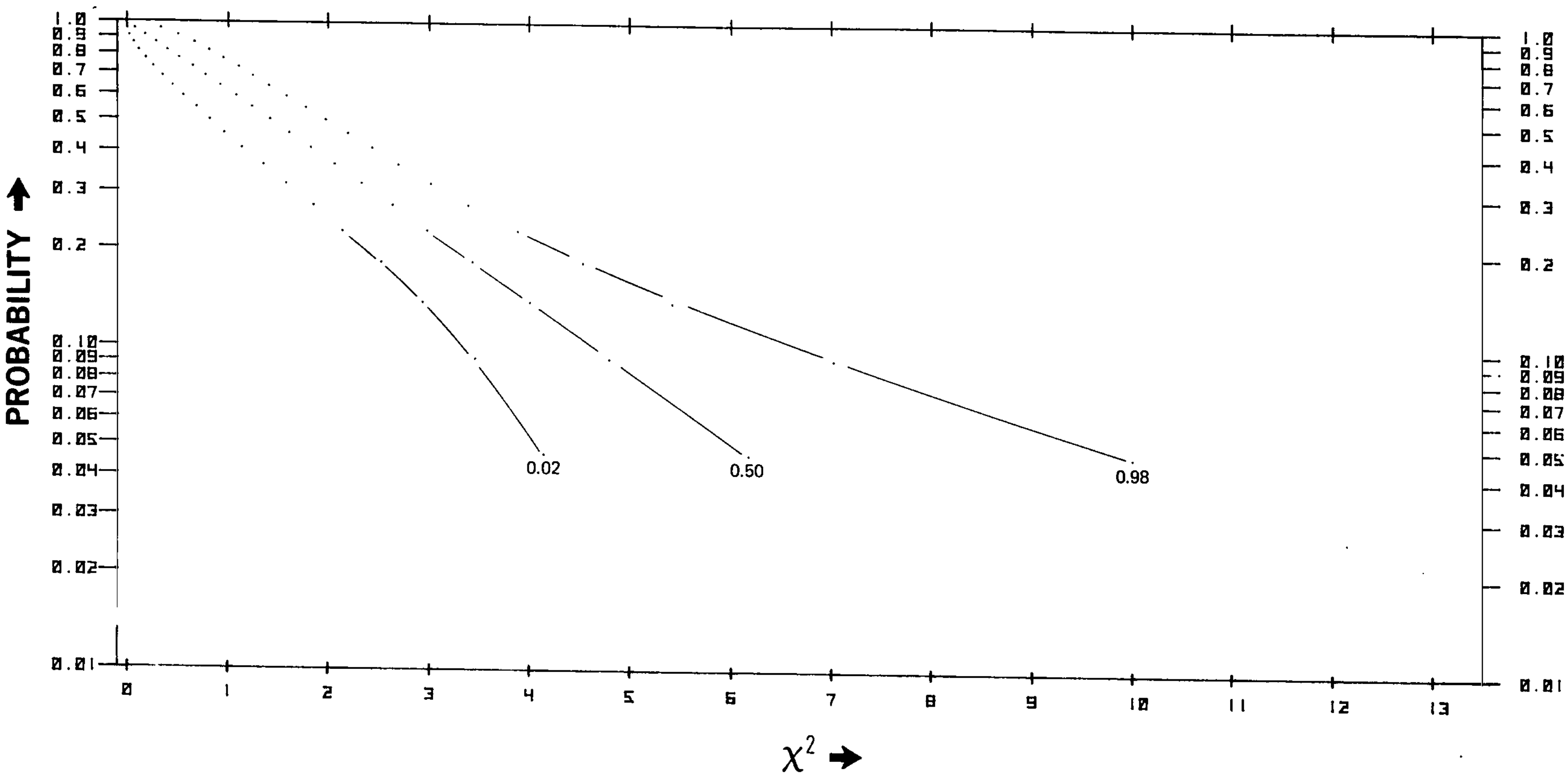

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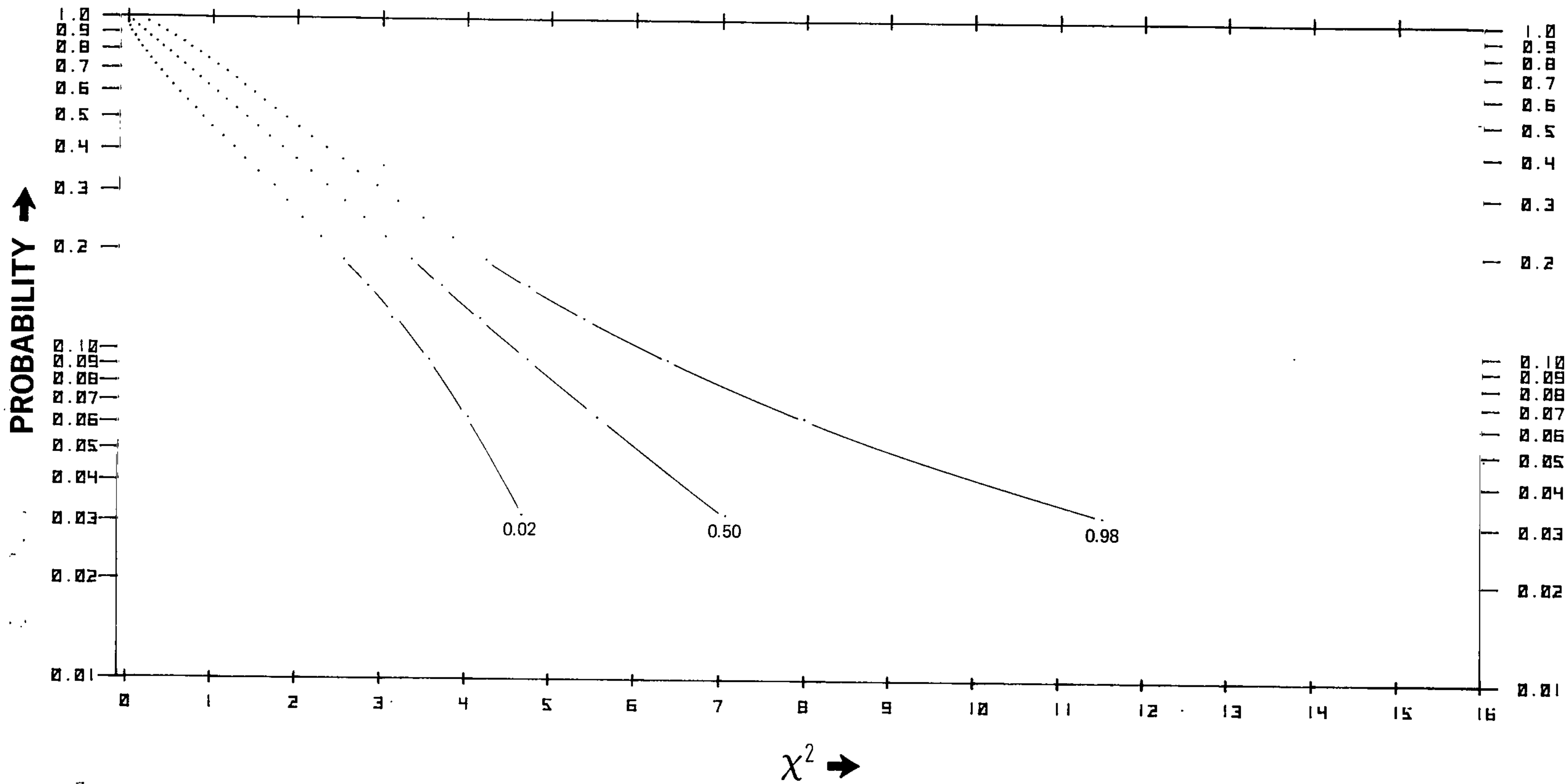

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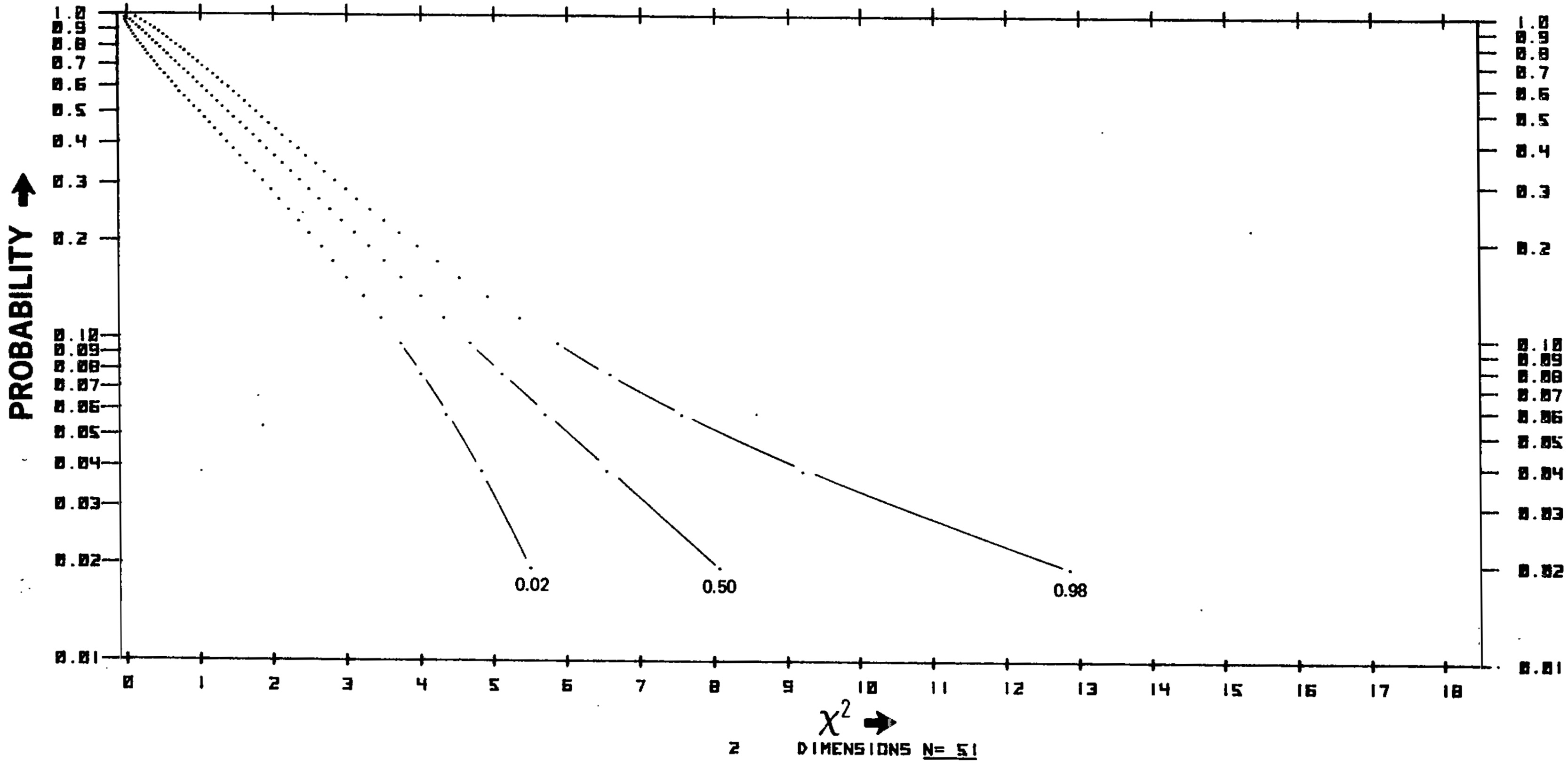
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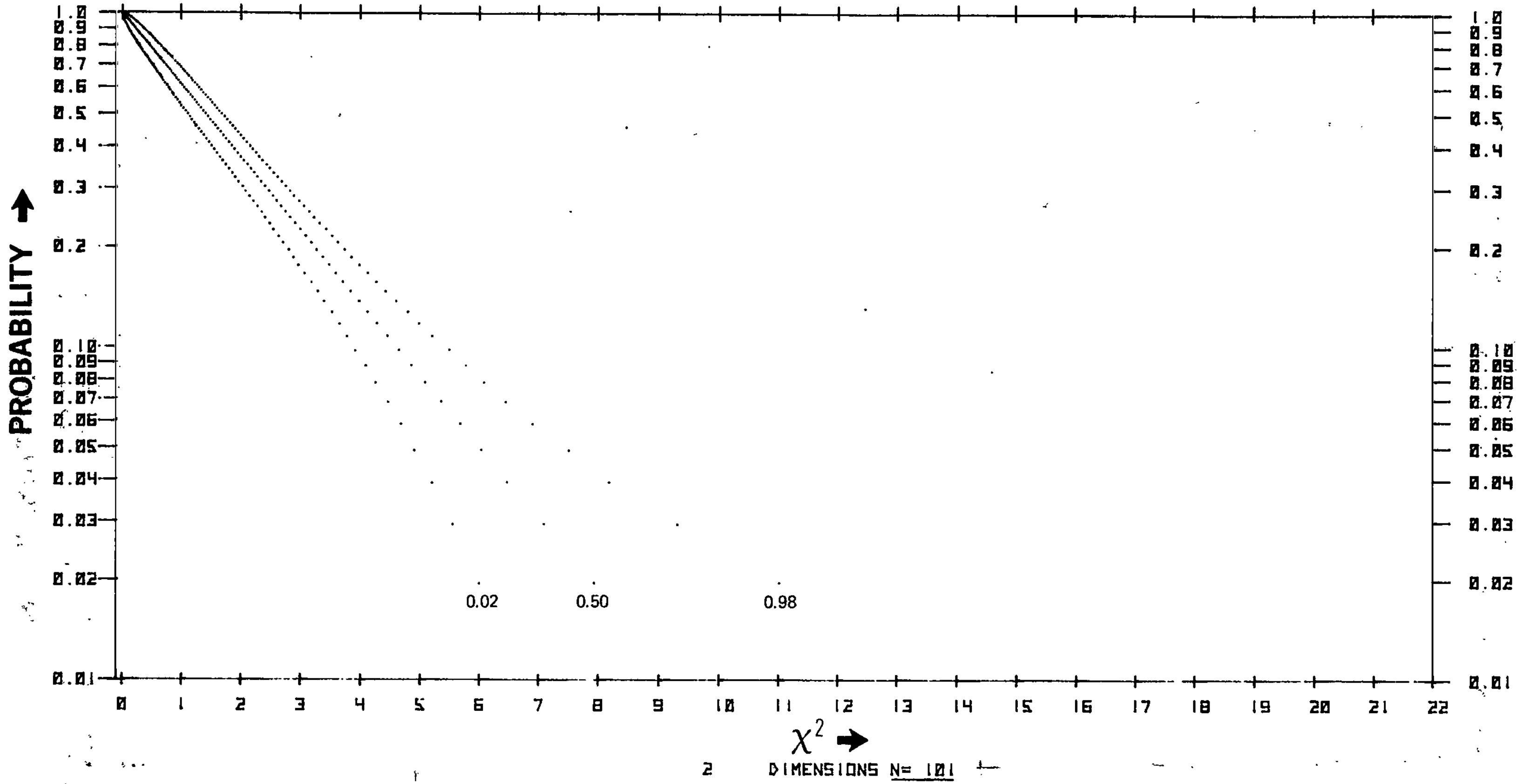
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
χ^2
2 DIMENSIONS N= 51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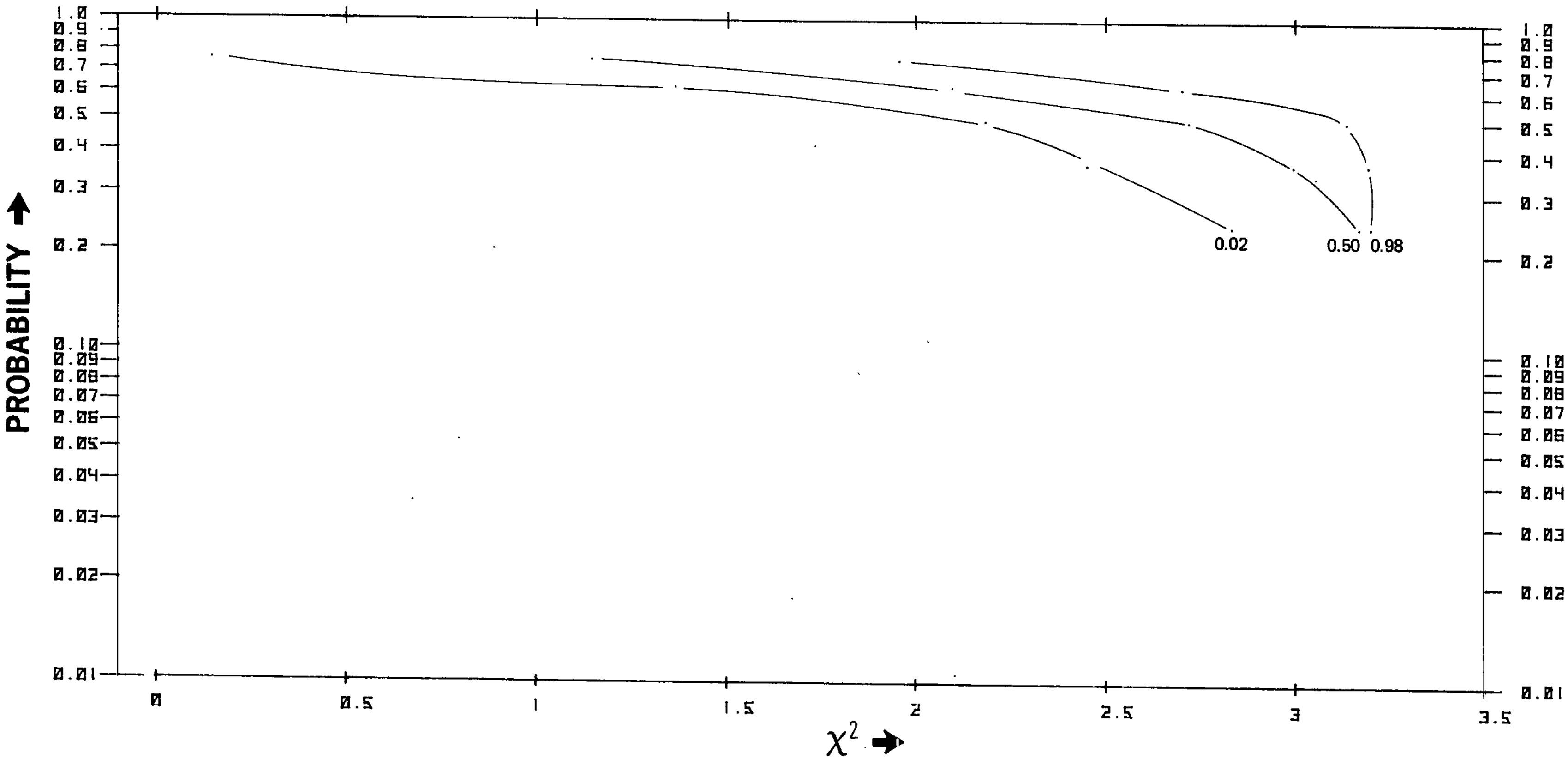
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22
χ^2
2 DIMENSIONS N= 101



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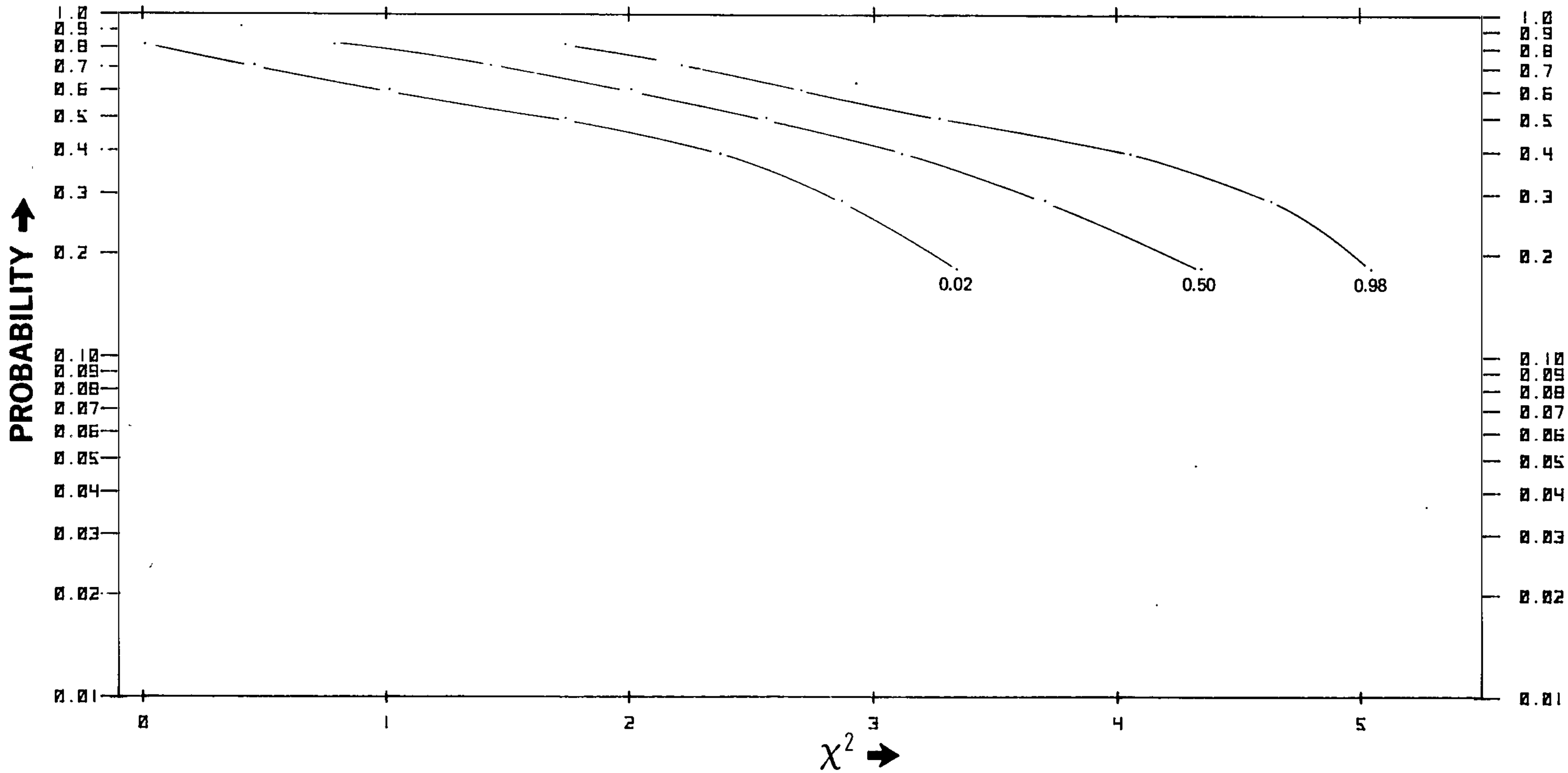

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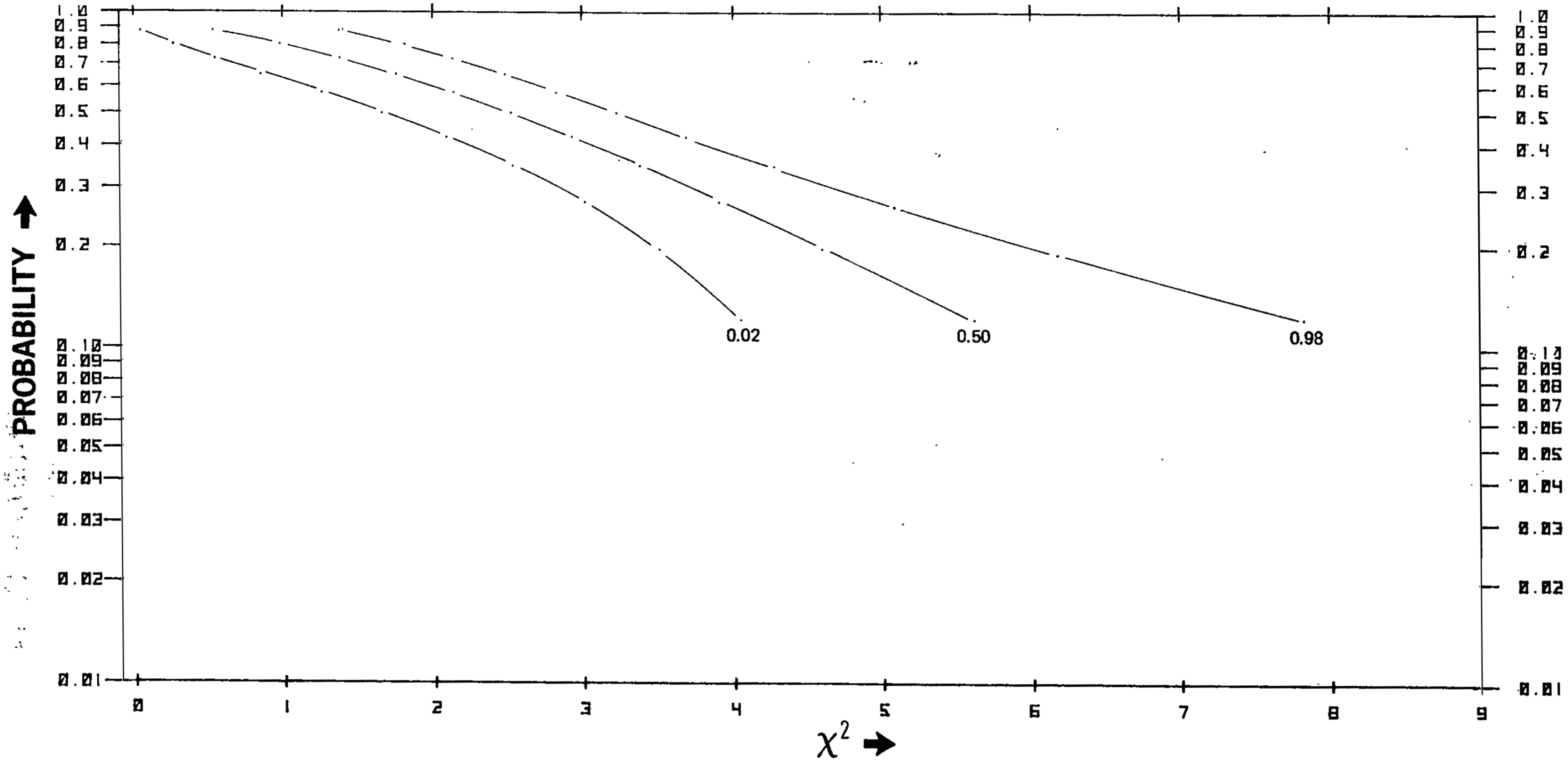

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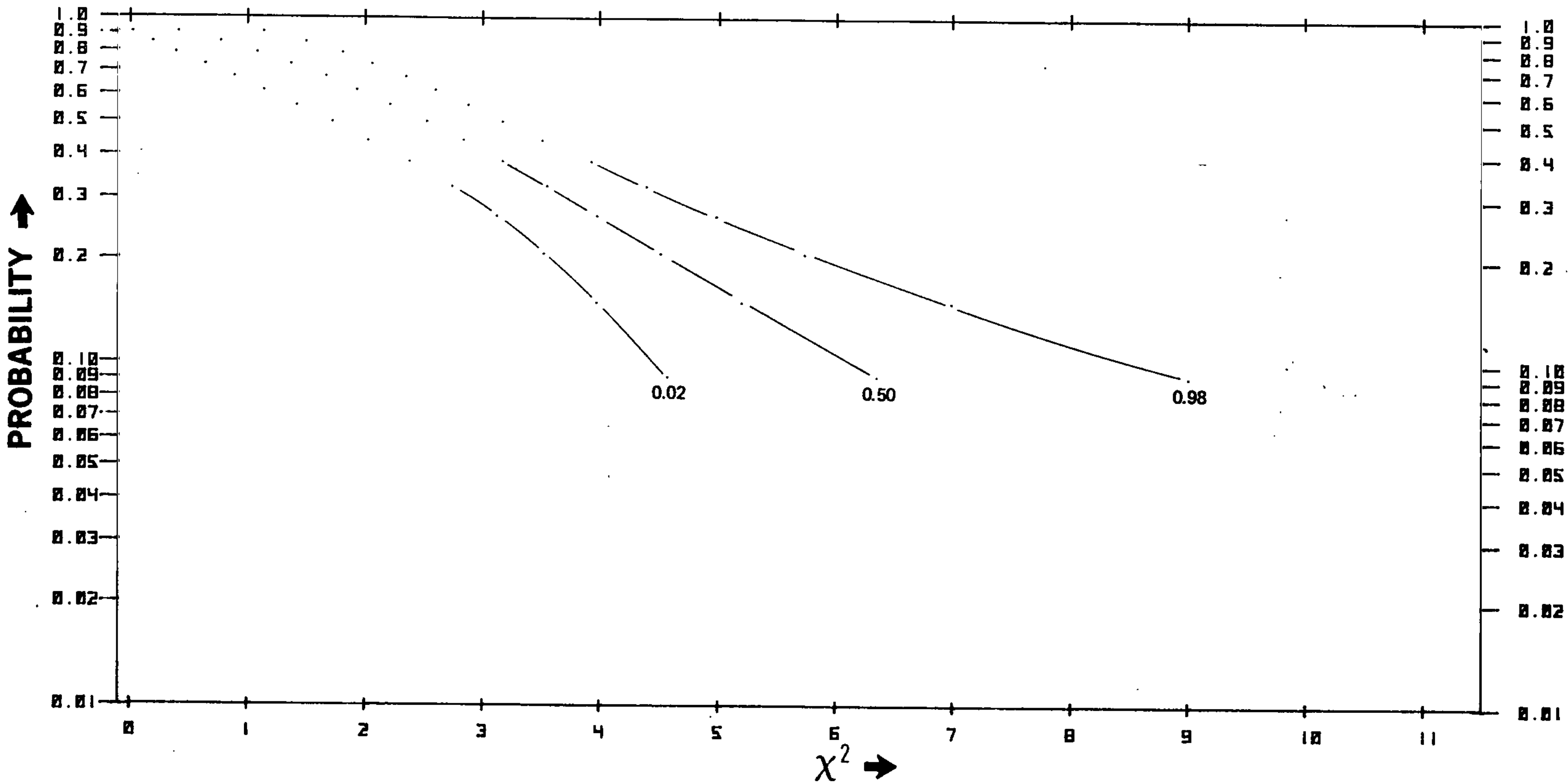

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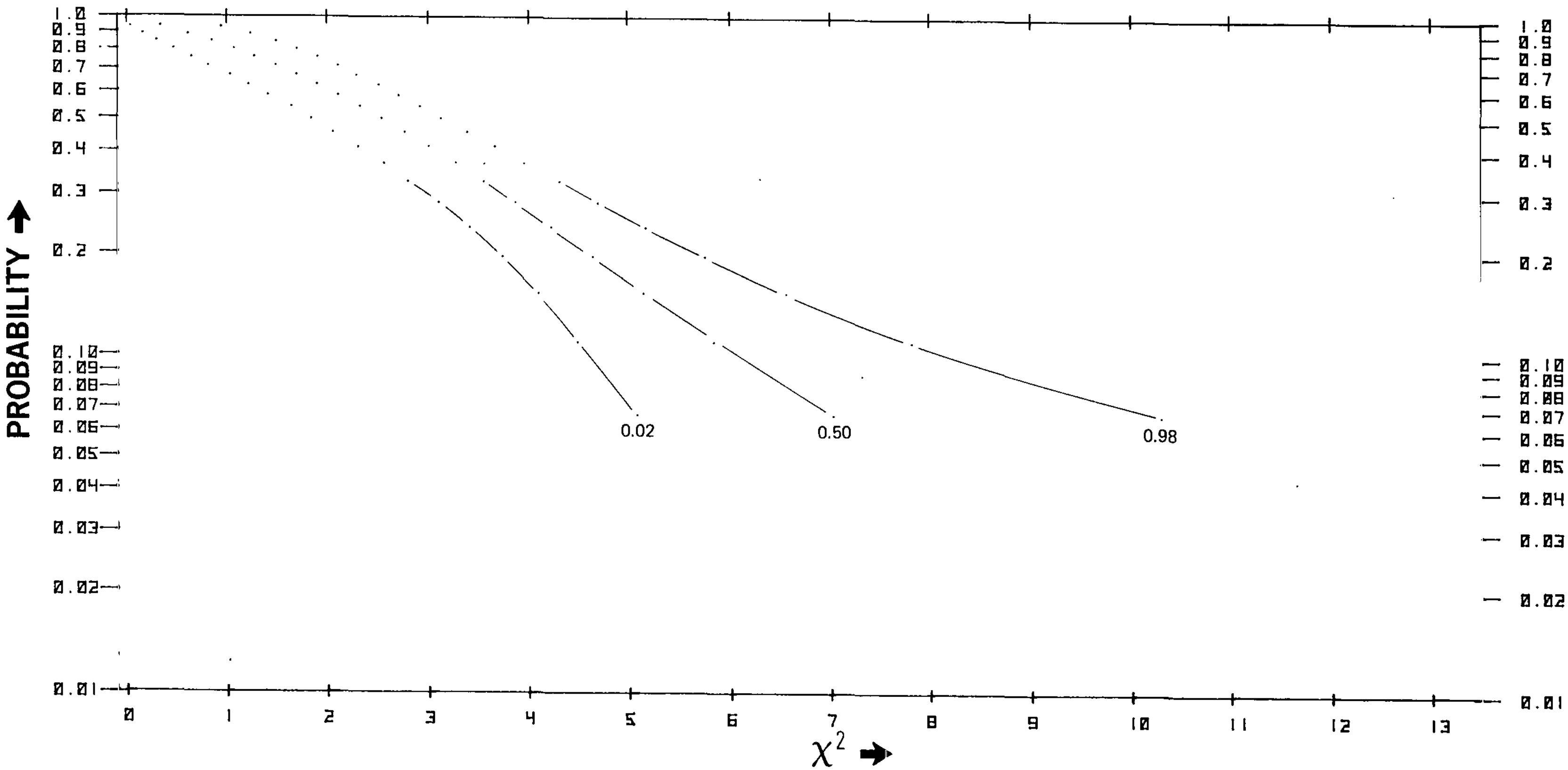

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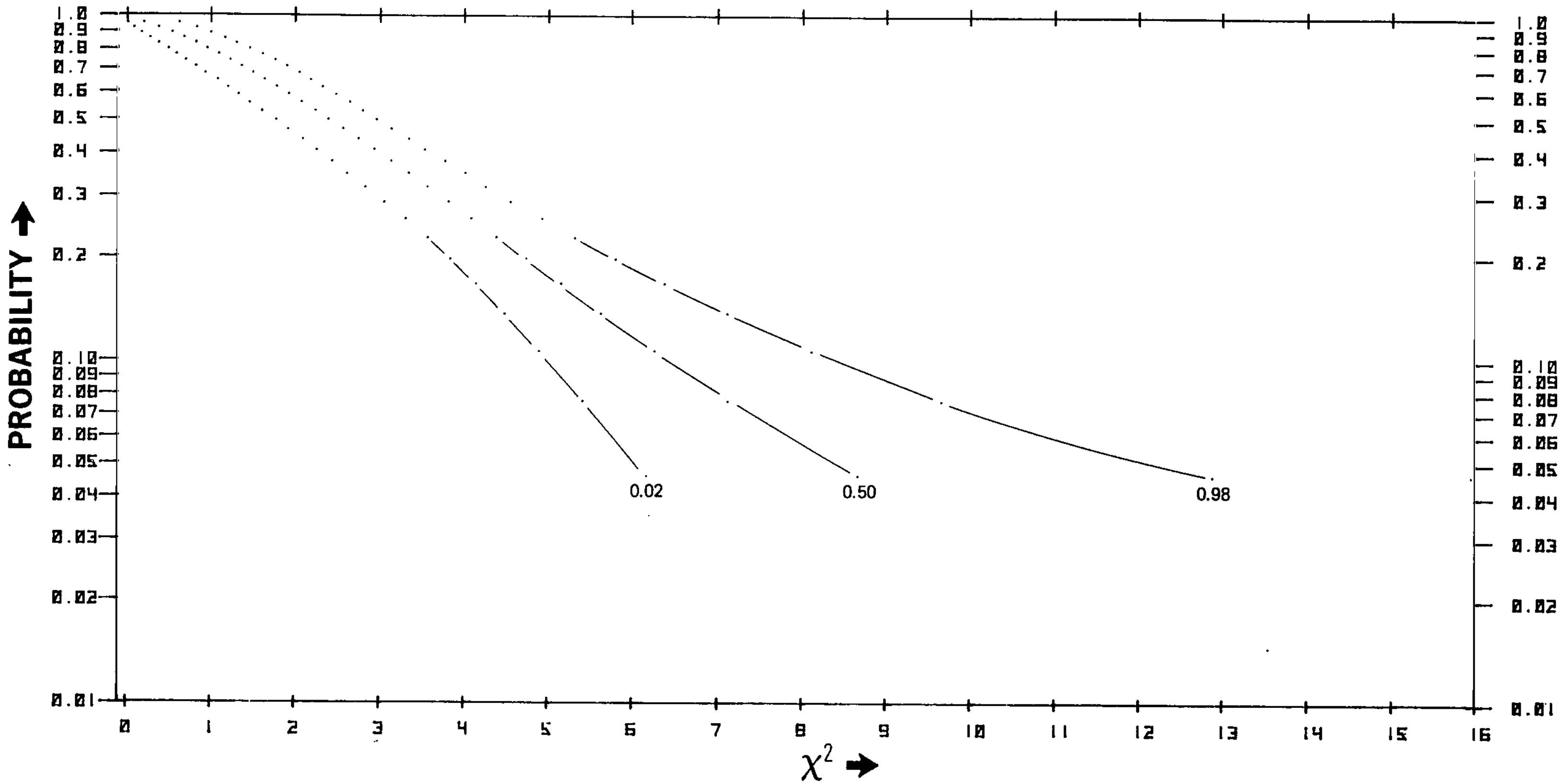

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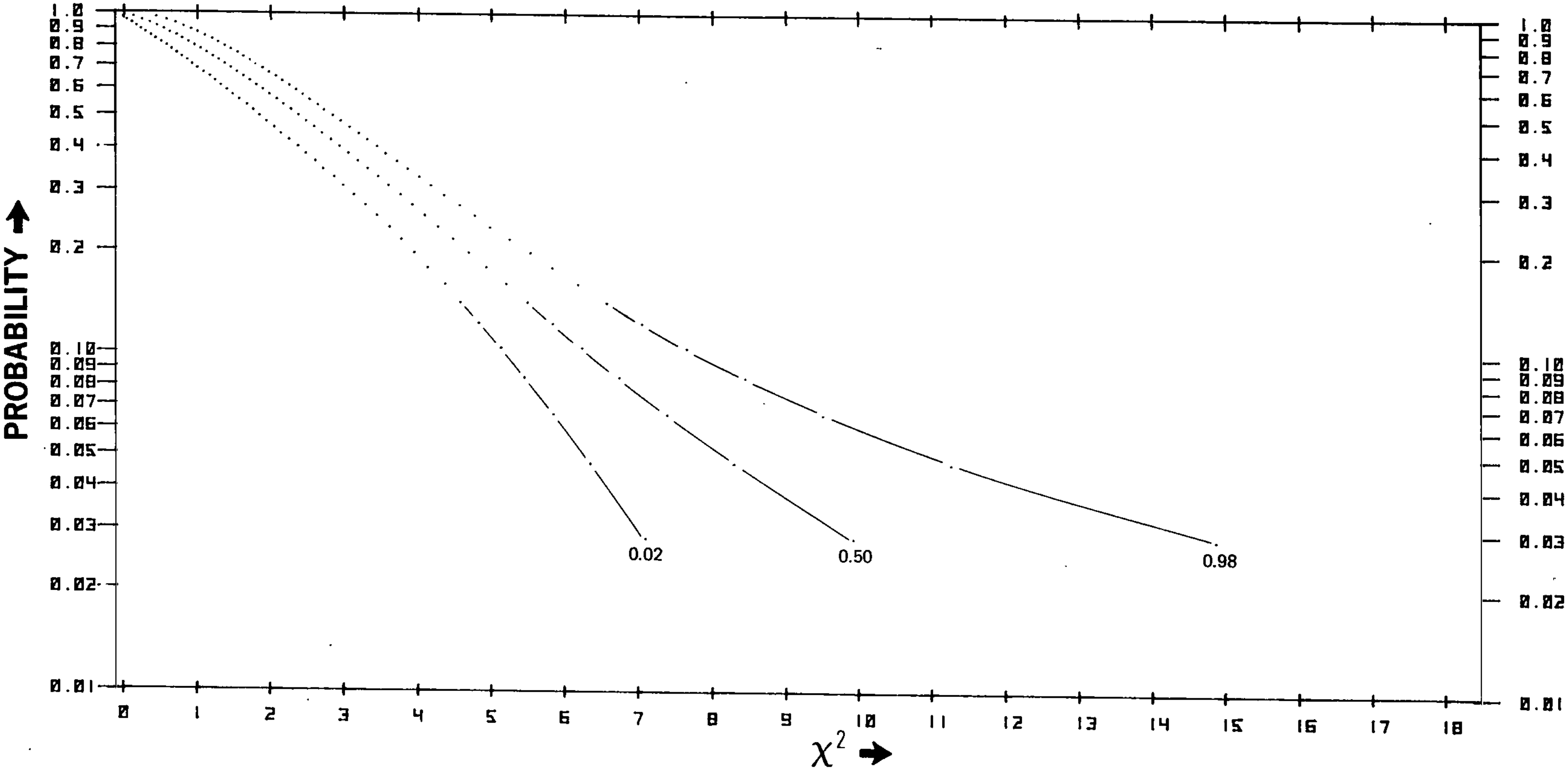

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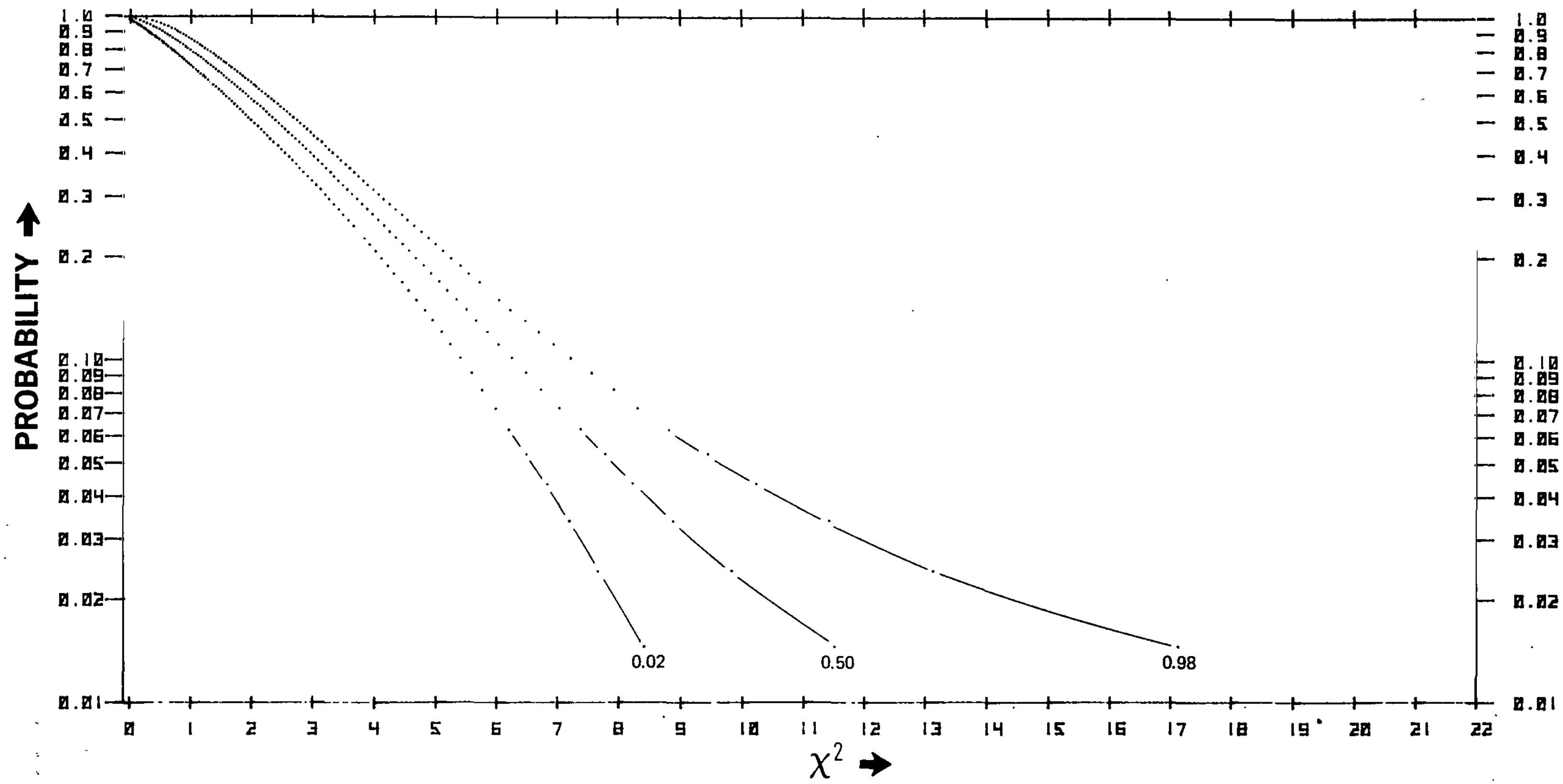

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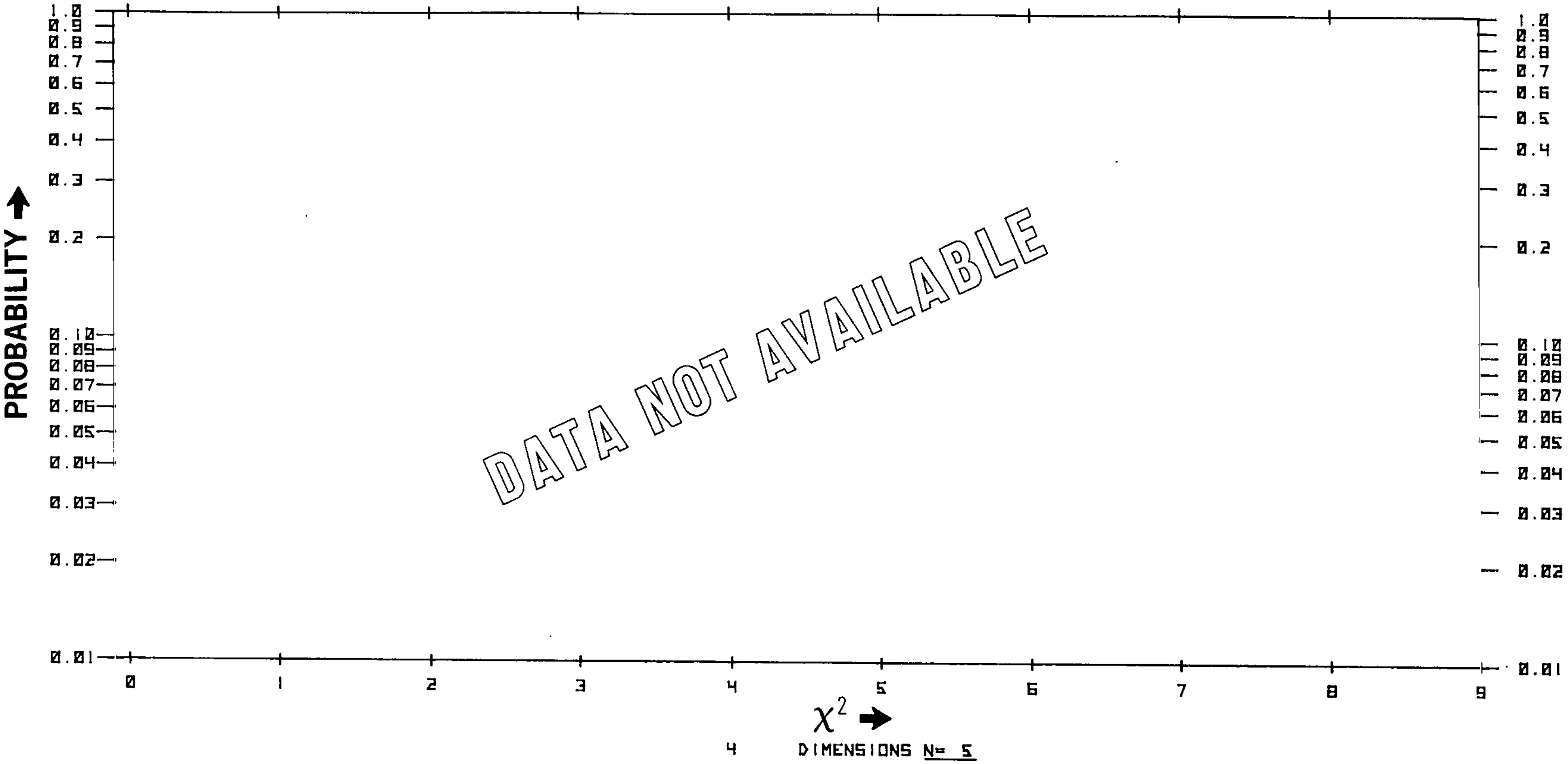
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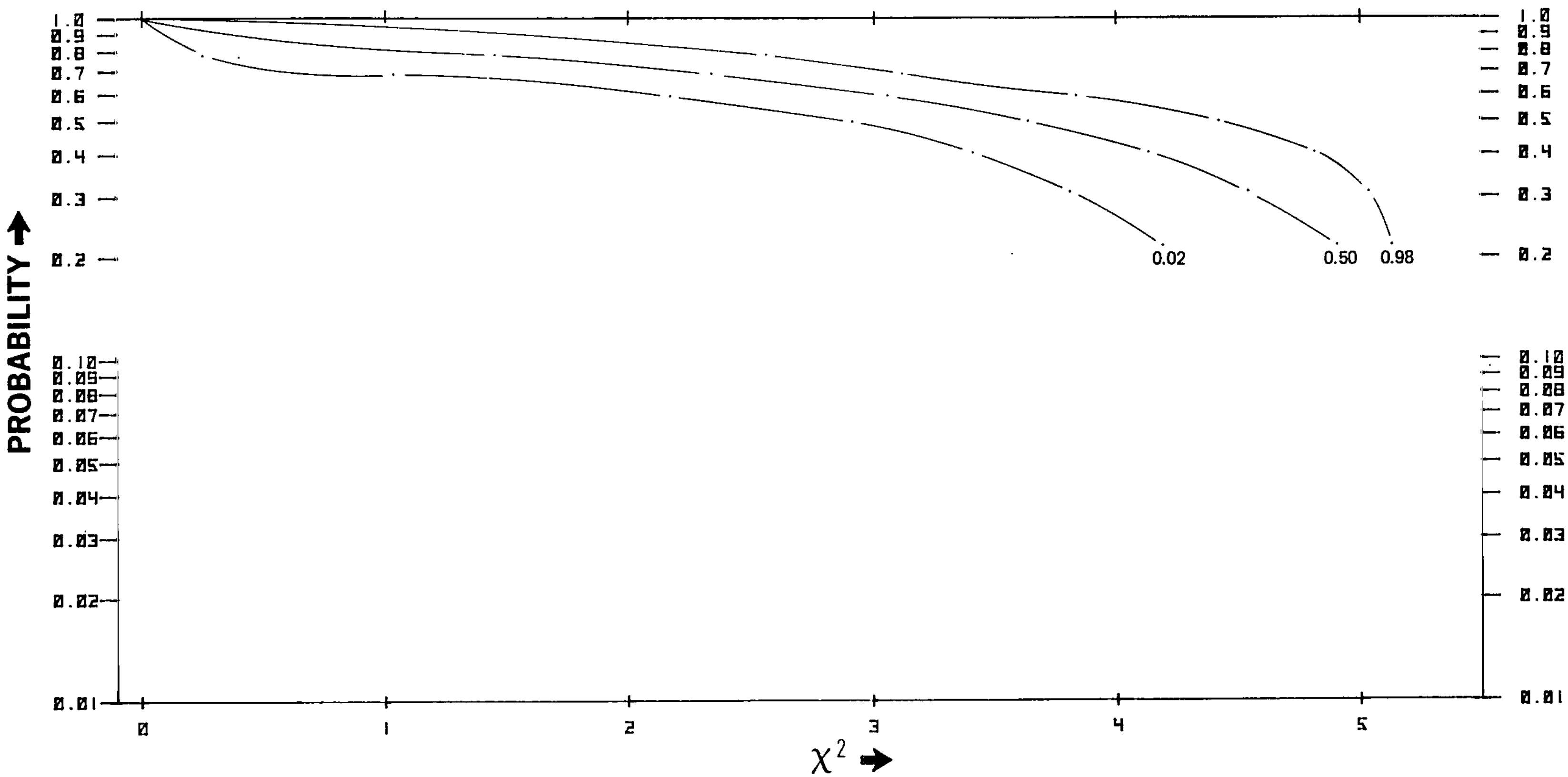
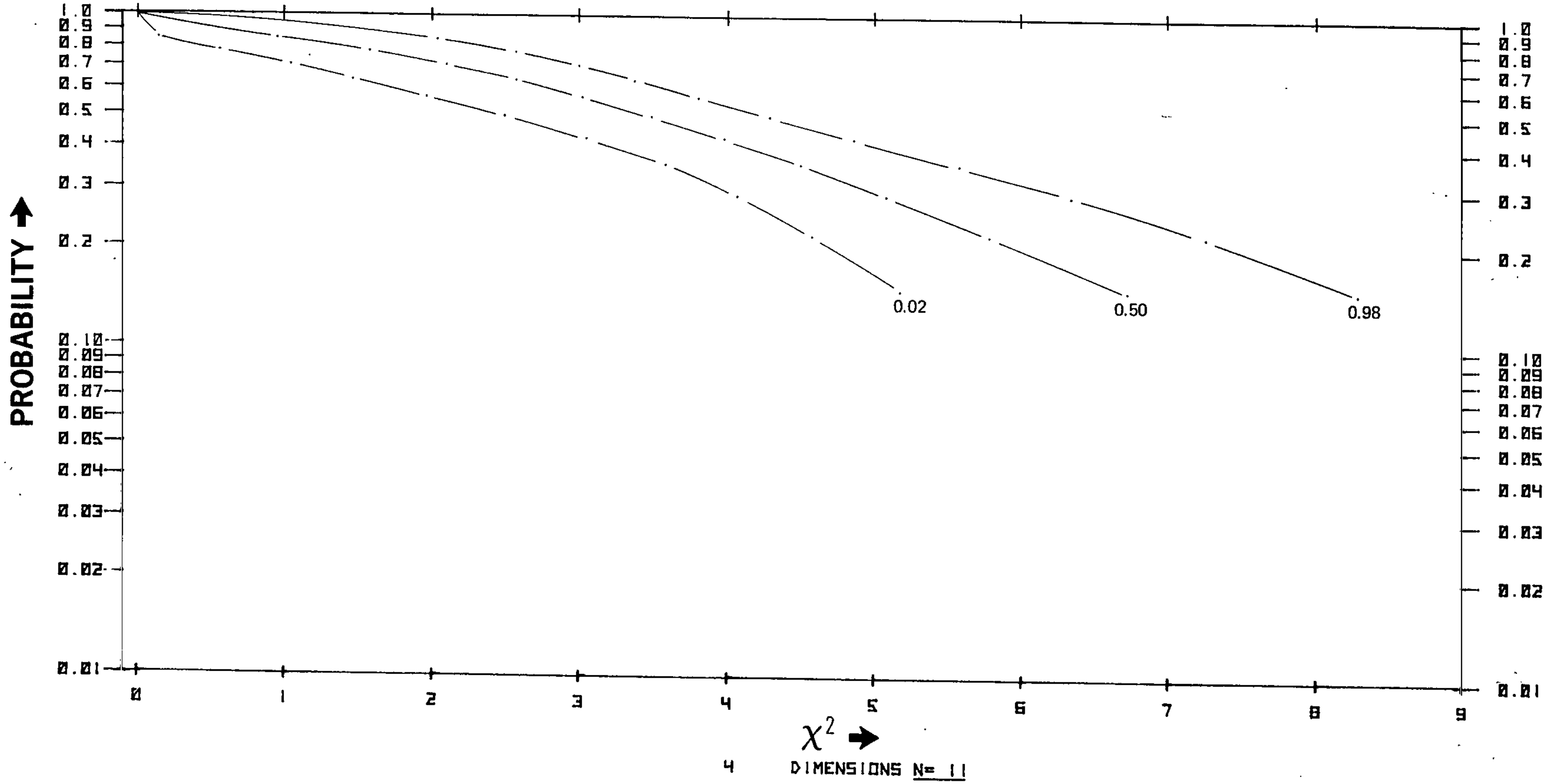
0 1 2 3 4 5 6 7 8 9
χ^2
DATA NOT AVAILABLE
4 DIMENSIONS N= 5



4 DIMENSIONS <u>N= 7</u>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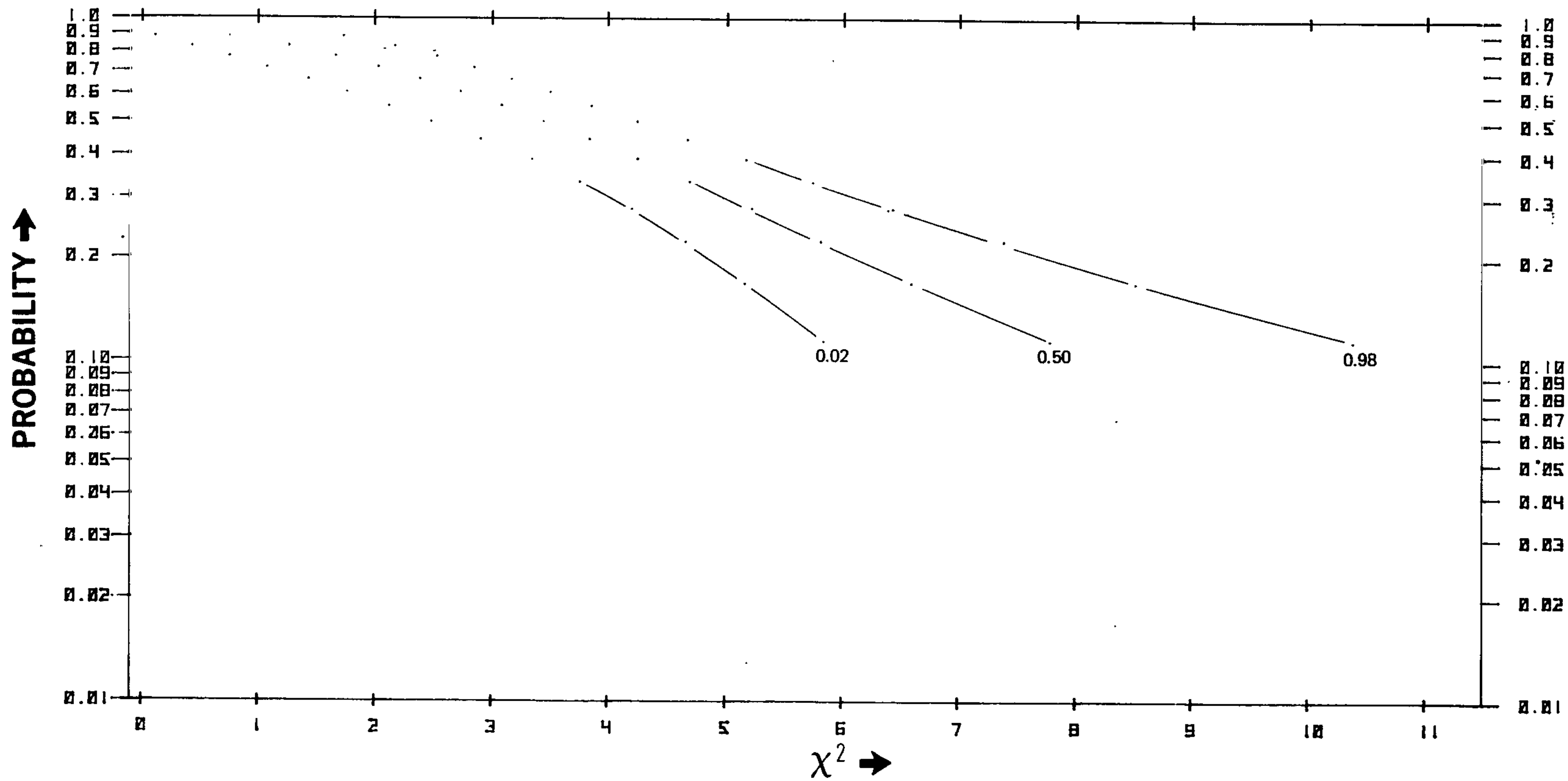
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0 1 2 3 4 5 6 7 8 9
0.02
0.50
0.98
χ^2
4 DIMENSIONS N= 11



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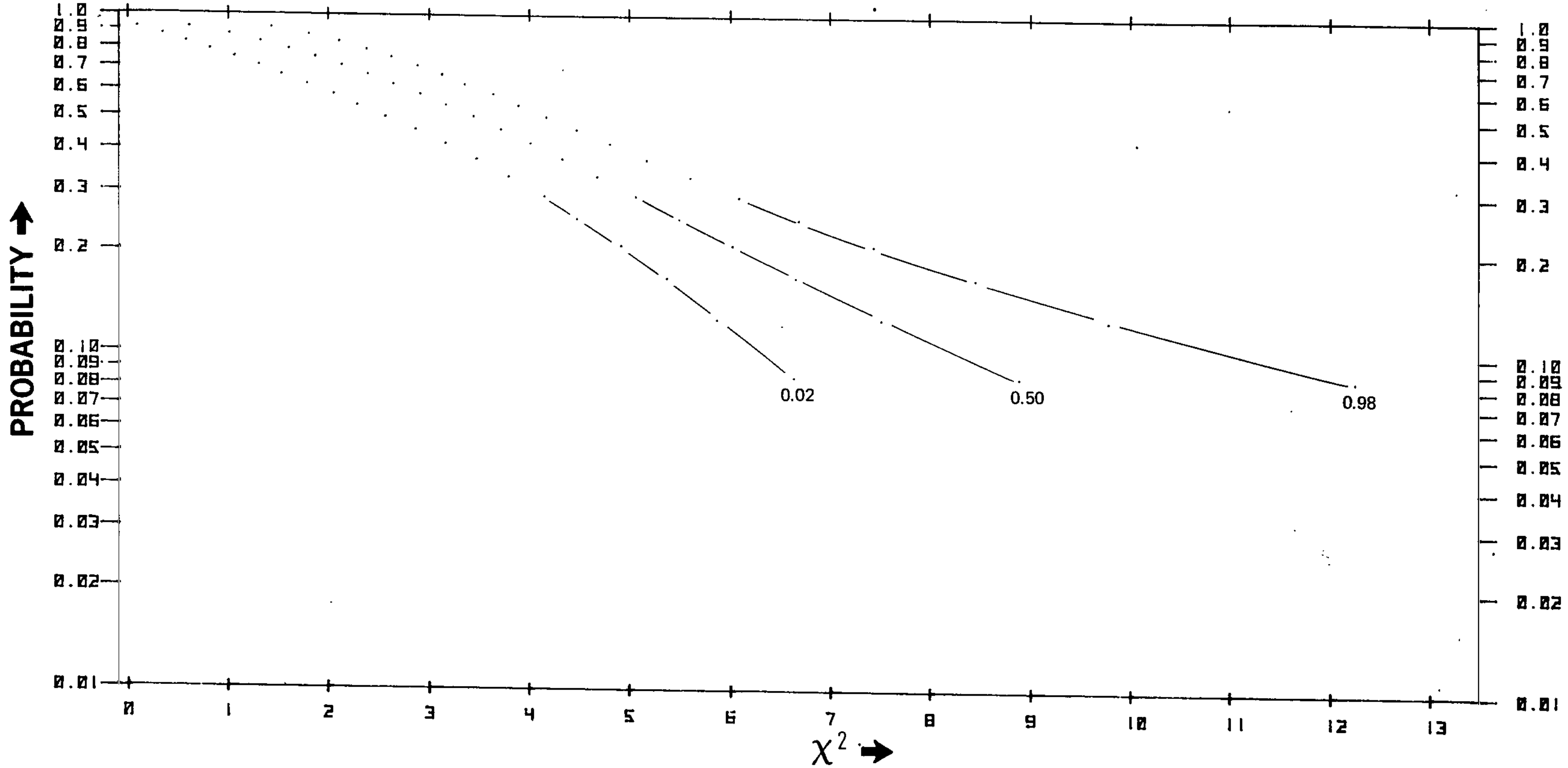

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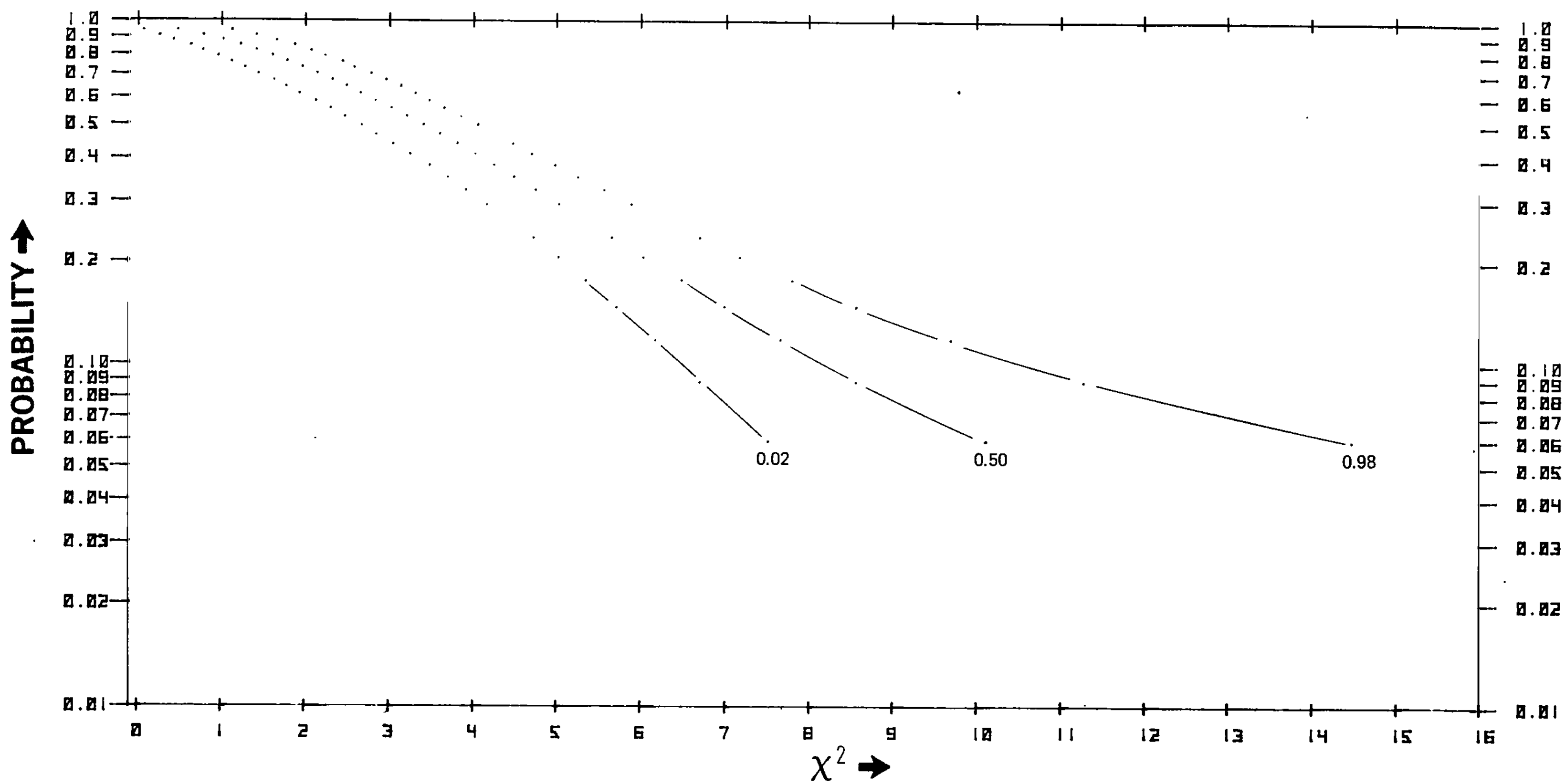

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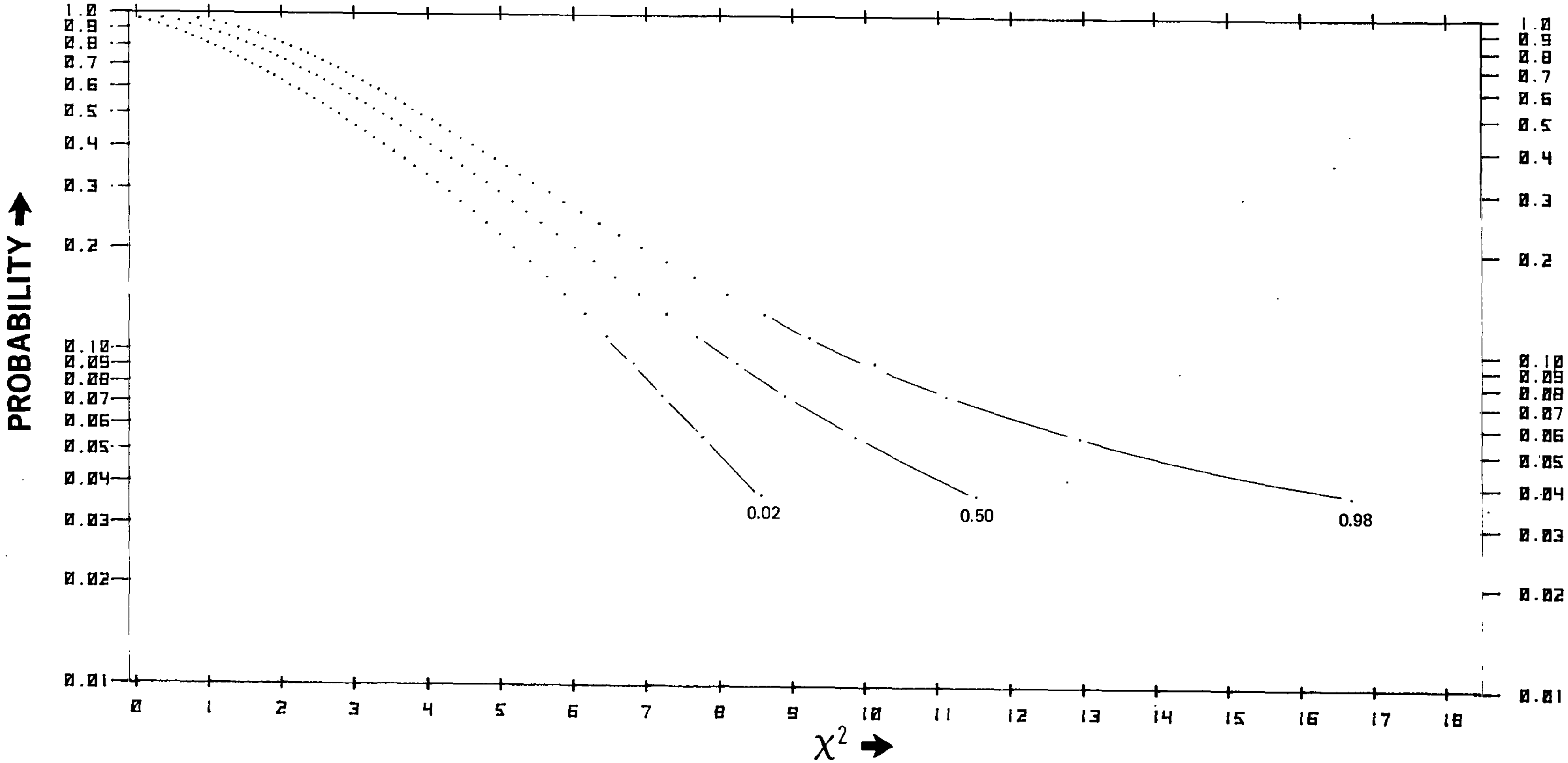

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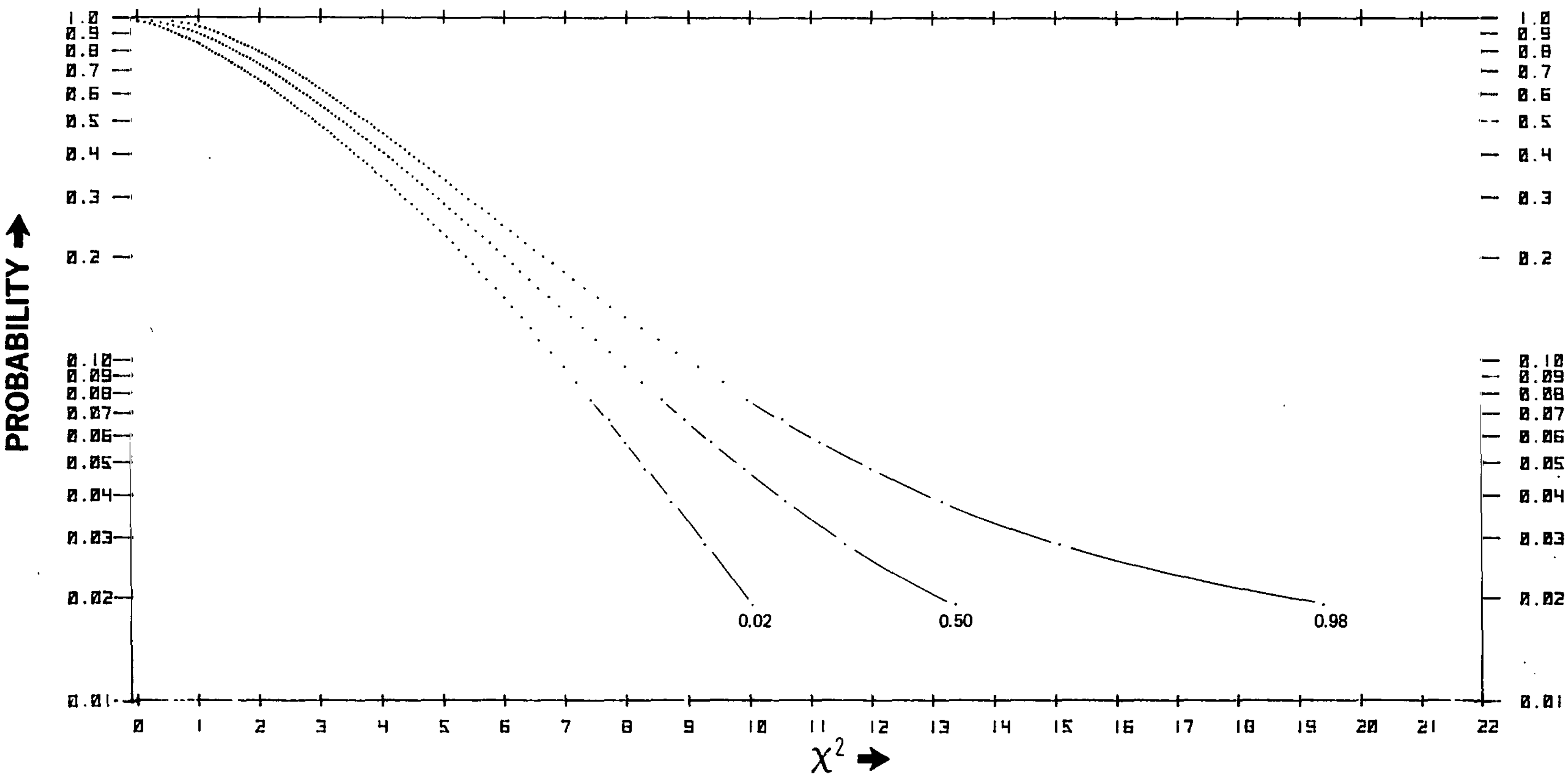

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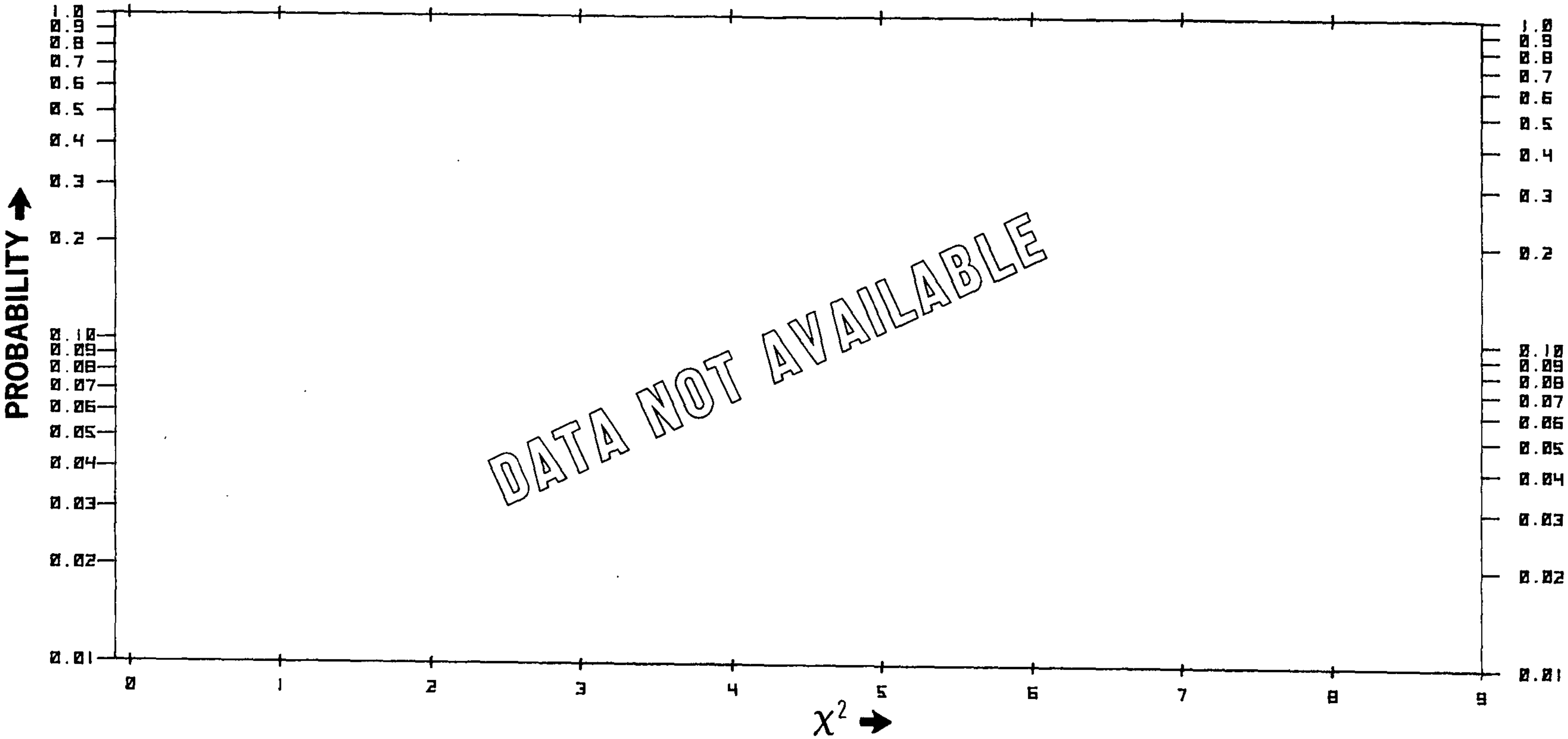
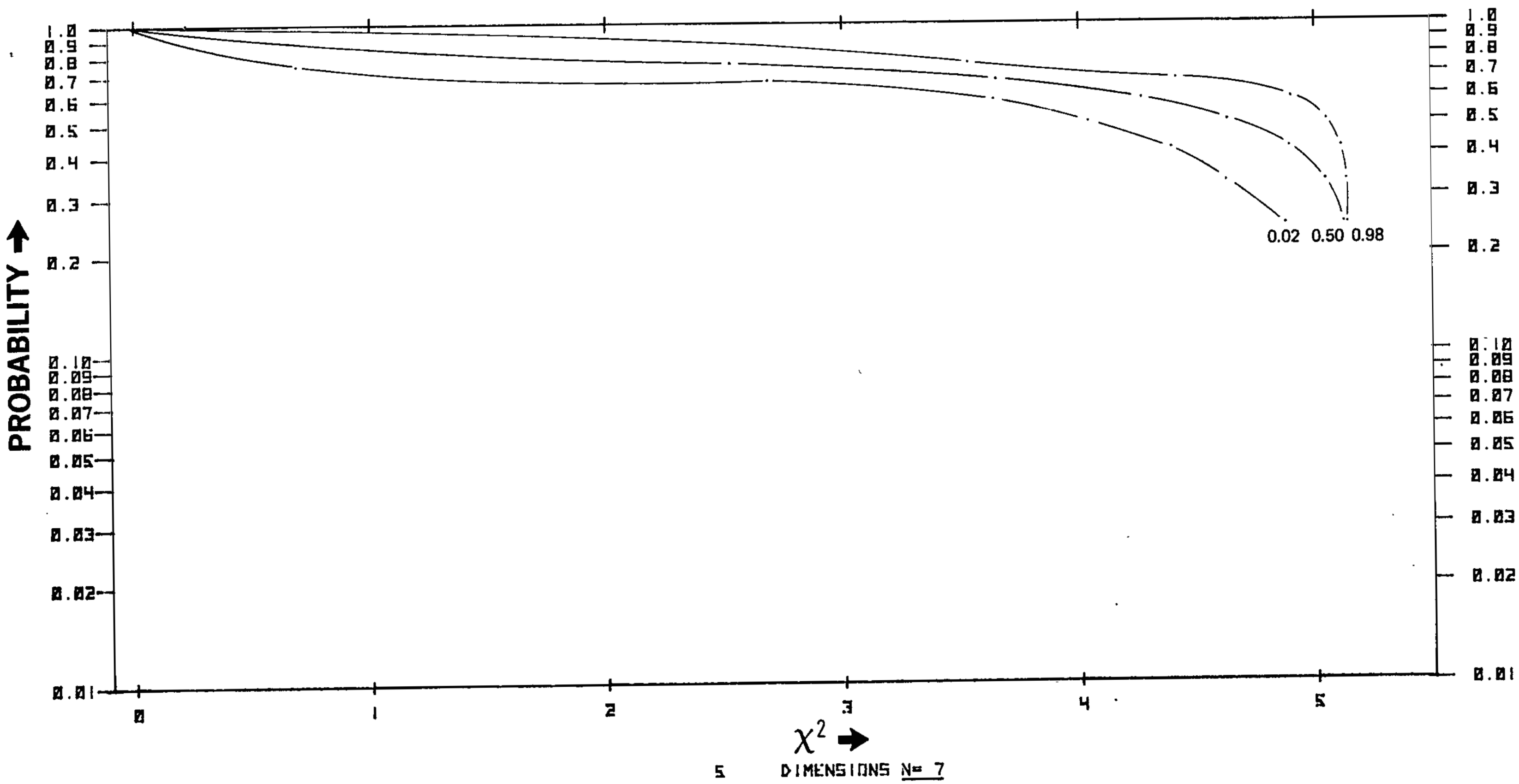


5 DIMENSIONS N= 5


PROBABILITY
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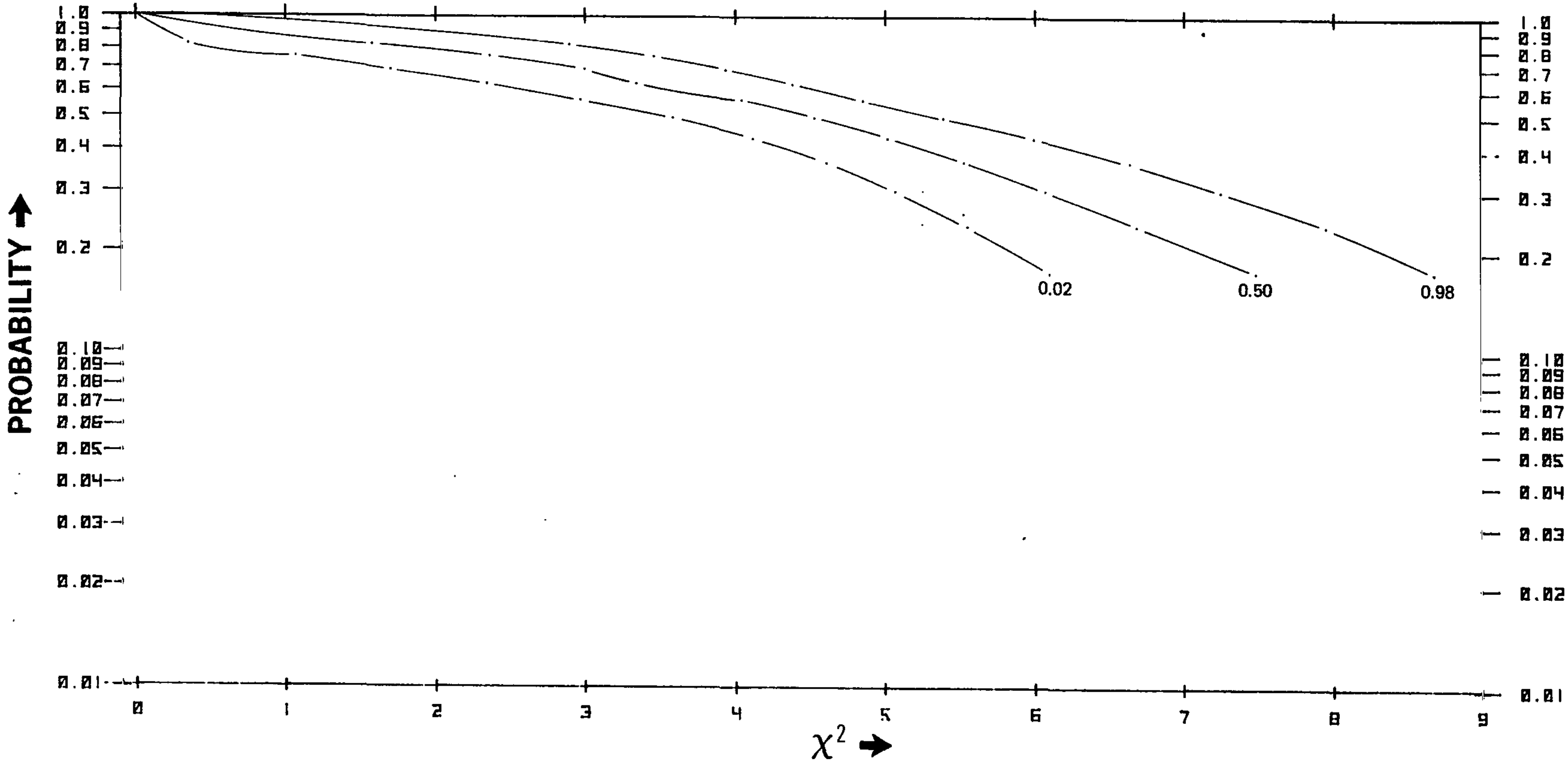
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.02 0.50 0.98
0
1
2
3
4
5
χ^2
5 DIMENSIONS N= 7



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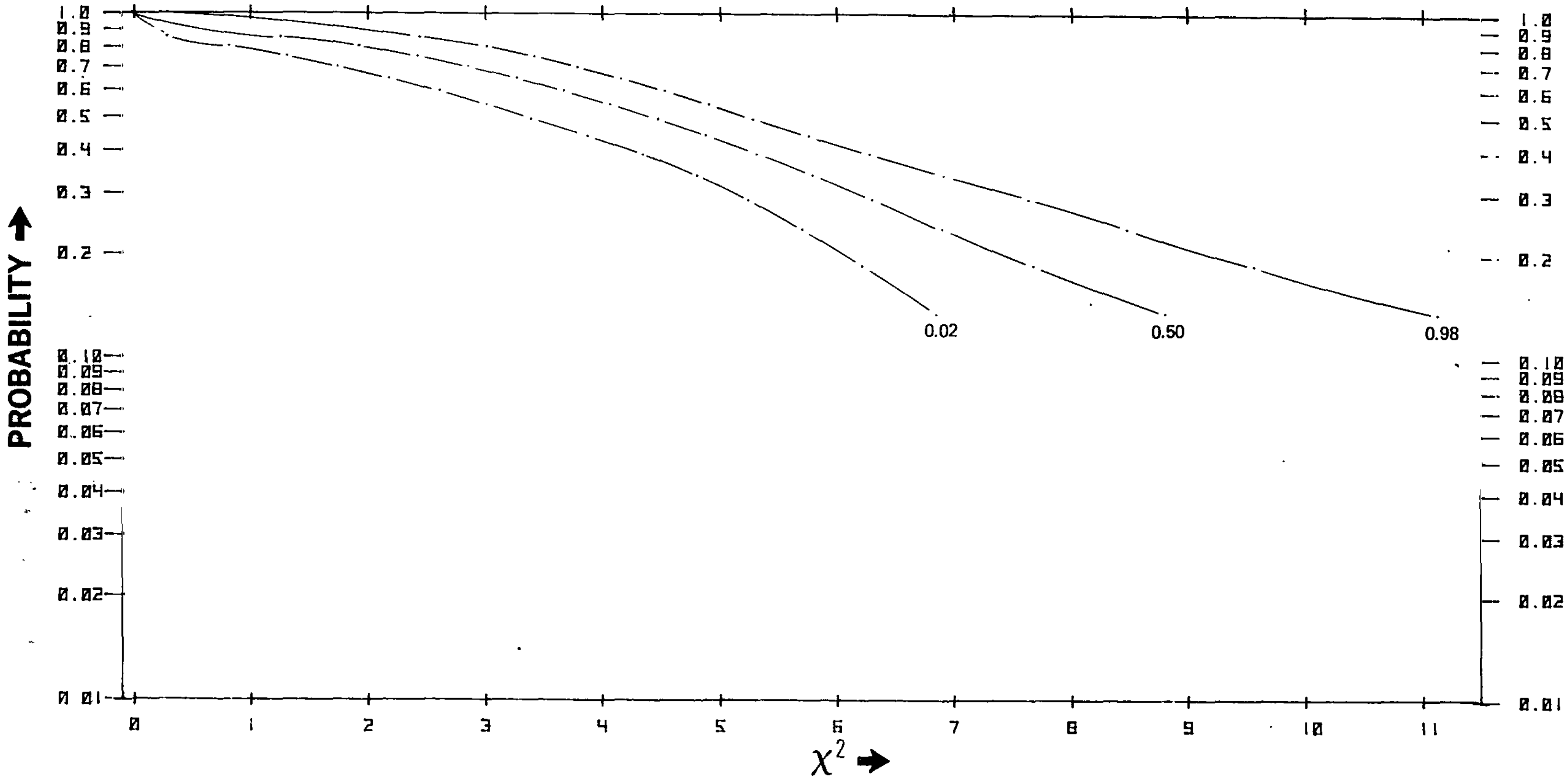

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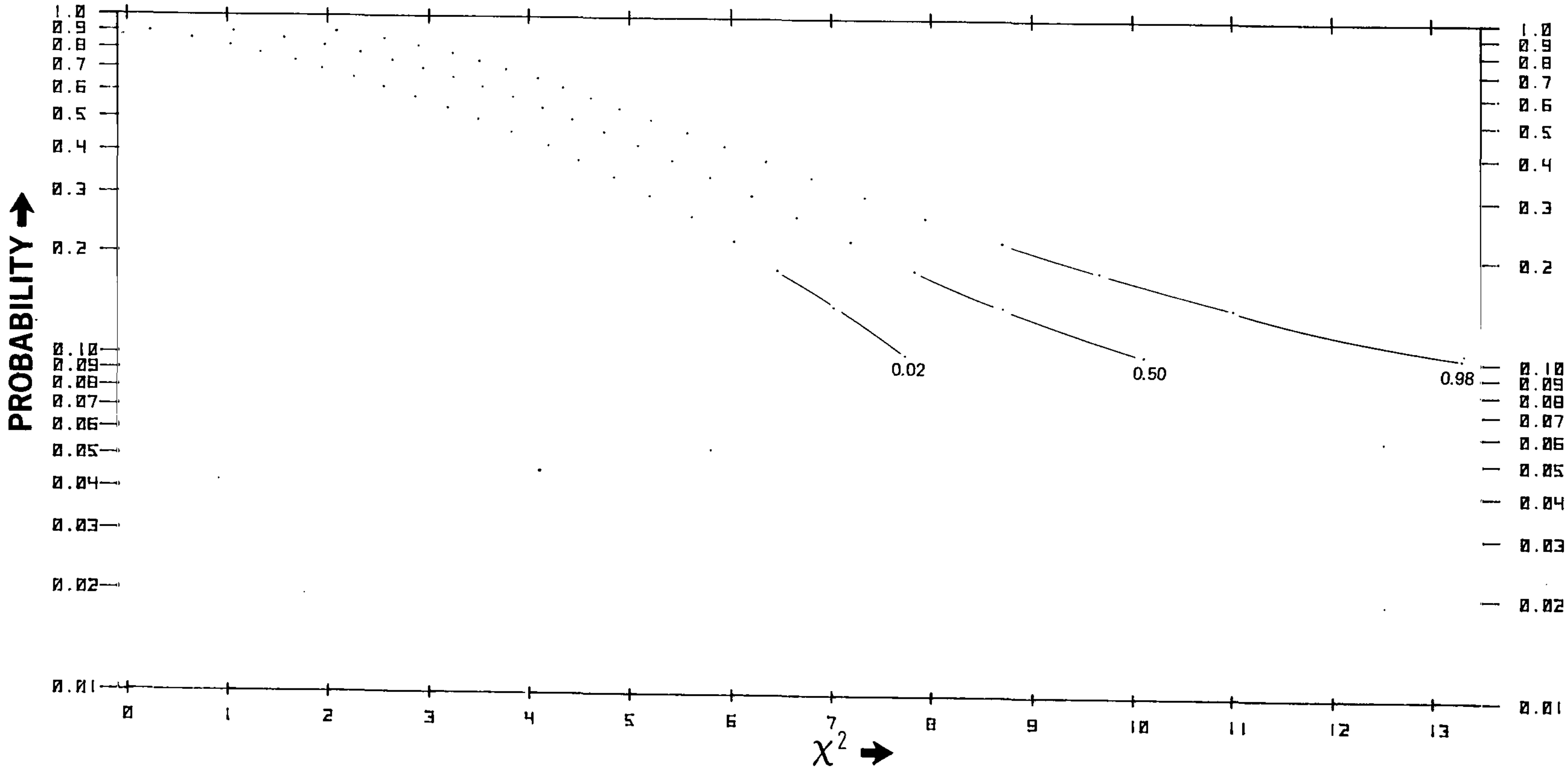

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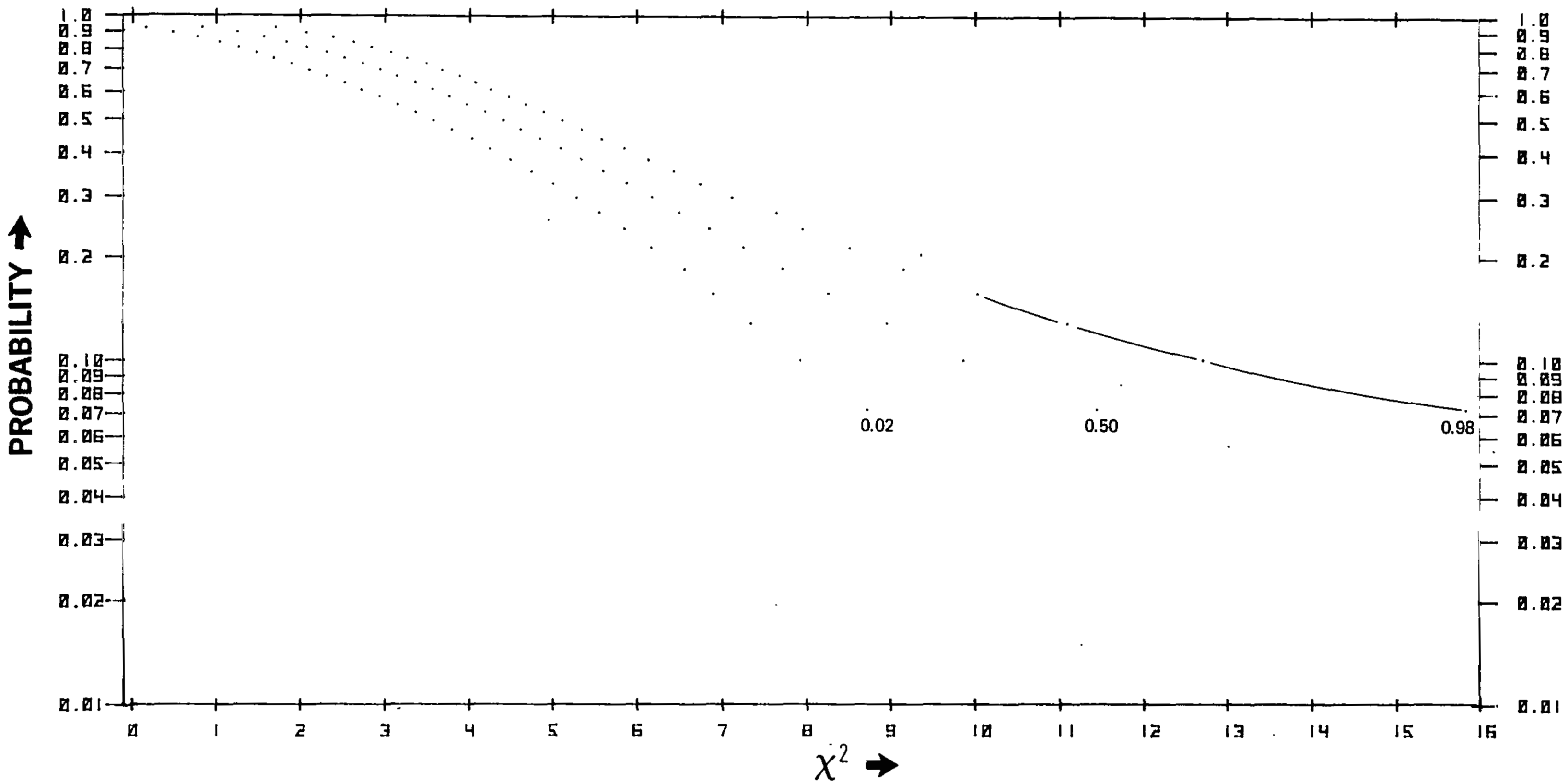

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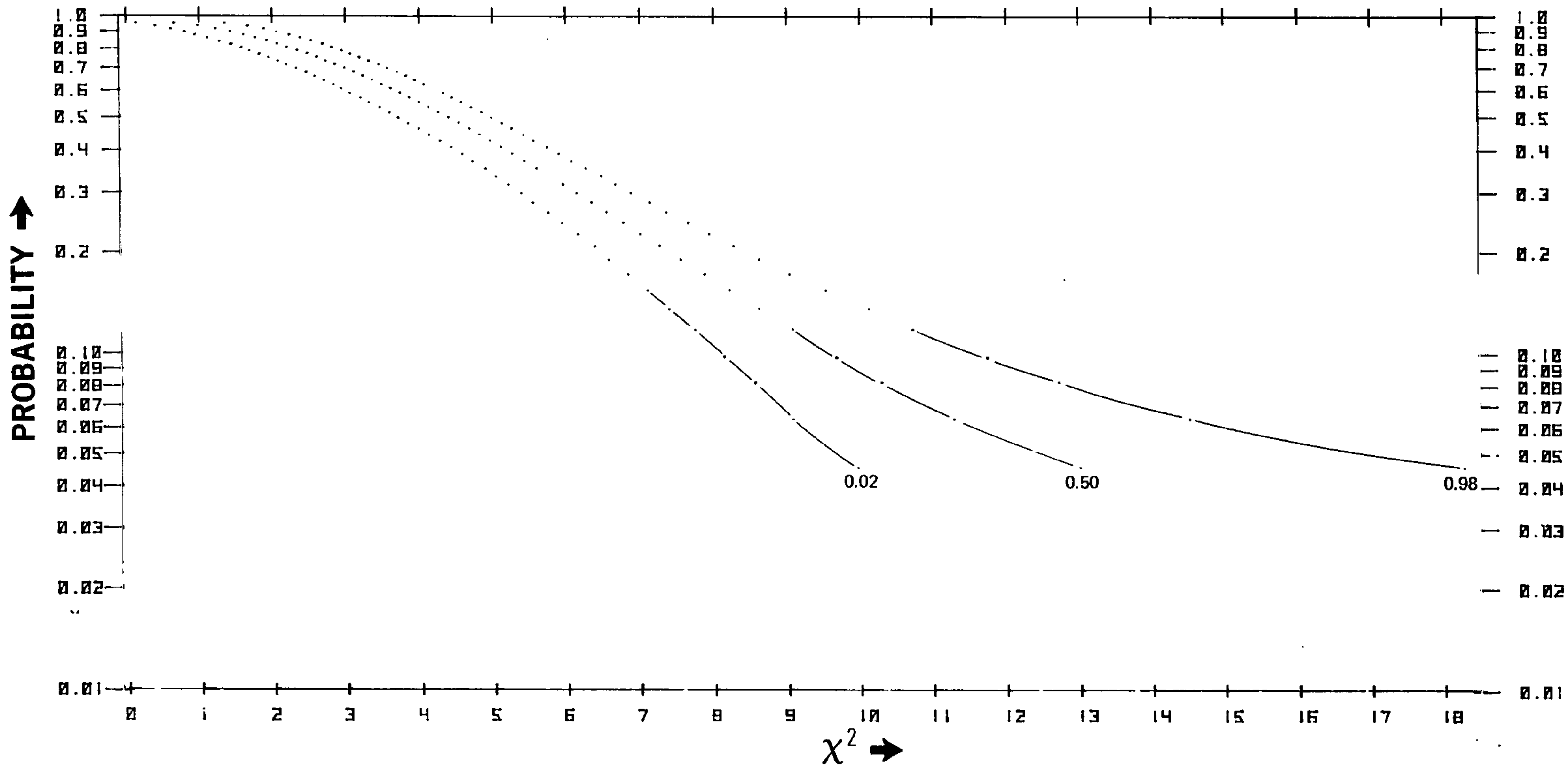

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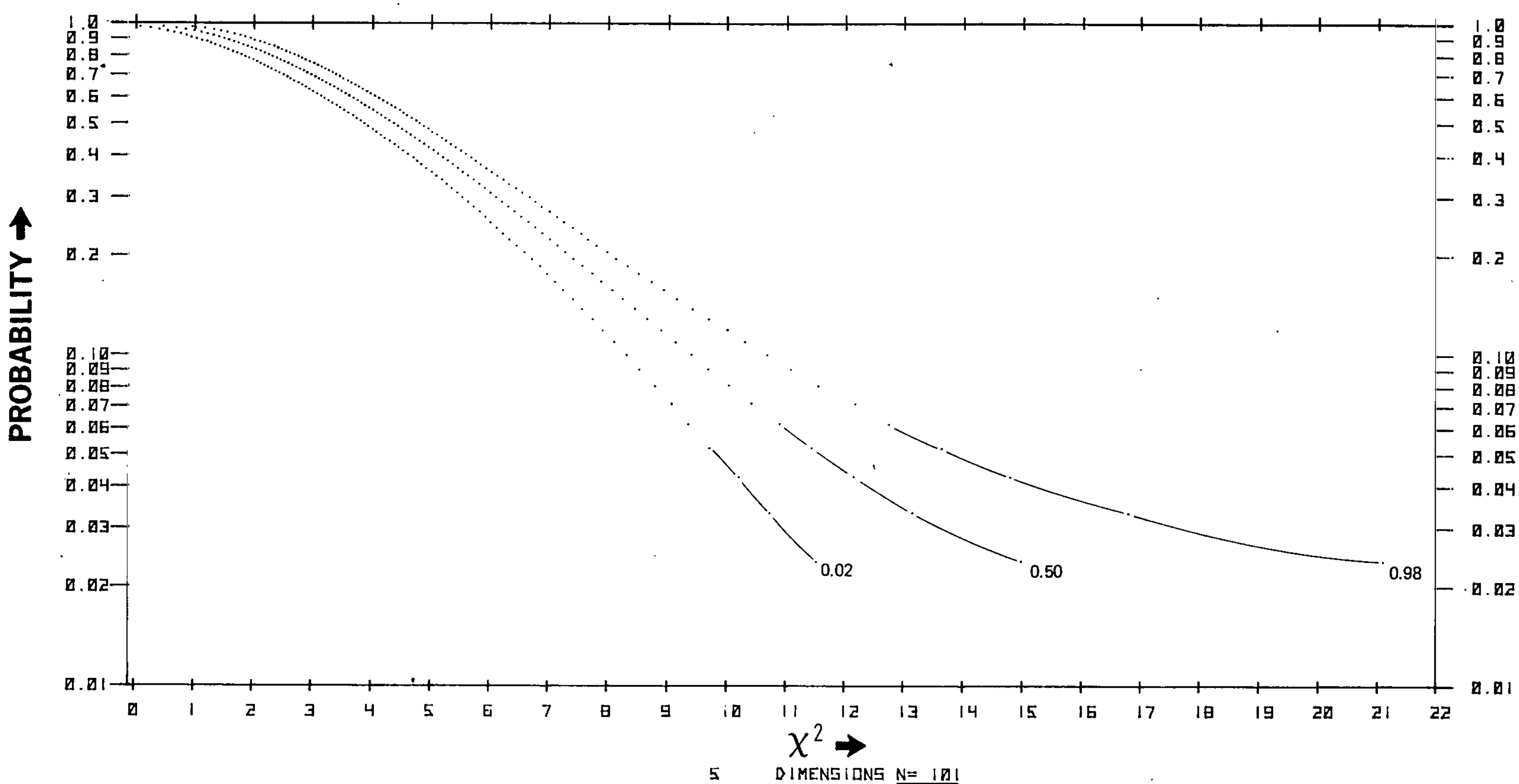

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